Chapter 16: Portfolio Selection

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- An asset is an investment instrument that can be bought and sold. Its return is the percentage of value increased from time bought to time sold. By return rate it means return per unit time.
- A portfolio is a collection of shares of assets. The proportions in value of assets in a portfolio are called the weights.
- A portfolio's return is the percentage of value increased from time bought to time sold.
- The basic assumption here is that R_i is a random variable, with mean μ_i and standard deviation σ_i

$$\mu_i = E(R_i) \qquad \sigma_i = \sqrt{Var(R_i)}$$

- We call μ_i the expected return and in the current context σ_i the risk of the asset i.
- Also we denote the covariance and correlation between the returns of the asset i and asset j

$$\sigma_{ij} = E((R_i - \mu_i)(R_j - \mu_j)) \quad \rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

Introduction

- We now consider a portfolio that consists of a collection of the above assets.
- Since the sizes of units of these assets are quite different, we shall not pay any attention on the particular numbers of units, rather, we are concerned about the percentage of each asset value in the portfolio.
- Suppose the total value of the portfolio is V_0 and the value in asset i is $V_i, i = 1, 2, \ldots, m$. Then the weight of the asset i in the portfolio is

$$\omega_i = \frac{V_i}{V_0}$$

We denote the portfolio's weight by the row vector

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_m)$$

Then

$$\sum_{i=1}^{m} \omega_i = \sum_{i=1}^{m} \frac{V_i}{V_0} = 1.$$

- In general, the weight is a function of time, since the returns of different assets are different.
- In this chapter, we shall consider only two times, the time when the portfolio is bought and the time when it is sold.
- Denote by R the portfolio's return:

$$R = \frac{\text{portfolio value at time sold - initial portfolio value initial portfolio value}}{\text{intial portfolio value}}$$

- A simple arithmetic gives the relation among portfolio return, asserts return and weight:

$$R = \sum_{i=1}^{m} \omega_i R_i$$

Introduction

- By varying the weight, one obtains different portfolios of different risk-return balances.
- There are people who are willing to take high risk expecting high returns, whereas there are also people who want security thus are willing to accept moderate returns with small risks.
- Mathematically, we are going to find optimal weights that minimizes risk with given expected return or maximizes the expect return with given risk.
- These two problems are dual to each other. Since the mean is a linear and the variance is a quadratic function of the weights, as one shall see, the problem can be solved explicitly.

- When a weight ω_i is positive, it means to buy (long) asset i certain units worth $V_0\omega_i$. When $\omega_i < 0$, it means selling (short) the asset certain units to generate $|V_0\omega_i|$ cash that can be used to buy other assets.
- By doing that, one owes certain shares of assets i which has to be paid back, with the same amount of units, at time the portfolio is sold

- Portfolios with only a few assets may be subject to a high degree of risk represented by a relatively high standard deviation. As a general rule, the variance of the return of a portfolio can be reduced by including additional assets in the portfolio, a process referred to as diversification
- Example: Consider the following simple yet illustrative situation. Suppose there are m assets each of which has return μ and variance σ^2. Suppose also that all these assets are mutually uncorrelated. One then construct a portfolio by investing equally into these assets, namely, taking $\omega_i = 1/m$ for all $i = 1, 2, \ldots, m$. The overall expected rate of return is still μ. Nevertheless, the overall risk becomes

$$Var[R] = \sigma^2/m$$

which decays rapidly as m increases.

Trading off expected return and risk

A key question investors ask is "How should we invest our wealth"? Portfolio theory provides an answer to this question based on the following two principles:

1. maximize expected return
2. minimize risk which we define to be the standard deviation of the return.

These goals however are somewhat at odds because:

- In general, riskier assets have higher expected return
- investors demand a reward for bearing risk The difference between the expected return of a risky asset and the risk-free rate of return is called the risk premium.
- there are optimal compromises between expected return and risk

In this chapter we seek to do the following:

- maximize expected return with upper bound on the risk
- or minimize risk with lower bound on expected return.

A key concept here is reduction of risk by diversification. We start with a simple example of one risky asset and one risk-free asset.

- Suppose we have one risky asset A (which could be a portfolio, e.g., a mutual fund) with
  - expected return is $\mu_1 = 0.15$
  - standard deviation of the return is $\sigma_1 = .25$
- One risk-free asset (e.g., a 30-day T-bill) and
  - expected value of the return is $\mu_f = .06$
  - standard deviation of the return is 0 by definition of risk free

Trading off expected return and risk

Problem: construct an investment portfolio

- Suppose we invest a fraction ω of our wealth in the risky asset A
- and we invest the remaining fraction $1-\omega$ in the risk-free asset
- The expected return of our investment is then given by
- The return of our portfolio is

$$R = \omega R_1 + (1-\omega)\mu_f$$

where R_1 is the return of asset A.

$$\mu_R = E(R) = 0.15\omega + 0.06(1-\omega) = .06 + .09\omega.$$

- and its variance is given by

$$\sigma_R^2 = Var(R) = \omega^2(.25)^2 + (1-\omega)^2(0)^2 = \omega^2(.25)^2.$$

and hence its standard deviation is

$$\sigma_R = .25\omega.$$

Trading off expected return and risk

We need to choose w.

- To do so, we can either choose the expected return $E(R)$ or the amount of risk σ_R desired.
- Once either E(R) or σ_R is chosen, w can then be determined.
- Question: Suppose you want an expected return of .10. What should ω be?
  Answer: $.10 = .06\omega + .09(1-\omega)$ this gives $\omega = 4/9$
- Question: Suppose you want $\sigma_R = .05$. What should ω be?
  Answer: $.05 = \omega(.25) \Longrightarrow \omega = 0.2$

In general, if

- the expected returns on the risky and risk-free assets are μ_1 and μ_f
- and the standard deviation of the risky asset is σ_1, then
  - the expected return on the portfolio is $\omega\mu_1 + (1 - \omega)\mu_f$
  - the standard deviation of the portfolios return is $\omega\sigma_1$.

