Economics 361
Problem Set #1 (Suggested Solutions)

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Fall 2022

The answers for many of these problems can be found in various texts/websites. But that's no fun (and I'll "get you" come exam time). Consider these practice and solve them in good faith.

Question 1: Warm-up

Using just Kolmogorov's Axioms of Probability and the properties of set operation, prove the following properties of a probability function $P(\cdot)$

- $P(\emptyset) = 0$
- $P(A^c) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If $A \subset B$ then $P(A) \leq P(B)$

Note: there are many possible answers. The ones below are examples

$P(\emptyset) = 0$

- S and $\emptyset$ are pairwise disjoint; by the 3^{rd} Axiom, $P(S \cup \emptyset) = P(S) + P(\emptyset)$
- But $S \cup \emptyset = S$
- So $P(S \cup \emptyset) = P(S) + P(\emptyset) = P(S)$
- Therefore $P(\emptyset) = 0$

$P(A^c) = 1 - P(A)$

- A and A^c are pairwise disjoint; by the 3^{rd} Axiom, $P(A \cup A^c) = P(A) + P(A^c)$
- $A \cup A^c = S$
- So $P(A) + P(A^c) = P(A \cup A^c) = P(S) = 1$
- Therefore $P(A^c) = 1 - P(A)$

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$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- $B = (B \cap A) \cup (B \cap A^c)$ as A and A^c are pairwise disjoint and $A \cup A^c = S$
- So $P(B) = P(B \cap A) + P(B \cap A^c)$ and $P(B \cap A^c) = P(B) - P(A \cap B)$
- Similarly, $A \cup B = A \cup \{(B \cap A) \cup (B \cap A^c)\} = \{A \cup (B \cap A)\} \cup \{A \cup (B \cap A^c)\} = A \cup (B \cap A^c)$
- $P(A \cup B) = P(A \cup \{B \cap A^c\}) = P(A) + P(B \cap A^c)$ as A and $B \cap A^c$ are pairwise disjoint
- Combining all of the above, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If $A \subset B$ then $P(A) \leq P(B)$

- $B = A + (B \cap A^c)$ as $A \subset B$
- $P(B) = P(A + \{B \cap A^c\}) = P(A) + P(B \cap A^c)$ as A and $B \cap A^c$ are pairwise disjoint
- $P(B \cap A^c) \geq 0$ by 1^{st} Axiom
- Therefore $P(A) \leq P(B)$

Question 2: Choice of Uncertainty

(a) ANS: The key to solving this problem is in recognizing that you do not want the probability of drawing some colored ball – say, red – to be the same for the two urns. Without such probability differences, observing a particular draw provides little information about the identity of the urn.

Prior to any draws, the probability of drawing a red ball is $\frac{2}{3} \approx 0.67$ for urn A and $\frac{101}{201} \approx 0.50$ for urn B. But if a red ball is drawn and not replaced, the probability changes to $\frac{1}{2}$ for both urns (calculated using classical methods, as the balls represent distinct and equally likely outcomes that span the sample space). So, if a red ball is drawn, you should opt to replace the ball such that the urns once again have different probabilities of drawing a red (green) ball. However, if a green ball is drawn, the probability of drawing a red ball changes to 1 for urn A but $\frac{101}{200} \approx 0.50$ for urn B. Therefore, drawing a green ball amplifies the probability difference between the two urns – you should choose *not* to replace if the first draw is green.

With the above replacement strategy, the probability of getting red on the second draw is higher for urn A than for urn B. Similarly, the probability of getting green on the second draw (the complement of getting red on the second draw) is higher for urn B. So, guess urn A if the second draw is red and urn B if the second draw is green.

