LECTURE 21

STOCHASTIC VOLATILITY

6 classes: 2 on Stochastic vol; 2 on Jump diffusion;
Please Attend For
Mon May 24: Mike Kamal of Citadel;
Wed May 25: Chris Delong of Taconic

21.1 The PDE for Valuing Options With Stochastic Volatility

Extending the Black-Scholes riskless-hedging argument.

$$
\begin{aligned}
dS &= \mu S dt + \sigma S dW \\
d\sigma &= p(S, \sigma, t)dt + q(S, \sigma, t)dZ \\
dWdZ &= \rho dt
\end{aligned}
\qquad \text{Eq.21.1}
$$

Now consider two options $V(S, \sigma, t)$ and $U(S, \sigma, t)$

$\Pi = V - \Delta S - \delta U$, short Δ shares of S and short δ options U to hedge V.

$$
d\Pi = dt\left[\begin{array}{l}
\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}q^2 + \dfrac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho \\
-\delta\left(\dfrac{\partial U}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}q^2 + \dfrac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho\right)
\end{array}\right]
$$

$$
+ dS\left(\frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} - \Delta\right) + d\sigma\left(\frac{\partial V}{\partial \sigma} - \delta\frac{\partial U}{\partial \sigma}\right)
$$

We can eliminate all risk by choosing Δ and δ to satisfy

$$\frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} - \Delta = 0 \qquad \left(\frac{\partial V}{\partial \sigma} - \delta\frac{\partial U}{\partial \sigma} = 0\right)$$

$$\Delta = \frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} \qquad \delta = \frac{\partial V}{\partial \sigma}\Big/\frac{\partial U}{\partial \sigma} \qquad \text{Eq.21.2}$$

$$\text{Then } d\Pi = dt\begin{bmatrix} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}q^2 + \dfrac{\partial^2 V}{\partial S \partial \sigma}\sigma q S\rho \\ -\delta\left(\dfrac{\partial U}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}q^2 + \dfrac{\partial^2 U}{\partial S \partial \sigma}\sigma q S\rho\right) \end{bmatrix} \qquad \text{Eq.21.3}$$

No riskless arbitrage: $d\Pi = r\Pi dt = r[V - \Delta S - \delta U]dt$

Leads to:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S\rho + rS\frac{\partial V}{\partial S} + \phi(S, \sigma, t)\frac{\partial V}{\partial \sigma} - rV = 0 \qquad \text{Valuation PDE}$$

This is the PDE for the value of an option with stochastic volatility σ.

Notice: we don't know the value of the function ϕ!

21.2 The Sharpe-ratio meaning of $\phi(S, \sigma, t)$

See what the PDE says about expected risk and return of the option V using Ito's lemma:

$$\begin{aligned} dS &= \mu S dt + \sigma S dW \\ d\sigma &= p(S, \sigma, t)dt + q(S, \sigma, t)dZ \\ dWdZ &= \rho dt \end{aligned}$$

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 dt + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho dt \\ &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\mu S dt + \frac{\partial V}{\partial \sigma}p dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 dt + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho dt \\ &\quad + \frac{\partial V}{\partial S}\sigma S dW + \frac{\partial V}{\partial \sigma}q dZ \\ &\equiv \mu_V V dt + V\sigma_{V_S}dW + V\sigma_{V_\sigma}dZ \end{aligned}$$ Eq.21.4

$$\mu_V = \frac{1}{V}\left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial \sigma}p + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho\right]$$ Eq.21.5

$$\sigma_{V_S} = \frac{S}{V}\frac{\partial V}{\partial S}\sigma \qquad \sigma_{V_\sigma} = \frac{1}{V}\frac{\partial V}{\partial \sigma}q \qquad \sigma_V \equiv \sqrt{\sigma_{V_S}^2 + \sigma_{V_\sigma}^2 + 2\rho\sigma_{V_S}\sigma_{V_\sigma}}$$

where σ_{V_S} and σ_{V_σ} are the partial volatilities of option V with total volatility σ_V.

The stochastic volatility PDE was

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} + \phi(S, \sigma, t)\frac{\partial V}{\partial \sigma} - rV = 0$$

Leads to:

$$\frac{(\mu_V - r)}{\sigma_V} = \frac{\sigma_{V_S}}{\sigma_V}\frac{(\mu - r)}{\sigma} + \frac{\sigma_{V_\sigma}}{\sigma_V}\frac{(p - \phi)}{q}$$

Excess return per unit of risk for the option = the excess return per unit of risk for the stock and the excess return per unit of risk for volatility, weighted by each one's contribution to volatility of V.

ϕ plays the role for stochastic volatility that the riskless rate r plays for a stochastic stock price.

In the Black-Scholes case, r is the rate at which the stock price must grow in order that option payoffs can be discounted at the riskless rate.

Similarly, ϕ is the drift p that volatility must undergo in order that option prices with stochastic volatility be discounted at the riskless rate (or values can grow risk-neutrally).

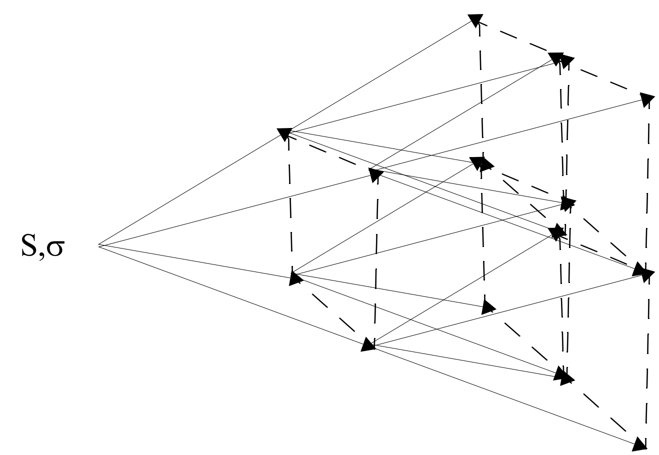
ϕ is not equal to r because σ is not traded. ϕ is the rate at which volatility σ must grow in order that the price of the option V grows at the rate r when you can hedge away all risk.

From a calibration point of view, ϕ or p must be chosen to make option prices grow at the riskless rate.

If we know the market price of just one option U, and **we assume an evolution process for volatility**, $d\sigma = \phi(S, \sigma, t)dt + q(S, \sigma, t)dZ$, then we can choose/calibrate the effective drift ϕ of volatility so that the calculated discounted expected value of U matches its market price.

Then we can value all other options from the same pde.

In a quadrinomial picture of stock prices where volatility and stock prices are stochastic, as illustrated in the figure below, we much calibrate the drift of volatility ϕ so that the value of an option U is given by the expected risklessly discounted value of its payoffs.



Once we've chosen ϕ to match that one option price, then, assuming we have the correct model for volatility, all other options can be valued risk-neutrally by discounting their expected payoffs.

Of course, **it may be naive** to assume that just one option can calibrate the entire volatility evolution process, just as one bond cannot determine the whole yield curve.)

Note that even though the payoffs of the option are the same as in the Black-Scholes world, the evolution process of the stock is different, and so the option price will be different too. The distribution is NOT lognormal any more.

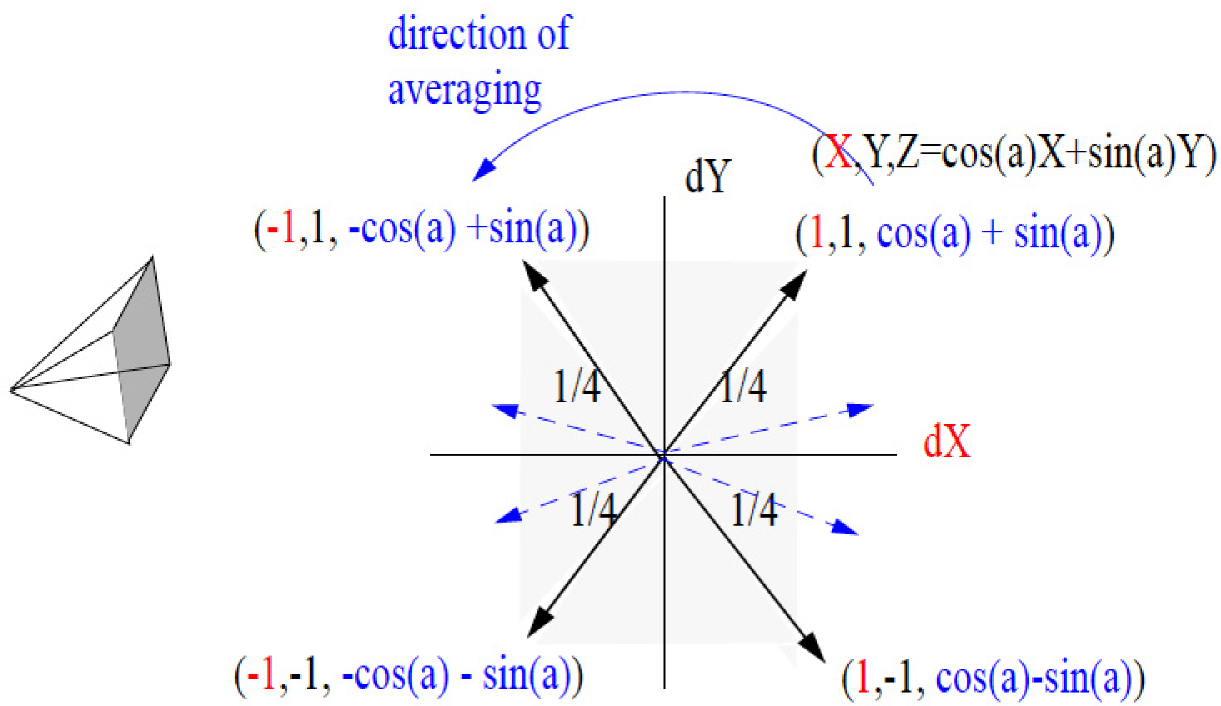
The homework problems give you a stochastic process, an *r* and a drift and then you have to calculate the expected discounted value of the payoff assuming it grows risk-neutrally.