Although σ is a measure of risk, a more direct measure of risk is actual monetary loss. Next we choose w to control the maximum size of the loss. Suppose $R \sim N(\mu_R = \omega\mu_1 + (1-\omega)\mu_f, \sigma_R^2 = \omega^2\sigma_1^2)$ and we wish to choose w so that $P(R < r_0) = \alpha$ for some given r_0 (so that the probability that loss is more than $-r_0\times$ amount invested is α) Since

$$P(R < r_0) = \Phi\left(\frac{r_0 - (\omega\mu_1 + (1-\omega)\mu_f)}{\omega\sigma_1}\right)$$

we have

$$\frac{r_0 - (\omega\mu_1 + (1-\omega)\mu_f)}{\omega\sigma_1} = \Phi^{-1}(\alpha)$$

therefore

$$\omega = \frac{r_0 - \mu_f}{\sigma_1\Phi^{-1}(\alpha) + (\mu_1 - \mu_f)}$$

Example:

- Suppose a firm in planning to invest \$ 1,000,000 and has capital reserves that could cover \$ 150, 000 but no more.
- The firm would therefore like to be certain that if there is a loss, it is no more than 15% (150, 000 is 15% of 1, 000, 000). That is R is greater than -15%
- The only way to guarantee this is not to invest in the risky asset (or invest in the risky asset no more than \$ 150, 000)
- The firm chooses w so that $P(R < -0.15)$ is small. For example $P(R < -0.15) = 0.01$.
- The value \$ 150, 000 is called the value at risk (= VaR) and 1-0.01=0.99 is called the confidence coefficient
- What say that the portfolio has a VaR of \$ 150, 000 with 0.99 confidence

Answer: Suppose $r_0 = -0.15, \mu_1 = 0.06, \mu_2 = 0.15, \sigma_1 = 0.25$ and $\alpha = 0.01$, then

$$\omega = \frac{-0.15 - 0.06}{0.25\Phi^{-1}(0.01) + (0.15 - 0.06)} = 0.4264$$

Selling short is a way to profit if a stock price goes down.

- To sell a stock short, one sells the stock without owning it.
- Suppose a stock is selling at $25/share and you sell 100 shares short.
- This gives you $2,500.
- If the stock price goes down to $17 a share, you can buy the 100 shares for $1,700 and close out your short position. You made $800 (net transaction costs). If the stock goes up, you would have a loss.

- Suppose that you have \$100 and there are two risky assets.
  - With your money you could buy \$150 worth of risky asset 1 and sell \$50 short of risky asset 2.
  - the net cost would be exactly \$100.
  - the return on your portfolio would be

$$\frac{3}{2}R_1 + (-\frac{1}{2})R_2.$$

  - Your portfolio weights are $\omega_1 = 3/2$ and $\omega_2 = -1/2$.

- The model of one risk-free asset and one risky asset is very simple but not useless
- In general, finding an optimal portfolio can be achieved in two steps.
    1. find the portfolio of risky assets. This portfolio is called the tangency portfolio
    2. find the appropriate mix of the risk-free asset and the tangency portfolio determined in step one (We now know how to do the second step, see above).
- What we need to learn is how to mix optimally a number of risky assets. This easily understood when there are only two risky assets. We therefore start with two risky assets.

Two risky assets

Suppose that

- we have two risky assets with returns R_1 and R_2
- we mix them in proportions w and $1 - w$ so the return on our portfolio is $R_P = \omega R_1 + (1 - \omega) R_2$.
- The expected return on the portfolio is

$$E(R_P) = \mu_{R_P} = \omega \mu_1 + (1 - \omega) \mu_2$$

- if ρ_{12} is the correlation so that $\sigma_{R_1,R_2} = \rho_{12}\sigma_1\sigma_2$, then the variance of the return on the portfolio is

$$\sigma_{R_P}^2 = \omega^2 \sigma_1^2 + (1 - \omega)^2 \sigma_2^2 + 2\omega(1 - \omega)\rho_{12}\sigma_1\sigma_2.$$

Example: If $\mu_1 = .14, \mu_2 = .08, \sigma_1 = .2, \sigma_2 = .15$ and $\rho_{12} = 0$, then

$$\mu_{R_P} = .14\omega + 0.08(1-\omega) = 0.08 + 0.06\omega$$

and

$$\sigma^2_{R_P} = \omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2. = (.2)^2\omega^2 + (.15)^2(1-\omega)^2$$

since $\rho_{12} = 0$. Using calculus, one can show that the portfolio with the minimum risk corresponds to

$$\omega = .025/.125 = .36.$$

For this portfolio $\mu_{R_P} = .1016$ and $\sigma_{R_P} = .12$.

Explanation:To find the minimum variancel (MVP) portfolio, we need to find w that minimizes $\sigma^2_{R_P}$. To do so we use calculus.

$$\begin{aligned}\frac{d\sigma^2_{R_P}}{dw} &= 2w\sigma_1^2 - 2(1-w)\sigma_2^2 \\ &= 2w(\sigma_1^2+\sigma_2^2) - 2\sigma_2^2 = 0\end{aligned}$$

this gives

$$w = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}.$$

In general, for n uncorrelated asset returns $R_1, ..., R_n$, the MVP is

$$(\frac{1/\sigma_1^2}{1/\sigma_1^2 + ... + 1/\sigma_n^2}, ..., \frac{1/\sigma_n^2}{1/\sigma_1^2 + ... + 1/\sigma_n^2})$$

Values of μ_{R_P} and σ_{R_P} for some values of w

w	μ_{R_P}	σ_{R_P}
0.00	0.080	0.150
0.25	0.095	0.123
0.50	0.110	0.125
0.75	0.125	0.155
1.00	0.140	0.200

- We plot the two assets at coordinates (σ_1, R_1) and (σ_2, R_2), respectively.
- The curve is the set of points $(\sigma_P\ ,\ 0.08 + 0.06\omega)$, $0 \leq \omega \leq 1$
- Notice that MV has a volatility less than either asset

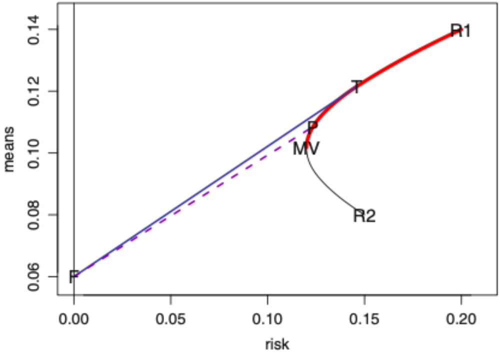


Fig. 16.1. *Expected return versus risk for Example 16.2. F = risk-free asset. T = tangency portfolio. R1 is the first risky asset. R2 is the second risky asset. MV is the minimum variance portfolio. The efficient frontier is the red curve. All points on the curve connecting R2 and R1 are attainable with $0 \leq w \leq 1$, but the ones on the black curve are suboptimal. P is a typical portfolio on the efficient frontier.*

Markowitz Optimal Portfolio Problem

- Goal: minimize the risk while the expected return achieves a fixed value.