To make the above intuition abundantly clear, here is the brute force method involving the relevant joint and conditional probabilities. Events are characterized by three aspects: the urn chosen, color of the first ball, color of the second ball. So, there are 8 distinct joint events we need to consider. If the above replacement strategy is chosen, then

	Urn A	Urn B	Choice?
Red, Red	$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{2}{9}$	$\frac{1}{2} \cdot \frac{101}{201} \cdot \frac{101}{201} \approx \frac{1}{8}$	Urn A
Red, Green	$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{9}$	$\frac{1}{2} \cdot \frac{101}{201} \cdot \frac{100}{201} \approx \frac{1}{8}$	Urn B
Green, Red	$\frac{1}{2} \cdot \frac{1}{3} \cdot 1 = \frac{1}{6}$	$\frac{1}{2} \cdot \frac{100}{201} \cdot \frac{101}{200} \approx \frac{1}{8}$	Urn A
Green, Green	$\frac{1}{2} \cdot \frac{1}{3} \cdot 0 = 0$	$\frac{1}{2} \cdot \frac{100}{201} \cdot \frac{99}{200} \approx \frac{1}{8}$	Urn B

Note that (assuming the above replacement strategy)

- P(First Draw, Second Draw, Urn Type) = P(Second Draw | First Draw, Urn Type) × P(First Draw | Urn Type) × P(Urn Type) by iterated application of the definition of conditional probability
- P(Urn A) = P(Urn B) = $\frac{1}{2}$
- P(First Draw Red | Urn A) = $\frac{2}{3}$ and P(First Draw Red | Urn B) = $\frac{101}{201}$
- P(First Draw Green | Urn A) = $\frac{1}{3}$ and P(First Draw Green | Urn B) = $\frac{100}{201}$
- P(Second Draw Red | First Draw Red, Urn A) = $\frac{2}{3}$ and P(Second Draw Red | First Draw Red, Urn B) = $\frac{101}{201}$

- P(Second Draw Red | First Draw Green, Urn A) $= 1$ and P(Second Draw Red | First Draw Green, Urn B) $= \frac{101}{200}$
- P(Second Draw Green | First Draw Red, Urn A) $= \frac{1}{3}$ and P(Second Draw Green | First Draw Red, Urn B) $= \frac{100}{201}$
- P(Second Draw Green | First Draw Green, Urn A) $= 0$ and P(Second Draw Green | First Draw Green, Urn B) $= \frac{99}{200}$

Note that with a different replacement strategy, the conditional probabilities for the second draw (given first draw and urn type) differs.

For example, if the balls were always replaced after the first draw

	Urn A	Urn B	Choice?
Red, Red	$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{2}{9}$	$\frac{1}{2} \cdot \frac{101}{201} \cdot \frac{101}{201} \approx \frac{1}{8}$	Urn A
Red, Green	$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{9}$	$\frac{1}{2} \cdot \frac{101}{201} \cdot \frac{100}{201} \approx \frac{1}{8}$	Urn B
Green, Red	$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{1}{9} < \frac{1}{6}$	$\frac{1}{2} \cdot \frac{100}{201} \cdot \frac{101}{201} \approx \frac{1}{8}$	Urn B
Green, Green	$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{18} > 0$	$\frac{1}{2} \cdot \frac{100}{201} \cdot \frac{100}{201} \approx \frac{1}{8}$	Urn B

But the probability of choosing the right urn with the earlier replacement strategy is, approximately, $\frac{2}{9} + \frac{1}{8} + \frac{1}{6} + \frac{1}{8} = 0.64$ but with the strategy of always replacing approximately $\frac{2}{9} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 0.60$.

You can similarly show that with any other replacement strategy, the probability of choosing the right urn is less than the "optimal replacement strategy" that was initially proposed.

(b) ANS: There are 8 possible ways to schedule three tests, given two types of tests (**A,B**): (AAA),(AAB), (ABB), (BBB), (BBA), (BAA), (ABA), (BAB).

