Class 6: Equally Likely Outcomes & Counting Problems

Thursday, September 19, 2019 11:37 AM

- HW 3 due Thurs, Sept 26, 1pm on Gradescope
- Recitations: Fri, 3-4pm, Mudd 834
  starting this week!

Equally Likely Outcomes:

- Say Ω has n outcomes, each equally likely,
  then each has prob. $\frac{1}{n}$.
- So prob. of an event E is then $\dfrac{|E|}{n} = \dfrac{|E|}{|\Omega|}$.

Basic Counting Principles: ordered/unordered?

① N "objects," each can be in one of k different "states."

Ex: N dice, $k=6$ different numbers possible for each die.

$1,2,\ldots,k$ $1,2,\ldots,k$ $1,2,\ldots,k$ ... $1,2,\ldots,k$

$\sqcup$ $\sqcup$ $\sqcup$ ... $\sqcup$

1   2   3   ... N

(with replacement)

↳ k^N different ways to configure the N objects

Ex: → 6^N ways to roll N dice.

→ We are keeping track of order

② N objects, choose k objects from the N, without replacement.

don't put it back

Q1: How many ways to pick k ordered objects out of N?

(without replacement)

$$\underset{1^{st}\text{ choice}}{N}\ \underset{2^{nd}}{(N-1)}\ \underset{3^{rd}}{(N-2)}\ \underset{4^{th}}{(N-3)}\ \underset{\ldots}{\ldots}\ \underset{k^{th}}{(N-k+1)}$$

$N \quad (N-1)(N-2)(N-3) \cdots (N-k+1)$

1st choice 2nd 3rd 4th ... kth

Note: If $k=N$, this says there are

$$N(N-1)\cdots 2\cdot 1 = N!$$

ways to order our N objects

Q2: How many ways to pick k unordered objects from N?

Ex: $N=52$, $k=5$, poker hands (order doesn't matter)

$$\binom{N}{k} = \frac{N!}{k!(N-k)!} = \frac{N(N-1)\cdots(N-k+1)(N-k)\cdots 2\cdot 1}{k!\ (N-k)(N-k-1)\cdots 2\cdot 1}$$

$$= \frac{N(N-1)\cdots(N-k+1)}{k!}$$

$$= \frac{\#\text{ ways to pick } k \text{ ordered elements out of } N}{\#\text{ ways to order those } k \text{ elements}}.$$

Summary: 2 important questions to ask:

1. ordered / unordered?
2. replacement / no replacement?

ordered → factorial

unordered → binomial coefficients

replacement → exponent

no replacement → factorial

Ex 1: Birthday Problem.

- What is the probability that some 2 people out of a group of N people share the out of a group of N people share the same birthday?

- Could write sample space: $\Omega = \{1, 2, \dots, 365\}^N$.

  Each outcome equally likely, prob. is $\dfrac{1}{365^N}$.

- Let $p_N = \mathbb{P}(\text{some 2 people share bday out of } N)$

$$= \frac{\#\text{ ways to have 2 people share bday out of } N}{365^N}.$$

- Let $q_N = 1 - p_N = \mathbb{P}(\text{ no 2 people share bday out of } N)$.

- <u>First cases:</u>

<u>$N=2$:</u> $q_2 = \mathbb{P}(\text{2 people out of 2 have diff. bdays})$

$$= \underbrace{\frac{365}{365}}_{1^{st}\text{ person}} \cdot \underbrace{\frac{364}{365}}_{2^{nd}\text{ person}}$$

