Math Methods – Financial Price Analysis

Mathematics GR5360

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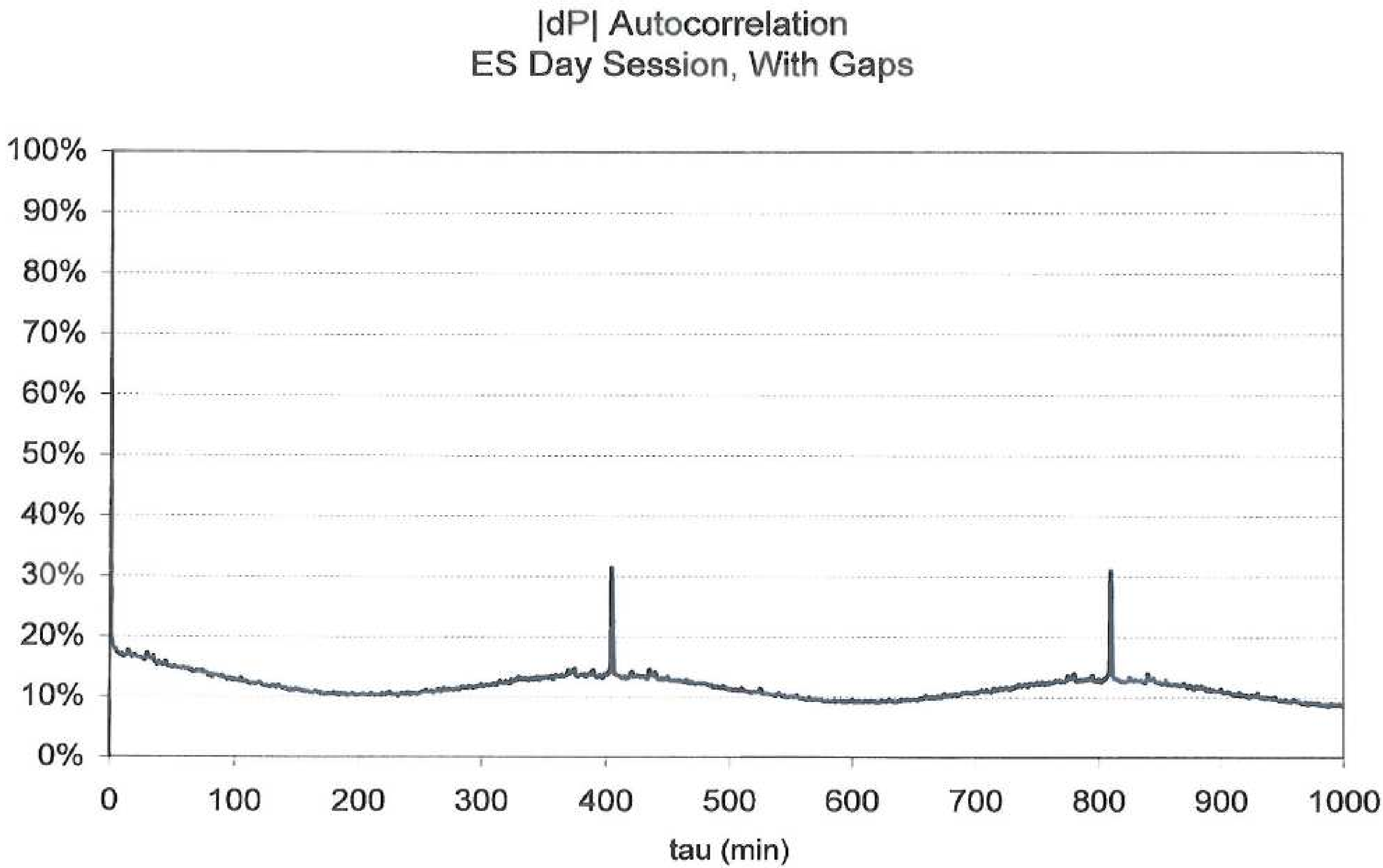
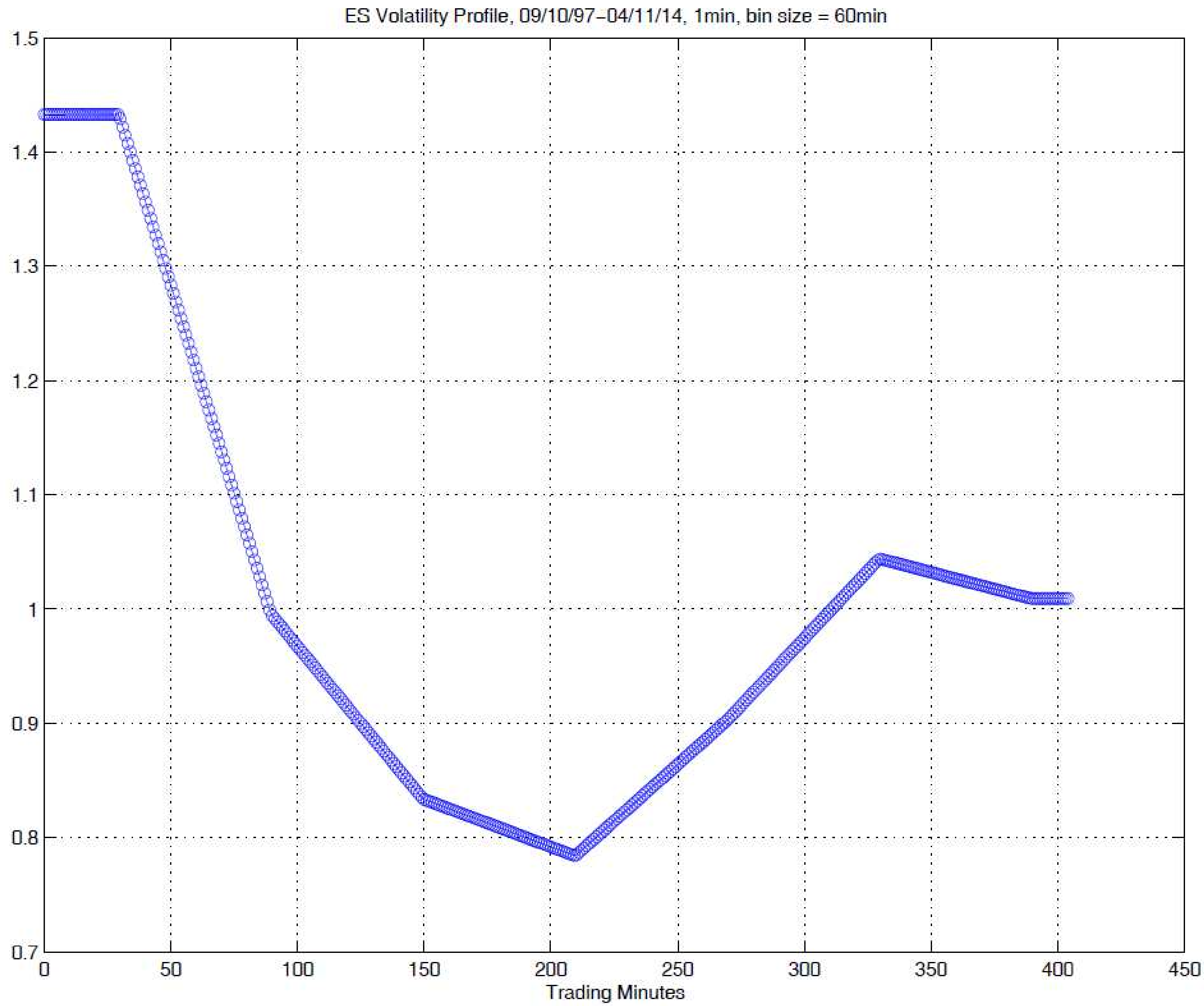
Intra-day Seasonality

- Naïve measurements of correlation functions of local volatility $|\Delta p|$ reveal highly repetitive structure (see next chart);
- This indicates that the price changes have some dynamic (predictable) in addition to fluctuating or random structure with a period of 1 day;
- Furthermore, this indicates that there exists a repetitive pattern of "intraday seasonality" of volatility as a regular (non-random) function of time within a day;
- If measured, it can be used to normalize the price changes through a multiplicative transformation, with the remaining price fluctuations free from such seasonality;
- Such de-seasonalization procedure can reveal deeper statistical properties which would remain "invisible" without such transformation.

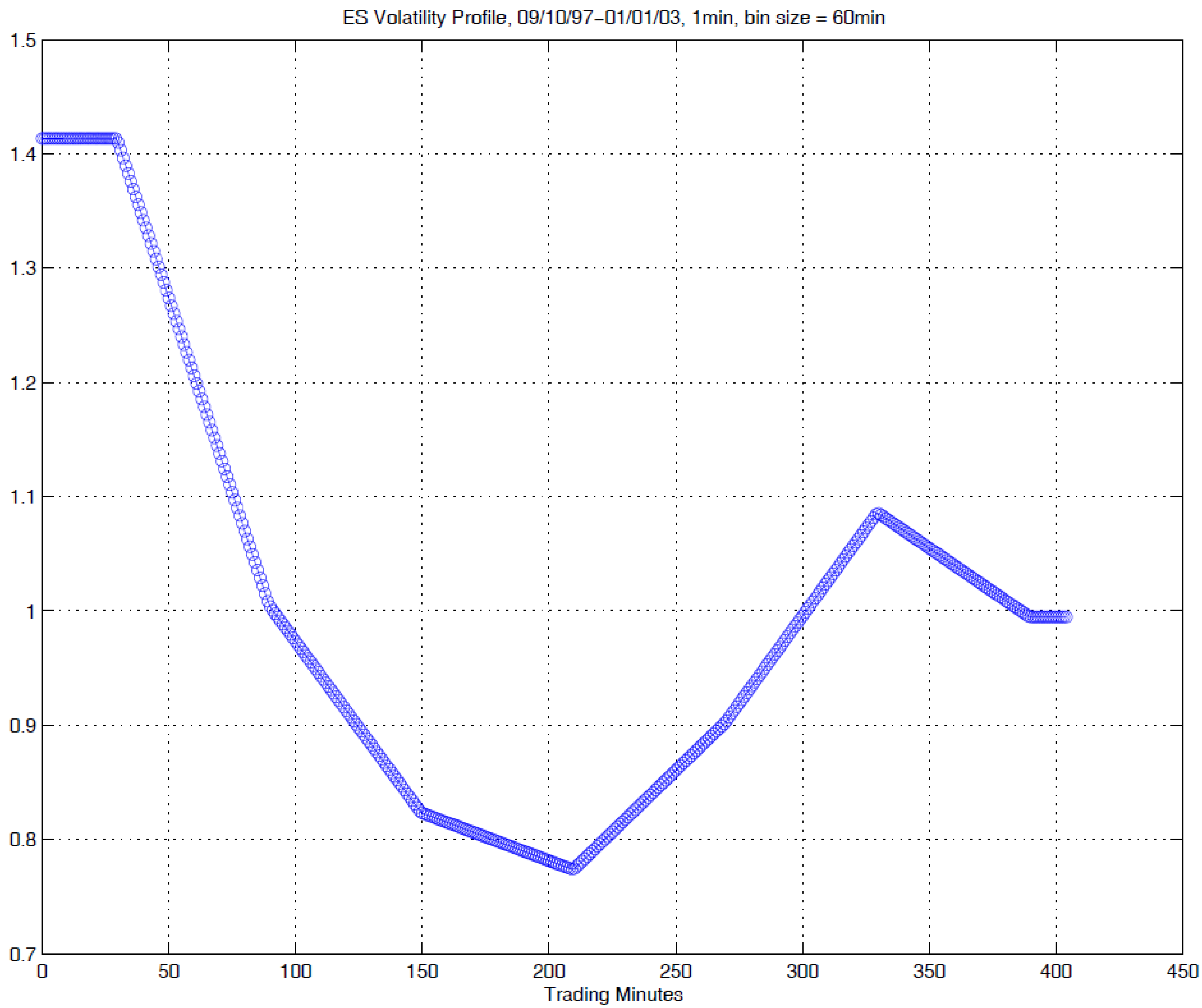
Naïve Measurements of Auto-Correlation Function of Local Volatility |ΔP|



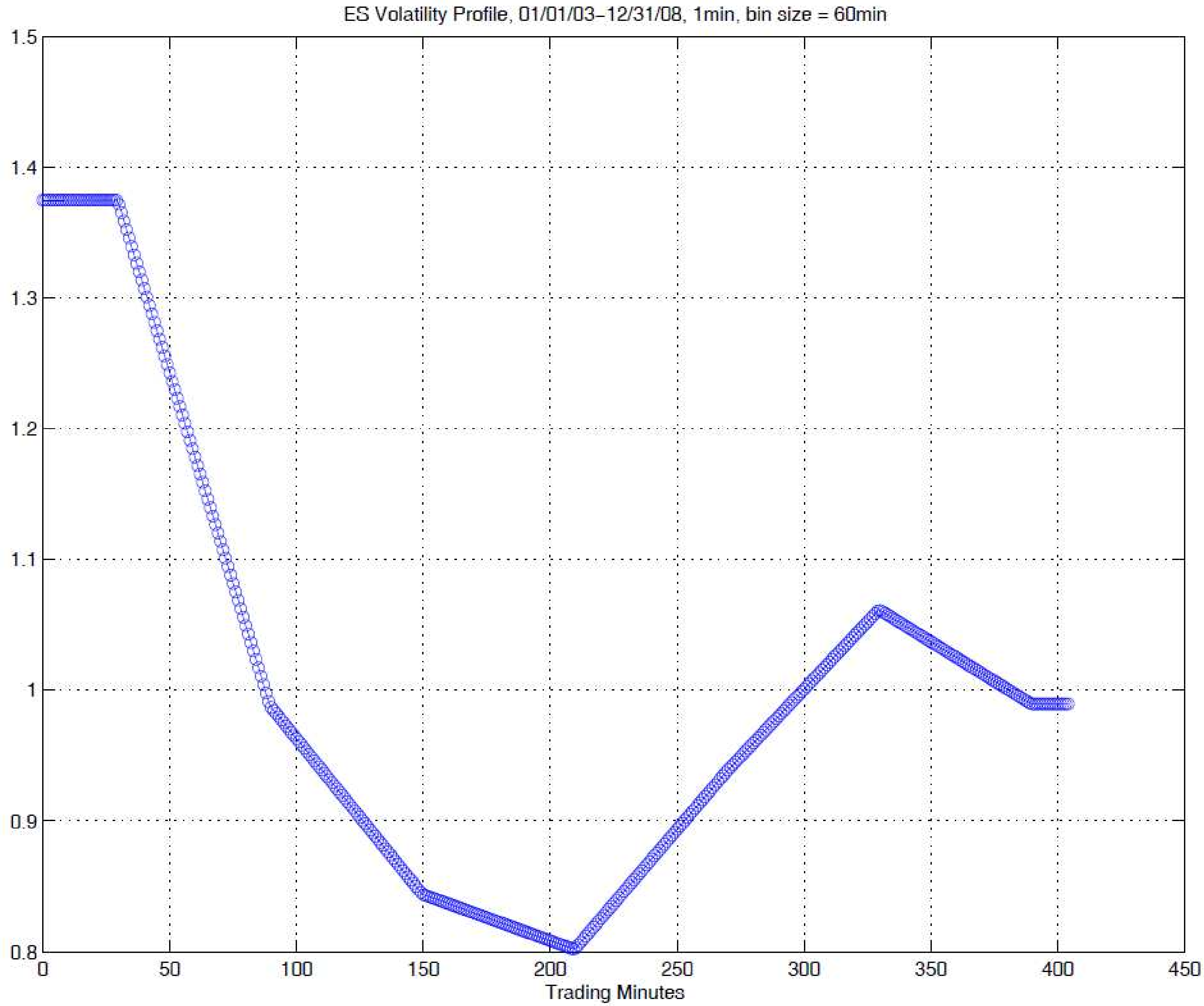
Intra-day Seasonality Profile for |ΔP| at 1-hour bins, sample 1997-2014



