Handout. The Black-Scholes Model for Options Prices. Parameters of the Model.

The theoretical price of European calls and puts is given by Black-Scholes formula. There are six parameters included in the formula.

- X is the current price of the underlying asset,
- K is the strike price of the option,
- r is the current risk-free interest rate,
- q is the continuous dividend yield on the underlying asset,
- σ is the volatility of the underlying asset, and
- T is the time remaining until the expiration of the option.

The Black-Scholes call and put formulas give the prices of options as functions of these parameters: The exact form of these functions will be given later.

$$\text{Black - Scholes call price} = f_{\text{Black-Scholes call}}(X, K, r, q, \sigma, T)$$

$$\text{Black - Scholes put price} = f_{\text{Black-Scholes put}}(X, K, r, q, \sigma, T)$$

Example. Suppose we have European call and put options expiring in 60 days with a strike price $K=20$\$. If we know that the current stock price $X=19$\$, the volatility of the stock price $\sigma=31\%$, the risk-free interest rate $r=6\%$, and the continuous dividend yield $q=0$, then we can calculate the Black-Scholes prices of the options.

$$f_{\text{Black-Scholes call}}(19,\ 20,\ 0.06,\ 0.0,\ 0.31,\ 60/365) = 0.62$$

$$f_{\text{Black-Scholes put}}(19,\ 20,\ 0.06,\ 0.0,\ 0.31,\ 60/365) = 1.43$$

It is useful to remember the functional expressions for and , but it is even more crucial to understand the qualitative behavior of Black-Scholes prices with respect to the six parameters included in the formula.

The estimation of all parameters except the volatility σ is straightforward.

The current stock price X can be taken from the most recent available quote.

The strike price K and time to maturity T are given with the option.

For the risk-free interest rate r, the most recent available data can be used. It is not very important to have a value for r be updated during the course of a single day because the magnitude of change in r during the day does not usually much affect the price of an option. The change in the price of underlying X has overwhelming effect.

The dividend yield can be estimated using previous history and some projections, but the effect of a change in the dividend yield will not be very significant.

Volatility as the Most Important Parameter.

Volatility σ is by far the most important parameter affecting the option price because it is the parameter that is difficult to estimate.

There are two main approaches to estimating volatility. The first method uses a series of historical stock prices (usually 10 to 90 days of closing or high/low prices) to determine **the historical volatility** of the stock price.

we defined **historical volatility** as

$$\sigma = (\text{Standard deviation of daily returns } r_i \text{ for the last } n \text{ days}) \times \sqrt{\text{number of trading days in a year}}$$

Implied Volatility

Volatility σ can also be calculated by using prices of traded options. In this case σ is called **implied volatility**.

If all other parameters necessary for the Black-Scholes model (underlying stock price *X*, strike price *K*, riskless interest rate *r*, dividend yield *q*, time to maturity *T*) are known and the price of the option is known (because the option was traded), then one can find a unique volatility σ_{IMP} that makes the equality

$$f_{\text{Black-Scholes}}(X, K, r, q, \sigma_{IMP}, T) = \text{Price Observed in the Market}$$

Example. Calculation of Implied Volatility

The current stock price $X=19$.

The strike price of the option $K=20$.

The risk-free interest rate $r=5.6\%=0.056$.

The continuous dividend yield $q=2\%=0.02$.

Time to expiration T=30 days 1/12 year.

The observed price of 20 call is 0.50.

To calculate the implied volatility we have to solve the equation

$$f_{\text{Black-Scholes Call}}(19, 20, 0.056, 0.02, \sigma_{IMP}, 1/12) = 0.5$$

with respect to unknown parameter σ_{IMP}. This can be done numerically.

Try to use **European Options Calculator.**

It is interesting to note that historical volatility is not a good predictor of true volatility. That means that volatility calculated using say previous 20-day prices can be quite different from the volatility in the following 20 days.

The implied volatility is supposed to be a better predictor of the true volatility of the underlying during the remaining life of the option. That is because implied volatility in the market with the absence of arbitrage is the consensus of the market participants about the volatility of the underlying during the remaining life of the option. There is no conclusive evidence how well the implied volatility predicts realized future volatility. On the other hand, for many hedging purposes it is reasonable to use implied volatility.

Volatility Smile. Correspondence between Volatility and Option Price

The standard Black-Scholes options pricing model assumes that volatility σ is constant at any one time for any given underlying stock. So the implied volatility σ of an asset is the same for any option with any strike and any time to expiration.

