Practice Final

Exercise 1 [3+2=5 points]
Suppose that $X_1, \ldots, X_n$ is a random sample from a distribution probability mass function

$$f(x \mid \theta) = (1-\theta)^{x-1}\theta$$

for $x = 1, \ldots$. Suppose that the prior for θ is a Beta(a, b) distribution.

(a) Find the posterior distribution of θ.

(b) Find the Bayes estimator of θ.

Exercise 2 [5+2=7 points]
Assume that $X_1, \ldots, X_n$ are a random sample from the a distribution with probability density function

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

We want to test

$$H_0: \ \lambda = 1 \qquad \text{vs.} \qquad H_1 : \lambda = 3.$$

(a) Fix $\alpha_0 \in (0,1)$. Find the test which minimises $\beta(\delta)$ among all test procedures for which $\alpha(\delta) \leq \alpha_0$.

(b) For $\alpha_0 = 0.1$ and $n = 10$ specify the rejection region of the test.
Hint: The sum of n i.i.d. exponential random variables follows a Gamma(n, $1/\lambda$) distribution and you might need the quantile $F^{-1}_{\text{Gamma}(10,1)}(0.1) = 6.22$.

Exercise 3 [3+3+5=11 points]
Suppose that we have one sample $X_1, \ldots, X_5$ of a normal distribution with unknown mean μ_1 and unknown σ_1^2 and another sample $Y_1, \ldots, Y_{10}$ of a normal distribution with unknown mean μ_2 and variance σ_2^2.

(a) Describe how to test the hypothesis that the variances are the same.

(b) You are given the numbers $\bar{X}_5 = 3$, $\bar{Y}_{10} = 2.8$, $S_X^2 = 32$, $S_Y^2 = 71$. Carry out the test with $\alpha_0 = 0.05$.
Hint: you might need the quantiles of the F-distribution: $G^{-1}_{4,9}(0.025) = 0.112$, $G^{-1}_{4,9}(0.975) = 4.718$.

(c) What is the distribution of $(S_X^2 + S_Y^2)/\sigma_1^2$ under the null hypothesis?

Exercise 4 [2+2+5=9 points]
Consider the simple linear regression model of Chapter 10. In particular $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ for $i = 1, \ldots, n$.

(a) Is the estimator

$$\hat{\gamma}_1 = \frac{1}{\bar{x}^3} \sum_{i=1}^{n} (Y_i - a_i \bar{Y})$$

linear in Y_i? Determine its value if $a_i = 1$ for all $i = 1, \ldots, n$.

(b) Compute $\mathrm{Cov}(Y_1 + 4Y_2, Y_2 + 5Y_3)$.

(c) In a normal simple linear regression model, determine the distribution of $Y_1 + 4Y_2$.

Exercise 5 [3+1+3+4=11 points] Assume the normal simple linear regression model. You are given the following data:

i	1	2	3	4	5
x_i	1	5	9	11	13
Y_i	9	4	10	3	2

(a) Fit a least-squares line to the data.

(b) Use the least-squares line to estimate the mean value of Y for $x_h = 5$.

(c) Determine the 90% CI for β_1.
Hint: you might need the quantile $t_3(0.95) = 2.35$.

(d) Test the null hypothesis $\beta_0 = 0$ for $\alpha_0 = 0.05$.
Hint: you might need the quantile $t_3(0.975) = 3.18$.