The tests are independent of each other. Let X_1, X_2, and X_3 represent the outcome for test 1, 2, and 3, respectively. We can show that

$$\begin{aligned}
P(X_1, X_2, X_3) &= P(X_1, X_2 \mid X_3)\ P(X_3) \ \text{ from def of conditional probability} \\
&= P(X_1, X_2)\ P(X_3) \ \text{ from independence of } X_1, X_2, X_3 \\
&= P(X_1 \mid X_2)\ P(X_2)\ P(X_3) \\
&= P(X_1)\ P(X_2)\ P(X_3)
\end{aligned}$$

Therefore, P(fail A then pass B then pass A) = P(fail A) $\times$ P(pass B) $\times$ P(pass A)

- (AAA) and (BBB) may be eliminated as the government requires the drug to pass both types of tests. There are no testing outcomes where (AAA) (BBB) yield drug approval
- Remaining schedules involving consecutive trials of the same test – (AAB), (ABB), (BBA), (BAA) – may be eliminated as you can show that each is dominated by one of the "alternating" schedules. For example, (AAB) has a lower probability of approval than (BAB) as (AAB) yields approval only if the latter two tests pass but (BAB) also yields approval when the former two tests pass

  P(AAB yielding approval) = P(pass A) P(pass A) P(pass B) + P(fail A) P(pass A) P(pass B) = P(pass A) P(pass B)

  P(BAB yielding approval) = P(pass B) P(pass A) P(pass B) + P(fail B) P(pass A) P(pass B) + P(pass B) P(pass A) P(fail B) = P(pass A) P(pass B) + P(pass B) P(pass A) P(fail B)

  Note that P(fail A) = 1 - P(pass A) and P(fail B) = 1 - P(pass B)

- So, the choice is between the two alternating schedules (ABA) and (BAB)

  P(ABA yielding approval) = P(pass A) P(pass B) + P(fail A) P(pass B) P(pass A)

  P(BAB yielding approval) = P(pass A) P(pass B) + P(pass B) P(pass A) P(fail B)

- Note that P(pass A) > P(pass B) and, therefore, P(fail A) < P(fail B)
- The firm should choose (BAB) as P(ABA yielding approval) < P(BAB yielding approval)

Question 3: Bayes' Rule and Monty Hall

One of the more useful properties of a probability function is **Bayes' Rule**:

$$P(A|B) \quad = \quad \frac{P(B|A)P(A)}{P(B)}$$

(a) Using the definition of conditional probability, prove the above Bayes' Rule.

From the definition of conditional probability, $P(A \cap B) = P(A|B)P(B)$**. Similarly,** $P(A \cap B) = P(B|A)P(A)$**. Setting the two equal to each other,** $P(A|B)P(B) = P(B|A)P(A)$**. Therefore** $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$

(b) Calculate the following probabilities

- P(Opens B | A has prize)

  If A **has the prize, then the host opens either** B **or** C **with equal probability**

  So $P(\textbf{Opens B} \mid \textbf{A has prize}) = \frac{1}{2}$

- P(Opens B | B has prize)

  If B **has the prize, then the host never opens** B

  So $P(\textbf{Opens B} \mid \textbf{B has prize}) = 0$

- P(Opens B | C has prize)

  If C **has the prize, then the host must open** B

  So $P(\textbf{Opens B} \mid \textbf{C has prize}) = 1$

- P(Opens B)

  $\textbf{Opens B} = (\textbf{Opens B} \ \cap \ \textbf{A has prize}) \ \cup \ (\textbf{Opens B} \ \cap \ \textbf{B has prize})$

  $\cup \ (\textbf{Opens B} \ \cap \ \textbf{C has prize})$

  Note that $P(\textbf{A has prize}) = P(\textbf{B has prize}) = P(\textbf{C has prize}) = \frac{1}{3}$

  So $P(\textbf{Opens B}) = \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3} = \frac{1}{2}$ **using def of cond. prob.**