From the late 1980's to the present, however, Black-Scholes constant volatility model has been inconsistent with the observed market reality. Implied volatilities of options on the same asset with same maturity but with different strikes can be different. That effect is called volatility smile.

Volatility Smile

It is the relationship between implied volatility and strike price for options with a certain maturity

The volatility smile for European call options should be exactly the same as that for European put options

The same is at least approximately true for American options

Volatility Smile is the Same for European Calls and Puts

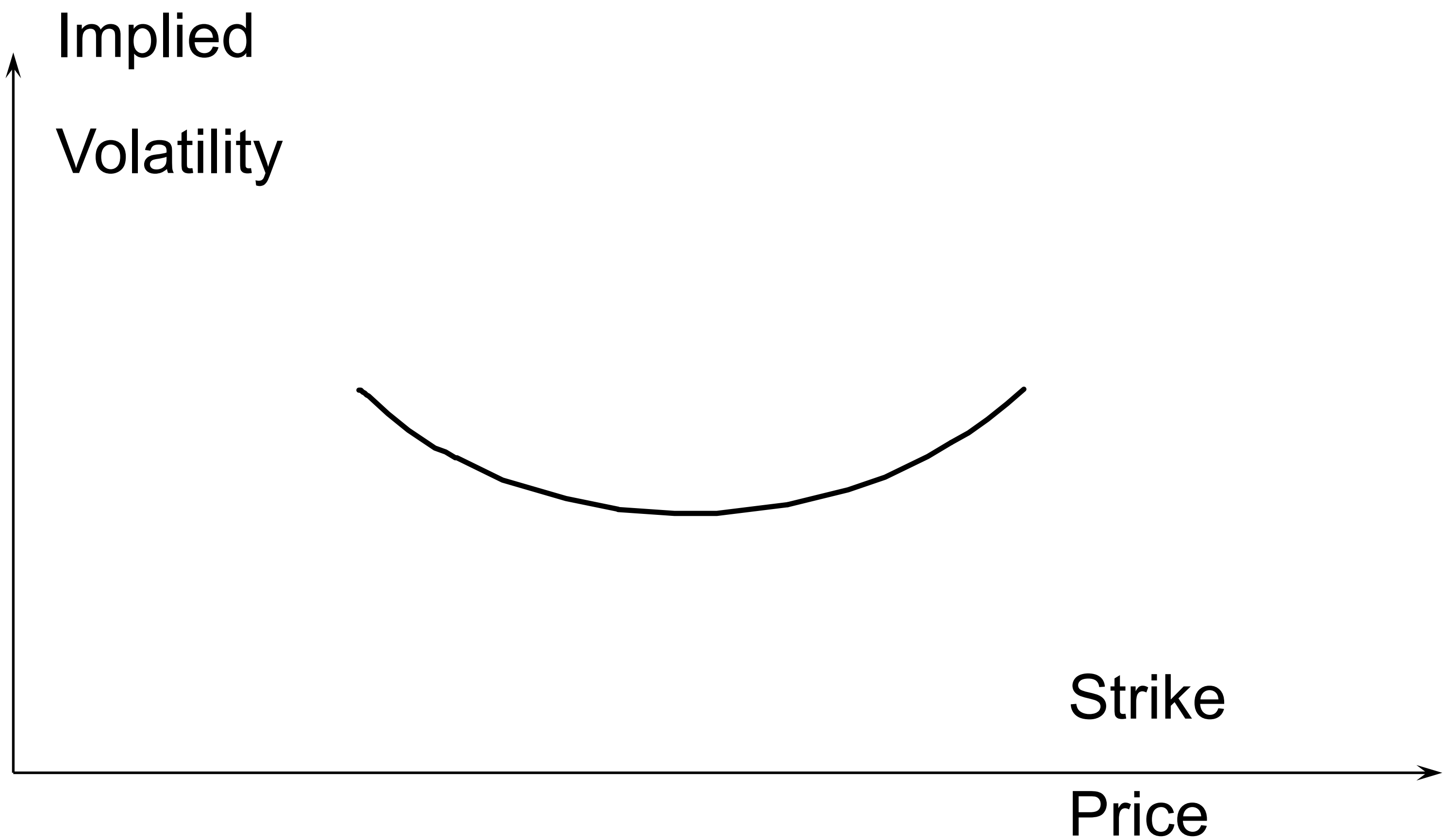
Put-call parity $p + X_0 e^{-qT} = c + K e^{-rT}$ holds for market prices (p_{mkt} and c_{mkt}) and for Black-Scholes-Merton prices (p_{bs} and c_{bs})

