The Local Volatility Model Continued …

Rules of Thumb for Implied Volatilities
Values of Hedge Ratios
Values of Exotic Options

Local vol and its consequences;
homework on local vol

(Weighted variance swaps later)
(Programming local volatility trees)

Stochastic vol …

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17.1 An Exact Relationship Between Local and Implied Volatilities and Its Consequences

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For zero interest rates and dividend yields, we derived $\sigma^2(K, T) = \left(2\frac{\partial C}{\partial T}\Big|_K\right) \Big/ \left(K^2 \frac{\partial^2 C}{\partial K^2}\Big|_T\right)$

Quoting in terms of BS implied vols: $C(S, t, K, T) = C_{BS}(S, t, K, T, \Sigma(S, t, K, T))$

By carefully using the chain rule for differentiation and the formulas for the Black-Scholes Greeks:

$$\sigma^2(K, \tau) = \frac{2\frac{\partial \Sigma}{\partial \tau} + \frac{\Sigma}{\tau}}{K^2\left[\frac{\partial^2 \Sigma}{\partial K^2} - d_1\sqrt{\tau}\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \frac{1}{\Sigma}\left(\frac{1}{K\sqrt{\tau}} + d_1\frac{\partial \Sigma}{\partial K}\right)^2\right]}$$

where $d_1 = \frac{\ln(S/K)}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2}$, and $\Sigma = \Sigma(S, t, K, T)$ is a function of S, t, K, T.

This formula is the generalization of the notion of forward volatilities in a no-skew world to local volatilities in a skewed world.

We can now prove rigorously the previous relations we intuited between implied volatility and local volatility.

17.1.1 Implied variance is average of local variance over life of the option if there is no skew.

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$\Sigma(S, t, K, T)$ is independent of strike K, $\frac{\partial\Sigma}{\partial K} = 0$ with no skew at all. Then, writing $\tau = T - t$

$$\frac{1}{2}\sigma^2(K, T) = \frac{\frac{\partial\Sigma}{\partial\tau} + \frac{\Sigma}{2\tau}}{K^2\frac{1}{\Sigma}\left\{\frac{1}{K\sqrt{\tau}}\right\}^2} = \tau\Sigma\frac{\partial\Sigma}{\partial\tau} + \frac{\Sigma^2}{2}$$

$$\tau\Sigma^2(\tau) = \int_0^\tau \sigma^2(u)du$$

the standard result that expresses the total variance as an average of forward variances.

17.1.2 The Rule of Two: Near the money, for flat term structure, the slope of the skew w.r.t strike is 1/2 the slope of the local volatility w.r.t. spot, *for weak skew close to atm*

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$\Sigma = \Sigma(K)$ alone, independent of expiration, and $\dfrac{\partial\Sigma}{\partial\tau} = 0$, weak linear skew, close to atm, small volatility. Then:

$$\sigma(K) = \frac{\Sigma(K)}{1 + d_1 K\sqrt{\tau}\dfrac{\partial\Sigma}{\partial K}}$$

Close to at-the-money, $K = S + \Delta K$. Then approximately

$$\sigma(S) + \frac{\partial\sigma(S)}{\partial S}\Delta K \approx \Sigma(S) + 2\frac{\partial\Sigma(S)}{\partial S}\Delta K \quad \text{and so} \quad \frac{\partial}{\partial S}\sigma(S) \approx 2\left(\frac{\partial\Sigma}{\partial K}\right)\Bigg|_{K=S}$$

The local volatility $\sigma(S)$ grows twice as fast with stock price *S* as the implied volatility $\Sigma(K)$ grows with strike!

17.1.3 Implied volatility is an harmonic average over local volatility *at short expirations*.

For zero rates and dividends:
$$\sigma^2(K,T) = \frac{2\frac{\partial\Sigma}{\partial\tau}+\frac{\Sigma}{\tau}}{K^2\left(\frac{\partial^2\Sigma}{\partial K^2} - d_1\sqrt{\tau}\left(\frac{\partial\Sigma}{\partial K}\right)^2 + \frac{1}{\Sigma}\left\{\frac{1}{K\sqrt{\tau}} + d_1\frac{\partial\Sigma}{\partial K}\right\}^2\right)}$$

Multiplying top and bottom by τ:
$$\sigma^2(K,T) = \frac{\cancel{2\tau\frac{\partial\Sigma}{\partial\tau}}+\Sigma}{K^2\left(\cancel{\tau\frac{\partial^2}{\partial K^2}}(\Sigma) - \cancel{d_1\tau\sqrt{\tau}\left(\frac{\partial\Sigma}{\partial K}\right)^2} + \frac{1}{\Sigma}\left\{\frac{1}{K} + \sqrt{\tau}d_1\frac{\partial\Sigma}{\partial K}\right\}^2\right)}$$

A s $\tau \to 0$, this becomes the o.d.e.
$$\sigma^2(K,T) = \frac{\Sigma}{K^2\left(\frac{1}{\Sigma}\left\{\frac{1}{K} + \sqrt{\tau}d_1\frac{d\Sigma}{dK}\right\}^2\right)} = \frac{\Sigma^2}{\left\{1 + \sqrt{\tau}Kd_1\frac{d\Sigma}{dK}\right\}^2}$$

Now $\sqrt{\tau}Kd_1 \to \frac{K\ln(S/K)}{\Sigma}$ as $\tau \to 0$, and we obtain $\sigma(K) = \frac{\Sigma}{1 + \frac{K}{\Sigma}\frac{d\Sigma}{dK}\ln(S/K)}$ as a function of K for fixed S.