Consider the following optimization problem,

$$\min_w \left\{ \sigma^2_{R_P} \right\}$$

subject to two conditions,

$$\sum_i \omega_i \mu_i = \mu,$$

$$\sum_i \omega_i = 1.$$

Here μ is a pre-determined value. Therefore, the solution to the Markowitz problem is a function of μ. Note that the MVP is a solution to the Markowitz problem when $\mu = \mu_{MVP}$ or when there is no constraint on the mean. The Markowitz problem can be solved by Lagrange-multipliers method if allows short selling.

Example: Let $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$, $\sigma_1 = \sigma_2 = \sigma_3 = 1$, and $\mu_1 = 1, \mu_2 = 2, \mu_3 = 3$. For this example, the Markowitz problem becomes

$$\min_{\omega_1,\omega_2,\omega_3} \left\{\omega_1^2 + \omega_2^2 + \omega_3^2\right\}, \quad \text{s.t.}$$

$$\omega_1 + 2\omega_2 + 3\omega_3 = \mu, \quad \omega_1 + \omega_2 + \omega_3 = 1.$$

By Lagrange multipliers, we have the following linear system

$$2\omega_1 + \lambda_1 + \lambda_2 = 0,$$

$$2\omega_2 + 2\lambda_1 + \lambda_2 = 0,$$

$$\omega_3 + 3\lambda_1 + \lambda_2 = 0,$$

$$\omega_1 + 2\omega_2 + 3\omega_3 = \mu, \quad \text{and} \omega_1 + \omega_2 + \omega_3 = 1.$$

The last five equations lead to the following optimal solution,

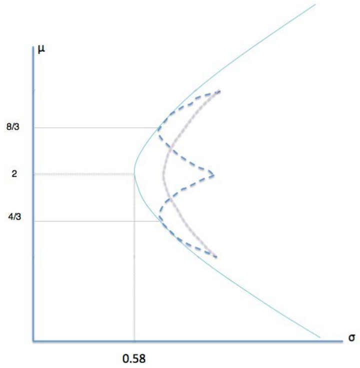
$$\omega_1 = \frac{4}{3} - \frac{\mu}{2}, \quad \omega_2 = \frac{1}{3}, \quad \omega_3 = \frac{\mu}{2} - \frac{2}{3}.$$

Then the standard deviation of this efficient portfolio becomes

$$\sigma_{min} = \sqrt{\frac{7}{3} - 2\mu + \frac{\mu^2}{2}} = \sqrt{\frac{1}{3} + \frac{1}{2}(\mu - 2)^2},$$

which is a function (curve) of μ. When $\mu = 2$, it gives us the MVP.

The trajectory of (σ_{min}, μ) is called the efficient frontier and is plotted as follows. All possible pairs of σ and μ are to the right of the solid blue curve when short-selling is allowed. $(\sigma_{MVP}, \mu_{MVP})$ is the left endpoint of the efficient frontier. When short-selling is not allowed, the frontier is segmented by $4/3 \leq \mu \leq 8/3$.



- estimates of μ_1 and σ_1 can be obtained from past returns on first risky asset
- If $R_{i,1}, \dots, R_{i,n}$, $i = 1, 2$, denote the time series of the returns of the two assets, then $\overline{R}_i, i = 1, 2$, (the sample means) and $s_i, i = 1, 2$, (the sample standard deviations) are estimates of $\mu_i, i = 1, 2$, and $\sigma_i, i = 1, 2$.
- the covariance σ_{12} can be estimated by the sample covariance

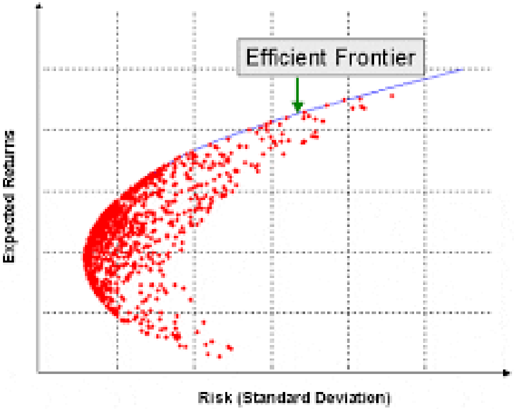
$$s_{12} = \frac{1}{n} \sum_{t=1}^{n} (R_{1t} - \overline{R}_1)(R_{2t} - \overline{R}_2).$$

- the correlation coefficient ρ_{12} can be estimated by the sample correlation

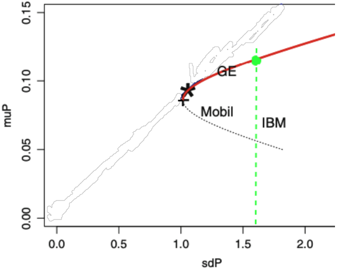
$$\hat{\rho}_{12} = \frac{s_{12}}{s_1 s_2}$$

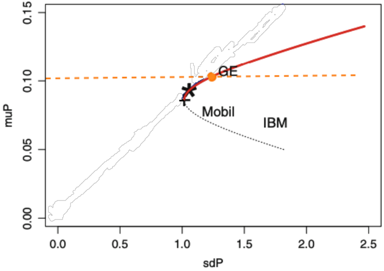
We can extend this concept to portfolios of many assets:

- Combine the assets in all possible weights (summing to 1)
- Plot the expected return against standard deviation on the x-axis
- All the possible combinations make up a cloud of points that fill what looks like a parabola
- Only the superior border of this cloud is the efficient frontier

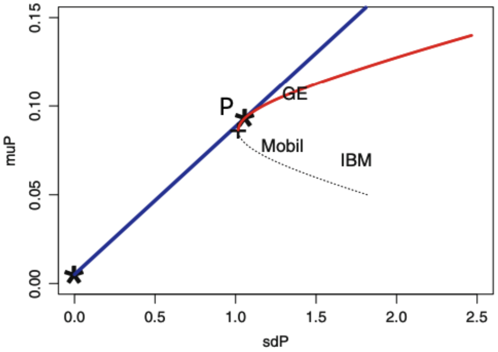


- For a given volatility, there is at most one portfolio that has the highest return (green)
- For a given return, there is at most one portfolio that has the lowest volatility (orange)
- Those are the **efficient portfolios**