(c) Now, using Bayes' Rule, calculate

- P(A has prize | Opens B)

  $$P(\textbf{A has prize} \mid \textbf{Opens B}) = \frac{P(\textbf{Opens B} \mid \textbf{A has prize}) \ P(\textbf{A has prize})}{P(\textbf{Opens B})}$$

  $$= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2}} \ = \ \frac{1}{3}$$

- P(C has prize | Opens B)

  $$P(\textbf{C has prize} \mid \textbf{Opens B}) = \frac{P(\textbf{Opens B} \mid \textbf{C has prize}) \ P(\textbf{C has prize})}{P(\textbf{Opens B})}$$

  $$= \frac{1 \times \frac{1}{3}}{\frac{1}{2}} \ = \ \frac{2}{3}$$

(d) Should you stick with A or switch to C?

Absolutely. Given that the host opened door B, you double your chance to win by switching. By switching, you are essentially getting to pick two of the three doors, as opposed to just one. (The host tells you which of the two has the prize – if any – through his reveal.)

(e) Now try answering this variant of the Monty Hall problem (credited to the late Harvard statistician Frederick Mosteller) often dubbed as "Prisoner's Dilemma meets Monty Hall"

> "Three prisoners, A, B, and C, with apparently equally good records have applied for parole. The parole board has decided to release two of the three, and the prisoners know this but not which two. A warder friend of prisoner A knows who are to be released. Prisoner A realizes that it would be unethical to ask the warder if he, A, is to be released, but thinks of asking for the name of one prisoner *other than himself* who is to be released. He thinks that before he asks, his chances of release are $\frac{2}{3}$. He thinks that if the warder says, 'B will be released,' his own chances have gone down to $\frac{1}{2}$, because either A and B or B and C are to be released. And so A decides not to reduce his chances by asking. However, A is mistaken in his calculations. Explain."

ANS: Let us first consider the relevant joint probabilities prior to A approaching the warder.

- P(A and B released) = P(A and C released) = P(B and C released) = $\frac{1}{3}$ as the prisoners are equally likely to be chosen

Now consider the warder's decision. If A had been chosen, the warder will with probability 1 mention the other released prisoner. But if A had not been chosen, the warder is equally like to mention either released prisoner:

- P(Warder says B | A and B released) = P(Warder says C | A and C released) = 1
- P(Warder says B | B and C released) = P(Warder says C | B and C released) = $\frac{1}{2}$

Therefore, using the definition of conditional probability

- P(A and B released and Warder says B) = P(A and C released and Warder says C) = $\frac{1}{3} \times 1 = \frac{1}{3}$
- P(B and C released and Warder says B) = P(B and C released and Warder says C) = $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
- Other four combinations (e.g. A and B released and Warder says C) have a joint probability of zero

And therefore

- $\text{P(A released | Warder says B)} = \dfrac{\text{P(A released and Warder says B)}}{\text{P(Warder says B)}}$
  $= \dfrac{\text{P(A and B released and Warder says B)}}{\text{P(A and B released and Warder says B)} + \text{P(B and C released and Warder says B)}} = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{6}} = \frac{2}{3}$

So, his chances of being released remains the same before and after asking the Warder!

You should be able to figure out the intuition underlying this result after having solved the "regular" Monty Hall problem. Note that the mathematical calculations for the above prisoner's problem and that of the earlier, traditional Monty Hall problem are analogous, with "NOT being chosen for release" corresponding to "winning the prize." (So, in the prisoner's problem, you win by losing ...)

Question 4: Current Events

This question concerns a debate between two prominent economists, Sendhil Mullainathan and Rajiv Sethi, concerning racial bias and police shootings.

