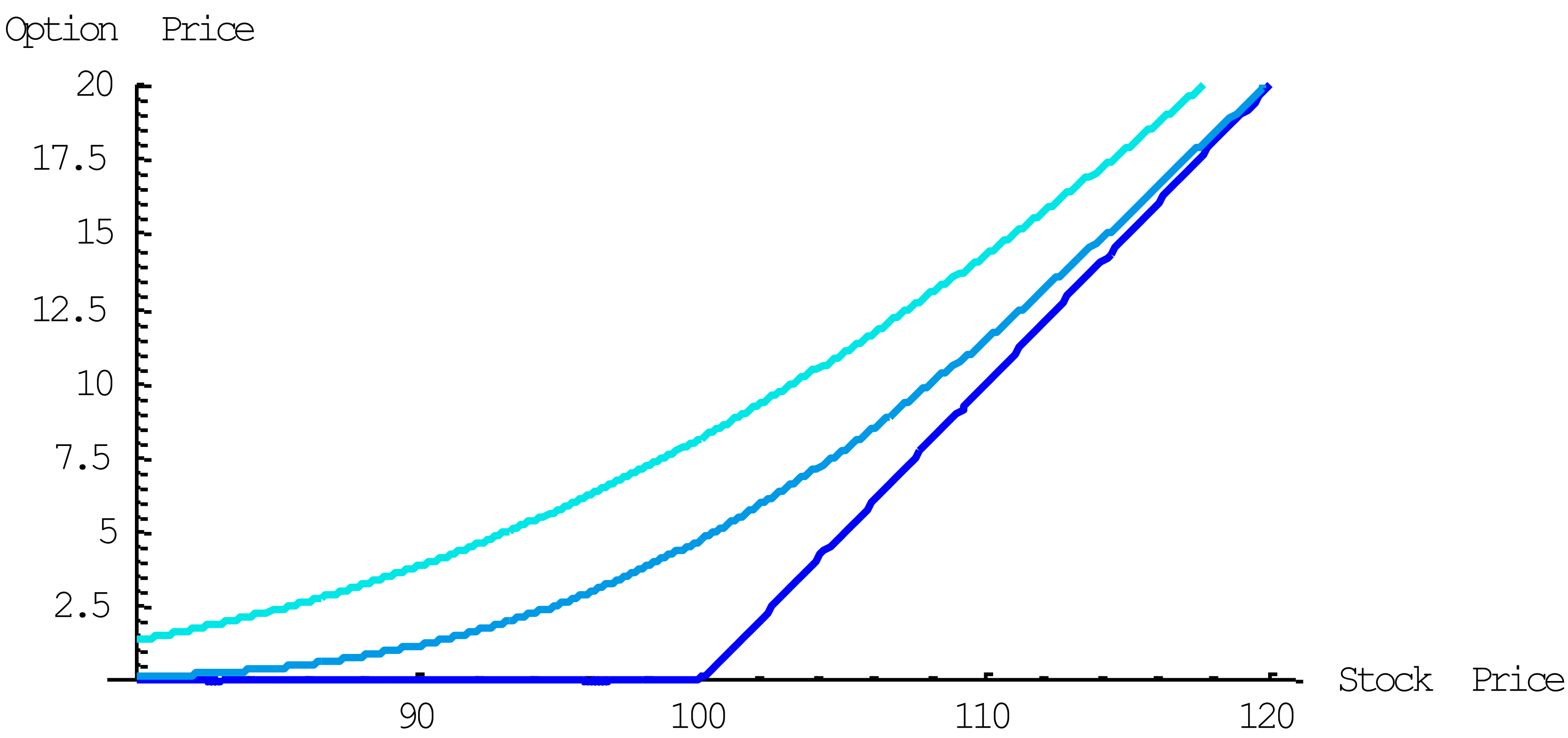
Greeks or Options Sensitivities.

Delta Hedging.

Hedging of Other Greeks.

The graphs below show call price 6 month, 2 month, and 2 hours before expiration.

The call has strike price *K=100,* volatility s=*30%,* interest rate *r=6%,* and no dividends.



One can see that before expiration there is a **convexity** in the graph. The convexity is described by the second derivative of the call price with respect to stock price *X* that is also called Gamma and denoted by a greek letter Γ. This second derivative is positive for call owner, (and put owner too).

The holder of the call makes more money when stock price goes 1$ up then loses less money when stock price goes 1$ down.

However for that positive convexity the holder of the option pays with **time decay**.

If the stock price stays the same as time passes, the call will lose value. The price curve for a call 6 months before expiration is above the curve 2 moths before expiration etc.

Option sensitivities. Greeks: Delta, Gamma, Vega, Theta, Rho

Formulas expressing theoretical European option (call or put) price *f* depend on three "main" variables:

Time remaining to expiry *T,*

Current price of the underlying asset *X,*

Option strike price *K*.

They also depend the following "secondary" variables:

Volatility of the underlying asset σ,

Current risk-free rate *r,*

Continuous dividend yield *q.*

$$\mathbf{f} = \mathbf{f}(\mathbf{X}, \mathbf{K}, \mathbf{r}, \mathbf{q}, \sigma, \mathbf{T})$$

We define “**Greeks**” or sensitivities of option price with respect to input variables:

$$\theta = \frac{\partial f}{\partial t}$$ is “Theta”, measuring time-decay of an option,

$$\Delta = \frac{\partial f}{\partial X}$$ is “Delta”, responsible for sensitivity to changes in underlying,

$$\Lambda = \frac{\partial f}{\partial \sigma}$$ is “Vega”, responsible for sensitivity towards volatility changes,