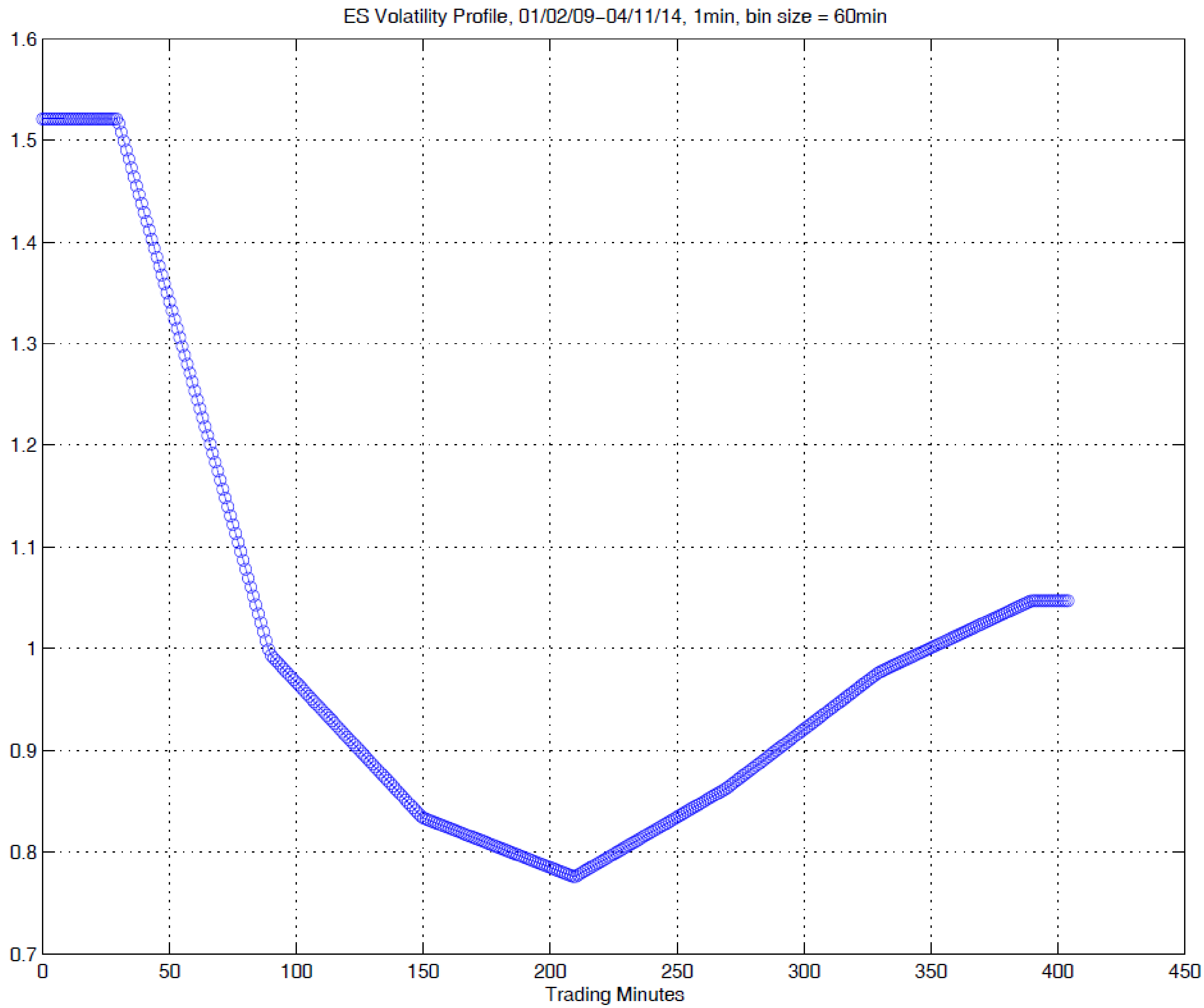
Intra-day Seasonality Profile for |ΔP| at 1-hour bins, sub-sample 1997-2002



Intra-day Seasonality Profile for |ΔP| at 1-hour bins, sub-sample 2003-2008



Intra-day Seasonality Profile for |ΔP| at 1-hour bins, sub-sample 2009-2014



Long Memory of a Random Variable and Relationship to Energy Spectrum

To remind you, a stochastic process $x(t)$ is stationary if its PDF $P(x)$ is independent of the time shift $t \to t + \Delta t$. Then autocorrelation function

$$R(t_1, t_2) \equiv \overline{x(t_1) \cdot x(t_2)} = R(\tau), \tau = t_2 - t_1.$$

Physical meaning of autocorrelation :

$$\tau^* = \int_0^{+\infty} R(\tau) \cdot d\tau = \begin{cases} \text{finite - "short" memory;} \\ \text{infinite - "long" memory;} \end{cases}$$

is a characteristic auto - correlation time scale. For example, a power - law autocorrelation function $R(\tau) = \tau^{\eta - 1}$, with $0 < \eta \le 1$ has a "long memory".

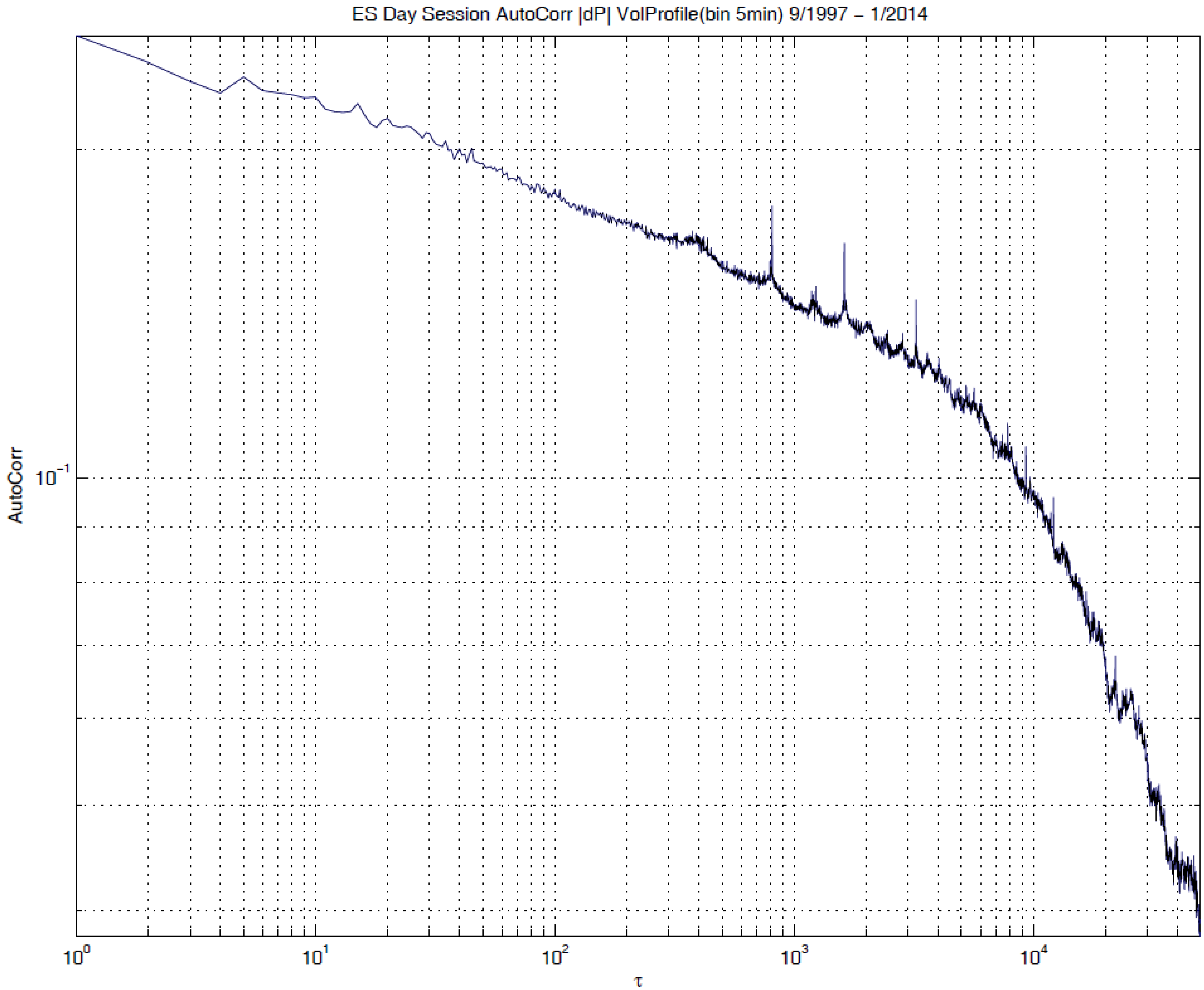
Relationship of Autocorrelation to Energy Spectrum or the Wiener - Khinchin theorem :

$$E(\nu) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} R(\tau) \cdot e^{-i\nu\tau} \cdot d\tau,$$ or the energy spectrum is the Fourier transform of the autocorrelation function. For example, for a short - range autocorrelated process with

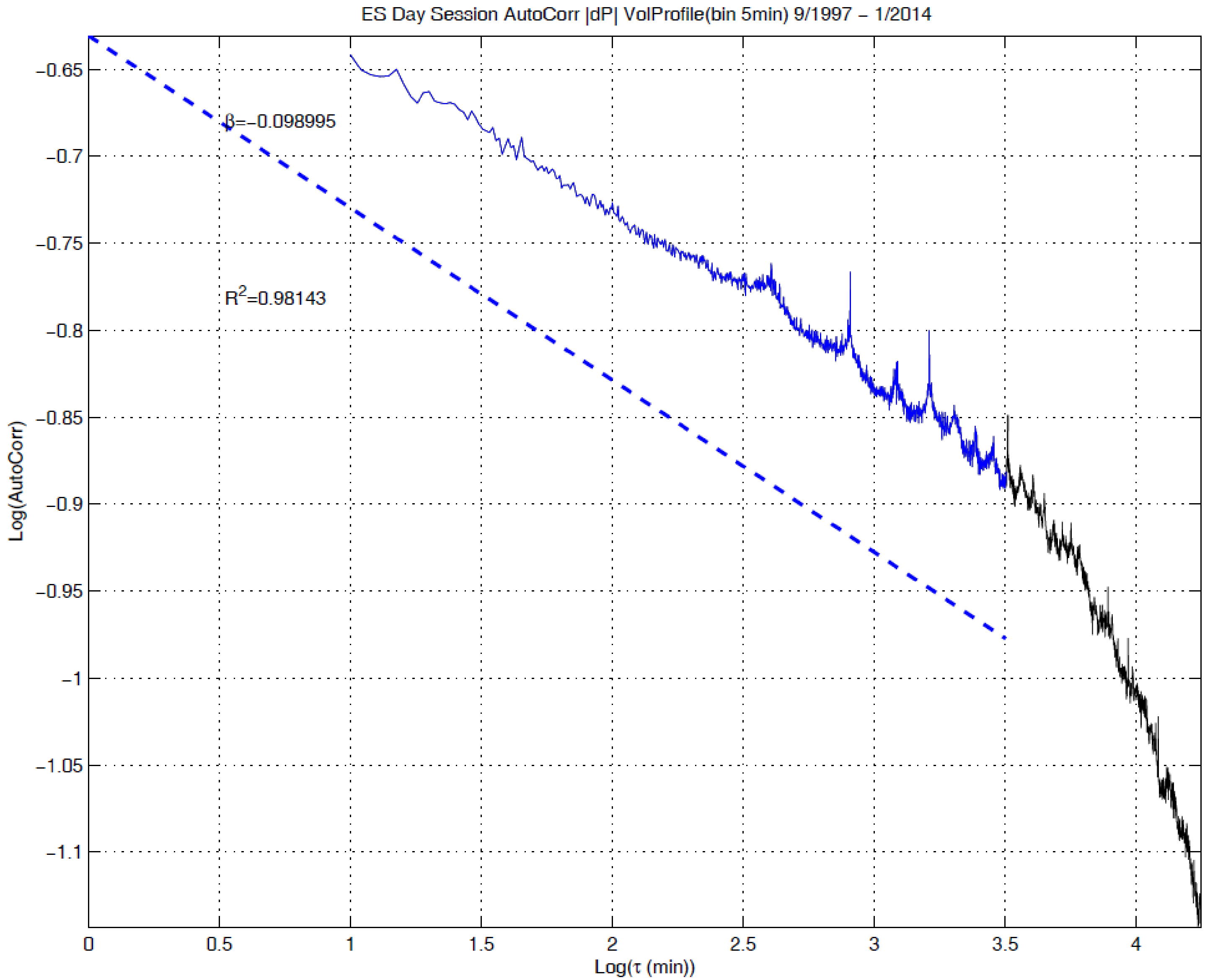
$$R(\tau) = \sigma^2 \cdot e^{-\frac{|\tau|}{\tau_c}},$$ we get $E(\nu) =$

$$= \frac{2\sigma^2 \tau_c}{1 + (2\pi\nu\tau_c)^2} \to \begin{cases} 2\sigma^2 \tau_c, \text{ for } \nu \to 0+, \text{"white noise";} \\ \dfrac{\sigma^2}{2\pi^2 \tau_c} \cdot \dfrac{1}{\nu^2}, \text{ for } \nu \to +\infty, \text{"Random Walk".} \end{cases}$$

