1. The Mean Equation

Before the model can analyze volatility, it establishes the expected behavior of the time series (the "average" return). This equation separates the predictable part of the return from the unpredictable shock.

r_t = mu + epsilon_t

Where:

1) **r_t (Return at time t):** actual return of the asset for the current day.
2) **mu(constant mean return):** model estimates this parameter, assuming the asset fluctuates around a stable long-term average.
3) **epsilon_t (Epsilon / Residual):** The "Shock" or error term for the day. This represents the unexpected movement in price that cannot be predicted by the average.

2. The Variance Equation (The GARCH Core)

This equation calculates the conditional variance (volatility) for the current day based on past information. It combines a baseline variance, the "shock" from yesterday, and the "volatility" from yesterday.

sigma_t^2 = omega + alpha *epsilon_{t-1}^2 + beta*sigma_{t-1}^2

Where:

1) **sigma_t^2 (Conditional Variance):** predicted variance for the current day (t). This is the measure of risk the model is trying to solve for.
2) **omega:** baseline or "floor" constant for variance. It ensures the variance never drops below a certain positive level.
3) **epsilon_{t-1}^2 (The ARCH Term / Squared Residual):** it represents the shock from yesterday. It is the squared error from the previous day (t-1).This term measures how the market reacts to new information. A large squared error (a big surprise yesterday) increases the variance today.
4) **sigma_{t-1}^2 (The GARCH Term / Lagged Variance):** it represents the memory of the system. It is the estimated variance from the previous day (t-1). This term measures persistence. If the market was risky yesterday, this term ensures the model predicts it will likely remain risky today.
5) **alpha:** weight assigned to yesterday's shock.
6) **beta :** weight assigned to yesterday's variance.

High ARCH Alpha (alpha > 0.1): The asset is jumpy. It reacts violently to recent news.

Low ARCH Alpha (alpha < 0.05): The asset is stable. It generally ignores daily noise.

3.Parameter estimation through MLE

Our goal is to find out value of alpha,beta,mu,omega such that our probability of returns are maximised. If we assume the Gaussian Distribution of the residuals (zt) our log likelihood function is

$$L(\theta) = -\frac{1}{2} \sum_{t=1}^{T} \left[\ln(2\pi) + \ln(\sigma_t^2) + \frac{(r_t - \mu)^2}{\sigma_t^2} \right]$$

Where:

1) **T :** Number of observation in time series
2) **ln(sigma_t^2):** a penalty term for high variance
3) **(r_t-mu/sigma_t)^2:** standardised residual term