21.3 A Method for Simulating Two Correlated Stochastic Variables Step by Step

random increments X and Y are uncorrelated and Z is correlated with X

Correlated Normal Distributions: Method 1:Equal Probabilities

$$Z = \rho X + \sqrt{1-\rho^2}Y \equiv X\cos a + Y\sin a \text{ where } \cos a = \rho$$



The correlation between dZ and dX is cos(a).

Simulating Stochastic Vol Step by Step, An Example


```
function [call, implied] = stochastic_vol_option(vol, mu_vol, vol_vol, rho, r, strike, s, t, npaths,
nsteps, rseed);


dt = t/nsteps;
randn('seed', rseed);
call_vec = zeros(npaths,1);

for j=1:npaths
   s_value=s;
   vol_value = vol;
   for i=1:nsteps
      % generate the random increment for the stock
      z_s = randn(1,1);
      % generate the random increment for the volatility
      z_vol = sqrt(1-rho*rho) * randn(1,1) + rho * z_s;
      % generate the next stock price dt later
      s_value = s_value*exp((r-0.5*vol_value*vol_value)*dt + vol_value*sqrt(dt)*z_s);
       % generate the next volatility dt later
      vol_value = vol_value*exp((mu_vol-0.5*vol_vol*vol_vol)*dt + vol_vol*sqrt(dt)*z_vol);
   end
   call_vec(j) = max(0,s_value-strike); % option payoff on this path
end

% Discounted expected value for call
call_vec = exp(-r*t)*call_vec;
call     = mean(call_vec);
implied  = blsimpv(s, strike, r, t, call, 2);
```

21.4 The Simpler Characteristic Solution to the Stochastic Volatility Model with Zero Correlation

Before solving Stochastic Vol, First Recall: The continuous-time treatment of a time-dependent volatility:

$$dS = rSdt + \sigma(t)SdW_t \text{ in the risk-neutral world.}$$

$$d\ln S = \frac{dS}{S} - \frac{1}{2}\frac{1}{S^2}\sigma^2(t)S^2 dt = rdt + \sigma(t)dW_t - \frac{1}{2}\sigma^2(t)dt$$

$$\ln S(t) = \ln S(0) + rt - \frac{1}{2}\int_0^t \sigma^2(s)ds + \int_0^t \sigma(s)dW_s$$

The distribution is given by a sum of normals, which itself is normal, with total variance given by

$$E\int_0^t\int_0^t \sigma(u)\sigma(s)dW_u dW_s = \int_0^t\int_0^t \sigma(u)\sigma(s)E[dW_u dW_s]$$

$$= \int_0^t\int_0^t \sigma(u)\sigma(s)duds\delta(u-s) = \int_0^t \sigma^2(u)du = t\bar{\sigma}^2(t)$$

Thus at time t the mean of the distribution of $\ln S$ is at $\left(r - \frac{\bar{\sigma}^2}{2}\right)t$

where $\bar{\sigma}^2 = \frac{1}{t}\int_0^t \sigma^2(s)ds$ is the **path variance, known and deterministic.**

Thus $\log S(t)$ is distributed normally too, as follows.

$$\log S_t/S_0 \sim \mathcal{N}\left(\left(r - \frac{\bar{\sigma}^2}{2}\right)t, \bar{\sigma}^2 t\right)$$

Thus, calculating the value of the option as an expected value of the payoff in a risk-neutral world, we find

$$C = C_{BSM}(S, t, K, T, r, \bar{\sigma}^2(t))$$

The Black-Scholes-Merton value of the option is the discounted expected value of the payoff after a time-dependent volatility, discounted at the riskless rate.

$$V = V_{BSM}(S, t, K, T, r, \bar{\sigma}^2(t)) = \exp(-r\tau)E_Q[payoff]$$

Now let's look at stochastic volatility rather than deterministic time-dependent volatility

The price of the hedged European option is given by the expected risk-neutral value of the terminal payoff where the stock price $S(t)$ and the variance $v(t) = \sigma^2(t)$ are both stochastic at each instant:

$$V = e^{-r(T-t)} \sum_{\text{all paths}} p(\text{path}) \times \text{payoff}|_{\text{path}}$$

V is the integral over the payoff conditional on the two diffusions.

Hull and White (1987) cleverly characterize **each path** by its terminal stock price S_T and the average variance along that path

$$\overline{\sigma_T^2} = \frac{1}{T}\int_0^T \sigma_t^2 dt$$

We refer to $\overline{\sigma}_T$ as the path volatility -- the square root of the average path variance.

Then

$$V = e^{-r(T-t)} \sum_{\text{all } \sigma_T} \sum_{\substack{\text{paths of } S_T \\ \text{given } \overline{\sigma_T}}} p(\overline{\sigma_T}, S_T) \times \text{payoff}|_{\text{path}}$$

where $p(\overline{\sigma}_T, S_T)$ is the probability of one of these paths.

If the stock and its volatility are uncorrelated ($\rho = 0$), then the probability p can factor such that

$$p(\overline{\sigma_T}, S_T) = f(\overline{\sigma_T}) \times g(S_T)$$

and so

$$V = e^{-r(T-t)} \sum_{\text{all } \overline{\sigma_T}} f(\overline{\sigma_T}) \sum_{\substack{\text{paths of } S_T \\ \text{given } \overline{\sigma_T}}} g(S_T) \times \text{payoff}|_{\text{path}}$$

But because $g(\ )$ is the usual lognormal distribution for a known path volatility.t

$$V_{\text{BSM}}(S, t, K, T, r, \overline{\sigma_T}) = e^{-r(T-t)} \sum_{\substack{\text{paths of } S_T \\ \text{given } \overline{\sigma_T}}} g(S_T) \times \text{payoff}|_{\text{path}}$$

Combining these previous two equations we find that

$$V = \sum_{\text{all } \overline{\sigma_T}} f(\overline{\sigma_T}) \times V_{\text{BSM}}(S, t, K, T, r, \overline{\sigma_T})$$

When the correlation is zero, the stochastic volatility solution for a standard European option is the weighted sum over the BSM solutions with different path volatilities.

Mixing Theorem

The price of an option in a stochastic volatility model with zero correlation is the weighted integral/sum over BS prices over the distribution of path volatilities.

$$V = \sum_{\text{all } \overline{\sigma_T}} f(\overline{\sigma_T}) \times V_{\text{BSM}}(S, t, K, T, r, \overline{\sigma_T})$$

It doesn't matter what order the stochastic volatilities occur in -- as long as the variance along the path is the same, all paths with that variance have the same BS terminal distribution of the stock price.

What is the advantage of this?

The mixing theorem reduces this to a **one-dimensional simulation or integration in the model**.

IF the correlation is different from zero, this doesn't work: then all paths conditional on a definite variance still have a normal distribution of returns, BUT that variance depends on the stock price path, not just on time. The resultant formula is

$$V_t = E\left[V_{\text{BSM}}\left(S_t^*(\overline{\sigma_T}, \rho), K, r, \overline{\sigma_T}^*(\rho), T\right)\right]$$

where the stock price and volatility in the Black-Scholes formula are shifted to "fake" values that differ from actual values by something related to the correlation. Much less useful.

[Refs: Fouque, Papanicolaou and Sircar book, and Roger Lee, *Implied and Local Volatilities under Stochastic Volatility*, International Journal of Theoretical and Applied Finance, 4(1), 45-89 (2001).]

21.5 The zero-correlation smile depends on log moneyness

- When the stock and its stochastic volatility are uncorrelated, the smile is a symmetric function of the log moneyness $\ln(K/S)$.
- For small volatility of volatility, the mixing theorem leads to approximate analytic expressions for the smile as a function of moneyness.

Mixing: average BS solutions over the volatility distribution to get the stochastic volatility solution.
Example: path volatility can be one of two values, either high or low, with equal probability:

$$C_{SV} = \frac{1}{2}[C_{BS}(S, K, \sigma_H) + C_{BS}(S, K, \sigma_L)] \qquad \text{Eq.21.6}$$

Homogeneity:

$$C_{SV} = \frac{1}{2}\left[SC_{BS}\left(1, \frac{K}{S}, \sigma_H\right) + SC_{BS}\left(1, \frac{K}{S}, \sigma_L\right)\right] = Sf\left(\frac{K}{S}\right)$$

Now, defining BS Σ by

$$C_{SV} = Sf\left(\frac{K}{S}\right) \equiv SC_{BS}\left(1, \frac{K}{S}, \Sigma\right)$$

and so

$$\Sigma = g\left(\frac{K}{S}\right)$$

Implied volatility is a function of moneyness in stochastic volatility models with zero correlation (conditional on not knowing the future volatility).

Deriving Euler's equation: $$\frac{\partial\Sigma}{\partial S} = \left(-\frac{K}{S^2}\right)g' \qquad \frac{\partial\Sigma}{\partial K} = \frac{1}{S}g' \qquad S\frac{\partial\Sigma}{\partial S} + K\frac{\partial\Sigma}{\partial K} = 0$$

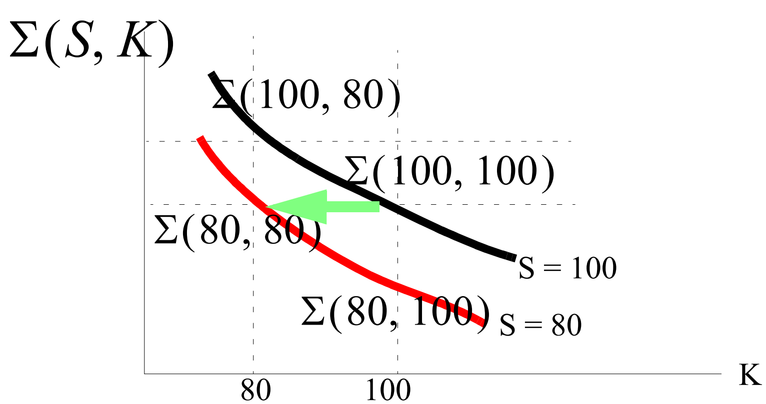
Close to at-the-money, $$\frac{\partial\Sigma}{\partial S} \approx -\frac{\partial\Sigma}{\partial K}$$

just the opposite of what we got with local volatility models.

