Lecture 9:

Back To The Smile:
Plotting The Skew
Consequences for Trading
Bounds on the Smile

The Relationship between Δ and Strike

As the stock moves you more likely quote a certain moneyness or delta rather than a strike.

A standard measure of the skew is the *risk reversal*: difference in volatility between an out-of-the-money call option with a 25% Δ and an out-of-the-money put with a –25% Δ.

Moneyness rather than strike because it's more general to talk about moneyness as markets move.

What percentage of moneyness corresponds to a given Δ?

For simplicity set $r = 0$ and assume small volatility.

$$\Delta = N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} \exp\left(-\frac{y^2}{2}\right)dy$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} \exp\left(-\frac{y^2}{2}\right)dy + \int_{0}^{d_1} \exp\left(-\frac{y^2}{2}\right)dy\right] \approx \frac{1}{2} + \frac{d_1}{\sqrt{2\pi}}$$

$$d_{1,2} = \frac{\ln\frac{S}{K}}{\Sigma\sqrt{\tau}} \pm \frac{\Sigma\sqrt{\tau}}{2} \text{ and } \tau = T - t$$

At the money $S = K$ $$\Delta \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\frac{\Sigma\sqrt{\tau}}{2} \approx 0.5 + (0.4)(0.5)\Sigma\sqrt{\tau}$$

For 20% volatility 1 year expiration $$\Delta \approx 0.5 + 0.04 = 0.54.$$

Slightly out of the money: $K = S + \delta S$ $\ln\left(\frac{S}{S+\delta S}\right) = -\ln(1+\delta S/S) \approx -\frac{\delta S}{S}$

$$d_1 = \frac{\ln\frac{S}{K}}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2} \approx -\frac{(\delta S)/S}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2}$$

Then for a slightly out-of-the-money option,

$$\Delta \approx \frac{1}{2} + \frac{d_1}{\sqrt{2\pi}} \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(\frac{\Sigma\sqrt{\tau}}{2} - \frac{(\delta S)/S}{\Sigma\sqrt{\tau}}\right)$$

percent move of

total variance

Suppose $(\delta S)/S = 0.01$, $T = 1$ year $\Sigma = 0.2$.

Then $\Delta \approx 0.54 - \frac{(0.4)(0.01)}{0.2} = 0.54 - 0.02 = 0.52$

Thus, Δ decreases by two basis points for every 1% that the strike moves out of the money.

The difference between a 50-delta and a 25-delta option therefore corresponds to about a 12% or 13% move in the strike price.

The move δS to decrease the delta from atm 0.54 to 0.25 is approximately given by

$$\frac{1}{\sqrt{2\pi}}\frac{(\delta S)/S}{\Sigma\sqrt{\tau}} \approx 0.29 \text{ or } (\delta S)/S = 0.29\sqrt{2\pi}\Sigma\sqrt{\tau} \approx 0.29 \times 2.5 \times 0.2 \approx 0.15$$

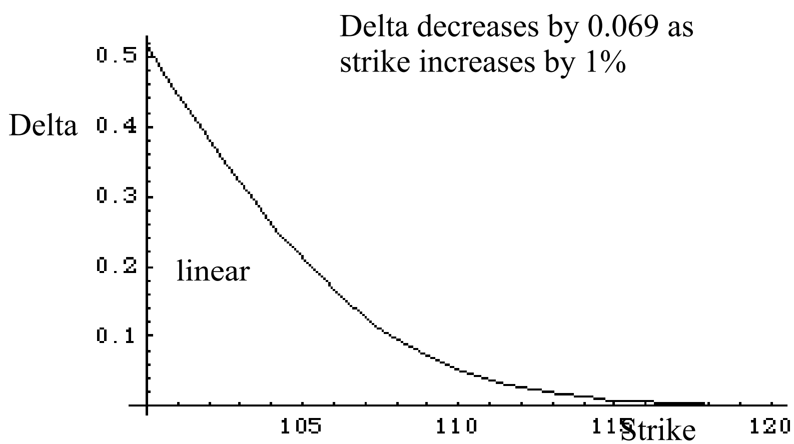
Thus the strike of the 25-delta call is about 115. Actually it's about 117 if you use the exact Black-Scholes formula to compute deltas.

More generally $$\text{change in Delta} \approx \frac{1}{\sqrt{2\pi}}\left(-\frac{(\delta S)/S}{\Sigma\sqrt{\tau}}\right)$$

and a one-basis point change in Δ corresponds to a change in $(\delta S)/S$ of about $0.025\Sigma\sqrt{\tau}$.

Delta depends upon the percent move in stock price divided by the square root of the total variance. For a greater volatility or time to expiration and you need a bigger move in the strike to get to the same Δ.

A 1-month call with S = 100 and with zero interest rates, 20% volatility, a 1% move in strike produces $(\sqrt{12})2 = 6.9 \text{ b.p.}$, much bigger because the total variance is smaller.



9.1 No-Arbitrage Bounds on the Smile

An analogy:

Yield to maturity: the quoted parameter that determines bond prices: $B_T = 100\exp(-y_T T)$

Implied volatility Σ: the quoted parameter that determines options prices in Black-Scholes.

When you deal with parameters in models, you have to be careful that they don't produce arbitrageable prices.

There are no-arbitrage bounds on bond yields. Suppose we choose yields for 1-year and 2-years such that $B_1 = 90$ and $B_2 = 91$. The two-year bond worth more than one-year.

$\pi = \frac{91}{90}B_1 - B_2$ has zero cost.

After one year the long position is worth more than \$100, so if you wait for B_2 to mature and pay off the face value you have a riskless profit so there is something wrong with these yields. They produce negative forward prices. You cannot choose yields that produce these prices.

There are similar constraints on option implied volatilities

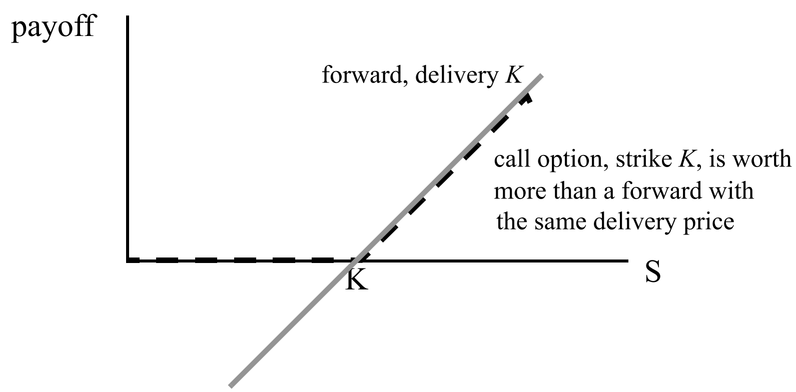
Some of the Merton Inequalities for Strike

Assume zero dividends, European calls.