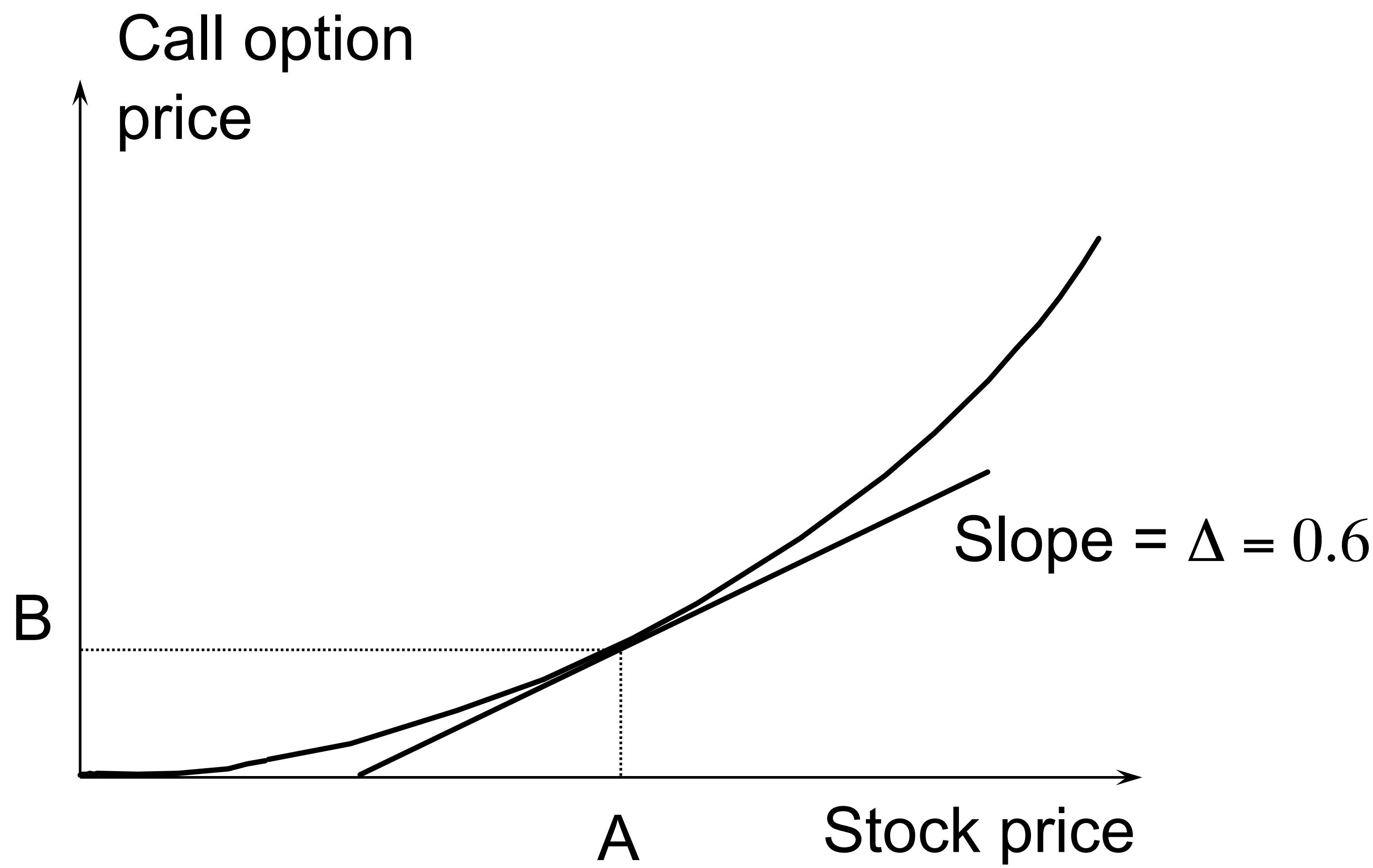
$$\rho = \frac{\partial f}{\partial r}$$ is “Rho”, measures sensitivity to changes in interest rate,

$$\rho_1 = \frac{\partial f}{\partial q}$$ is “Rho 1”, measures sensitivity to changes in dividend yield,

$$\Gamma = \frac{\partial^2 f}{\partial X^2}$$ is “Gamma”, measures the convexity of the option price.

Delta

Delta (Δ) is the rate of change of the option price with respect to the underlying



Delta of an option is often measured in percents.

For example, if an option to buy 100 shares of XYZ stock has Delta of 45% then in the first approximation the value of the option for the small price move in the stock will change the same way as the value of 45 shares of XYZ.

Delta can also be measured in units of the underlying, for stocks in shares.

Delta hedging involves maintaining a Delta neutral portfolio i.e. maintaining of total Delta of option+(stock hedge) zero when stock price change.

Delta hedging of a written (sold) option involves a “**buy high, sell low**” trading rule

The delta of a European call on a non-dividend paying stock is

$$N(d_1)$$

The delta of a European put on the non-dividend paying stock is

$$N(d_1) - 1$$

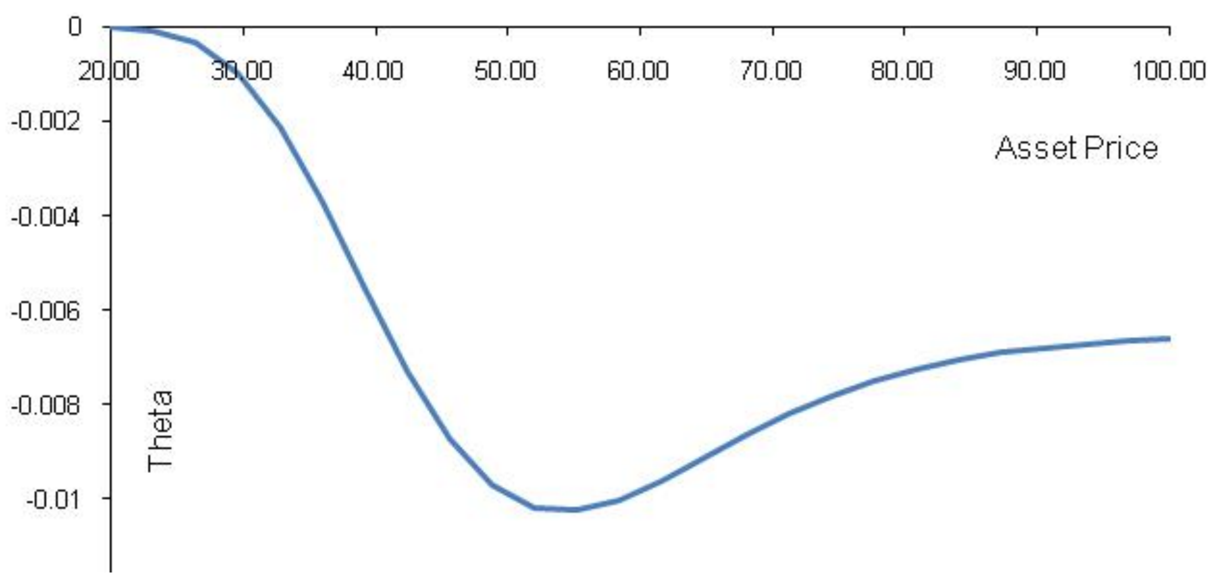
Theta Θ or Time Decay of a derivative security is the rate of change of the value with respect to the passage of time