Close to at-the-money

$$\Sigma \approx \Sigma(S-K)$$

In stochastic volatility models, *conditioned on not knowing the future volatility,*



Note that the volatility of all options drops when the stock price drops. Of course if the volatility itself changes, then the whole curve can move stochastically.

21.5.1 The zero correlation smile is symmetric

The mixing theorem: $$C_{SV} = \int_0^{\infty} C_{BSM}(\overline{\sigma_T})\phi(\overline{\sigma_T})d\overline{\sigma_T}$$

Taylor expansion about the average value $\bar{\bar{\sigma}}$ of the path volatility, dropping subscript T.

$$\begin{aligned} C_{SV} &= \int_0^{\infty} C_{BSM}\left(\bar{\bar{\sigma}} + \bar{\sigma} - \bar{\bar{\sigma}}\right)\phi(\bar{\sigma})\,d\bar{\sigma} \\ &\approx \int_0^{\infty}\left[C_{BSM}(\bar{\bar{\sigma}}) + \left.\frac{\partial C_{BSM}}{\partial\bar{\sigma}}\right|_{\bar{\bar{\sigma}}}(\bar{\sigma} - \bar{\bar{\sigma}}) + \frac{1}{2}\left.\frac{\partial^2 C_{BSM}}{\partial\bar{\sigma}^2}\right|_{\bar{\bar{\sigma}}}(\bar{\sigma} - \bar{\bar{\sigma}})^2\right]\phi(\bar{\sigma})\,d\bar{\sigma} \\ &\approx C_{BSM}(\bar{\bar{\sigma}}) + 0 + \frac{1}{2}\left.\frac{\partial^2 C_{BSM}}{\partial\bar{\sigma}^2}\right|_{\bar{\bar{\sigma}}}\mathrm{var}[\bar{\sigma}] \\ &\approx C_{BSM}\left(\bar{\bar{\sigma}}\right) + \frac{1}{2}\left.\frac{\partial^2 C_{BSM}}{\partial\bar{\sigma}^2}\right|_{\bar{\bar{\sigma}}}\mathrm{var}\left[\bar{\sigma}\right] \end{aligned} \tag{21.13}$$

where $var[\bar{\sigma}]$ is the *variance of the path volatility* $\bar{\sigma}$ *over the life* τ *of the option.*

Let $\bar{\bar{\sigma}}$ be the average of the path volatility.

Define BS implied volatility Σ by

$$\begin{aligned} C_{\mathrm{SV}} &= C_{\mathrm{BSM}}(\Sigma) \\ &= C_{\mathrm{BSM}}(\bar{\bar{\sigma}} + \Sigma - \bar{\bar{\sigma}}) \\ &= C_{\mathrm{BSM}}(\bar{\bar{\sigma}}) + \left.\frac{\partial C_{\mathrm{BSM}}}{\partial \bar{\sigma}}\right|_{\bar{\bar{\sigma}}} (\Sigma - \bar{\bar{\sigma}}) + \dots \\ &\approx C_{\mathrm{BSM}}(\bar{\bar{\sigma}}) + \left.\frac{\partial C_{\mathrm{BSM}}}{\partial \bar{\sigma}}\right|_{\bar{\bar{\sigma}}} (\Sigma - \bar{\bar{\sigma}}) \end{aligned}$$

Then equating two expressions

$$\Sigma \approx \bar{\bar{\sigma}} + \frac{\frac{1}{2} \left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \bar{\sigma}^2}\right|_{\bar{\bar{\sigma}}} \operatorname{var}[\bar{\sigma}]}{\left.\frac{\partial C_{\mathrm{BSM}}}{\partial \bar{\sigma}}\right|_{\bar{\bar{\sigma}}}}$$

Eq.21.7

Use BS derivatives to find the functional form of $\Sigma(S, K)$.

$$V = \frac{\partial C_{\mathrm{BSM}}}{\partial \sigma} = \frac{S\sqrt{\tau}}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \qquad \frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2} = V\frac{d_1 d_2}{\sigma} = \frac{V}{\sigma}\left[\left(\frac{1}{v}\ln\left(\frac{S}{K}\right)\right)^2 - \frac{v^2}{4}\right] \qquad v = \sigma\sqrt{\tau}\ \mathrm{i}$$

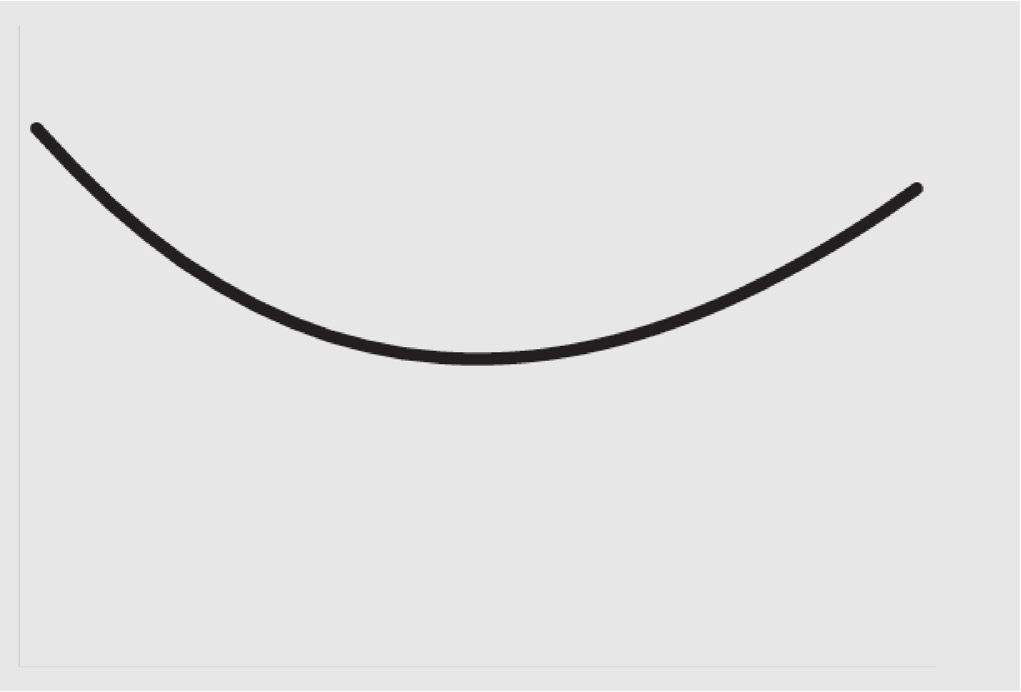
Eq.21.8

Substituting the derivatives for zero interest rates we get

$$\Sigma \approx \bar{\bar{\sigma}} + \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\left(\frac{1}{\bar{\bar{v}}}\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\bar{\bar{v}}^2}{4}\right] \qquad \bar{\bar{v}} = \bar{\bar{\sigma}}\sqrt{\tau}.$$

$$\approx \bar{\bar{\sigma}} + \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\frac{1}{\bar{\bar{\sigma}}^2\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\bar{\bar{\sigma}}^2\tau}{4}\right]$$

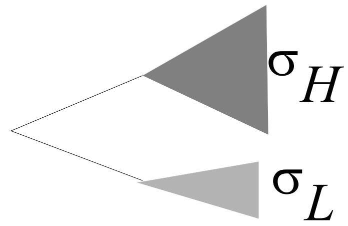
Quadratic function of $\ln S/K$, parabolically shaped smile that varies as $(\ln S/K)^2$ or $(K-S)^2$. Sticky moneyness smile, no scale, a function of K/S.



21.6 A Simple Two-State Stochastic Volatility Model

Mixing two path

$$C_{SV} = \frac{1}{2}[C_{BSM}(S, K, \sigma_H) + C_{BSM}(S, K, \sigma_L)]$$



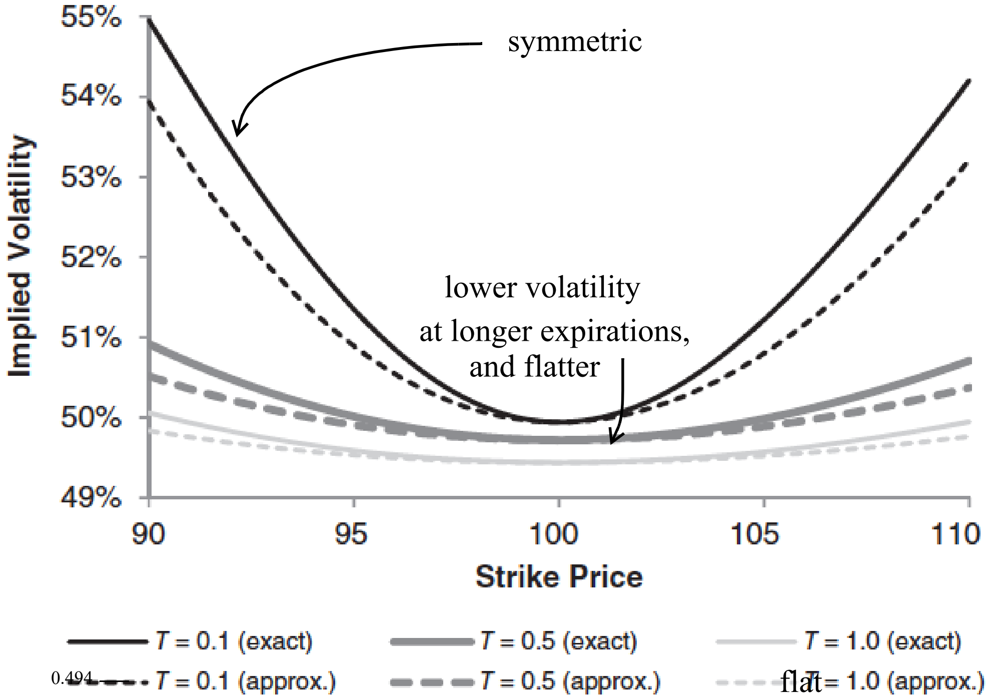
volatilities

Low volatility be 20% and the high volatility 80% with a mean volatility of 50%.

