Lecture 10:

Problems Caused by the Skew; Static Replication.

10.1 Problems Caused by the Smile for Trading

- You can regard liquid standard call and put options prices as being simply *quoted* via the Black-Scholes formula, *so the model doesn't really matter for pricing*. (Analogy: bond prices are quoted by a single yield to maturity even though you may have calculated the PV of the future payments via a different method.)

- The model does matter if you want to generate your own idea of fair standard options values and then arbitrage them against market prices, but that is a very risky long-term business. Deciding which standard options are too cheap or too rich is a buy-side view.

- The model does matter for calculating hedge ratios, for market makers.

- The model matters for pricing illiquid OTC exotic options, for market makers or buy side.

- The question in both of these cases is of course: **which** model?

We will therefore now try first **to estimate** the effect that the skew has on hedging and valuation, **without using a particular model**.

10.1.1 Fluctuations in the P&L from incorrect hedging of standard options

If we have the wrong model, then, even if liquid vanilla options prices are forced to be correct, the hedge ratio is wrong. A bad hedge causes a distribution in the P&L as we saw in our simulations.

.We write the market price of an option in terms of BS as follows:

$$C_{\text{mkt}}(S, t, K, T) \equiv C_{\text{BSM}}(S, t, K, T, \Sigma), \qquad \Sigma = \Sigma(S, t, K, T)$$

Estimate the hedge ratio using the chain rule,

$$\Delta = \frac{\partial C_{\text{mkt}}(S, t, K, T)}{\partial S} = \frac{\partial C_{\text{BSM}}}{\partial S} + \frac{\partial C_{\text{BSM}}}{\partial \Sigma}\frac{\partial \Sigma}{\partial S} = \Delta_{\text{BSM}} + \cancel{\frac{\partial C_{\text{BSM}}}{\partial \Sigma}\frac{\partial \Sigma}{\partial S}}$$

At the money, vega for the S&P 500 index assuming S ~4000 and $\boldsymbol{T = 1}$ **year**, $\sigma = 0.2$, is given by

$$\frac{\partial C_{BSM}}{\partial \Sigma} = \frac{S\sqrt{\tau}}{\sqrt{2\pi}} \approx 1600$$

If $K\frac{\partial \Sigma}{\partial K} \sim \frac{2}{5} = 0.4$ vol point per moneyness point, let's **assume** on dimensional grounds roughly

the same for the dependence of volatility on stock price: $S\frac{\partial \Sigma}{\partial S} \approx 0.4$, so $\frac{\partial \Sigma}{\partial S} \approx \frac{0.4}{4000} = 0.0001$

$\Delta - \Delta_{BSM} \approx 1600 \times 0.0001 = 0.16$. That's a big potential correction to the hedge ratio and causes a large fluctuation in the daily P&L from the mismatch in Delta.

10.1.2 Errors in the Valuation of Exotic Options

Even if we know the price of vanillas, finding the value of an exotic options needs a model.

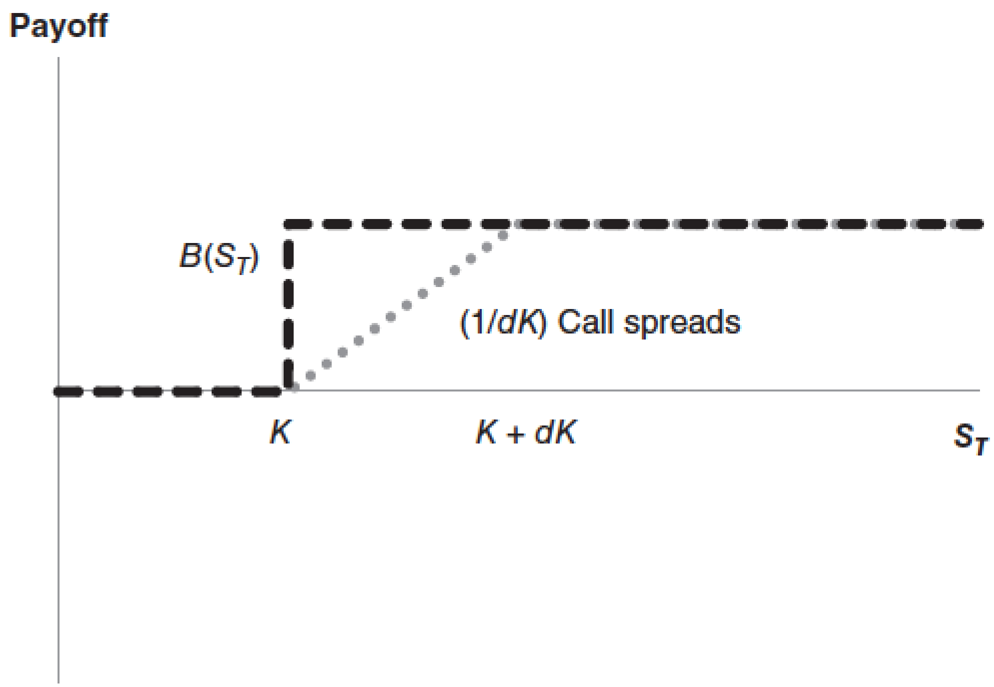
European-style pseudo-exotic digital option D which pays \$1 if $S \geq K$ at time T, and 0 otherwise:

This serves as insurance against a fixed loss above the strike K, but not against a proportional loss as in the case of a vanilla call.

It is very hard to hedge this because the payoff oscillates between 0 and 1.

Approximately replicate D with $1/(dK)$ call spreads with strikes separated by dK.

In the limit as $dK \to 0$ the call spread's payoff converges to that of the exotic option.



The value of the digital option D is approximately given by

$$D \approx \frac{C_{\mathrm{BSM}}(S, K, \Sigma(K)) - C_{\mathrm{BSM}}(S, K + dK, \Sigma(K + dK))}{dK}$$

$$\begin{aligned} D &= \lim_{dK \to 0} \frac{C_{\mathrm{BSM}}(S, K, \Sigma(K)) - C_{\mathrm{BSM}}\left(S, K + dK, \Sigma\left(K + dK\right)\right)}{dK} \\ &= -\frac{dC_{\mathrm{BSM}}(S, K, \Sigma(K))}{dK} \end{aligned}$$

The **total derivative** with respect to K includes the change of all variables with K, including that of the implied volatility.

Chain rule:

$$D = -\frac{\partial C_{\mathrm{BSM}}}{\partial K} - \frac{\partial C_{\mathrm{BSM}}}{\partial \Sigma} \frac{\partial \Sigma}{\partial K}$$

This is model-independent: a static hedge. All values are observable. If we know $\Sigma(K)$ we know the value in terms of the slope of the smile.

