Applications of Convex Optimization in Quantitative Trading and Portfolio Management

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Setting it up

In Quantitative Trading Industry, various firms participate in the markets by making decisions based on mathematical and statistical analysis using proprietary software in order to find and invest in new trends or just to make markets more efficient by sourcing inefficiencies and getting rid of them by providing liquidity.

So there are many trading strategies that you could employ in the financial markets:

- Traditional: buy & hold, hold & rebalance, rank & long/short, momentum/reversion etc.
- Academic/Research-heavy: stochastic control, dynamic programming etc.
- **Optimization-based:** Markowitz Modern Portfolio theory, **Single-Period Optimization** (SPO), **Multi-Period Optimization** (MPO) etc.

Using Convex Optimization to exploit the predictions we have to the fullest!

Optimization-based Trading Preliminaries

Portfolio Definitions

- $\underline{\text{Holdings vector}}\ (\vec{h}_t) = \begin{bmatrix} h_{1t} \\ h_{2t} \\ \vdots \\ h_{nt} \\ h_{(n+1)t} \end{bmatrix}$

(n+1) column vector

$h_{1t}, \dots, h_{nt}$: \$ in each of n-assets

$h_{(n+1)t}$: \$ is in associated cash account

- $\underline{\text{Post trade Holdings}}\ (\vec{h}_t^{+}) = \vec{h}_t + \vec{u}_t$

- $\underline{\text{Total value}}\ (v_t) = \mathbf{1}^T \vec{h}_t$

- $\underline{\text{weights vector}}\ (\vec{\omega}_t) = \dfrac{\vec{h}_t}{v_t}$

- $\underline{\text{trade vector}}\ (\vec{u}_t)$ is (n+1) column vector with i^{th} entry >0 if buy, <0 if sell.

[let $\vec{z}_t = \dfrac{\vec{u}_t}{v_t}$, be normalized trade vector]

Optimization-based Trading Preliminaries...(contd.)

Transaction cost

$$\phi_t^{trade}(\vec{u}_t) : \mathbb{R}^{n+1} \to \mathbb{R}$$

Holding Cost

$$\phi_t^{hold}(\vec{h}_t^{+}) : \mathbb{R}^{n+1} \to \mathbb{R}$$

Self-financing condition

$$1^T \vec{u}_t + \phi_t^{trade}(\vec{u}_t) + \phi_t^{hold}(\vec{h}_t^{+}) = 0$$

(Normalized version)

$$1^T \vec{z}_t + \phi_t^{trade}(\vec{z}_t) + \phi_t^{hold}(\vec{w}_t + \vec{z}_t) = 0$$

OR

$$(u_t)_{n+1} = -\left[1^T (\vec{u}_t)_{1:n} + \phi_t^{trade}((\vec{h}_t + \vec{u}_t)_{1:n}) + \phi_t^{hold}((\vec{u}_t)_{1:n})\right]$$

(Normalized version)

$$(z_t)_{n+1} = -\left[1^T (\vec{z}_t)_{1:n} + \phi_t^{trade}((\vec{w}_t + \vec{z}_t)_{1:n}) + \phi_t^{hold}((\vec{z}_t)_{1:n})\right]$$

Optimization-based Trading Preliminaries...(contd.)

Investment: we will assume that the post trade portfolio is invested until start of next period with r_t return rate vector,

$$h_{t+1} = (1 + r_t) \circ h_t^+ \quad [\text{"}\circ\text{" is element-wise multiplication}]$$

Using self-financing, $V_{t+1} = V_t + r_t^T (h_t + u_t) - \phi_t^{trade}(u_t) - \phi_t^{hold}(h_t^+)$

$$\underline{\underline{\text{Realized return}}} \, (R_t^p) = \frac{V_{t+1} - V_t}{V_t} = \underbrace{r_t^T w_t}_{\substack{\text{return}\\\text{without}\\\text{trades}}} + \underbrace{r_t^T z_t}_{\substack{\text{trade}\\\text{returns}}} - \underbrace{\phi_t^{trade}(z_t)}_{\substack{\text{transaction}\\\text{cost}}} - \underbrace{\phi_t^{hold}(w_t + z_t)}_{\substack{\text{holding}\\\text{cost}}}$$

$\xrightarrow{}$ some tedious algebra

$$\underline{\underline{w_{t+1}}} = \left(\frac{1}{1 + R_t^p}\right) (\vec{1} + \vec{r}_t) \circ (\vec{w}_t + \vec{z}_t) \approx \underline{\underline{\vec{w}_t + \vec{z}_t}}$$

(when $r_t \to 0$ & $R_t^p << 1$)

Aspects ignored: External cash, Dividends, Imperfect execution, Mergers, etc.

Optimization-based Trading Preliminaries...(contd.)