1. A call is always worth more than a forward: $C \geq S - Ke^{-r(T-t)}$:

   Proof: An option is always worth more than a forward, because it has the same payoff when $S_T > K$, and is worth more when $S_T < K$.

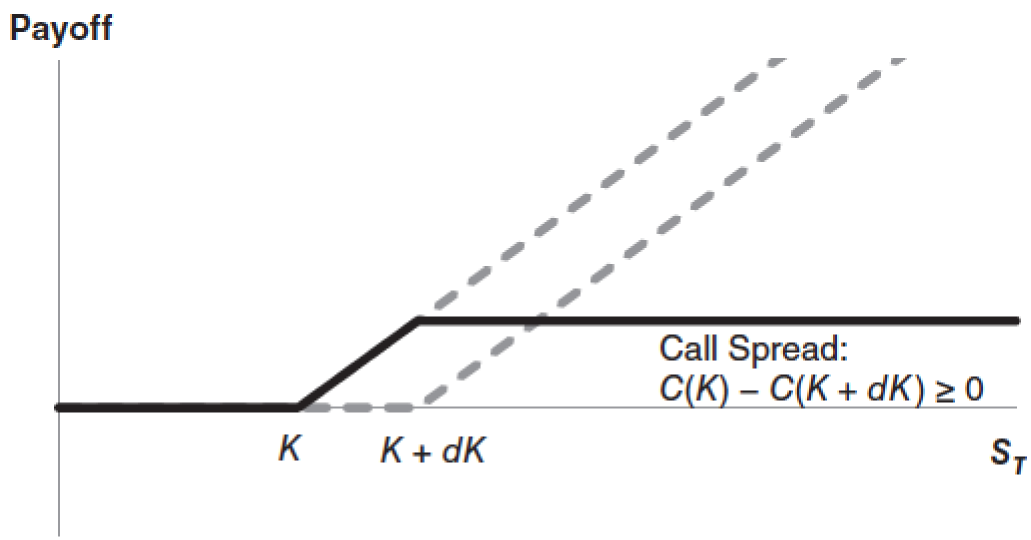
   Diagrammatically:



2. For the same expiration, options prices satisfy two constraints on their derivatives:

$$\frac{\partial C}{\partial K} \le 0 \text{ and } \frac{\partial^2 C}{\partial K^2} \ge 0,$$

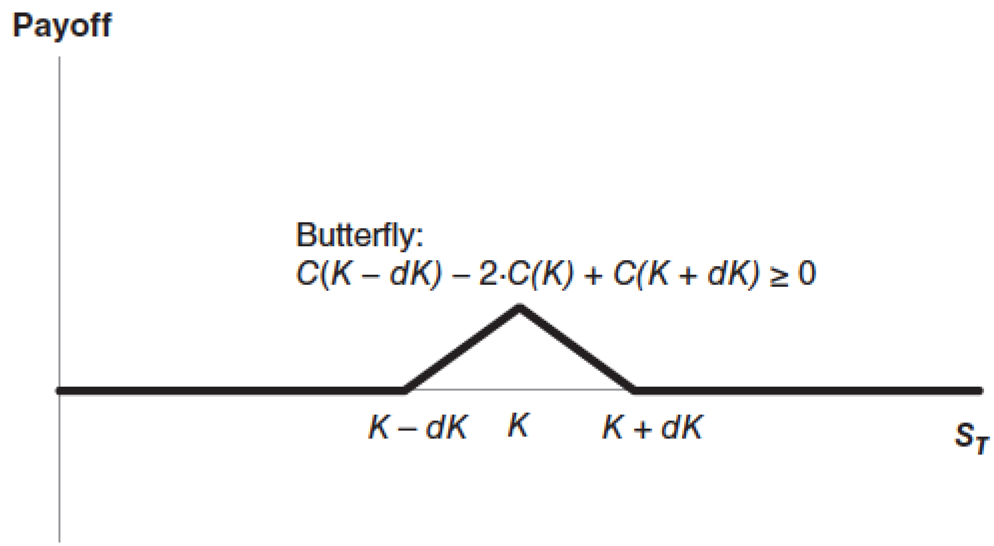
Proof 1: Look at payoff of a *call spread* at expiration:



In the limit,

$$\frac{\partial C}{\partial K} \le 0$$

3. Look at the payoff of a *butterfly spread* at expiration:



$$\pi_B = C(K-dK) - 2C(K) + C(K+dK)$$
$$= [C(K+dK) - C(K)] - [C(K) - C(K-dK)]$$

$$\lim_{dK\to 0}\frac{\pi_B}{dK^2} = \frac{\partial^2 C}{\partial K^2}$$

From the law of no-riskless-arbitrage,

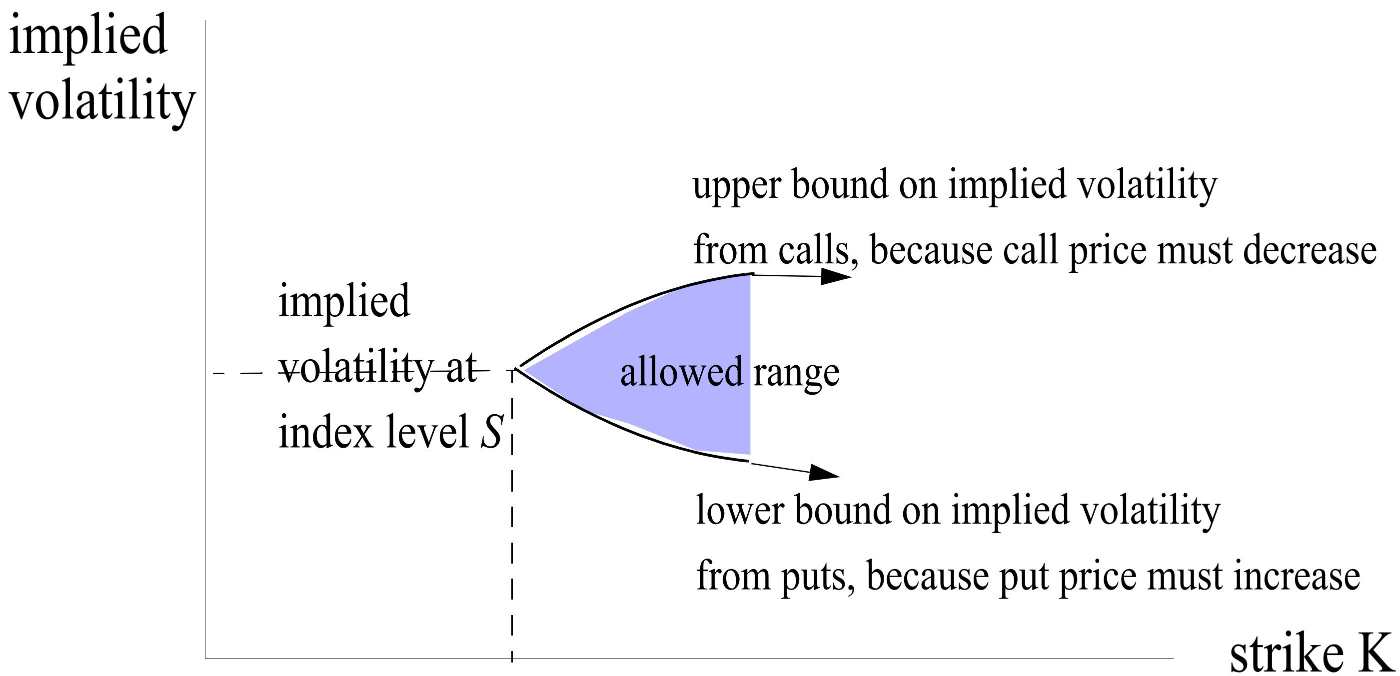
$$\frac{\partial^2 C}{\partial K^2} \geq 0$$

Similarly for puts, $\frac{\partial P}{\partial K} \geq 0$ $\qquad\qquad$ $\frac{\partial^2 P}{\partial K^2} \geq 0$

Inequalities for the slope of the smile

The constraints on $\frac{\partial C}{\partial K} < 0$ and $\frac{\partial P}{\partial K} \geq 0$ put limits on the slope of the smile independent of model.

Therefore they put constraints on the implied volatility parameters in the BS formula as a function of strike.



Now let's develop this idea more quantitatively.

$$C = C_{BS}(S, t, K, T, r, \Sigma)$$

$$\frac{\partial C}{\partial K} = \frac{\partial C_{BS}}{\partial K} + \frac{\partial C_{BS}}{\partial \Sigma}\frac{\partial \Sigma}{\partial K} \le 0$$

$$\frac{\partial C_{BS}}{\partial \Sigma} = S\sqrt{\tau}N'(d_1) \equiv Ke^{-r\tau}\sqrt{\tau}N'(d_2) \qquad \text{Eq.9.1}$$

$$\frac{\partial \Sigma}{\partial K} \le -\frac{\frac{\partial C_{BS}}{\partial K}}{\frac{\partial C_{BS}}{\partial \Sigma}} = \frac{e^{-r\tau}N(d_2)}{Ke^{-r\tau}\sqrt{\tau}N'(d_2)} = \frac{N(d_2)}{K\sqrt{\tau}N'(d_2)}$$

For small volatility, at the money: $d_2 \approx 0$, $N(d_2) \approx 0.5$ and $N'(d_2) \approx \frac{1}{\sqrt{2\pi}}$:

$$\frac{\partial \Sigma}{\partial K} \le \sqrt{\frac{\pi}{2}}\frac{1}{K\sqrt{\tau}} \approx \frac{1.25}{K\sqrt{\tau}} \qquad \text{Eq.9.2}$$

For 1-month options on the S&P with $S = K = 4000$ $\qquad K\frac{\partial \Sigma}{\partial K} \le 4.3$

For a 5% change in moneyness $\Delta K = 200$) volatility must change less than 0.22 or 22 vol points. Recall: the S&P skew slope for one-month options changed by about 1 vol point for a 1% change in moneyness, so it's not too far from the no-arbitrage limit.

Asymptotically short-term expiration

$$\frac{\partial\Sigma}{\partial K} \le \frac{N(d_2)}{K\sqrt{\tau}N'(d_2)}$$

At-the-money forward, as $\tau \to 0$

$$d_2 \to -\frac{\Sigma\sqrt{\tau}}{2} \to 0$$

$$N(d_2) \to \frac{1}{2}$$

$$N'(d_2) \to \frac{1}{\sqrt{2\pi}}$$

and so

$$\frac{\partial\Sigma}{\partial K} \le O(\tau^{-1/2}) \qquad \text{as } \tau \to 0.$$

As the time to expiration $\tau \to 0$, the slope steepness can increase no faster than $O(\tau^{-1/2})$.

Asymptotically long expiration

At the other extreme, as $\tau \to \infty$, $d_2 \to -\infty$, and therefore

$$\frac{\partial \Sigma}{\partial K} \le \frac{1}{K\sqrt{\tau}} \frac{N(d_2)}{N'(d_2)} \sim O\left(\frac{1}{\sqrt{\tau}}\frac{1}{\sqrt{\tau}}\right) \sim O\left(\frac{1}{\tau}\right)$$

To prove the line above we have made use of the asymptotic relation

$N(d_2)/N'(d_2) \sim O(\tau^{-0.5})$ as $\tau \to \infty$.

The area under the tail gets smaller faster than the height of the tail.

Thus, the slope of the smile can decrease with time to expiration no more slowly than $O(\tau^{-1})$.

Reference: *Arbitrage Bounds on the Implied Volatility Strike and Term Structures of European-Style Options*. Hardy M. Hodges, Journal of Derivatives, Summer 1996, pp. 23-35.

9.2 Some Behavioral Reasons for an Implied Volatility Skew

There are two approaches to options trading:

1. Consumers buying protection from or seeking exposure to the underlier.
2. Sophisticated people trading volatility as an asset.

- Knowledge of past behavior in options markets suggests a skew in options would be wise. (How much, though? What's the fair value?) Implied and realized volatilities go up after a crash.
- Expectation of future changes in volatility naturally gives rise to a term structure.
- Expectation of changes in volatility as support or resistance levels in currencies and interest rates suggests that realized volatility will decrease as those levels are approached.

•Expectation of an increase in the cross-sectional correlation between the returns of constituent stocks in the index as the market drops can cause an increase in the volatility of the entire index.

$$\text{Index:} \quad r = \sum w_i r_i \qquad \sigma^2 = \sum w_i w_j \sigma_{ij}$$

- Dealers' tend to be short options because they sell zero-cost collars (short otm call - long otm put) to investors who want protection against a decline.