For $r = 0$, $\Sigma = 20\%$, $T - t = 1$ year, $K = S = 2000$, and a skew slope given by

$$\left.\frac{\partial \Sigma}{\partial K}\right|_{K=2,000} = -0.0001$$

At the money,

$$\begin{aligned}\frac{\partial C_{\mathrm{BSM}}}{\partial K} &= -N(d_2) \\ &= -N\left(-\frac{\Sigma}{2}\right) \\ &\approx -\left(0.5 - \frac{1}{\sqrt{2\pi}}\frac{\Sigma}{2}\right) \\ &\approx -0.46\end{aligned}$$

$$\frac{\partial C_{BS}}{\partial \Sigma} \sim \frac{S\sqrt{T}}{\sqrt{2\pi}} \sim 800$$

The replicated value of the digital is

$$\begin{aligned}D &\approx 0.46 + 800(0.0001) \\ &\approx 0.46 + 0.08 \\ &\approx 0.54\end{aligned}$$

The non-zero slope of the skew adds about 17% to the value of the option. This is a significant difference.

Why does the negative skew *add* to the value of the derivative D?

How can we extend Black-Scholes to match the skew and allow us to calculate all these quantities correctly? What changes can we make? Or, how, as we did in the above example, can we tread carefully and so avoid our lack of knowledge about the right model and still get reasonable estimates of value by replication? Those are the questions we will tackle later.

NEW TOPIC: STATIC REPLICATION AND IMPLIED DISTRIBUTIONS

VALUING EUROPEAN OPTIONS IN THE PRESENCE OF A SKEW, EXACTLY, WITHOUT A MODEL

10.2 Static Replication and Implied Distributions

The Black-Scholes formula calculates options prices as the expected risklessly discounted value of the risky payoff over *a lognormal stock distribution in a risk-neutral world*, and corresponds to a flat smile.

But in reality there is a non-flat smile. That poses the inverse question: for a fixed expiration, what stock distribution (the so-called ***option implied distribution***) matches the observed smile when options prices are computed as expected risk-neutrally discounted payoffs?

Discrete States: State security with price π_i that pays $1 only when the stock is in state *i* with price S_i at time *T*, zero otherwise. Suppose you know the current market price π_i for each of these securities. Assume a frequentist view of the world in which we can imagine all possibilities and their probabilities remain stable.

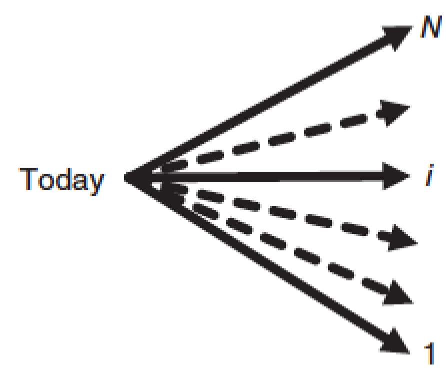


FIGURE 11.1 A World with *N* Possible Future States

Sum of all π_i is a riskless bond because it pays off $1 in every future state:

$$\sum_{1}^{N} \pi_i = \exp[-r\tau] \equiv \frac{1}{R}$$

Define Pseudo-probabilities $p_i \equiv R\pi_i$ and we can write $\pi_i = \frac{p_i}{R}$, $\sum p_i = 1$

Complete Market Constraint

If there is one state-contingent security with price π_i for payoff of \$1 in each state *i* at time *T,* then these securities provide a complete basis that span the space of future payoffs, and the market is "complete".

For any European payoff $V(i, T)$ dollars in each state *i* at time *T*, replication tells us that

$$\begin{aligned} V(t) &= \sum_{i=1}^{N} \pi_i V(i, T) \\ &= \sum_{i=1}^{N} p_i e^{-r\tau} V(i, T) \\ &= e^{-r\tau} \sum_{i=1}^{N} p_i V(i, T) \end{aligned}$$

Actual probabilities are never known but it's convenient to think in terms of *pseudo*-probabilities.

In continuous-state notation

$$V(S,t) = e^{-r\tau}\int_0^\infty p(S,t,S_T,T)V(S_T,T)\,dS_T$$

Here $p(S, t, S_T, T)$ is the risk-neutral (pseudo-) probability density for S_T at time T when the current stock price is S at time t.

Define $$\pi(S, t, S_T, T) = e^{-r\tau}p(S, t, S_T, T)$$

$\pi(S, t, S_T, T)dS_T$ is the price at time t of a state-contingent security that pays \$1 if the stock price at time T lies between S_T and $S_T + dS_T$:

$$\int_0^\infty \pi(S, t, S_T, T)dS_T = e^{-r\tau} \text{ and } \int_0^\infty p(S, t, S_T, T)dS_T = 1$$

Knowing prices or pseudo-probabilities $p(S, t, S_T, T)$ at time *t* when stock price is *S* at time *t* determines the value of all European-style options with payoffs only at time *T*.

How can we find $p(S, t, S_T, T)$? From all standard call and put prices:

In particular for a standard call option C with strike K,

$$C(S_T, T) = [S_T - K]_+ = max(S_T - K, 0) = [S_T - K]\theta(S_T - K)$$

where $\theta(x)$ is the Heaviside/indicator function, equal to 1 when x is greater than 0, and 0 otherwise.

$$\begin{aligned} C(S, t, K, T) &= e^{-r\tau}\int_K^{\infty} p(S, t, S_T, T)(S_T - K)(dS_T) \\ &= e^{-r\tau}\int_0^{\infty} dS_T (S_T - K)\theta(S_T - K)p(S, t, S_T, T) \end{aligned} \qquad \text{Eq 10.1}$$

We'll see that a knowledge of call prices (or put prices) for all strikes K at expiration time T are enough to determine the density $p(S, t, S_T, T)$ for all S_T.

Then we'll show that one can statically replicate any known European-style payoff at time T through a combination of zero-coupon bonds, forwards, calls and puts of all strikes.

Note well. The risk-neutral distribution $p(S, t, S_T, T)$ **is insufficient for valuing all options** on the underlier. The risk-neutral distribution at expiration tells you nothing about the evolution of the stock price on its way to expiration. Hence, implied distributions are not useful in determining dynamic hedges at any time.

10.3 Some Math: The Heaviside and Dirac Delta functions

The Heaviside function: $\theta(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$

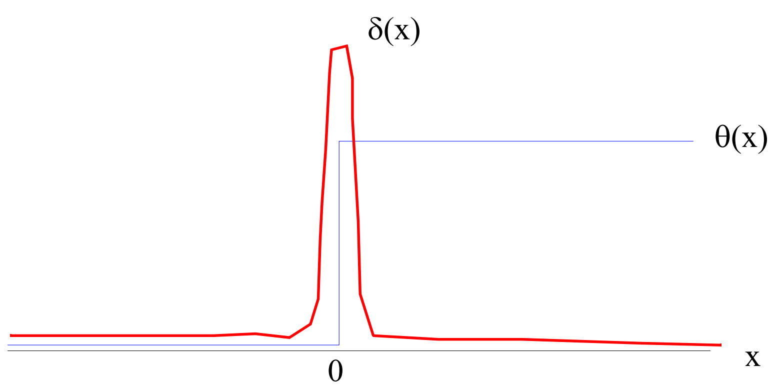
The derivative of the Heaviside function is the Dirac *delta function*: $\frac{\partial}{\partial x}\theta(x) = \delta(x)$

$\delta(x)$ is a distribution, a very singular function that only makes sense when used within an integral.
$\delta(x)$ is zero everywhere except at $x = 0$, where its value is infinite. Its integral over all x is 1.