The theta of a long call or long put is usually negative. If time passes and the price of the underlying asset and its volatility remain the same, the value of a long call or long put option declines

In practice Theta of an option is often calculated for 1 day

$$\Theta = \textbf{Opt price Tomorrow} - \textbf{Opt price Today} = \textbf{Opt price }(T - \tfrac{1}{365})\textbf{ to expiry} - $$
$$- \textbf{Opt price } T\textbf{ to expiry}$$

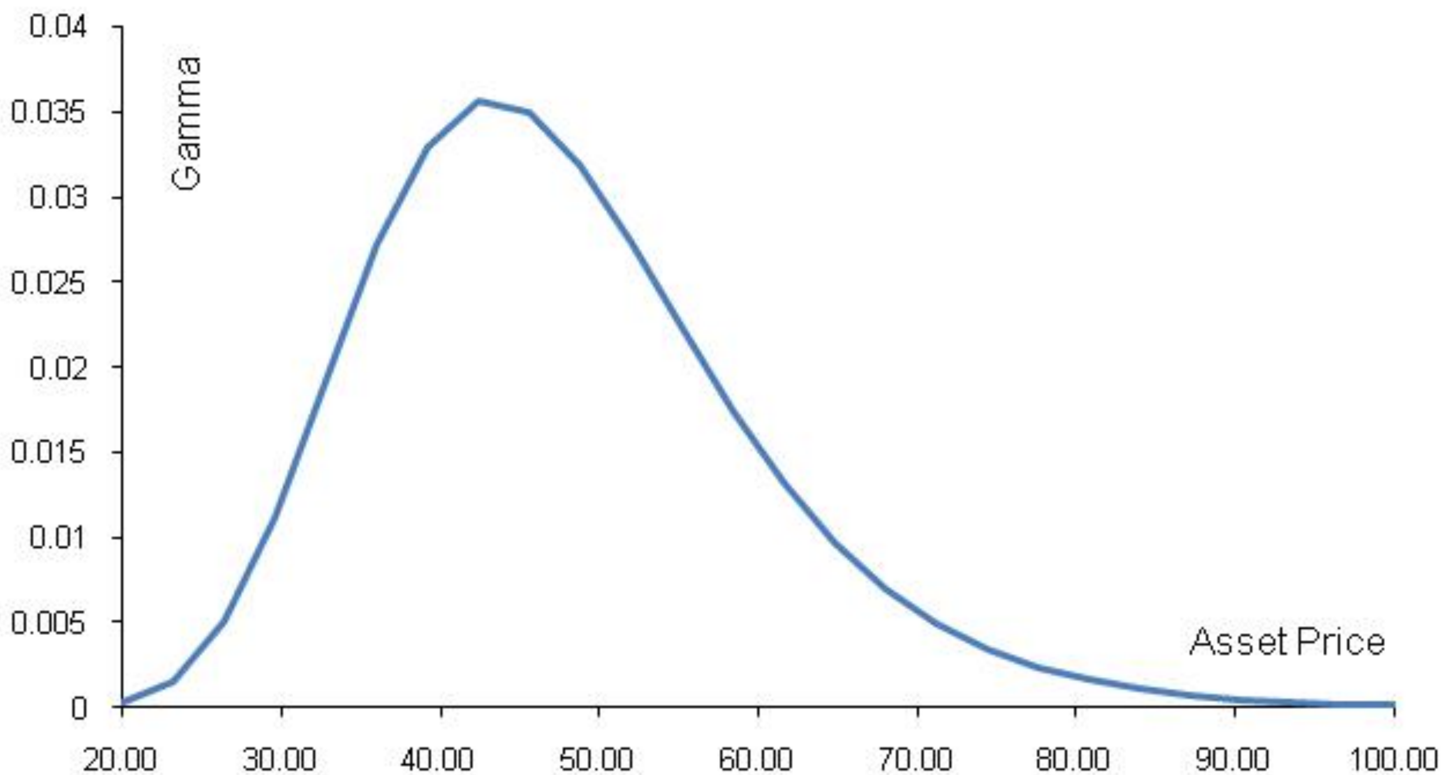
Theta for Call Option: K=50, σ = 25%, r = 5% T = 1