9.3 An Overview of Smile-Consistent Models

Three choices:

(i) Model the realistic stochastic evolution of the underlying asset S and its realized volatility, and then deduce effective BS $\Sigma(S, t, K, T)$: this is a fundamental approach, we avoid arbitrage violations; but it's hard to get the right equation for the evolution.

(ii) instead, directly model the dynamics of the parametric surface $\Sigma(S, t, K, T)$.
Stochastic Implied Volatility: it is more intuitive but we are modeling a parameter in a bad model, not a price, and it's hard to avoid arbitrage violations.

(iii) Pragmatic vanna-volga models to value an exotic option by replicating it with a portfolio of vanilla options in the BS world, by making the exotic and the vanillas have the same vega, vanna, and volga. Then adjust the value for a skewed world by how much the portfolio of vanillas change as the skew is turned on. It's analogous to using the Black-Scholes world as a control variate.

Local Volatility Models -- the first smile models.

Black-Scholes: $\Sigma(S, t, K, T) = \sigma$ is independent of strike and expiration.

Local volatility models: $$\frac{dS}{S} = \mu(S, t)dt + \sigma(S, t)dZ$$

$\sigma(S, t)$ is a *deterministic* function of a *stochastic variable* S.

This is a one-factor model so our replication strategy and risk-neutral valuation still works. But does it truly describe the behavior of S?

The issue: Calibration: how to choose $\sigma(S, t)$ to match market values of $\Sigma(S, t, K, T)$? The model provides great intuition. (Cf. forward prices and the yield curve).

People make use of it for trading and as a proxy for other models, and extend it with some stochastic volatility.

What might account for local volatility being a function $\sigma(S, t)$?

***The leverage effect*:** leverage makes volatility increase as the company moves towards bankruptcy:

$$S = A - B \qquad \text{assets - liabilities}$$

$$\frac{dA}{A} = \sigma dZ$$

$$\frac{dS}{S} = \frac{dA}{S} = \frac{A\sigma dZ}{S} = \frac{(S+B)}{S}\sigma dZ$$

$$\sigma_S = \sigma(1 + B/S)$$

Constant Elasticity of Variance (CEV) models (Cox and Ross):

$$\frac{dS}{S} = \mu(S,t)dt + \sigma S^{\beta-1}\, dZ$$

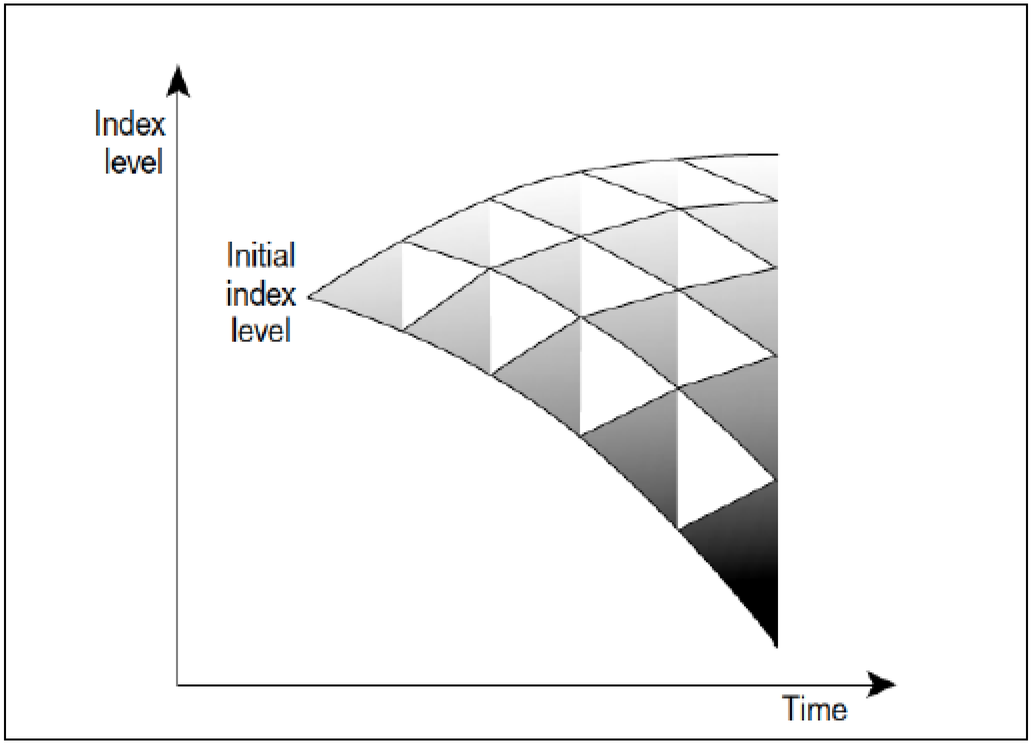
$$\text{Vol} = \sigma S^{\beta - 1}$$

$$\frac{S}{V}\frac{\partial V}{\partial S} = \text{constant}$$

$\beta = 1$ lognormal; $\beta = 0$ normal evolution. β needs to be large and negative, but then model has mathematical problems.

CEV is a parametric model and cannot fit an arbitrary smile; local volatility models are non-parametric and $\sigma(S, t)$ can be calibrated numerically.

Schematic view of a local volatility model



Stochastic Volatility Models

Volatility is actually random too.