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

$$x\delta(x) = 0 \qquad \forall x$$



10.4 The Breeden-Litzenberger Formula: Finding the risk-neutral probability density from call prices:

$$\exp(r\tau) \times C(S, t, K, T) \equiv \int_K^\infty dS_T (S_T - K) p(S, t, S_T, T)$$

$$\equiv \int_0^\infty dS_T (S_T - K) \theta(S_T - K) p(S, t, S_T, T)$$

$$\frac{\partial C(S,t,K,T)}{\partial K} = -e^{-r\tau} \int_K^\infty p(S,t,S_T,T)\, dS_T$$

$$\frac{\partial^2 C(S,t,K,T)}{\partial K^2} = e^{-r\tau} p(S,t,K,T) \qquad p(S,t,K,T) = e^{r\tau} \frac{\partial^2 C(S,t,K,T)}{\partial K^2}$$

Breeden
Litzenberger
Formula

The second derivative with respect to K of call prices is the risk-neutral probability distribution.

Exactly the same formula holds for puts. **Note the dual role** of K in the LHS and RHS:.

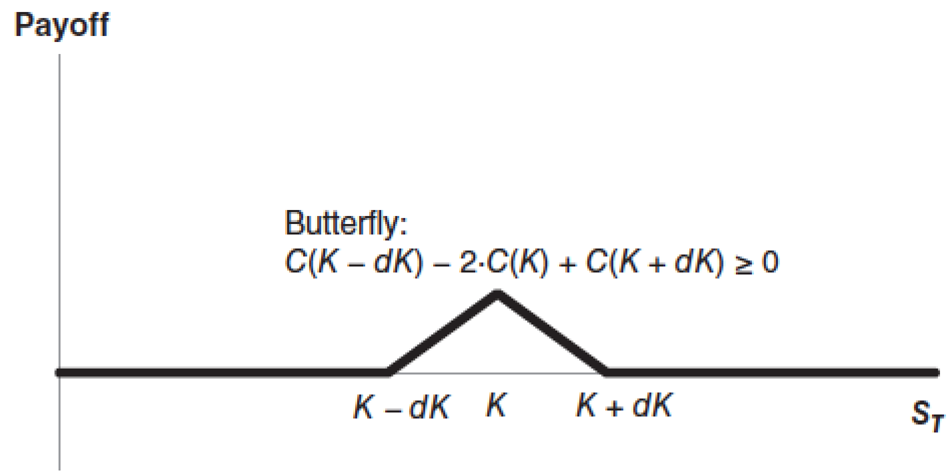
$$p(S,t,K,T) = e^{r\tau} \frac{\partial^2 C(S,t,K,T)}{\partial K^2}$$

stock price strike

Intuition: The Butterfly Spread is a State Contingent Security

An infinitesimal butterfly spread is

$$d^2C_K = C_{K+dK} - 2C_K + C_{K-dK} = (C_{K+dK} - C_K) - (C_K - C_{K-dK})$$



The maximum payoff is dK. By owning $1/(dK)^2$ spreads, i.e. d^2C_K/dK^2, we obtain a portfolio with height $1/dK$ and width $2dK$, with area 1, like a Dirac delta function. It behaves like a state-contingent security.

Consistency of the Probability Interpretation

Note that at any time *t:*

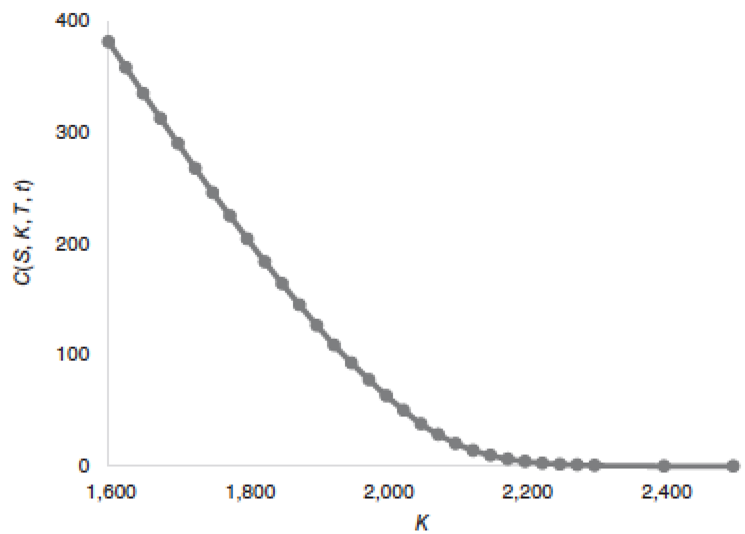
$$\int_0^{\infty} p(S, t, K, T)dK = e^{r\tau}\int_0^{\infty} \frac{\partial^2 C}{\partial K^2}dK = e^{r\tau}\left[\left.\frac{\partial C}{\partial K}\right|_{\infty} - \left.\frac{\partial C}{\partial K}\right|_0\right] \equiv 1$$

- $\left.\frac{\partial C}{\partial K}\right|_{\infty} = 0$ as the strike gets very large and calls become worthless; and

- for $K \to 0$ the call becomes a forward with value $S - Ke^{-r\tau}$, so that $\left.\frac{\partial C}{\partial K}\right|_0 = -e^{-r\tau}$.

Examining the Risk-Neutral Probability Distribution

S&P 500 six-month call prices on 9/10/2014, S ~ 2000.



