PROBABILITY GU4155: Spring 2023

ASSIGNMENT # 2
DO EXERCISES 1, 2, 4, 5, 8, 10 AND RETURN BY
9:00 PM TUESDAY, JANUARY 31, 2023

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Read Chapter 3 in Walsh (2012), as well as Sections 4.1 – 4.3 in Stirzaker (2003).

Exercise #1: Construct a distribution function which is discontinuous at every rational point, and continuous at all irrational points on the real line.

Conversely: is there a distribution function which is discontinuous at every irrational point, and continuous at all rational points on the real line?

Exercise #2: (i) Suppose μ is a probability measure on the Borel subsets of the real line. We use it to define a function $F : \mathbb{R} \to [0,1]$ via

$$F(x) := \mu\big((-\infty, x]\big), \quad x \in \mathbb{R}.$$

Show that this function is nondecreasing and right continuous, that it satisfies $F(-\infty) = 0$ and $F(\infty) = 1$, and that for any real numbers $a < b$ we have

$$\mu((a,b]) = F(b) - F(a), \quad \mu([a,b)) = F(b-) - F(a-), \quad \mu(\{a\}) = F(a) - F(a-), \quad (0.1)$$

$$\mu([a,b]) = F(b) - F(a-), \quad \mu((a,b)) = F(b-) - F(a). \quad (0.2)$$

(ii) Given a random variable $X : \Omega \to \mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider the induced measure $\mu_X = \mathbb{P} \cdot X^{-1}$. Show that the function

$$F_X(x) := \mu_X\big((-\infty, x]\big) = \mathbb{P}(X \le x), \qquad x \in \mathbb{R}$$

is nondecreasing and right continuous, and satisfies $F_X(-\infty) = 0$, $F_X(\infty) = 1$ as well as the properties (0.1), (0.2) above.

Exercise #3: De Moivre-Laplace for Coin Tossing. Let $X_1, X_2, \cdots$ be independent random variables with $\mathbb{P}(X_j = 1) = 1 - \mathbb{P}(X_j = 0) = p \in (0,1)$ for all $j \in \mathbb{N}$, and denote by $S_n = \sum_{j=1}^n X_j$ the "number of successes in the first n tosses". Show that

$$\mathbb{P}\left[a \le \frac{S_n - np}{\sqrt{np(1-p)}} \le b \right] \longrightarrow \Phi(b) - \Phi(a)\,, \qquad a < b \quad \text{in } \mathbb{R} \tag{0.3}$$

as $n \to \infty$, where

$$\Phi(x) := \int_{-\infty}^{x} \varphi(\xi)\, d\xi\,, \qquad \varphi(x) := \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

is the so-called *standard Normal distribution function.*
(*Hint:* With the help of the STIRLING formula $n! \sim \sqrt{2\pi n}\, n^n\, e^{-n}$, actually in its stronger form

$$n\,! = \sqrt{2\pi}\; n^{n+1/2} e^{-n+\varepsilon_n} \quad \text{with} \quad \frac{1}{12n+1} < \varepsilon_n < \frac{1}{12n} \tag{0.4}$$

from the previous assignment, show first the "local" form

$$\lim_{n\to\infty} \left(\sqrt{npq} \cdot \frac{n!}{k_n!\,(n-k_n)!}\, p^{k_n} (1-p)^{n-k_n} \right) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} = \varphi(x)$$

of this result, where $x \in \mathbb{R}$ is fixed and

$$k_n := x\sqrt{np(1-p)} + np\,;$$

then observe that this convergence is uniform over x in the bounded interval $[a, b]$, as the next step towards (0.3).)

Exercise #4: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and consider a set $\mathcal{J}$ which is (at most) countable. For every $j \in \mathcal{J}$ we are given two events $B_j \subseteq A_j$ in $\mathcal{F}$. Show that

$$\mathbb{P}\Big(\bigcup_{j\in\mathcal{J}} A_j \Big) - \mathbb{P}\Big(\bigcup_{j\in\mathcal{J}} B_j \Big) \le \sum_{j\in\mathcal{J}} \big[\mathbb{P}(A_j) - \mathbb{P}(B_j) \big]\,.$$

Exercise #5: Coin Tossing (cont'd). Use (0.3) to establish the BERNOULLI *Weak Law of Large Numbers*

$$\lim_{n\to\infty} \mathbb{P}\big(|\,\overline{X}_n - p| > \varepsilon \big) = 0\,, \qquad \forall \;\; \varepsilon > 0$$

in the Coin-Tossing context of Exercise 3. Here $\overline{X}_n := S_n/n$ is the relative frequency of 1's in the first n independent tosses of the coin.

Exercise #6: Do Problem 3.46 in WALSH (2012).

Exercise #7: Do Problem 3.47 in WALSH (2012).

Exercise #8: Do Problem 3.37 in WALSH (2012).

Exercise #9: Do Problem 2.51 in WALSH (2012).

Exercise #10: In the context of Exercise 3, compute the probability $\mathbb{P}(S_n \text{ is even})$.