Homework 4: Monday February 13, 2023

Due Wednesday Feb 22, 2023

Problem 1. Simulating Hedging at Different Replication Volatilities [40 points]

Consider a call with stock price 100, strike 100, time to expiration = 1/12 of a year, r = 0, $\mu = 0$, zero dividend yield. Assume zero transactions costs. **The realized stock volatility is always 30%**.

Use this formula for the replication of the option where Δ is calculated at the **replication** volatility:

$$C_0 = \Delta_0 S_0 + e^{-r\tau}(C_T - \Delta_T S_T) + \int_0^{\tau} e^{-rx} S_x [d\Delta_x]_b$$

Use a Monte Carlo program with 10,000 paths (always begin **with the same seed for each simulation a - f below**) to find the expected value of the call and its standard deviation from the values on the distribution of paths in the following cases:

a. replication vol = 30%; rebalance the Delta at regular intervals 30 times.

b. replication vol = 30%; rebalance at regular intervals 120 times.

c. replication vol = 35%,; rebalance at regular intervals 30 times.

d. replication vol = 35%; rebalance at regular intervals 120 times.

e. replication vol = 40%; replication at regular intervals 30 times.

f. replication vol =40%; rebalance at regular intervals 120 times.

Now let $HE_m(\sigma_h)$ be the standard deviation of the call distribution for m rebalancings at replication volatility σ_h.

Plot the ratio $\frac{HE_{30}(\sigma_h)}{HE_{120}(\sigma_h)}$ of the replication errors as a function of σ_h for the 3 values of hedging volatility used above.

Solution 1: approx mean, std dev, ratio

a. 3.43, 0.54,

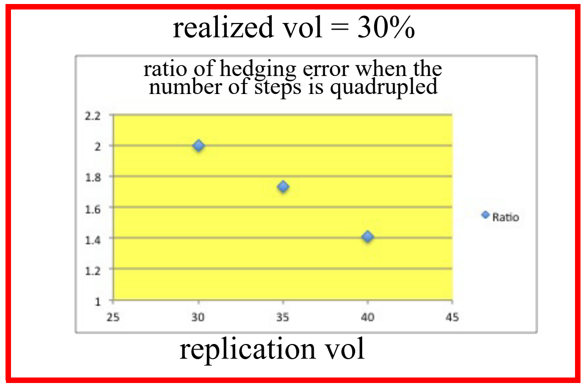
b. 3.43, 0.27, ratio 2.0

c. 3.4, 0.57,

d. 3.44, 0.33, ratio 1.73

e. 3.44, 0.65,

f. 3.43, 0.46, ratio 1.41



Problem 2: Strike and Delta **[20 points]**

Consider a call option C with stock price = 100, an annual stock volatility of 0.1 (i.e. 10% per annum) and three months to expiration, and assume all interest rates are equal to zero.

(i) Using the formula for the delta of a call, what is the value of the strike K_1 for which the call's delta is $\Delta = 0.5$. [5 points]

(ii) What is the approximate value of the strike K_2 for which the call's delta is $\Delta = 0.25$? Don't use a Black-Scholes calculator, use the formula, approximately, but check your answer with a Black-Scholes calculator. [15 points]

Solution 1: Delta and Strike

The BS call formula is given by $\boxed{C(S, K, T-t, r, \Sigma)}$.

Here we have S = 100, T - t = 0.25 years, r = 0, $\boxed{\Sigma = 0.10}$ and $\boxed{\Sigma\sqrt{T-t} = 0.05}$.

$$\boxed{\begin{aligned} \Delta &= N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} \exp\left(-\frac{x^2}{2}\right)dx \\ d_1 &= \frac{\ln S/K}{\Sigma\sqrt{T-t}} + \frac{\Sigma\sqrt{T-t}}{2} \end{aligned}}$$

(i) $\boxed{\Delta = 0.5}$ **when** $\boxed{d_1 = 0}$**.**

That is $\boxed{\ln\frac{S}{K} = \frac{-\Sigma^2\tau}{2} = -\frac{(0.1)^2(0.25)}{2} = -0.0013}$