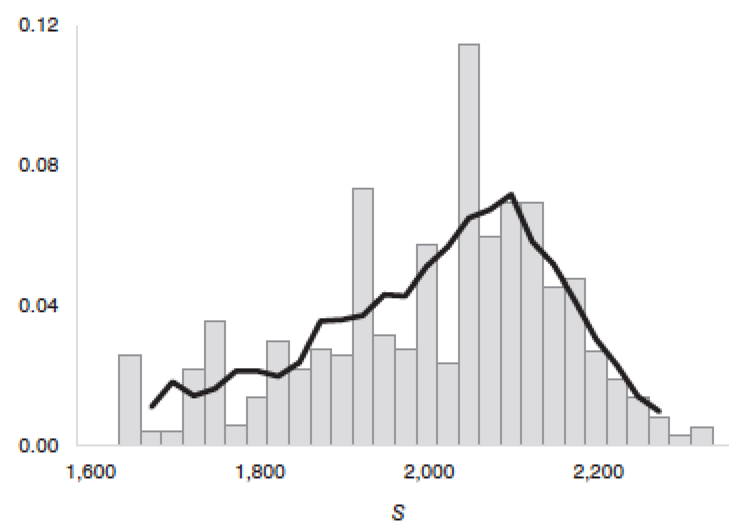
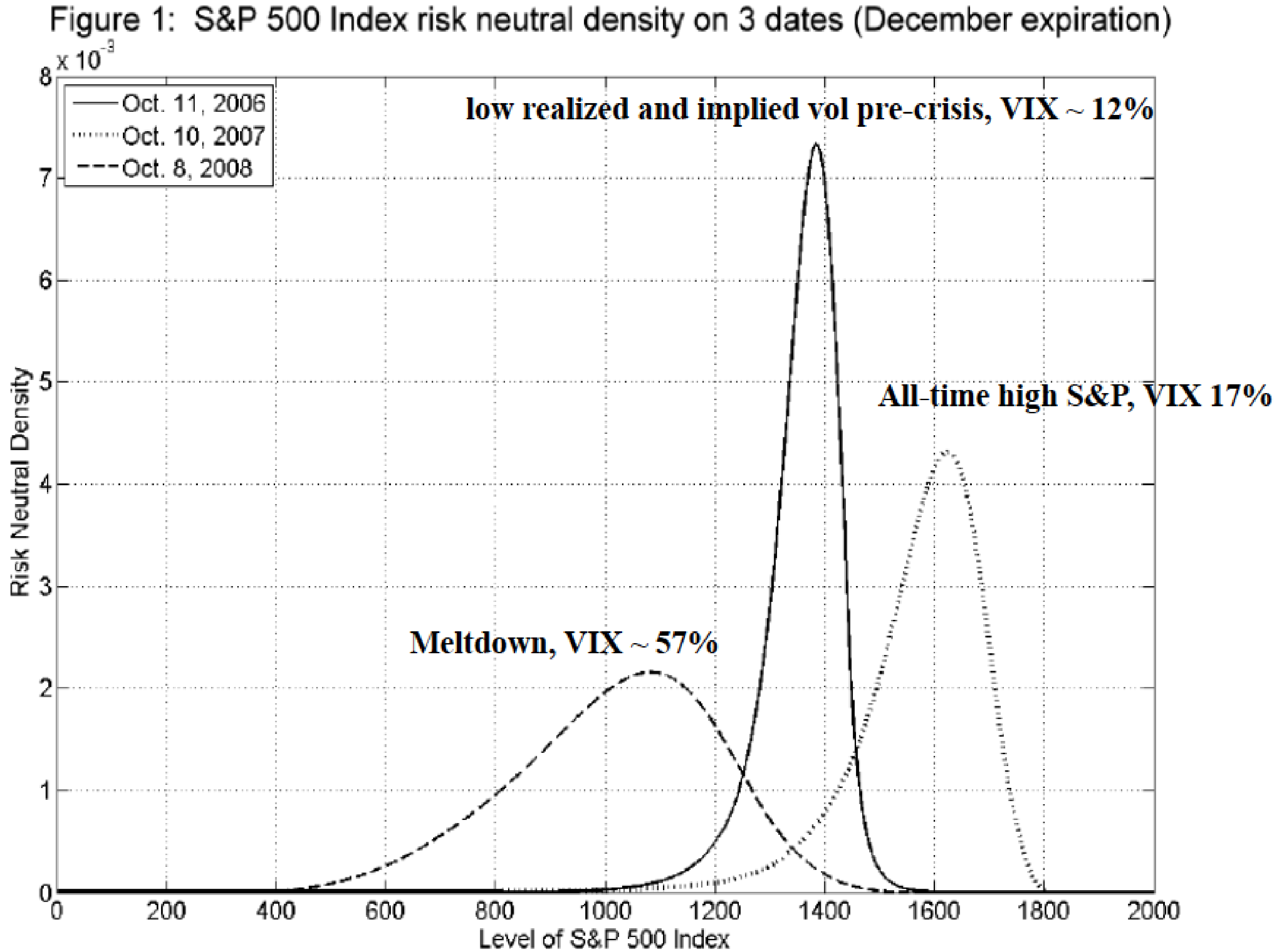


FIGURE 11.3 Risk-Neutral PDF from the Call Price Function

Jaggedness: Discrete prices, some may be stale. Interpolation.

During Great Financial Crisis: Birru and Figlewski (2012).

71 days to expiration



The shape is relevant. The mean is related to the riskless rate at that time.

10.5 Static Replication: Valuing Arbitrary European Payoffs at a Fixed Expiration Without Any Model

For any V(K,t):

$$V(S,t) = \int_0^{\infty} \frac{\partial^2 C(S,t,K,T)}{\partial K^2} V(K,T)\, dK \qquad V(S,t) = \int_0^{\infty} \frac{\partial^2 P(S,t,K,T)}{\partial K^2} V(K,T)\, dK$$

If we know call prices (or put prices) and their derivatives for all strikes at a fixed expiration, we can find the value of any other European-style derivative security at that expiration.

This **involves no use of option theory at all, and no use of the Black-Scholes equation**. It works even if there is a smile or skew or jumps. As long as the options' payoffs are honored.

Replicating by standard options

Integration by parts to get V as the sum of portfolios of zero coupon bonds, forwards, puts & calls.

European payoff $V(K, T)$. K represents **the terminal stock price**.
Use puts below strike A and calls above strike A.

$$V(S, t) = \int_0^A \frac{\partial^2}{\partial K^2} P(S, t, K, T) V(K, T) dK + \int_A^{\infty} \frac{\partial^2}{\partial K^2} C(S, t, K, T) V(K, T) dK$$

Integrate by parts twice to get

$$\begin{aligned}
V(S,t) &= \int_0^A \frac{\partial^2 P(S,t,K,T)}{\partial K^2} V(K,T)\,dK + \int_A^\infty \frac{\partial^2 C(S,t,K,T)}{\partial K^2} V(K,T)\,dK \\
&= \int_0^A \frac{\partial^2 V(K,T)}{\partial K^2} P(S,K)\,dK + \int_A^\infty \frac{\partial^2 V(K,T)}{\partial K^2} C(S,K)\,dK \\
&\quad + \left[V(K,T)\frac{\partial P(S,K)}{\partial K} - P(S,K)\frac{\partial V(K,T)}{\partial K} \right]_{K=0}^{K=A} \\
&\quad + \left[V(K,T)\frac{\partial C(S,K)}{\partial K} - C(S,K)\frac{\partial V(K,T)}{\partial K} \right]_{K=A}^{K=\infty}
\end{aligned}$$

where $P(S,K)$ and $C(S,K)$ are shorthand for the current values at time t and stock price S of a put and call with strike K and expiration T. Use the following conditions for the current call and put prices.