Performance metrics

Absolute

- **Return Rate** $(\bar{R}^p) = \frac{1}{T}\sum_{t=1}^{T} R_t^p$
- **Growth Rate** $(\bar{G}_t^p) = \log\left(\frac{V_{t+1}}{V_t}\right) = \log(1+R_t^p)$
- **Realized Volatility** $(\sigma^p) = \sqrt{\frac{1}{T}\sum_{t=1}^{T} (R_t^p - \bar{R}^p)^2}$ ← risk is $(\sigma^p)^2$

Relative

- **Benchmark weights**: eg $\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \rightarrow$ risk free cash benchmark
- **Active/excess return**: $R_t^a = R_t^p - R_t^b$, $R_t^e = R_t^p - (r_t)_{n+1}$
- **Information/Sharpe Ratio**: $IR = \bar{R}^a / \sigma^a$, $SR = \bar{R}^e / \sigma^e$

Single-Period Trading (SPO)

Risk-return Optimization Strategy

Initially inspired by Markowitz

Forming a Convex-Optimization Problem!

We need to establish...

- Risk Measures
- Forecast Error Risk
- Objective & Constraints

Initial Formulation and Risk Types

Markowitz introduced...

maximize $\left[\left(\hat{R}_t^p - \gamma_t \psi_t (\vec{\omega}_t + \vec{z}_t) \right)\right.$, which by substituting $\hat{R}_t^p$ is...

$$\left. \left(\hat{r}_t^T \vec{z}_t - \hat{\phi}_t^{trade}(\vec{z}_t) - \hat{\phi}_t^{hold}(\vec{\omega}_t + \vec{z}_t) - \gamma_t \psi_t (\vec{\omega}_t + \vec{z}_t) \right) \right]$$

s.t. $\left(\mathbf{1}^T z_t + \phi_t^{trade}(z_t) + \phi_t^{hold}(w_t + z_t) = 0 \right)$ $\boxed{\text{OR}}$ $\left(\mathbf{1}^T z_t = 0 \right)$

where $\psi_t, z_t, (z_t + w_t), \phi_t^{trade}, \phi_t^{hold}$ are $\boxed{\text{Convex or } \in \text{Convex sets}}$

and γ_t is risk-aversion parameter & ψ_t is risk function

Risk Measures

Absolute or Traditional quadratic risk	$\psi_t(x) = x^T \Sigma_t x$ (covariance of stochastic returns)
Active risk	$\psi_t(x) = (x - w_t^b)^T \Sigma_t (x - w_t^b)$

...Other models available based on factors & worst-case

- γ_t (risk-aversion parameter): estimated constant based on maximizing expected portfolio growth rate for example $(= \frac{1}{2})$, by J. Kelly

Forecast Error Risk

(1^{st} & 2^{nd} order moments)

return forecast risk

$$\boxed{\psi_t(x) = \rho^T |x - w_t^b|}$$

where ρ is column vector of uncertainities in our returns forecast

Covariance forecast risk

perturbations covariance matrix (Δ)

$|\Delta_{ij}| \le \kappa (\Sigma_{ii} \Sigma_{jj})^{1/2}$

$\Rightarrow$ using idea of worst case quadratic risk...

$$\boxed{\psi_t(x) = (x - w_t^b)^T \Sigma_t (x - w_t^b) + \kappa \left(\sigma^T |x - w_t^b| \right)^2}$$

where $\sigma = (\Sigma_{11}^{1/2}, \dots, \Sigma_{nn}^{1/2})$

$\hookrightarrow$ asset volatilities

SPO Formulation

$\boxed{\text{Other Constraints}}$: Holding constraints and Trading constraints such as no-hold & no-trade respectively

can be made "<u>soft</u>" by assigning priorities, so they are only violated when they are needed to be.

$\boxed{\text{SPO Model}}$

$$\underline{\underline{\text{maximize}}} \quad \left(\hat{r}_t^T \vec{z}_t - \gamma_t^{trade} \hat{\phi}_t^{trade}(\vec{z}_t) - \gamma_t^{hold} \hat{\phi}_t^{hold}(\vec{w}_t + \vec{z}_t) - \gamma_t^{risk} \psi_t(\vec{w}_t + \vec{z}_t) \right)$$

$$\underline{\underline{\text{s.t.}}} \quad \left(\mathbf{1}^T z_t = 0 \right) \Leftarrow \text{approximated self-financing constraint}$$

where $\psi_t, z_t, (z_t + w_t), \phi_t^{trade}, \phi_t^{hold}$ are $\boxed{\text{Convex or} \in \text{Convex sets}}$
and γ_t^{risk} is risk-aversion parameter & ψ_t is risk function

[Here, γ_t^{trade} & γ_t^{hold} can be treated as hyperparameters
Or, simply, knobs/dials which we will turn through
<u>Backtesting</u> to better model the Data Generating Process]

Multi-Period Trading (MPO)

Motivation behind

SPO → MPO

Initially inspired by Merton

The Greedy Strategy of SPO fails to give most optimal results when current performance depends on previous holdings

We need to establish...