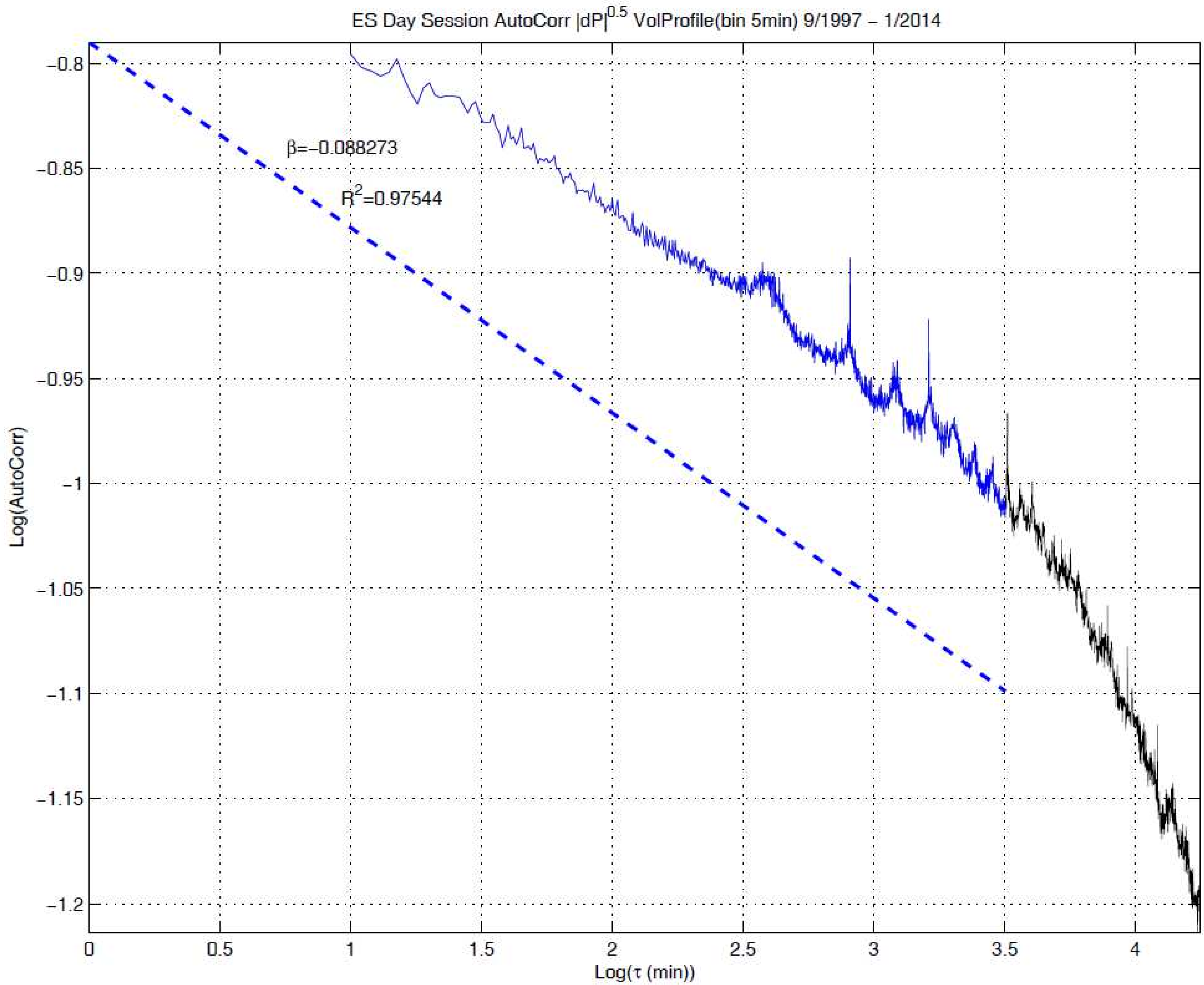
Autocorrelation Function for Local Volatility |ΔP|



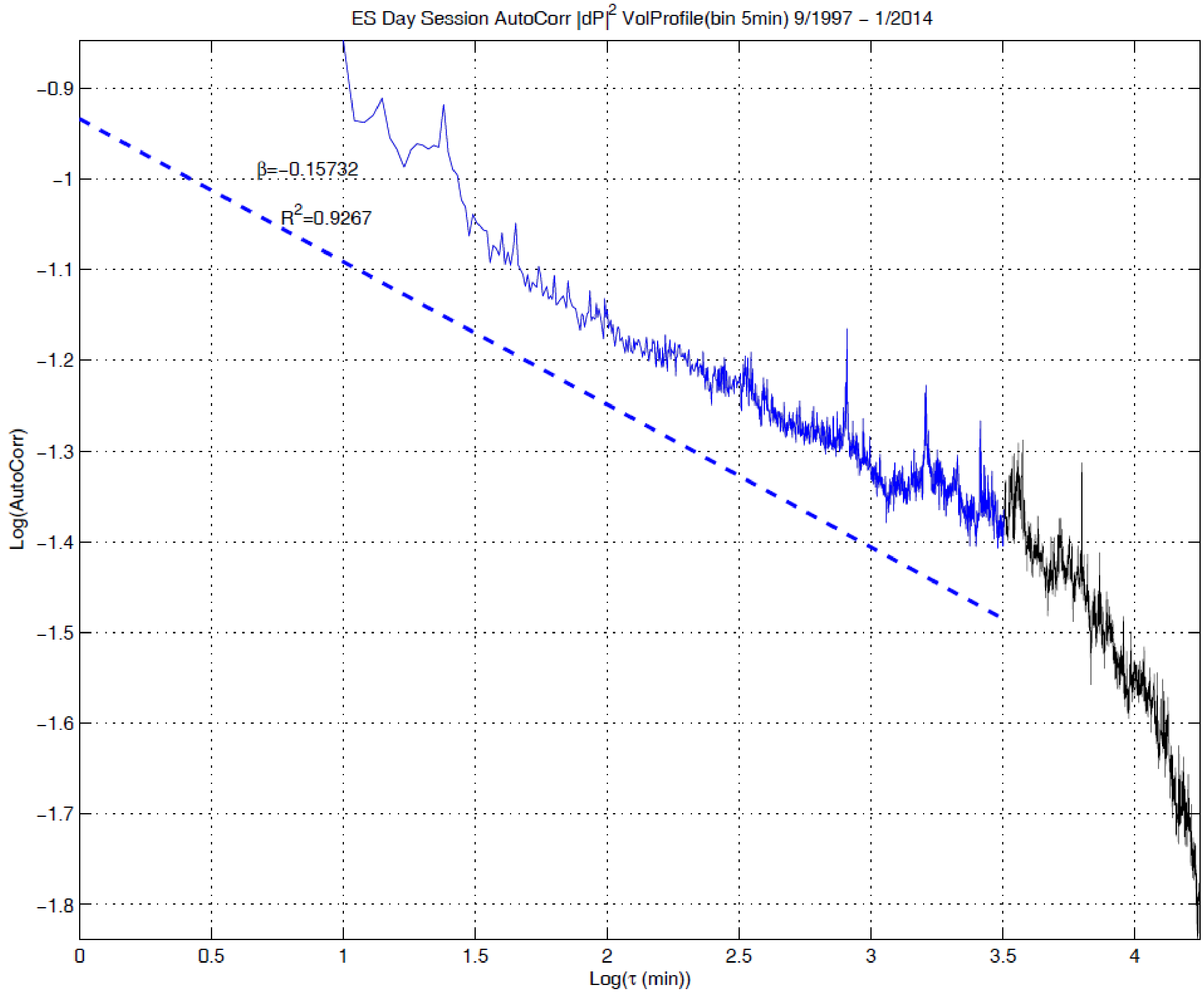
Long Memory of Local Volatility |ΔP|



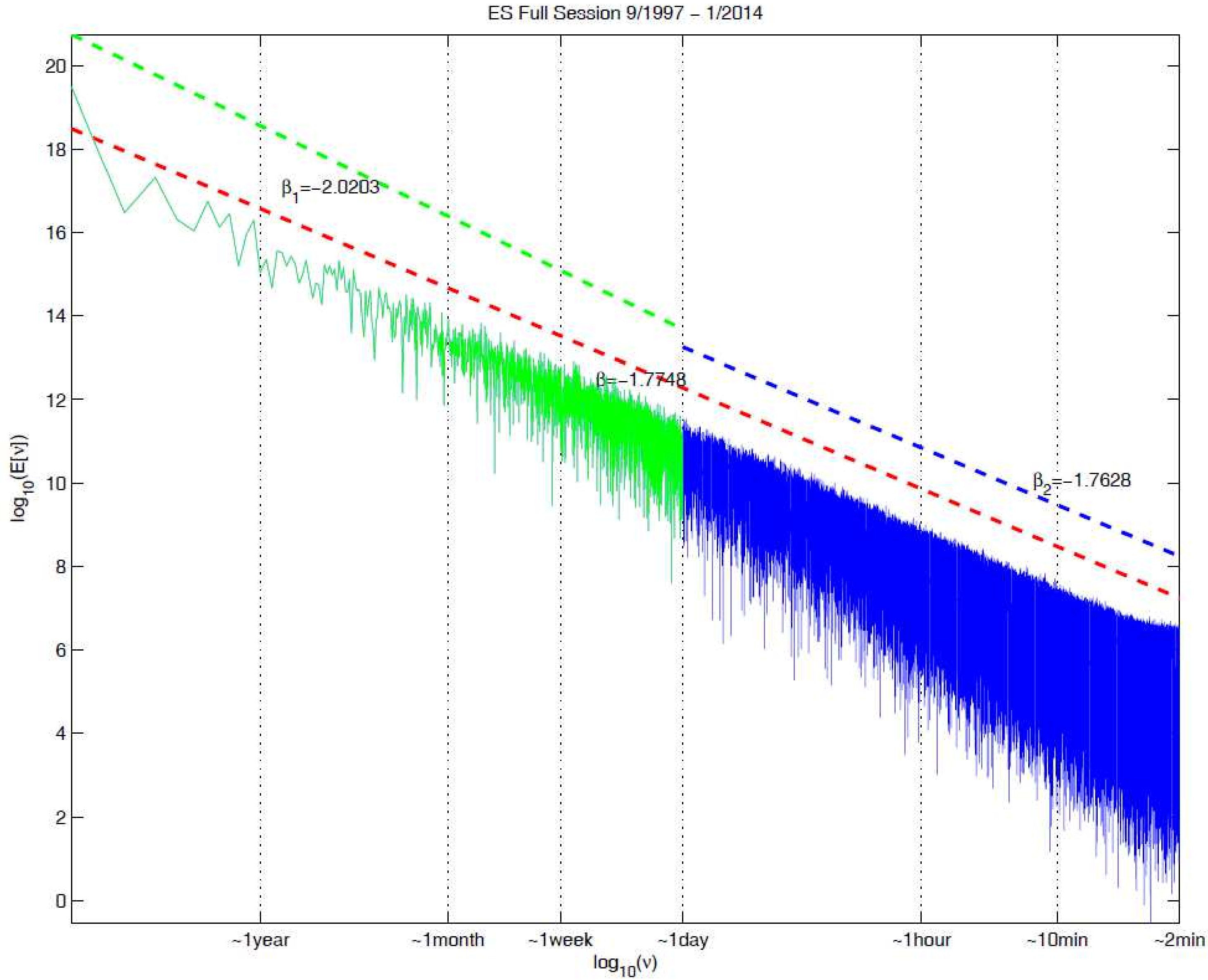
Long Memory of Local Volatility $|\Delta P|^{0.5}$



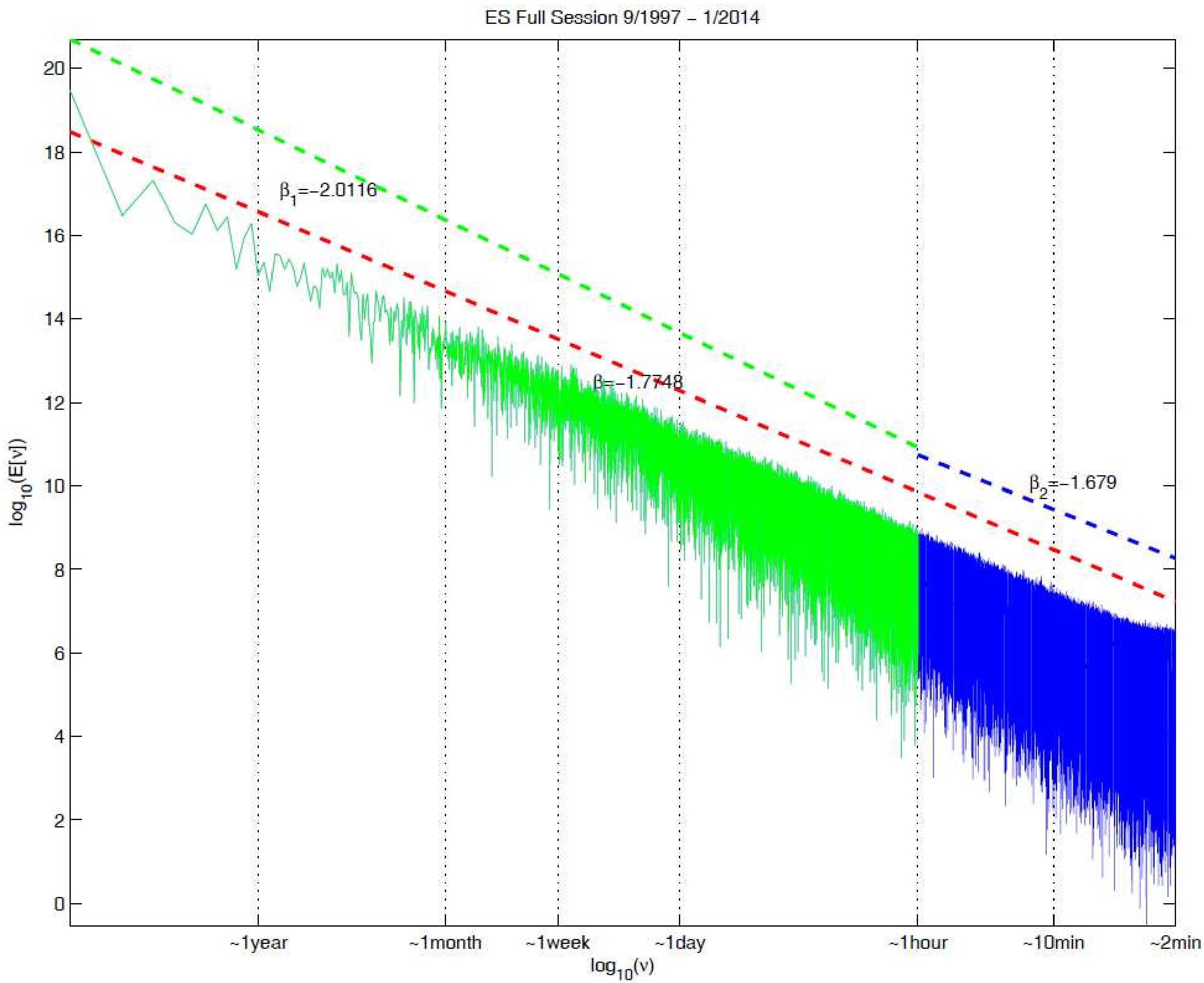
Long Memory of Local Volatility $|\Delta P|^2$



