Class 16: Continuous joint distributions

Thursday, October 31, 2019 11:39 AM

- No class Tues, Nov 5
- HW 6 due Thurs, Nov 7, 1pm
- Midterm 2 Tues, Nov 12

Recap: X a continuous RV with PDF f, CDF F, then:

- $F(x) = \mathbb{P}(X \le x)$ for $x \in \mathbb{R}$
- $f(x) = F'(x) = \frac{dF}{dx}(x)$
- $\mathbb{P}(a \le X \le b) = F(b) - F(a) = \int_a^b f(x)\,dx$
- $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx$

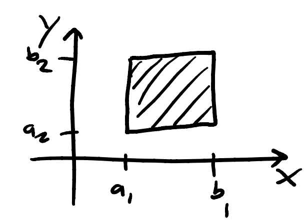
  $\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\,dx$ for functions h

---

Definition: A pair of random variables (X, Y) has a continuous joint distribution if there is a function f (called the joint density function) such that

$$\mathbb{P}(a_1 \le X \le b_1,\ a_2 \le Y \le b_2) = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dy\,dx$$

$$= \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y)\,dx\,dy$$



for all $a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $a_1 \le b_1$, $a_2 \le b_2$.

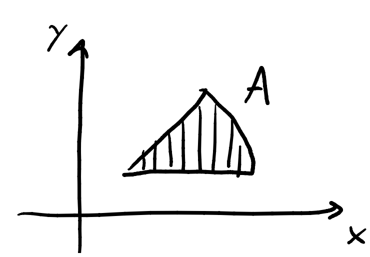
Facts:

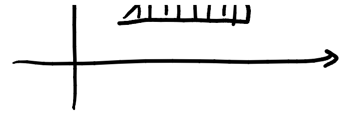
① $f \ge 0$

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$ (total prob. $= 1$)

③ For any region $A \subseteq \mathbb{R}^2$

$$\iint_A f(x,y)\,dy\,dx = \mathbb{P}((X,Y) \in A)$$





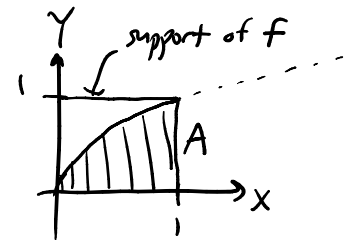
$$\iint_A f(x,y)\,dy\,dx = \mathbb{P}\big((X,Y)\in A\big).$$

<u>Terminology</u>: The <u>support</u> of <u>f</u> is the region where $f>0$, i.e.

$$\{(x,y)\in\mathbb{R}^2 : f(x,y)>0\}.$$

We only need to integrate over support of f.

<u>Ex 1</u>: Let $f(x,y) = \begin{cases} 1 & \text{if } 0\le x\le 1,\ 0\le y\le 1 \\ 0 & \text{otherwise.}\end{cases}$



↪ Uniform distribution on the square.

<u>Check</u>: $f\ge 0$, $\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^1\int_0^1 1\,dy\,dx = 1.$

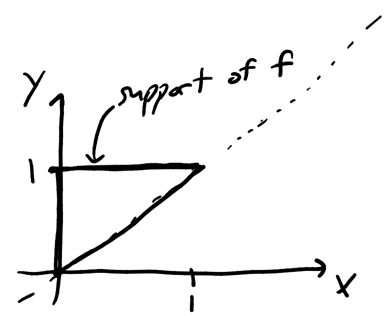
<u>E.g.</u>

$$\mathbb{P}(Y\le\sqrt{X}) = \iint_A f(x,y)\,dy\,dx \quad \text{where } A=\{(x,y)\in\mathbb{R}^2 : y\le\sqrt{x}\}.$$

$$= \int_0^1\int_0^{\sqrt{x}} 1\,dy\,dx$$

$$= \int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}.$$

<u>Ex 2</u>: Let $f(x,y) = \begin{cases} 8xy & \text{if } 0\le x\le y\le 1 \\ 0 & \text{otherwise.}\end{cases}$



