Economics 361
Simultaneous Equations Model: Supply & Demand

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Supply & Demand

Suppose we are interested in estimating the supply and demand function for gasoline. We believe that the following **structural** (or *behavioral*) relationship holds:

$$\begin{aligned} \text{Supply :} \quad & P_t = \alpha_0 + \alpha_1 Q_t^s + \delta C_t + \eta_t \\ \text{Demand :} \quad & Q_t^d = \beta_0 + \beta_1 P_t + \gamma I_t + \nu_t \\ \text{Assume} \quad & \eta_t \overset{i.i.d.}{\sim} N(0, \sigma_\eta^2) \quad \nu_t \overset{i.i.d.}{\sim} N(0, \sigma_\nu^2) \end{aligned}$$

t indexes observations. P_t is the price of gasoline, (Q_t^s, Q_t^d) quantity supplied and quantity demanded, C_t the cost of providing gasoline, and I_t consumer income. The above system of equations presumes that firm sets gasoline prices (P_t) keeping in mind the quantity it will supply at the price and the cost of providing the gasoline and the consumers buy gasoline (Q_t) keeping in mind the price of gasoline and the amount of income she has. (η_t, ν_t) will be interpreted as "shocks" that induce parallel shifts in the supply and demand functions, respectively. An example of a supply shock would be a transitory cost increase (war on Iraq) and an example of a demand shock would be a transitory demand increase (summertime road trips).

Assuming that the gasoline market is in equilibrium, we know that $Q_t^s = Q_t^d = Q_t^e$. Consequently, we know that P_t is determined in part by Q_t^d and vice versa, leading to a **simultaneity** bias

$$\begin{aligned} E(\eta_t | Q_t^s, C_t) &= E(\eta_t | Q_t^d, C_t) \\ &= E(\eta_t | \beta_0 + \beta_1 \mathbf{P_t} + \gamma I_t + \nu_t, C_t) \\ E(\nu_t | P_t, I_t) &= E(\nu_t | \alpha_0 + \alpha_1 Q_t^s + \delta C_t + \eta_t, I_t) \\ &= E(\nu_t | \alpha_0 + \alpha_1 \mathbf{Q_t^d} + \delta C_t + \eta_t, I_t) \end{aligned}$$

Note that P_t is a function of η_t and Q_t^d is a function of ν_t. Consequently, even though $E(\eta_t) = 0$ and $E(\nu_t) = 0$, $E(\eta_t | Q_t^s, C_t) \neq 0$ and $E(\nu_t | P_t, I_t) \neq 0$ in general. So the supply and demand equations,

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individually, do not satisfy the classical OLS assumptions. This is *not* an example of the SUR model.

Note that the problem arises because of the equilibrium condition $Q_t^s = Q_t^d$. If Q_t^s and Q_t^d were independent of each other, then the supply and demand equations would each satisfy the classical OLS assumptions. This corresponds to the case where we survey the firms (consumers) of the quantity they would supply (demand), **not** taking into consideration the quantity that would be demanded (supplied) by the consumers (firms) at the given price. It is the **simultaneous** determination of quantity supplied and quantity demanded in equilibrium that lead to the OLS coefficients being "biased" for $(\alpha_0, \alpha_1, \gamma, \beta_0, \beta_1, \delta)$.

Suppose instead of focusing on an equation for supply and an equation for demand, we focus on an equation for the market equilibrium price and an equation for the market equilibrium quantity. We can derive this from the above structural supply and demand equations by setting quantity demanded (Q_t^d) equal to quantity supplied (Q_t^s) and re-arranging terms

$$\begin{aligned}
P_t &= \alpha_0 + \alpha_1 Q_t^s + \delta C_t + \eta_t \\
&= \alpha_0 + \alpha_1 Q_t^d + \delta C_t + \nu_t \\
&= \alpha_0 + \alpha_1 [\ \beta_0 + \beta_1 P_t + \gamma I_t + \nu_t\ ] + \delta C_t + \eta_t \\
P_t - \alpha_1\beta_1 P_t &= \alpha_0 + \alpha_1\beta_0 + \alpha_1\gamma I_t + \delta C_t + \alpha_1\nu_t + \eta_t \\
P_t &= \underbrace{\frac{\alpha_0 + \alpha_1\beta_0}{1 - \alpha_1\beta_1}}_{\Pi_{11}} + \underbrace{\frac{\alpha_1\gamma}{1 - \alpha_1\beta_1}}_{\Pi_{12}} I_t + \underbrace{\frac{\delta}{1 - \alpha_1\beta_1}}_{\Pi_{13}} C_t + \underbrace{\frac{\alpha_1\nu_t + \eta_t}{1 - \alpha_1\beta_1}}_{\epsilon_{1t}}
\end{aligned}$$