Variance of the volatility is $0.5(0.8 - 0.5)^2 + 0.5(0.5 - 0.2)^2 = 0.09$ per year.

In the figure below we show the smile corresponding to the exact mixing formula together with the approximation $\bar{\bar{\sigma}} + \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\frac{1}{\bar{\bar{\sigma}}^2\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\bar{\bar{\sigma}}^2\tau}{4}\right]$

The Volatility Smile in a Discrete Two-Volatility Model With Zero Correlation



- The smile with zero correlation is symmetric;
- the long-expiration smile is relatively flat, while the short expiration skew is more curved (note the τ^{-1} coefficient of $(\ln S/K)^2$ in the formula; and
- at the forward price of the stock, the at-the-money implied volatility decreases monotonically with time to expiration, and lies below the mean volatility of 50%, because of the negative convexity of the Black-Scholes options price at the money.

The approximate solution works quite well.

At-the-money, with these parameters, the approximation reduces to

$$\Sigma_{SV}^{ATM} \approx \bar{\bar{\sigma}} + \frac{1}{2} var[\bar{\sigma}] \left[\frac{-\left(\bar{\bar{\sigma}}^4 \tau^2\right)/4}{\bar{\bar{\sigma}}^3 \tau} \right] \approx \bar{\sigma} - \frac{1}{8} var[\bar{\sigma}][\bar{\bar{\sigma}}\tau] \approx 0.5 - \frac{1}{8}(0.09)\bar{\bar{\sigma}}\tau \approx 0.5 - 0.0056\tau$$

For $\tau = 1$, the at-the-money volatility is 0.4944, which agrees well with the figure above.

21.7 Next: The Smile for GBM Continuous Stochastic Volatility with No Mean Reversion and Zero Correlation

A more sophisticated continuous distribution of stochastic volatilities. $d\sigma = a\sigma dt + b\sigma dZ$ $\rho = 0$, an initial volatility of 0.2 and a volatility of volatility of 1.0, straightforward Monte Carlo simulation of stock paths.From the mixing formula:

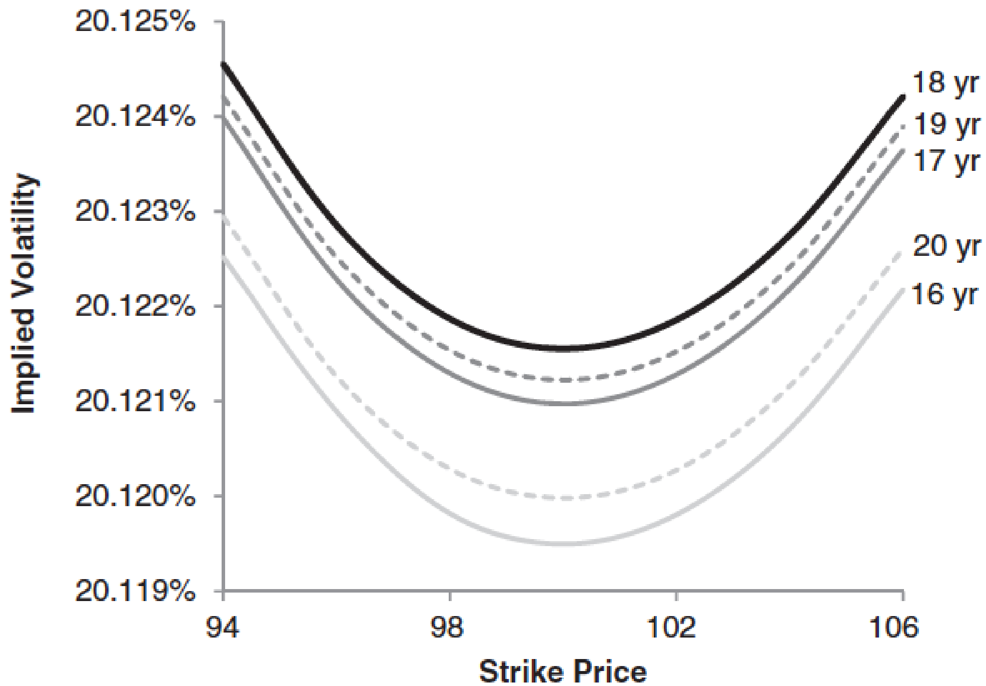


FIGURE 21.4 The Smile for a Variety of Expirations in a GBM Stochastic Volatility Model with Zero Correlation, $a = 0$, $b = 0.1$

Still symmetric but now not monotonic in expiration. $\overset{\Rightarrow}{\sigma}$ is a function of time because of the Brownian motion, which changes the time dependence from the two-state case.

The curvature of the smile in the Figure is approximately independent of time to expiration because of the GBM of stochastic volatility. You can understand this from the formula

$$\Sigma \approx \bar{\bar{\sigma}} + \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\frac{1}{\bar{\bar{\sigma}}^2\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\bar{\bar{\sigma}}^2\tau}{4}\right]$$

$$\approx \bar{\bar{\sigma}} + \frac{1}{2}\frac{\mathrm{var}\,[\bar{\sigma}]}{\bar{\bar{\sigma}}^3\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\mathrm{var}\,[\bar{\sigma}]\,\bar{\bar{\sigma}}\tau}{8}$$

The quadratic skew term is

$$\frac{1}{2}\frac{\mathrm{var}\,[\bar{\sigma}]}{\bar{\bar{\sigma}}^3\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 \approx \frac{1}{2}\frac{\frac{b^2}{3}\sigma^2\tau}{\bar{\bar{\sigma}}^3\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2$$

$$\approx \frac{1}{6}\frac{b^2}{\sigma}\left(\ln\left(\frac{S}{K}\right)\right)^2$$

roughly independent of time to expiration.

This is different from the case where the volatility distribution is two-state discrete.

An Analytic Approximation to Understand Non-monoticity

one term grows with time, one decreases

$$\Sigma_{\mathrm{atm}} \approx \bar{\bar{\sigma}} \left(1 - \frac{1}{8} \mathrm{var}\,[\bar{\sigma}]\,\tau \right) \quad \textbf{Equation GBM} \qquad \text{Eq.1.9}$$

Let's estimate the time-to-expiration dependence of these path-volatility quantities in a geometric Brownian motion model for the *instantaneous* volatility σ.

$$d\sigma = a\sigma dt + b\sigma dZ$$

σ^2, therefore, satisfies a similar stochastic differential equation with

$$\mathrm{drift}[\sigma^2] = 2a + b^2$$

$$\mathrm{vol}[\sigma^2] = 2b$$

The extra b^2 term in the drift arises from Ito's Lemma for the square of a Wiener process.

Now let's consider the path variance $\bar{\sigma}^2$ which is relevant to the mixing formula. The path variance is an arithmetic average of the instantaneous variances out to time T, but the variance itself evolves geometrically, and so there is no closed-form expression for its value. Nevertheless, it is well known that the average has approximately 1/2 the drift and $1/\sqrt{3}$ the volatility of the non-averaged variable.

Thus approximately

$$\mathrm{drift}[\bar{\sigma}^2] \approx a + \frac{1}{2}b^2$$

$$\mathrm{vol}[\bar{\sigma}^2] \approx \frac{2b}{\sqrt{3}}$$

But Equations 1.9 involves the square root of $\bar{\sigma}^2$ (i.e., $\bar{\sigma}$), which from Ito's Lemma for the square root of a variable undergoing geometric Brownian motion is

$$\begin{aligned}\mathrm{drift}[\bar{\sigma}] &\approx \frac{1}{2}\left(a + \frac{1}{2}b^2\right) - \frac{1}{8}\left(\frac{2b}{\sqrt{3}}\right)^2 \\ &\approx \frac{a}{2} + \frac{1}{12}b^2\end{aligned}$$

$$\mathrm{vol}[\bar{\sigma}] \approx \left(\frac{b}{\sqrt{3}}\right)$$

This drift means that there is a time-dependence to the coefficient $\bar{\bar{\sigma}}$ which grows by GBM in Equations 1.9.

$$\begin{aligned}\bar{\bar{\sigma}}(\tau) &\approx \sigma e^{\left(\frac{a}{2}+\frac{1}{12}b^2\right)\tau} \\ &\approx \sigma\left[1+\left(\frac{a}{2}+\frac{b^2}{12}\right)\tau+\frac{1}{2}\left(\frac{a}{2}+\frac{b^2}{12}\right)^2\tau^2\right]\end{aligned}$$

The volatility of the path volatility is $b/\sqrt{3}$. The variance of the path volatility is therefore

$$\mathrm{var}[\bar{\sigma}] \approx \frac{b^2}{3}\sigma^2\tau$$

Thus

$$\begin{aligned}\Sigma_{\mathrm{atm}} &\approx \bar{\bar{\sigma}}\left(1-\frac{1}{8}\mathrm{var}[\bar{\sigma}]\,\tau\right) \\ &\approx \sigma\left[1+\left(\frac{a}{2}+\frac{b^2}{12}\right)\tau+\frac{1}{2}\left(\frac{a}{2}+\frac{b^2}{12}\right)^2\tau^2\right]\left(1-\frac{1}{8}\frac{b^2}{3}\sigma^2\tau^2\right) \\ &\approx \sigma\left[1+\left(\frac{a}{2}+\frac{b^2}{12}\right)\tau+\left(\frac{1}{2}\left(\frac{a}{2}+\frac{b^2}{12}\right)^2-\frac{b^2}{24}\sigma^2\right)\tau^2\right]\end{aligned}$$

For a = 0 as in **Figure 21.4** above

$$\Sigma_{\text{atm}} \approx \sigma \left[1 + \frac{b^2}{12}\tau + \frac{b^2}{24}\left(\frac{b^2}{12} - \sigma^2\right)\tau^2\right]$$

For b = 0.1 and $\sigma = 0.2$ the τ^2 term has a negative coefficient and explains the non-monotonicity in the Figure. The maximum occurs at $\tau = \frac{1}{\sigma^2 - \frac{b^2}{12}}$, about 25, roughly accounting for the Figure.