<u>Check</u>: $f\ge 0$ ✓

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^1\int_x^1 8xy\,dy\,dx$$

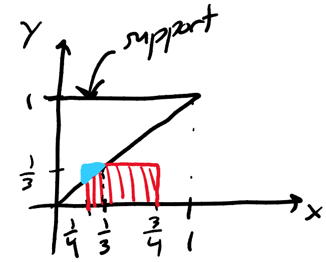
$$= \int_0^1\int_0^y 8xy\,dx\,dy$$

$$= \int_0^1 4x^2y\Big|_{x=0}^{y}\,dy$$

$$= \int_0 4xy\big|_{x=0}\, dy$$

$$= \int_0^1 4y^3\,dy$$

$$= y^4\big|_{y=0}^{1} = 1. \quad ✓$$



only integrate over where region and support meet

1. Calculate $\mathbb{P}\left(\frac{1}{4} \le X \le \frac{3}{4},\ 0 \le Y \le \frac{1}{3}\right)$

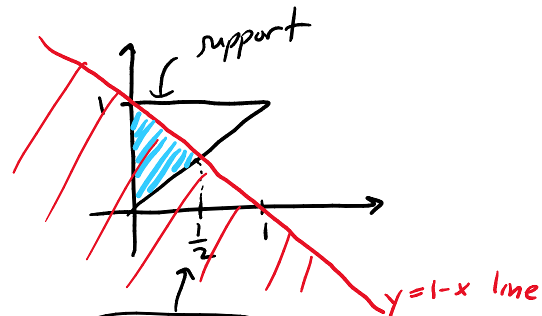
$$= \int_0^{\frac{1}{3}} \int_{\frac{1}{4}}^{\frac{3}{4}} f(x,y)\,dx\,dy$$

$$= \int_{\frac{1}{4}}^{\frac{1}{3}} \int_x^{\frac{1}{3}} 8xy\,dy\,dx$$

$$= \int_{\frac{1}{4}}^{\frac{1}{3}} 4xy^2\Big|_{y=x}^{\frac{1}{3}}\,dx$$

$$= \int_{\frac{1}{4}}^{\frac{1}{3}} 4x\left(\frac{1}{9} - x^2\right)dx = \cdots$$

2. Calculate $\mathbb{P}(X+Y \le 1)$



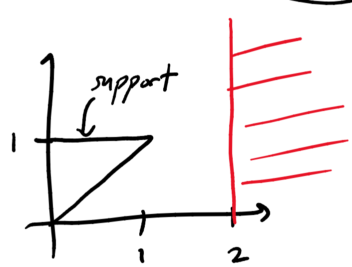
$x = y = 1-x$
$\to x = \frac{1}{2}$

$$= \iint_{\text{blue region}} f(x,y)\,dy\,dx$$

$$= \int_0^{\frac{1}{2}} \int_x^{1-x} 8xy\,dy\,dx$$

$$= \cdots$$

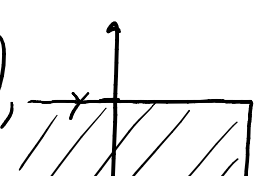
3. Calculate $\mathbb{P}(X \ge 2)$



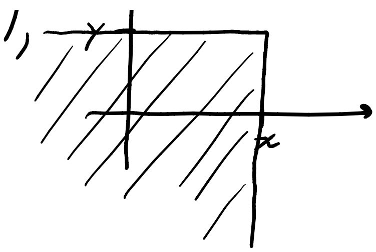
$$= \iint_{\text{blue}} f(x,y)\,dy\,dx = 0$$

since blue = (support) $\cap$ red $= \emptyset$

Def: The joint CDF of (X,Y) is the function $F(x,y) = \mathbb{P}(X \le x, Y \le y)$, * not often used

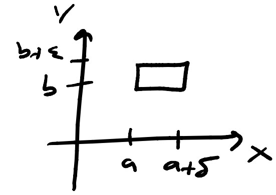


the function $F(x,y) = \mathbb{P}(X \le x, Y \le y)$,



or $F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\,dv\,du$.

Note $\frac{\partial}{\partial x}\frac{\partial}{\partial y} F(x,y) = f(x,y)$.



$\mathbb{P}(a \le X \le a+\delta,\ b \le Y \le b+\varepsilon) = \int_a^{a+\delta}\int_b^{b+\varepsilon} f(x,y)\,dy\,dx \approx \delta\varepsilon\, f(a,b)$

for $\varepsilon, \delta > 0$ small.

Note $\mathbb{P}(X=a, Y=b) = 0$ again!

Again $f(a,b)$ is not a probability!