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Transforming from K into the new variable $x = \ln(K/S)$ we can rewrite this as the o.d.e.

$$\frac{\Sigma}{1-\frac{x}{\Sigma}\frac{d\Sigma}{dx}} = \sigma(x) \qquad \text{Regard } \sigma \text{ as function of } x$$

We can solve this as follows. Define $L = \frac{1}{\Sigma}$. Then $\frac{d\Sigma}{dx} = -\frac{1}{L^2}\frac{dL}{dx}$ and so you can write the equation

$$\frac{1}{L\left[1+\frac{x}{L}\frac{dL}{dx}\right]} = \sigma(x)$$

or

$$L + x\frac{dL}{dx} = \frac{1}{\sigma(x)} \qquad \text{Eq.17.1}$$

which is relatively simple. In fact you can rewrite this as

$$\frac{d}{dx}[Lx] = \frac{1}{\sigma(x)}$$

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Integrating from $x = 0$, i.e. $K = S$, to $x = \ln\frac{K}{S}$ we obtain

$$\left(\ln\frac{K}{S}\right)L(S, K) = \int_{o}^{\ln\frac{K}{S}} \frac{1}{\sigma(x)}dx \qquad \text{Eq.17.2}$$

So

$$\frac{\ln\left(\frac{K}{S}\right)}{\Sigma\left(\frac{K}{S}\right)} = \int_0^{\ln\left(\frac{K}{S}\right)} \frac{1}{\sigma(x)}dx$$

$$\frac{1}{\Sigma\left(\frac{K}{S}\right)} = \frac{1}{\left(\ln\frac{K}{S}\right)} \int_{o}^{\ln\frac{K}{S}} \frac{1}{\sigma(x)}dx$$

In other words, at very short times to expiration, the implied volatility is the harmonic mean of the local volatility as a function of $\ln K/S$ between spot and strike.

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This is intuitively reasonable, more sensible than an arithmetic mean.

Suppose that $\sigma(y)$ falls to zero above a certain level K, so that the stock price can never diffuse higher. Then the implied volatility of any option with a strike above that level should be zero.

If $\Sigma(x) = \frac{1}{x}\int_{o}^{x}\sigma(y)dy$, an ordinary arithmetic mean, then its value would be non-zero, which is impossible if the stock can never reach the strike.

In contrast, for the harmonic mean, if $\sigma(y)$ becomes zero anywhere in the range between spot and strike, then the implied volatility for that strike, $\Sigma(x)$, becomes zero too, which is as to be expected.

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There is an intuitive way to understand this by thinking of volatility as the speed of log diffusion.

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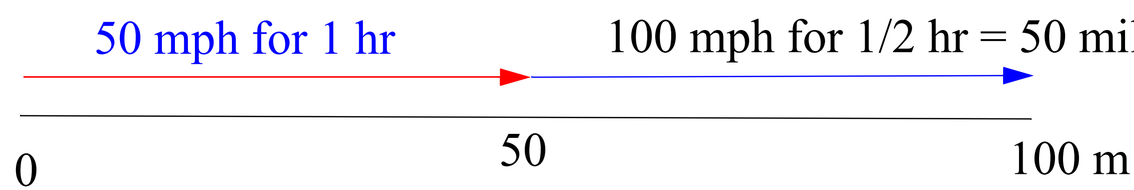
Think of a car with a speed $v(x)$ that varies locally with position x. $v(x)$ is a local velocity.

Specifically, say a car travels at 50 mph for the first 50 miles, and 100 mph for the second 50 miles to cover 100 miles in total.



Total distance = 100m. Total time = 1.5 hrs. Average velocity is not 75 mph because the car spends less time traveling at 100 mph to cover the distance. The average velocity is 100m/1.5 hrs = 66.7.

Average velocity is *not* average of local velocities because high velocity means less time spent.

The **average velocity *V* is the total distance *D* divided by the total time *T***.

Times are additive. The **total time** *T* for the trip is the sum of the local times = dist/speed

$$T = \int_0^D \frac{dx}{v(x)} \equiv \frac{D}{V} \quad \text{by definition}$$

$$\frac{1}{V} = \frac{1}{D}\int_0^D \frac{dx}{v(x)}$$

Average velocity is the harmonic average of the local velocity!

That was for linear motion, not diffusion.

Now think about diffusion. Think of the stock as diffusing through as a medium with a volatility that is analogous to speed. The greater the volatility, the faster and wider the diffusion. (Brownian motion). If the stock's volatility were infinite, the medium would immediately be transparent to diffusion.

σ has the dimension $time^{-1/2}$. $1/\sigma$ has the dimensions of sqrt(time) taken for *dlnS* to diffuse at each stock price *S*.

In velocity, it's the times that are additive. In diffusion it's the square roots of time that effectively are additive.

Think of $\int_{\ln S}^{\ln K} 1/\sigma$ as the sum of the *square roots* of the total diffusion time from ln(S) to ln(K), because it's square roots that matter in Brownian motion.

Let $x = \ln K/S$. Then $\int_{o}^{x} \frac{1}{\sigma(y)} dy$ is *roughly* the total $\sqrt{\text{diffusion time}}$ computed from the sum of local $\sqrt{\text{diffusion times}}$ under lognormal diffusion.