- Add the risk-free asset
- Draw a line that goes through the risk-free asset and tangents the efficient frontier
- Let's call P the tangency portfolio



Tangency portfolio with two risky assets

- each point on the efficient frontier is (σ_R, μ_R) for some value of w
- if we fix w, then we have a fixed portfolio of the two risky assets.
- Now let us mix that portfolio of risky assets with the risk-free asset.
- The slope of the line connecting the risk-free with a portfolio of risky assets is called the Sharpe ratio

$$\frac{E(R_P) - \mu_f}{\sigma_{R_P}}$$

- Sharpes ratio = reward-to-risk ratio = ratio of excess expected return to standard deviation
- the bigger the Sharpe ratio the better

- The point T with the highest Sharpe ratio and is called the tangency portfolio
- Efficient portfolios mix the tangency portfolio with the risk-free asset.
- all efficient portfolios use the same mix of the two risky assets, namely the tangency portfolio

let

$$V_1 = \mu_1 - \mu_f, \quad V_2 = \mu_2 - \mu_f$$

be the excess returns. Let

$$\omega_T = \frac{V_1\sigma_2^2 - V_2\rho_{12}\sigma_1\sigma_2}{V_1\sigma_2^2 + V_2\sigma_1^2 - (V_1 + V_2)\rho_{12}\sigma_1\sigma_2}.$$

The tangency portfolios allocates

- ω_T to the first risky asset
- $1 - \omega_T$ to the second risky asset
- Let R_T, μ_T and σ_T be the return, the expected return and the standard deviation corresponding to the tangency portfolio.

Example : Suppose $\mu_1 = 0.14, \mu_2 = 0.08, \sigma_1 = 0.2, \sigma_2 = 0.15$ and $\sigma_{12} = 0$. Suppose also that $\mu_f = 0.06$. then

$$V_1 = 0.14 - 0.06 = 0.08, \; V_2 = 0.08 - 0.06 = 0.02$$

and

$$\omega_T = \frac{0.08(0.15)^2 - 0}{0.08(0.15)^2 + 0.02(0.20)^2 - 0} = 0.693.$$

Therefore

$$\mu_T = 0.693(0.14) + (1 - 0.693)(0.08) = 0.122$$

and

$$\sigma_T = \sqrt{(0.693)^2(0.2)^2 + (1 - 0.693)^2(0.15)^2} = 0.146.$$

R_P is the return of the portfolio that allocates w to the tangency portfolio and $1 - w$ to the risk free asset then

$$R_P = \omega R_T + (1 - \omega)\mu_f$$

and

$$\mu_{R_P} = \omega\mu_T + (1 - \omega)\mu_f, \quad \sigma_{R_P} = \omega\sigma_T.$$

Example (continued): What is the optimal investment with $\sigma_{R_P} = 0.05$?
Answer: Since $\sigma_{R_P} = \omega\sigma_T$ and

$$\omega = \frac{\sigma_{R_P}}{\sigma_T} = \frac{0.05}{0.146} = 0.343.$$

Therefore 34.3% should be invested in the tangency portfolio and the remaining 65.7% is the risk free asset. The 34% should be allocated as follows between the two risky assets

- 0.693(0.343)=23.7% should be the first risky asset
- (1-0.693)(0.343) = 10.5% should be in the second risky asset

the asset allocation is therefore as follows

- 65.7% risk free
- 23.7 % first risky asset
- 10.5% second risky asset

Now suppose that you want a 10% expected return. Compare

1. The best portfolio of only risky assets
2. the best portfolio of the risky assets and the risk-free asset

Answer: 1. Best portfolio of risky assets. We have

$$0.10 = \omega\mu_1 + (1-\omega)\mu_2 = \omega(0.14) + (1-\omega)(0.08) = 0.08 + (0.14 - 0.08)\omega$$

therefore

$$\omega = (0.10 - 0.08)/(0.14 - 0.08) = 0.02/0.06 = 1/3.$$

Since this is the only portfolio of risky assets with $\mu_R = 0.10$, so by default it is best. A direct computation of σ_R gives

$$\sigma_R = \sqrt{\omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2} = \sqrt{(1/9)(.2)^2 + 4/9(.15)^2} = .120.$$

2. Best portfolio of the two risky assets and the risk-free asset. We have

$$\begin{aligned} 0.10 = E(R) &= \omega\mu_T + (1-\omega)\mu_f \\ &= \frac{\sigma_R}{\sigma_T}\mu_T + (1 - \frac{\sigma_R}{\sigma_T})\mu_f \\ &= \sigma_R \frac{0.122}{0.146} + (1 - \frac{\sigma_R}{0.146})(0.06) \\ &= 0.06 + 0.425\sigma_R. \end{aligned}$$

This implies that

$$\sigma_R = 0.04/0.425 = 0.094.$$

Therefore

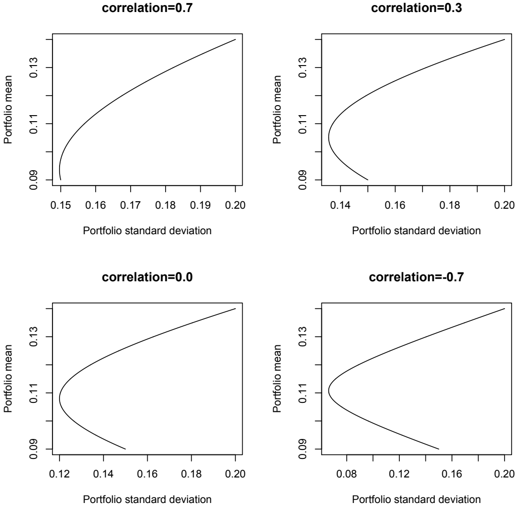
$$\omega = \sigma_R/\sigma_T = 0.094/0.146 = 0.644$$

Conclusion: if we combine the risk-free asset with the two risky assets, σ_R reduces from .120 to .094 while maintaining $\mu_R = 0.10$. The reduction in risk is 28%.