Marginal densities:

Let (X,Y) have joint density $f = f(x,y)$, we define the marginal PDFs of X and Y by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Recall discrete case!

$p_X(x) = \sum_y p(x,y)$

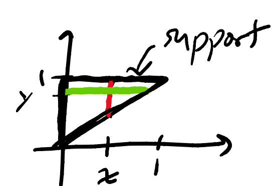
$p_Y(y) = \sum_x p(x,y)$

Expectations:

$$\mathbb{E}[h(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f(x,y)\,dy\,dx$$

$$\mathbb{E}[h(X,Y)] = \sum_x \sum_y h(x,y)\,p(x,y)$$

Ex 2 revisited: $f(x,y) = \begin{cases} 8xy & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{else.} \end{cases}$



$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^1 8xy\,dy$$

$$= 4xy^2\Big|_{y=x}^{1} = 4x(1-x^2) \quad \text{for } 0 \le x \le 1.$$

For x fixed, the range of possible y values is from x to 1.

For y fixed, the

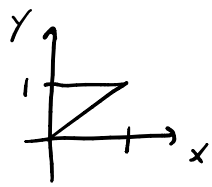
$= 4xy^2\Big|_{y=x}^{1} = 4x(1-x^2)$ for $0 \le x \le 1$. from x to 1.

For y fixed, the range of x values is 0 to y.

Complete answer: $f_X(x) = \begin{cases} 4x(1-x^2) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y 8xy\,dx$$
$$= 4x^2y\Big|_{x=0}^{y} = 4y^3 \quad \text{for } 0 \le y \le 1,$$

Complete answer: $f_Y(y) = \begin{cases} 4y^3 & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$



Calculate: $\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dy\,dx$

$$= \int_0^1\int_x^1 xy\cdot 8xy\,dy\,dx \overset{\text{(alt.)}}{=} \int_0^1\int_0^y xy\cdot 8xy\,dx\,dy$$
$$= \int_0^1\int_0^y 8x^2y^2\,dx\,dy = \int_0^1 \tfrac{8}{3}x^3y^2\Big|_{x=0}^{y} dy$$
$$= \int_0^1 \tfrac{8}{3}y^5\,dy = \tfrac{8}{18}y^6\Big|_{y=0}^{1} = \tfrac{4}{9}.$$

Def: Two joint continuous r.v.'s X and Y are independent if $f(x,y) = f_X(x)f_Y(y)$ for all x, y.

joint PDF = (X marginal) $\cdot$ (Y marginal)

Facts: If X and Y are indep:

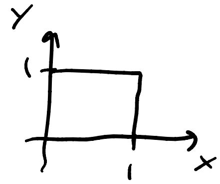
① $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$ for regions $A, B \subseteq \mathbb{R}$.

"and"

why?

$$\mathbb{P}(X \in A, Y \in B) = \int_A\int_B f(x,y)\,dy\,dx$$
$$= \int_A\int_B f_X(x)f_Y(y)\,dy\,dx$$
$$= \left(\int_A f_X(x)\,dx\right)\left(\int_B f_Y(y)\,dy\right)$$
$$= \mathbb{P}(X \in A)\mathbb{P}(Y \in B).$$

② $\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\,\mathbb{E}[h(Y)]$ for functions g and h.



Ex 1 revisited: $f(x,y) = \begin{cases} 1 & \text{if } 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$

Can read off directly that X and Y are independent!
↳ Factorizes

Marginals:
$$f_X(x) = \begin{cases} \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 1\,dy = 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

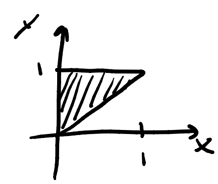
Similarly, $f_Y(y) = \begin{cases} 1 & \text{if } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$

Check: $f(x,y) = f_X(x) f_Y(y)$ for all x, y. ✓

so X and Y are independent.

Calculate: $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$ by independence
$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Or directly: $\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dy\,dx$
$= \int_0^1 \int_0^1 xy\,dy\,dx = \cdots = \frac{1}{4}$.



Ex 2 revisited: $f(x,y) = \begin{cases} 8xy & \text{if } 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

These are not independent.
Two ways to show this:

① Use marginals from before and check that $f_X(x) f_Y(y) \ne f(x,y)$, for some x, y.