21.8 The Smile in Mean-Reverting Stochastic Volatility Models

Finally, we explore the smile when volatility mean reverts:

$$\frac{dS}{S} = \mu dt + \sigma dZ \qquad d\sigma = \alpha(m-\sigma)dt + \beta\sigma dW \qquad dZdW = \rho dt$$

The following pages show the results of a double Monte Carlo for BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation.

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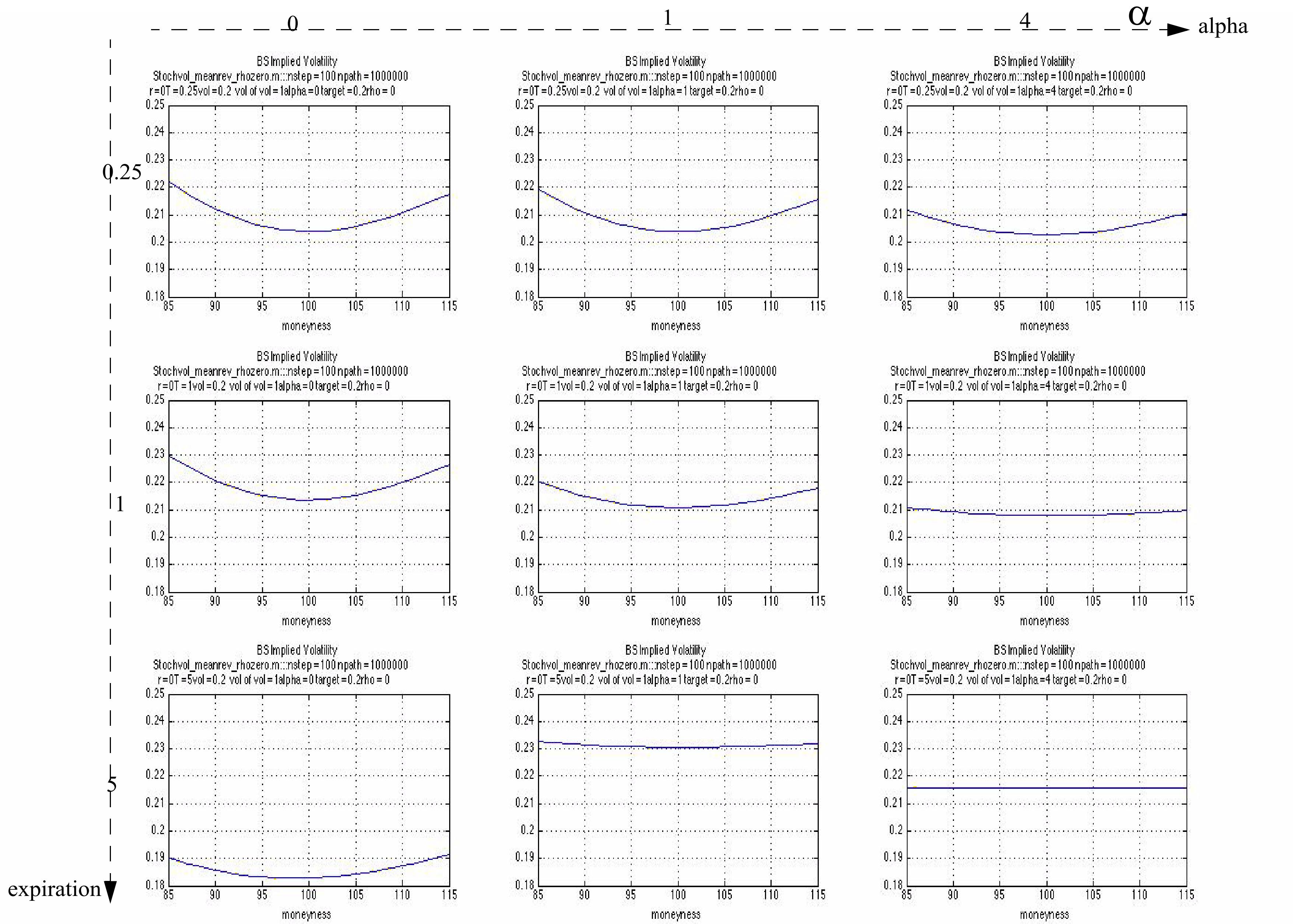
BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation, initial volatility 0.2 and target volatility 0.3.The target volatility and the initial volatility are both 0.2



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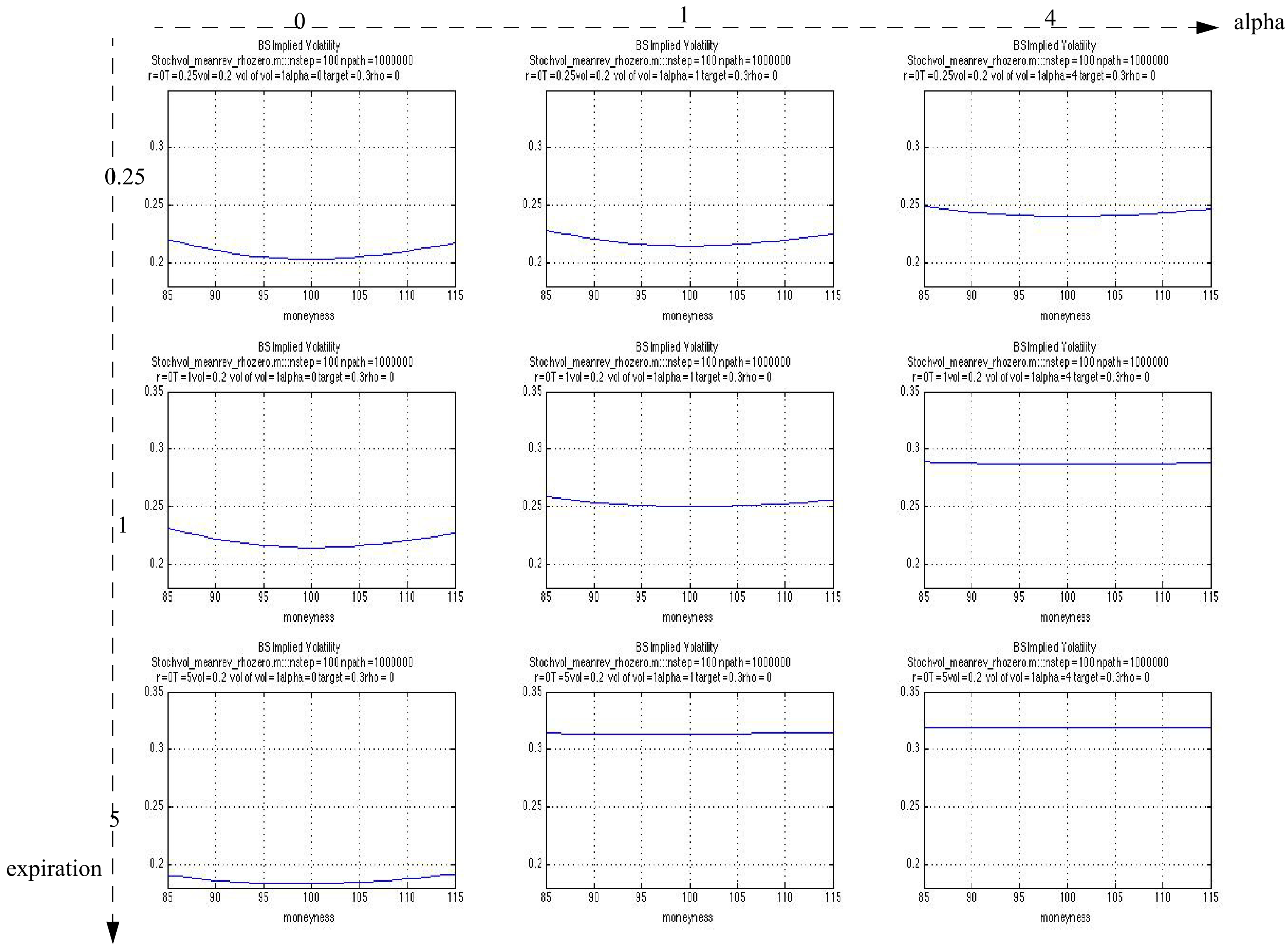
BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation, initial volatility 0.2 and target volatility 0.3.



21.1 Mean-Reverting Stochastic Volatility and the Asymptotic Behavior of the Smile.

$$\Sigma \approx \bar{\bar{\sigma}} + \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\frac{1}{\bar{\bar{\sigma}}^2\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\bar{\bar{\sigma}}^2\tau}{4}\right]$$

From p19, approximately for small vol of vol Eq.21.10

Now employ intuition about mean reversion for σ.

