MATH-365 Stochastic Processes Final Project

Black Scholes Equation and Formula for Options Pricing:

Application of Exponential Brownian Motion and Ito's Calculus

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Options

Definitions

- **Derivative:** A financial asset that derives its value from another financial asset
- **Option:** A derivative that provides the opportunity, but not the obligation to buy or sell an asset at a predetermined **(Strike)** price in the future.
- We will be dealing with the options that have specified future dates of expiry called **European Options.**

Types

- For a **call option**, if the market price rises above the strike price, the investor will be willing to buy.
- For a **put option**, if the market price falls below the strike price, the investor will want to sell the underlying asset.

Notations

- Assuming the stock price is $S(t)$
- Let's say option Price $O(t) = f(t,S(t))$
- Also, $dS = AS(t)dt + BS(t)dw(t)$

Risk Hedging Portfolio and Deriving Black Scholes Equation

By Ito's lemma:

$$dO = [f_t + ASf_s + \frac{B^2S^2f_{SS}}{2}]dt + [BSf_S]dw(t) = [f_t + \frac{B^2S^2f_{SS}}{2}]dt + f_S dS$$

$$\Downarrow$$

$$dP = N_1 dS + N_2 dO = N_1 dS + N_2[f_t + \frac{B^2S^2f_{SS}}{2}]dt + N_2 f_S dS$$

Risk Hedging Portfolio and Deriving Black Scholes Equation

$$\frac{dP}{P} = \frac{N_2[f_t + \frac{B^2 S^2 f_{SS}}{2}]dt}{N_1 S + N_2} = R_b dt$$