$$dS = \mu S dt + \sigma S dZ_t$$

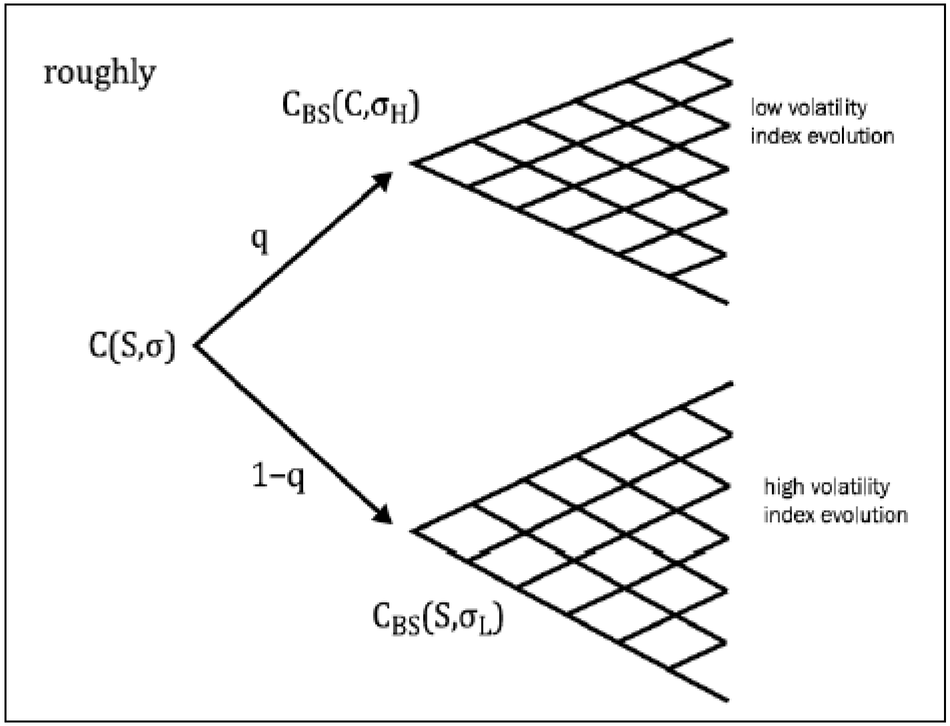
$$d\sigma = p\sigma dt + q\sigma dW_t$$

$$E[dWdZ] = \rho dt$$

In that case perfect replication is impossible if you allow yourself to hedge only with the stock.

Analogy: to hedge a long position in a bond that is exposed to interest rates, you have to short another bond. You cannot short an interest rate, only a security. The hedged portfolio, long one bond, short another, must earn the riskless rate. Hence one can derive the PDE for interest-rate sensitive securities and their prices. (Vasiçek)

Similarly, to hedge an option's exposure to volatility, you have to short another option. You cannot short σ. ***Assuming (!?)***you know the stochastic process for volatility, then you can hedge one option's exposure to volatility with another option and derive an arbitrage-free formula for options values. Stochastic volatility models assume that the correlation ρ is constant but that is stochastic too.

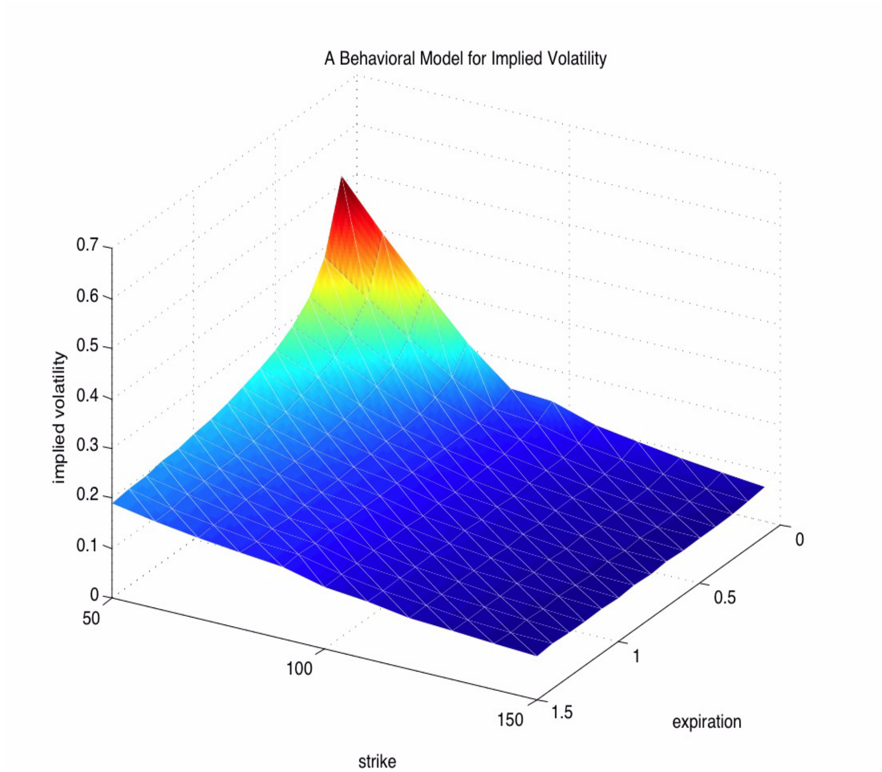

roughly
$C_{BS}(C,\sigma_H)$
low volatility
index evolution
q
$C(S,\sigma)$
1−q
high volatility
index evolution
$C_{BS}(S,\sigma_L)$

Jump-Diffusion Models

Black-Scholes ignores discontinuous jumps. Merton model allows an arbitrary number of jumps plus diffusion.

With a finite number of jumps of known size in the model, one can hedge the jumps perfectly by dynamic trading in a finite number of options, the stock and the bond, and so achieve risk-neutral pricing.

With an arbitrary or infinite possible number of jumps, one cannot hedge all the jumps, but people use risk-neutral pricing anyway.



Jumps affect the short-expiration part of the skew more than the long expiration part of the skew.

Schematic view of a jump-diffusion model

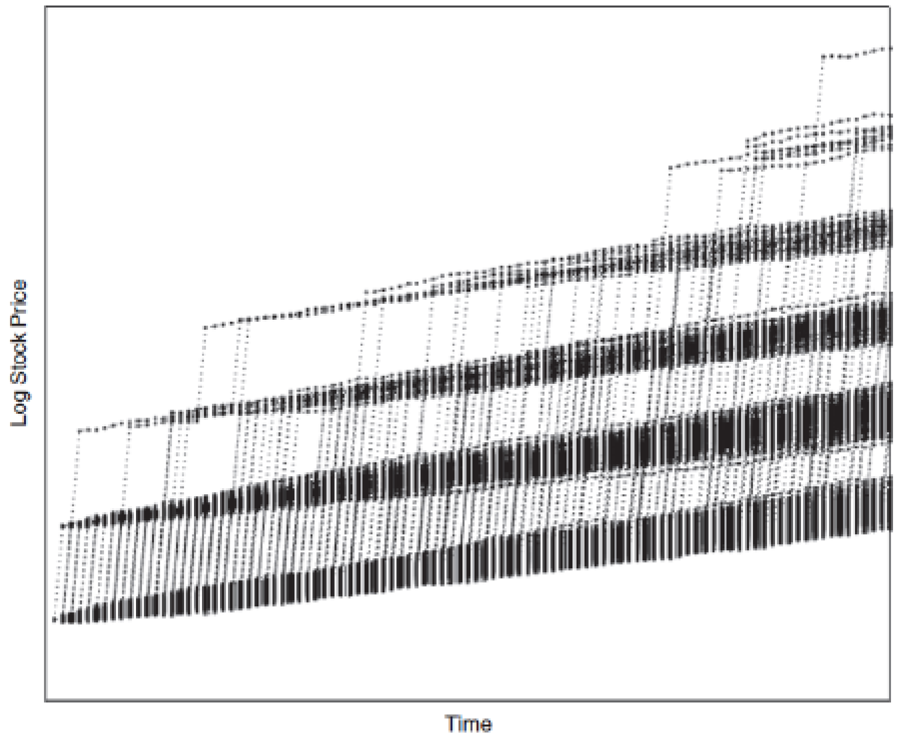


FIGURE 24.3 A Monte Carlo Simulation of the Log Stock Prices in the Jump-Diffusion Model

A Plenitude of Other Models

Local stochastic volatility: use local volatility to match the skew, stochastic volatility to add realism.

