E4718 Sample Final Exam Questions

Formula sheet is on the last page.

Please sign the honor pledge with your name and UNI:

I pledge that I have neither given nor received unauthorized aid during this examination.

Student's Name:_______________________________________UNI _________________

Signature:_____________________________________

Hand back **this entire examination** with your calculations and answers written on the blank pages after each question.

Problem 1 **[35 points]**

In 2024 the U.S. market is in crisis, and in order to save the market the Federal Reserve Bank has dictated that **all rates and dividends are zero, and that there will be no volatility smile.** To do this they regulate all market-makers **to use the Black-Scholes formula with the same fixed volatility** σ. The central bank supplies the same Black-Scholes software to everyone.

(i) Unfortunately, the software has been hacked by foreign governments who want to destroy the market. Internally, it **replaces** the correct value of the stock price S **by the slightly shifted values** $0.95S$. All options prices are therefore slightly incorrect.

However you have written your own correct BS calculator that provides the correct Black-Scholes formula $C_{BS}(S, K, \tau, \sigma)$ where K is the strike, τ the time to expiration and σ the volatility.

When you enter U.S. market-maker's options prices into *your* B-S calculator and back out the implied BS volatilities from your calculator, you see a volatility smile.

(i) The hacked calculator replaces S by $0.95S$. Will the hacked prices lead to implied volatilities in the correct calculator that will be lower or greater than σ? Explain your reasoning. **[5 points]**

(ii) For options very close to at-the-money and with small variance at expiration, prove that the smile is approximately described by the implied volatility function

$$\Sigma_{impl}(S, K, \tau) \approx \sigma - (0.05)\sqrt{2\pi}\frac{\Delta_{BS}(S, K, \tau, \sigma)}{\sqrt{\tau}}$$

where $\Delta_{BS}(S, K, \tau, \sigma)$ is the usual Black-Scholes hedge ratio for a call for zero interest rates.

[25 points]

(iii) Assume $S = 100$. On the same set of axes, sketch two rough graphs of the shape of this implied volatility smile as a function of strike price K for fixed S, one for small time to expiration and one for large time to expiration.

[5 points]

Solution 1:

(i) Let $\varepsilon = 0.05$ which we regard as small. The correct formula for a call option is $C_{BS}(S, K, \tau, \sigma)$. The hacked calculator produce a price $C_{BS}(S(1-\varepsilon), K, \tau, \sigma)$. Since the hacked stock price is lower with the same volatility, the hacked call price is lower. Therefore the correct calculator will interpret this call price when the correct stock price is used as requiring a lower volatility.

(ii) The implied volatilities $\Sigma(S, K, \tau)$ of the hacked call prices are defined by

$$C_{BS}(S, K, \tau, \Sigma(S, K, \tau)) = C_{BS}(S(1-\varepsilon), K, \tau, \sigma)$$

Since ε is small, we can write $\Sigma(S, K, \tau) = \sigma + f(S, K, \tau)$ where *f* is small too.

Thus

$$C_{BS}(S, K, \tau, \sigma + f(S, K, \tau)) = C_{BS}(S(1-\varepsilon), K, \tau, \sigma)$$

Using a Taylor series to first order for small *f* and ε,

$$\cancel{C_{BS}(S, K, \tau, \sigma)} + \left[\frac{\partial C_{BS}(S, K, \tau, \sigma)}{\partial \sigma}\right] f(S, K, \tau) \approx \cancel{C_{BS}(S, K, \tau, \sigma)} + \left[\frac{\partial C_{BS}(S, K, \tau, \sigma)}{\partial S}\right][-\varepsilon S]$$

$$f(S, K, \tau) \approx -\varepsilon S \frac{\dfrac{\partial C_{BS}(S, K, \tau, \sigma)}{\partial S}}{\dfrac{\partial C_{BS}(S, K, \tau, \sigma)}{\partial \sigma}} = -\varepsilon S \frac{N(d_1)}{S e^{-d_1^2/2} \sqrt{\dfrac{\tau}{2\pi}}} = -\varepsilon \sqrt{2\pi} \frac{N(d_1)}{e^{-d_1^2/2} \sqrt{\tau}}$$

