Price Impact Models and Applications

Introduction to Algorithmic Trading

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Plan

Last Week
Back testing and statistical arbitrage.

For this Week
Applying price impact to risk management.

(a) Distinguishing between position entering and exiting trades.
(b) How price impact distorts mark-to-market P&L.
(c) Application to portfolio scaling.
(d) Simulating fire sales.

Next Week
Module 2: Measuring Price Impact.

Last Week's Summary

Waelbroeck's backtest algorithm

"To make a good assessment of alternative strategies, one may wish to first subtract out the impact of those strategies to then be able to simulate accurately alternative strategies."

Statistical arbitrage theory
Consider alpha decay in alpha research and trading strategies.

Statistical arbitrage implementation

(a) Given alpha level and decay, implement target impact state.

(b) Invert the map to translate impact into trades.

Introduction

Motivation on Liquidity Risk

Emphasis mine

Bernanke (2010)

> *The notion that financial assets can always be sold at* **prices close to their fundamental values** *is built into most economic analysis, and before the crisis, the liquidity of major markets was often taken for granted by financial market participants and regulators alike.*

Caccioli, Bouchaud and Farmer (CFM, 2012)

> *The practice of valuation by marking-to-market with current trading prices is seriously flawed. [...] We propose an alternative accounting procedure based on the estimated* **market impact** *of liquidation that removes* **the illusion of profit.**

The Illusion of Profit

Liquidity risk begins when positions are entered

(a) when entering a trade, price impact pushes prices away from fundamentals

(b) this creates the illusion of profit

(c) which either slowly erodes of time or crashes during times of stress

How to mathematically formalize this?
Caccioli, Bouchaud and Farmer (CFM, 2012) provide the framework.

Definition of a Liquidity Stress Test

ESMA's guidelines on liquidity stress testing (2020)

Managers using LST should simulate assets being liquidated in a way that reflects how the manager would liquidate assets during a period of exceptional market stress.

Mock liquidity stress test simulating a fire sale

Scenario	Target Position	Target risk exposure	Hedging trade	Price dislocation due to market reaction	Price dislocation due to hedging trade	Half-life of total price dislocation
No hedging	100 mil	10 mil	0 mil	40bps	0bps	5 days
Moderate hedging	50mil	5mil	50mil	45bps	30bps	10 days
Full hedging	0mil	0mil	100mil	50bps	60bps	15 days

Exposure measures downside risk of keeping the position

Hedging the risk leads to trading costs due to price dislocation

Price impact of the market reaction will cause a dislocation regardless of our trading

Our hedging trade causes additional price impact

The effect of the dislocation on our position is temporary

Liquidity Stress Test Literature

Framework and implications for risk management

(a) Caccioli, Bouchaud and Farmer (2012) model the effect of price impact on accounting P&L and leverage, with applications to risk management.

(b) The Basel Committee (Stress testing principle, 2018) defines liquidity stress tests as a key risk management tool that should inform business decisions.

(c) The ESMA (Liquidity stress testing guidelines, 2020) outlines requirements for liquidity stress tests, such as taking into account price dislocations during liquidations.

(d) Roncelli et al. (2021) implement, both empirically and in simulation, a liquidity risk management framework in the spirit of Caccioli, Bouchaud and Farmer (2012).

Arrival Slippage vs Mark-to-Market (1/3)

A Straightforward Scenario
Consider two orders with identical trades but different starting positions.

(a) Position entering order: $Q_0 = 0$, $Q_T = \Delta Q$.

(b) Position exiting order: $Q_0 = -\Delta Q$, $Q_T = 0$.

How traders see the trade's P&L
The two order's have the same P&L, as measured by their *arrival slippage*.

How portfolio managers see the trade's P&L
The position entering order lifts the final position's value. Therefore, position entering and exiting trades have asymmetric mark-to-market P&L.

Arrival Slippage vs Mark-to-Market (2/3)

Simulation results

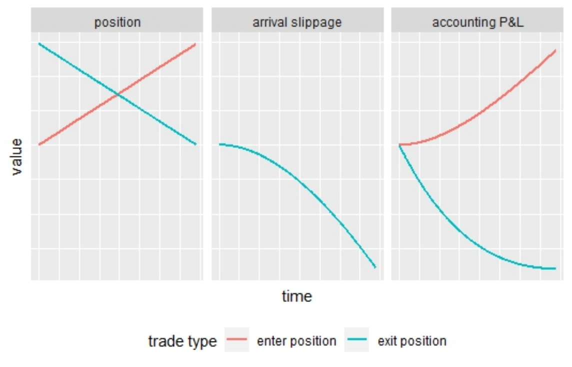


Figure 1: Panel (a) depicts the positions of two equal trades, one entering (in red) and the other exiting (in blue). Panel (b) shows identical arrival slippages. However, their mark-to-market prices in panel (c) are asymmetric.