As a result, $p_{\text{mkt}} - p_{\text{bs}} = c_{\text{mkt}} - c_{\text{bs}}$

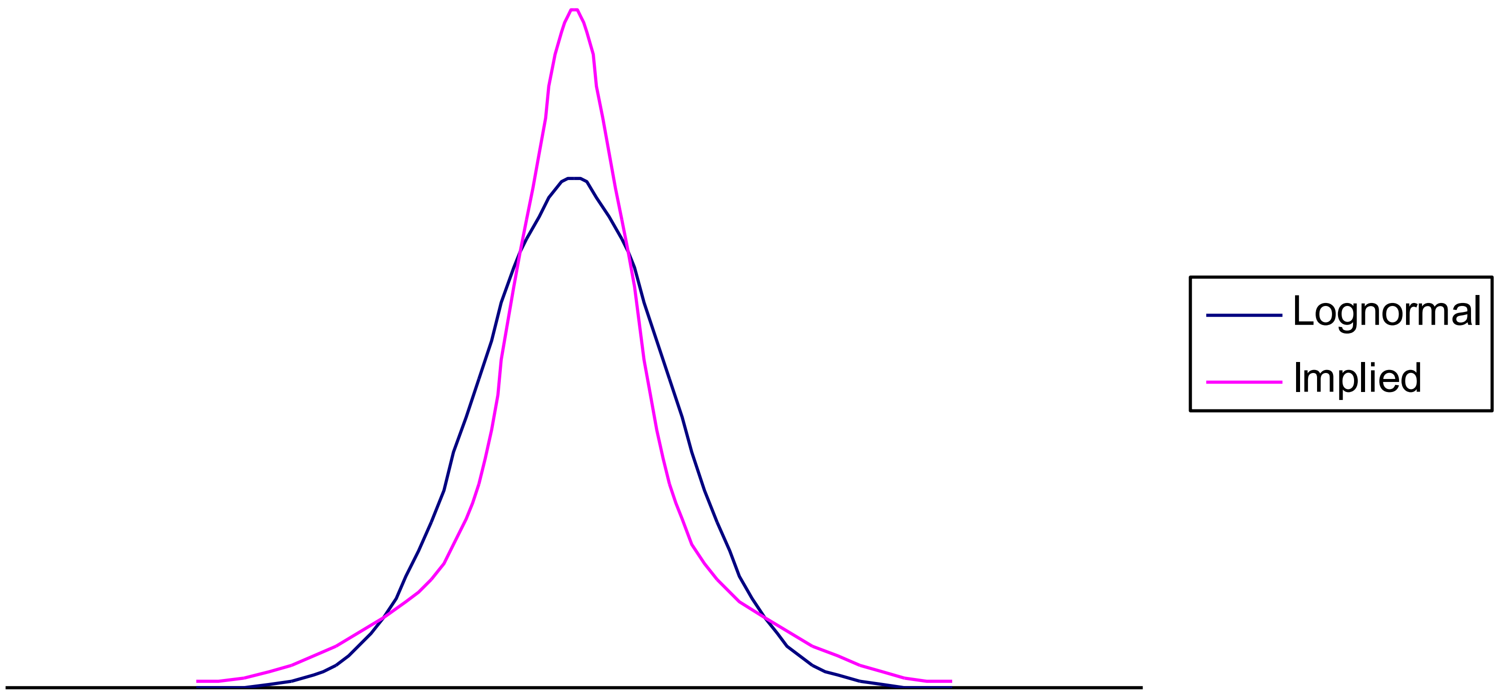
When $p_{\text{bs}} = p_{\text{mkt}}$, it must be true that $c_{\text{bs}} = c_{\text{mkt}}$

It follows that the implied volatility calculated from a European call option should be the same as that calculated from a European put option when both have the same strike price and maturity