$$\begin{aligned}
&P(S,0) = 0 \\
&\frac{\partial P(S,0)}{\partial K} = 0 \\
&C(S,\infty) = 0 \\
&\frac{\partial C(S,\infty)}{dK} = 0 \\
&P(S,K) - C(S,K) = Ke^{-r(T-t)} - S \\
&\frac{\partial P(S,K)}{\partial K} - \frac{\partial C(S,K)}{\partial K} = e^{-r(T-t)}
\end{aligned}$$

with dividends: $P(S,K) - C(S,K) = Ke^{-r(T-t)} - Se^{-q(T-t)}$

Then

$$V(S,t) = V(A,T)e^{-r(T-t)} + \left.\frac{\partial V(K,T)}{\partial K}\right|_{K=A}(S - Ae^{-r(T-t)})$$

$$+ \int_0^A \frac{\partial^2 V(K,T)}{\partial K^2} P(S,K)\,dK + \int_A^\infty \frac{\partial^2 V(K,T)}{\partial K^2} C(S,K)\,dK \qquad \text{Eq 10.2}$$

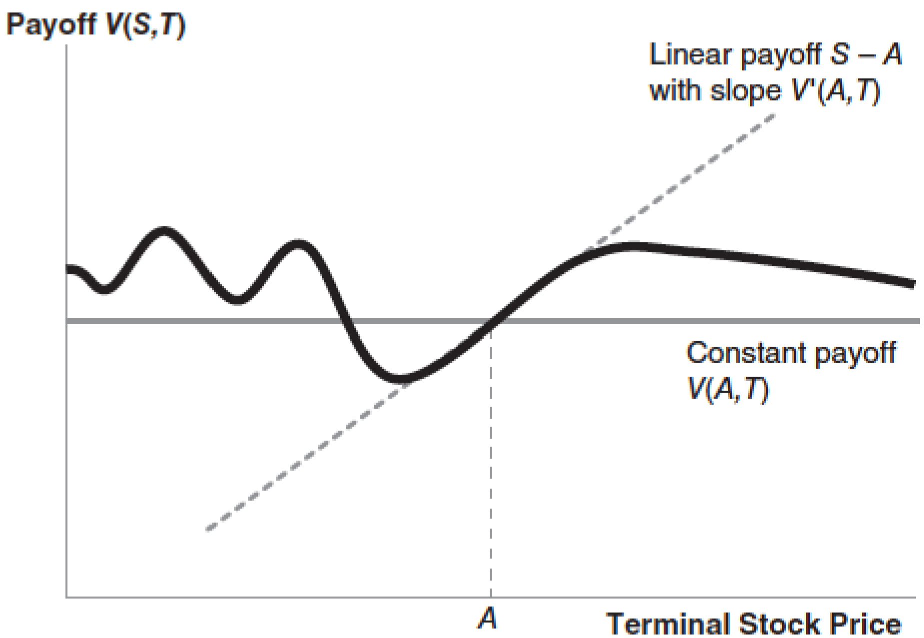


FIGURE 11.5 Replication of Exotic European Payoff

If we choose A to be the forward price $Se^{r(T-t)}$, the second term vanishes.

Two views of static replication.

- If you know the risk-neutral density ρ then you can write the value $V(S, t)$ today as a discounted integral over the terminal payoff $V(K, T)$ time the density ρ.

- Alternatively, if you know the derivatives up to $\frac{\partial^2}{\partial K^2} V(K, T)$ of the terminal payoff, you can write V(S,t) today as an integral over today's call and put prices with different strikes.

If you can buy every option in the continuum you need from someone who will never default on their payoff, then you have a perfect static replication. No math involved.

If you cannot buy every single option, then you have only an approximate replicating portfolio whose value will deviate from the value of the target option's payoff. Picking a reasonable or tolerable replicating portfolio is up to you.

This works even if there is a volatility skew.

Note: The Black-Scholes risk-neutral implied probability density corresponding to a flat skew

In the BS evolution, returns $\ln S_T/S_t$ are normal with a risk-neutral mean $r\tau - \frac{1}{2}\sigma^2\tau$ and a standard deviation $\sigma\sqrt{\tau}$, where $\tau = T - t$.

Therefore,

$$x = \frac{\ln S_T/S_t - (r\tau - \frac{1}{2}\sigma^2\tau)}{\sigma\sqrt{\tau}} \qquad \text{Eq.10.1}$$

is normally distributed with mean 0 and standard deviation 1, with a probability density $h(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$. The returns $\ln S_T/S_t$ can range from $-\infty$ to ∞. Differentiation Eq.10.1,

$$\frac{dS_T}{S_T} = \sigma\sqrt{\tau}dx$$

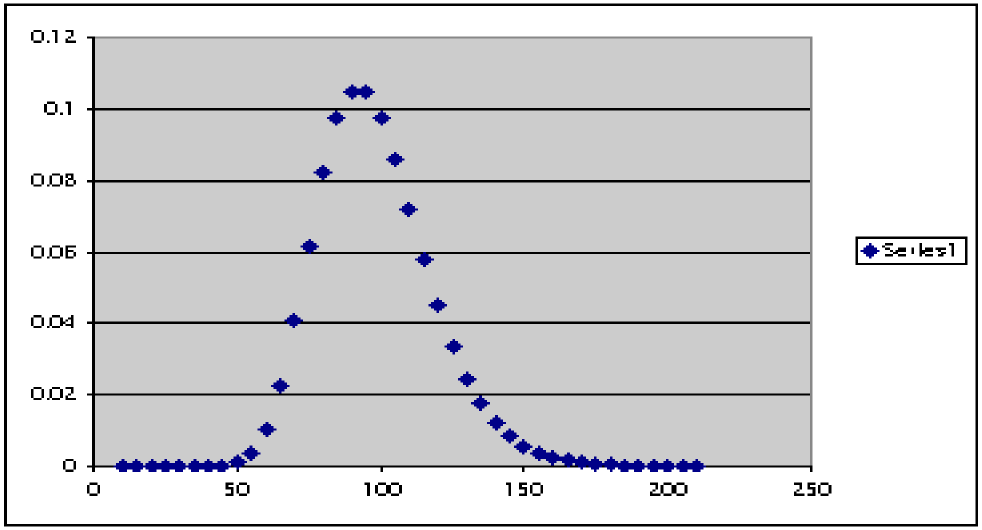
The risk-neutral value of the option is given by

$$e^{r\tau}C = \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty}(S_T - K)\exp\left(\frac{-x^2}{2}\right)dx = \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{K}^{\infty}(S_T - K)\exp\left(\frac{-x^2}{2}\right)\frac{dS_T}{S_T}$$

where

$$\frac{\exp\left(\frac{-x^2}{2}\right)}{\sqrt{2\pi\tau}\sigma S_T}$$

is the risk-neutral density function to be used in integrating payoffs over S_T, plotted below



10.6 A Static Replication Example in the Presence of a Skew

Consider an exotic option, strike B that pays one share of stock for every dollar in the money:

$$V(S_T) = S_T \times \max[S_T - B, 0] = S_T \times (S_T - B)H(S_T - B)$$

Quadratic curved payoff, which we must replicate.

