COLUMBIA UNIVERSITY
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Bayesian Statistics

Dobrin Marchev

Generalized Linear Mixed Effects

- As the name suggests, a generalized linear mixed effects model combines aspects of linear mixed effects models with those of generalized linear models.
- Such models are useful when we have a hierarchical data structure but the normal model for the within-group variation is not appropriate.
- For example, if the variable Y were binary or a count, then more appropriate models for within-group variation would be logistic or Poisson regression models, respectively.

GLMM

A basic generalized linear mixed model is as follows:

$$f(\boldsymbol{y}_j|X_j, \beta_j, \gamma) = \prod_{j=1}^{n_j} f(y_{ij}|\beta_j' x_{ij}, \gamma)$$

$$\beta_1, \dots, \beta_m \sim iid\ N(\theta, \Sigma)$$

with observations from different groups also being conditionally independent.

Note that $f(y|\boldsymbol{\beta}'\boldsymbol{x}, \gamma)$ is a density from the Exponential family whose mean depends on $\boldsymbol{\beta}'x$ via a link function, and γ is an additional parameter, often representing scale.

For example, in the Poisson model $f(y|\boldsymbol{\beta}'\boldsymbol{x}, \gamma)$ is Poisson pmf with parameter (average) $= e^{\boldsymbol{\beta}'\boldsymbol{x}}$ and $\gamma = 1$ (there is no γ).

Metropolis - Gibbs for GLMM

Estimation for the linear mixed effects model was straightforward because the full conditional distribution of each parameter was standard, allowing for the easy implementation of a Gibbs sampling algorithm. In contrast, for non- normal generalized linear mixed models, typically only θ and Σ have standard full conditional distributions.

This suggests we use a Metropolis-Hastings algorithm to approximate the posterior distribution of the parameters, using a combination of Gibbs steps for updating (θ, Σ) with a Metropolis step for updating each β_j.

The full conditional distributions of θ and Σ depend only on $\beta_1, \ldots, \beta_m$. The form of the specific exponential family doesn't affect this step.

Metropolis step on each β_j can be done with a multivariate normal proposal. If there is γ it has to be handled with one extra Gibbs.

Example 1: pulp data

Data concerns the brightness of paper which may vary between operators of the production machinery.

We fit ANOVA with random effects.

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, i = 1, \ldots, a, j = 1, \ldots, n_i$$

Questions:

1. Is there a difference between operators in general?
2. How much is the difference between operators in general?
3. How does the variation between operators compare to the variation within operators?
4. What is the difference between these four operators?

Example 2: nitrofen data

In Davison and Hinkley, 1997, the results of a study on Nitrofen, a herbicide, are reported. Due to concern regarding the effect on animal life, 50 female water fleas were divided into five groups of ten each and treated with different concentrations of the herbicide. The number of offspring in three subsequent broods for each flea was recorded.

We fit Poisson GLMM :

$$Y_i \sim Poisson(e^{\eta_i}), i = 1, \ldots, 150$$
$$\eta_i = x_i'\beta + u_j, i = 1, \ldots, 150, j = 1, \ldots, 50$$

There are two fixed effects: concentration and brood and we assume they interact.

There is one random effect: the water flea

Example 3: tumor location data

A certain population of laboratory mice experiences a high rate of intestinal tumor growth. One item of interest to researchers is how the rate of tumor growth varies along the length of the intestine. To study this, the intestine of each of 21 sample mice was divided into 20 sections and the number of tumors occurring in each section was recorded.

We fit a Poisson model

$$Y_{xj} \sim Poisson\left(e^{f_j(x)}\right)$$
$$f_j(x) = \beta_{1j} + \beta_{2j}x + \beta_{3j}x^2 + \cdots + \beta_{pj}x^{p-1}$$