Or $\boxed{K_1 = S\exp(0.0013) \sim 100 \times 1.0013 \approx 100.13}$

(b) $\boxed{\Delta = 0.25}$ when $\boxed{N(d_1) = 0.25}$.

$$\boxed{\begin{aligned} N(d_1) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} \exp\left(-\frac{x^2}{2}\right)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \exp\left(-\frac{x^2}{2}\right)dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{d_1} \exp\left(-\frac{x^2}{2}\right)dx \\ 0.25 &\approx 0.5 + (0.39)d_1 \approx 0.5 + (0.39)\left(\frac{\ln S/K_2}{\Sigma\sqrt{T-t}} + \frac{\Sigma\sqrt{T-t}}{2}\right) \end{aligned}}$$

assuming $\boxed{\exp(-0.5d_1^2)}$ is close to 1. Therefore

$$\boxed{\frac{-0.25}{0.39} \approx \frac{\ln 100/K_2}{0.05} + 0.025}$$

or

$$\boxed{\begin{aligned} -0.64 &\approx \frac{\ln 100/K_2}{0.05} + 0.025 \\ \frac{\ln 100/K_2}{0.05} &\approx -0.641 - 0.025 = -0.666 \end{aligned}}$$

and so $\boxed{K_2 = 100\exp(0.033) \approx 103.4}$

Checking our assumptions:

For this value of $\boxed{K_2 = 103.4}$, $\boxed{d_1 \sim -0.64}$ and $\boxed{\exp(-0.5d_1^2) \sim 0.82}$ which is not too far from 1.____________

Problem 3: Arbitrage bounds on the smile **[20 points]**

(i) Show by examining the payoffs of a put with strike K and a put with strike L > K that the put with the higher strike must always be worth at least as much as the put with strike K. [5]

(ii) Assume S = 100, K = 100, r = 0, div yield = 0, and that the implied volatility for a one-year put with strike 100 is 20%. Given only this information, find an upper bound on the implied volatility of a one-year put with strike 90. You can use a Black-Scholes calculator if you like. [10]

(iii) Repeat case (ii) above with the only difference that all statements refer to a one-month put rather than a one-year put. [5]

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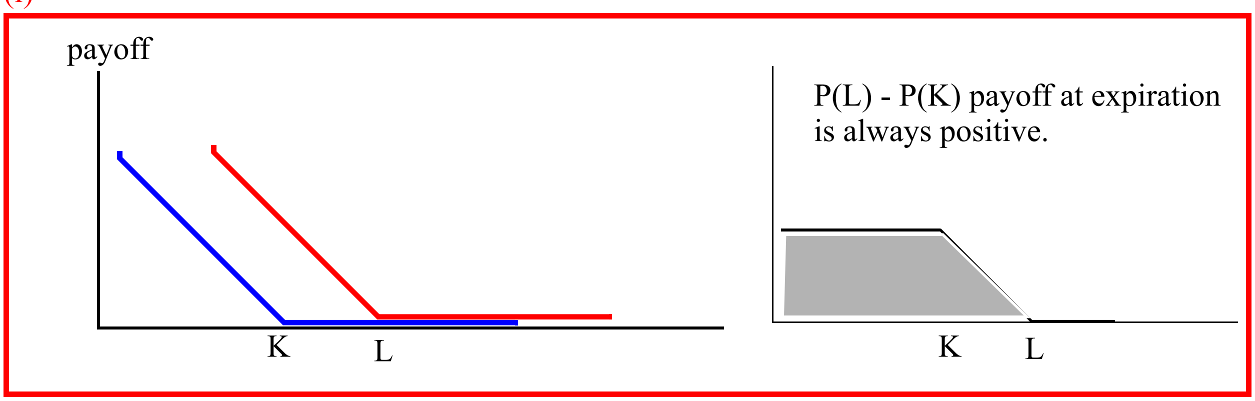
Solution 3: Arbitrage bounds

(i)



(ii) A one-year put with strike 100 is worth 7.97 at 20% vol. Therefore a one-year put with strike 90 must be worth no more. The greatest corresponding implied vol is about 32.6%

(iii) A one-month at-the-money put is worth 2.29 at 20% vol. A one-month put at strike 90 can be worth no more. The greatest implied vol is about 55.4%. The implied vol is so much bigger here because 90 is much more out-of-the-money in units of $\boxed{\Sigma\sqrt{T-t}}$ or in terms of $\boxed{\Delta}$ at one month than it is at one year, and so it needs a much bigger implied volatility to match the limiting price.