There are many other smile models too, which we may discuss later: fractional Brownian motion for volatility, forward variance models, mixing models, variance gamma models, stochastic implied volatility models …

In practice, one has to see which model best describes the market one is working in.

In the real world there is indeed diffusion, jumps and stochastic volatility!

There are too many different ways of fitting the observed smile that the model is non-parsimonious and offers too many choices.

In the end, you want to model the market with reasonable (but not perfect) accuracy via a fairly simple model that captures most of the important behavior of the asset.

A model is only a model, not the real thing.

9.4 Problems Caused by the Smile for Trading

- You can regard liquid standard call and put options prices as being simply *quoted* via the Black-Scholes formula, *so the model doesn't really matter for pricing*. (Analogy: bond prices are quoted by a single yield to maturity even though you may have calculated the PV of the future payments via a different method.)

- The model does matter if you wanted to generate your own idea of fair options values and then arbitrage them against market prices, but that is a very risky long-term business. Deciding which vanilla options are too cheap or too rich is a buy-side view.

- The model does matter for calculating hedge ratios, for market makers.

- The model matters for pricing illiquid OTC exotic options, for market makers or buy side.

- The question in both of these cases is of course: which model?

We now try to estimate the effect that the skew has on hedging and valuation, without using a particular model.

9.4.1 Fluctuations in the P&L from incorrect hedging of standard options

If we have the wrong model, then, even if liquid vanilla options prices are forced to be correct, the hedge ratio is wrong. A bad hedge causes a distribution in the P&L as we saw in our simulations.

.We write the market price of an option in terms of BS as follows:

$$C_{\mathrm{mkt}}(S,t,K,T) \equiv C_{\mathrm{BSM}}(S,t,K,T,\Sigma), \qquad \Sigma = \Sigma(S,t,K,T)$$

Estimate the hedge ratio using the chain rule,

$$\Delta = \frac{\partial C_{\mathrm{mkt}}(S,t,K,T)}{\partial S} = \frac{\partial C_{\mathrm{BSM}}}{\partial S} + \frac{\partial C_{\mathrm{BSM}}}{\partial \Sigma}\frac{\partial \Sigma}{\partial S} = \Delta_{\mathrm{BSM}} + \cancel{\frac{\partial C_{\mathrm{BSM}}}{\partial \Sigma}\frac{\partial \Sigma}{\partial S}}$$

At the money, vega for the S&P 500 index assuming S ~4000 and $\boldsymbol{T} = \mathbf{1}$ **year**, $\sigma = 0.2$, is given by

$$\frac{\partial C_{BSM}}{\partial \Sigma} = \frac{S\sqrt{\tau}}{\sqrt{2\pi}} \approx 1600$$

If $K\frac{\partial \Sigma}{\partial K} \sim \frac{2}{5} = 0.4$ vol point per moneyness point, let's **assume** on dimensional grounds roughly

the same for the dependence of volatility on stock price: $S\frac{\partial \Sigma}{\partial S} \approx 0.4$, so $\frac{\partial \Sigma}{\partial S} \approx \frac{0.4}{4000} = 0.0001$

$\Delta - \Delta_{BSM} \approx 1600 \times 0.0001 = 0.16$. That's a big potential correction to the hedge ratio and causes a large fluctuation in the daily P&L from the mismatch in Delta.

9.4.2 Errors in the Valuation of Exotic Options

Even if we know the price of vanillas, finding the value of an exotic options needs a model.

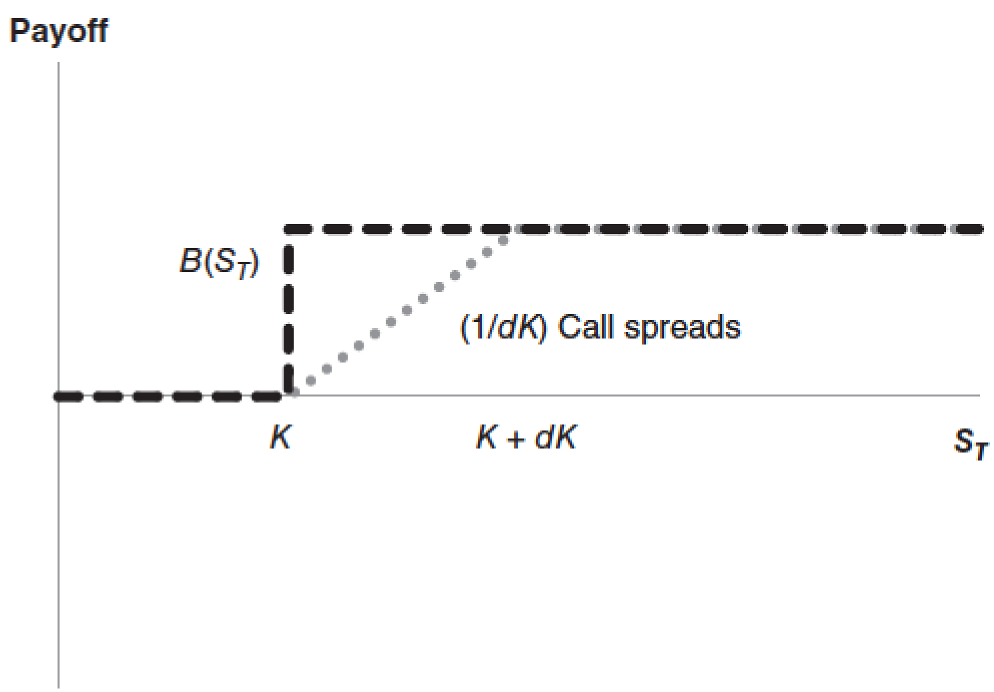
European-style pseudo-exotic digital option D which pays \$1 if $S \geq K$ at time T, and 0 otherwise:

This serves as insurance against a fixed loss above the strike K, but not against a proportional loss as in the case of a vanilla call.

