

# Mathematics of Finance

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## **The Role of Stocks and Bonds in Corporate Capital Formation**

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- Companies need capital (i.e. money) to purchase inventory, open new plants, etc.
- Companies w/o an established name usually raise capital by:
  - getting a loan from a bank, and/or
  - getting money from some private equity source
- Companies w/an established name can raise capital through the stock and the bond markets at a more attractive level

# Stocks and Bonds

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- A stock is a security representing an ownership interest in a company
- A bond is a security whereby the issuer borrows money, called the principal, and agrees to:
  - pay the lender (the bond holder) interest payments based on the outstanding amount of principal
  - return the principal through a “lump sum” payment, periodic payments over time, or in the case of a “perpetual” bond, not at all
- Publicly traded stocks are actively traded at exchanges as well as OTC (Over the Counter) markets (independent market makers)
- Publicly traded bonds are actively traded mainly in the OTC market (i.e., wall street bond dealers)

# Investment Banks' Role in Corporate Funding

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- Traditionally, investment banking refers to Corporate Finance,
  - which is the process of raising money for corporate clients (or public institutions) in the form of equity/stocks, debt, or convertible securities.
- This process involves two steps:
  - determining the most efficient funding for the client
    - type, amount, and structure
  - finding investors to supply those funds
    - for larger investment banks, this step will involve other areas of the firm, such as sales, trading, research and a syndicate function

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Example:

- ABC Co. needs money to build a new plant
- ABC Co. decides to employ Investment Bank IBK in this pursuit
- Services IBK provides ABC Co.:

(1) Corporate Finance expertise:

+ evaluates funding options (amount, type, and structure); options considered:

- common equity
- debt (fixed-rate and floating rates with various maturities)
- convertible debt

+ helps decide that a \$200MM 7 yr. fixed-rate bond offering is the best option

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- Services IBK provides ABC Co. (continued):

(2) Pricing & Underwriting, and Distribution Capability:

- + IBK's sales/trading expertise in the secondary market helps determine fair pricing of the bond (new issues are sold in the primary market, issues are subsequently traded in the secondary market)
- + IBK leads an underwriting group, which purchases the bonds and sells them to investors (over days and sometimes weeks)
- + IBK (and other dealers) are expected to provide 2-way markets to buy/sell ABC's bonds in the secondary market, an important function that would
  - keep institutional investors happy which in turn satisfies ABC Co., and
  - keep IBK on “top of the market” and remain competitive

# **Governments' Need to Borrow Money**

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The U.S. Government, federal agencies, and state/local governments (municipalities)

raise money in the capital markets to fund expenditures and other activities, for example:

- the U.S. Government issues Treasury Bills/Notes/Bonds to fund federal programs
- federal agencies such as FNMA issue debt (bearing its name) so as to help provide funds to home buyers
- state and local governments issue municipal bonds to fund building roads, etc.

# Investment Banks' Role in Government Related Debt

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The main roles of investment banks in these debt markets are:

- U.S. Treasury Debt Market: bid in U.S. Treasury auctions (if primary dealers)
- Agency and Municipal Debt Market:
  - advise on efficient financing: e.g., maturities, call features, the need for credit enhancement (muni bonds)
  - underwrite and distribute debt
- Secondary Market for the above: provide 2-way markets for investors to buy/sell



# **Who are the Institutional Investors?**

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Institutional Investors are clients on the sales/trading side, including:

- Pension Funds
- Money/Asset/Investment Managers (& Hedge Funds)
- Insurance companies
- Banks
- Mutual Funds
- Central Banks (in various nations)

# Front Office Functions

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- Sales:
  - Talk to institutional investors; facilitate trades between customers and traders
  - Sometimes specialize in a major product (e.g., government, mortgage sales)
- Trading:
  - Commit the firm's capital buying/selling securities
  - A “Market Maker” has the obligation and the privilege to trade with customers
    - usually trade a specific sector within a product (e.g., 1-5 yr. Treasuries )
    - also trades with other dealers via brokers
  - A “Proprietary Trader”: has neither the obligation nor privilege to trade with customers

# 1. Basic traded assets.

There are many financial instruments that are traded every day on exchanges and among dealers around the world. **Stocks** of individual companies are probably best known in the general public.

Other actively traded assets are **currencies, government and corporate bonds, physical commodities**: crude oil, oil products, natural gas, electricity, precious metals, base metals, agricultural commodities, and others.

Many assets are traded on organized exchanges, like New York Stock Exchange, Chicago Mercantile Exchange, EUREX etc.

The mechanics of trading can be different on different exchanges, and exchanges themselves can be fully electronic or include human brokers in the trading process. We do not go here in the details of mechanics of trading.