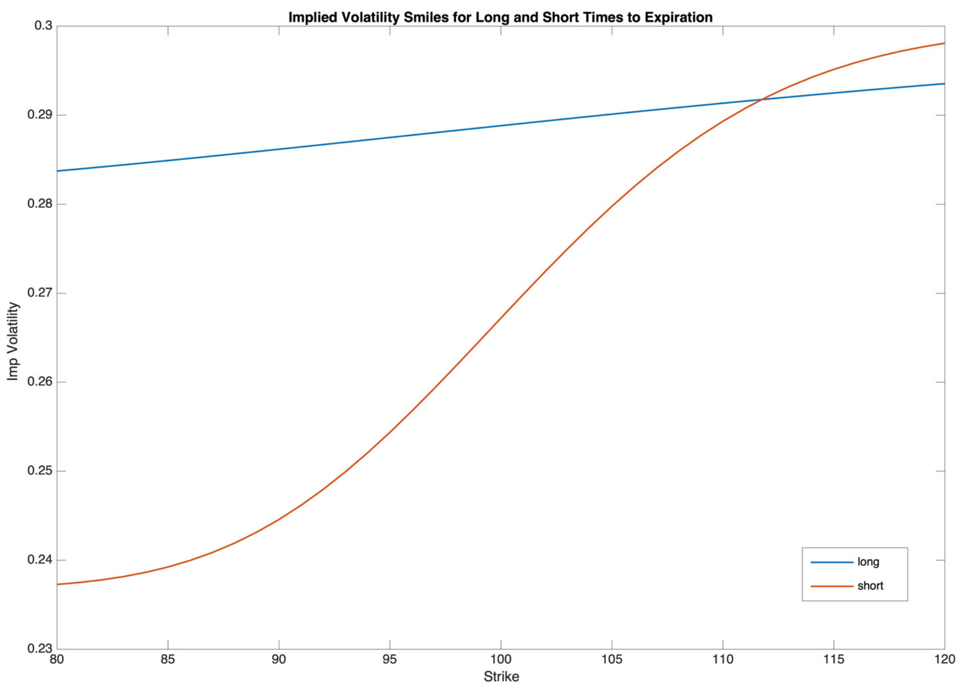
Now $\Delta_{BS}(S, K, \tau, \sigma) \equiv N(d_1)$ and so $f(S, K, \tau) = -\varepsilon \sqrt{2\pi} \dfrac{\Delta_{BS}(S, K, \tau, \sigma)}{e^{-d_1^2/2} \sqrt{\tau}}$ and is always negative.

Now $d_1 = \dfrac{\ln S/K}{\sigma\sqrt{\tau}} + \dfrac{\sigma\sqrt{\tau}}{2}$. Furthermore for options close to at the money $\dfrac{\ln S/K}{\sigma\sqrt{\tau}}$ is small, and for small variance at expiration $\sigma\sqrt{\tau}$ is small so that $d_1 \approx 0$. Therefore

$$\Sigma = \sigma + f \approx \sigma - \varepsilon\sqrt{2\pi}\frac{\Delta_{BS}(S, K, \tau, \sigma)}{\sqrt{\tau}} = \sigma - (0.05)\sqrt{2\pi}\frac{\Delta_{BS}(S, K, \tau, \sigma)}{\sqrt{\tau}} \quad \text{for } \varepsilon = 0.05$$

(iii) For low K, Delta is close to 1. For high K, Delta is close to 0.
For high $\boxed{\tau}$, the contribution of Delta decreases.

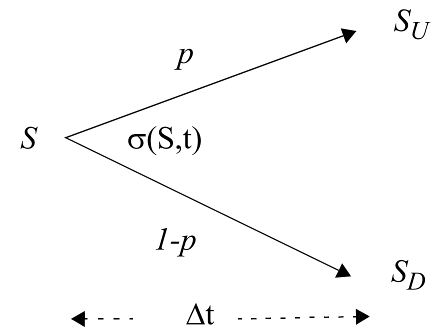
Getting the shape right is good enough in each case.