$$\Downarrow$$

$$f_t + \frac{B^2 S^2 f_{SS}}{2} + R_b f_S S = R_b f$$

Risk Hedging Portfolio and Deriving Black Scholes Equation

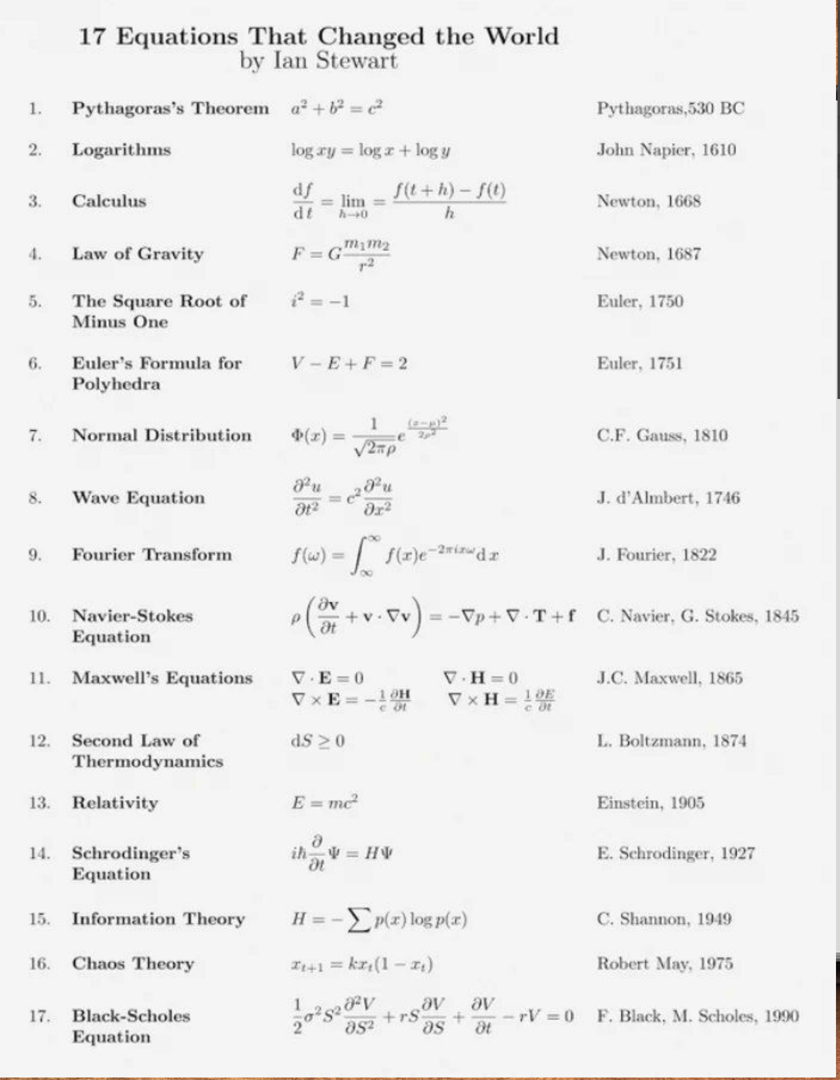
Definitions

- Considering an investor who holds a **portfolio of stock and its option**: P(t) = N1(t)S(t) + N2(t)O(t)
- *Mallaris* makes a clever argument that holding a ratio of N1/N2 in portfolio equal to -df/dS, this is called **delta-hedging.**
- With above-mentioned ratio, the **rate of return of our portfolio** turns out to be **equal** to the **rate of return of riskless bonds.**

Equations

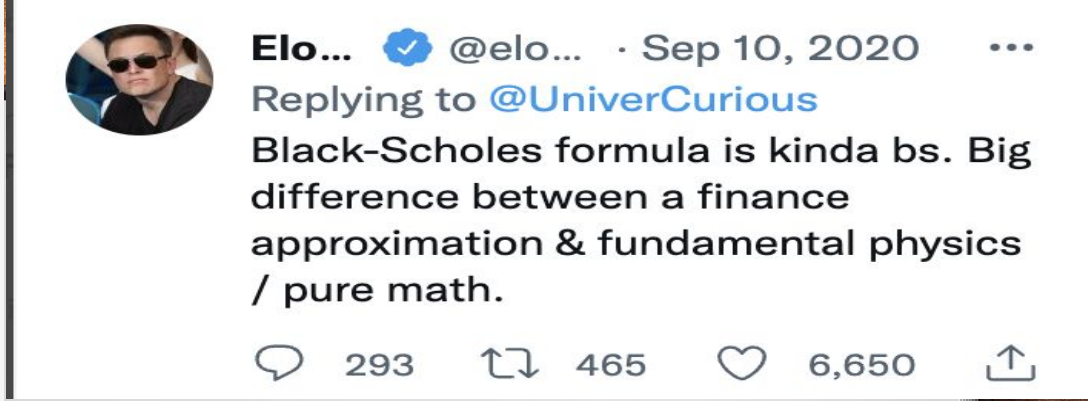
- dP/P = (Rb)dt
- This gives us the **Black Scholes Equation** for Options Pricing:

$$f_t + \frac{B^2 S^2 f_{SS}}{2} + R_b f_S S = R_b f$$



17 Equations That Changed the World
by Ian Stewart

1.	**Pythagoras's Theorem**	$a^2 + b^2 = c^2$	Pythagoras,530 BC
2.	**Logarithms**	$\log xy = \log x + \log y$	John Napier, 1610
3.	**Calculus**	$\frac{df}{dt} = \lim_{h \to 0} = \frac{f(t+h) - f(t)}{h}$	Newton, 1668
4.	**Law of Gravity**	$F = G\frac{m_1 m_2}{r^2}$	Newton, 1687
5.	**The Square Root of Minus One**	$i^2 = -1$	Euler, 1750
6.	**Euler's Formula for Polyhedra**	$V - E + F = 2$	Euler, 1751
7.	**Normal Distribution**	$\Phi(x) = \frac{1}{\sqrt{2\pi\rho}} e^{\frac{(x-\mu)^2}{2\rho^2}}$	C.F. Gauss, 1810
8.	**Wave Equation**	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$	J. d'Almbert, 1746
9.	**Fourier Transform**	$f(\omega) = \int_{\infty}^{\infty} f(x) e^{-2\pi i x \omega} dx$	J. Fourier, 1822
10.	**Navier-Stokes Equation**	$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \nabla \cdot \mathbf{T} + \mathbf{f}$	C. Navier, G. Stokes, 1845
11.	**Maxwell's Equations**	$\nabla \cdot \mathbf{E} = 0$ $\quad \nabla \cdot \mathbf{H} = 0$ $\quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$ $\quad \nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial E}{\partial t}$	J.C. Maxwell, 1865
12.	**Second Law of Thermodynamics**	$dS \geq 0$	L. Boltzmann, 1874
13.	**Relativity**	$E = mc^2$	Einstein, 1905
14.	**Schrodinger's Equation**	$i\hbar \frac{\partial}{\partial t} \Psi = H \Psi$	E. Schrodinger, 1927
15.	**Information Theory**	$H = -\sum p(x) \log p(x)$	C. Shannon, 1949
16.	**Chaos Theory**	$x_{t+1} = kx_t(1 - x_t)$	Robert May, 1975
17.	**Black-Scholes Equation**	$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - rV = 0$	F. Black, M. Scholes, 1990