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Define **Average Volatility** $\Sigma(S, K)$ by the relation between total $\sqrt{\text{diffusion time}}$ and total log distance $\ln K/S$:

Total $\sqrt{\text{diffusion time}}$ $= \dfrac{\text{total log distance}}{average\Sigma} = \dfrac{\ln K/S}{\Sigma} \equiv \dfrac{x}{\Sigma(x)}$ - this is definition of the average volatility $\Sigma(S, K)$

So total $\sqrt{\text{diffusion time}}$ is the integral of the local $\sqrt{\text{diffusion time}}$: $\int_o^x \frac{1}{\sigma(y)} dy$, and this must equal the total ln( ) distance divided by the average volatility.

$\dfrac{x}{\Sigma(x)} = \displaystyle\int_o^x \frac{1}{\sigma(y)} dy$ The average volatility is found from the total time and total distance.

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17.2 Why Does the Implied Vol as the Average of Local Vols from Spot to Strike Work Quite Well? More Rigorously: Spot-to-Strike is the Path With The Highest Gamma, Most P&L From Hedging. ...

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Assume zero rates for simplicity. Value an option with strike K, expiration T.

Black-Scholes PDE for price $C_\sigma(S, t, K, T)$ with variable volatility is $dS = S\sigma(S, t)dZ$ for the actual evolution of the stock:

$$\frac{\partial C_\sigma}{\partial t} + \frac{1}{2}\frac{\partial^2 C_\sigma}{\partial S^2}S^2\sigma^2(S, t) = 0 \qquad \text{The BS PDE for the call price with variable volatility}$$

Let this solution value **be quoted** in terms of the Black-Scholes implied volatility Σ_{KT}, different for different strikes *K, T*, so that

$$C_\sigma(S, t, K, T) \equiv C_{\Sigma_{KT}}(S, t, K, T) \quad \text{using Black-Scholes solution}$$

$C_\Sigma(S, t, K, T)$ satisfies the BS equation for each strike, by definition:

$$\frac{\partial C_{\Sigma_{KT}}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{\Sigma_{KT}}}{\partial S^2}S^2\Sigma_{KT}^2 = 0 \qquad \text{BS Equation, different for each strike} \qquad \text{Eq 17.1}$$

Now use Ito to figure out how $C_{\Sigma_{K,T}}$ varies as the stock price does the actual local vol evolution:

$$dC_{\Sigma} = \frac{\partial C_{\Sigma}}{\partial t}dt + \frac{\partial C_{\Sigma}}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C_{\Sigma}}{\partial S^2}S^2\sigma^2(S,t)dt = \frac{\partial C_{\Sigma_{KT}}}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C_{\Sigma_{KT}}}{\partial S^2}S^2\left[\sigma^2(S,t) - \Sigma_{KT}^2\right]dt \quad \text{Eq 17.2}$$

because from Equations 17.1 $\frac{\partial C_{\Sigma_{KT}}}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Sigma_{K,T}}{\partial S^2}S^2\Sigma^2_{KT}$

Now take expected value E_{σ} of both sides of Equations 17.2 over the distribution of S given by local vols $dS = S\sigma(S,t)dZ$, i.e. over the realized paths, where remember that Σ means Σ_{KT}:

$$E_{\sigma}[dC_{\Sigma}] = E_{\sigma}\left\{\cancel{\frac{\partial C_{\Sigma}}{\partial S}dS} + \frac{1}{2}\frac{\partial^2 C_{\Sigma}}{\partial S^2}S^2[\sigma^2(S,t) - \Sigma^2]dt\right\} \text{ where } E_{\sigma}[dS] = 0$$

Now integrate from 0 to expiration T:

$$E_{\sigma}[C_{\Sigma}(S_T,T) - C_{\Sigma}(S_0,0)] = E_{\sigma}[C_{\Sigma}(S_T,T)] - \underbrace{C_{\Sigma}(S_0,0)}_{\text{known}} = E_{\sigma}\left\{\int_0^T \frac{1}{2}\frac{\partial^2 C_{\Sigma}}{\partial S^2}S^2[\sigma^2(S,t) - \Sigma^2]dt\right\}$$

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The LHS is zero because the value at expiration of $C_{\Sigma}(S_T, T)$ is independent of volatility, and by definition of Σ the expected value of the payoffs equals the current Black-Scholes value $C_{\Sigma}(S_0, 0)$.

$$E_{\sigma}\left\{\int_0^T \frac{\partial^2 C_{\Sigma}}{\partial S^2} S^2 [\sigma^2(S, t) - \Sigma^2] dt\right\} = 0$$

So

$$\Sigma^2 = \left(E_{\sigma}\left\{\int_0^T \frac{\partial^2 C_{\Sigma}}{\partial S^2} S^2 [\sigma^2(S, t)] dt\right\}\right) \Big/ \left(E_{\sigma}\left\{\int_0^T \frac{\partial^2 C_{\Sigma}}{\partial S^2} S^2 dt\right\}\right)$$

So we see that the implied vol squared, $\Sigma^2{}_{KT}$, is average of local vol squared over all actual paths of the stock, weighted by **implied** vol gamma x S^2. This is an implicit equation because the LHS and the RHS both involve $\Sigma^2{}_{K,\, T}$.