Typical Volatility Smile for Foreign Currency Options



Typical Implied Distribution for Foreign Currency Options



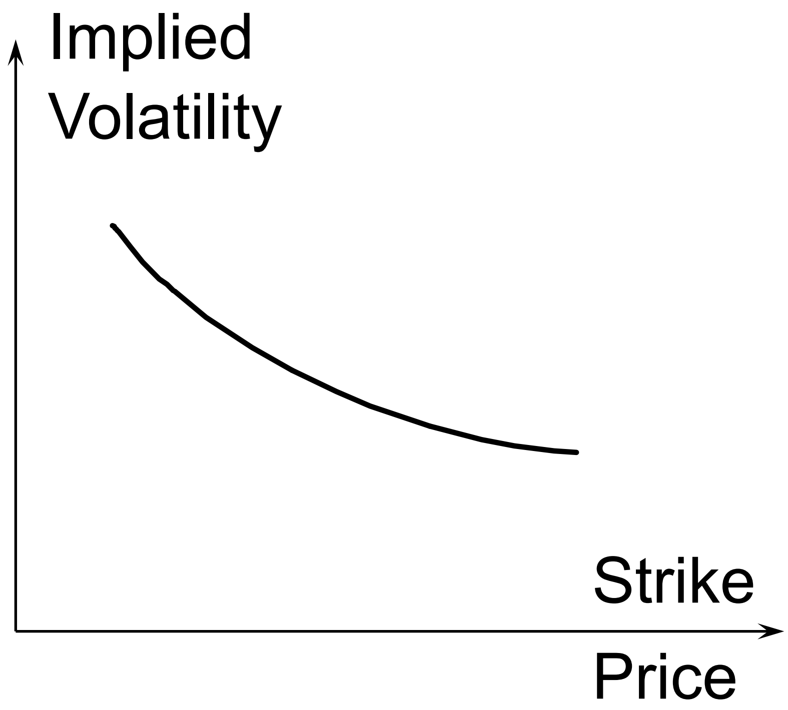
Both tails are heavier than the lognormal distribution
Distribution is also “more peaked” than the lognormal distribution

Possible causes of volatility smile for foreign currencies:
1.Exchange rate exhibits jumps rather than continuous changes,
2.Volatility of exchange rate is stochastic

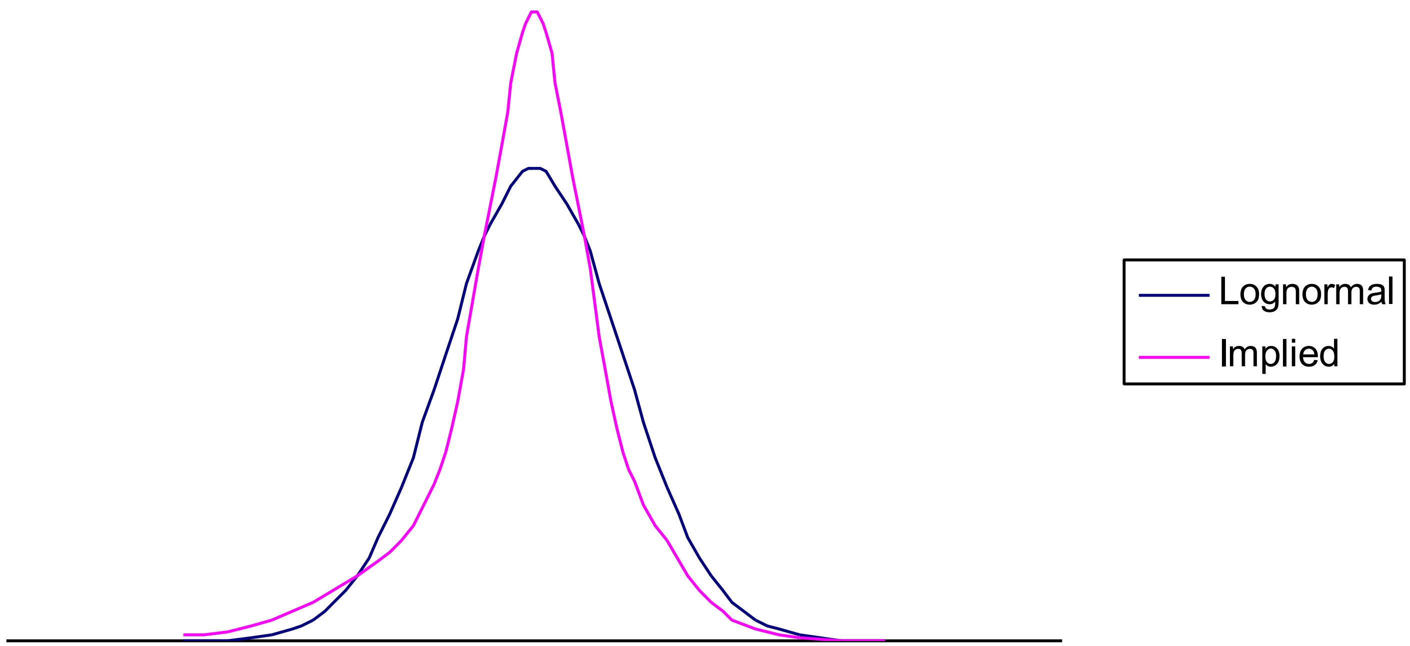
Historical Analysis of Exchange Rate Changes