It is very hard to hedge this because the payoff oscillates between 0 and 1.

Approximately replicate D with $1/(dK)$ call spreads with strikes separated by dK.

In the limit as $dK \to 0$ the call spread's payoff converges to that of the exotic option.



The value of the digital option D is approximately given by

$$D \approx \frac{C_{\mathrm{BSM}}(S, K, \Sigma(K)) - C_{\mathrm{BSM}}(S, K + dK, \Sigma(K + dK))}{dK}$$

$$\begin{aligned} D &= \lim_{dK \to 0} \frac{C_{\mathrm{BSM}}(S, K, \Sigma(K)) - C_{\mathrm{BSM}}\left(S, K + dK, \Sigma\left(K + dK\right)\right)}{dK} \\ &= -\frac{dC_{\mathrm{BSM}}(S, K, \Sigma(K))}{dK} \end{aligned}$$

The total derivative with respect to K includes the change of all variables with K, including that of the implied volatility.

Chain rule:

$$D = -\frac{\partial C_{\mathrm{BSM}}}{\partial K} - \frac{\partial C_{\mathrm{BSM}}}{\partial \Sigma}\frac{\partial \Sigma}{\partial K}$$

This is model-independent: a static hedge. If we know $\Sigma(K)$ we know the value in terms of the slope of the smile.

For $r = 0$, $\Sigma = 20\%$, $T - t = 1$ year, $K = S = 2000$, and a skew slope given by

$$\left.\frac{\partial \Sigma}{\partial K}\right|_{K=2,000} = -0.0001$$

At the money,

$$\begin{aligned}\frac{\partial C_{\mathrm{BSM}}}{\partial K} &= -N(d_2) \\ &= -N\left(-\frac{\Sigma}{2}\right) \\ &\approx -\left(0.5 - \frac{1}{\sqrt{2\pi}}\frac{\Sigma}{2}\right) \\ &\approx -0.46\end{aligned}$$

$$\frac{\partial C_{BS}}{\partial \Sigma} \sim \frac{S\sqrt{T}}{\sqrt{2\pi}} \sim 800$$

The replicated value of the digital is

$$\begin{aligned}D &\approx 0.46 + 800(0.0001) \\ &\approx 0.46 + 0.08 \\ &\approx 0.54\end{aligned}$$

The non-zero slope of the skew adds about 17% to the value of the option. This is a significant difference.

Why does the negative skew *add* to the value of the derivative D?

How can we extend Black-Scholes to match the skew and allow us to calculate all these quantities correctly? What changes can we make? Or, how, as we did in the above example, can we tread carefully and so avoid our lack of knowledge about the right model and still get reasonable estimates of value by replication? Those are the questions we will tackle later.

New Topic: Static Replication and Implied Distributions

Valuing European Options in the Presence of a Skew, Exactly, Without a Model

9.5 Static Replication and Implied Distributions

The Black-Scholes formula calculates options prices as the expected risklessly discounted value of the risky payoff over *a lognormal stock distribution in a risk-neutral world*, and corresponds to a flat smile.

But in reality there is a non-flat smile. That poses the inverse question: for a fixed expiration, what stock distribution (the so-called ***option implied distribution***) matches the observed smile when options prices are computed as expected risk-neutrally discounted payoffs?

Discrete States: State security with price π_i that pays $1 only when the stock is in state i with price S_i at time T, zero otherwise. Suppose you know the current market price π_i for each of these securities. Assume a frequentist view of the world in which we can imagine all possibilities and their probabilities remain stable.

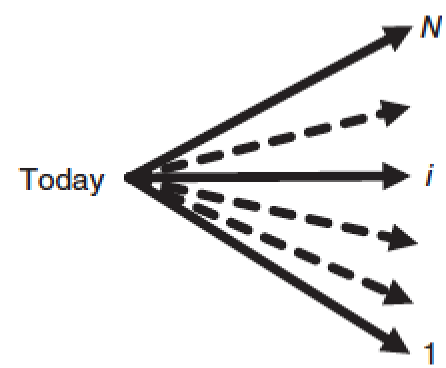


FIGURE 11.1 A World with *N* Possible Future States

Sum of all π_i is a riskless bond because it pays off $1 in every future state:

$$\sum_{1}^{N} \pi_i = \exp[-r\tau] \equiv \frac{1}{R}$$

Define Pseudo-probabilities $p_i \equiv R\pi_i$ and we can write $\pi_i = \frac{p_i}{R}$, $\qquad \sum p_i = 1$

Complete Market Constraint

If there is one state-contingent security with price π_i for payoff of \$1 in each state *i* at time *T*, then these securities provide a complete basis that span the space of future payoffs, and the market is "complete".

For any European payoff $V(i, T)$ dollars at time *T*, replication tells us that

$$\begin{aligned} V(t) &= \sum_{i=1}^{N} \pi_i V(i, T) \\ &= \sum_{i=1}^{N} p_i e^{-r\tau} V(i, T) \\ &= e^{-r\tau} \sum_{i=1}^{N} p_i V(i, T) \end{aligned}$$

Actual probabilities are never known but it's convenient to think in terms of *pseudo*-probabilities.

In continuous-state notation

$$V(S,t) = e^{-r\tau}\int_0^{\infty} p(S,t,S_T,T)V(S_T,T)\,dS_T$$

Here $p(S, t, S_T, T)$ is the risk-neutral (pseudo-) probability density for S_T at time T.

Define $$\pi(S, t, S_T, T) = e^{-r\tau}p(S, t, S_T, T)$$

$\pi(S, t, S_T, T)dS_T$ is the price at time t of a state-contingent security that pays \$1 if the stock price at time T lies between S_T and $S_T + dS_T$:

$$\int_0^{\infty}\pi(S, t, S_T, T)dS_T = e^{-r\tau} \text{ and } \int_0^{\infty}p(S, t, S_T, T)dS_T = 1$$

Knowing prices or pseudo-probabilities $p(S, t, S_T, T)$ at time *t* when stock price is *S* at time *t* determines the value of all European-style options with payoffs only at time *T*.