Read the following short articles/blog posts

- "Police Killings of Blacks: Here is What the Data Says," Sendhil Mullainathan, NY Times 10/16/2015 https://www.nytimes.com/2015/10/18/upshot/police-killings-of-blacks-what-the-data-says.html
- "Threats Perceived When There Are None," Rajiv Sethi, blog post, 10/16/2015 http://rajivsethi.blogspot.com/2015/10/threats-perceived-when-there-are-none.html
- "It's all about the denominator: Rajiv Sethi and Sendhil Mullainathan in a statistical debate on racial bias in police killings," Andrew Gelman, blog post 10/21/2015 http://andrewgelman.com/2015/10/21/its-all-about-the-denominator-and-rajiv-sethi-and-sendhil-mullainathan-in-a-statistical-debate-on-racial-bias-in-police-killings/

For the purposes of this question, let the relevant random experiment be defined as an encounter between a police officer and some community resident. Some of the relevant events associated with outcomes of this random experiment are

- `Black`: outcomes where the resident was African American
- `Arrested`: outcomes where the resident was arrested by the police officer
- `Shot`: outcomes where the resident was shot by the police officer

For simplicity, assume that $\texttt{Shot} \subset \texttt{Arrested}$; all residents who were shot were also arrested.

Within this framework, the "31.8 percent of people shot by the police were African-American" quoted by Sendhil Mullainathan may be interpreted as an estimate of $\mathrm{P}(\texttt{Black} \mid \texttt{Shot})$. Similarly, the "28.9 percent of arrestees were African-American" as an estimate of $\mathrm{P}(\texttt{Black} \mid \texttt{Arrested})$

(a) Express $\mathrm{P}(\texttt{Black} \mid \texttt{Arrested})$ in terms of $\mathrm{P}(\texttt{Black} \mid \texttt{Shot})$, $\mathrm{P}(\texttt{Black} \mid \texttt{Arrested and Not Shot})$, and $\mathrm{P}(\texttt{Shot} \mid \texttt{Arrested})$

ANS: Note from the definition of conditional probability

- $\mathrm{P}(\texttt{Black} \mid \texttt{Arrested}) = \dfrac{\mathrm{P}(\texttt{Black and Arrested})}{\mathrm{P}(\texttt{Arrested})}$
  $= \dfrac{\mathrm{P}(\texttt{Black and Arrested and Shot}) + \mathrm{P}(\texttt{Black and Arrested and Not Shot})}{\mathrm{P}(\texttt{Arrested})}$
  $= \dfrac{\mathrm{P}(\texttt{Black and Arrested and Shot})}{\mathrm{P}(\texttt{Arrested and Shot})} \cdot \dfrac{\mathrm{P}(\texttt{Arrested and Shot})}{\mathrm{P}(\texttt{Arrested})}$
  $+ \dfrac{\mathrm{P}(\texttt{Black and Arrested and Not Shot})}{\mathrm{P}(\texttt{Arrested and Not Shot})} \cdot \dfrac{\mathrm{P}(\texttt{Arrested and Not Shot})}{\mathrm{P}(\texttt{Arrested})}$
  $= \mathrm{P}(\texttt{Black} \mid \texttt{Arrested and Shot}) \cdot \mathrm{P}(\texttt{Shot} \mid \texttt{Arrested})$
  $+ \mathrm{P}(\texttt{Black} \mid \texttt{Arrested and Not Shot}) \cdot \mathrm{P}(\texttt{Not Shot} \mid \texttt{Arrested})$
  $= \mathrm{P}(\texttt{Black} \mid \texttt{Shot}) \cdot \mathrm{P}(\texttt{Shot} \mid \texttt{Arrested})$
  $+ \mathrm{P}(\texttt{Black} \mid \texttt{Arrested and Not Shot}) \cdot (1\text{-}\ \mathrm{P}(\texttt{Shot} \mid \texttt{Arrested}))$

The last step uses $\texttt{Shot} \subset \texttt{Arrested}$ and recognizes that ($\texttt{Shot}$ and $\texttt{Not Shot}$) are complements

Note: **(b)** and **(c)** are fairly open-ended problems, with several acceptable answers. For each, I provide one possible answer

(b) Use the above framework to explain Mullainathan's hypothesis concerning why African Americans are seemingly disproportionately likely to be police shooting victims.