Intuitively: We can replicate by adding together a collection of vanilla calls with strikes starting at B, and then adding successively more of them to create a quadratic payoff, as illustrated below.

$$\mathrm{V(S)} = \int_0^\infty q(K)\theta(K-B)\mathrm{C(S,K)dK}$$

where $q(K)$ is the unknown density of calls with strike K.

More formally, we can choose A in Equations 10.2 to be zero.

$$q(K) = \frac{\partial^2}{\partial K^2}V(K, T)$$

$$
\begin{aligned}
\frac{\partial V}{\partial K}(K) &= \frac{\partial}{\partial K}[K \times (K-B)\theta(K-B)] \\
&= (K-B)\theta(K-B) + K\theta(K-B) + K(K-B)\delta(K-B) \\
&= (K-B)\theta(K-B) + K\theta(K-B)
\end{aligned}
$$

Second derivative:

$$
\begin{aligned}
\frac{\partial^2 V}{\partial K^2} &= (K-B)\delta(K-B) + 2\theta(K-B) + K\delta(K-B) \\
&= 2\theta(K-B) + K\delta(K-B)
\end{aligned}
$$

Integrate over calls with this density: $V(S, t) = \int_B^{\infty} \frac{\partial^2}{\partial K^2} V(K, T) C(S, K) dK$

$$
\begin{aligned}
V(S,t) &= \int_B^{\infty} \frac{\partial^2 V(K,T)}{\partial K^2} C(S,K)\, dK \\
&= \int_B^{\infty} K \times \delta(K-B) C(S,K)\, dK + 2\int_B^{\infty} H(K-B) C(S,K)\, dK \\
&= BC(S,B) + 2\int_B^{\infty} C(S,K)\, dK \qquad (11.26)
\end{aligned}
$$

Security **V** in terms of call options $C(K)$ of various strikes K:

Security: $\boldsymbol{V} = B\mathbf{C}(\mathbf{B}) + \int_B^{\infty} 2\mathbf{C}(\mathrm{K})d\mathrm{K}$ Value: $V(S, t) = BC(S, t, B, T) + 2\int_B^{\infty} C(S, t, K, T)dK$

Payoff of 50 calls with strikes equally spaced and \$1 apart between 100 and 150.

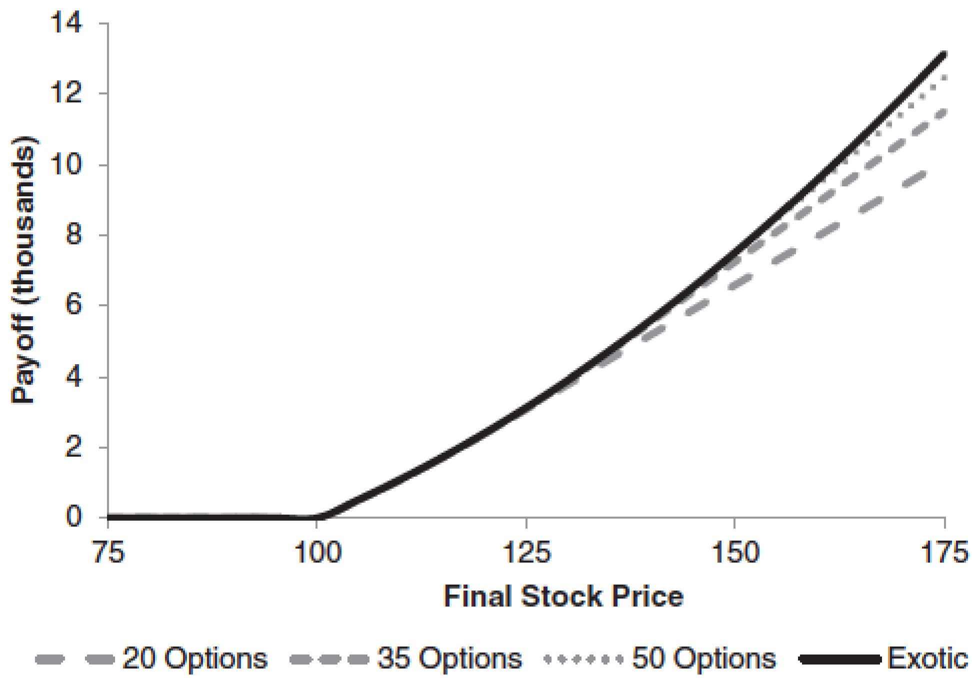


FIGURE 11.7 Approximation to Quadratic Payoff with Calls Spaced \$1 Apart

Convergence of the value of the replicating formula to the correct no-arbitrage value for two different smiles.

$$\Sigma(K) = 0.2\left(\frac{K}{100}\right)^{\beta}$$

$\beta = -0.5$ "negative" skew. Implied volatility increases with decreasing strike.

$\beta = 0$ corresponds to no skew at all.

$\beta = 0.5$ corresponds to a positive skew.

For $\beta = 0$ the fair value of V when replicated by an infinite number of calls is 1033. With 10 strikes the value has almost converged.

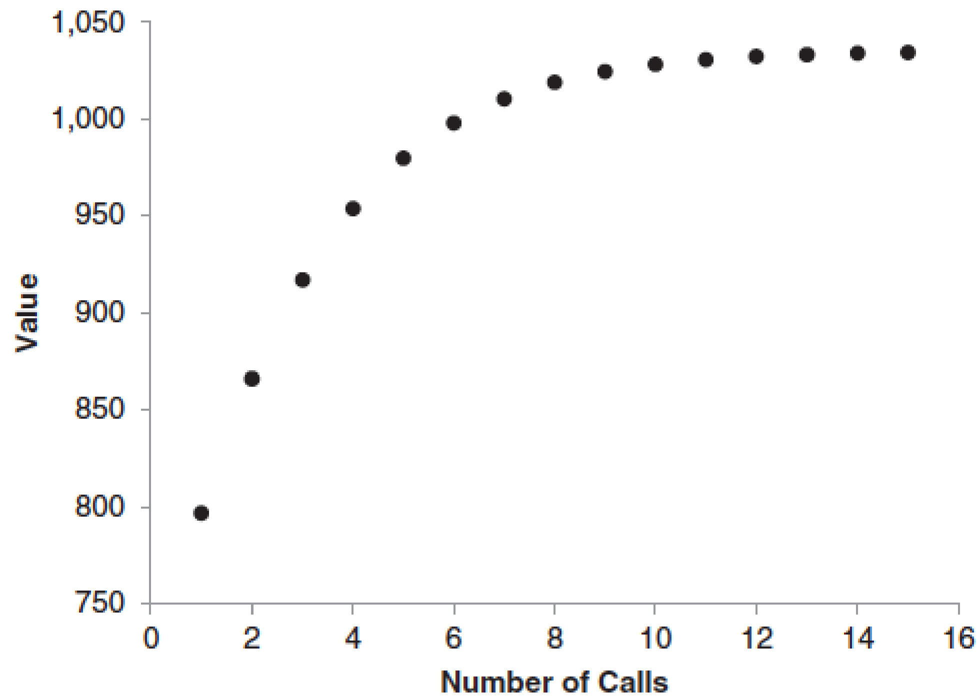


FIGURE 11.8 Convergence for No Skew, $\beta = 0$

Positive skew $\beta = 0.5$: Convergence for a positive skew to a fair value of 1100 is slower and requires more strikes