This gives insight but doesn't make computation easy because of the implicitness.

What is the distribution of the weight $\frac{\partial^2 C_{\Sigma}}{\partial S^2} S^2$ under the evolution of S and σ?

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Graphing The Gamma Weighting Function

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Assume zero rates. Value an option with strike K, expiration T. We showed

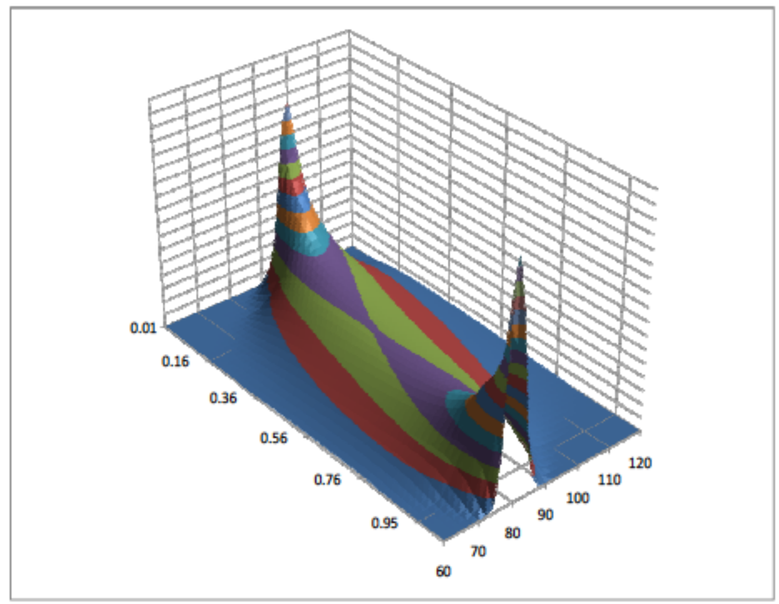
$$\Sigma^2 = \left(E_\sigma \left\{ \int_0^T \frac{\partial^2 C_\Sigma}{\partial S^2} S^2 [\sigma^2(S,t)] dt \right\} \right) \Big/ \left(E_\sigma \left\{ \int_0^T \frac{\partial^2 C_\Sigma}{\partial S^2} S^2 dt \right\} \right)$$

Implied vol squared is average of local vol squared weighted by implied vol gamma. $dS = S\sigma(S,t)dZ$ describes the evolution of local vol in E_σ.

Where does this peak? The Brownian evolution in E_σ starts out from S_0 with probability distribution function:

$$\frac{\exp\left(-\frac{\left(\frac{\ln S_t/S_0 - (rt - \frac{1}{2}\sigma^2 t)}{\sigma\sqrt{t}}\right)^2}{2} \right)}{\sqrt{2\pi t}\sigma S_t}$$

and is a Dirac delta function at $t = 0$ at S_0 and gets wider at later times.



Gamma plotted over distribution of lognormal paths for the stock price

FIGURE 1. Graph of $q_\sigma(t,f)$ when the forward is lognormal; $\sigma = 30\%$, $f_0 = 100$, $K = 80$, $T = 1$.

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The Gamma in the integral for the implied volatility is

$$\frac{\partial^2 C_\Sigma}{\partial S^2} = \frac{\exp\left(-\frac{d_1^2}{2}\right)}{S\Sigma\sqrt{2\pi\tau}}$$

$$d_1 = \frac{\ln S/K}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2}$$

which peaks and is singular at $S = K$ as the time to expiration $\tau \to 0$, and gets broader at earlier times.

So path from S to K is the dominant one for determining the implied volatility.

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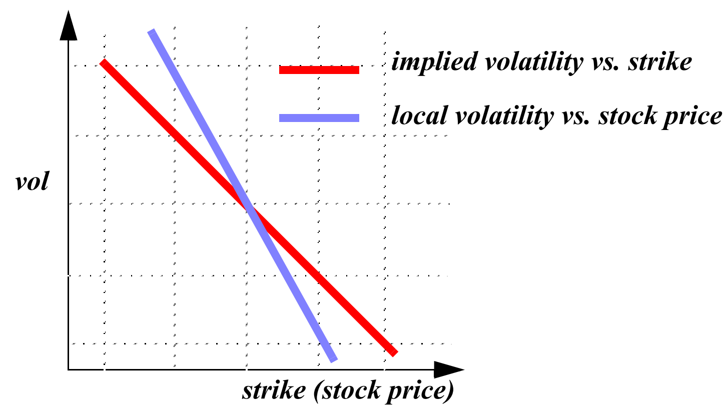
17.3 Hedge Ratios in Local Volatility Models

We observe $\frac{\partial \Sigma}{\partial K}$. This is a static quantity. We'd like to know $\frac{\partial \Sigma}{\partial S}$, predicting the future. The connection between statics and dynamics takes a model. We will later. categorize different models by their **Skew Stickiness Ratio**.

Black-Scholes in inconsistent with the smile. It makes no sense to hedge or value an exotic in an inconsistent model that cannot value vanillas correctly.

Local volatility models are consistent. Now we can use the **rules of thumb** to make things easier, based on theory, assuming that the options are close to at the money with a small linear skew.