Short Expirations, Zero Correlation

In the limit that $\tau \to 0$

$$\lim_{\tau \to 0}\Sigma \approx \bar{\bar{\sigma}} + \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}^3\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2$$

For small times, $var[\sigma] = \beta'\tau$. Substituting this relation into Equation leads to the expression

$$\lim_{\tau \to 0}\Sigma \approx \bar{\bar{\sigma}} + \frac{1}{2}\beta'\frac{1}{\bar{\bar{\sigma}}^3}\left(\ln\left(\frac{S}{K}\right)\right)^2$$

$\tau \to 0$ limit Eq.21.11

Smile is quadratic and finite as $\tau \to 0$ for short expirations, and depends on volatility of volatility.

Long Expirations

As $\tau \to \infty$ $\quad \Sigma_{SV} \approx \bar{\bar{\sigma}} - \frac{1}{8} var[\bar{\sigma}][\bar{\bar{\sigma}}\tau]$

where $\bar{\sigma}$ is the path volatility over the life of the option and is itself a function of the time to expiration due to the stochastic nature of the instantaneous volatility.

For Ornstein-Uhlenbeck processes the path volatility to expiration $\bar{\sigma}$ converges to a constant along all paths as $\tau \to \infty$, and so $\bar{\sigma}$ has zero variance as $\tau \to \infty$, $var[\bar{\sigma}] \to \text{const}/\tau$.

$$\Sigma_{SV} \approx \bar{\bar{\sigma}} - \frac{\text{const}}{8} \bar{\bar{\sigma}} \qquad \text{Eq.21.12}$$

NO smile at large expirations.

Why is the correction term negative? The option price $C_{BS}(\sigma)$ has negative convexity, and for a concave function $f(x)$, the average of the function $\overline{f(x)}$ is less than the function $f(\bar{x})$ of the average.

Thus, for zero correlation, we expect to see stochastic volatility smiles that look like this:

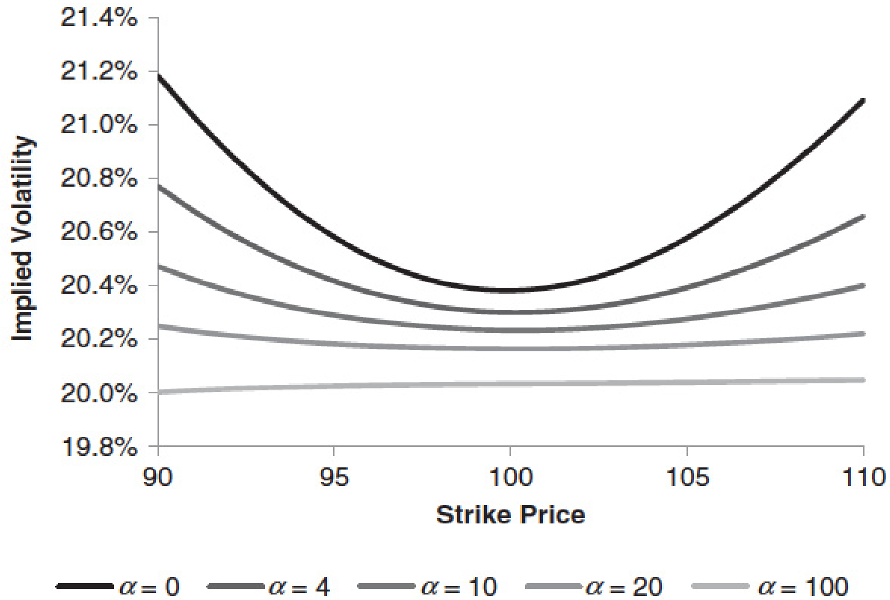


FIGURE 22.2 The Smile for a Mean-Reverting Stochastic Volatility Model with $\rho = 0$ and Varying Mean Reversion Strength

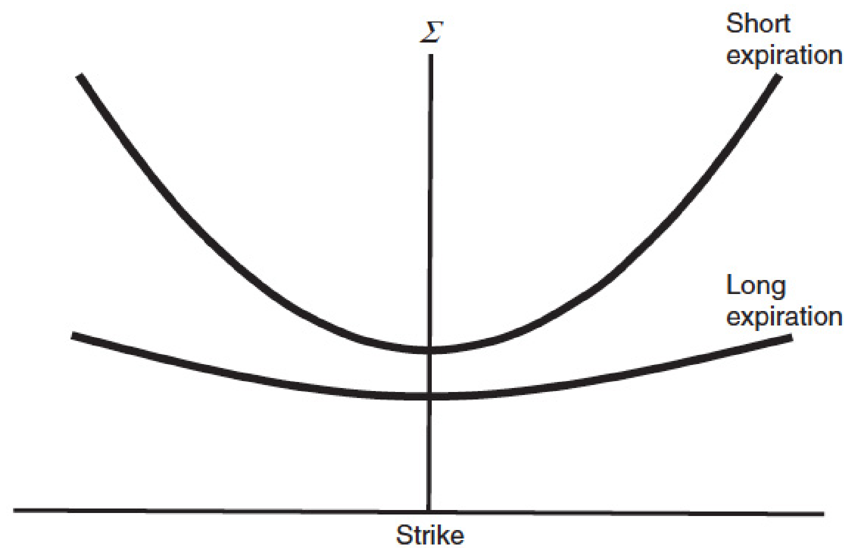


FIGURE 22.1 The Smile for Stochastic Volatility Model with $\rho = 0$

We can understand this intuitively as follows. In the long run, all paths will have the same volatility if volatility mean reverts, and so the long-term skew is flat. In the short run, bursts of high volatility act almost like jumps, and induce fat tails

21.2 Non-zero correlation ρ in stochastic volatility models

No correlation lead to a symmetric smile.

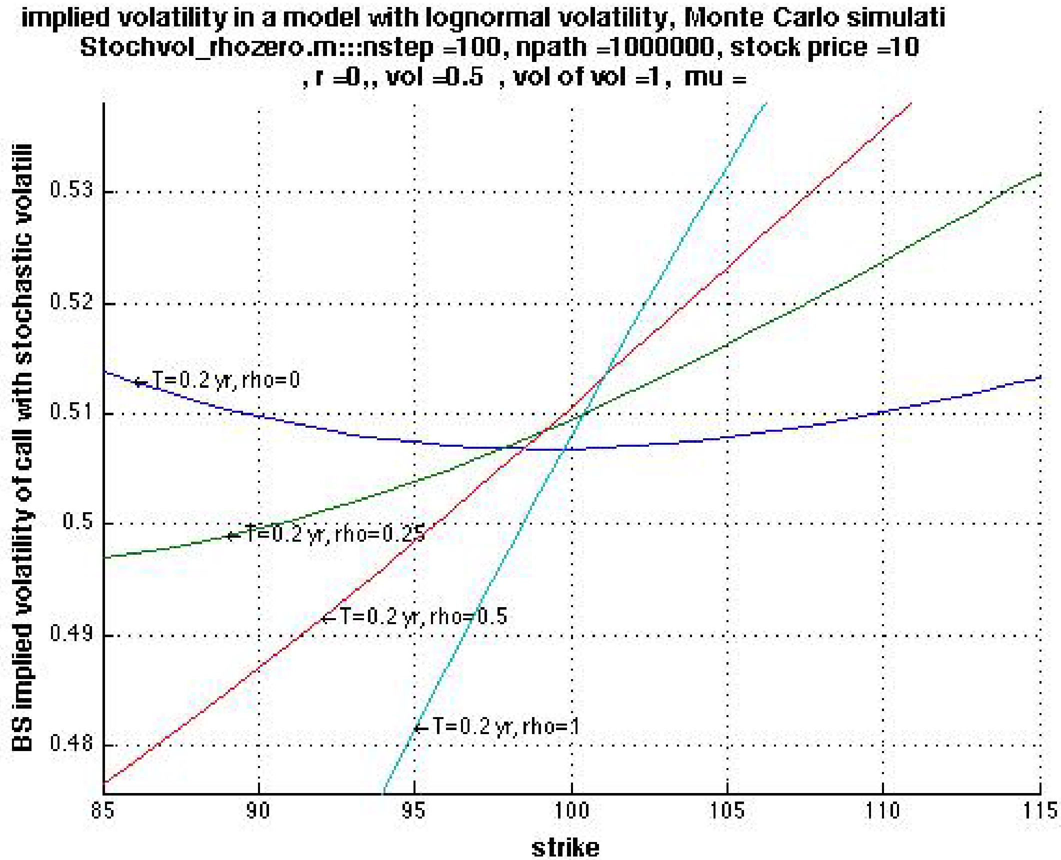
With correlation the smile still depends on (K/S_F) but the dependence is not quadratic.