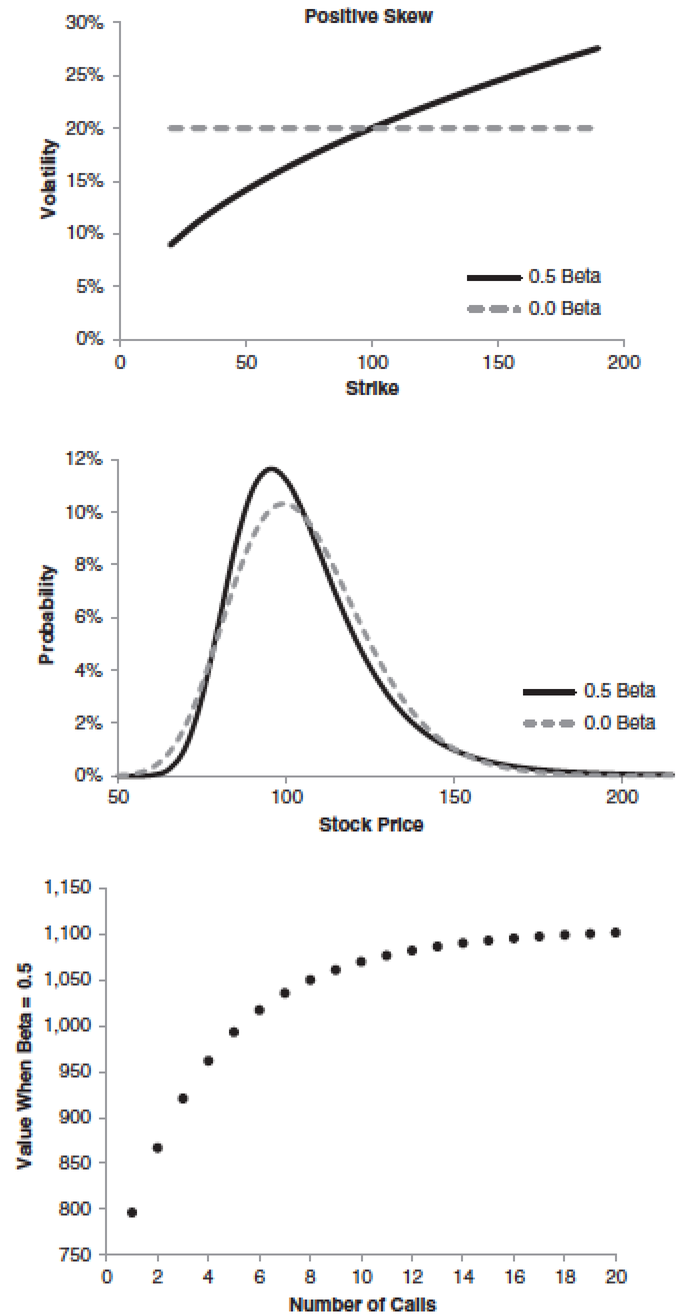
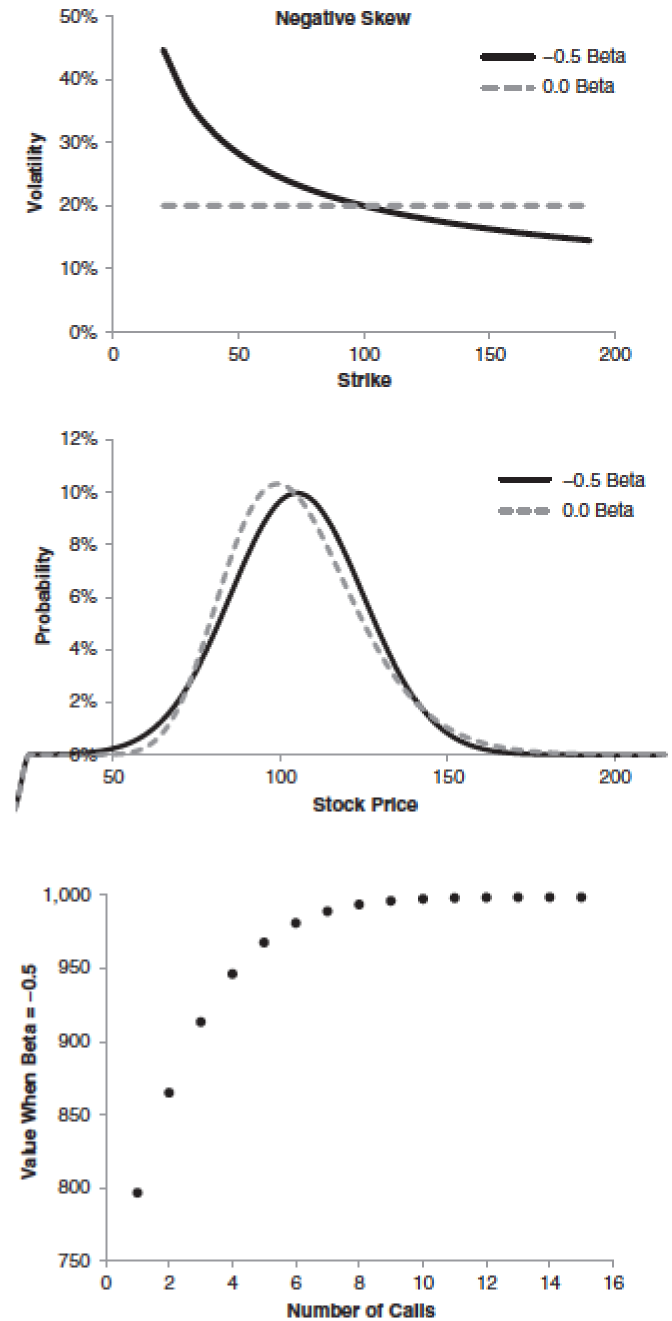


FIGURE 11.9 Convergence for Negative and Positive Skews

Negative skew $\beta = -0.5$ **:** Convergence for a positive skew to a fair value of 996 is faster and requires less strike



Weak Static Replication and Non-European Options: Some Tricks …

- Dynamic replication of exotic options requires frequent and sometimes expensive rebalancing.
- Weak static replication tries to match the payoffs of an exotic option on all its boundaries using portfolios of standard options.
- The weights of the static replication portfolio depend on the model used (as does the hedge ratio in dynamic replication).
- The portfolio often has to be unwound as the option approaches a barrier.
- There is no unique static replication portfolio. It takes art and a knowledge of valuation to find a good one.