Black Scholes Equation vs Formula

Image(Left) Credit: *Ian Stewart, 17 Equations That Changed The World (book)*
Image(Right) Credit: *Elon Musk, Twitter.com (website)*

Black Scholes Formula

Mathematical view:

- The formula:

$$C_0 = S_0\Phi(d_1) - Xe^{-rT}\Phi(d_2)$$

- where,

$$d_1 = \frac{ln(\frac{S_0}{X}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, d_2 = \frac{ln(\frac{S_0}{X}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

Notation:

- C_0 : call option price
- S_0 : underlying stock price
- X : exercise price
- T : time of expiry
- σ : standard deviation of log returns
- r : risk-free interest rate
- Φ(x) : represents cumulative distribution function for the normally distributed random variable x

Deriving Black Scholes Formula using Ito's Calculus

$$C_0 = e^{-rT} E^Q[(S_0 - X)^+ | F_0]$$

$$\Downarrow$$

$$C_0 = e^{-rT} \int_X^{\infty} (S_0 - X) dF(S_0)$$

$$C_0 = e^{-rT} \int_X^{\infty} S_0 dF(S_0) - e^{-rT} X \int_X^{\infty} dF(S_0)$$

Deriving Black Scholes Formula using Ito's Calculus

Using the log-normal distribution,

$$E[X|X > x] = e^{\mu + \frac{\sigma^2}{2}} \Phi(\frac{-lnK + \mu + \sigma^2}{\sigma})$$

$$C_0 = e^{-rT} \int_X^{\infty} S_0 dF(S_0) - e^{-rT} X \int_X^{\infty} dF(S_0)$$

$$\Downarrow$$

$$C_0 = S_0 \Phi(d_1) - e^{-rT} X[1 - F(X)]$$

$$\Downarrow$$

$$C_0 = S_0 \Phi(d_1) - e^{-rT} X \Phi(d_2)$$

Applying the Black Scholes Formula

An Example of Using Black Scholes Formula in real-life scenario

Let's try to find a price of an European call option with stock price of $90, time of expiration being 6 months, risk-free interest rate is 8%, standard deviation of stock is 23%, exercise price is $80.

Hence, $S_0 = 90$, $T = 0.5$, $r = 0.08$, $\sigma = 0.23$, and $X = 80$, plug in those values in the Black-Scholes formula to get

$$C_0 = 90 * \Phi(d_1) - 80 * e^{-0.08*0.5} * \Phi(d_2)$$

where,

$$d_1 = \frac{ln(\frac{90}{80}) + (0.08 + \frac{0.23^2}{2})0.5}{0.23\sqrt{0.5}} = 1.0515$$

$$d_2 = \frac{ln(\frac{90}{80}) + (0.08 - \frac{0.23^2}{2})0.5}{0.23\sqrt{0.5}} = 0.8889$$

Using standard normal distribution we can get the following values:

$$\Phi(d_1) = 0.8535 \Phi(d_2) = 0.813$$

Therefore, the value of the option C_0 can be calculated as follows:

$$C_0 = 90 * 0.8535 - 80 * e^{-0.08*0.5} * 0.813 = 14.33$$

An interesting thing to ponder about the results generated by the Black Scholes formula...

Option buyers pay prices consistently higher than those predicted by the formula.

Let’s think about the reason behind such a discrepancy.

- In the real market, **real interest rates are not constant** as assumed in Black-Scholes model.
- Most **stocks pay some form of distributions including dividends.**

Due to such factors, **volatility (σ) in Black-Scholes formula may be underestimated.**

Since the price of an option (C_0) is proportional to volatility (σ), such a difference in volatility could be one of the reasons for underestimation of option prices.

Estimating Volatility...

After requesting student connection from **Nasdaq Data Link,** I was able to gain access to financial data in R through the Quandl package

This allowed me to access and play around with the data to get a better sense of how we can estimate volatility at a given time. I am still working on this!!

The market-making firms like Optiver, Akuna Capital and Jane Street have dedicated option-traders and researchers who trade options and develop complex strategies to make huge profits through making better approximations of volatility in market.

Thanks!

References are at the end of the write-up!

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Feel free to contact me! I will be happy to share my project write-up after the end of the semester.