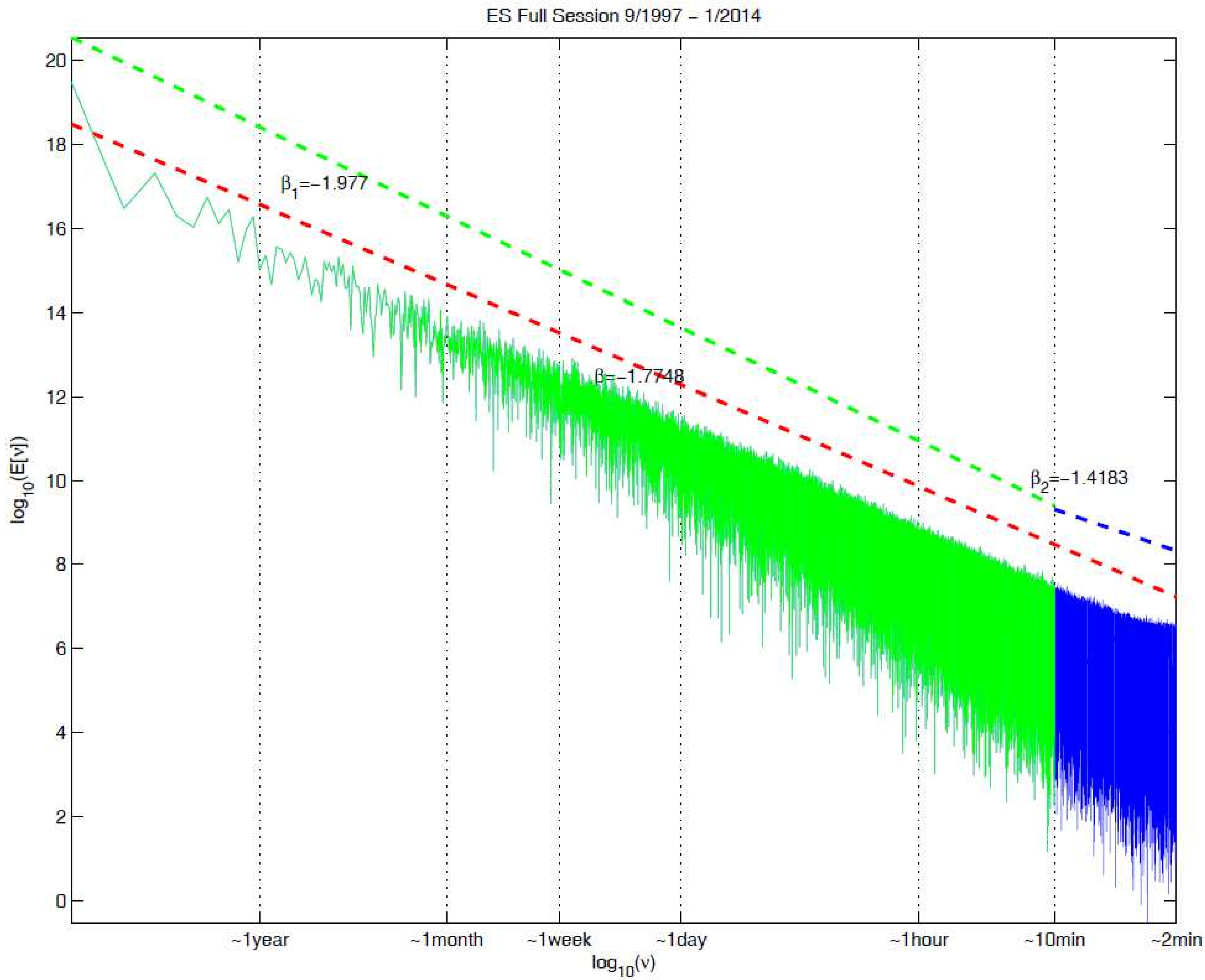
Energy Spectrum Measurements for ΔP and Slopes Divide at 1 Day



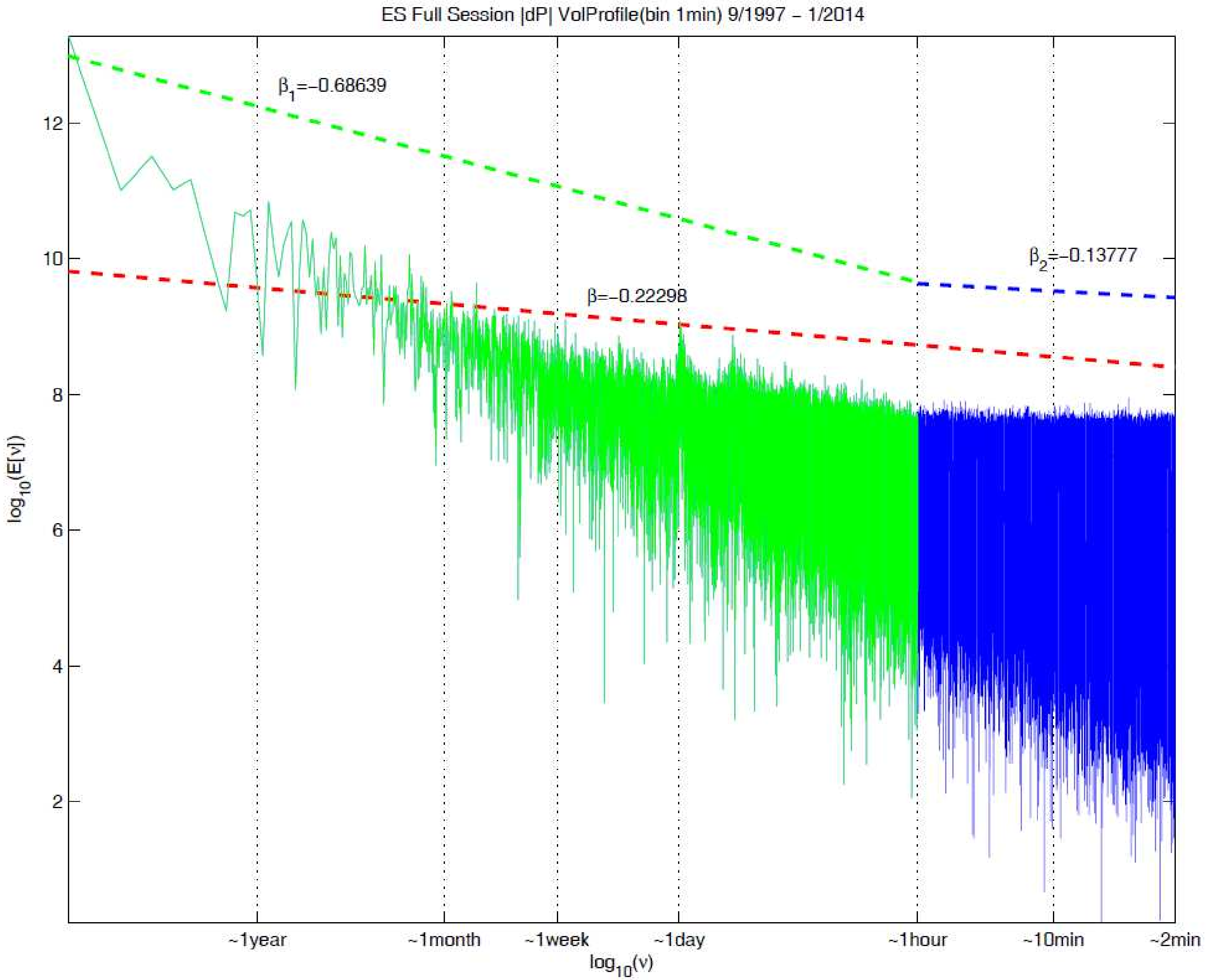
Energy Spectrum Measurements for ΔP and Slopes Divide at 1 Hour



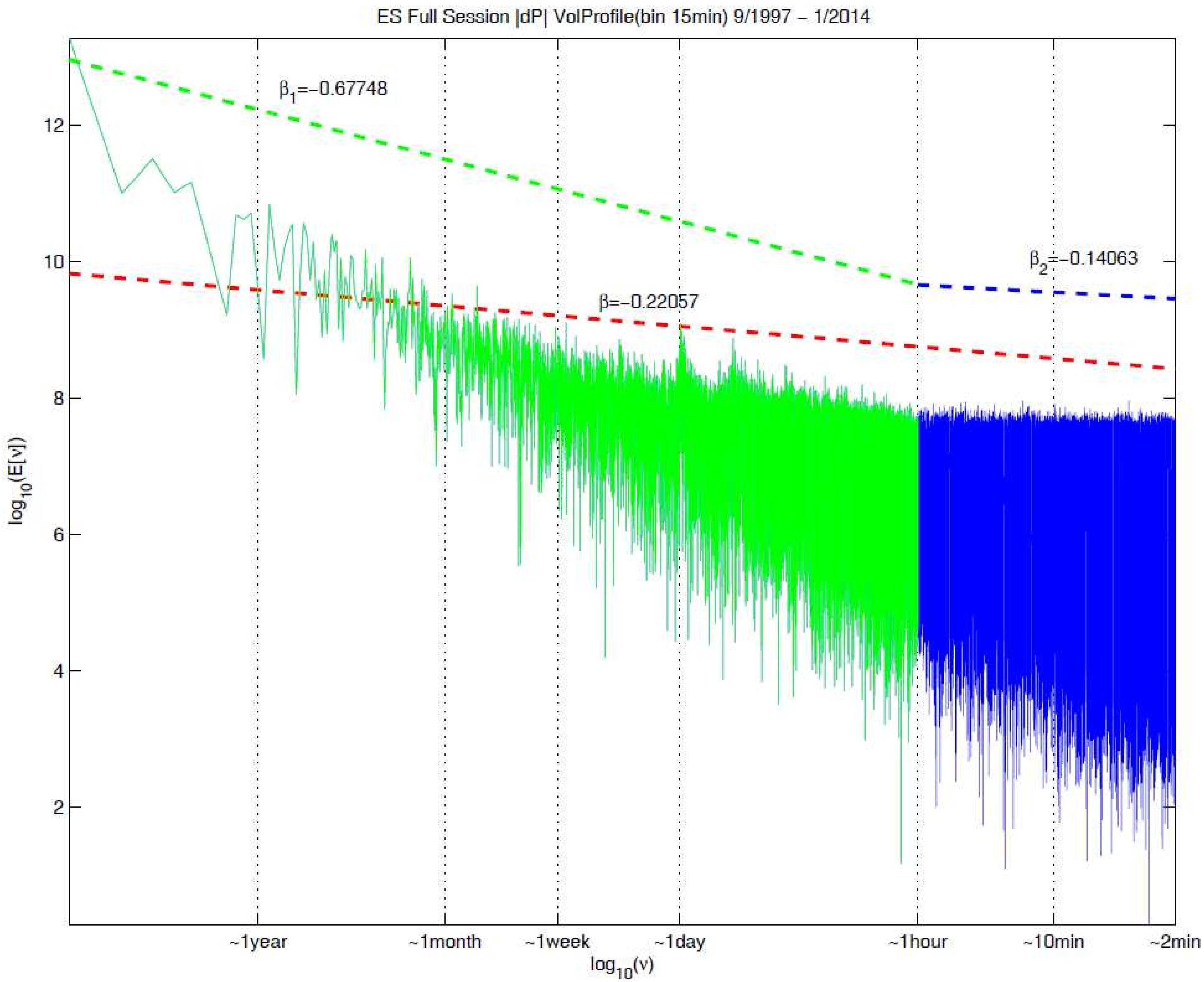
Energy Spectrum Measurements for ΔP and Slopes Divide at 10 Minutes



Energy Spectrum Measurements for de-seasonized |ΔP| and Intraday Volatility Profile Binned at 1 Minute



Energy Spectrum Measurements for de-seasonized |ΔP| and Intraday Volatility Profile Binned at 15 Minutes



Influence of Mean-Reversion on Variance Behavior

To remind you, for a Random Walk: $\mathrm{V}(\tau)=\overline{(\Delta \mathrm{x})^2} \propto \tau$, or $\sigma(\tau)=\sqrt{\mathrm{V}(\tau)} \propto \sqrt{\tau}.$

Consider a discrete mean-reverting process:

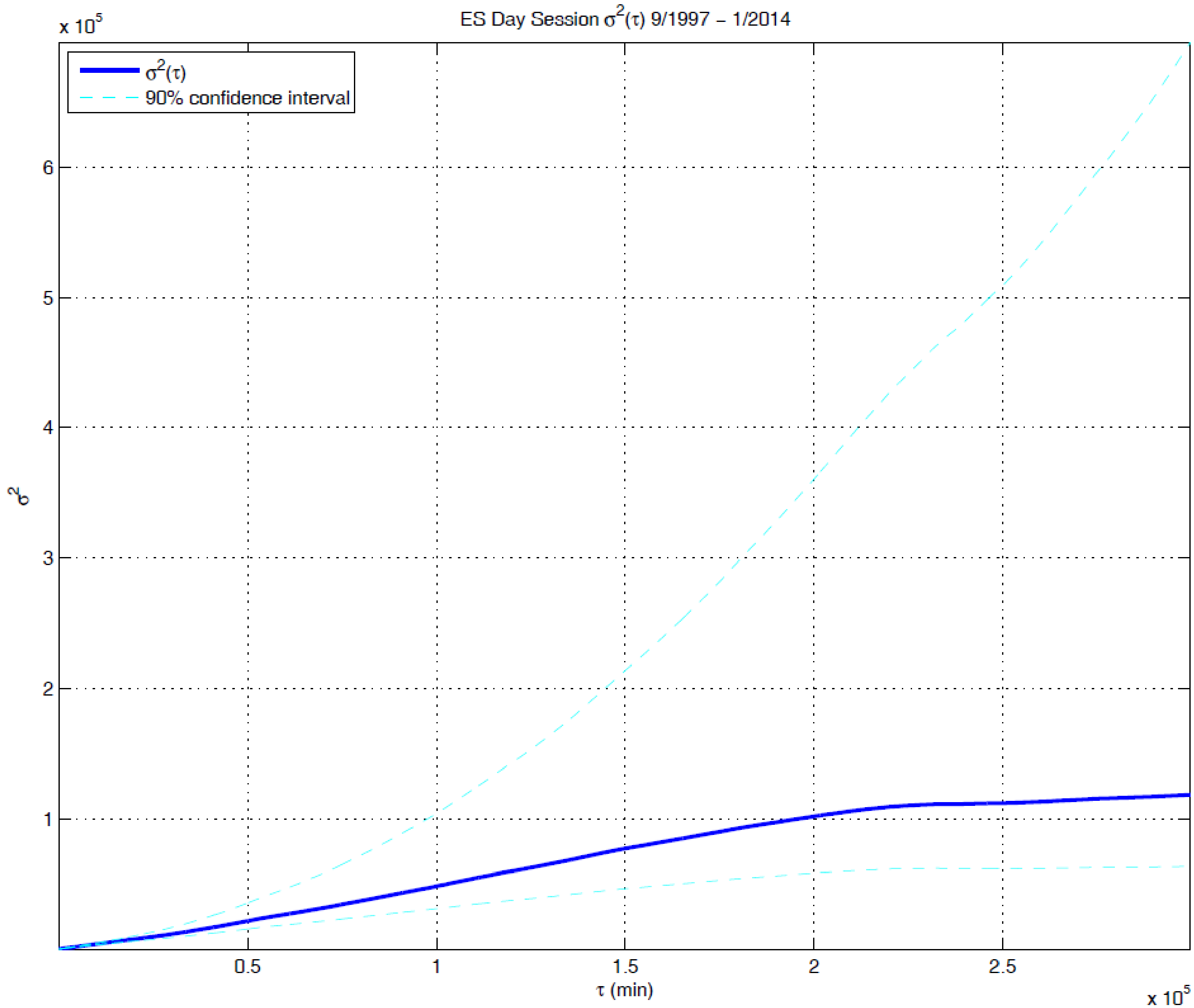
$$\begin{cases} x_{\mathrm{k}+1}=\alpha \cdot x_k+\xi_k, \\ \qquad x_0=0, \end{cases}$$

for $\alpha \leq 1$, and ξ_{k} - i.i.d. random variables. The exact solution is:

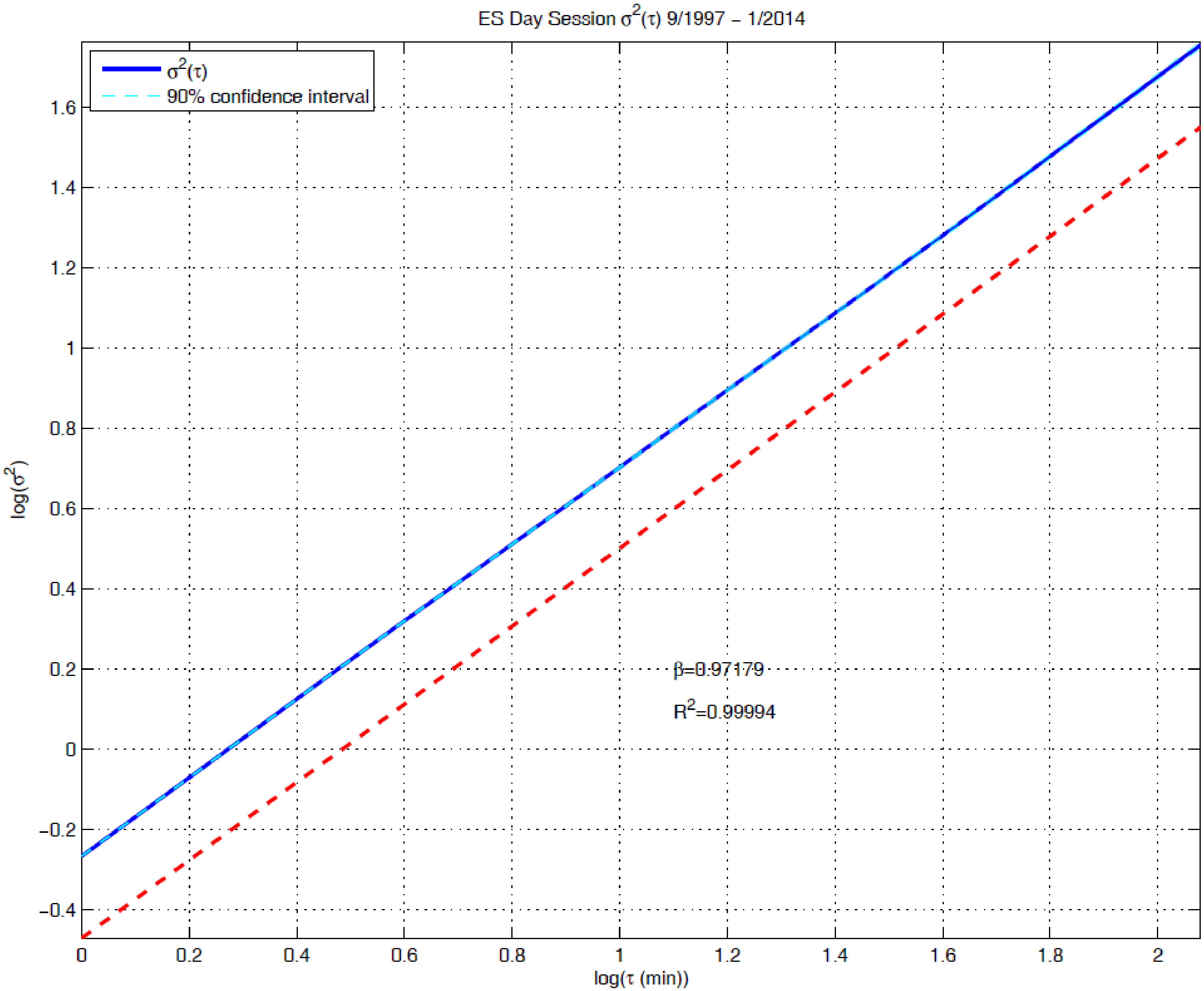
$x_k=\sum_{l=0}^{k-1} \alpha^{k-1-l} \cdot \xi_l$. Using it we get for variance:

$$\mathrm{V}(\tau)=2 \sigma^2 \cdot \frac{1-\alpha^\tau}{1-\alpha^2} \rightarrow \begin{cases} \sigma^2 \tau, \text{ for } \alpha=1\text{-}\varepsilon,\ \tau<<\dfrac{1}{\varepsilon}; \text{(Random Walk)} \\ \qquad \dfrac{\sigma}{\varepsilon}, \text{ for } \tau>>\dfrac{1}{\varepsilon}. \text{(mean reversion)} \end{cases}$$

Variance $\sigma^2(\tau)$ Influenced by Short-Term Mean-Reversion



Variance $\sigma^2(\tau)$ Influenced by Short-Term Mean-Reversion



Selected Levy Probability Density Function Calculus

Symmetric Levy Probability Denisity Function (PDF):

$$P_{\alpha,\gamma}(x,\tau)=\frac{1}{\pi}\int_0^{+\infty} e^{-\gamma\tau q^{\alpha}}\cos(qx)dq, \text{ where parameters } \{\alpha,\gamma\} \text{ for Levy}$$

correspond to $\{2,\sigma\}$ for Gaussian PDF and the integrand $e^{-\gamma\tau q^{\alpha}}$ is the characteristic function $\chi(q)$ of the Levy PDF.

Here we denoted: $x=\Delta p, \tau=\Delta t$.

In new self-similar variables: $\tilde{q}=q\cdot(\gamma\tau)^{1/\alpha}, \tilde{x}=x/(\gamma\tau)^{1/\alpha}$ we have:

$$P_{\alpha,\gamma}(x,\tau)=(\gamma\tau)^{-1/\alpha}\cdot F_{\alpha}(\tilde{x}), \text{ where } F_{\alpha}(\tilde{x})=\frac{1}{\pi}\int_0^{+\infty} e^{-\tilde{q}^{\alpha}}\cos(\tilde{q}\tilde{x})d\tilde{q}.$$

For $x\to 0+$ we can use the Taylor series expansion:

$$\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^n}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots$$

and using the notation for Gamma function $\Gamma(z)=\int_0^{\infty} t^{z-1}e^{-t}dt$ we get:

$$\int_0^{+\infty} q^{2n}e^{-q^{\alpha}}dq=\frac{1}{\alpha}\int_0^{+\infty} p^{\frac{2n+1}{\alpha}-1}\cdot e^{-p}\cdot dp=\frac{1}{\alpha}\Gamma\left(\frac{2n+1}{\alpha}\right), \text{ for } p=q^{\alpha}.$$

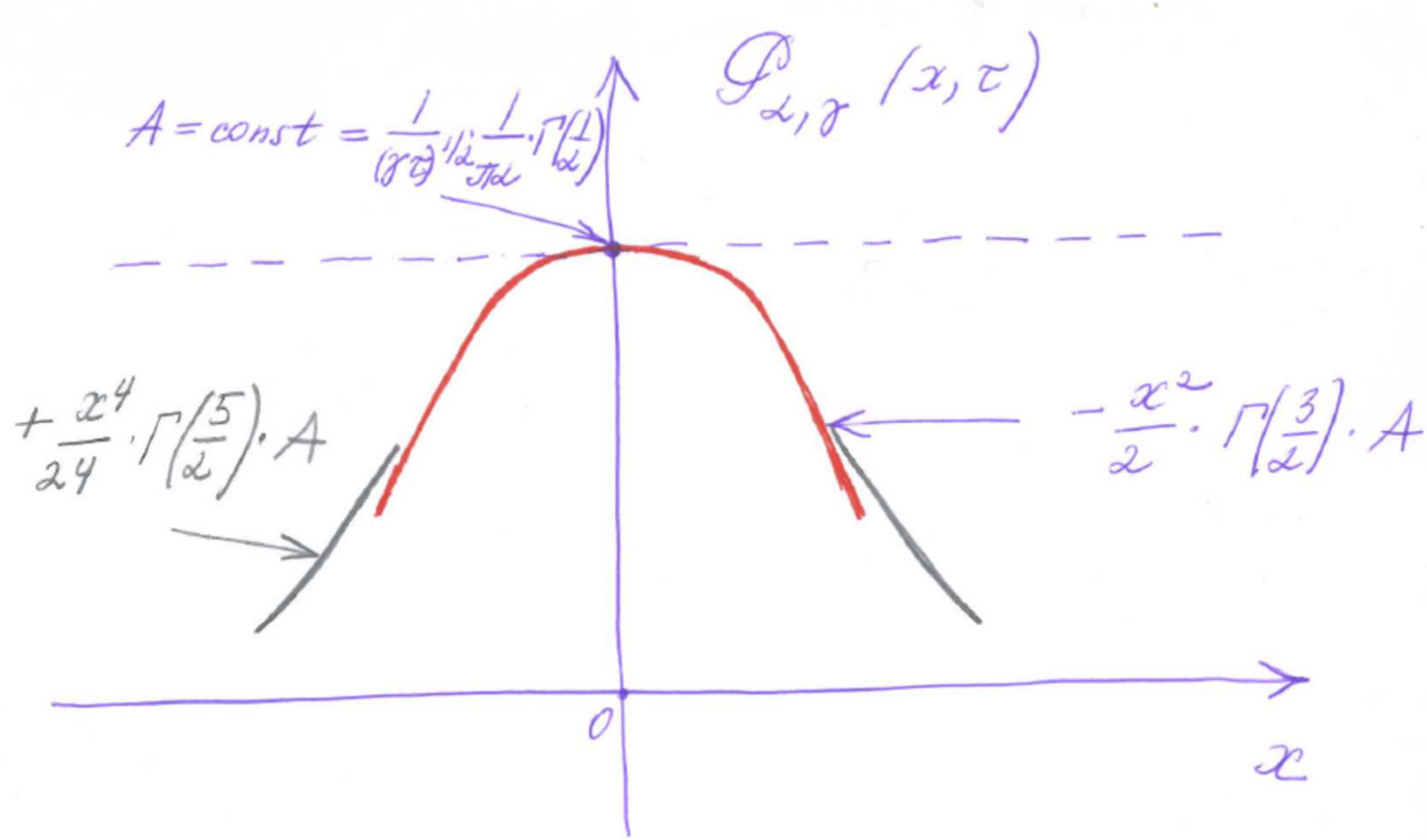
Therefore, we get the following asymptotic series expansion for Levy PDF:

$$P_{\alpha,\gamma}(x,\tau)=(\gamma\tau)^{-1/\alpha}\cdot\frac{1}{\pi\alpha}\cdot\sum_{n=0}^{\infty}\tilde{x}^{2n}\cdot\Gamma\left(\frac{2n+1}{\alpha}\right)\cdot\frac{(-1)^n}{(2n)!}=$$

$$=(\gamma\tau)^{-1/\alpha}\cdot\frac{1}{\pi\alpha}\cdot\left\{\Gamma\left(\frac{1}{\alpha}\right)-\frac{\tilde{x}^2}{2}\cdot\Gamma\left(\frac{3}{\alpha}\right)+\frac{\tilde{x}^4}{24}\cdot\Gamma\left(\frac{5}{\alpha}\right)-\ldots\right\}.$$

Selected Levy Probability Density Function Calculus

Then, for fixed variable τ and parameters α, γ we can schematically plot the symmetric Levy Probability Denisity Function (PDF) as follows:



Selected Levy Probability Density Function Calculus

Further, for ν - th order moments of structure functions we get :

$$S_\nu(\tau) = \overline{|x|^\nu} = \int_{-\infty}^{+\infty} |\,x\,|^\nu \cdot P_{\alpha,\gamma}(x,\tau) \cdot dx = \frac{2}{\pi} \int_0^{+\infty} dx \int_0^{+\infty} dq \cdot x^\nu \cdot e^{-\gamma\tau q^\alpha} \cdot \cos(qx) =$$

using substitutions $q = \dfrac{\tilde{q}}{(\gamma\tau)^{1/\alpha}}$ and $x = \tilde{x}(\gamma\tau)^{1/\alpha}$

$$= \frac{2}{\pi} (\gamma\tau)^{\nu/\alpha} \int_0^{+\infty} d\tilde{x} \int_0^{+\infty} d\tilde{q} \cdot \tilde{x}^\nu \cdot e^{\tilde{q}^\alpha} \cdot \cos(\tilde{q}\tilde{x}) = \frac{2}{\pi} (\gamma\tau)^{\nu/\tau} \cdot B, \text{ where :}$$

$$B = \int_0^{+\infty} d\tilde{x} \cdot \int_0^{+\infty} d\tilde{q} \cdot \tilde{x}^\nu \cdot e^{\tilde{q}^\alpha} \cdot \cos(\tilde{q}\tilde{x}), \text{ if it is finite.}$$

For example, for variance we get :

$$S_2(\tau) \propto \tau^{2/\alpha} \propto \tau^{1+\varepsilon/2}, \text{ for } \alpha = 2 - \varepsilon \text{ and } 0 < \varepsilon << 1.$$

Selected Levy Probability Density Function Calculus

Now, let us develop the large x asymptotic behavior for Levy PDF.