ANS: For notational simplicity, let

- $\mathrm{A} = \mathrm{P}(\texttt{Black} \mid \texttt{Arrested})$
- $\mathrm{B} = \mathrm{P}(\texttt{Black} \mid \texttt{Shot}) = \mathrm{P}(\texttt{Black} \mid \texttt{Arrested and Shot})$
- $\mathrm{C} = \mathrm{P}(\texttt{Black} \mid \texttt{Arrested and Not Shot})$
- $\mathrm{D} = \mathrm{P}(\texttt{Shot} \mid \texttt{Arrested})$
- Therefore $\mathrm{A} = \mathrm{B} \cdot \mathrm{D} + \mathrm{C} \cdot (1\text{-}\mathrm{D})$

A is therefore a proper mixture of B and C. In order for A to be similar (same) as B, there are two possibilities. First, D is large (close to 1); so A is largely based on B. This translates into the situation where a high proportion of arrests involve shootings. So, Mullainathan's view would be correct in this possibility as the large share of African Americans involved in police shootings is more a consequence of the large share of African Americans involved in arrests.

The other possibility is that B and C are similar; A is a mixture of two similar values. This translates into the situation that the share of African Americans involved in police shootings is not that different from the share of African Americans involved in arrests that do not involve police shootings. So, conditional on an arrest, knowing whether the resident was shot does not provide any more statistical information about whether the resident was African American. This is also consistent with Mullainathan's view that the seemingly larger share of police shootings involving African American residents is driven by racial differences in arrests.

(c) Use the above framework to explain Sethi's critique of Mullainathan's hypothesis.

ANS: This problem is even more open-ended. Consider the following that builds off Sethi's narrative concerning safe/risky encounters. Let us introduce `Risky` as a possible encounter event

- `Risky`: outcomes where the encounter is "risky" for the police officer (e.g. resident is armed)

Using Bayes' Theorem

- $\mathrm{P}(\texttt{Black} \mid \texttt{Arrested}) = \mathrm{P}(\texttt{Arrested} \mid \texttt{Black}) \cdot \dfrac{\mathrm{P}(\texttt{Black})}{\mathrm{P}(\texttt{Arrested})}$
- $\mathrm{P}(\texttt{Black} \mid \texttt{Shot}) = \mathrm{P}(\texttt{Shot} \mid \texttt{Black}) \cdot \dfrac{\mathrm{P}(\texttt{Black})}{\mathrm{P}(\texttt{Shot})}$

So, $\mathrm{P}(\texttt{Black} \mid \texttt{Arrested}) \approx \mathrm{P}(\texttt{Black} \mid \texttt{Shot})$ implies

- $\mathrm{P}(\texttt{Arrested} \mid \texttt{Black}) \cdot \dfrac{\mathrm{P}(\texttt{Black})}{\mathrm{P}(\texttt{Arrested})} \approx \mathrm{P}(\texttt{Shot} \mid \texttt{Black}) \cdot \dfrac{\mathrm{P}(\texttt{Black})}{\mathrm{P}(\texttt{Shot})}$
- $\mathrm{P}(\texttt{Shot} \mid \texttt{Black}) \approx \mathrm{P}(\texttt{Arrested} \mid \texttt{Black}) \cdot \dfrac{\mathrm{P}(\texttt{Shot})}{\mathrm{P}(\texttt{Arrested})}$

Note that Risky and Not Risky are complements. So

- $\text{P}(\texttt{Shot and Risky} \mid \texttt{Black}) + \text{P}(\texttt{Shot and Not Risky} \mid \texttt{Black})$
  $\approx [\text{P}(\texttt{Arrested and Risky} \mid \texttt{Black}) + \text{P}(\texttt{Arrested and Not Risky} \mid \texttt{Black})] \cdot \dfrac{\text{P}(\texttt{Shot})}{\text{P}(\texttt{Arrested})}$