FIGURE 22.3 The Smile as a Function of Correlation in a Stochastic Volatility Model with Zero Mean Reversion

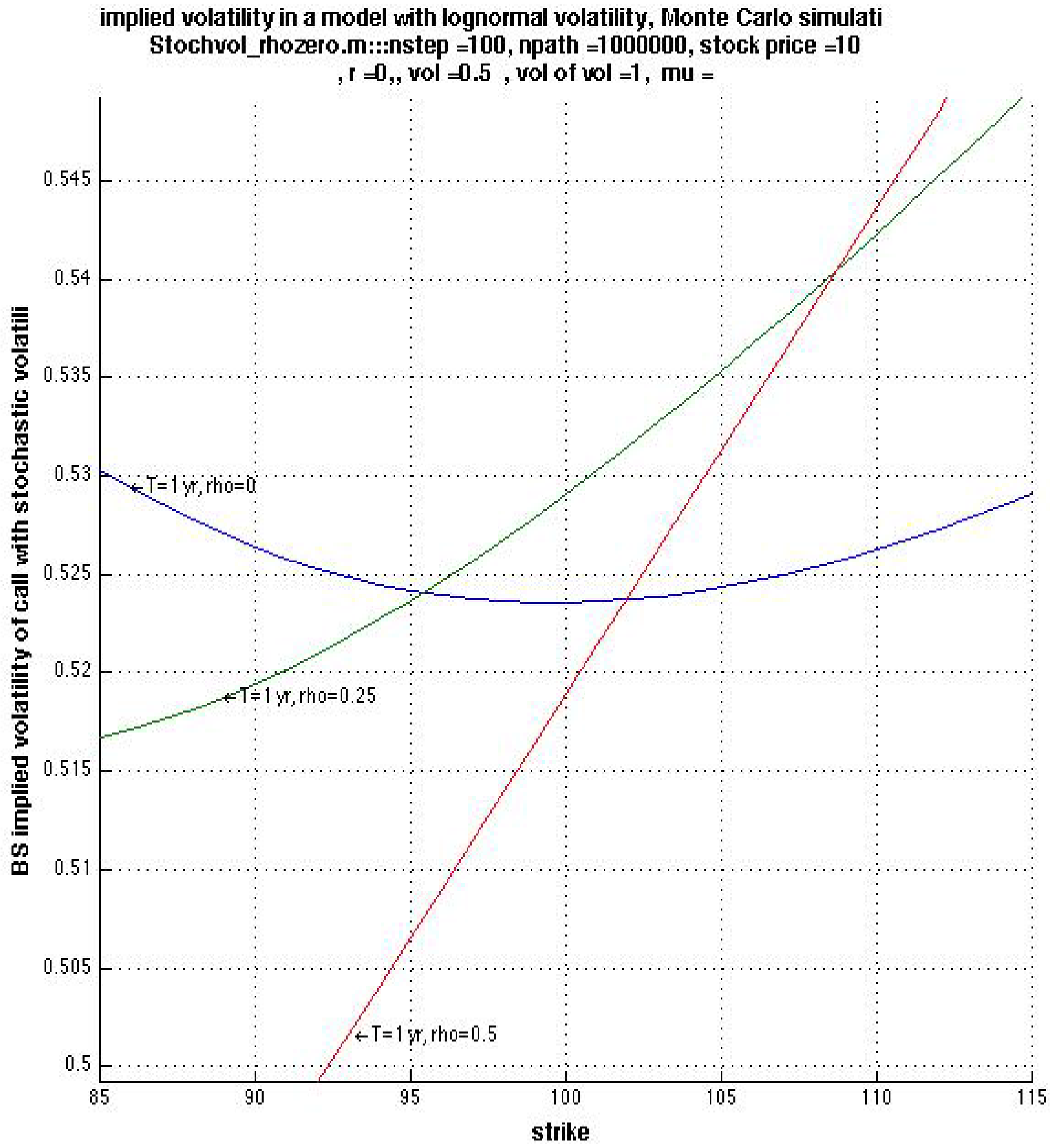
A very steep short-term skew is difficult in these models; since volatility diffuses continuously in these models, at short expirations volatility cannot have diffused too far. A very high volatility of volatility and very high mean reversion are needed to account for steep short-expiration smiles.

21.3 Simulation of Non-Zero Correlation, No Mean Reversion

Monte Carlo simulation for $\tau = 0.2\text{yrs}$ with non-zero ρ. You can see that increasing the value of the correlation steepens the slope of the smile.



$\tau = 1yr$.


implied volatility in a model with lognormal volatility, Monte Carlo simulati
Stochvol_rhozero.m:::nstep =100, npath =1000000, stock price =10
, r =0,, vol =0.5 , vol of vol =1 , mu =
BS implied volatility of call with stochastic volatili
strike
T=1 yr, rho=0
T=1 yr, rho=0.25
T=1 yr, rho=0.5
0.545
0.54
0.535
0.53
0.525
0.52
0.515
0.51
0.505
0.5
85
90
95
100
105
110
115

21.4 Simulation with Mean Reversion and Correlation

volatility is 20%, the long-term volatility *m* is 20%, the volatility of volatility is 50%, and the correlation between the stock and its volatility is –30%.

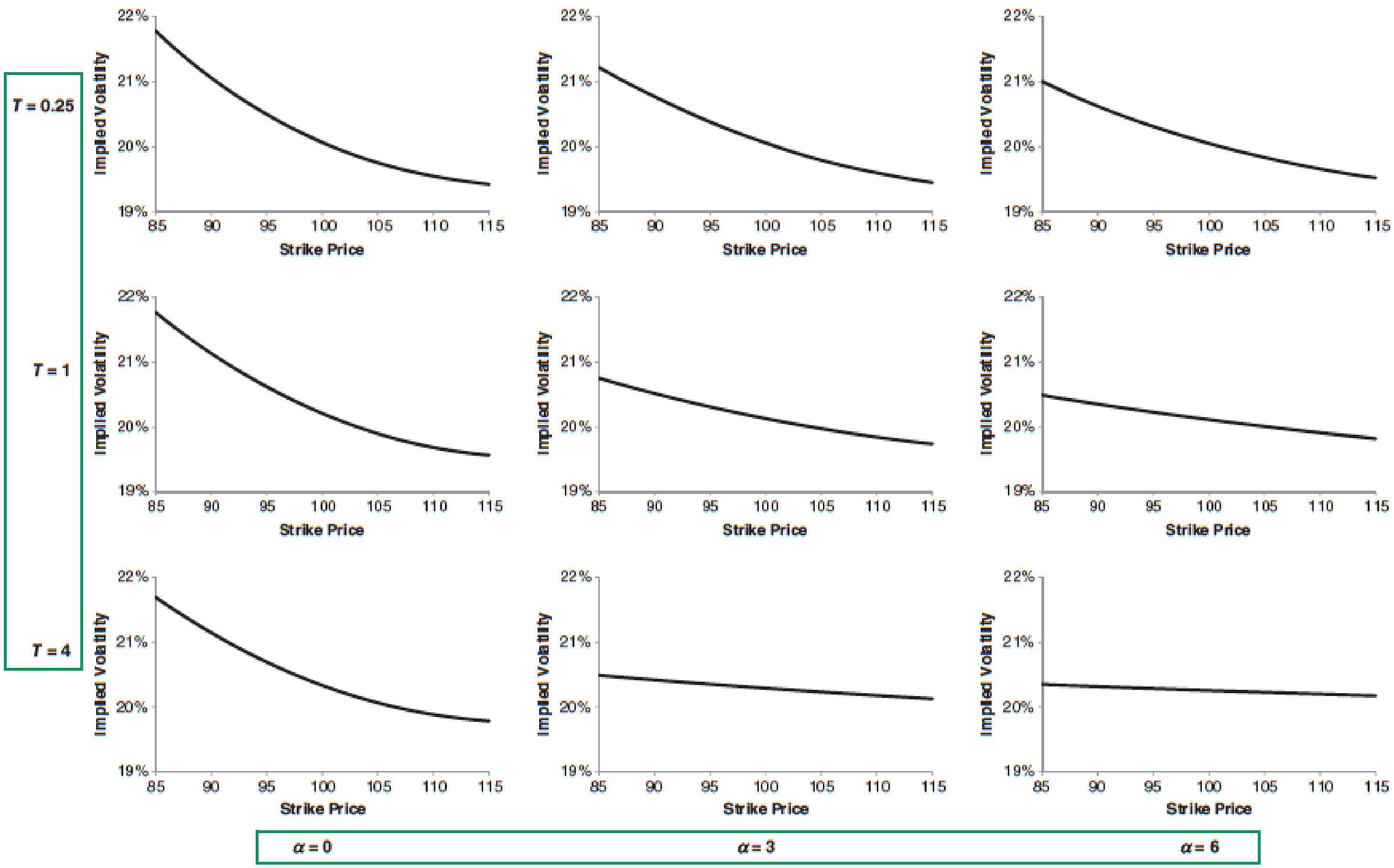


FIGURE 22.4 The Smile in a Mean-Reverting Stochastic Volatility Model and Its Variation with Time to Expiration and Mean Reversion Strength (Long-Term Volatility = 20%)

volatility is 20%, the long-term volatility m is **40%** the volatility of volatility is 50%, and the correlation between the stock and its volatility is –30%.