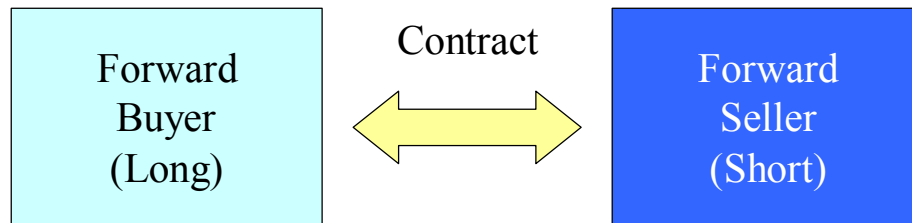
It is important to understand that many assets are traded not on organized exchanges but “**over the counter**”. That is when the trading is done between interested counterparties, often one of the parties is a dealer having some inventory of the asset.

## 2. Forward Contract.

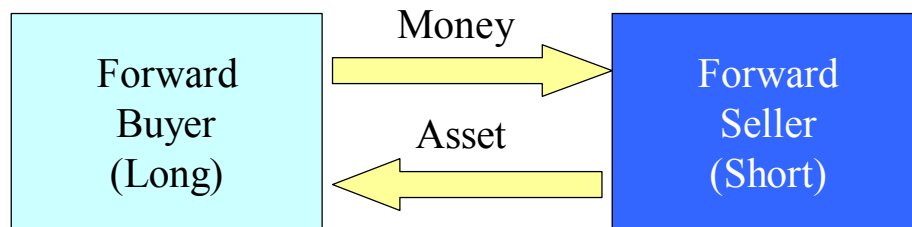
A **forward contract** is an agreement made between two counterparties to buy or sell an asset on a certain future date for an agreed price, called the **delivery price**.

The party who agrees to buy the asset is said to hold the **long position**, and the party who agrees to sell holds the **short position** in the contract.

$T = \text{Now}$



$T = \text{Later Delivery Time}$



A forward contract is usually transacted not on an organized exchange but “over the counter” by agreement of two counterparties. When a forward contract is negotiated, the delivery price that denoted here by  $K$  is usually set in such way that neither counterparty owes money initially. Such price  $K$  is called **forward price** of the underlying asset for delivery time  $T$ .

In the simplest case, the forward price is the current price of the asset plus an additional amount that reflects interest earned on the asset's value. We assume first for simplicity that holding of the asset provides no income or other benefits from holding. If that is not true, i.e. when the asset provides income or convenience benefits from holding, we would need to reflect it in calculating forward price. But for now we would stick to assets providing no income and convenience yields.

Suppose  $X$  is the current asset price,  $r$  is the continuously compounded risk-free interest rate,  $T$  is delivery time and  $t$  is current time so that  $T-t$  is the time length to delivery. Then the forward price  $F$  of the asset at time  $t$  is

$$F = X \exp(r(T-t))$$

If we have to finance holding of asset by borrowing  $X$  dollars at time  $t$ ,

$F$  is exactly equal to  **$X + \text{Accrued interest from } t \text{ to } T$** .

As we shall see later there is a deep economic reason for the last identity.

So when the forward contract is entered, the value of the forward contract is zero and the forward price is equal to the delivery price  $F=K$ . The delivery price  $K$  will remain constant over the life of the forward contract, while the forward price  $F$  may change as spot price and remaining time to delivery change.

## EXAMPLE

Suppose that the current price of a stock is 100. Stock is paying no dividends. The risk-free interest rate is 5% per year. The forward price of the stock for 1 year is

$$F=100 \exp(0.05*1)=105.13.$$

There is a reason why the delivery price  $K$  for a forward contract should be  $F=X \exp(r(T-t))$ .

We can show that if  $K$  is less than or greater than  $F$ , then an investor can easily make an arbitrage, i.e. risk-free, profit.



### **3.Arbitrage.**

#### **DEFINITION**

**Arbitrage** is exploiting of discrepancies between prices of the same or related securities in different markets. By trading securities in these markets, a profit is realized without taking a risk.

#### **EXAMPLE**

Buying XYZ stock in London for 100 British Pounds and simultaneous selling it in New York for 151 Dollars when  $1 \text{ Pound} = 1.50 \text{ Dollars}$  will yield a riskless profit of 1 Dollar. That would be an arbitrage.

## EXAMPLE

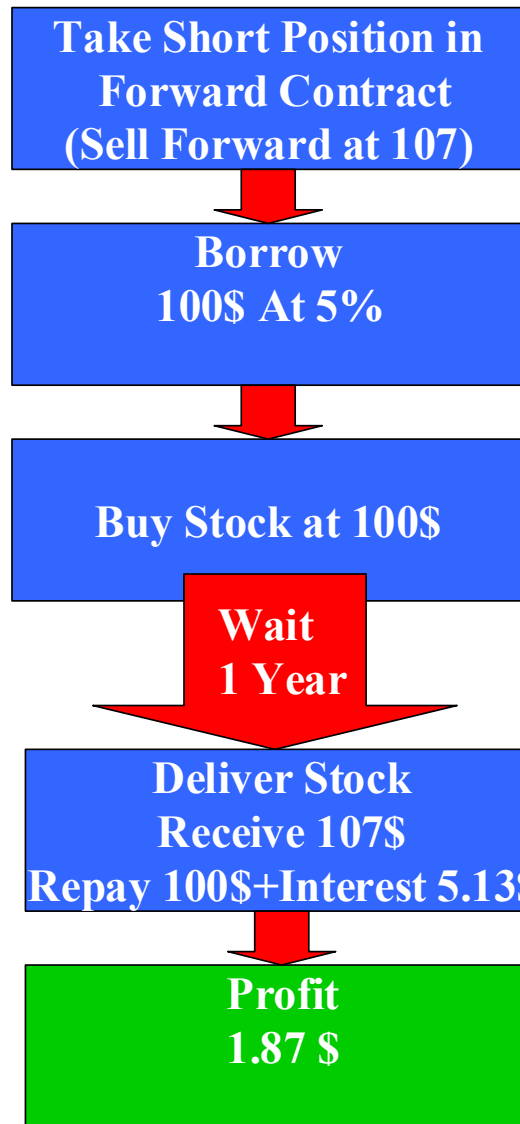
When the stock price is 100, stock is paying no dividends, and interest rate is 5% per year with continuous compounding the theoretical forward price of the stock for 1 year is  **$F=100 \exp(0.05*1)=105.13$** .