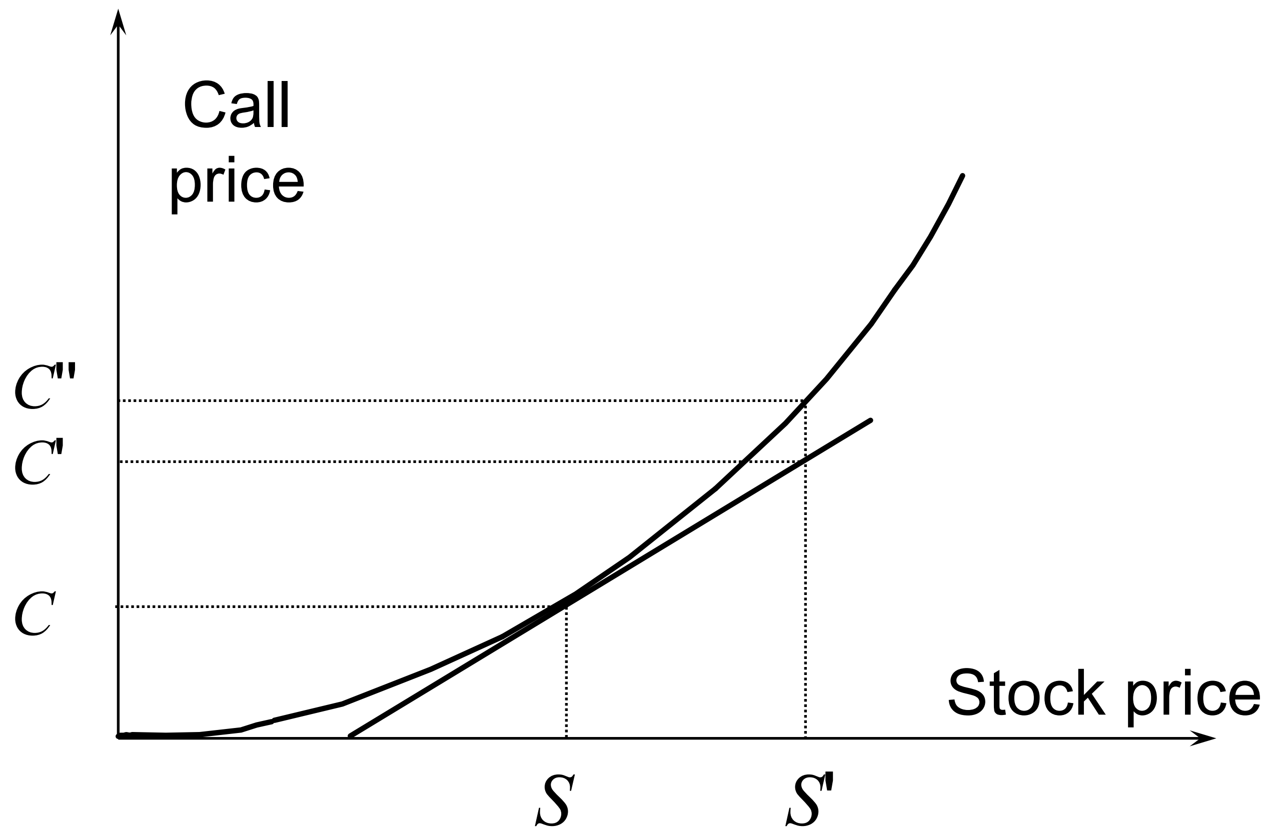
Gamma Γ is the rate of change of delta Δ with respect to the price of the underlying asset

Gamma is greatest for options that are close to the money

Gamma for Call or Put Option: K=50, σ = 25%, r = 5% T = 1

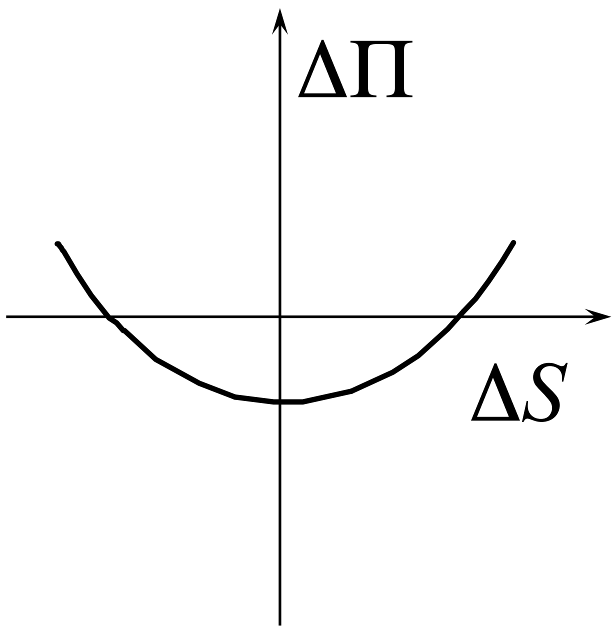


Gamma Corrects Delta Hedging Errors Caused By Curvature

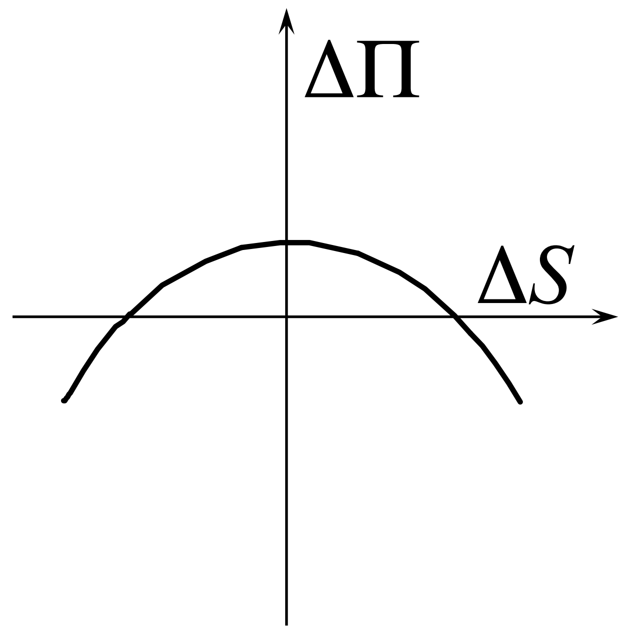


Interpretation of Gamma

For a delta neutral portfolio, $\Delta\Pi \approx \Theta\,\Delta t + \frac{1}{2}\Gamma\Delta S^2$



Positive Gamma



Negative Gamma

Relationship Between Delta, Gamma, and Theta

For a portfolio Π of options on a stock paying a continuous dividend yield at rate q it follows from the Black-Scholes partial differential equation that