Problem 4. Another tighter arbitrage bound on the smile **[20 points]**

(i) Show that if the strike $K_1 < K_2$, the value of a European put $P(K)$ with strike K on a non-dividend-paying stock must satisfy $\frac{P(K_1)}{K_1} < \frac{P(K_2)}{K_2}$ in order not to violate the principle of no riskless arbitrage. [10]

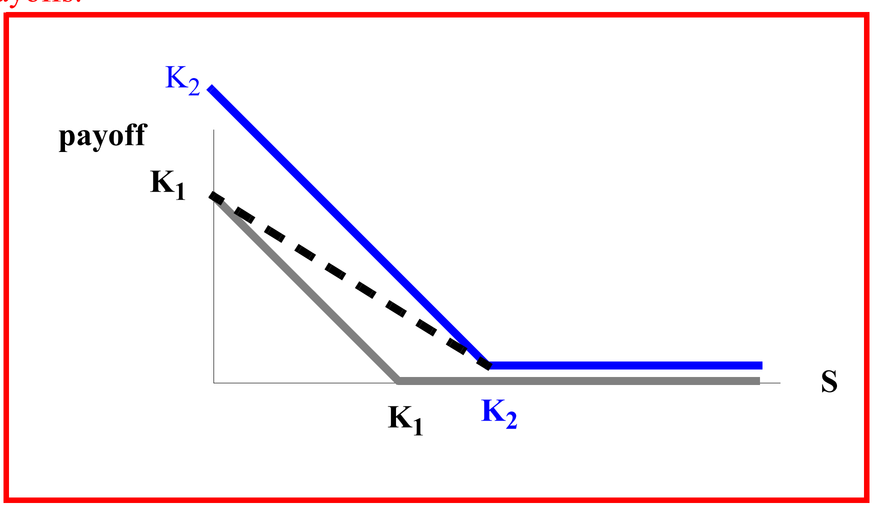
(ii) In the continuum limit for the strikes close to each others, i.e. $K_2 = K_1 + \delta K$, show that

$$K\frac{\partial \Sigma_{BS}}{\partial K} > -\frac{N(-d_1)}{\sqrt{\tau}N'(d_1)}$$

where τ is the time to expiration of the put, N(x) the cumulative normal distribution, d_1 is the standard definition, and N'(x) is the derivative of N(x) w.r.t. x. [10]

Solution 4: Another arbitrage bound on the smile

Here are put payoffs:



The put $\boxed{P(K_2)}$ has a greater payoff than $\boxed{P(K_1)}$ and therefore is always worth more. But one can do better. By looking at the dotted line payoff above which still dominates $\boxed{P(K_1)}$, you can see that $\boxed{K_1/K_2}$ puts with strike $\boxed{K_2}$ dominates $\boxed{P(K_1)}$ too.

Therefore $\boxed{\dfrac{P(K_1)}{K_1} < \dfrac{P(K_2)}{K_2}}$

In the continuum limit as the strikes approach each other, we obtain $\boxed{\dfrac{\partial}{\partial K}\left(\dfrac{P(K)}{K}\right) > 0}$

Thus in terms of the Black-Scholes parametrization:

$$\boxed{\frac{1}{K}\left(\frac{\partial P_{BS}}{\partial K} + \frac{\partial P_{BS}}{\partial \Sigma}\frac{\partial \Sigma}{\partial K}\right) - \frac{1}{K^2}P_{BS} > 0}$$

or

$$\boxed{\frac{\partial \Sigma}{\partial K} > \frac{\dfrac{P_{BS}}{K} - \dfrac{\partial P_{BS}}{\partial K}}{\dfrac{\partial P_{BS}}{\partial \Sigma}} = \frac{e^{-r\tau}N(-d_2) - \dfrac{S}{K}N(-d_1) - e^{-r\tau}N(-d_2)}{SN'(d_1)\sqrt{\tau}} = \frac{-N(-d_1)}{KN'(d_1)\sqrt{\tau}}}$$