Solutions Assignment 3

Part 1

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8.8.4

$$
\begin{aligned}
f(x \mid \sigma) &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}, \\
\lambda(x \mid \sigma) &= -\log\sigma - \frac{x^2}{2\sigma^2} + const. \\
\lambda'(x \mid \sigma) &= -\frac{1}{\sigma} + \frac{x^2}{\sigma^3}, \\
\lambda''(x \mid \sigma) &= \frac{1}{\sigma^2} - \frac{3x^2}{\sigma^4}.
\end{aligned}
$$

Therefore,

$$
I(\theta) = -E_\theta[\lambda''(X \mid \theta)] = -\frac{1}{\sigma^2} + \frac{3E(X^2)}{\sigma^4} = -\frac{1}{\sigma^2} + \frac{3}{\sigma^2} = \frac{2}{\sigma^2}.
$$

Alternative:

$$
I(\theta) = Var(\lambda'(X,\sigma)) = Var\left(-\frac{1}{\sigma} + \frac{X^2}{\sigma^3}\right) = Var\left(\frac{1}{\sigma^3}X^2\right) = \frac{1}{\sigma^6}Var(X^2)
$$

as seen in class $= \frac{1}{\sigma^6} \times 2\sigma^4 = \frac{2}{\sigma^2}$.

8.8.14

NOT GRADED

$$
\begin{aligned}
f(x \mid \alpha) &= \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\exp(-\beta x), \\
\lambda(x \mid \alpha) &= \alpha\log\beta - \log\Gamma(\alpha) + (\alpha-1)\log x - \beta x, \\
\lambda'(x \mid \alpha) &= \log\beta - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \log x, \\
\lambda''(x \mid \alpha) &= -\frac{\Gamma(\alpha)\Gamma''(\alpha) - [\Gamma'(\alpha)]^2}{[\Gamma(\alpha)]^2}
\end{aligned}
$$

Therefore,

$$
I(\alpha) = \frac{\Gamma(\alpha)\Gamma''(\alpha) - [\Gamma'(\alpha)]^2}{[\Gamma(\alpha)]^2}
$$

The distribution of the M.L.E. of α will be approximately the normal distribution with mean α and variance $1/[nI(\alpha)]$.
It should be noted that we have determined this distribution without actually determining the M.L.E. itself.

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8.9.15

In the notation of Sec. 8.8,

$$\begin{aligned}
\lambda(x \mid \theta) &= \log\theta + (\theta - 1)\log x, \\
\lambda'(x \mid \theta) &= \frac{1}{\theta} + \log x, \\
\lambda''(x \mid \theta) &= -1/\theta^2.
\end{aligned}$$

Hence, by Eq. (8.8.3), $I(\theta) = 1/\theta^2$ and it follows that the asymptotic distribution of $\hat{\theta}_n$ is

$$\cancel{\frac{n^{1/2}}{\theta}(\hat{\theta}_n - \theta)} \qquad \hat{\theta}_n \overset{\text{approx.}}{\sim} N\left(\theta, \frac{1}{nI(\theta)}\right) = \boxed{N\left(\theta, \frac{\theta^2}{n}\right)}$$

~~is standard normal.~~

NOT GRADED

Exercise I

Note that if $X_i \overset{iid}{\sim} \text{Poisson}(\theta)$, then $\mathbb{E}[X_i] = \text{Var}(X_i) = \theta$

$$\overset{CLT}{\Longrightarrow} \sqrt{n}(\bar{X}_n - \theta) \overset{\text{approx.}}{\sim} N(0,\theta) \quad \text{or} \quad \frac{\bar{X}_n - \theta}{\sqrt{\theta/n}} \overset{\text{approx.}}{\sim} N(0,1)$$

Thus, for $\alpha \in (0,1)$,

$$\mathbb{P}\left(\underbrace{q\left(N(0,1), \frac{\alpha}{2}\right)}_{= -q(N(0,1), 1-\frac{\alpha}{2})} \le \frac{\bar{X}_n - \theta}{\sqrt{\theta/n}} \le q\left(N(0,1), 1 - \frac{\alpha}{2}\right)\right) \approx 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha$$

I write $q = q(N(0,1), 1 - \frac{\alpha}{2})$ in the following. Then

$$\begin{aligned}
-q \le \frac{\bar{X}_n - \theta}{\sqrt{\theta/n}} \le q &\iff -q\sqrt{\theta/n} \le \bar{X}_n - \theta \le q \iff (\bar{X}_n - \theta)^2 \le q^2\frac{\theta}{n} \\
&\iff \bar{X}_n^2 - 2\theta\bar{X}_n + \theta^2 - q^2\frac{\theta}{n} \le 0 \\
&\iff \theta^2 - \left(2\bar{X}_n + \frac{q^2}{n}\right)\theta + \bar{X}_n^2 \le 0 \\
&\iff \theta \in \frac{2\bar{X}_n + \frac{q^2}{n} \pm \sqrt{(2\bar{X}_n + \frac{q^2}{n})^2 - 4\bar{X}_n^2}}{2} \\
&= \bar{X}_n + \frac{q^2}{2n} \pm \sqrt{\frac{q^2\bar{X}_n}{n} + \frac{q^4}{4n^2}} = \bar{X}_n + \frac{q^2}{2n} \pm \frac{q}{\sqrt{n}}\sqrt{\bar{X}_n + \frac{q^2}{4n}}
\end{aligned}$$

For $\alpha = 0.05$, $q(N(0,1), 1-\frac{\alpha}{2}) = q(N(0,1), 0.975) = 1.96$

$\Longrightarrow$ 95%-CI: $(3.11, 3.31)$

NOT GRADED

Exercise II

Take: $\hat{V}_n := \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$

Then $\mathbb{E}[\hat{V}_n] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[(X_i-\mu)^2] = \frac{1}{n} n \, Var(X_1) = V$