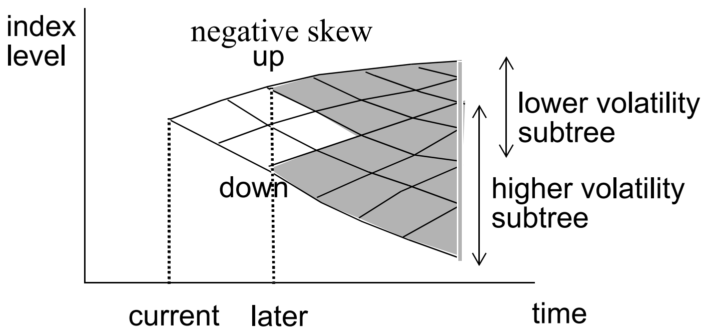
The Rule of 2: Local volatility varies with market level about twice as rapidly as implied volatility varies with strike.



As a result, $\frac{\partial}{\partial S}\Sigma(S, t, K, T) \approx \frac{\partial}{\partial K}\Sigma(S, t, K, T)$

The change in implied volatility of a given option for a change in market level is about the same as the change in implied volatility for a change in strike level.

Here is a rough argument for linear skew:



Implied volatility is average of local volatility between spot and strike.

For a linear local volatility, changing strike is same as changing spot in terms of calculating an average.

Therefore:

$$\Sigma(S, K) \approx \sigma_0 - \beta(S + K) + 2\beta S_0 \qquad \text{symmetry}$$

How does this affect hedge ratios?

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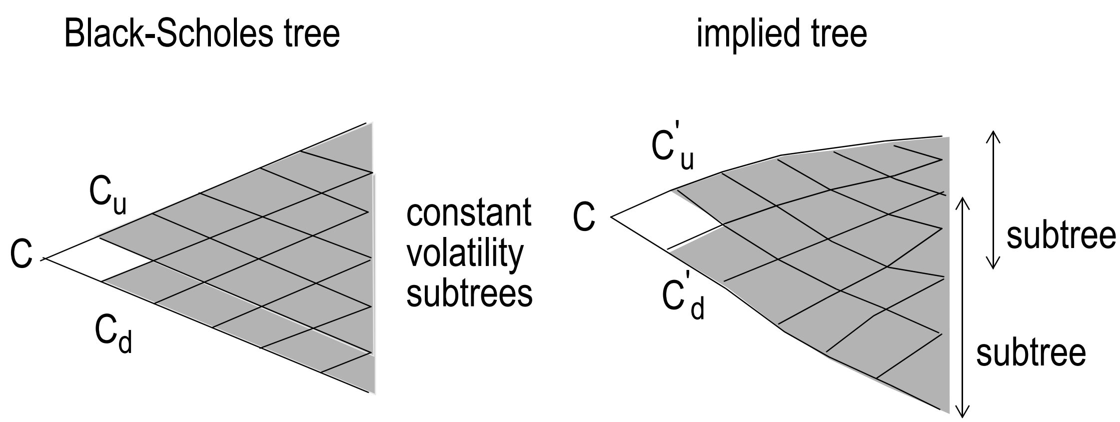
17.3.1 The correct exposure Δ of an option is approximately given by the chain rule formula

$$\Delta = \Delta_{BS} + Vega_{BS} \times \beta \qquad \text{Eq.17.3}$$

One-year S&P option with 20% volatility has a B-S hedge ratio of 54% but probably has a true hedge ratio of 46%, because volatility moves down as the market moves up.

S = 2000; $Vega_{BS}$ = 800 dollars; β = -0.0001 vol point per strike pt.: $Vega_{BS}\beta \sim -0.08$



Biggest effect for large vega.

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17.4 Exotics: The Theoretical Value of Barrier Options in Local Volatility Models

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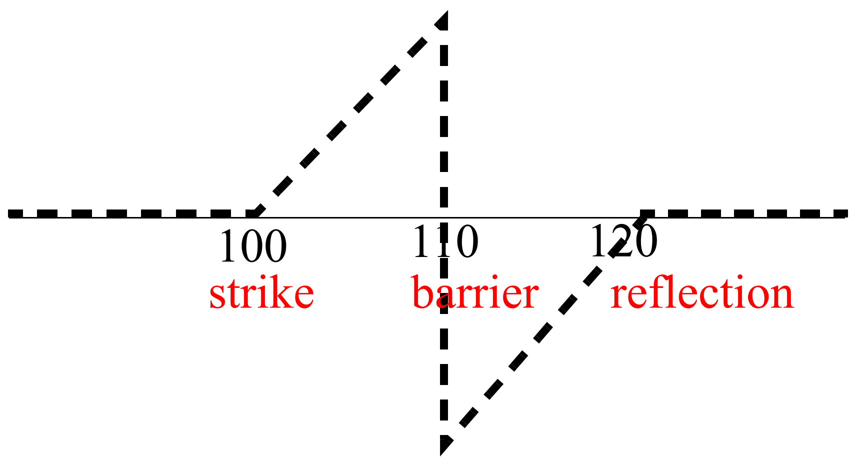
Barrier options depend on the risk-neutral probability of index remaining in the region between the strike and the barrier, and hence on the local volatility in this region, which depends on the skew.

Example 1: An Up-and-Out Call, S = 100, with Strike 100 and Barrier 110

• You can approximately replicate an up and out call by means of a European payoff so the present value earlier on the barrier is zero, something like this:.



• In a skewed world, each call has an implied volatility which is approximately the average of the local volatilities between spot and strike.