How can we find $p(S, t, S_T, T)$? From all standard call and put prices:

In particular for a standard call option C with strike *K*,

$$C(S_T, T) = [S_T - K]_+ = max(S_T - K, 0) = [S_T - K]\theta(S_T - K)$$

where $\theta(x)$ is the Heaviside/indicator function, equal to 1 when *x* is greater than 0, and 0 otherwise.

$$\begin{aligned} C(S, t, K, T) &= e^{-r\tau}\int_K^{\infty} p(S, t, S_T, T)(S_T - K)(dS_T) \\ &= e^{-r\tau}\int_0^{\infty} dS_T(S_T - K)\theta(S_T - K)p(S, t, S_T, T) \end{aligned} \qquad \text{Eq 9.1}$$

We'll see that a knowledge of call prices (or put prices) for all strikes K at expiration time T are enough to determine the density $p(S, t, S_T, T)$ for all S_T.

Then we'll show that one can statically replicate any known European-style payoff at time T through a combination of zero-coupon bonds, forwards, calls and puts of all strikes.

Note well. The risk-neutral distribution $p(S, t, S_T, T)$ **is insufficient for valuing all options** on the underlier. The risk-neutral distribution at expiration tells you nothing about the evolution of the stock price on its way to expiration. Hence, implied distributions are not useful in determining dynamic hedges.

9.6 Some Math: The Heaviside and Dirac Delta functions

The Heaviside function: $\theta(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$

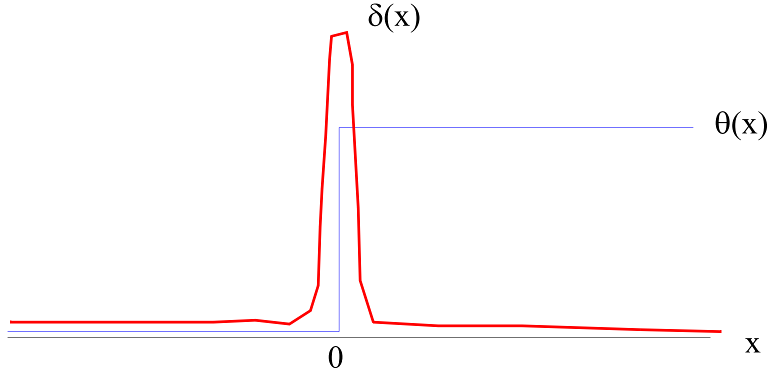
The derivative of the Heaviside function is the Dirac *delta function*: $\frac{\partial}{\partial x}\theta(x) = \delta(x)$

$\delta(x)$ is a distribution, a very singular function that only makes sense when used within an integral.
$\delta(x)$ is zero everywhere except at $x = 0$, where its value is infinite. Its integral over all x is 1.

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

$$x\delta(x) = 0 \qquad \forall x$$



9.7 The Breeden-Litzenberger Formula: Finding the risk-neutral probability density from call prices:

$$\exp(r\tau) \times C(S, t, K, T) \equiv \int_K^\infty dS_T(S_T - K)p(S, t, S_T, T)$$

$$\equiv \int_0^\infty dS_T(S_T - K)\theta(S_T - K)p(S, t, S_T, T)$$

$$\frac{\partial C(S,t,K,T)}{\partial K} = -e^{-r\tau} \int_K^\infty p(S,t,S_T,T)\, dS_T$$

$$\frac{\partial^2 C(S,t,K,T)}{\partial K^2} = e^{-r\tau} p(S,t,K,T) \qquad p(S,t,K,T) = e^{r\tau} \frac{\partial^2 C(S,t,K,T)}{\partial K^2}$$

Breeden
Litzenberger
Formula

The second derivative with respect to K of call prices is the risk-neutral probability distribution.

Exactly the same formula holds for puts. Note the dual role of K in the LHS and RHS:.

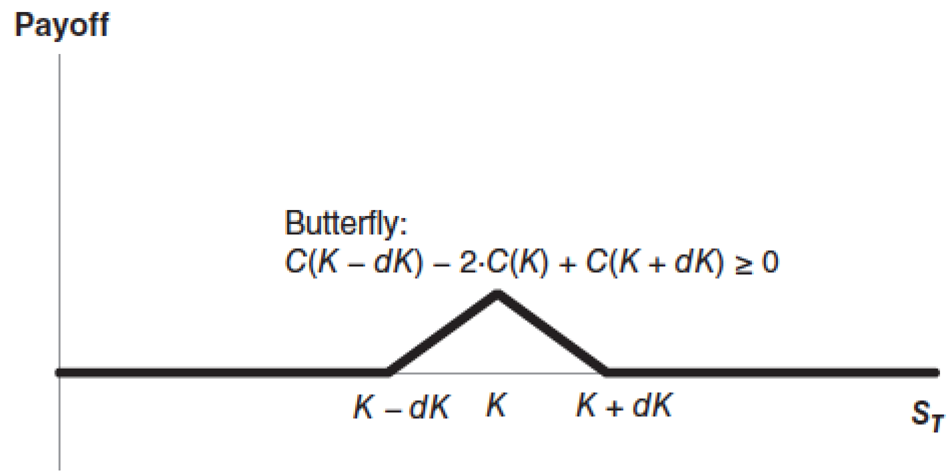
$$p(S,t,K,T) = e^{r\tau} \frac{\partial^2 C(S,t,K,T)}{\partial K^2}$$

stock price strike

Intuition: The Butterfly Spread is a State Contingent Security

An infinitesimal butterfly spread is

$$d^2C_K = C_{K+dK} - 2C_K + C_{K-dK} = (C_{K+dK} - C_K) - (C_K - C_{K-dK})$$



The maximum payoff is dK. By owning $1/(dK)^2$ spreads, i.e. d^2C_K/dK^2, we obtain a portfolio with height $1/dK$ and width $2dK$, with area 1, like a Dirac delta function. It behaves like a state-contingent security.

Consistency of the Probability Interpretation

Note that at any time *t:*

$$\int_0^\infty p(S, t, K, T)dK = e^{r\tau}\int_0^\infty \frac{\partial^2 C}{\partial K^2}dK = e^{r\tau}\left[\left.\frac{\partial C}{\partial K}\right|_\infty - \left.\frac{\partial C}{\partial K}\right|_0\right] \equiv 1$$

- $\left.\frac{\partial C}{\partial K}\right|_\infty = 0$ as the strike gets very large and calls become worthless; and

- for $K \to 0$ the call becomes a forward with value $S - Ke^{-r\tau}$, so that $\left.\frac{\partial C}{\partial K}\right|_0 = -e^{-r\tau}$.