10 + 7 + ~~[illegible]~~ = 17

Solutions Assignment 5

9.6.2

2. In this exercise, $m = 8$, $n = 6$, $\overline{x}_m = 1.5125$, $\overline{y}_n = 1.6683$, $S_X^2 = 0.18075$, and $S_Y^2 = 0.16768$. When $\mu_1 = \mu_2$, the statistic U defined by Eq. (9.6.3) will have the t distribution with 12 degrees of freedom. The hypotheses are as follows:

$$\begin{aligned} H_0 &: \quad \mu_1 \geq \mu_2, \\ H_1 &: \quad \mu_1 < \mu_2. \end{aligned}$$

Since the inequalities are reversed from those in (9.6.1), the hypothesis H_0 should be rejected if $U < c$. It is found from a table that $c = -1.356$. The calculated value of U is -1.692. Therefore, H_0 is rejected.

7

9.6.6

6. If $\mu_1 - \mu_2 = \lambda$, the following statistic U will have the t distribution with $m + n - 2$ degrees of freedom:

$$U = \frac{(m+n-2)^{1/2}(\overline{X}_m - \overline{Y}_n - \lambda)}{\left(\dfrac{1}{m} + \dfrac{1}{n}\right)^{1/2}(S_X^2 + S_Y^2)^{1/2}}.$$

The hypothesis H_0 should be rejected if either $U < c_1$ or $U > c_2$.

3

9.7.8

8. For any values of σ_1^2 and σ_2^2, the random variable

$$\frac{S_1^2/(15\sigma_1^2)}{S_2^2/(9\sigma_2^2)}$$

has the F distribution with 15 and 9 degrees of freedom. Therefore, if $\sigma_1^2 = 3\sigma_2^2$, the following statistic V will have that F distribution:

$$V = \frac{S_1^2/45}{S_2^2/9}.$$

As before, H_0 should be rejected if $V > c$, where $c = 3.01$ if the desired level of significance is 0.05.

Not graded

Exercise I 4

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{S_X'^2} = \sum_{i=1}^n \frac{(x_i - \bar{x})}{S_X'^2} y_i = \sum_{i=1}^n c_i y_i \text{ for } c_i = \frac{(x_i - \bar{x})}{S_X'^2} \quad 2$$

$$\hat{\beta}_0 = \frac{1}{n} \sum_{i=1}^n y_i - \bar{x} \sum_{i=1}^n \frac{(x_i - \bar{x})}{S_X^2} y_i = \sum_{i=1}^n \left(\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{S_X^2} \right) y_i = \sum_{i=1}^n a_i y_i \quad 2$$

$$\text{for } a_i = \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_X^2}$$

Exercise II 3

Equation (23) yields $\hat{\beta}_0 = \frac{1}{n}\left(\sum_{i=1}^n y_i - \hat{\beta}_1 \sum_{i=1}^n x_i \right) = \bar{y} - \hat{\beta}_1 \bar{x}$ 1

(24) $- \bar{x} \cdot 23$ yields:

$$\sum_{i=1}^n (x_i - \bar{x}) y_i = \hat{\beta}_0 \sum_{i=1}^n (x_i - \bar{x}) + \hat{\beta}_1 \sum_{i=1}^n (x_i^2 - \bar{x} x_i)$$

$$= \hat{\beta}_0 \cdot 0 + \hat{\beta}_1 \left(\sum_{i=1}^n x_i^2 - n\bar{x}^2 \right) = \hat{\beta}_1 \sum_{i=1}^n (x_i - \bar{x})^2 = \hat{\beta}_1 S_X'^2$$

$$\Rightarrow \hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{S_X^2}. \quad 2$$

NOT GRADED

7. (a) By the Neyman-Pearson lemma, H_0 should be rejected if $f_1(\boldsymbol{X})/f_0(\boldsymbol{X}) > k$. Here,

$$f_0(\boldsymbol{X}) = \frac{1}{(2\pi)^{n/2}2^{n/2}} \exp\left[-\frac{1}{4}\sum_{i=1}^{n}(x_i-\mu)^2\right]$$

and

$$f_1(\boldsymbol{X}) = \frac{1}{(2\pi)^{n/2}3^{n/2}} \exp\left[-\frac{1}{6}\sum_{i=1}^{n}(x_i-\mu)^2\right].$$

Therefore,

$$\log\frac{f_1(\boldsymbol{X})}{f_0(\boldsymbol{X})} = \frac{1}{12}\sum_{i=1}^{n}(x_i-\mu)^2 + (const.).$$

It follows that the likelihood ratio will be greater than a specified constant k if and only if $\sum_{i=1}^{n}(x_i - \mu)^2$ is greater than some other constant c. The constant c is to be chosen so that

$$\Pr\left[\sum_{i=1}^{n}(X_i-\mu)^2 > c \,\middle|\, H_0\right] = 0.05.$$

The value of c can be determined as follows. When H_0 is true, $W = \sum_{i=1}^{n}(X_i-\mu)^2/2$ will have χ^2 distribution with n degrees of freedom. Therefore,

$$\Pr\left[\sum_{i=1}^{n}(X_i-\mu)^2 > c \,\middle|\, H_0\right] = \Pr\left(W > \frac{c}{2}\right).$$

If this probability is to be equal to 0.05, then the value of $c/2$ can be determined from a table of the χ^2 distribution.

(b) For $n = 8$, it is found from a table of the χ^2 distribution with 8 degrees of freedom that $c/2 = 15.51$ and $c = 31.02$.

NOT GRADED

12. In the notation of this section, $f_i(\boldsymbol{x}) = \theta_i^n \exp\left(-\theta_i \sum_{j=1}^n x_j\right)$ for $i = 0, 1$. The desired test has the following form: reject H_0 if $f_1(\boldsymbol{x})/f_0(\boldsymbol{x}) > k$ where k is chosen so that the probability of rejecting H_0 is α_0 given $\theta = \theta_0$. The ratio of f_1 to f_0 is

$$\frac{f_1(\boldsymbol{x})}{f_0(\boldsymbol{x})} = \frac{\theta_1^n}{\theta_0^n} \exp\left([\theta_0 - \theta_1] \sum_{i=1}^n x_i\right).$$

Since $\theta_0 < \theta_1$, the above ratio will be greater than k if and only if $\sum_{i=1}^n x_i$ is less than some other constant, c. That c is chosen so that $\Pr\left(\sum_{i=1}^n X_i < c \middle| \theta = \theta_0\right) = \alpha_0$. The distribution of $\sum_{i=1}^n X_i$ given $\theta = \theta_0$ is the gamma distribution with parameters n and θ_0. Hence, c must be the α_0 quantile of that distribution.