• If the local vol varies linearly with spot, then the average of the locals is approximately the barrier implied volatility.

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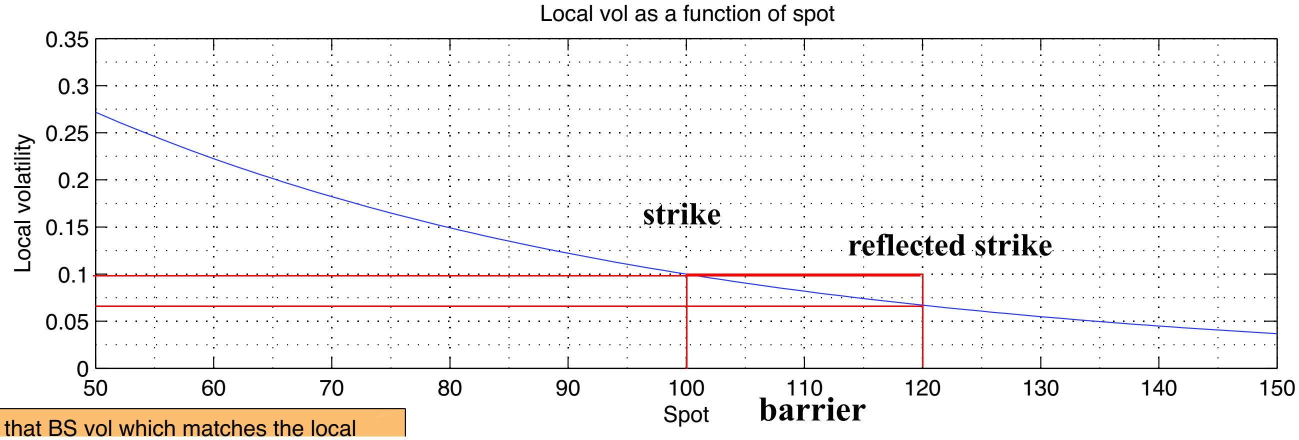
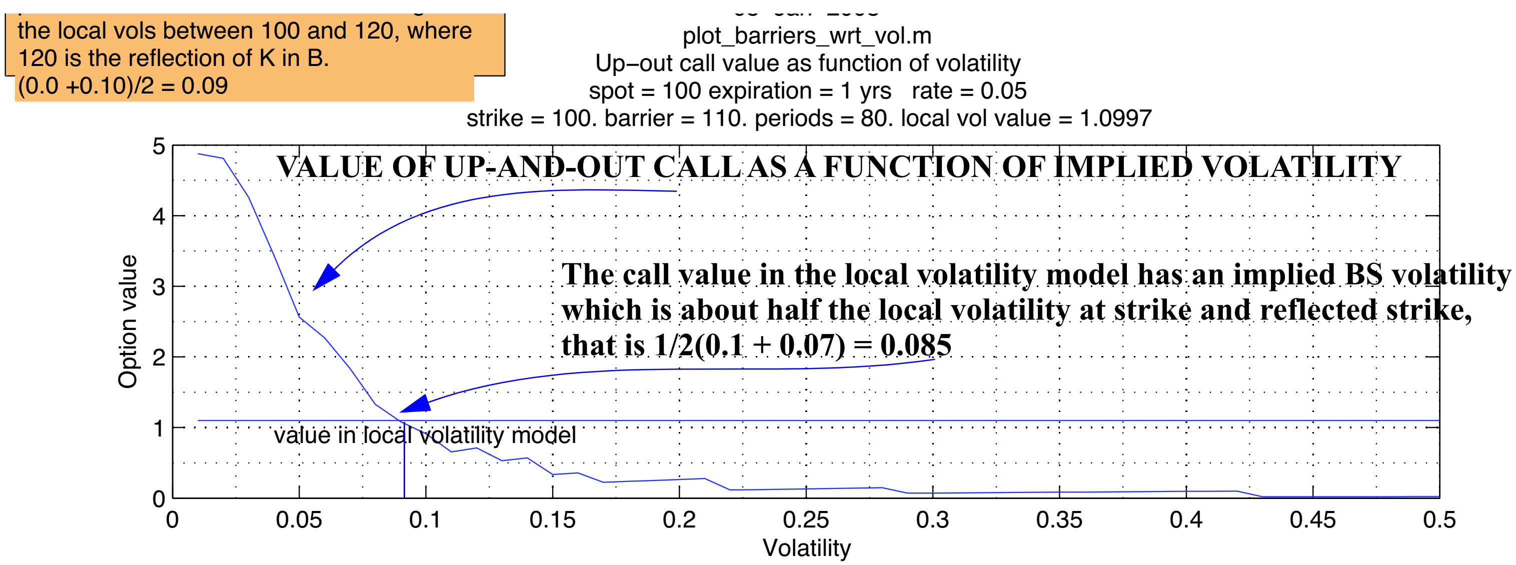
• Thus the approximate value of the Black-Scholes implied volatility for the up-and-out call is the average of the local volatilities between 100 and 120.



• Local volatility varies between 0.1 and 0.07 in this range, with an average of a about 0.085.

.VALUE OF UP-AND-OUT CALL AS A FUNCTION OF IMPLIED VOLATILITY IN BLACK-SCHOLES



The value of the up and out option in the local volatility model is about 1.1, which corresponds to a Black-Scholes implied volatility of about 0.09, so this intuition about averaging works reasonably.