Effect of ρ_{12}

- Correlation effects only the risk, not the expected return.
- Positive correlation between the two risky assets is bad. With positive correlation, the two assets tend to move together which increases the volatility of the portfolio.
- Negative correlation is good. If the assets are negatively correlated, a negative return of one tends to occur with a positive return of the other

Effect of ρ_{12}



Assume that we have N risky assets and that the return on the *ith* risky asset is μ_i. Define

$$\mathbf{R} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{pmatrix}$$

be the vector of the random vector of returns. Then

$$E(R) = \boldsymbol{\mu} = \begin{pmatrix} E(R_1) \\ E(R_2) \\ \vdots \\ E(R_N) \end{pmatrix}.$$

Suppose there is a target value μ_P of the expected return on the portfolio.

- When $N = 2$ we saw before that the target μ_P is achieved by only one portfolio.
- For $N \geq 3$, there will be an infinite number of portfolios achieving this target mean μ_P.
- The one with the smallest variance is called the efficient portfolio.
- Our goal next is to find the efficient portfolio.

Let

- $\sigma_i = \sqrt{\Omega_{ii}}$ be the standard deviation of R_i
- Ω_{ij} be the covariance between R_i and R_j.
- $$\rho_{ij} = \frac{\Omega_{ij}}{\sigma_i \sigma_j}$$
  be the correlation coefficient between R_i and R_j.

Let

$$\Omega = \mathrm{COV}(\mathbf{R})$$

be the covariance matrix of **R** Let

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_N \end{pmatrix}$$

be a vector of portfolio weights and let

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

be an $N \times 1$ vector of ones. We assume that $\mathbf{1}^T\boldsymbol{\omega} = 1$.

- To find the mvp we need to solve

$$\min \frac{1}{2}\boldsymbol{\omega}^T \Omega \boldsymbol{\omega}$$

  subject to $\boldsymbol{\omega}^T \mathbf{1} = 1$.

- The Lagragian corresponding to this situation is

$$L(\boldsymbol{\omega}, \lambda) = \frac{1}{2}\boldsymbol{\omega}^T \Omega \boldsymbol{\omega} + \lambda(1 - \boldsymbol{\omega}^T \mathbf{1})$$

- To find the solution which we denote by $\boldsymbol{\omega}_{mvp}$, we must solve

$$\begin{aligned} \mathbf{0} &= \frac{\partial}{\partial \boldsymbol{\omega}} L(\boldsymbol{\omega}, \delta_1, \delta_2) = \Omega \boldsymbol{\omega} - \lambda \mathbf{1} && (1) \\ 1 &= \mathbf{1}^T \boldsymbol{\omega} && (2) \end{aligned}$$

  where

$$\frac{\partial}{\partial \boldsymbol{\omega}} L(\boldsymbol{\omega}, \lambda) = \begin{pmatrix} \frac{\partial}{\partial w_1} L(\boldsymbol{\omega}, \lambda) \\ \frac{\partial}{\partial w_2} L(\boldsymbol{\omega}, \lambda) \\ \vdots \\ \frac{\partial}{\partial w_N} L(\boldsymbol{\omega}, \lambda) \end{pmatrix}$$

- The solution is given by

$$\boldsymbol{\omega}_{mvp} = \frac{\Omega^{-1}\mathbf{1}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}$$

  and its return is

$$R = \boldsymbol{\omega}_{mvp}^T\mathbf{R}$$

- Its mean and variance are

$$\mu_{mvp} = \frac{\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}} \qquad \sigma^2_{mvp} = \frac{1}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}$$

- To find the efficient portfolio with mean μ_p, we need to solve

$$\min \frac{1}{2}\boldsymbol{\omega}^T\Omega\boldsymbol{\omega}$$

subject to $\boldsymbol{\omega}^T\mathbf{1} = 1$ and $\boldsymbol{\omega}^T\boldsymbol{\mu} = \mu_p$.

- The Lagragian corresponding to this situation is

$$L(\boldsymbol{\omega}, \lambda_1, \lambda_2) = \frac{1}{2}\boldsymbol{\omega}^T\Omega\boldsymbol{\omega} + \lambda_1(1 - \boldsymbol{\omega}^T\mathbf{1}) + \lambda_2(\mu_p - \boldsymbol{\omega}^T\boldsymbol{\mu})$$

- To find the solution which we denote by $\boldsymbol{\omega}_{mvp}$, we must solve

$$\begin{aligned} \mathbf{0} &= \frac{\partial}{\partial\boldsymbol{\omega}}L(\boldsymbol{\omega}, \lambda_1, \lambda_2) = \Omega\boldsymbol{\omega} - \lambda_1\mathbf{1} - \lambda_2\boldsymbol{\mu} && (3) \\ 1 &= \mathbf{1}^T\boldsymbol{\omega} && (4) \\ \mu_p &= \boldsymbol{\omega}^T\boldsymbol{\mu} && (5) \end{aligned}$$

where

$$\frac{\partial}{\partial\boldsymbol{\omega}}L(\boldsymbol{\omega}, \lambda_1, \lambda_2) = \begin{pmatrix} \frac{\partial}{\partial w_1}L(\boldsymbol{\omega}, \lambda_1, \lambda_2) \\ \frac{\partial}{\partial w_2}L(\boldsymbol{\omega}, \lambda_1, \lambda_2) \\ \vdots \\ \frac{\partial}{\partial w_N}L(\boldsymbol{\omega}, \lambda_1, \lambda_2) \end{pmatrix}$$

- The solution is given by

$$\begin{aligned}
\boldsymbol{\omega}^* &= \lambda_1\Omega^{-1}\mathbf{1} + \lambda_2\Omega^{-1}\boldsymbol{\mu} \\
&= \lambda_1\mathbf{1}^T\Omega^{-1}\mathbf{1}\frac{\Omega^{-1}\mathbf{1}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}} + \lambda_2\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}\frac{\Omega^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}} \\
&= \theta\boldsymbol{\omega}_1 + (1-\theta)\boldsymbol{\omega}_2
\end{aligned}$$

where $\theta = \lambda_1\mathbf{1}^T\Omega^{-1}\mathbf{1}, 1-\theta = \lambda_2\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}$ and

$$\boldsymbol{\omega}_1 = \frac{\Omega^{-1}\mathbf{1}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}$$

and

$$\boldsymbol{\omega}_2 = \frac{\Omega^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}}$$

Note that $\boldsymbol{\omega}^{*T}\mathbf{1} = 1$ implies

$$\lambda_1\mathbf{1}^T\Omega^{-1}\mathbf{1} + \lambda_2\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu} = 1$$

- Note also that $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_{mvp}$ and that $\boldsymbol{\omega}_2^T 1 = 1$.
- Assuming that the Lagrange multiplier approach gave the correct answer, to invest efficiently and achieve mean μ_p, a portion θ of our money should be put in the mvp and the proportion $(1-\theta)$ is the portfolio with weight given by $\boldsymbol{\omega}_2$.
- The proportion θ is determined by the equation

$$\mu_p = \theta \boldsymbol{\omega}_1^T \boldsymbol{\mu} + (1-\theta)\boldsymbol{\omega}_2^T \boldsymbol{\mu}.$$

- This gives

$$\theta = \frac{\mu_p - \boldsymbol{\omega}_2^T \boldsymbol{\mu}}{\boldsymbol{\omega}_1^T \boldsymbol{\mu} - \boldsymbol{\omega}_2^T \boldsymbol{\mu}}$$

- Next we show that the solution is the efficient portfolio with mean μ_p.