	Real World (%)	Normal Model (%)
>1 SD	25.04	31.73
>2SD	5.27	4.55
>3SD	1.34	0.27
>4SD	0.29	0.01
>5SD	0.08	0.00
>6SD	0.03	0.00

Typical Volatility Smile for Equity Options



Implied Distribution for Equity Options



The left tail is heavier than the lognormal distribution

The right tail is less heavy than the lognormal distribution

Reasons for smile in equity options

1. Leverage
2. Fear of Crash

Other Volatility Smiles

What is the volatility smile

True distribution has a less heavy left tail and heavier right tail   (right upward slope)

True distribution has both a less heavy left tail and a less heavy right tail
(right and left downward slope also called smirk)

Plotting Volatility Smiles

Plot implied volatility against strike/spot K/S_0

Plot implied volatility against strike/forward K/F_0

Note: traders frequently define an option as at-the-money when K equals the forward price, F_0, not when it equals the spot price S_0

Plot implied volatility against delta of the option

Note: traders sometimes define at-the money as a call with a delta of 0.5 or a put with a delta of −0.5. These are referred to as "50-delta options"

Volatility Term Structure

1. In addition to calculating a volatility smile, traders also calculate a volatility term structure

2. This shows the variation of implied volatility with the time to maturity of the option

3. The volatility term structure tends to be downward sloping when volatility is high (later volatility is lower than recent maturities) and upward sloping when it is low (later volatility is higher than recent maturities)

The implied volatility as a function of the strike price and time to maturity is known as a volatility surface.

Example of a Volatility Surface Table

	K/S_0				
	0.90	0.95	1.00	1.05	1.10
1 mnth	14.2	13.0	12.0	13.1	14.5
3 mnth	14.0	13.0	12.0	13.1	14.2
6 mnth	14.1	13.3	12.5	13.4	14.3
1 year	14.7	14.0	13.5	14.0	14.8
2 year	15.0	14.4	14.0	14.5	15.1
5 year	14.8	14.6	14.4	14.7	15.0

Smile Effect on Delta

If the Black-Scholes price, c_{BS} is expressed as a function of the stock price, X, and the implied volatility, $\sigma_{imp}(X)$ the total delta of a call is

$$\frac{\partial c_{BS}}{\partial X} + \frac{\partial c_{BS}}{\partial \sigma_{imp}} \frac{\partial \sigma_{imp}}{\partial X}$$

Is the delta higher or lower than

for equities? $\dfrac{\partial c_{BS}}{\partial X}$

Volatility Smiles When a Large Jump is Expected

At the money implied volatilities are higher that in-the-money or out-of-the-money options (so that the smile is a frown)

Determining the Implied Distribution Density p(X)

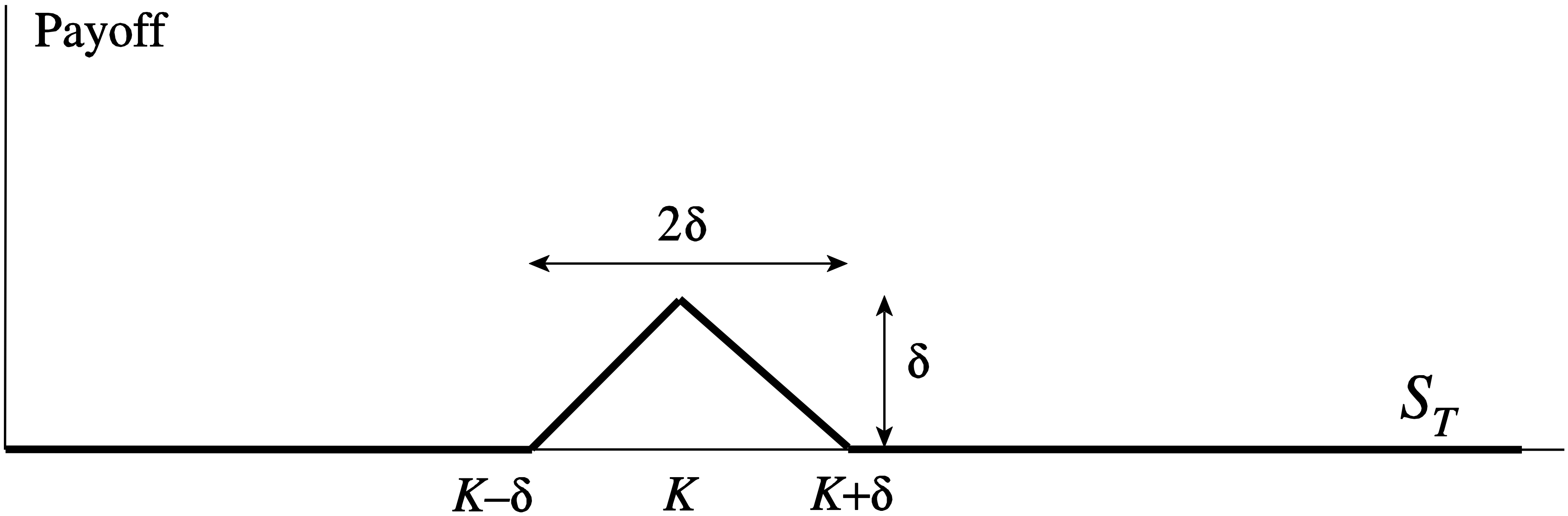
$$c = e^{-rT}\int_{X_T=K}^{\infty}(X_T - K)p(X_T)dX_T$$