$\Longrightarrow$ $\hat{V}_n$ is unbiased

$Var(\hat{V}_n) = \frac{1}{n^2}\sum_{i=1}^{n} Var(\underbrace{(X_i-\mu)}_{\sim N(0,V)}{}^2) \overset{\text{as seen in class}}{=} \frac{1}{n^2}\sum_{i=1}^{n} 2V^2 = \frac{2V^2}{n}$

Fisher information:

$f_n(X_n; V) = (2\pi V)^{-\frac{n}{2}} e^{-\frac{1}{2V}\sum_{i=1}^{n}(X_i-\mu)^2}$; $\ell_n(X_n; V) = -\frac{n}{2}\log(2\pi V) - \frac{1}{2V}\sum_{i=1}^{n}(X_i-\mu)^2$

$\dot{\ell}_n(X_n; V) = -\frac{n}{2V} + \frac{1}{2V^2}\sum_{i=1}^{n}(X_i-\mu)^2$

$I_n(\theta) = Var(\dot{\ell}_n(X_n; V)) = \frac{1}{4V^4}\sum_{i=1}^{n} Var((X_i-\mu)^2) \overset{\text{as above}}{=} \frac{n}{4V^4} \times 2V^2 = \frac{n}{2V^2}$

Cramér-Rao bound: Every unbiased estimator T of V has variance $Var(T) \geq \frac{1}{I_n(\theta)} = \frac{2V^2}{n}$. Thus, $\hat{V}_n$ is an MVUE of V.

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Exercise III

To find a candidate, we compute the MLE:

$L_n(\theta) = \prod_{i=1}^{n} f(X_i; \theta) = (2\theta^3)^{-n}\prod_{i=1}^{n} X_i^2 \, e^{-\frac{1}{\theta}\sum X_i}$

$\log L_n(\theta) = -n\log(2\theta^3) + 2\sum_{i=1}^{n}\log X_i - \frac{1}{\theta}\sum X_i$

$\frac{d}{d\theta}\log L_n(\theta) = -\frac{n}{\theta^3}\times 3\theta^2 + \frac{1}{\theta^2}\sum X_i = -\frac{3n}{\theta} + \frac{1}{\theta^2}\sum X_i = 0$

$\Longleftrightarrow$ $\hat{\theta}_n = \frac{1}{3n}\sum X_i = \frac{1}{3}\overline{X}_n$ ✓✓✓

For this exercise, the students don't have to justify how they found $\hat{\theta}_n$.

$\frac{d^2}{d\theta^2}\log L_n(\theta) = \frac{3n}{\theta^2} - \frac{2}{\theta^3}\sum X_i \overset{\theta=\frac{1}{3}\overline{X}_n}{=} \frac{27n}{\overline{X}_n^2} - \frac{2\times 27}{\overline{X}_n^3}\overline{X}_n = -\frac{27}{\overline{X}_n^2} < 0$

$\Longrightarrow$ $\hat{\theta}_n$ is an MLE.

$\mathbb{E}[\hat{\theta}_n] = \frac{1}{3n}\sum_{i=1}^{n}\mathbb{E}[X_i] = \frac{1}{3}\mathbb{E}[X_1] = \frac{1}{3}\times 3\theta = \theta \Longrightarrow \hat{\theta}_n$ is unbiased ✓

$Var(\hat{\theta}_n) = \frac{1}{9n^2}\sum_{i=1}^{n} Var(X_i) = \frac{1}{9n}(12\theta^2 - 9\theta^2) = \frac{\theta^2}{3n}$. ✓

$I_n(\theta) = Var(\dot{\ell}_n(X_n; \theta)) = Var(\frac{d}{d\theta}\log L_n(\theta; X_n)) = Var(-\frac{3n}{\theta} + \frac{1}{\theta^2}\sum X_i) = \frac{1}{\theta^4}\sum Var(X_i)$

$= \frac{n}{\theta^4}\times 3\theta^2 = \frac{3n}{\theta^2}$. ✓

Cramér-Rao bound: Every unbiased estimator T of θ has variance

$Var(T) \geq \frac{1}{I_n(\theta)} = \frac{\theta^2}{3n}$. ✓ Thus, $\hat{\theta}_n$ is an MVUE of θ. ✓

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Exercise IV

MLE of v is $\hat{v}_n := \frac{1}{n}\sum_{i=1}^{n} X_i^2$ (we have seen this before) ✓

$\Longrightarrow$ Invariance principle: MLE of $\log v$: $T(\underline{X}_n) := \log \hat{v}_n$ ✓

By the CLT,

$$\sqrt{n}\,(\hat{v}_n - v) \overset{\text{approx.}}{\sim} N(0, Var(X_1^2)) = N(0, 2v^2)$$ ✓

[Some students may get this result by computing $I(\theta)$ and then noting $\sqrt{n}(\hat{v}_n - v) \overset{\text{approx}}{\sim} N(0, \frac{1}{I(\theta)})$]

δ-method $\Longrightarrow$

$g(x) = \log x$ ✓

$g'(x) = \frac{1}{x}$

$$\underbrace{\sqrt{n}\,\frac{\log \hat{v}_n - \log v}{g'(v)}}_{= \sqrt{n}\, v\,(\log \hat{v}_n - \log v)} \overset{\text{approx.}}{\sim} N(0, 2v^2)$$ ✓

$$\Longrightarrow \quad \log \hat{v}_n \overset{\text{approx}}{\sim} N\left(\log v, \frac{2v^2}{nv^2}\right) = N\left(\log v, \frac{2}{n}\right)$$ ✓ ✓

$\Sigma = 27$