Example 2. An Up-and Out Call that has no Black-Scholes Implied Volatility

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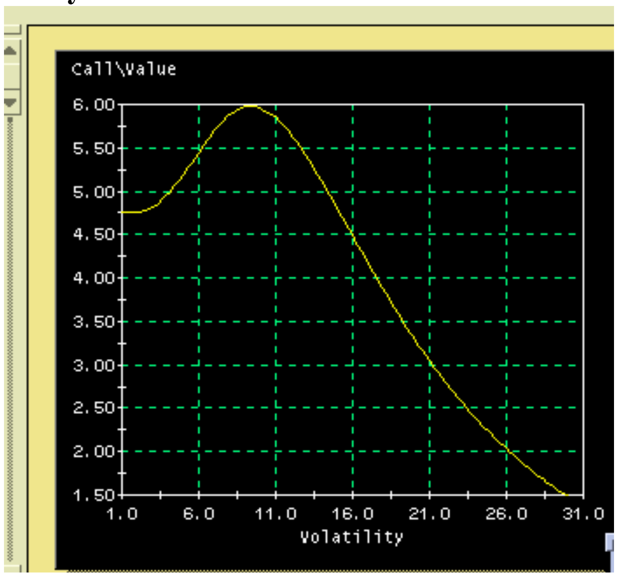
- In some cases, the local volatilities can produce options values that cannot be matched by *any* Black-Scholes implied volatility. No amount of intuition can get you the exactly correct value.
- Up and out call, spot and strike at 100, and the barrier at 130, and the skew as shown below.
- The value of the barrier option in this local volatility model is 6.46.

A hypothetical volatility skew for options of any expiration. We assume r = 5% d = 0%

- The maximum Black-Scholes value in a no-skew world is 6.00 corresponding to a 9.5% implied volatility.
- The BS value is smaller than the "correct" value in the local volatility model.
- There is NO Black-Scholes implied volatility which gives the local-volatility "correct" option value.
- The implied volatility that comes closest to it is about 10%. Why?

No skew: Up-and-Out call value as a function of Black-Scholes Implied Volatility



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- The slope of the skew is 1 vol pt. per 10 strike points. The rule of 2 then indicates that the slope of the local volatilities will be about 1 vol pt. per 5 strike points.
- An up-and-out call with strike 100 and barrier 130 as being replicated by an ordinary call with strike 100 and a reflected put with strike 160 using reflection.
- The local volatility that is relevant to valuation ranges between spot prices 100 and 160 with a slope of approximately 1 vol pt. per 5 strike points, that is from values of 15% to $15 - (60/5) = 3\%$. The average local volatility in this range is about (15 + 3)/2 = 9%, which substantiates the approximate claim the implied volatility is the average of the local volatilities between spot and strike

More Exotics in Local Volatility Models

Lookback Call Options With A Smile

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Path-dependent options (averages, lookbacks) are sensitive to local volatility in multiple regions. No single constant volatility is correct for valuing a path-dependent option in a skew. You can simulate the index evolution over all future market levels and their corresponding local volatilities to calculate fair value.

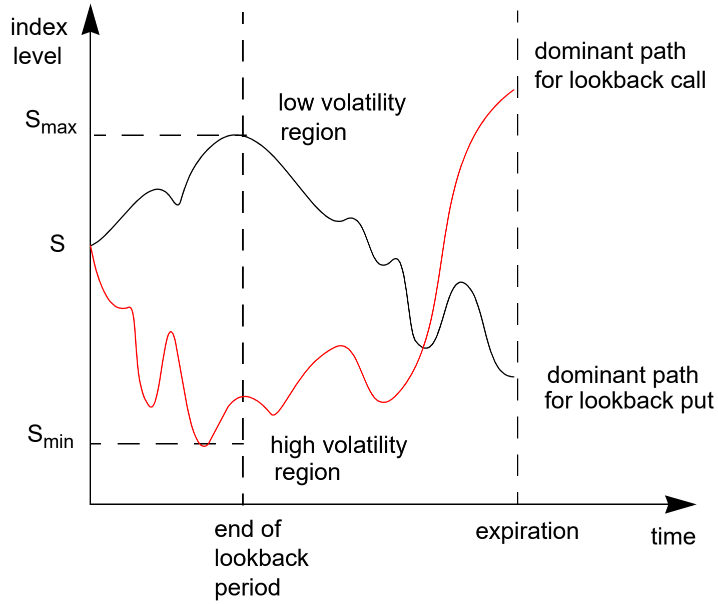
A lookback call pays out $[S_T - S_{min}]$

One-year lookback call or put with a initial three-month lookback period on the strike.

Lookback Call: $max[S_T - S_{min}]$ Lookback Put: $max[S_{max} - S_T]$

Value by simulating index paths whose local volatilities are extracted from the relevant implied volatility smile.

Dominant paths contributing to the value of a lookback call and put. The local volatilities are negatively skewed.



The **most important path** for a lookback call sets a low strike S_{min} during the first three months, and then rises to achieve a high payoff.

In a negatively skewed world, low S_{min} has higher index local volatility from then on, and therefore the option is worth more than in a flat world.

Conversely, the dominant path for a lookback put is more likely to have a high strike S_{max}, and a low subsequent volatility.