Problem 2: **[30 points]**

A stock S that pays no dividends evolves according to the risk-neutral local volatility model $\frac{dS}{S} = rdt + \sigma(S, t)dZ$ where r is the riskless interest rate.

Shown at right is a one-period binomial tree for the stock *S*.



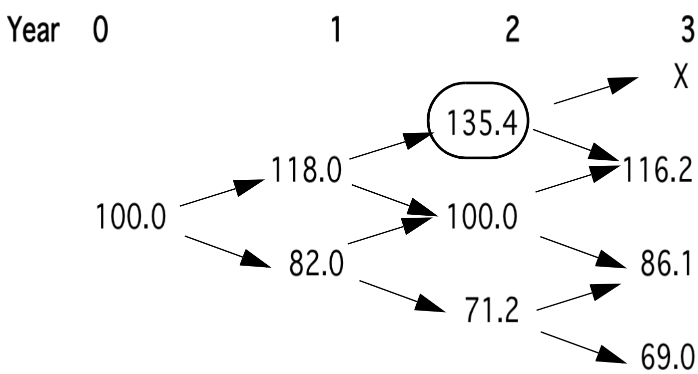
(i) Show that, for the tree to match the stochastic differential equation, we must require that

$$p = \frac{F - S_D}{S_U - S_D} \qquad \text{and}$$

$$(S_U - F)(F - S_D) \approx \{S\sigma(S, t)\}^2 \Delta t$$

where F is the forward price of S at time Δt in the future. **[10 points]**

(ii) Below is a binomial tree of stock prices with $\Delta t = 1$ year. The riskless rate r is zero. The stock price X is unknown.



Find the volatility $\sigma(S, t)$ at each node in year 0, 1 and 2, except for the circled node with price 135.4. Draw a tree that shows the values of the volatilities at each node. **[10 points]**

(iii) On a similar Cox-Ross-Rubinstein tree with **constant volatility**, the implied CRR constant volatility of a three-year European call with strike 135.4 is 15%. **Estimate** the local volatility at the circled node with stock price 135.4 using what you know about the **approximate** relation between implied and local volatilities. **[5 points]**

(iv) Estimate the stock price X in year 3. **[5 points]**

Solution to Problem 2.

(i): Risk-neutral condition on returns:

$$F = pS_u + (1-p)S_d$$

From Ito, the variance of the stock over time dt is $(dS)^2 = \sigma^2(S,t)S^2 dt$, so that we must require *approximately*, to leading order in Δt, that

$$S^2\sigma^2\Delta t = p(S_u - F)^2 + (1-p)(S_d - F)^2.$$

Solving we obtain

$$p = \frac{F - S_d}{S_u - S_d}$$

$$(F - S_d)(S_u - F) = S^2\sigma^2(S,t)\Delta t$$

(ii) vol tree from these formulas:

		v
	0.150	
0.180		0.150
	0.170	
		0.080

(iii) Let **v** be the volatility at the circled node. According to the statement that implied volatility is approximately the average over the local volatilities (or the Rule of 2) between spot and strike, (you don't need to do harmonic averages here, this is good enough), the implied volatility 15% = 1/3(18% + 15% + v%) and therefore $v \approx 12\%$.

(iv) At the circled node, $(S_U - F)(F - S_D) \approx \{S\sigma(S,t)\}^2\Delta t$ where $S = F = 135.4$, $S_D = 116.2$, $\sigma(135.4, 2) = 0.12$ and $S_U = X$.

Therefore $(X - 135.4)(135.4 - 116.2) \approx (135.4 \times 0.12)^2$ and so $X \approx 149.1$

Problem 3: **[30 points]**

Consider the Black-Scholes solution $C_{BS} = SN(d_1) - KN(d_2)$ for zero interest rates and zero dividend yields.