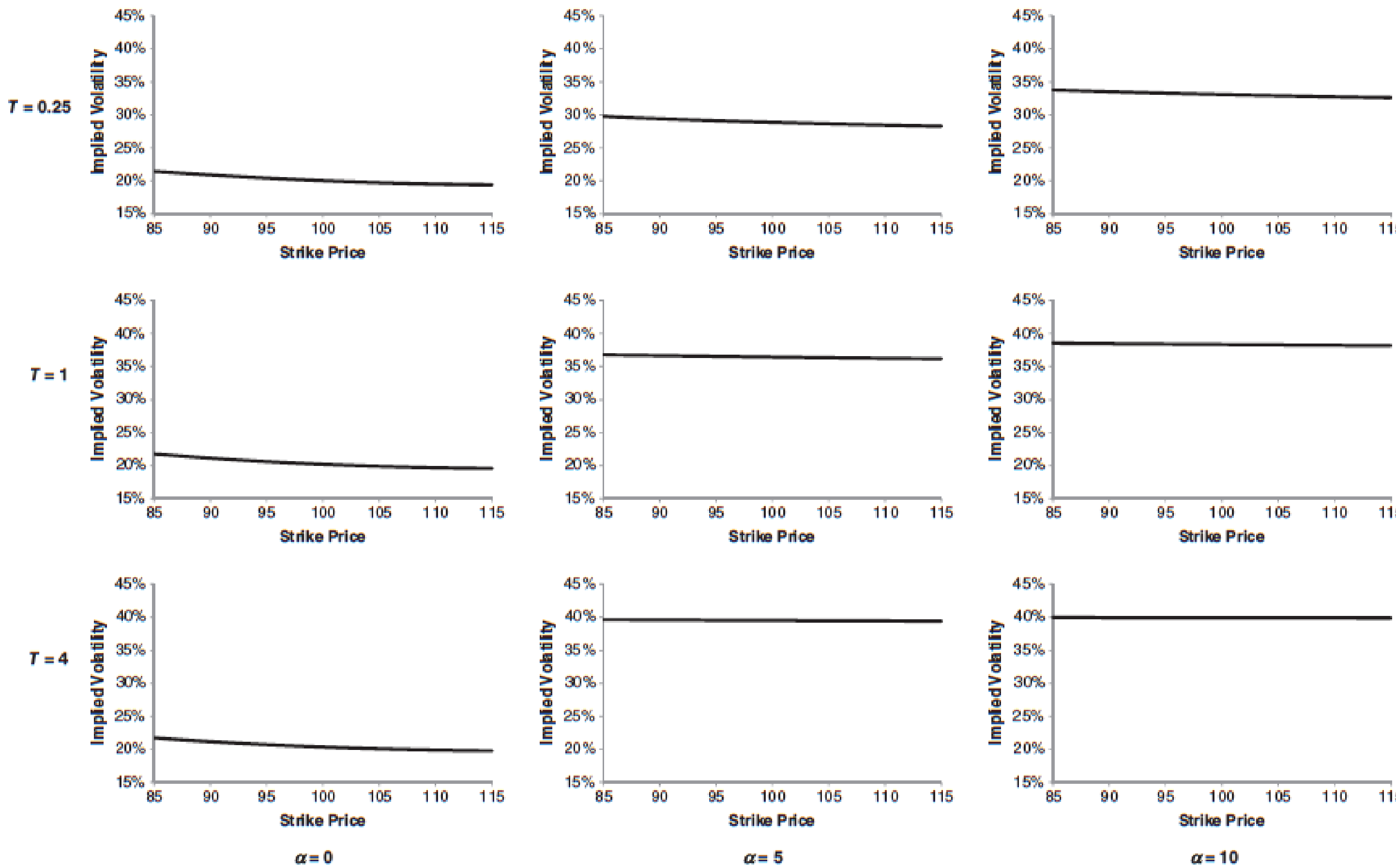
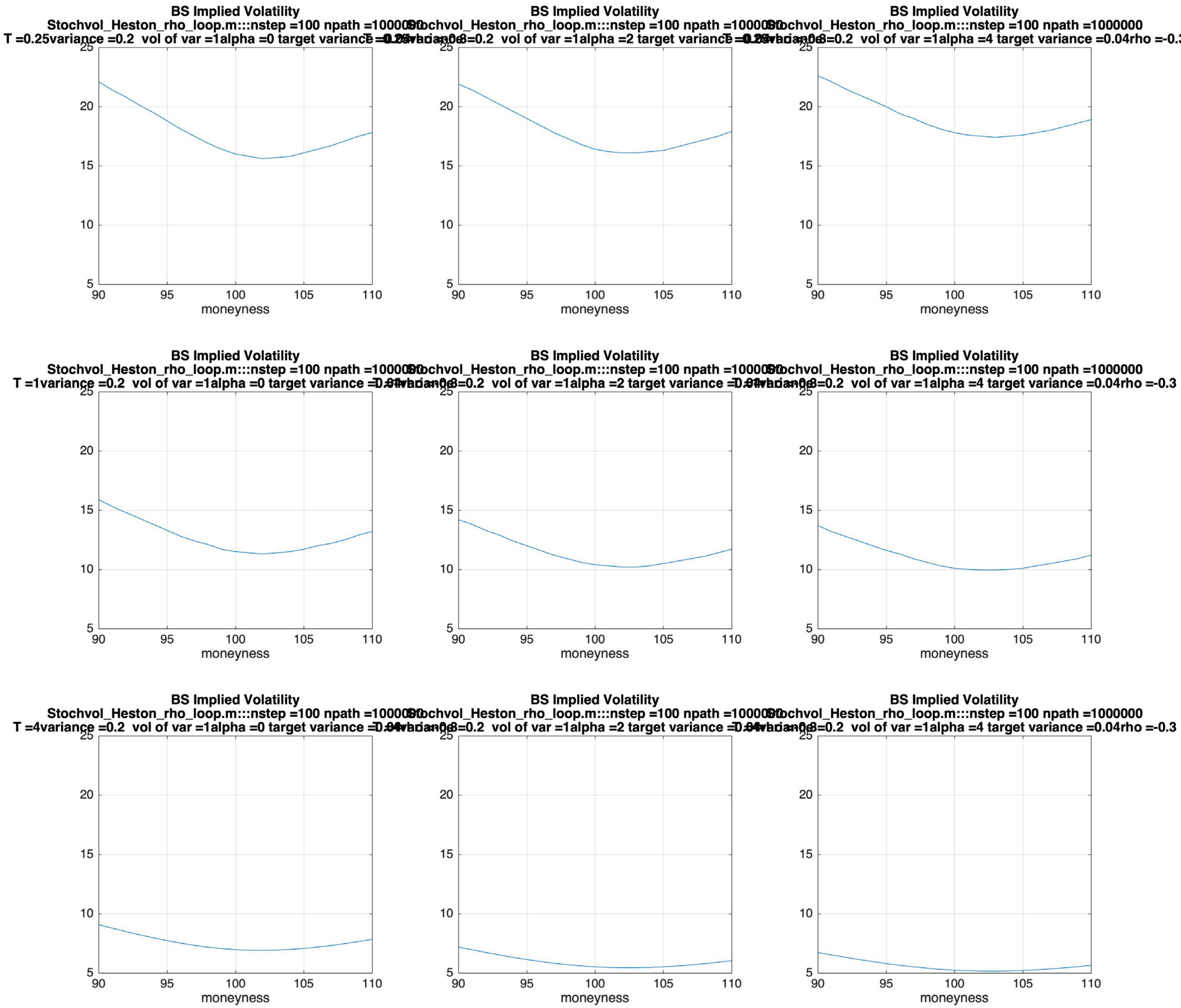


FIGURE 22.5 The Smile in a Mean-Reverting Stochastic Volatility Model and Its Variation with Time to Expiration and Mean Reversion Strength (Long-Term Volatility = 40%)

Heston with correlation -30% and mean reversion -- note flattening and downward drift of skew



Comparison of vanilla hedge ratios under Black-Scholes, Local Volatility and Stochastic Volatility models when all are calibrated to the same negative skew

Calibrated to the same current negative skew for the S&P, different models have different evolutions of volatility, different hedge ratios, different deltas, different forward skews, different exotic options values.

Black-Scholes: Implied volatility is independent of stock price. The correct delta is the Black-Scholes delta.

Local Volatility: Local volatility goes down as market goes up, so the correct delta is smaller than Black-Scholes.

Stochastic Volatility:

Implied volatility is a function of K/S,
Negative skew means that implied volatility goes up as K goes down
Then implied volatility must go up as S goes up.

Therefore, the st**ock-only hedge ratio will be greater than Black-Scholes**, contingent on the level of the stochastic volatility remaining the same. But, remember, in a stochastic volatility model there are two hedge ratios, a delta for the stock and another hedge ratio for the volatility, so just knowing how one hedge ratio behaves doesn't tell the whole story anymore.

21.5 Best stock-only hedge in a stochastic volatility model is like a local vol model

Although stochastic volatility models suggest a hedge ratio greater than Black-Scholes in a negative skew environment, that hedge ratio is only the partial hedge ratio w.r.t. the stock degree of risk, and doesn't mitigate the volatility risk.

What is the best stock-only hedge, best in the sense that you don't hedge the volatility but try to hedge away as much risk as possible with the stock alone, by minimizing the P&L volatility?

Best stock-only hedge is a lot like a local volatility hedge ratio, and is indeed smaller than hedge ratio in a Black-Scholes model.

Simplistic **stochastic implied volatility** model

$$\begin{aligned}\frac{dS}{S} &= \mu dt + \Sigma dZ \\ d\Sigma &= p dt + q dW \\ dZ dW &= \rho dt\end{aligned}$$

We have for simplicity assumed that the stock evolves with a realized volatility equal to the implied volatility of the particular option itself. Then for an option $C_{BS}(S, \Sigma)$ where both S and Σ are stochastic, we can find the hedge that minimizes the instantaneous variance of the hedged portfolio. That's as good as we can do with stock alone.

This partially hedged portfolio is $\pi = C_{BS} - \Delta S$

Then in the next instant $d\pi = \left(\frac{\partial C_{BS}}{\partial S} - \Delta\right) dS + \frac{\partial C_{BS}}{\partial \Sigma} d\Sigma = (\Delta_{BS} - \Delta) dS + \kappa d\Sigma$

The instantaneous variance of this portfolio is defined by $(d\pi)^2 = var[\pi]dt$ where

$$var[\pi] = (\Delta_{BS} - \Delta)^2 (\Sigma S)^2 + \kappa^2 q^2 + 2(\Delta_{BS} - \Delta)\kappa S \Sigma q \rho$$

The value of Δ that minimizes the residual variance of this portfolio is given by

$$\frac{\partial}{\partial \Delta} var[\pi] = -2(\Delta_{BS} - \Delta)(\Sigma S)^2 - 2\kappa S \Sigma q \rho = 0$$

$$\Delta = \Delta_{BS} + \rho\left(\frac{\kappa q}{\Sigma S}\right)$$

The second derivative $\frac{\partial^2}{\partial\Delta} var[\pi]$ is positive, so that this hedge produces a minimum variance.

The hedge ratio Δ is less than Δ_{BS} when ρ is negative. The best stock-only hedge in a stochastic volatility model tends to resemble the local volatility hedge ratio.