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT}p(K)$$

If $c_1, c_2,$ and c_3 are call prices for strikes $K-\delta, K,$ and $K+\delta$ then

$$p(K) = e^{rT}\frac{c_1 + c_3 - 2c_2}{\delta^2}$$

Geometric Interpretation



Assuming that density is $p(K)$ from $K-\delta$ to $K+\delta$, $c_1 + c_3 - c_2 = e^{-rT}\,\delta^2\, p(K)$

No arbitrage principle. Equating stock growth rate a risk-free rate.

No arbitrage principle states that a correct price of a derivative security can be calculated as the price that does not allow for arbitrage.
Based on that fundamental principle is the **risk-neutral valuation.**

Risk-neutral valuation states that if one replaces growth rate of all securities by risk free rate ***r***, and calculates probability distributions of returns of securities based on that artificial assumption then the no-arbitrage price of derivative security would be just expected payoff.

The so-called risk-neutral valuation is by all means one of the most important principles in theoretical finance and derivatives theory. It reflects one of the properties of the Black-Scholes(-Merton) equation which does not contain the expected return on the stock, μ, and, instead, contains only the risk-free rate, r.

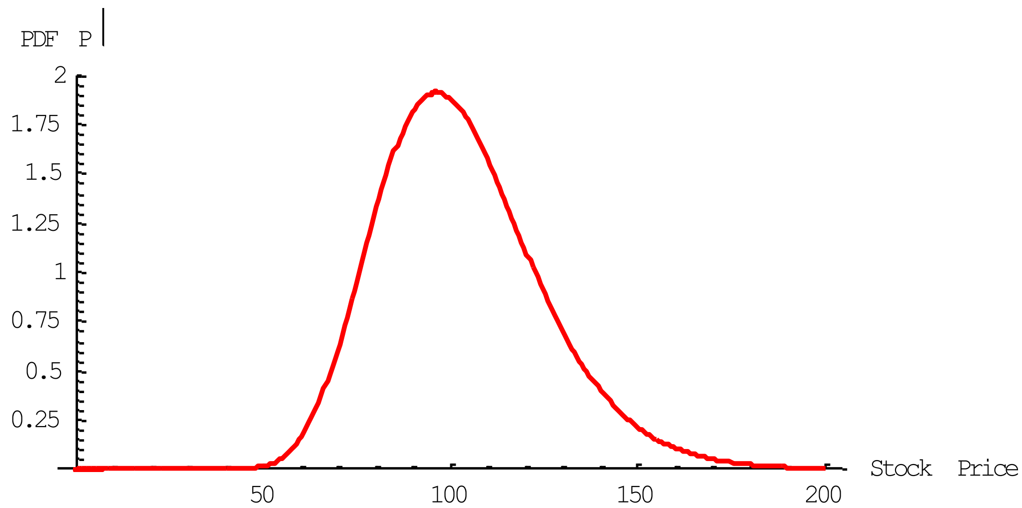
Calculating price of a European option through expected payoff

Let us remind, that the risk-neutral valuation principle allows one to replace the stock growth rate μ with a risk-free rate r when calculating no arbitrage price.

