STAT GU4261/GR5261 Statistical Methods in Finance

Spring 2022

Homework 7 Suggested Solutions

Question 1

We note that:

$$c_Y(u_1, u_2) = \frac{f_Y(F_{Y_1}^{-1}(u_1), F_{Y_2}^{-1}(u_2))}{f_{Y_1}(F_{Y_1}^{-1}(u_1)) f_{Y_2}(F_{Y_2}^{-1}(u_2))}.$$

Now notice that the joint is given by:

$$f_Y(y_1, y_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp(-\frac{1}{2(1-\rho^2)}[\frac{(y_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(y_1-\mu_1)(y_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(y_2-\mu_2)^2}{\sigma_2^2}]).$$

Whereas the marginals are given by:

$$f_{Y_1}(y_1) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp(-\frac{(y_1-\mu_1)^2}{2\sigma_1^2}),$$

$$f_{Y_2}(y_2) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp(-\frac{(y_2-\mu_2)^2}{2\sigma_2^2}).$$

Next, note that:

$$F_{Y_1}(y_1) = P(Y_1 \leq y_1) = P(\mu_1 + \sigma_1 Z \leq y_1) = P(Z \leq (y_1 - \mu_1)/\sigma_1) = \Phi(\frac{y_1-\mu_1}{\sigma_1}),$$

where Φ denotes the CDF of a standard normal. By inverting this function, we find that:

$$F_{Y_1}^{-1}(u_1) = \mu_1 + \sigma_1\Phi^{-1}(u_1),$$

and similarly,

$$F_{Y_2}^{-1}(u_2) = \mu_2 + \sigma_2\Phi^{-1}(u_2).$$

Putting everything together, we find that:

$$c_Y(u_1, u_2) = \frac{1}{\sqrt{1-\rho^2}} \exp(-\frac{1}{2(1-\rho^2)}[\rho^2(\Phi^{-1}(u_1)^2 + \Phi^{-1}(u_2)^2) - 2\rho\Phi^{-1}(u_1)\Phi^{-1}(u_2)]).$$

Question 2

Problem 1

(a) This is a t copula with 1 degree of freedom and the following covariance matrix:

$$\begin{pmatrix} 1 & -0.6 & 0.75 \\ -0.6 & 1 & 0 \\ 0.75 & 0 & 1 \end{pmatrix} \tag{1}$$

(b) The sample size is 500.

Problem 2

(a) They do not appear independent as the data is tightly clustered in the corners whereas the middle part of the plot has more dispersion. In particular, when either component is extremal (on either end), the other component tends to also be extremal (on either end). This contradicts independence.

(b) The tendency for extremes to co-occur along both components is a sign of tail dependence. This can be seen in the corners of the plot.

(c) See previous answers.

(d) The correlation coefficient in the copular refers to the original t variates in unconstrained real space. When transformed onto the unit interval (in this case, making a 3-dimensional cube), those coefficients are no longer equal to the correlation of the transformed random variables. This is evident from the fact that the CDF transformation applied to obtain the copula is a nonlinear transformation, and correlation is not preserved under general nonlinear transformations.

Problem 3

(a) They are exponentially distributed with rate parameter 2, 3, and 4, respectively.

(b) The second and third components are indeed independent, since they are formed by taken univariate transformations (the inverse exponential CDF) of independent random variables. Note that in this case the second and third components of the copula themselves are independent random variables, since they correspond to univariate transformations of independent normal random variables (they are formed by taking the normal CDF of two multivariate normal random variables with zero covariance, which we know are independent).

Question 3

One can use code of the following kind:

```
library(readxl)
library(copula)

data = read_excel("~/Downloads/CAPM-DATA.xlsx")
GE = data$GE
GM = data$GM
n=length(GE)

edata=cbind(rank(GE)/(n+1), rank(GM)/(n+1))

ncop=normalCopula(param=c(0), dim=2, dispstr="un")
fit1=fitCopula(data=edata, copula=ncop, method="ml")
AIC(fit1)

tcop=tCopula(param=c(0), dim=2, dispstr="un", df=5)
fit2=fitCopula(data=edata, copula=tcop, method="ml")
```

```
AIC(fit2)

gcop=archmCopula(family="gumbel", dim=2, param=2)
fit3=fitCopula(data=edata, copula=gcop, method="ml")
AIC(fit3)
```

Question 4

(a) We can use the following code to get an answer of 32679 dollars:

```
C = function(u, v) {
  return(1 - sqrt((1-u)^2 + (1-v)^2 - (1-u)^2 * (1 - v)^2))
}

Fa = function(x) {return(pexp(x, 0.1))}
Fb = function(x) {return(pexp(x, 0.07))}

price = 2*C(Fa(1), Fb(1)) + C(Fa(2), Fb(2)) - C(Fa(2), Fb(1)) -
C(Fa(1), Fb(2)) + C(Fa(1), Fb(1))
```

(b) The fair price is the same value from the previous part, but in addition we have to add:

$$\begin{aligned}
1P(1 < T_A < 2, 2 < T_B) + 1P(2 < T_A, 1 < T_B < 2) &= P(1 < T_A < 2) + P(1 < T_B < 2) + \\
&\quad - 2P(T_A < 2, T_B < 2) \\
&\quad + P(T_A < 1, T_B < 2) + P(T_A < 2, T_B < 1) \\
&= F_A(2) - F_A(1) + F_B(2) - F_B(1) \\
&\quad - 2C(F_A(2), F_B(2)) \\
&\quad + C(F_A(1), F_B(2)) + C(F_A(2), F_B(1)).
\end{aligned}$$

This can be evaluated with the following code (143376 dollars):

```
price + Fa(2) - Fa(1) + Fb(2) - Fb(1) - 2C(Fa(2), Fb(2)) + C(Fa(1), Fb(2)) + C(Fa(2), Fb(1
```