For that we need to look at the integral behavior

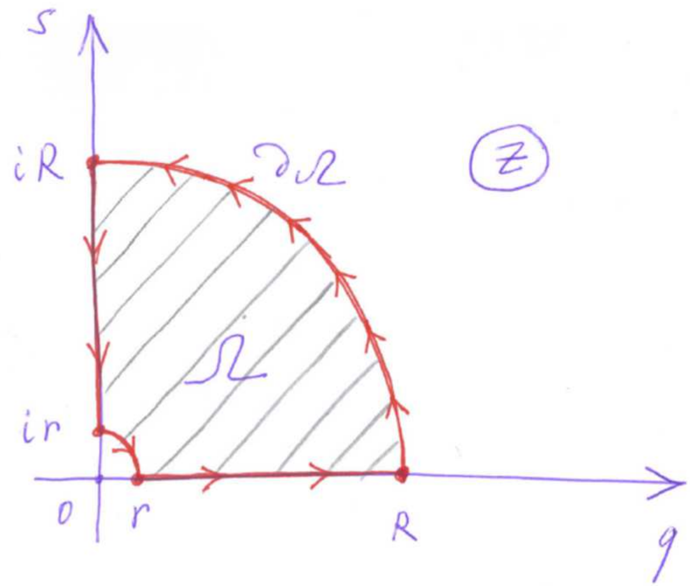
$$\frac{1}{\pi}\int_0^\infty \underbrace{e^{-\gamma\tau q^\alpha}\cdot\cos(qx)}_{F(q)}\cdot dq$$

for, generally, complex values of integration variable q:

$z = q + is$. Then in the complex plane for variable z the function F becomes:

$$F(z) = \frac{1}{2\pi}\cdot e^{-\gamma\tau z^\alpha}\cdot e^{izx}.$$

This function is an analytical function in the following area Ω:



Then, according to Cauchy's integral theorem, we have:

$$\int_{\partial\Omega} F(z)dz = 0.$$

Schematically, we can show that:

$$\underbrace{\int_{C_r}}_{\to 0,\text{ for } r\to 0+} + \underbrace{\int_{C_R}}_{\to 0,\text{ for } R\to +\infty} + \int_r^R + \int_{iR}^{ir} = 0.$$

Selected Levy Probability Density Function Calculus

From this follows that integral over the real axis can be replaced with the integral over the imaginary axis:

$$\int_r^R F(z)dz = \int_{ir}^{iR} F(z)dz.$$

Then our integral of interest becomes:

$$\mathrm{Re}\left[\frac{1}{\pi}\int_{ir}^{iR} e^{-\gamma\tau z^\alpha}\cdot e^{izx}\cdot dz\right],$$ which along the imaginary axis $z = is$ becomes:

$$\frac{1}{\pi}i\int_r^R \underbrace{e^{-\gamma\tau i^\alpha s^\alpha}}_{\substack{\text{Taylor series}\\ \text{expansion for}\\ \text{small } s}}\cdot e^{-sx}\cdot ds = \frac{1}{\pi}i\sum_{k=1}^{\infty}\int_r^R ds\cdot\frac{(-1)^k(\gamma\tau)^k i^{k\alpha}s^{k\alpha}}{k!}\cdot e^{-sx} =$$

the limit $x \to +\infty$ is equivalent to limit $s \to 0+$ if we use a transformation $t = sx$ for finite t,

$$= \frac{1}{\pi}i\sum_{k=1}^{\infty}\frac{(-1)^k(\gamma\tau)^k e^{i\frac{\pi}{2}k\alpha}}{x^{k\alpha+1}}\cdot\underbrace{\int_{rx}^{Rx} t^{k\alpha}e^{-t}dt}_{\substack{\text{approaches }\Gamma(k\alpha+1)\\ \text{in the limit } r\to 0+,\, R\to+\infty}} = \quad \mathrm{Re}\left[\frac{1}{\pi}i\sum_{k=1}^{\infty}\frac{(-1)^k(\gamma\tau)^k e^{i\frac{\pi}{2}k\alpha}}{x^{k\alpha+1}}\cdot\Gamma(k\alpha+1)\right].$$

Some Analytics of Levy Probability Density Function

Now, using

$$\mathrm{Re}\left[i e^{i\frac{\pi}{2}k\varepsilon} \right] = -\sin\left(\frac{\pi}{2}k\alpha\right),$$

we obtain :

$$\frac{1}{\pi}\int_0^{\infty} e^{-\gamma\tau q^{\alpha}} \cos(qx)dq \underbrace{\to}_{\text{as } x\to+\infty} -\frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k (\gamma\tau)^k \sin\left(\frac{\pi k\alpha}{2}\right)}{|x|^{k\alpha+1}} \cdot \Gamma(k\alpha+1) =$$

which in the limit $|x| \to +\infty$ leads to the following asymptotic series expansion :

$$P_{\alpha,\gamma}(x,\tau) \propto \frac{1}{\pi}\cdot\gamma\tau\cdot\sin\left(\frac{\pi\alpha}{2}\right)\cdot\Gamma(\alpha+1)\cdot \underbrace{\frac{1}{|x|^{\alpha+1}}}_{\text{power law}} + \text{h.o.t.}$$

For example, the PDF normalization condition :

$$\int_R^{+\infty} \frac{dx}{x^{\alpha+1}} \text{ converges for } \alpha > 0, \text{ and diverges for } \alpha = 0.$$

Similarly, the ν - th order moment (structure function) :

$$S_{\nu}(\tau) \propto \int_R^{+\infty} \frac{dx}{x^{-\nu+\alpha+1}} \text{ converges for } \nu < \alpha.$$

A particularly interesting case $\nu = 2$, of variance $S_2(\tau)$, formally diverges if $\alpha = 2 - \varepsilon$ and $\varepsilon > 0$ is a small number.

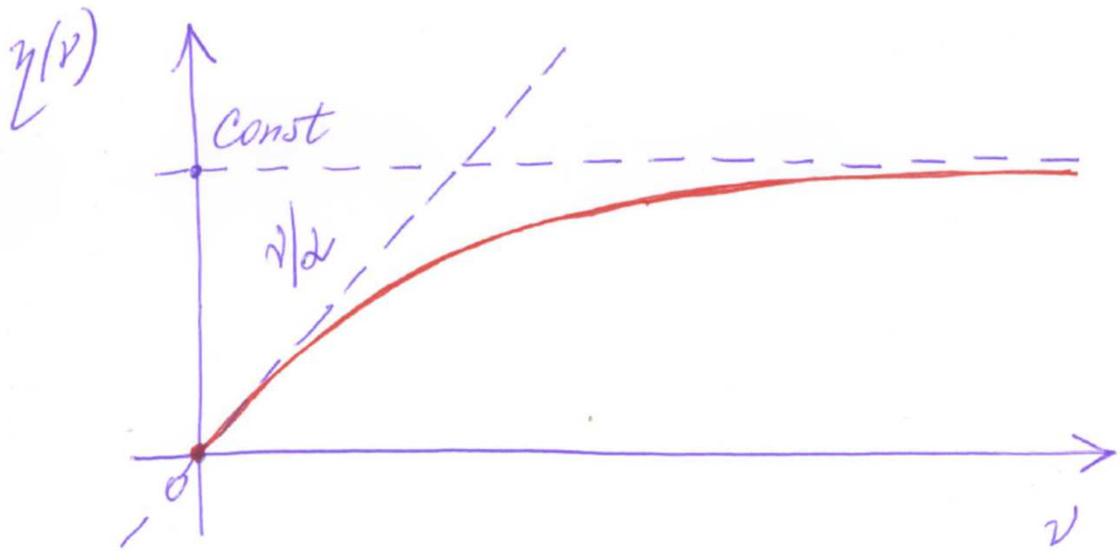
Some Analytics of Levy Probability Density Function

We have just therefore showed what in turbulence is called "intermittency" or bi-scaling bahavior of the structure functions.

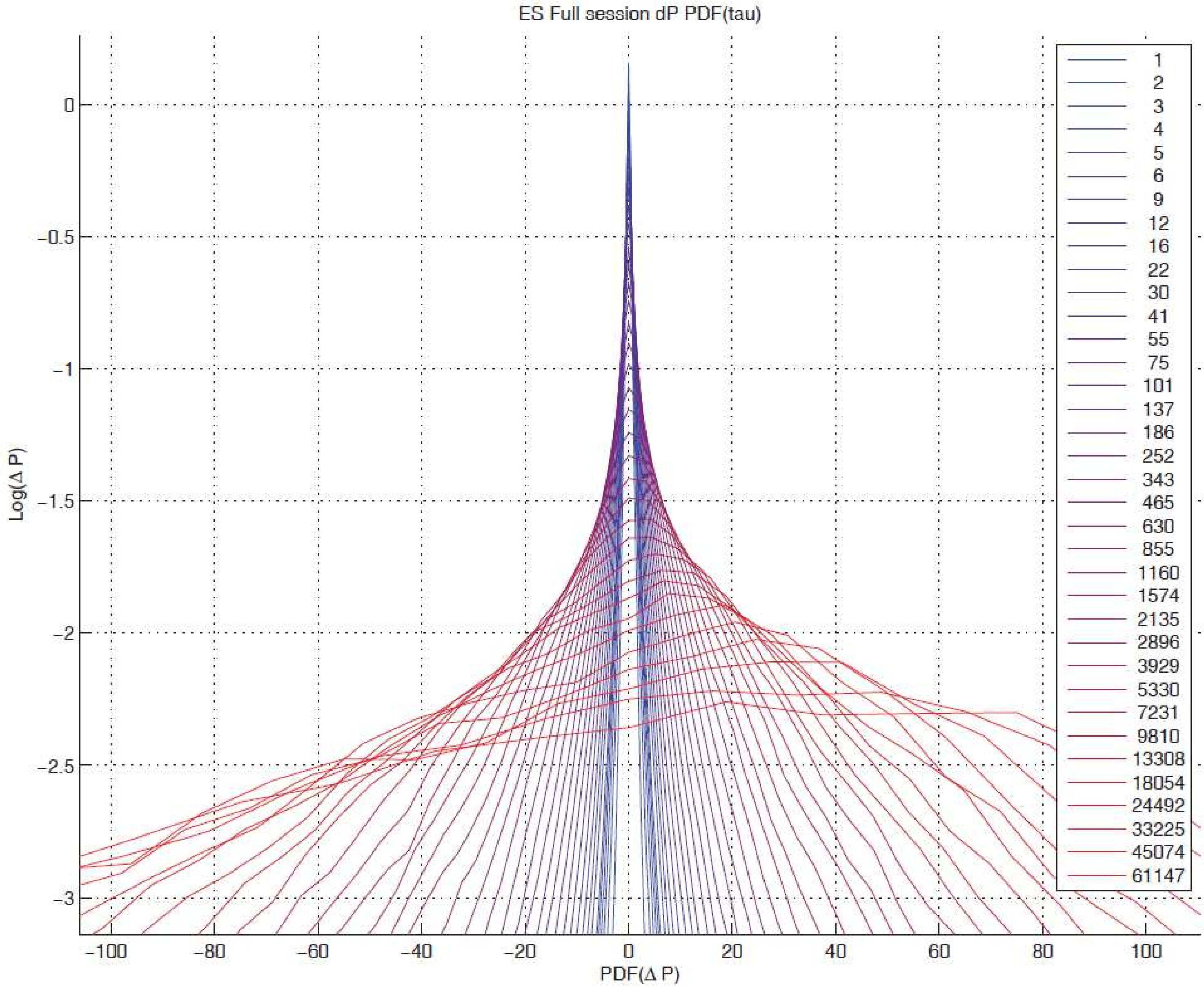
It means that the ν-th order moment (structure function):

$$S_\nu(\tau) \propto \begin{cases} \dfrac{2}{\pi}(\gamma\tau)^{\nu/\alpha} \cdot B, \text{ for } \nu < \alpha, \\ Const \quad (\text{tail-dependent}), \text{ for } \nu \geq \alpha \end{cases},$$

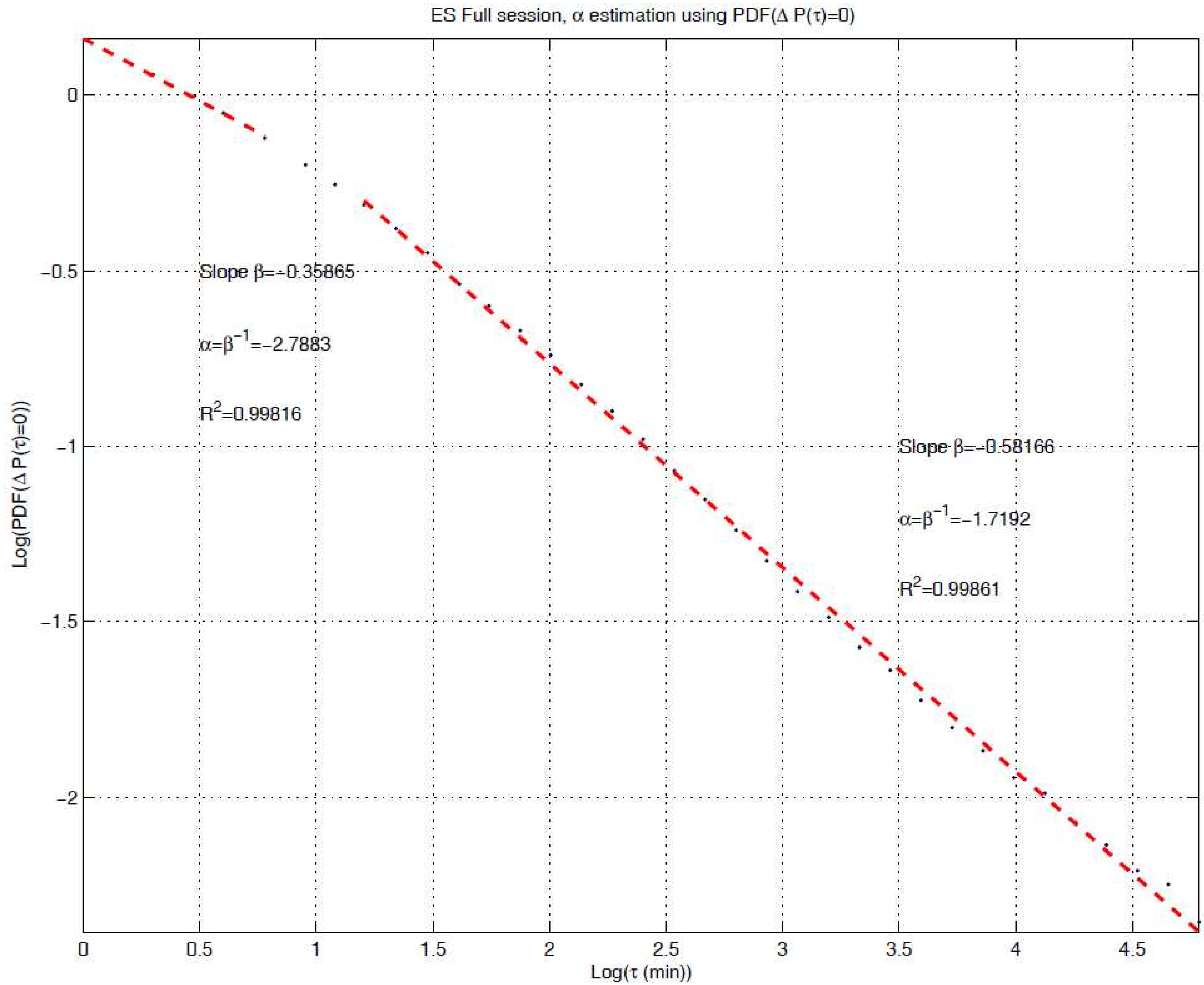
which can be graphically shown through a "critical exponent" $\eta(\nu)$ such that as $S_\nu(\tau) \propto \tau^{\eta(\nu)}$, as the following "critical diagram":