Similarly, we can calculate the market equilibrium quantity equation as

$$\begin{aligned}
Q_t &= \beta_0 + \beta_1 P_t + \gamma I_t + \nu_t \\
&= \beta_0 + \beta_1 [\ \alpha_0 + \alpha_1 Q_t + \delta C_t + \eta_t\ ] + \gamma I_t + \nu_t \\
Q_t - \alpha_1\beta_1 Q_t &= \beta_0 + \alpha_0\beta_1 + \gamma I_t + \beta_1\delta C_t + \beta_1\eta_t + \nu_t \\
Q_t &= \underbrace{\frac{\beta_0 + \alpha_0\beta_1}{1 - \alpha_1\beta_1}}_{\Pi_{21}} + \underbrace{\frac{\gamma}{1 - \alpha_1\beta_1}}_{\Pi_{22}} I_t + \underbrace{\frac{\beta_1\delta}{1 - \alpha_1\beta_1}}_{\Pi_{23}} C_t + \underbrace{\frac{\beta_1\eta_t + \nu_t}{1 - \alpha_1\beta_1}}_{\epsilon_{2t}}
\end{aligned}$$

Together, these two market equilibrium equations can be considered the **reduced form** system

$$\begin{aligned}
P_t &= \Pi_{11} + \Pi_{12} I_t + \Pi_{13} C_t + \epsilon_{1t} \\
Q_t &= \Pi_{21} + \Pi_{22} I_t + \Pi_{23} C_t + \epsilon_{2t}
\end{aligned}$$

Note two things. First, each of the equation excludes the other (no quantity in the price equation and no price in the quantity equation). Second, the implied disturbance terms $(\epsilon_{1t}, \epsilon_{2t})$ are a function of both the supply and demand shocks and are mean zero.

$$\begin{aligned}
E(\epsilon_{1t}) = \frac{1}{1 - \alpha_1\beta_1} E(\alpha_1\nu_t + \eta_t) = 0 \\
E(\epsilon_{2t}) = \frac{1}{1 - \alpha_1\beta_1} E(\beta_1\eta_t + \nu_t) = 0
\end{aligned}$$

Assuming that C_t and I_t are independent of the equilibrium price and quantity, then we know that

$$E(\epsilon_{1t}|C_t, I_t) = E(\epsilon_{1t}) = 0$$
$$E(\epsilon_{2t}|C_t, I_t) = E(\epsilon_{2t}) = 0$$

The assumption that C_t and I_t are independent of our two endogenous variables, P_t and Q_t, is referred to as an **exogeneity** assumption. We are assuming that the cost of gasoline production and consumer income are determined **exogenously** from the market price and quantity of gasoline. This seems reasonable. Of course, the assumption can be violated in special circumstances (e.g. consumer works for the gasoline firm).

Given an exogenous C_t and I_t, we can show that the above **reduced form** regression equations individually satisfy the classical OLS assumptions. Hence, running OLS on each individual reduced form equation yields us B.L.U.E. for $(\Pi_{11}, \Pi_{12}, \Pi_{13}, \Pi_{21}, \Pi_{22}, \Pi_{23})$.

Moreover, note that we can use the reduced form parameters to back out the structural parameters

$$\begin{aligned} \alpha_1 &= \frac{\Pi_{12}}{\Pi_{22}} & \beta_1 &= \frac{\Pi_{23}}{\Pi_{13}} \\ \gamma &= \Pi_{22}(1 - \frac{\Pi_{12}}{\Pi_{22}}\frac{\Pi_{23}}{\Pi_{13}}) & \delta &= \Pi_{13}(1 - \frac{\Pi_{12}}{\Pi_{22}}\frac{\Pi_{23}}{\Pi_{13}}) \\ \alpha_0 &= \Pi_{11} - \frac{\Pi_{12}}{\Pi_{22}}\Pi_{21} & \beta_0 &= \Pi_{21} - \frac{\Pi_{23}}{\Pi_{13}}\Pi_{11} \end{aligned}$$

Given consistent estimates of $(\Pi_{11}, \Pi_{12}, \Pi_{13}, \Pi_{21}, \Pi_{22}, \Pi_{23})$ we can derive consistent estimates of $(\alpha_0, \alpha_1, \gamma, \beta_0, \beta_1, \delta)$ via Slutsky's Theorem and the above algebraic transformations.