Using the value of θ above, we have

$$\begin{aligned}
\boldsymbol{\omega}^* &= \theta\boldsymbol{\omega}_1 + (1-\theta)\boldsymbol{\omega}_2 \\
&= \boldsymbol{\omega}_2 + \theta(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \\
&= \boldsymbol{\omega}_2 + \frac{\mu_p - \boldsymbol{\omega}_2^T\boldsymbol{\mu}}{\boldsymbol{\omega}_1^T\boldsymbol{\mu} - \boldsymbol{\omega}_2^T\boldsymbol{\mu}}(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \\
&= \frac{(\boldsymbol{\omega}_1^T\boldsymbol{\mu})\boldsymbol{\omega}_2 - (\boldsymbol{\omega}_2^T\boldsymbol{\mu})\boldsymbol{\omega}_1}{\boldsymbol{\mu}^T\Delta\boldsymbol{\omega}} + \mu_p\frac{\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2}{\boldsymbol{\mu}^T\Delta\boldsymbol{\omega}} \\
&= \mathbf{e}_1 + \mu_p\mathbf{e}_2
\end{aligned}$$

where

$$\Delta\boldsymbol{\omega} = \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2$$

and

$$\mathbf{e}_1 = \frac{(\boldsymbol{\omega}_1^T\boldsymbol{\mu})\boldsymbol{\omega}_2 - (\boldsymbol{\omega}_2^T\boldsymbol{\mu})\boldsymbol{\omega}_1}{\boldsymbol{\mu}^T\Delta\boldsymbol{\omega}} \quad \text{and} \quad \mathbf{e}_2 = \frac{\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2}{\boldsymbol{\mu}^T\Delta\boldsymbol{\omega}}$$

- Note that
$$\mathbf{e}_1^T\mathbf{1} = 1, \quad \mathbf{e}_1^T\mu = 0, \quad \mathbf{e}_2^T\mathbf{1} = 0, \quad \mathbf{e}_2^T\boldsymbol{\mu} = 1$$
- Next we show that if $\boldsymbol{\omega}$ is such that
$$\boldsymbol{\omega}^T\mathbf{1} = 1 \qquad \text{and} \qquad \mathrm{E}(\boldsymbol{\omega}^{\mathrm{T}}\mathbf{R}) = \mu_{\mathrm{p}},$$
  then
$$Var(\boldsymbol{\omega}^{*T}\mathbf{R}) \leq Var(\boldsymbol{\omega}^T\mathbf{R})$$
  That is, the solution we obtained using Lagrange multipliers is the efficient portfolio with mean μ_p
- Let $\tilde{\mathbf{w}} = \mathbf{w} - \boldsymbol{\omega}^*$. Then
  1. $\tilde{\mathbf{w}}^T\mathbf{1} = (\mathbf{w} - \boldsymbol{\omega}^*)^T\mathbf{1} = \mathbf{w}^T\mathbf{1} - \boldsymbol{\omega}^{*T}\mathbf{1} = 0$
  2. $\tilde{\mathbf{w}}^T\boldsymbol{\mu} = (\mathbf{w} - \boldsymbol{\omega}^*)^T\boldsymbol{\mu} = \mathbf{w}^T\boldsymbol{\mu} - \boldsymbol{\omega}^{*T}\boldsymbol{\mu} = \mu_p - \mu_p = 0$

  This implies that
  1. $\tilde{\mathbf{w}}^T\Omega\omega_1 = \frac{\tilde{\mathbf{w}}^T\mathbf{1}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}} = 0$
  2. $\tilde{\mathbf{w}}^T\Omega\omega_2 = \frac{\tilde{\mathbf{w}}^T\boldsymbol{\mu}}{\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}} = 0$

- Since $\boldsymbol{\omega}^* = \theta\boldsymbol{\omega}_1 + (1-\theta)\boldsymbol{\omega}_2$, we have

$$\tilde{\boldsymbol{\omega}}^T \Omega \boldsymbol{\omega} = 0.$$

- Notice that

$$\boldsymbol{\omega} = \boldsymbol{\omega}^* + \tilde{\boldsymbol{\omega}}$$

Therefore

$$\boldsymbol{\omega}^T \mathbf{R} = \boldsymbol{\omega}^{*T} \mathbf{R} + \tilde{\boldsymbol{\omega}}^T \mathbf{R}$$

and as a result

$$\begin{aligned} Var(\boldsymbol{\omega}^T \mathbf{R}) &= Var(\boldsymbol{\omega}^{*T}\mathbf{R} + \tilde{\boldsymbol{\omega}}^T\mathbf{R}) \\ &= Var(\boldsymbol{\omega}^{*T}\mathbf{R}) + Var(\tilde{\boldsymbol{\omega}}^T\mathbf{R}) + 2Cov(\boldsymbol{\omega}^{*T}\mathbf{R}, \tilde{\boldsymbol{\omega}}^T\mathbf{R}) \\ &= \boldsymbol{\omega}^{*T}\Omega\boldsymbol{\omega}^* + \tilde{\boldsymbol{\omega}}^T\Omega\tilde{\boldsymbol{\omega}} + 2Cov(\boldsymbol{\omega}^{*T}\mathbf{R}, \tilde{\boldsymbol{\omega}}^T\mathbf{R}) \\ &= \boldsymbol{\omega}^{*T}\Omega\boldsymbol{\omega}^* + \tilde{\boldsymbol{\omega}}^T\Omega\tilde{\boldsymbol{\omega}} \end{aligned}$$