The above can be re-written using the definition of conditional probability

- $\text{P}(\texttt{Shot} \mid \texttt{Black and Risky}) \cdot \text{P}(\texttt{Risky} \mid \texttt{Black}) + \text{P}(\texttt{Shot} \mid \texttt{Black and Not Risky}) \cdot \text{P}(\texttt{Not Risky} \mid \texttt{Black})$
  $\approx [\ \text{P}(\texttt{Arrested} \mid \texttt{Black and Risky}) \cdot \text{P}(\texttt{Risky} \mid \texttt{Black})$
  $+ \text{P}(\texttt{Arrested} \mid \texttt{Black and Not Risky}) \cdot \text{P}(\texttt{Not Risky} \mid \texttt{Black})\ ] \cdot \dfrac{\text{P}(\texttt{Shot})}{\text{P}(\texttt{Arrested})}$ ... (1)

Note that P(Not Black | Arrested) = 1 - P(Black | Arrested) as ((Black, Not Black) are complements.. Similarly, P(Not Black | Shot) = 1 - P(Not Black | Shot). Therefore, P(Black | Arrested) ≈ P(Black | Shot) implies P(Not Black | Arrested) ≈ P(Not Black | Shot).

So

- $\text{P}(\texttt{Shot} \mid \texttt{Not Black and Risky}) \cdot \text{P}(\texttt{Risky} \mid \texttt{Not Black})$
  $+ \text{P}(\texttt{Shot} \mid \texttt{Not Black and Not Risky}) \cdot \text{P}(\texttt{Not Risky} \mid \texttt{Not Black})$
  $\approx [\ \text{P}(\texttt{Arrested} \mid \texttt{Not Black and Risky}) \cdot \text{P}(\texttt{Risky} \mid \texttt{Not Black})$
  $+ \text{P}(\texttt{Arrested} \mid \texttt{Not Black and Not Risky}) \cdot \text{P}(\texttt{Not Risky} \mid \texttt{Not Black})\ ] \cdot \dfrac{\text{P}(\texttt{Shot})}{\text{P}(\texttt{Arrested})}$ ... (2)

For simplicity, assume that there are no racial differences in the riskiness of an encounter:

- $\text{P}(\texttt{Risky} \mid \texttt{Black}) = \text{P}(\texttt{Risky} \mid \texttt{Not Black})$
- $\text{P}(\texttt{Not Risky} \mid \texttt{Black}) = \text{P}(\texttt{Not Risky} \mid \texttt{Not Black})$

Then Sethi's conjectures concerning police officers "perceiving threats when there are none" can be interpreted as a claim of

- $\text{P}(\texttt{Shot} \mid \texttt{Black and Not Risky}) > \text{P}(\texttt{Shot} \mid \texttt{Not Black and Not Risky})$
- $\text{P}(\texttt{Arrested} \mid \texttt{Black and Not Risky}) > \text{P}(\texttt{Arrested} \mid \texttt{Not Black and Not Risky})$

The first inequality is police officers more likely to shoot an unarmed African American resident than an unarmed non-African American ("white") resident. The second inequality is related to Sethi's observation concerning the Henry Louis Gates arrest.

Note that the above inequalities do not necessarily violate (1) and (2) and, as such, may be consistent with P(Black | Arrested) ≈ P(Black | Shot). The two inequalities help "balance" the two sides of P(Black | Arrested) ≈ P(Black | Shot). For racial bias in shooting that vary by encounter type, racial bias in arrests may "cover up" racial bias in shootings.

If you find this composition argument difficult to follow, try reading a follow-up blog post that Rajiv Sethi wrote in July 2016 where he discusses a similar issue:

http://rajivsethi.blogspot.com/2016/07/a-fallacy-of-composition.html

For more information concerning the related debate centered on Roland Fryer's high-profile study:

https://fivethirtyeight.com/features/why-are-so-many-black-americans-killed-by-police/