$$\Theta + (r - q)S\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

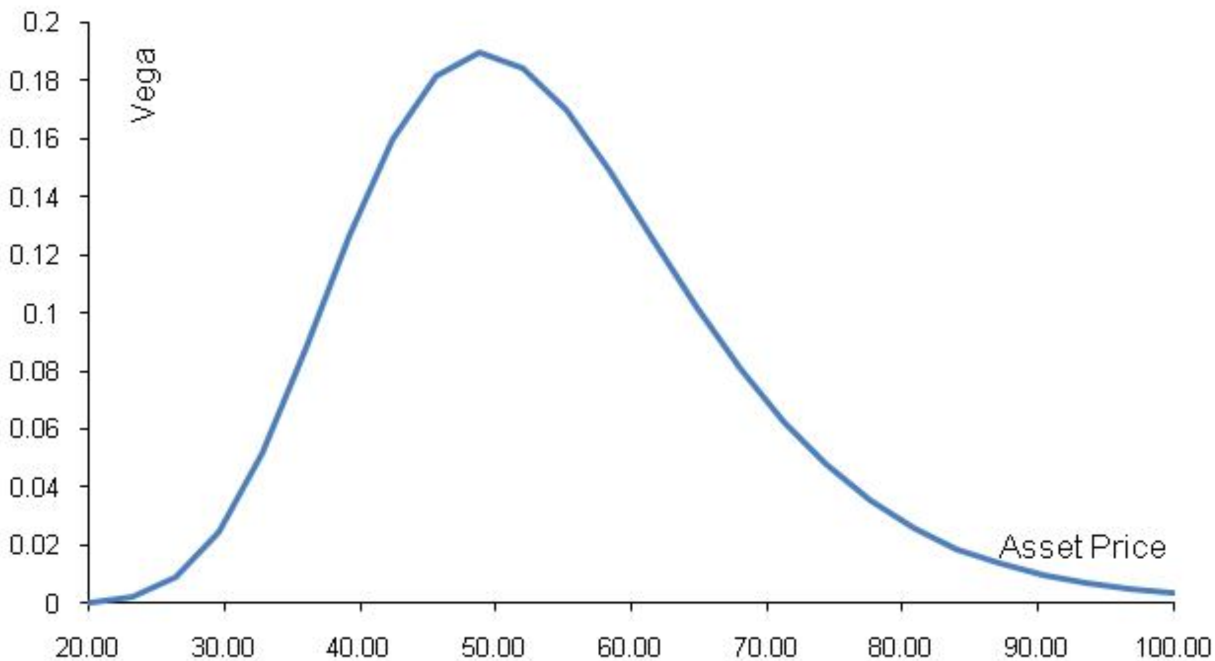
Where S is a stock price also denoted by X sometimes

Vega usually denoted by greek letters Lambda Λ or Nu ν, as there is no greek letter vega, is the rate of change of the value of a derivatives portfolio with respect to volatility

In practice Vega of an option is often calculated for increase in volatility from current value σ, to σ +1%= σ + 0.01.

$$Vega = \text{Option price with } (\sigma + 0.01) \text{ - Option price with } \sigma$$

Vega for Call or Put Option: K=50, σ = 25%, r = 5% T = 1



Taylor Series Expansion of Derivatives Portfolio Change

The change in value of a portfolio of derivatives dependent on an asset as a function of of the asset price S, its volatility σ, and time t

$$\Delta\Pi = \frac{\partial\Pi}{\partial S}\Delta S + \frac{\partial\Pi}{\partial\sigma}\Delta\sigma + \frac{\partial\Pi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2\Pi}{\partial S^2}(\Delta S)^2 + \ldots$$

$$= \text{Delta}\times\Delta S + \text{Vega}\times\Delta\sigma + \text{Theta}\times\Delta t + \frac{1}{2}\text{Gamma}\times(\Delta S)^2 + \ldots$$

Delta × (Change in Underlying Price). This part depends on the delta of the overall position, and, is a directional component. A positive delta means that the position is bullish and benefits from an increase in the underlying price. A negative delta means that the position is bearish and benefits from a decrease in the underlying price.

Vega × Change in Volatility. If the volatility suddenly increases, then, a long derivative position will benefit from it. If the volatility suddenly decreases, then, a long derivative position will lose money as a result of it.

Theta ×(Number of Days Gone by) is the time decay. For a long position in option, theta is negative, and this term is loosing money as time passes. For a short position in option, this term is, respectively, making money.

Gamma" × ½ (Change in Underlying Price)^2. The so-called, "gamma term" which reflects the convexity of the option prices. This is term is very important, since it is arises in hedging.

Greeks for European Options on an Asset that Provides a Continuous Yield at a Rate q, T time to expiration

Greek Letter	Call Option	Put Option
Delta	$e^{-qT}N(d_1)$	$e^{-qT}\left[N(d_1)-1\right]$
Gamma	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$
Theta	$-S_0N'(d_1)\sigma e^{-qT}/\left(2\sqrt{T}\right)$ $+qS_0N(d_1)e^{-qT}-rKe^{-rT}N(d_2)$	$-S_0N'(d_1)\sigma e^{-qT}/\left(2\sqrt{T}\right)$ $+qS_0N(-d_1)e^{-qT}+rKe^{-rT}N(-d_2)$
Vega	$S_0\sqrt{T}N'(d_1)e^{-qT}$	$S_0\sqrt{T}N'(d_1)e^{-qT}$
Rho	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$

Managing Delta, Gamma, and Vega in a Portfolio of Derivatives

Delta can be changed by taking a position in the underlying asset

To adjust gamma and vega it is necessary to take a position in an option or other derivative

Example

	Delta	*Gamma*	*Vega*
Portfolio	0	−5000	−8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

What position in option 1 and the underlying asset will make the portfolio delta and gamma neutral? Answer: Long 10,000 options, short 6000 of the asset

What position in option 1 and the underlying asset will make the portfolio delta and vega neutral? Answer: Long 4000 options, short 2400 of the asset

Example

	Delta	*Gamma*	*Vega*
Portfolio	0	−5000	−8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

What position in option 1, option 2, and the asset will make the portfolio delta, gamma, and vega neutral?
We solve

$$-5000 + 0.5\, w_1 + 0.8\, w_2 = 0$$
$$-8000 + 2.0\, w_1 + 1.2\, w_2 = 0$$

to get $w_1 = 400$ and $w_2 = 6000$. We require long positions of 400 and 6000 in option 1 and option 2. A short position of 3240 in the asset is then required to make the portfolio delta neutral

Hedging in Practice

Traders usually ensure that their portfolios are delta-neutral at least once a day

Whenever the opportunity arises, they improve gamma and vega

There are economies of scale

- As portfolio becomes larger hedging becomes less expensive per option in the portfolio

Delta Hedging.

Every trader having an option portfolio can hedge the directional component of his portfolio, i.e. reduce his local “Delta” to zero.

Once faced with this option he has a choice to do it very frequently, less frequently, or, not do it at all depending on his situation.