(i) Prove $\text{vega} \equiv \frac{\partial C_{BS}}{\partial \sigma} = S\exp\left(-\frac{d_1^2}{2}\right)\frac{\sqrt{\tau}}{\sqrt{2\pi}}$ where $\tau = T - t =$ time to expiration **[5 points]**

(ii) Prove that $\text{volga} \equiv \frac{\partial^2 C_{BS}}{\partial \sigma^2} = \frac{1}{\sigma}\frac{\partial C_{BS}}{\partial \sigma}d_1 d_2$ **[5 points]**

(iii) Suppose we model σ to be stochastic, so that it can take on with equal probability the values $\sigma_0 \pm \varepsilon$ where ε is small. Use the mixing theorem to show that the approximate value of a call option C_{SV} in this model is

$$C_{SV} \approx C_{BS}(\sigma_0) + \frac{\varepsilon^2}{2}\frac{\partial^2}{\partial \sigma^2}C_{BS}(\sigma_0)$$

where $C_{BS}(\sigma_0)$ is the Black-Scholes formula with volatility set equal to σ_0. **[10 points]**

(iv) Use the result of (ii) above to show that approximately for small ε the BS implied volatility in this model is

$$\Sigma_{SV} \approx \sigma_0 + \frac{\varepsilon^2}{2}\frac{d_1 d_2}{\sigma_0}$$

[5 points]

(v) Sketch the skew as a function of $\ln\frac{S}{K}$ in this model. **[5 points]**

Solution 3:

(i) $C = SN(d_1) - KN(d_2)$

$$\frac{\partial C}{\partial \sigma} = SN'(d_1)\frac{\partial d_1}{\partial \sigma} - KN'(d_2)\frac{\partial d_2}{\partial \sigma}$$

$$= SN'(d_1)\left(\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right)$$

$$= SN'(d_1)\sqrt{\tau}$$

$$= S\frac{e^{-d_1^2/2}}{\sqrt{2\pi}}\sqrt{\tau}$$

from formula sheet

$SN'(d_1) = KN'(d_2)$

$$\frac{\partial d_{1,2}}{\partial \sigma} = -\frac{\ln S/K}{\sigma^2\sqrt{\tau}} \pm \frac{\sqrt{\tau}}{2}$$

$$= -\frac{d_{2,1}}{\sigma}$$

(ii) $$\frac{\partial^2 C}{\partial \sigma^2} = \frac{S\sqrt{\tau}}{\sqrt{2\pi}}e^{-d_1^2/2}\left[-d_1\right]\frac{\partial d_1}{\partial \sigma}$$

$$= \frac{\partial C}{\partial \sigma}\left[-d_1\right]\left[\frac{-d_2}{\sigma}\right] = \frac{1}{\sigma}\frac{\partial C}{\partial \sigma}d_1 d_2$$

(iii)

$$C_{SV} = \frac{1}{2}C_{BS}(\sigma_0+\varepsilon) + \frac{1}{2}C_{BS}(\sigma_0-\varepsilon)$$

$$= C_{BS}(\sigma_0) + \frac{\partial^2 C}{\partial \sigma^2}(\sigma_0)\frac{\varepsilon^2}{2} + \cdots$$

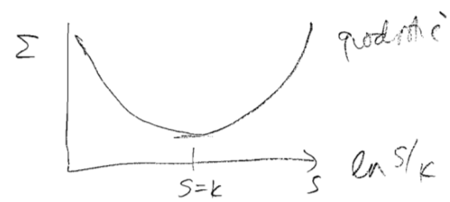
(iv) $$C_{SV} \doteqdot C_{BS}(\sigma_0) + \frac{1}{\sigma}\frac{\partial C}{\partial \sigma}d_1 d_2\frac{\varepsilon^2}{2} + \cdots$$ using (ii) in (iii)

$$\simeq C_{BS}\left(\sigma_0 + \frac{d_1 d_2}{\sigma_0}\frac{\varepsilon^2}{2}\right)$$ using Taylor series in σ

So $$\Sigma = \sigma_0 + \frac{d_1 d_2 \varepsilon^2}{2\sigma_0}$$

$$d_1 d_2 = \frac{\ln^2 S/K}{\sigma^2\tau} - \frac{\sigma^2\tau}{4}$$

(v)



Problem 4. **[30 points]**

Assume that a stock price S satisfies ***arithmetic Brownian motion*** *with zero risk-neutral interest rates and zero dividends:*