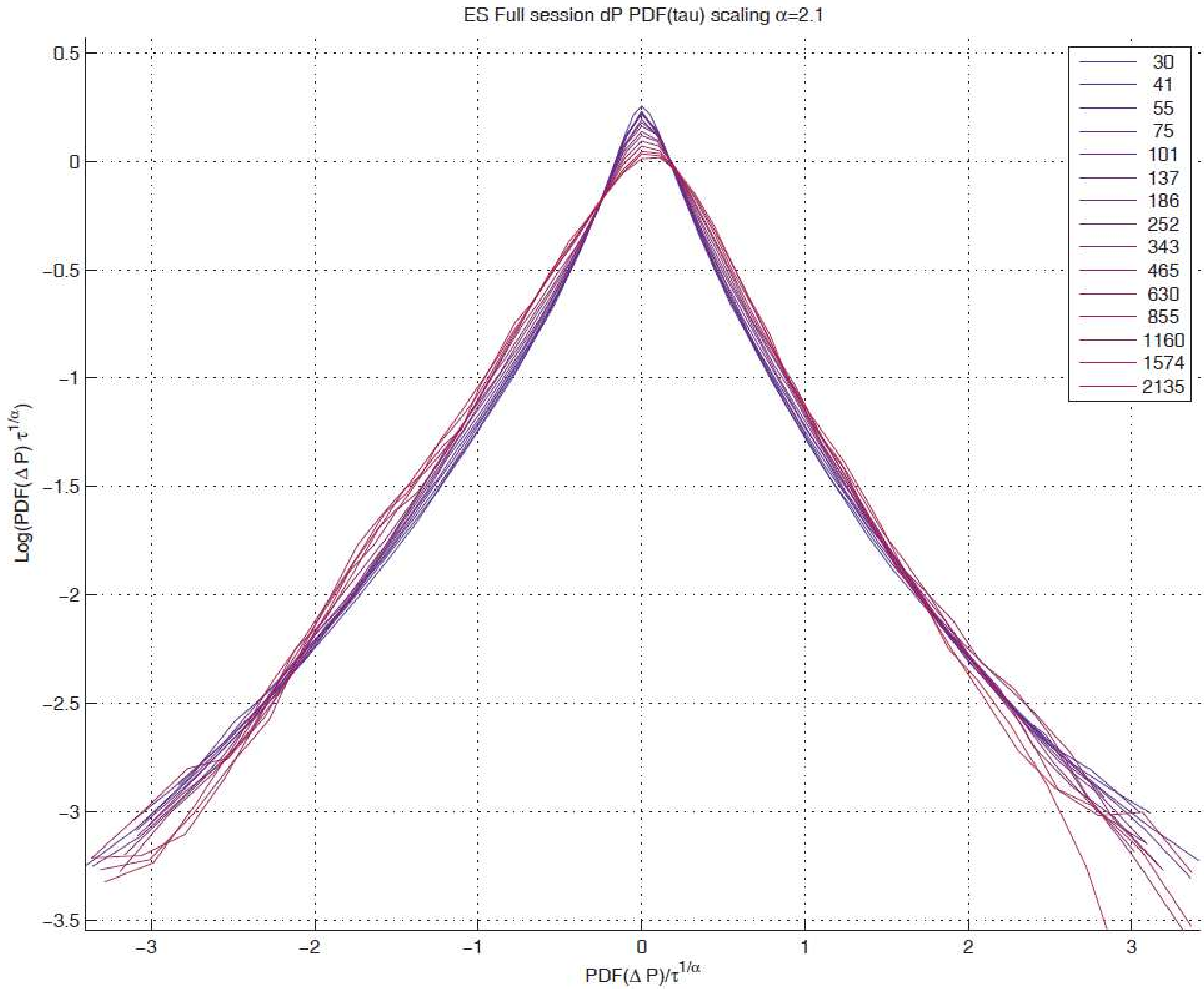
Raw ΔP Histograms



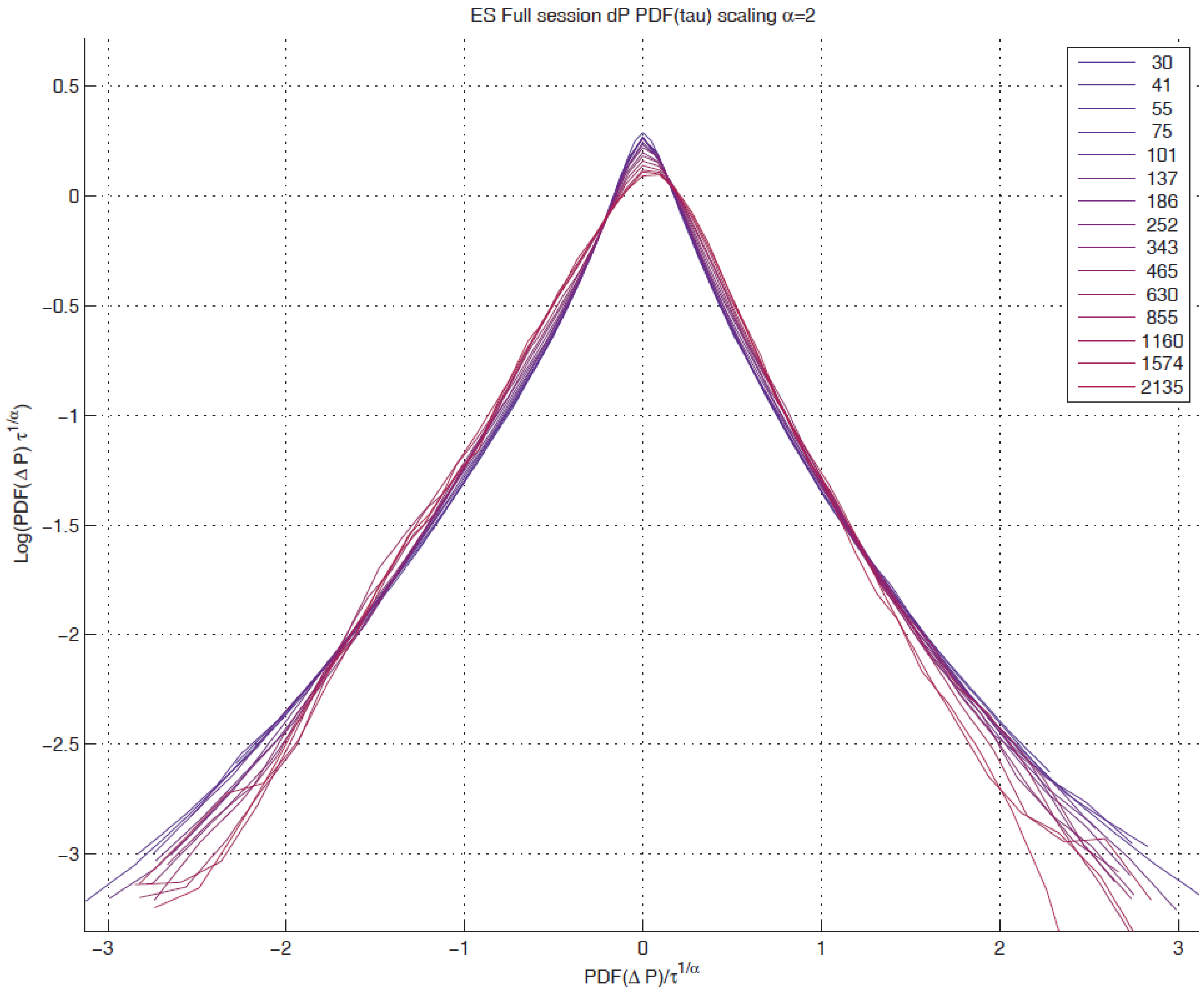
Levy Scaling α Estimation from the Tip of the Histogram



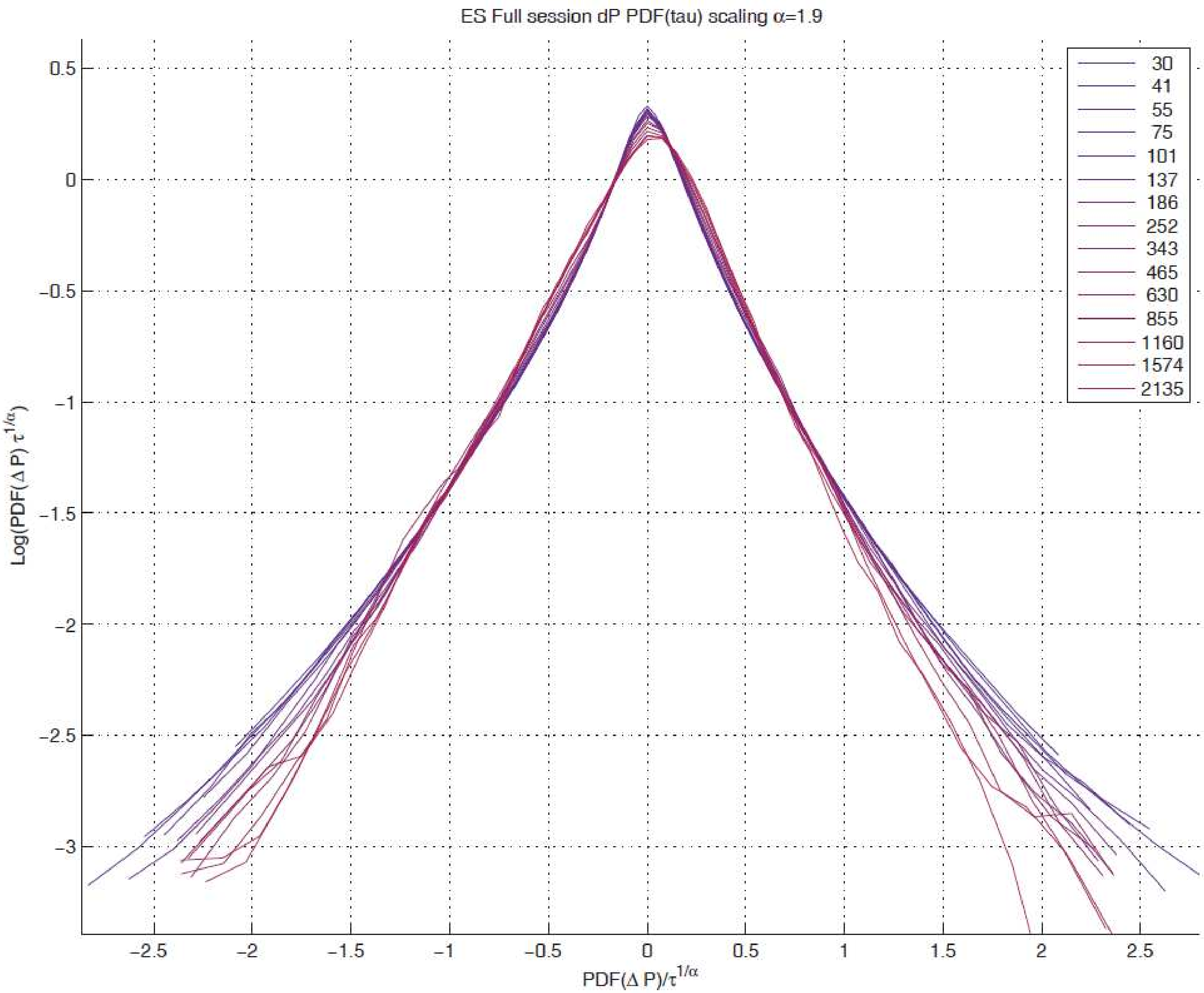
Levy Scaling α Estimation Using PDFs Collapse, α=2.1



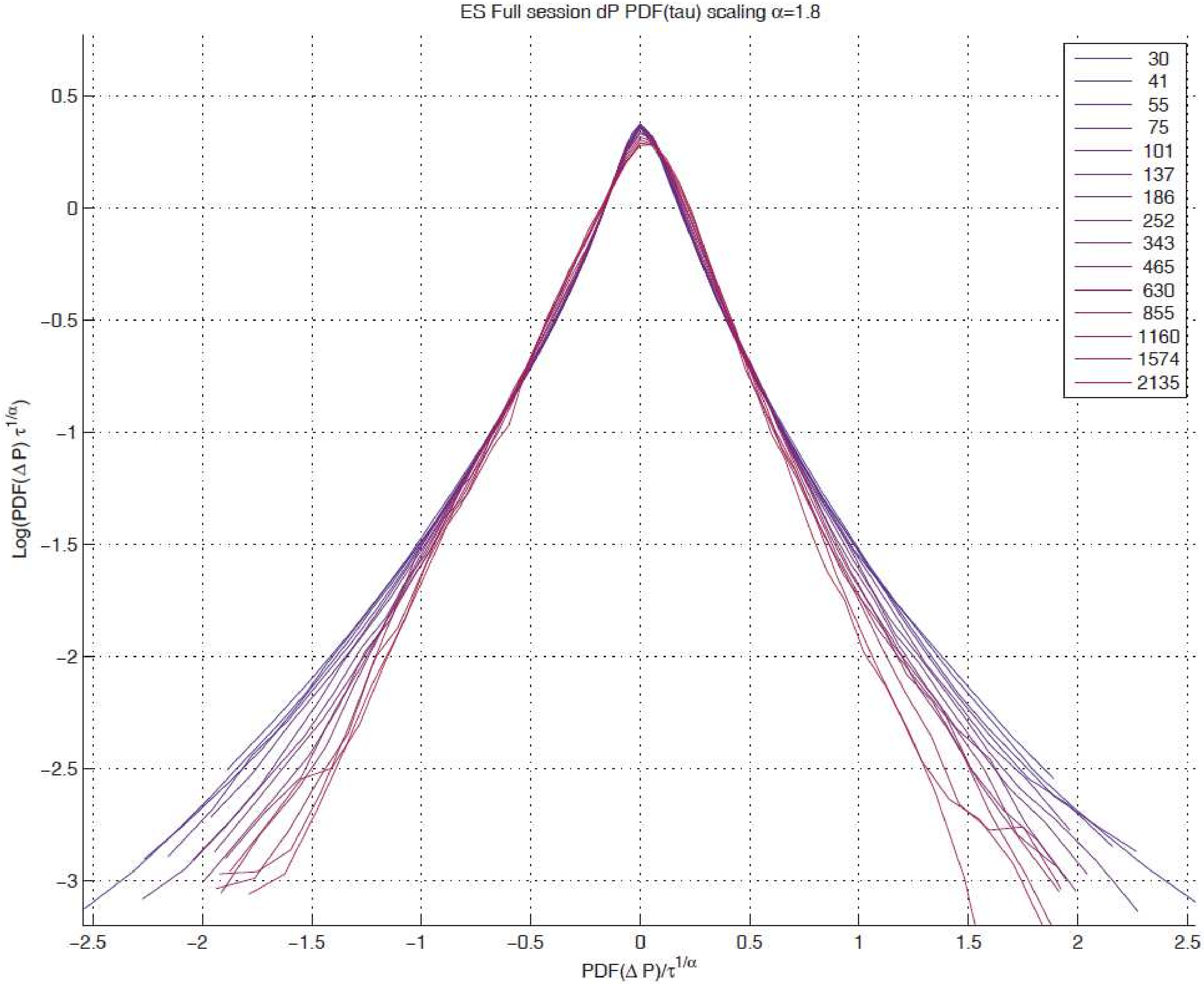
Levy Scaling α Estimation Using PDFs Collapse, α=2.0



