$\underbrace{6+14}_{20}+\underbrace{8+5+10}_{23}=43$

Solutions Assignment 4

Ex. 1: Who invented the t-distribution? What was his occupation?

William Gosset, (Head) Brewer (of Guinness)

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Ex. 2: Let X denote the sample of undergraduate students taking the midterm and Y the sample of graduates students taking the midterm. Assume that both samples are i.i.d. normal with means mu_1 and mu_2 respectively and the same (unknown) variance. Test the hypthesis H_0: mu_1 = mu_2 against the alternative using the test statistic from the lectures. You are given the following values: m=90, n=8, X_bar= 29.405, Y_bar= 31.38, S_X^2= 7875.41, S_Y^2=671.88, alpha_0=0.25. What do you conclude? What is the p-value in this case?

We conduct a 2-sample t-test. We calculate

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$$U_{90,8}=\frac{(90+8-2)^{1/2}(29.405-31.38)}{\sqrt{\left(\frac{1}{90}+\frac{1}{8}\right)(7875.41+671.88)}}=-0.558$$

$$T^{-1}_{90+8-2}(1-0.125)=T^{-1}_{96}(0.875)=1.157$$

$\Rightarrow$ Fail to reject H_0 $\Rightarrow$ p-value $T_{96}(-0.558)+1-T_{96}(0.558)$

$=0.56$!!!

Ex. 3: Show that the two-sample test statistic U has t-distribution with m+n-2 degrees of freedom, if mu_1=mu_2.

See textbook Theorem 9.6.1 & proof

Not graded

1 point for each sentence
(If students directly conclude correctly, 2 pts. each) 4

9.1.2

(a) We know that if $0 < y < \theta$, then $\Pr(Y_n \le y) = (y/\theta)^n$. Also, if $y \ge \theta$, then $\Pr(Y_n \le y) = 1$. Therefore, if $\theta \le 1.5$, then $\pi(\theta) = \Pr(Y_n \le 1.5) = 1$. If $\theta > 1.5$, then $\pi(\theta) = \Pr(Y_n \le 1.5) = (1.5/\theta)^n$.

(b) The size of the test is

$$\alpha = \sup_{\theta \ge 2} \pi(\theta) = \sup_{\theta \ge 2} \left(\frac{1.5}{\theta}\right)^n = \left(\frac{1.5}{2}\right)^n \underbrace{= \left(\frac{3}{4}\right)^n}_{\text{not expected}}.$$

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8

9.1.4

The null hypothesis H_0 is simple. Therefore, the size α of the test is $\alpha = \Pr(\text{Rejecting } H_0 \,|\, \mu = \mu_0)$. 1
When $\mu = \mu_0$, the random variable $Z = n^{1/2}(\overline{X}_n - \mu_0)$ will have the standard normal distribution. 1
Hence, since $n = 25$,

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$$\alpha = \Pr(|\overline{X}_n - \mu_0| \ge c) = \Pr(|Z| \ge 5c) = 2[1 - \Phi(5c)].$$

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Thus, $\alpha = 0.05$ if and only if $\Phi(5c) = 0.975$. It is found from a table of the standard normal distribution that $5c = 1.96$ and $c = 0.392$.

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9.1.6

If H_0 is true, then X will surely be smaller than 3.5. If H_1 is true, then X will surely be greater than 3.5. Therefore, the test procedure which rejects H_0 if and only if $X > 3.5$ will have probability 0 of leading to a wrong decision, no matter what the true value of θ is.

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6

9.2.8

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(a) The p.d.f.'s $f_0(x)$ and $f_1(x)$ are as sketched in Fig. S.9.3. Under H_0 it is impossible to obtain a value of X greater than 1, but such values are possible under H_1. Therefore, if a test procedure rejects H_0 only if $x > 1$, then it is impossible to make an error of type 1, and $\alpha(\delta) = 0$. Also, 1

$$\beta(\delta) = \Pr(X < 1 \,|\, H_1) = \frac{1}{2}.$$

(Any value (1,2) instead of 1 is also fine.) 1

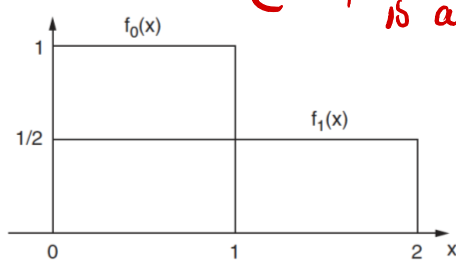

Figure S.9.3: Figure for Exercise 8a of Sec. 9.2.

(b) To have $\alpha(\delta) = 0$, we can include in the critical region only a set of points having probability 0 under H_0. Therefore, only points $x > 1$ can be considered. To minimize $\beta(\delta)$ we should choose this set to have maximum probability under H_1. Therefore, all points $x > 1$ should be used in the critical region.

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9.5.2 (a,b only)

When $\mu_0 = 20$, the statistic U given by Eq. (9.5.2) has a t distribution with 8 degrees of freedom. The value of U in this exercise is 2.

(a) We would reject H_0 if $U \geq 1.860$. Therefore, we reject H_0.

(b) We would reject H_0 if $U \leq -2.306$ or $U \geq 2.306$. Therefore, we don't reject H_0.

(a) $U = (9)^{1/2} \dfrac{22-20}{\left(\frac{72}{8}\right)^{1/2}} = 2$ 1 $T_8^{-1}(0.95) = 1.860$ 1 $\Rightarrow$ Reject 1

(b) $T_8^{-1}(0.975) = 2.306$ 1 $\Rightarrow$ Fail to reject 1

9.7.7

(a) Here, $\overline{X}_m = 84/16 = 5.25$ and $\overline{Y}_n = 18/10 = 1.8$. Therefore, $S_1^2 = \sum_{i=1}^{16} X_i^2 - 16(\overline{X}_m^2) = 122$ and $S_2^2 = \sum_{i=1}^{10} Y_i^2 - 10(\overline{Y}_n^2) = 39.6$. It follows that 6

$$\hat{\sigma}_1^2 = \frac{1}{16} S_1^2 = 7.625 \quad \text{and} \quad \hat{\sigma}_2^2 = \frac{1}{10} S_2^2 = 3.96.$$

(b) If $\sigma_1^2 = \sigma_2^2$, the following statistic V will have the F distribution with 15 and 9 degrees of freedom:

$$V = \frac{S_1^2/15}{S_2^2/9}.$$ 4

(b) If the test is to be carried out at the level of significance 0.05, then H_0 should be rejected if $V > 3.01$. It is found that $V = 1.848$. Therefore, we do not reject H_0.