$q_3 = \mathbb{P}(\text{3 people have distinct bdays})$

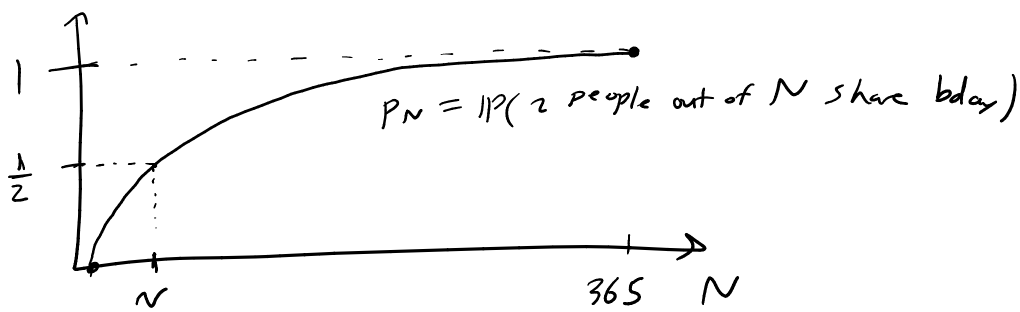
$$= \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \qquad \text{(with replacement)}$$

$\vdots$

$$q_N = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \dots \frac{365-N+1}{365} \qquad \text{for } N \leq 365$$

$$= \frac{365!}{(365-N)!\, 365^N}.$$

<u>Plot $p_N = 1 - q_N$ versus N:</u>



Smallest N such that $p_N \geq \frac{1}{2}$ is $N = 23$.

Poker Hands:

- Deck of 52 cards. $(2, 3, \dots, 10, J, Q, K, A)$ (club, spade, diamond, heart)
- Draw a 5-card hand. (without replacement!)

Q1: How many distinct 5-card hands are there?

$$\binom{52}{5}$$

Q2: How many ways could I draw a pair?

"best hand" → not 3 of a kind
not full house
not 2 pair

J♦ 3♥ 7♣ 3♦ 4♣

There are 13 different non-suited cards we could pair.
After we pick 3 to be paired there are 12 more non-suited cards to choose from,

$$13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4 \cdot 4 \cdot 4$$

choose 3 to be the paired card

ways to pick 2 of the 4

pick 3 distinct or non-suited cards

4 J's to choose from

4 7's to choose from

4 4's to choose from

choose 5 to be the paired card

ways to pick 2 of the 4 3's

pick 3 distinct or non-suited cards of the remaining 12, say J, 7, 4.

Q2: How many ways to get 3 of a kind?

7♠ J♠ J♦ J♥ 5♠

$$\underbrace{13}_{\text{which card to get 3 of: J}} \cdot \underbrace{\binom{4}{3}}_{\text{ways to draw 3 of the 4 J's (without order)}} \cdot \underbrace{\binom{12}{2}}_{\text{pick 2 distinct non-suited cards of remaining 12 say 7 and 5}} \cdot \underbrace{4}_{\text{4 5's in deck}} \cdot \underbrace{4}_{\text{4 7's in deck}}$$

Ex: Probability of getting 3 of a kind

$$\frac{13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4 \cdot 4}{\binom{52}{5}}$$

Infinite Sample Spaces $\quad |\Omega| = \infty$

Two cases: countable vs. uncountable

Ex: Countable case:

- Choose a random number from $\{1, 2, \ldots\} = \mathbb{N} = \Omega$.
- Model the number of coin flips until getting first heads. (include last flip in count)

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(include last flip in count)

$$P(k) = P(\text{first } k-1 \text{ flips are T and } k^{th} \text{ is heads})$$

$$P(1) = \frac{1}{2}$$

$$P(2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^2}$$

$$P(3) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^3}$$

$$\ddots$$

$$P(k) = \frac{1}{2^k}.$$

- Is this a valid probability measure? $P(k) \geq 0$ ✓

should add up to 1:

$$\boxed{\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}}$$

$$\sum_{k=1}^{\infty} P(k) = \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)^k = 1. \quad ✓$$

- <u>Note</u>: On infinite sample spaces, we <u>cannot</u> have equally likely outcomes!

<u>Ex.</u> If $P(k) = c$ for all k then $\sum_{k=1}^{\infty} P(k) = \sum_{k=1}^{\infty} c = \begin{cases} 0 & \text{if } c=0 \\ \infty & \text{if } c>0. \end{cases}$

This cannot add up to 1!