② The function f does not factorize because, even though $8xy = (\text{function of } x)\cdot(\text{function of } y)$, the range of x depends on the value of y, and vice versa.

2 The support of f is not rectangular.

Ex 3: Let $X \sim \text{Unif}[0,1]$ and $Y \sim \text{Exponential}(\lambda)$, where $\lambda > 0$,

Ex 3: Let $X \sim \mathrm{Unif}[0,1]$ and $Y \sim \mathrm{Exponential}(\lambda)$, where $\lambda > 0$, and suppose X and Y are independent.

Q.1: Find joint PDF of X and Y.

$$X \sim \mathrm{Unif}[0,1] \longrightarrow f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$Y \sim \mathrm{Exp}(\lambda) \longrightarrow f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & \text{if } y \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

Independence implies joint PDF is

$$f(x,y) = f_X(x) f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & \text{if } 0 \le x \le 1,\ y \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

Q.2: Let $Z = XY$. Find the PDF of Z.

↳ Always start with CDF, since PDF $\neq$ probability, and then differentiate to get PDF.

For $r > 0$, (note $Z > 0$ since $X > 0$ and $Y > 0$)

(CDF of Z)

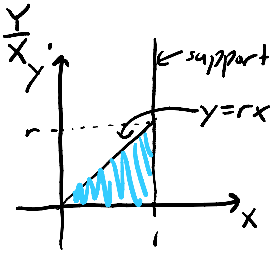
$$\mathbb{P}(Z \le r) = \mathbb{P}(XY \le r)$$
$$= \int_0^1 \int_0^{\frac{r}{x}} \lambda e^{-\lambda y}\, dy\, dx$$
$$= \int_0^1 -e^{-\lambda y}\Big|_{y=0}^{\frac{r}{x}}\, dx = \int_0^1 \left(1 - e^{-\frac{\lambda r}{x}}\right) dx$$
$$= \text{some function of } r \quad \text{(no closed form for integral).}$$

PDF of Z is then $f_Z(r) = \frac{d}{dr}\mathbb{P}(Z \le r) = \frac{d}{dr}\int_0^1 \left(1 - e^{-\frac{\lambda r}{x}}\right) dx$

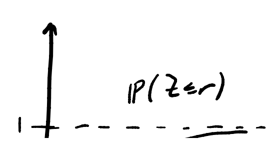
$$= \text{some function } r \ldots$$

* **EDIT:** The better (more tractable) example is $Z = \frac{Y}{X}$.

Then, for $r > 0$, the CDF of Z is

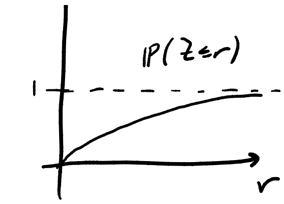


$$\mathbb{P}(Z \le r) = \mathbb{P}\left(\frac{Y}{X} \le r\right) = \mathbb{P}(Y \le rX)$$
$$= \int_0^1 \int_0^{rx} \lambda e^{-\lambda y}\, dy\, dx$$
$$= \int_0^1 -e^{-\lambda y}\Big|_{y=0}^{rx}\, dx$$
$$= \int_0^1 \left(1 - e^{-\lambda r x}\right) dx$$



$$= \int_0^1 \left(1 - e^{-\lambda r x}\right) dx$$

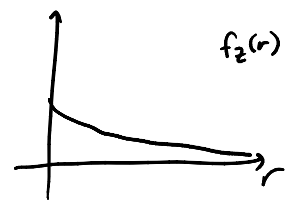
$$= \left(x + \frac{1}{\lambda r} e^{-\lambda r x}\right)\Big|_{x=0}^{1} = 1 - \frac{1}{\lambda r}\left(1 - e^{-\lambda r}\right).$$



Hence, the PDF of Z is

$$f_Z(r) = \frac{d}{dr}\,\mathbb{P}(Z \le r) = \frac{d}{dr}\left(1 - \frac{1}{\lambda r}\left(1 - e^{-\lambda r}\right)\right)$$

$$= \frac{1}{\lambda r^2}\left(1 - e^{-\lambda r}\right) - \frac{1}{r} e^{-\lambda r}, \quad \text{for } r > 0.$$



For a complete answer, one should also specify

$$\mathbb{P}(Z \le r) = 0 \quad \text{and} \quad f_Z(r) = 0 \quad \text{for } r \le 0.$$