Some important factors for deciding on frequency of hedging, are: the distribution of potential profits and losses (or, more particularly, how much risk is he willing to carry), and, total cost of hedging.

Continuous hedging may be very costly since the trader will have to buy or sell the underlying (depending on the concrete path of it) paying the commissions and experiencing bid/offer costs.

It is possible that the underlying will go up and down a lot during a day, causing a lot of trading but finish the day where it started, thus only forcing the trader to pay all the transaction costs without gaining anything end-of-the-day.

Some of the ways the delta-hedging may be implemented are: in constant increments of price change (for example, every 100 shares in Delta); in constant increments in time (end-of-the-day every day), etc.

Another possible cost of delta-hegding is not knowing exactly what the volatility is. If a trader is using his delta-hedging rule with a volatility error then he will also accumulate the P/L changes due to this discrepancy.

As one can see from the previous formulas, once the option is delta-hedged, under normal circumstances and on short time changes the P/L becomes dominated by the gamma term. This is why the delta-hedging is, sometimes called, “trading gamma”.

Delta Hedging Example. Spreadsheet and Graph.

The hedger owns a call with strike price 100 on a stock paying no dividends. The call has originally 20 days to expiration (it expires at the end of day 20), volatility=45%, *r* =5%. The price of this call on day 1 is 4.333, Delta=53.1%.

The hedger owns 10 contracts for 100 shares each, i.e options for 1000 shares of stock.

On day 1 the hedgers Delta is equivalent to 531 shares so the hedger sell short 531 shares at 100 to hedge. (receiving proceeds of 53,100 and investing them at 5%). After the original sell combined Delta of options and short stock holdings is 0.

On day 2 stock price went to 102, Delta of an options increased to 60.6% equivalent to 606 shares, but the hedger has sold short only totally 531 share. So his combined Delta is equivalent to +75 shares. To maintain Delta neutrality the hedger must sell additional 75 shares at that day current price 102.

A standard term in finance to denote change in value of a portfolio is PL (pronounced Pee n’ El) which is short for Profit and Loss.

PL of the hedge part of portfolio (short 531 share) when the stock moves from 100 to 102 is equal to (–532)*(102-100)= - 1062. The hedge part of portfolio will lose money.

However the option part of portfolio would make money. When the stock moves from 100 to 102, the call price will increase from 4.333 to 5.358. The change 1.025 times 10 contracts for 100 shares per contract is 1025.

Additional PL will come to the portfolio from investing 53,100 dollars of short sale proceeds at 5% for 1 day (about 7 dollars).

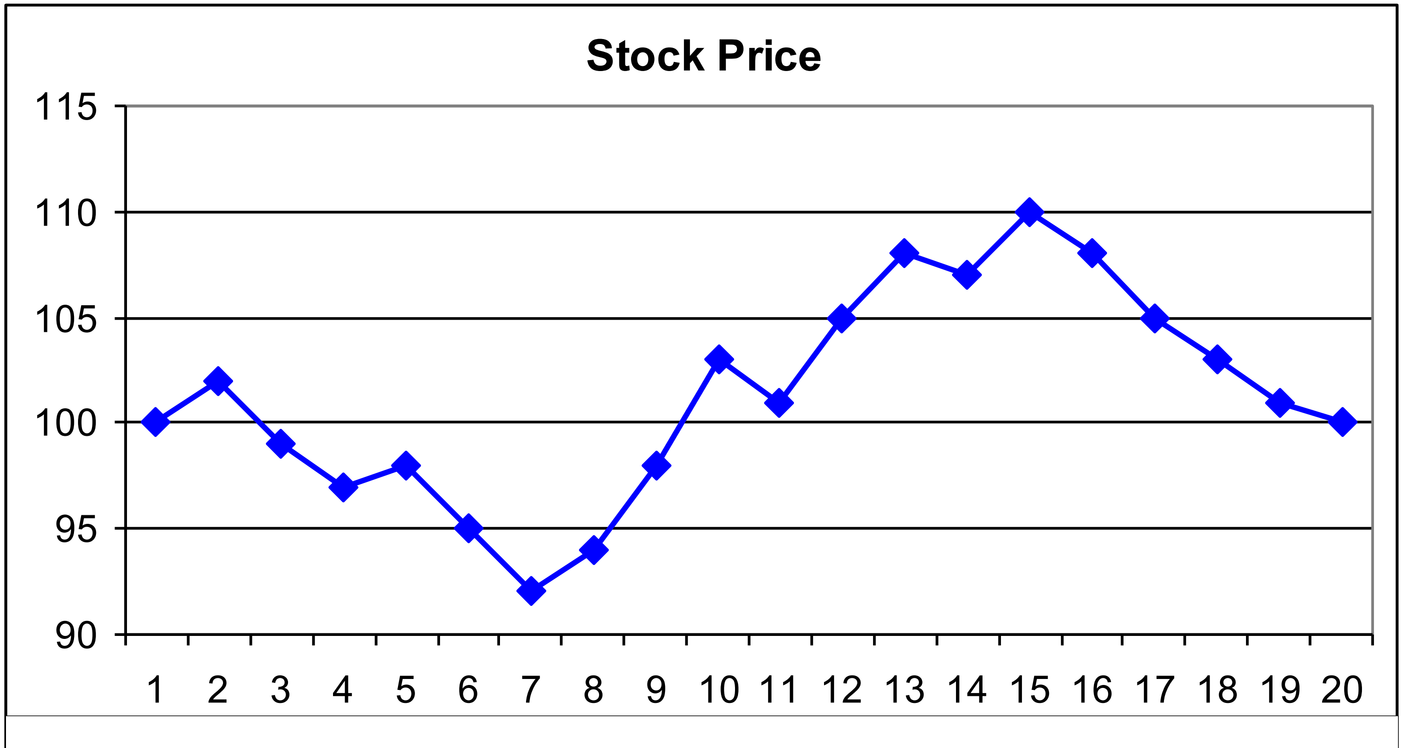
When the owner of the call readjust the hedge he will sell the stock after the stock went up and buy after the stock went down. Buying low and selling high will create a cumulative hedging PL.