Observe that each reduced form parameter was necessary in order to back out an estimate for the six structural parameters. No reduced form parameter was superfluous. This is an example of a structural model that is **just identified**. Each exogenous variable - the constant, gasoline cost, and consumer income - was necessary in order to obtain an estimate for the structural parameters. To see this, consider the matrix algebra formulation of the exercise we conducted above

$$\begin{aligned} \underbrace{\begin{pmatrix} P & Q \end{pmatrix}}_{Y'} \underbrace{\begin{pmatrix} 1 & -\beta_1 \\ -\alpha_1 & 1 \end{pmatrix}}_{\Gamma} &= \underbrace{\begin{pmatrix} 1 & I & C \end{pmatrix}}_{X'} \underbrace{\begin{pmatrix} \alpha_0 & \beta_0 \\ 0 & \gamma \\ \delta & 0 \end{pmatrix}}_{B} + \underbrace{\begin{pmatrix} \eta & \nu \end{pmatrix}}_{u'} \\ Y'\Gamma &= X'B + u' \\ Y' &= X' \underbrace{B\Gamma^{-1}}_{\Pi} + u'\Gamma^{-1} \\ Y' &= X'\Pi + u'\Gamma^{-1} \end{aligned}$$

The above equation is simply the reduced form equations in matrix (transposed) format. Note that

$$
\begin{aligned}
\Gamma^{-1} &= \begin{pmatrix} 1 & -\beta_1 \\ -\alpha_1 & 1 \end{pmatrix}^{-1} \\
&= \frac{1}{1-\alpha_1\beta_1}\begin{pmatrix} 1 & \beta_1 \\ \alpha_1 & 1 \end{pmatrix} \\
u'\Gamma^{-1} &= \left(\frac{\eta+\alpha_1\nu}{1-\alpha_1\beta_1} \quad \frac{\beta_1\eta+\nu}{1-\alpha_1\beta_1} \right) \\
\Pi &= B\Gamma^{-1} \\
&= \begin{pmatrix} \alpha_0 & \beta_0 \\ 0 & \gamma \\ \delta & 0 \end{pmatrix} \frac{1}{1-\alpha_1\beta_1}\begin{pmatrix} 1 & \beta_1 \\ \alpha_1 & 1 \end{pmatrix} \\
&= \frac{1}{1-\alpha_1\beta_1}\begin{pmatrix} \alpha_0+\alpha_1\beta_0 & \beta_0+\alpha_0\beta_1 \\ \alpha_1\gamma & \gamma \\ \delta & \delta\beta_1 \end{pmatrix}
\end{aligned}
$$

Therefore, we know that the path to backing out the structural parameters B from the reduced form parameters Π is simply $B = \Pi\Gamma$. A sufficient condition for the model to be **identified** is for the system of equations implied by $B = \Pi\Gamma$ to have at least as many linearly independent equations as there are unknown parameters.[1] In this case, we have six linearly independent equations for six unknown structural parameters - so the necessary condition for identification is satisfied. We know that they are sufficient as we can solve the system of equations to yield us representations for each of the structural parameters using only the "observed" reduced form parameters.

$$
\begin{aligned}
\underbrace{\begin{pmatrix} \alpha_0 & \beta_0 \\ 0 & \gamma \\ \delta & 0 \end{pmatrix}}_{B} &= \underbrace{\begin{pmatrix} \Pi_{11} & \Pi_{21} \\ \Pi_{12} & \Pi_{22} \\ \Pi_{13} & \Pi_{23} \end{pmatrix}}_{\Pi} \underbrace{\begin{pmatrix} 1 & -\beta_1 \\ -\alpha_1 & 1 \end{pmatrix}}_{\Gamma} \\
\begin{pmatrix} \alpha_0 & \beta_0 \\ 0 & \gamma \\ \delta & 0 \end{pmatrix} &= \begin{pmatrix} \Pi_{11}-\alpha_1\Pi_{21} & \Pi_{21}-\beta_1\Pi_{11} \\ \Pi_{12}-\alpha_1\Pi_{22} & \Pi_{22}-\beta_1\Pi_{12} \\ \Pi_{13}-\alpha_1\Pi_{23} & \Pi_{23}-\beta_1\Pi_{13} \end{pmatrix}
\end{aligned}
$$

Intuition

So what is the intuition behind identification? The key is not what exogenous variables are **included** in each structural equation but rather what exogenous variables are **excluded** from each structural equation. We want to use the exogenous shifts in the supply curve to trace the demand curve and exogenous shifts in the demand curve to trace the supply curve. What is excluded from the supply equation helps identify the supply equation and, similarly, what is excluded from the demand equation helps identify the demand equation. In the above example, gasoline cost helps identify demand, and consumer income, supply. As we will see, these "excluded" exogenous variables are natural candidates as "instruments."

[1] This is sufficient but not necessary. As will be discussed later, we might also get identification from the variance-covariance matrix. But it's "close" to necessary.