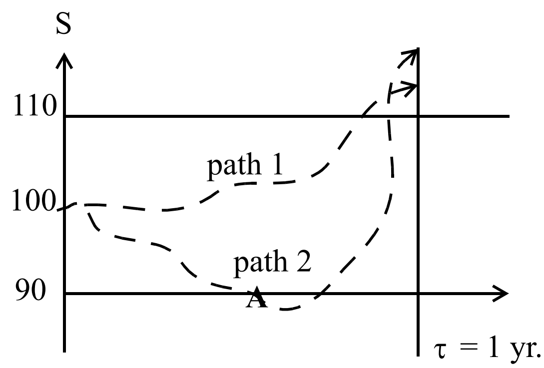
$$dS = \sigma dZ$$
$$S_t = S_0 + \sigma Z$$

where Z is a Brownian motion and σ is the annualized stock volatility.

(i) Prove or explain clearly why $C(S, S+h) = P(S, S-h)$, where $C(S, K)$ and $P(S, K)$ are the values of a call and put respectively. **[10 points]**

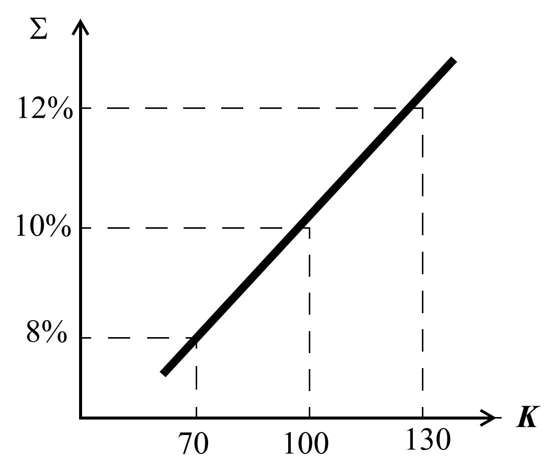
(ii) The current value of S is 100. Consider a one-year down-and-out call with strike $K = 110$ and knockout barrier at $B = 90$ and zero rebate. Now consider a portfolio Π **that is long a one-year call with strike 110 and short a one-year put with strike70**.

Explain clearly how Π can be used to replicate the payoff of the down-and-out call on paths 1 and 2 below.



10 points]

(iii) Suppose that after six months elapse there is a linear volatility skew as shown at right. Then, is the replicating portfolio worth more, less, or the same as the knockout, and why?
[10 points]



Solution 4:

(i) In arithmetic Brownian motion there is equal probability of moving up to S+h and down to S-h from S. They payoffs of the call on the upside is symmetric with the put on the downside of S. Therefore the expected risk-neutral values are equal too.

(ii) It has the payoff of the down and out call as long as stock never hits the barrier, because the call has the same strike as the down and out call, and the put will expire out of the money. If the stock hits the barrier, then by the symmetry of (i) above, the call and put have equal values anywhere on the barrier, and so the value of the portfolio is zero. As soon as the stock hits the barrier, you unwind the portfolio and are left with no position just when the barrier options knocks out.

(iii) $\boxed{\Pi = C_{110} - P_{70}}$ is the value of the portfolio on the barrier after six months. Because of the skew, the call struck at 110 is now worth more than the put struck at 70, and so the replicating portfolio would have positive value on the knockout barrier, and so be worth more than the knockout.

Problem 5. Multiple Choice Questions: Circle the Right Answer **[16 points]**

(5.1) An approximate no-arbitrage bound on the slope of the at-the-money volatility smile with respect to strike K at an expiration T is

-(a) $\frac{d\Sigma}{dK} \le \sqrt{\frac{\pi}{2}} \frac{1}{K\sqrt{T-t}}$ (b) $\frac{d\Sigma}{dK} \le \sqrt{2\pi} \frac{K}{\sqrt{T-t}}$ (c) $\frac{d\Sigma}{dK} \le \sqrt{\frac{\pi}{4}} \frac{\sqrt{T-t}}{K}$

[4 points]

