# STAT GU4221/GR5221 Homework 1 [100 pts]
# Due: Monday, February 9th at 11:59pm (ET)

**Problem 1**

Let $\{X_t : t \in \mathbb{N}\}$ be a stochastic process (or time series) defined by

$$X_t = \sum_{i=1}^{t} Z_i, \quad t \in \mathbb{N},$$

where $Z_i \sim WN(0, \sigma^2)$.

1.i Derive the mean function $\mu_X(t)$ of $\{X_t : t \in \mathbb{N}\}$.

1.ii Derive the covariance function $\gamma_X(t,s)$ of $\{X_t : t \in \mathbb{N}\}$.

1.iii Derive the correlation function $\rho_X(t,s)$ of $\{X_t : t \in \mathbb{N}\}$.

**Problem 2**

Let $\{X_t : t \in \mathbb{Z}\}$ be a time series defined by the *First-Order Autoregressive Model* (AR(1)):

$$X_t = \phi X_{t-1} + Z_t, \quad t \in \mathbb{N},$$

where $Z_t \sim WN(0, \sigma^2)$, $|\phi| < 1$.

2.i Derive the variance of $\{X_t\}$.

2.ii Show that if $\phi = 1$, the AR(1) is the random walk model (assume that and $X_0 = 0$).

**Problem 3**

Let $\{X_t : t \in \mathbb{Z}\}$ be a time series defined by the *First-Order Moving Average Model* (MA(1)):

$$X_t = Z_t + \theta Z_{t-1}, \quad t \in \mathbb{Z},$$

where $Z_t \sim WN(0, \sigma^2)$ and $\theta \in \mathbb{R}$.

3.i Derive the mean function $\mu_X(t)$ of $\{X_t : t \in \mathbb{Z}\}$.

3.ii Derive the covariance function $\gamma_X(t+h, t)$ of $\{X_t : t \in \mathbb{Z}\}$.

3.iii Is $\{X_t : t \in \mathbb{Z}\}$ stationary and why? Does the stationarity of $X_t$ depend on $\theta$?

3.iv Derive the correlation function $\rho_X(h)$ of $\{X_t : t \in \mathbb{Z}\}$.

**Problem 4**

Consider the csv file `HW1_problem4.csv`, which includes the realized time series $\{x_t : t = 1, 2, \ldots, 200\}$. Assume the classical decomposition

$$X_t = m_t + s_t + Y_t,$$

where $m_t$ is the deterministic trend, $s_t$ is the deterministic seasonal component and $\{Y_t\}$ is the *de-trended and de-seasonalized* time series.

4.i Display the raw time series $\{x_t : t = 1, 2, \ldots, 200\}$ in a scatterplot and plot its sample ACF. Do you see any notable features based on these two plots?

4.ii De-trend and de-seasonalize $\{x_t\}$, i.e., $y_t \approx x_t - \hat{m}_t - \hat{s}_t$. Show your resulting *residual plot* of $y_t$ and its sample ACF. Does this plot show an *iid* error structure and why?

4.iii Assume that $\{Y_t\}$ follows an AR(1) process, i.e.,

$$Y_t = \phi Y_{t-1} + Z_t,$$

where $Z_i \overset{iid}{\sim} WN(0, \sigma^2)$, $|\phi| \leq 1$ and $X_0 = 0$. Compute the ordinary least squares (OLS) estimator of $\phi$ by regressing $Y[2 : 200] \sim Y[1 : 199]$. What is the estimated AR(1) coefficient $\hat{\phi}$?

4.iv Estimate the noise variance $\sigma^2$ using techniques from your GU4205/GR5205 (or similar) class.

4.v Display the sample ACF of the *residuals* of your AR(1) model $Y_t$. Does this plot show an *iid* error structure and why?

**Problem 5** (Expectation exercise)

Let $Z \sim WN(0, \sigma^2)$. Show that $E|Z| \leq \sigma$.

Hint: Apply Jensen's inequality on concave function $h(u) = \sqrt{u}, u \geq 0$. Also note that $|u| = \sqrt{u^2}$

**Problem 6**

Let $\{X_t : t \in \mathbb{Z}\}$ be a time series defined by:

$$X_t = Z_t Z_{t-1},$$

where $Z_t \sim IID(0, \sigma^2)$.

6.i Derive the mean function $\mu_X(t)$ of $\{X_t : t \in \mathbb{Z}\}$.

6.ii Derive the covariance function $\gamma_X(t + h, t)$ of $\{X_t : t \in \mathbb{Z}\}$.

6.iii Is $\{X_t : t \in \mathbb{Z}\}$ a stationary time series?

**Problem 7**

Let $\{X_t : t \in \mathbb{Z}\}$ be a time series defined by:

$$X_t = Z_1 cos(\omega t) + Z_2 sin(\omega t)$$

where $Z_1, Z_2 \sim WN(0,1)$ and $\omega$ is a fixed frequency in the interval $[0, \pi]$.

7.i Derive the mean function $\mu_X(t)$ of $\{X_t : t \in \mathbb{Z}\}$.

7.ii Derive the covariance function $\gamma_X(t+h, t)$ of $\{X_t : t \in \mathbb{Z}\}$.

**Hint:** Look up the trig identities for $cos(A+B)$ and $sin(A+B)$. Note that this problem requires a lot of *"bookkeeping"*.

7.iii Is $\{X_t : t \in \mathbb{Z}\}$ a stationary time series?

7.iv Prove that the function $\kappa(h)$ is nonnegative definite, where

$$\kappa(h) = cos(\omega h), \quad h \in \mathbb{Z}.$$

**Hint:** Don't over complicate this problem.