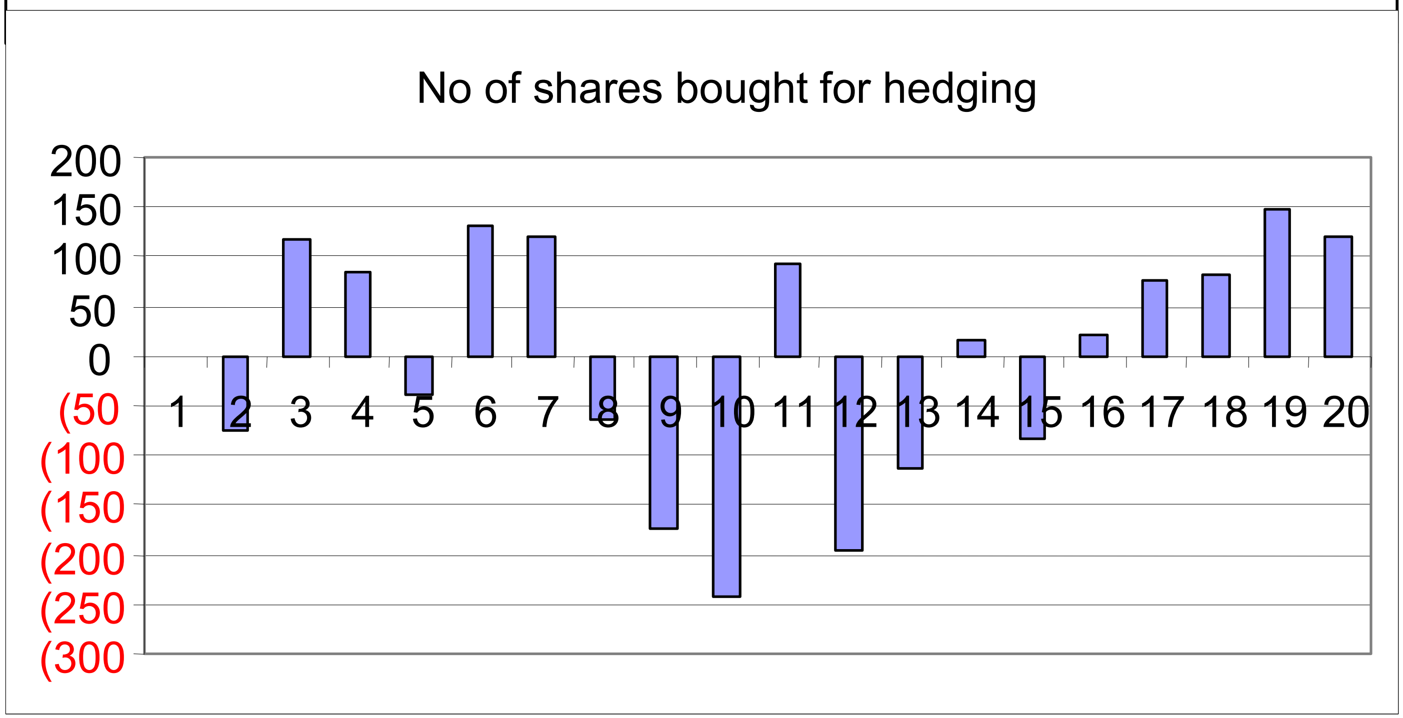
The spreadsheet below contains formulas and numbers for hedging that was explained here.

A3 B3 C3 D3 E3 F3 G3 H3

Day No	Stock Price	Option Price	Option Delta	Option Delta in Shares	No of shares Bought	Total hedge Shares	Cumulative Hedge PL	Option Value +Cum Hedge PL
1	100	4.3334	0.5313	=D6*1000		-531	0	=C6*1000+H6
2	102	5.3581	0.6061	=D7*1000	=E6-E7	=G6+F7	=H6+G6*(B7-B6)	=C7*1000+H7
3	99	3.5952	0.4896	=D8*1000	=E7-E8	=G7+F8	=H7+G7*(B8-B7)	=C8*1000+H8
4	97	2.5856	0.4047	=D9*1000	=E8-E9	=G8+F9	=H8+G8*(B9-B8)	=C9*1000+H9
5	98	2.8932	0.4427	=D10*1000	=E9-E10	=G9+F10	=H9+G9*(B10-B9)	=C10*1000+H10
6	95	1.6499	0.3107	=D11*1000	=E10-E11	=G10+F11	=H10+G10*(B11-B10)	=C11*1000+H11
7	92	0.8107	0.1895	=D12*1000	=E11-E12	=G11+F12	=H11+G11*(B12-B11)	=C12*1000+H12
8	94	1.1600	0.2531	=D13*1000	=E12-E13	=G12+F13	=H12+G12*(B13-B12)	=C13*1000+H13
9	98	2.3844	0.4260	=D14*1000	=E13-E14	=G13+F14	=H13+G13*(B14-B13)	=C14*1000+H14
10	103	4.9823	0.6687	=D15*1000	=E14-E15	=G14+F15	=H14+G14*(B15-B14)	=C15*1000+H15
11	101	3.5868	0.5749	=D16*1000	=E15-E16	=G15+F16	=H15+G15*(B16-B15)	=C16*1000+H16
12	105	6.1432	0.7711	=D17*1000	=E16-E17	=G16+F17	=H16+G16*(B17-B16)	=C17*1000+H17
13	108	8.5208	0.8856	=D18*1000	=E17-E18	=G17+F18	=H17+G17*(B18-B17)	=C18*1000+H18
14	107	7.5370	0.8709	=D19*1000	=E18-E19	=G18+F19	=H18+G18*(B19-B18)	=C19*1000+H19
15	110	10.202	0.9547	=D20*1000	=E19-E20	=G19+F20	=H19+G19*(B20-B19)	=C20*1000+H20
16	108	8.2385	0.9329	=D21*1000	=E20-E21	=G20+F21	=H20+G20*(B21-B20)	=C21*1000+H21
17	105	5.4219	0.8574	=D22*1000	=E21-E22	=G21+F22	=H21+G21*(B22-B21)	=C22*1000+H22
18	103	3.5987	0.7742	=D23*1000	=E22-E23	=G22+F23	=H22+G22*(B23-B22)	=C23*1000+H23
19	101	1.9115	0.6263	=D24*1000	=E23-E24	=G23+F24	=H23+G23*(B24-B23)	=C24*1000+H24
20	100	0.9465	0.5068	=D25*1000	=E24-E25	=G24+F25	=H24+G24*(B25-B24)	=C25*1000+H25