(a) by dimensional analysis. The other solutions do not have the correct dimensions on the RHS

(5.2) The mixing theorem says that the solution to certain models of the skew can be written as a mixture of Black-Scholes solutions *with the same stock price but different volatilities and/or stock drifts*. The mixing theorem applies to

(a) stochastic volatility models only
(b) jump-diffusion models only
(c) jump-diffusion and stochastic volatility models with zero correlation
(d) local volatility models and jump-diffusion models. **[4 points]**

(c)

(5.3) A recently invented Product Model (PM) for the volatility smile produces call prices C_{PM} that take the form

$$C_{PM}(S, t, K, T, a) = C_{BS}\left(S, t, K, T, \Sigma\left(\frac{SK}{a^2}\right)\right)$$

where the Black-Scholes implied volatility Σ in the above equation is a function of the single variable $\frac{SK}{a^2}$, and a is some constant.

The at-the-money slope of the skew with respect to K is observed to be positive.

The delta of an at-the-money call option is:

(a) less than the corresponding Black-Scholes delta;
(b) greater than the corresponding Black-Scholes delta;
(c) cannot say.

(a)

[4 points]

(5.4) You can best describe the relation between implied volatility $\Sigma(S, t, K, T)$ and local volatility $\sigma(S)$, where $K > S$ and $T - t$ is small, by the relation:

(a) $$(K - S)\Sigma^{-1}(S, t, K, T) = \int_S^K \sigma^{-1}(u)du$$

(b) $$(K - S)\Sigma^{2}(S, t, K, T) = \int_S^K \sigma^{2}(u)du$$

(c) $$\frac{\left(\ln\frac{K}{S}\right)}{\Sigma(K)} = \int_S^K \frac{1}{\sigma(u)}\frac{du}{u}$$ **[4 points]**

(c)

Formulas

1. The Black-Scholes formula for a European call option at time t with strike K expiring at time T on a non-dividend-paying stock of price S with future volatility σ and a continuously compounded riskless rate r is given by

$$C_{BS}(S, t, K, T, r, \sigma) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$d_{1,2} = \frac{\ln(S_F/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$S_F = e^{r(T-t)}S$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \exp\left(\frac{-y^2}{2}\right)dy$$

You may find these formulas useful too:

$$S_F N'(d_1) = KN'(d_2) \text{ where } N'(x) \equiv \frac{dN}{dx} = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right).$$

$$\frac{\partial C_{BS}}{\partial K} = -e^{-r(T-t)}N(d_2) \qquad \frac{\partial C_{BS}}{\partial S} = N(d_1) \qquad \frac{\partial C_{BS}}{\partial \sigma} = S\exp\left(-\frac{d_1^2}{2}\right)\sqrt{\frac{T-t}{2\pi}}$$

2. If you hedge a long position in an option V at the Black-Scholes implied volatility Σ, and the stock undergoes geometric Brownian motion with realized volatility σ over the next instant dt, then the P&L over that instant is given by

$$d\text{P\&L} = \frac{1}{2}\frac{\partial^2 V}{\partial S^2}S^2(\sigma^2 - \Sigma^2)dt$$

3. Dupire formula with zero interest rates and dividends

The local volatility $\sigma(K, T)$ at a future stock price K and time T is determined by the equation

$$\frac{\partial O(S, t, K, T)}{\partial T} = \frac{1}{2}\sigma^2(K, T)K^2\frac{\partial^2 O}{\partial K^2}(S, t, K, T)$$

where $O(S, t, K, T)$ is the market price for a standard European-style call or put option with strike K and expiration T when the stock price is S at time t.

4. Risk-neutral Probability Distributions

If the price at time t of a standard European call or put option with strike K and expiration T when the underlying stock price is S is represented by the $O(S, t, K, T)$, then the risk-neutral probability density $\rho(S, t, K, T)$ for the stock price to move from price S at time t to a future price between K and $K + dK$ at time T is given by

$$\rho(S, t, K, T) = \exp[r(T-t)]\frac{\partial^2 O(S, t, K, T)}{\partial K^2}$$

where r is the continuously compounded annual riskless rate.