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Therefore, in a negatively skewed world, compared to Black-Scholes, relative to each other, lookback puts are worth less and lookback calls are worth more.

Lookback calls will have higher **Black-Scholes implied volatilities** than lookback puts. In a Black-Scholes world with zero drift they would have the same implied volatilities.

Example: Index = 100, d = 2.5%, r = 6% per year.
Negative skew: ATM implied vol = 15%, decreases by 3 percentage points for each increase of 10 index strike points.

Monte Carlo simulation:

Lookback call value = 10.8 ($\Sigma_{BS} = 15.6\%$), Lookback put value = 5.8 ($\Sigma_{BS} = 13.0\%$).

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Practical Calibration of Local Volatility Models (Briefly)

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In practice we are given implied volatilities $\Sigma(K_i, T_i)$ and must calibrate a smooth local volatility function. To use Dupire's equation, we need a smooth implied volatility surface that is at least twice differentiable in the strike direction and once differentiable in the time direction, and that doesn't violate arbitrage.

But all we have is discrete points with noise. They have to be smoothed.

.The most straightforward way to do this is to write down a smooth parametric form for the implied volatilities, and then compute the parameters that minimize the distance between computed and observed standard options prices.

One can then calculate the local volatilities by taking the appropriate derivatives of the implieds. One difficulty with this method is how to determine a realistic form of the parametrization, particularly on the wings where prices are hard to obtain.

Other methods involve splines (nonparametric) or semiparametric interpolations.

There are many papers on this. Will perhaps have a talk later by Tim Klassens of Volardynamics.

SVI (Stochastic Volatility Inspired) Parametrization

In its original formulation (Gatheral, 2004), SVI model is defined at each maturity T in terms of the 5 parameters a, b, ρ, m, σ such that the square of the implied volatility $\theta(K,T)$ is

$$\begin{aligned} \theta^2(K,T) &= v(x,T) = a + b\Big(\rho(x-m) + \sqrt{(x-m)^2+\sigma^2}\Big) \qquad (1) \\ x &= \ln(K/F(T)) \end{aligned}$$

where $F(T)$ is the forward and the parameters lie in the following definition domain

$$\begin{aligned} b &> 0 \qquad (2) \\ \sigma &\geq 0 \qquad (3) \\ \rho &\in [-1,1] \qquad (4) \\ a &\geq -b\sigma\sqrt{1-\rho^2}. \qquad (5) \end{aligned}$$

- Increasing a increases the general level of variance, a vertical translation of the smile;
- Increasing b increases the slopes of both the put and call wings, tightening the smile;
- Increasing ρ decreases (increases) the slope of the left(right) wing, a counter-clockwise rotation of the smile;
- Increasing m translates the smile to the right;
- Increasing σ reduces the at-the-money (ATM) curvature of the smile.

Maybe rather parameterize local vols whose only constraint is to be positive.

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17.5 Benefits and Problems of Local Volatility Models

17.5.1 Local Vol May Provide Better Hedge Ratios For Index Volatility

Because it reflects the correlation between implied volatility and index level and results in a lower hedge ratio.

17.5.2 Inadequacy of the Short-Term Skew in the Model

For equity indexes, because the skew declines with increasing expiration, future local volatilities have less skew than current short-term implied volatilities.Therefore the short-term future skew in a local volatility model is too flat. One needs the threat of jumps or stochastic volatility in the near future to produce a short-term skew.

A good model would look more or less time-invariant and not require recalibration.

On the other hand, all financial models need recalibration; even in Black-Scholes, the implied volatility changes from day to day. Local volatility models, like Black-Scholes, must be recalibrated regularly; they allow the valuation of exotic options consistent with the volatility surface for vanilla options, and are widely used as a means of valuing exotics that will be hedged with vanillas.

The question is: to what extent do local vol models mirror the behavior of realized volatility?

17.5.3 Local vol models are not good for options on volatility.

Volatility is stochastic in real life. But in local volatility model the local volatility is not independent of the skew. It is too small, too determined, to represent stochastic volatility realistically. This is related to the short-term forward skew being relatively flat.

$$\frac{dS}{S} = \mu dt + \sigma(S, t)dZ$$

$$d\sigma = \frac{\partial\sigma}{\partial t}dt + \frac{\partial\sigma}{\partial S}dS + \frac{1}{2}\frac{\partial^2\sigma}{\partial S^2}\sigma^2 S^2 dt$$

The dS terms contains dZ and thus indicates the lognormal volatility of the volatility ξ.

$$d\sigma = \frac{\partial\sigma}{\partial S}\sigma S dZ + \ldots = \xi\sigma dZ + \ldots$$

$$\xi = S\frac{\partial\sigma}{\partial S} = \frac{\partial}{\partial \ln S}\sigma(S, t)$$

If the skew flattens out as time increases, becomes more Black-Scholes-like, then the local volatility becomes less variable as a function of S as time increases, and therefore the volatility of volatility ξ decreases when the skew flattens.

This is not good. You don't want volatility to *have to* decline when the skew declines. You probably want to be able to specify a volatility of volatility. It's too constrained. Local volatility models therefore often significantly undervalue structured products that depend on vol of volatility.

Nevertheless it is, by construction, a self-consistent model that is capable of producing the implied volatility surface observed in the market place. It's probably better than BS for exotic barrier options and vanilla hedge ratios.

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