Since

$$Cov(\boldsymbol{\omega}^{*T}\mathbf{R}, \tilde{\boldsymbol{\omega}}^T\mathbf{R}) = \boldsymbol{\omega}^{*T}\Omega\tilde{\boldsymbol{\omega}} = 0.$$

Since Ω is positive definite

$$\tilde{\boldsymbol{\omega}}^T\Omega\tilde{\boldsymbol{\omega}} \geq 0.$$

Therefore

$$Var(\boldsymbol{\omega}^T\mathbf{R}) \geq Var(\boldsymbol{\omega}^{*T}\mathbf{R})$$

Note also that since $\boldsymbol{\omega}^* = \mathbf{e}_1 + \mu_p \mathbf{e}_2$ we have

$$\begin{aligned}
Var(\boldsymbol{\omega}^{*T}\mathbf{R}) &= Var(\mathbf{e}_1^T\mathbf{R} + \mu_p \mathbf{e}_2^T\mathbf{R}) \\
&= \mathbf{e}_1^T\Omega\mathbf{e}_1 + \mu_p^2\mathbf{e}_2^T\Omega\mathbf{e}_2 + 2\mu_p\mathbf{e}_1^T\Omega\mathbf{e}_2 \\
&= \mathbf{e}_1^T\Omega\mathbf{e}_1 + \mathbf{e}_2^T\Omega\mathbf{e}_2(\mu_p^2 + 2\mu_p\frac{\mathbf{e}_1^T\Omega\mathbf{e}_2}{\mathbf{e}_2^T\Omega\mathbf{e}_2}) \\
&= \mathbf{e}_1^T\Omega\mathbf{e}_1 - \frac{(\mathbf{e}_1^T\Omega\mathbf{e}_2)^2}{\mathbf{e}_2^T\Omega\mathbf{e}_2} + \mathbf{e}_2^T\Omega\mathbf{e}_2\left(\mu_p^2 + \frac{\mathbf{e}_1^T\Omega\mathbf{e}_2}{\mathbf{e}_2^T\Omega\mathbf{e}_2}\right)^2
\end{aligned}$$

It turns out that

1. $$\mathbf{e}_1^T\Omega\mathbf{e}_1 - \frac{(\mathbf{e}_1^T\Omega\mathbf{e}_2)^2}{\mathbf{e}_2^T\Omega\mathbf{e}_2} = \frac{1}{\mathbf{1}^T\Omega^{-1}\mathbf{1}} = \sigma_{mvp}^2$$

   and

2. $$\frac{\mathbf{e}_1^T\Omega\mathbf{e}_2}{\mathbf{e}_2^T\Omega\mathbf{e}_2} = -\frac{\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}} = \mu_{mvp}.$$

Therefore

$$Var(\boldsymbol{\omega}^{*T}\mathbf{R}) = \sigma_{mvp}^2 + k^2(\mu_p - \mu_{mvp})^2$$

where

$$k^2 = \mathbf{e}_2^T\Omega\mathbf{e}_2$$

- As we change the value of μ_p, the variance of the efficient portfolio changes also. The curve
$$Var(\boldsymbol{\omega}^{*T}\mathbf{R}) = \sigma^2_{mvp} + k^2(\mu_p - \mu_{mvp})^2$$
or
$$\mu_p = \mu_{mvp} \pm \frac{1}{k}\sqrt{Var(\boldsymbol{\omega}^{*T}\mathbf{R}) - \sigma^2_{mvp}}$$
is a hyperbola with a tip at (σ_{mvp}, μ_p)
- The hyperbola is called the Markowitz curve.
- A portfolio is efficient if and only if its mean and standard deviation are on the Markowitz curve
- The unbounded region on the right hand side of the hyperbola is called the Markowitz bullet or the attainable region
- The top half of the hyperbola is called Markowitz efficient frontier

Example: Suppose $N = 3$,

$$\boldsymbol{\mu} = \begin{pmatrix} 0.08 \\ 0.08 \\ 0.12 \end{pmatrix}$$

and

$$\Omega = \begin{pmatrix} 0.02 & -0.01 & -0.02 \\ -0.01 & 0.04 & 0.01 \\ -0.02 & 0.01 & 0.09 \end{pmatrix}$$

$$\Omega^{-1} = \begin{pmatrix} 71.4 & 14.3 & 14.3 \\ 14.3 & 28.6 & 0 \\ 14.3 & 0 & 14.3 \end{pmatrix}$$

Now

$$\begin{aligned} \boldsymbol{\omega}_1 &= \frac{\Omega^{-1}\mathbf{1}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}} = (0.583, 0.250, 0.167)^T \\ \boldsymbol{\omega}_2 &= \frac{\Omega^{-1}\boldsymbol{\mu}}{\mathbf{1}^T\Omega^{-1}\boldsymbol{\mu}} = (0.577, 0.231, 0.192)^T \end{aligned}$$

If one invests a portion θ in the portfolio with weight $\boldsymbol{\omega}_1$ and $1-\theta$ in the the portfolio with with weight $\boldsymbol{\omega}_2$ then the mean

$$\mu_p = \theta \boldsymbol{\omega}_1^T \boldsymbol{\mu} + (1-\theta)\boldsymbol{\omega}_2^T \boldsymbol{\mu} = 0.0877 - 0.00106\theta$$

and

$$\begin{aligned} \sigma^2 &= \theta^2 \boldsymbol{\omega}_1^T \Omega \boldsymbol{\omega}_1 + 2\theta(1-\theta)\boldsymbol{\omega}_2^T \Omega \boldsymbol{\omega}_2 + (1-\theta)^2 \boldsymbol{\omega}_2^T \Omega \boldsymbol{\omega}_2 \\ &= 0.076^2 + 65.62(\mu_P - 0.087)^2 \end{aligned}$$

The Markowitz curve is

$$\sigma = \sqrt{0.076^2 + 65.62(\mu_P - 0.087)^2}$$

The mean and the variance of the minimum variance portfolio are 0.087 and 0.076, respectively.