Assume that the current stock price is X_0 , the stock growth rate is r, volatility of the stock is σ. Stock is paying no dividends. Then the Probability Density Function distribution of stock prices after time T is

$$P(x) = \frac{1}{x\sigma\sqrt{2\pi T}} e^{\frac{-(\ln(x)-\ln(x_0)-(r-\frac{\sigma^2}{2})T)^2}{2\sigma^2 T}}$$

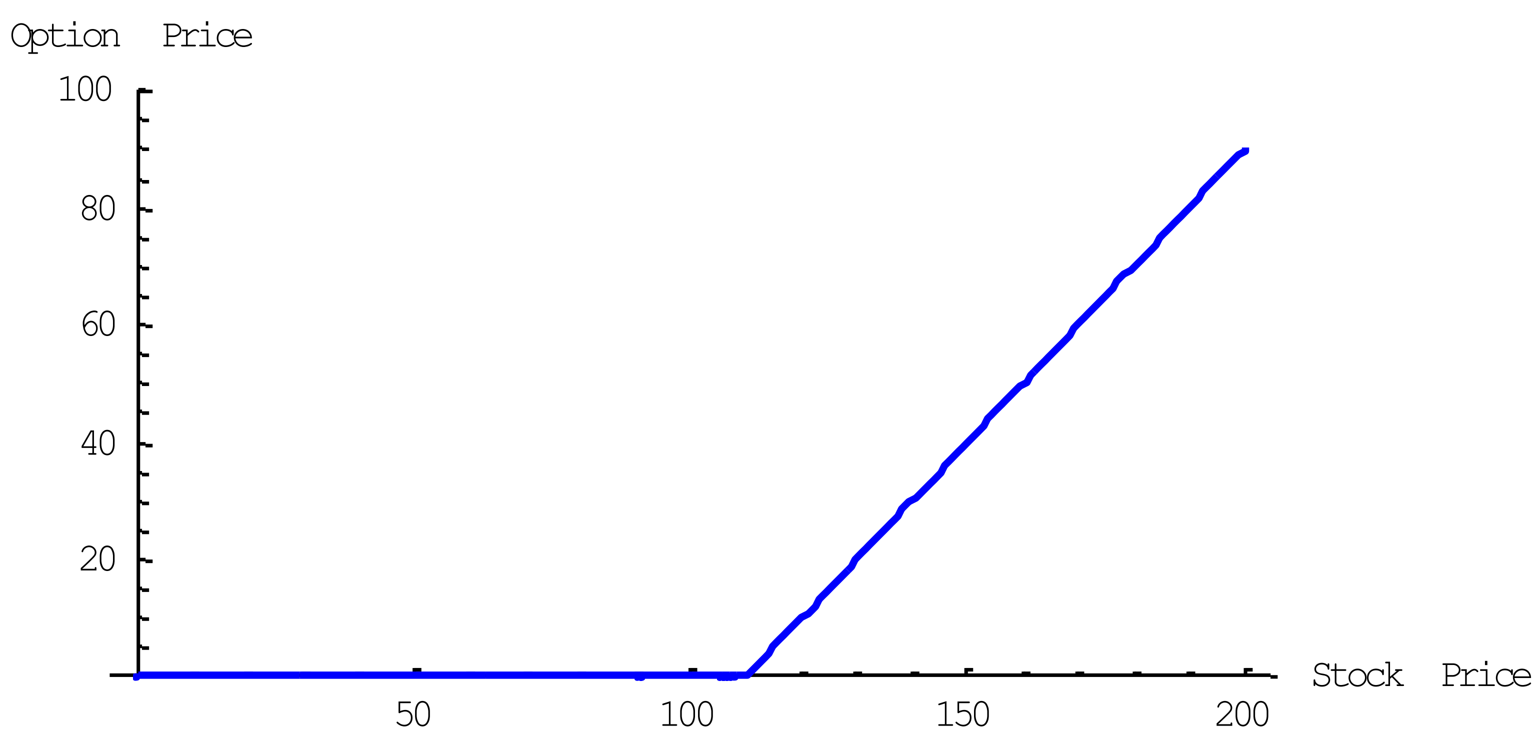
The graph below shows the Probability Density Function with X_0=100, r=0.06 i.e. 6%, σ =0.3 i.e. 30%.



The call option payoff function $C(x)$ is

$$C(x)=\max\{x - K, 0\} = \begin{cases} x-K & \text{if stock price } x >= K, \text{ where } K \text{ is strike price} \\ 0 & \text{if stock price } x < K \end{cases}$$

The payoff function with $K=110$ is shown below



The expected payoff of a call is $\int_{-\infty}^{+\infty} C(x)P(x)dx$

That is expected payoff at time T and to get the option value we must find its present value which can be obtained by multiplying by discount factor : e^{-rT}

$$\text{CallPrice} = e^{-rT} \int_{-\infty}^{+\infty} C(x)P(x)dx$$

The similar expression is true for a European put, one need to replace $C(x)$ with a put payoff max$\{K - x, 0\}$.

Note that expression for probability density function P(x) does not include growth rate of stock μ but has it replaced with a risk free rate r.