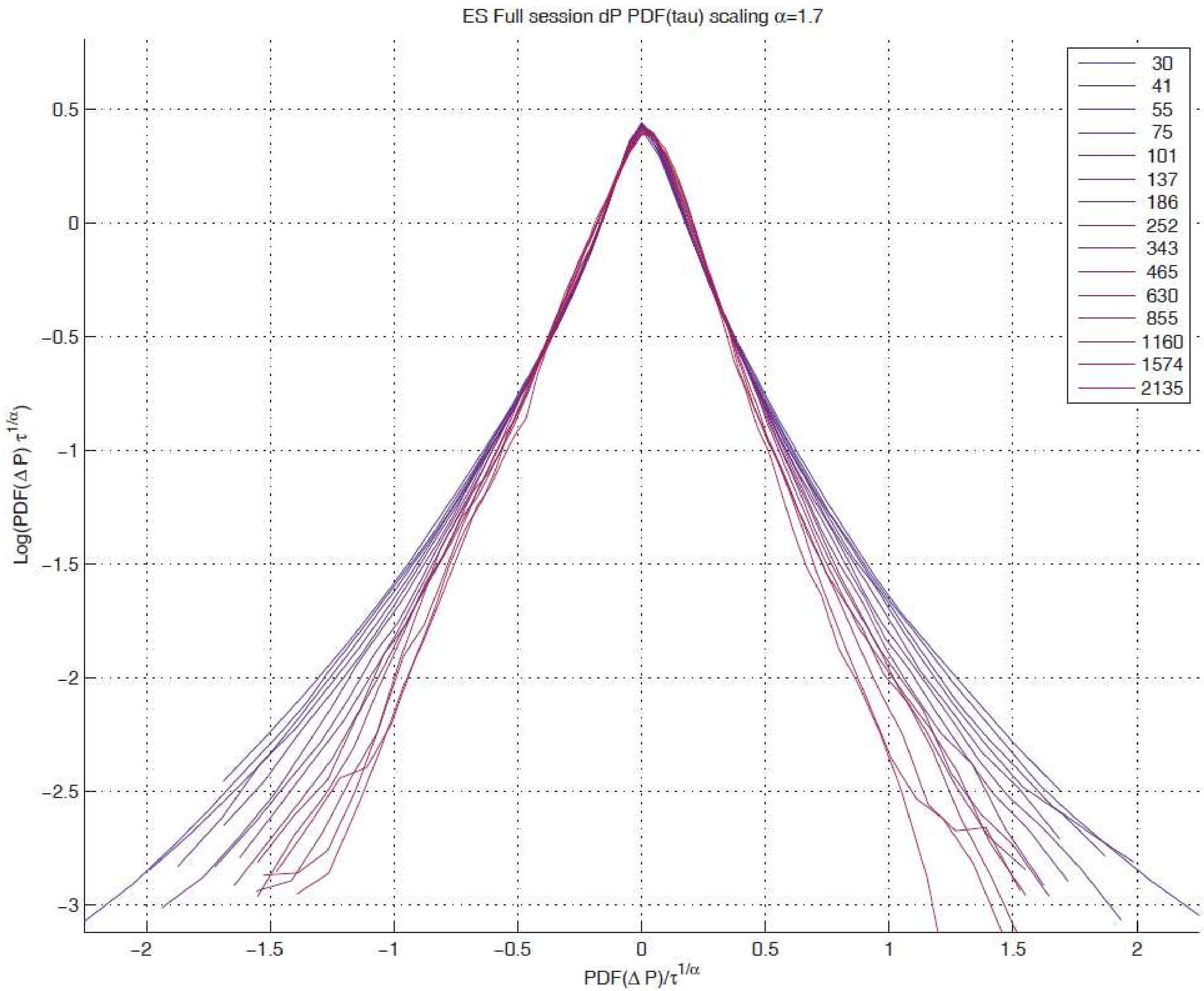
Levy Scaling α Estimation Using PDFs Collapse, α=1.9



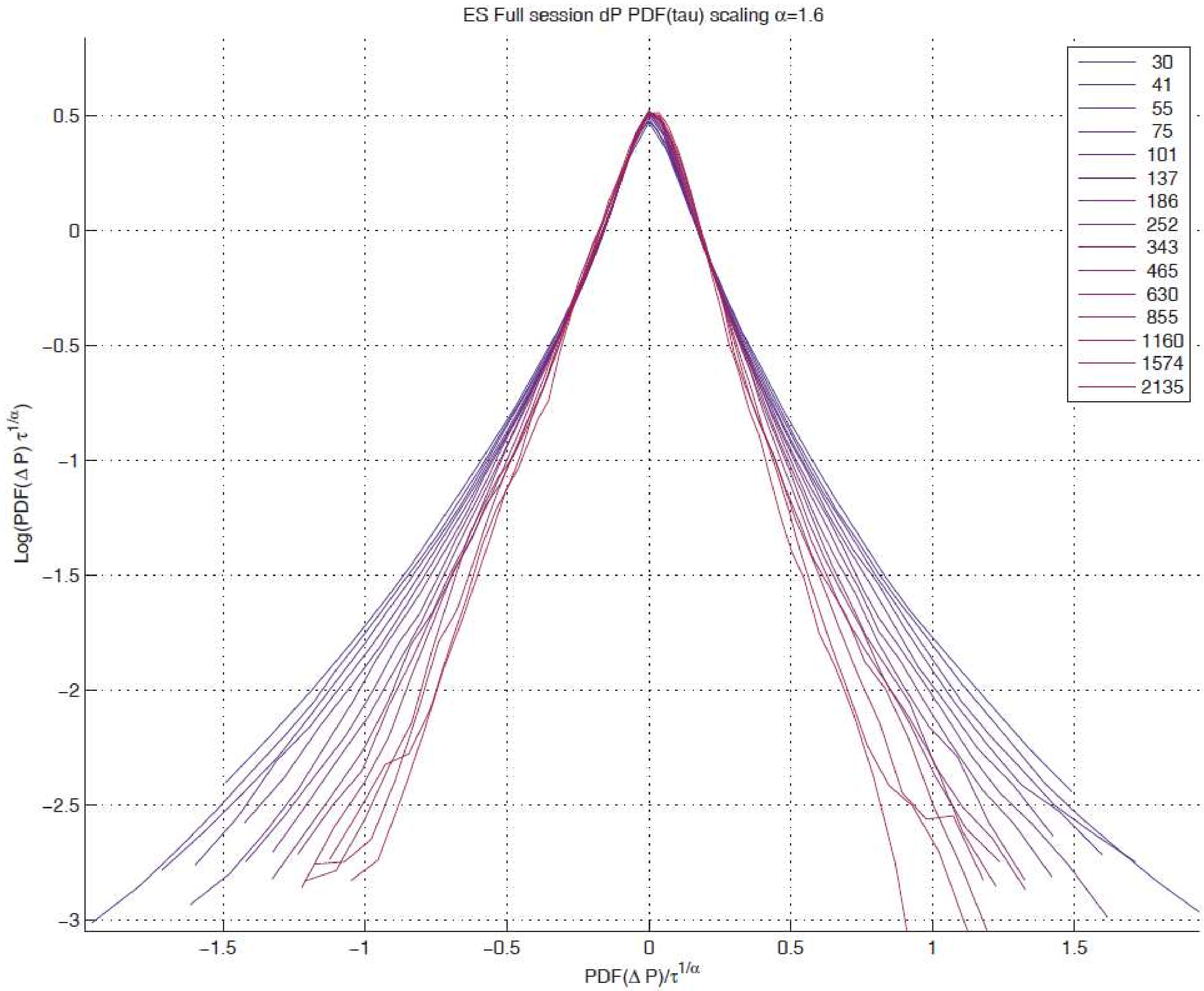
Levy Scaling α Estimation Using PDFs Collapse, α=1.8



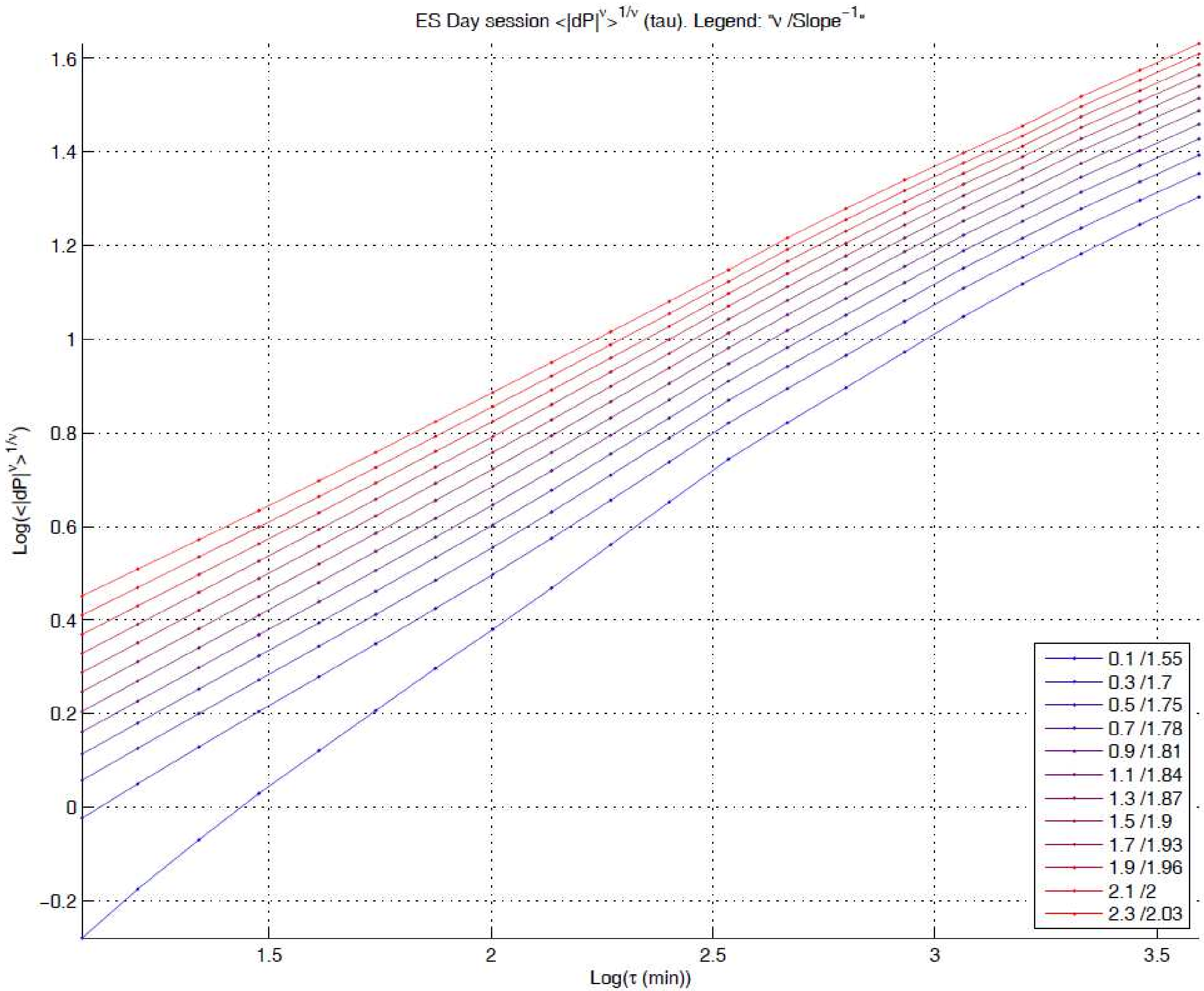
Levy Scaling α Estimation Using PDFs Collapse, α=1.7



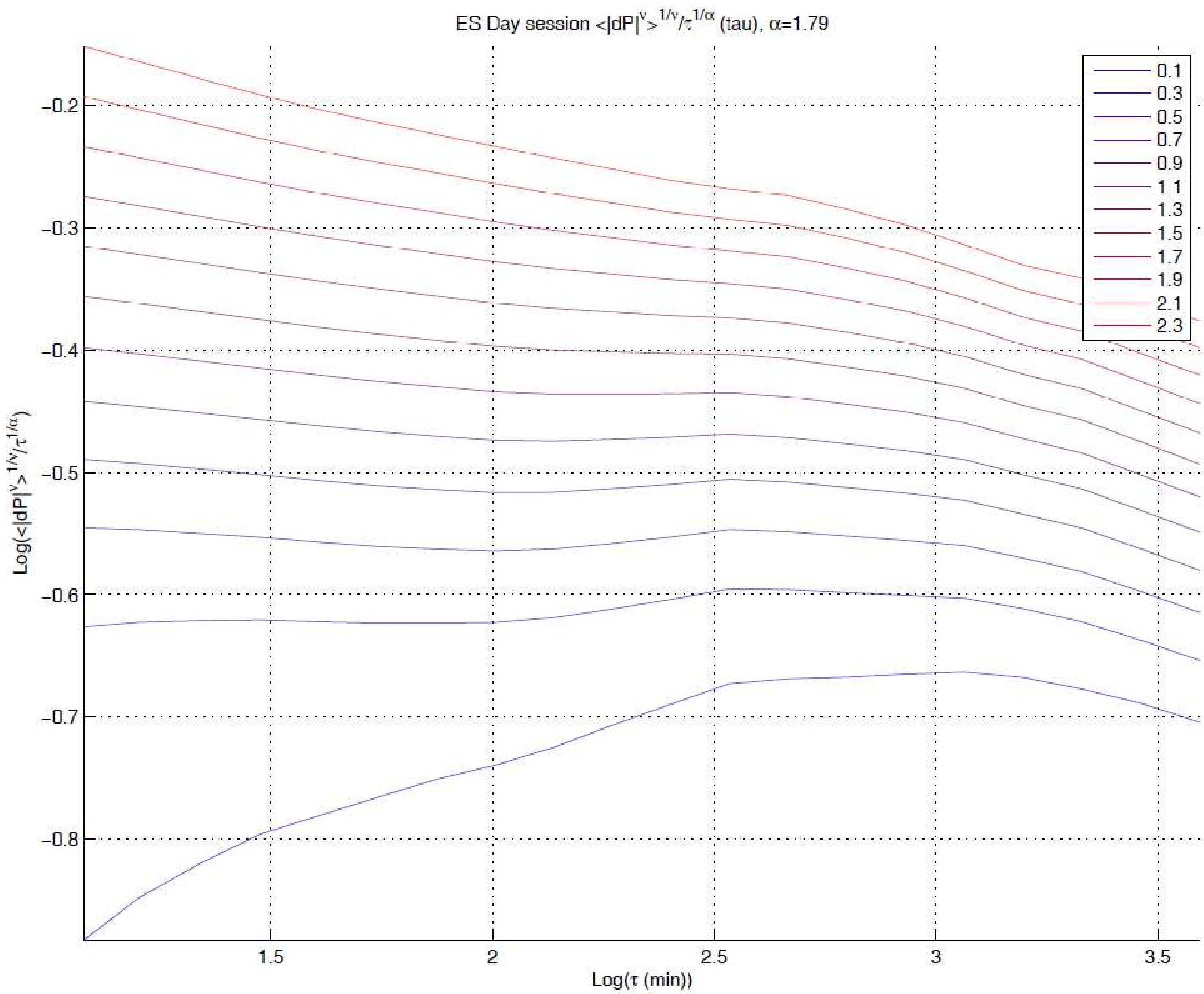
Levy Scaling α Estimation Using PDFs Collapse, α=1.6



Levy Scaling α Estimation Using Fractional Order Moments (Structure Functions)



Levy Scaling α Estimation, Fractional Order Moments (Structure Functions) compensated with α=1.79



Some Experimental Conclusions

- Properly constructed statistical tests of the financial data for most liquid financial instruments have revealed short-term mean-reversion.
- We have identified intraday seasonality – a predictable pattern of behavior of local volatility which allows one to de-seasonalize the price change data for further statistical analysis.
- In the de-seasonalized data we have found that the local volatility and its moments are long-memory random processes.
- We have found similar long-memory properties in the energy spectrum.
- We have related the anomalies in the variance of price changes to the short-term mean-reversion properties.
- We have parameterized the two-point probability density functions of price changes using a symmetric Levy probability density function with the approximate Levy exponent of 1.7-1.8.