Next we will combine the N assets with a risk free asset just like we did before with two risky assets and a risk free asset

- In this case the return of a portfolio is

$$
\begin{aligned}
R_p &= \omega_0 \mu_f + \sum_{i=1}^{N} \omega_i R_i \\
&= \omega_0 \mu_f + \sum_{j=1}^{N} \omega_j \sum_{i=1}^{N} \frac{\omega_i}{\sum_{j=1}^{N} \omega_j} R_i \\
&= (1-\theta)\mu_f + \theta \sum_{i=1}^{N} \tilde{\omega}_i R_i
\end{aligned}
$$

where

$$
\theta = \sum_{j=1}^{N} \omega_j \quad \text{and} \quad \tilde{\omega}_\mathrm{i} = \frac{\omega_\mathrm{i}}{\sum_{\mathrm{j}=1}^{\mathrm{N}} \omega_\mathrm{j}}, \mathrm{j} = 1, 2, \ldots, \mathrm{N}.
$$

- Note that

$$
\sum_{i=1}^{N} \tilde{\omega}_i = 1 \quad \text{and} \quad \mathrm{Var}(\mathrm{R_p}) = \theta^2 \tilde{\boldsymbol{\omega}}^\mathrm{T} \Omega \tilde{\boldsymbol{\omega}}
$$

- Suppose we want the efficient portfolio with mean μ_p. That is we want to find $\tilde{\boldsymbol{\omega}}^*$ that solves

$$\min \frac{1}{2}\theta^2 \tilde{\boldsymbol{\omega}}^T \Omega \tilde{\boldsymbol{\omega}}$$

subject to

$$\sum_{i=1}^{N} \tilde{\omega}_i = 1 \quad \text{and} \quad (1-\theta)\mu_\mathrm{f} + \theta \sum_{\mathrm{i}=1}^{\mathrm{N}} \tilde{\omega}_\mathrm{i}\mu_\mathrm{i} = \mu_\mathrm{p}$$

- We will show that

$$\tilde{\boldsymbol{\omega}}^* = \frac{\Omega^{-1}(\boldsymbol{\mu} - \mu_f \mathbf{1})}{\mathbf{1}^T \Omega^{-1}(\boldsymbol{\mu} - \mu_f \mathbf{1})}$$

This is what we called the Tangency Portfolio.

- Example: Suppose $N = 2$ in which case

$$\Omega = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}$$

and

$$\Omega^{-1} = \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}\sigma_{21}} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{pmatrix}$$

- Let $V_1 = \mu_1 - \mu_f$ and $V_2 = \mu_2 - \mu_f$. We then have

$$\Omega^{-1}(\boldsymbol{\mu} - \mu_f \mathbf{1}) = \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}\sigma_{21}} \begin{pmatrix} \sigma_2^2 V_1 - \sigma_{12} V2 \\ -\sigma_{21} V_1 + \sigma_2^2 V_2 \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{1}^T\Omega^{-1}((\boldsymbol{\mu} - \mu_f \mathbf{1}) &= \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}\sigma_{21}} \mathbf{1}^T \begin{pmatrix} \sigma_2^2 V_1 - \sigma_{12} V2 \\ -\sigma_{21} V_1 + \sigma_2^2 V_2 \end{pmatrix} \\ &= \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}\sigma_{21}} (\sigma_1^2 V_2 + \sigma_2^2 V_1 - \sigma_{12}(V_1 + V2)) \end{aligned}$$

- This gives

$$\tilde{\omega}_1^* = \frac{\sigma_2^2 V_1 - \sigma_{12} V2}{\sigma_1^2 V_2 + \sigma_2^2 V_1 - \sigma_{12}(V_1 + V2)}$$

  and

$$\tilde{\omega}_2^* = 1 - \tilde{\omega}_1^*$$

This is what we called before ω_T and $1 - \omega_T$.

The proof: The Lagrangian corresponding to this optimization problem is

$$L(\tilde{\boldsymbol{\omega}}, \theta, \lambda_1, \lambda_2) = \frac{1}{2}\theta^2 \tilde{\boldsymbol{\omega}}^T \Omega \tilde{\boldsymbol{\omega}} + \lambda_1(1 - \tilde{\boldsymbol{\omega}}^T \mathbf{1}) + \lambda_2(\mu_p - (1-\theta)\mu_f - \theta \tilde{\boldsymbol{\omega}}^T \boldsymbol{\mu})$$

By taking the derivative with respect to $\theta, \tilde{\boldsymbol{\omega}}, \lambda_1$ and λ_2, we get

$$\begin{cases} \theta \tilde{\boldsymbol{\omega}}^T \Omega \tilde{\boldsymbol{\omega}} - \lambda_2(-\mu_f + \tilde{\boldsymbol{\omega}}^T \boldsymbol{\mu}) = 0 \\ \theta^2 \Omega \tilde{\boldsymbol{\omega}} - \lambda_1 \theta \boldsymbol{\mu} - \lambda_2 \mathbf{1} = \mathbf{0} \\ 1 - \tilde{\boldsymbol{\omega}}^T \mathbf{1} = 0 \\ \mu_p - (1-\theta)\mu_f - \theta \tilde{\boldsymbol{\omega}}^T \boldsymbol{\mu} = 0 \end{cases}$$

The second equation gives

$$\tilde{w} = \frac{1}{\theta^2} \Omega^{-1} (\lambda_1 \theta \boldsymbol{\mu} + \lambda_2 \mathbf{1})$$

Now multiplying the second equation by $\tilde{w}^T$ and the first equation by θ and take the difference, we get

$$\lambda_2 = -\lambda_1 \theta$$

This implies that

$$\tilde{w} = \frac{\lambda_1}{\theta} \Omega^{-1} (\boldsymbol{\mu} - \mu_f \mathbf{1})$$

Using the fact that $\tilde{\boldsymbol{\omega}}^T\mathbf{1} = 1$ we get

$$\tilde{w} = \frac{\Omega^{-1}(\boldsymbol{\mu} - \mu_f\mathbf{1})}{\mathbf{1}^T\Omega^{-1}(\boldsymbol{\mu} - \mu_f\mathbf{1})}$$

which is the desired result. The solution is

$$\tilde{w}^* = \frac{\Omega^{-1}(\boldsymbol{\mu} - \mu_f\mathbf{1})}{\mathbf{1}^T\Omega^{-1}(\boldsymbol{\mu} - \mu_f\mathbf{1})}$$

this portfolio what we called the Tangency portfolio or the market portfilio

- The tangency portfolio contains all the assets. This makes sense since if asset i is not in this portfolio, then no one will want to purchase it and so the asset will wither and die and thus out of the market
- If everyone purchases the same portfolio of risky assets, the weights in this portfolio must match the capitalization weights, being the proportion of each individual asset's total capitalization value to the total market capital value. That is

$$\tilde{\omega}_i^* = \frac{V_i}{V}$$

for all i where V_i and V are as defined before