Expected value of payoff must be calculated using such artificial (called ***Risk Neutral***) probability distribution.

Such use of artificial Risk Neutral probability distribution is mathematically equivalent to the no arbitrage pricing principle.

We do not discuss here the proof and mathematical details.

Considering that a European call payoff $C(x)=\max\{x - K, 0\}$ we can express European call price as

$$\text{CallPrice} = e^{-rT}\int_{-\infty}^{+\infty} C(x)P(x)dx = e^{-rT}\int_{K}^{+\infty}(x-K)P(x)dx$$

The last integral can be calculated and a closed form solution for a European call price can be obtained.

Black-Scholes Formula

European Call option valuation. The main result in European option valuation, which at present serves as a keystone of quantitative finance, is known as Black-Scholes (-Merton) formula.

Consider, first, a case of a non-dividend-paying stock. The no arbitrage value of a European Call option with time to expiry *T* in an environment with risk-free rate *r*, and underlying stock volatility σ, is expressed by the following formulas.

$$c = X_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$d_1 = \frac{\ln\left(X_0 / K\right) + \left(r + \sigma^2 / 2\right)T}{\sigma\sqrt{T}} \qquad d_2 = \frac{\ln\left(X_0 / K\right) + \left(r - \sigma^2 / 2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

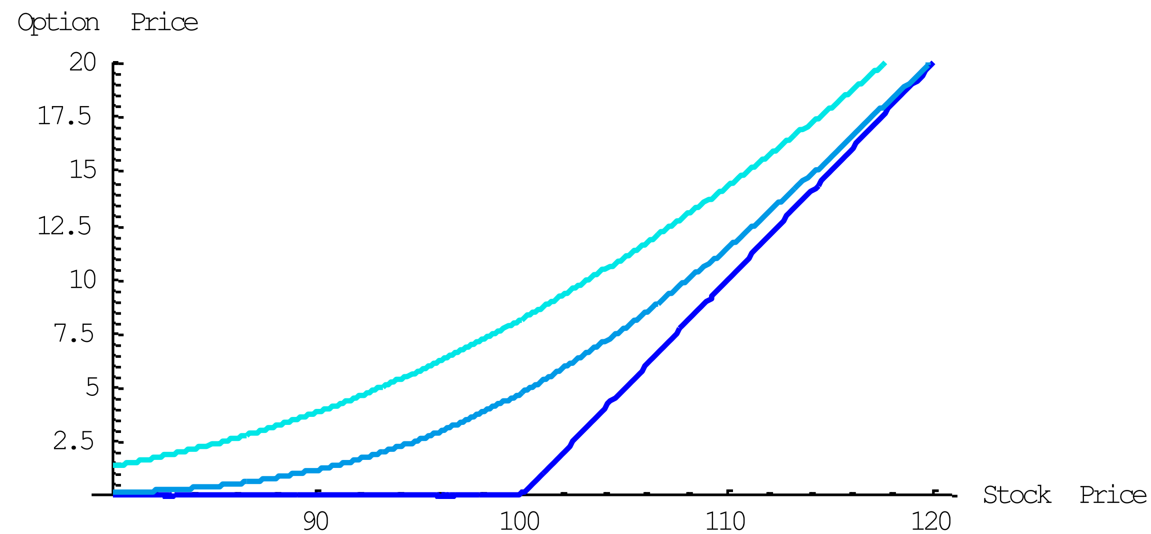
Formula for a European option may be obtained from the above using the Put-Call parity relationship: $\quad c - p = X_0 - Ke^{-rT}$

which implies $\quad p = Ke^{-rT} N(-d_2) - X_0 N(-d_1)$

Here, *N(x)* is a cumulative distribution function of a zero-mean and standard deviation *1* normal variable. As a reminder, by definition, it is the probability such that this variable is less than *x*.

Graphs of option price before expiration. Convexity.

The graphs below show call price 6 month, 2 month, and 2 hours before expiration. The call has strike price $K=100$, volatility $\sigma = 30\%$, interest rate $r=6\%$, and no dividends.



One can see that before expiration there is a **convexity** in the graph. The convexity is described by the second derivative of the call price with respect to stock price X that is also called Gamma and denoted by a greek letter Γ. This second derivative is positive for call owner, (and put owner too).

The holder of the call makes more money when stock price goes 1$ up and loses less money when stock price goes 1$ down.However for that positive convexity the holder of the option pays with **time decay**.

If the stock price stays the same as time passes, the call will lose value. The price curve for a call 6 months before expiration is above the curve 2 moths before expiration etc.