Day No	Stock Price	Option Price	Option Delta	Option Delta in Shares	No of shares Bought	Total hedge Shares	Cumulative Hedge PL	Option Value +Cum Hedge PL
1	100	4.33	53%	531		(531)	0	4,333
2	102	5.36	61%	606	(75)	(606)	(1,062)	4,296
3	99	3.60	49%	490	117	(489)	756	4,351
4	97	2.59	40%	405	85	(404)	1,734	4,320
5	98	2.89	44%	443	(38)	(442)	1,330	4,223
6	95	1.65	31%	311	132	(310)	2,657	4,307
7	92	0.81	19%	190	121	(189)	3,588	4,399
8	94	1.16	25%	253	(64)	(253)	3,210	4,370
9	98	2.38	43%	426	(173)	(426)	2,199	4,583
10	103	4.98	67%	669	(243)	(668)	70	5,053
11	101	3.59	57%	575	94	(575)	1,407	4,994
12	105	6.14	77%	771	(196)	(771)	(892)	5,252
13	108	8.52	89%	886	(115)	(885)	(3,204)	5,317
14	107	7.54	87%	871	15	(871)	(2,319)	5,219
15	110	10.20	95%	955	(84)	(954)	(4,931)	5,272
16	108	8.24	93%	933	22	(933)	(3,022)	5,217
17	105	5.42	86%	857	76	(857)	(224)	5,198
18	103	3.60	77%	774	83	(774)	1,491	5,089
19	101	1.91	63%	626	148	(626)	3,039	4,950


Stock Price
115
110
105
100
95
90
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20


No of shares bought for hedging
200
150
100
50
0
(50
(100
(150
(200
(250
(300
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

Option Price as a Cost of Delta Hedging.
Delta Hedging Cost is Independent on Trajectory of Stock Price

Consider a case of delta hedging of a European Call option under an assumption of zero transaction costs (no commissions, no bid/offer). Assume that the assumptions of the Black and Scholes formula are true: volatility is constant in time and across different strikes, stock distribution is log-normal, and the investment world is risk-neutral.

Let us, say, we delta hedge a call with different frequencies in time, for example, every 1024 minutes, 512 minutes, 256 minutes, etc. Consider a limit of hedging frequency going to zero. Then the error in delta hedging will shrink to zero.

Thus delta hedging of the long call will produce the so-called synthetic short call option. This argument does not depend on a path the stock is taking. Now, consider, also, the same convergence process on a sequence of admissible stock paths, i.e. all of those paths start at, obey the geometric Brownian motion with constant σ and μ.

Consider the multitude of hedging arising P/L’s at the final point. For every hedging frequency, the multitude of admissible paths will produce a distribution of P/L’s, which will be peaked at a single value, equal to namely the option price.

In the limit of hedging frequency going to zero this distribution of P/L’s will converge to a single value which is equal to the theoretical option price c.

In that sense, the cost of delta hedging (synthetic option) is converging to the actual option independently on the price path.

If the transaction costs are introduced, the whole procedure breaks down. Hedging an option in the presence of transaction costs of different is a more difficult problem.

Arbitrage. Option Model Price = Delta Hedging Cost

By the same token, consider two strategies: from some initial time to some final moment in time we will carry two instruments: one is an actual call option, another is a P/L of a dynamic delta hedging trading strategy according to the exact formula for delta with a certain hedging frequency.

If, in the limit of hedging frequency shrinking to zero, these two portfolios produce systematically different results, then there is an opportunity for an arbitrage.

A group of traders placing riskless arbitrage trades would eliminate the difference thus proving the result.

Futures Contract Can Be Used for Hedging

The delta of a futures contract on an asset paying a yield at rate q is $e^{(r-q)T}$ times the delta of a spot contract

The position required in futures for delta hedging is therefore $e^{-(r-q)T}$ times the position required in the corresponding spot contract

Portfolio Insurance

In 1987 many stock portfolio managers attempted to create a put option on a portfolio synthetically using futures

This involves initially selling enough of the portfolio (or of index futures) to match the Δ of the put option

As the value of the portfolio increases, the Δ of the put becomes less negative and some of the original portfolio is repurchased

As the value of the portfolio decreases, the Δ of the put becomes more negative and more of the portfolio must be sold

The strategy did not work well on October 19, 1987

Example of the Bank Option Hedge

A bank has sold for \$300,000 a European call option on 100,000 shares of a non-dividend paying stock

S_0 = 49, K = 50, r = 5%, σ = 20%, T = 20 weeks

The Black-Scholes-Merton value of the option is \$240,000

How does the bank hedge its risk to lock in a \$60,000 profit?

Naked and Covered Positions

Naked position

Take no action

Covered position

Buy 100,000 shares today

Bank Hedge

Bank would be hedged with the position:

short 1000 options for 100 shares

buy 60,000 shares as delta of option is 60%

Gain/loss on the option position is offset by loss/ gain on stock position

Delta changes as stock price changes and time passes

Hedge position must therefore be rebalanced

Quiz

1. An investor owns a call with delta 50% and the stock price goes up 2 dollars then the option price

    a. Will go up approximately 1 dollar.

    b. Will go up approximately 2 dollars.

    c. Will go down approximately 1 dollar.

    d. Will go down approximately 2 dollars.

2. An investor owns a put with delta 30% and the stock price goes up 2 dollars then the option price

a. Will go up approximately 60 cents.

b. Will go up approximately 30 cents.

c. Will go down approximately 60 cents.

d. Will go down approximately 30 cents.

3. An investor sold a put. His gamma is

a. Negative.

b. Positive.

c. Can be positive or negative.

d. Zero.

4. An investor sold a put. He is delta hedging it.

a. He will buy stock low sell high.

b. He will buy stock high and sell low.

c. He may do either a or b depending on other parameters.

d. He will buy stock at a strike price.