Let us show how to make an arbitrage in 2 cases

- There is a buyer of a stock 1 year forward at 107 (above theoretical forward price).
- There is a seller of a stock 1 year forward at 104 (below theoretical forward price).

Here we assume that stock is available for borrowing, and can be sold short.

(Short sale is the sale of shares borrowed from the broker. Short seller borrows shares from the broker, sells them in the market and receives cash proceeds. Short seller would eventually buy shares back and would benefit from share price going down.)



# Forward Price of Stock Paying no dividends

100 \$ = 1 Share Spot price

$r_{\text{usd}}=5\%$

Continuous  
Compounding



$$100 * e^{0.05 * 1} = 105.13\$ =$$

$r_{\text{stock}}=0\%$   
(Dividends=0%)

Continuous  
Compounding



1 Share Forward Price in 1 year

# Forward Price of Stock Paying continuous dividends at rate $d\%$

100 \$ = 1 Share Spot price

$r_{\text{usd}} = 5\%$

Continuous  
Compounding



$$100 * e^{0.05 * 1} = 105.13\$$$

$r_{\text{stock}} = d = 2\%$

Continuous  
Compounding

(Dividends reinvested in stock)



$$1 * e^{0.02 * 1} = 1.0202 \text{ Share}$$

$$100 * e^{(0.05 - 0.02) * 1} = 103.05\$ = 1 \text{ Share Forward Price in 1 year}$$

## FX Forward rate

$$1.30 \text{ USD} = 1 \text{ EUR} \quad \text{Spot rate}$$

$$r_{\text{usd}} = 1\%$$

Annual  
Compounding

$$r_{\text{eur}} = 2\%$$

Annual  
Compounding

$$1.30 \cdot (1 + 0.01) \text{ USD} = 1 \cdot (1 + 0.02) \text{ EUR}$$

$$\frac{1.30 \cdot (1 + 0.01)}{1 \cdot (1 + 0.02)} \text{ USD} = 1 \text{ EUR}$$

In 1 year

$$1.2873 \text{ USD} = 1 \text{ EUR} \quad \text{1 year Forward rate}$$

**If 1 year forward is 1.31 USD=1Eur  
(more than theoretical forward 1.2873)**

**Arbitrage:**

- 1. Sell 1 year forward 1 Euro at 1.31\$ per Euro**
- 2. Borrow  $1.2745\$ = 1.30\$ / (1+2\%)$  at 1%.**  
**Have to repay  $1.30\$ * (1+1\%) / (1+2\%) = 1.2873\$$  in 1 year**
- 3. Exchange 1.2745\$ at a spot rate 1.30\$=1 Euro. Get  $1 / (1+2\%) = 0.9804$  Euro now.**
- 4. Deposit 0.9804 Euro now at 2% for 1 year. Get 1 Euro in 1 year.**
- 5. In 1 year using forward that we entered exchange  
1 Euro into 1.31 \$  
Repay 1.2873 \$ that you have to repay**
- 6. Profit in 1 year  $1.31 - 1.2873 = 0.0227\$$**

**If 1 year forward is 1.28 USD=1Eur  
(less than theoretical forward 1.2873)**

**Arbitrage:**

- 1. Buy 1 year forward 1 Euro at 1.28\$ per Euro**
- 2. Borrow  $1/(1+2\%) = 0.9804$  Euro now at 2%.  
Have to repay 1Euro in 1 year**
- 3. Exchange  $1/(1+2\%) = 0.9804$  Euro at a spot rate 1.30\$=1 Euro.  
Get  $1.30\$/ (1+2\%) = 1.2745\$$  now.**
- 4. Deposit  $1.30\$/ (1+2\%) = 1.2745\$$  at 1%.  
Receive  $1.30\$ * (1+1\%) / (1+2\%) = 1.2873\$$  in 1 year**
- 5. In 1 year using forward that we entered exchange  
1.28 \$ into 1 Euro. Repay 1 Euro that you have to repay**
- 6. Profit in 1 year  $1.2873 - 1.28 = 0.0073\$$**