- Current Trades put us in Good/Bad future situations (Approx. Dynamic Programming)
- Considering a set of future time periods, but trade only the first one.
- Simplifying Investment equation and reduce down to a “feasibly” solvable MPO by approximations

Initial Formulation and Simplifications

Trying to solve for z_t over a planning horizon (H) number of future periods.

$\tau = t, t+1, \ldots, t+H-1$.

Notation used: $\hat{(\cdot)}_{\tau|t}$ means estimate of $(\cdot)$ at τ conditional on t time periods information.

Naturally we choose objective function as total risk-adjusted return over the horizon (H)

$$\sum_{\tau=t}^{t+H-1} \left(\hat{r}_{\tau|t}^{T} (w_\tau + z_\tau) - \gamma_\tau \psi_\tau (w_\tau + z_\tau) - \hat{\phi}_\tau^{hold} (w_\tau + z_\tau) - \hat{\phi}_\tau^{trade} (z_\tau) \right)$$

As mentioned previously

$$\left[\underline{\underline{w_{t+1}}} = \left(\frac{1}{1+R_t^p} \right) (\vec{1} + \vec{r}_t) \circ (\vec{w}_t + \vec{z}_t) \underset{\substack{\text{when } r_t \to 0 \\ \& \ R_t^p \ll 1}}{\approx} \underline{\underline{\vec{w}_t + \vec{z}_t}} \right] \rightarrow$$

we can get this objective solely in terms of w_τ's

Also, we will use the approximated self-financing constraint ($\mathbf{1}^T w_\tau = 1$), Getting rid of costs here doesn't undermine the purpose of MPO because we are just ignoring costs while propagating the portfolio.

MPO Formulation

MPO Model

$$\underline{\underline{\text{maximize}}} \quad \sum_{\tau=t+1}^{t+H} \left(\hat{r}_{\tau|t}^{T} \omega_\tau - \gamma_\tau \psi_\tau \omega_\tau - \hat{\phi}_\tau^{hold} \omega_\tau - \hat{\phi}_\tau^{trade} (\omega_\tau - \omega_{\tau-1}) \right)$$

$\underline{\underline{\text{s.t.}}}$ $\left(\mathbf{1}^T \omega_\tau = 1\right) \Leftarrow$ approximated self-financing constraint

where $\psi_\tau, \omega_\tau, (\omega_\tau - \omega_{\tau-1}), \phi_\tau^{trade}, \phi_\tau^{hold}$ are $\boxed{\text{Convex or} \in \text{Convex sets}}$

and γ_τ is risk-aversion parameter & ψ_τ is risk function

here $\tau = t+1, \dots, t+H$, where H is planning horizon over which we are considering the future impact of our trade.

Further, we can add <u>terminal constraints</u> such as $\omega_{t+H} = \omega^{term} = \omega^{b}_{t+H}$

If, for example, $\omega^{term} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \Rightarrow$ our optimization problem will give optimal solution which will have us in best spot considering our liquidation completely at time $t+H$.

Will help us stay away from investment decisions that look like positions of high-returns, but are expensive to get out of.

Further, we can do multi-scale optimization if we restrict the ability to trade to only certain periods over the bigger horizon.

Simulations Review & Experiments

Trading decisions made by MPO were of higher return and lower risk than SPO, although MPO is more computationally expensive.

So, it's a win-win, but there is computational trade-off!

Simulation and Experiments

Software and API Support used

- **CVXPortfolio** Python Package for object-oriented implementations of functions related to mentioned model.

  It serves as an add-on and is written by the authors of the original paper.

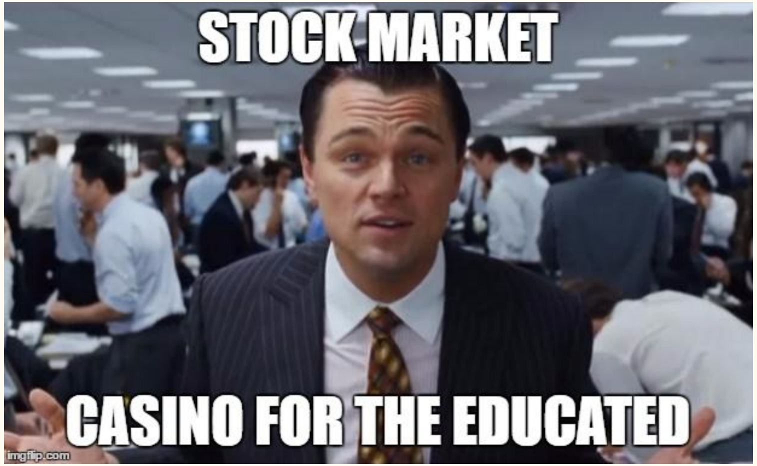
- **Quandl** Python package to get stock ticker's price data in Pandas friendly format.

Further work

- How long-term and short-term optimization based trading strategies differ at their core?
- Can we come up with better approximations of Dynamic Programming in terms of algorithmic complexity and running-time?

Thank you for your attention!

Bonus (Meme):



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