What We've Learned

1. The most reliable way to value a security is to replicate it, and static replication is best. If you cannot find a static replicating portfolio, use dynamic replication.

2. The Black-Scholes-Merton (BSM) model relies on continuous dynamic replication. Even if the model were correct in principle, hedging errors and transaction costs limit its practical implementation unless we can eliminate most of the hedging in the portfolio.

3. Even within the scope of the BSM model, we still need to pick a volatility to use for hedging. Hedging with implied volatility leads to an uncertain path-dependent total profit and loss (P&L); hedging with future realized volatility leads to a theoretically deterministic final P&L, but might involve large fluctuations in the P&L along the way to expiration. In practice, since future volatility cannot be known, significant P&L losses along the way might make it necessary to unwind the hedge before expiration in order to limit potential future losses.

4. We showed that you can statically replicate any European payoff with a portfolio of standard puts and calls, independent of any valuation model. This is called strong replication, because it involves no assumptions about the behavior of assets or markets except the absence of credit risk. While such perfect strong replication is possible in theory, it may require an infinite number of options. In practice, therefore, one can create only approximate replicating portfolios whose mismatch with the payoff of the actual security will lead to basis risk.

10.7 Weak Static Replication of Non-European Options

- **Strong static replication:** Replication is independent of model.

  That's what we just did.

- **Weak static replication:** Weak replication needs a model and an assumption about the future evolution of the stock and its volatility etc., i.e. about the future smile. The method relies on the assumptions behind the Black-Scholes theory, or any other theory you used to replace it.

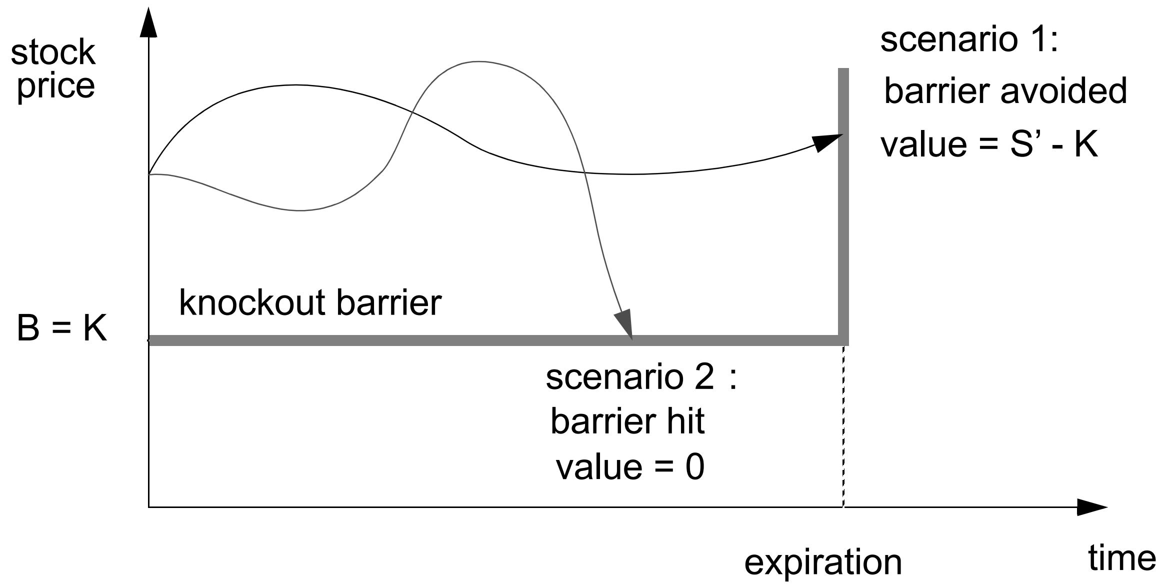
- In both cases, the costs of replication and transaction are embedded in the market prices of the standard options employed in the replication.

10.8 First Steps: Some Exact Static Hedges in Simple Cases

Sometimes you can statically replicate a barrier option with a position in stocks and bonds alone.

European Down-and-Out Call with Barrier at Strike

Scenario 1 in which the barrier is avoided and the option finishes in-the-money;
Scenario 2 in which the barrier is hit before expiration and the option expires worthless.

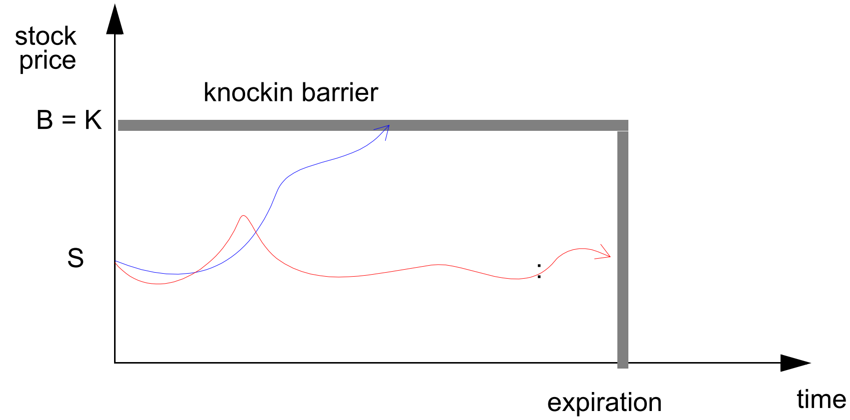


Replicate scenario 1 with a forward $F = Se^{-dt} - Ke^{-rt}$

If $d = r$ we also have perfect replication on the barrier, *if the stock moves continuously*.

An up-and-in European put with the strike K equal to the barrier. Assume r = d = 0.

Now consider an up-and-in put with strike K equal to the barrier B, as illustrated below.



Blue trajectories that hit the barrier generate a standard put $P(\mathrm{S{=}K}, K, \sigma, \tau)$
Red trajectories that avoid the barrier expire worthless.

To replicate we need a security that expires worthless if the barrier is avoided and has the value of the put $P(K, K, \sigma, \tau)$ on the barrier, so that we can buy the put.

A standard call $C(S, K, \sigma, \tau)$ bought at the beginning will expire worthless for all values of the stock price below K at expiration. And, on the boundary $S = K$, the value $C(\mathrm{S{=}K}, K, \sigma, \tau) = P(\mathrm{S{=}K}, K, \sigma, \tau)$ *if Black-Scholes with interest rates and dividend yields zero*. At the barrier, you must *sell* the standard call and *immediately buy* a standard put. This assumes the stock moves continuously, else it could jump across the barrier before you trade.

This is weak replication because it depends on the dynamics of the model (BSM with zero rates and dividends). If, for example, there is a smile when the stock touches the barrier, put-call symmetry will fail and you will not be able to exchange the call for the put at zero cost.