Arrival Slippage vs Mark-to-Market (3/3)

Why this matters
Traders and portfolio managers continuously monitor their P&L and take decisions accordingly.

Why Two Different P&L Metrics?
Both P&L metrics eventually agree when positions close.

(a) Traders want their P&L metric to fairly attribute position entering and exiting (no bias).

(b) Portfolio managers want their P&L metric to be completely unambiguous (e.g., choice of a starting time for the order).

Same trade-off as in fixed income or option pricing

(a) Mark-to-market P&L is model-free but potentially stale.

(b) Pricing-model based P&L updates frequently but introduces model risk.

Structure

Three sections

(a) *Mathematical framework*
This section formalizes accounting P&L in the presence of price impact.

(b) *Position inflation*
This section proves position inflation in two cases for the OW model.

(c) *A fire-sale model*
This section applies the framework to the liquidation costs during a fire-sale.

Mathematical Framework

Refresher: Three Prices

Let Q represent a given trader's cumulative trades.

Unperturbed price
$S_t(\omega)$ is the unperturbed price and is *unobserved*.

Observed price
$P_t(\omega, Q)$ is the observed price, *common to all market participants* (e.g. the mid price).

The trader causes the price impact $I = P - S$.

Transaction price
$\tilde{P}_t(\omega, Q)$ is *the trader's* transaction price (e.g. price in a dark pool).

The trader pays the instantaneous transaction costs $s = \tilde{P} - P$.

Establishing P&L

Define the cash position K_t as

$$K_t = K_0 - \int_0^t \tilde{P}_s dQ_s.$$

Two definitions of P&L
Define the *accounting* P&L X of the trader as

$$X_t = P_t Q_t + K_t.$$

Define the *fundamental* P&L Y of the trader as

$$Y_t = S_t Q_t + K_t.$$

Example of P&L Decomposition in Trading

What the optimal execution engine tracks
The optimal execution engine maximizes

$$Y_t = Y_0 + \underbrace{\int_0^T Q_t dS_t}_{\text{alpha capture}} - \underbrace{\int_0^T I_t dQ_t - \frac{1}{2}[I, Q]}_{\text{impact costs}}$$

What the portfolio manager tracks
The accounting system displays

$$X_t = Y_t + Q_t I_t.$$

Define the trading footprint F_t as

$$F_t = X_t - Y_t$$

Empirical Studies

Almgren (2005), Bouchaud (2012), Mackintosh (2022)...

(a) For typical day orders on large cap stocks, spread costs account for 5bps and impact costs for 30bps.

(b) For typical day orders on small cap stocks, impact costs increase to 75bps.

(c) Entering a position of 10x ADV on large cap stocks distorts prices by 2-9%.

(d) Half-life for the trading footprint range from days to months, depending on the size of the position.

But What's the "Correct" P&L of the Position?

Expected closing P&L
All definitions of P&L match when the position is closed. Let T be a stopping time such that $Q_T = 0$.

$$X_T = Y_T = K_T$$

A trader maximizes the expected closing P&L of a trade

$$\mathbb{E}\left[X_T \middle| \mathcal{F}_t\right].$$

Position inflation
Under reasonable assumptions, we prove that

$$X_t > \mathbb{E}\left[X_T \middle| \mathcal{F}_t\right].$$

We call this property *position inflation*.

The Obizhaeva and Wang Model

Dynamics of the price impact I

(a) for the original OW model:

$$dI_t = -\beta I_t dt + \lambda dQ_t.$$

(b) for the generalized OW model in the literature:

$$dI_t = -\beta_t I_t dt + \lambda_t dQ_t$$

with $\beta, \lambda > 0$.

Self-financing equation for Y under an OW model

$$dY_t = Q_t dS_t - I_t dQ_t - \frac{1}{2} d[I, Q]_t.$$

Expected Closing P&L in the OW Model

Lemma

Under the generalized OW model, assume given $Q \in \mathcal{L}^2$. Let T be such that $Q_T = 0$. Then

$$\mathbb{E}\left[X_T \middle| \mathcal{F}_t\right] = X_t - \underbrace{F_t}_{\textit{trading footprint}} + \underbrace{\mathbb{E}\left[\int_t^T Q_s dS_s \middle| \mathcal{F}_t\right]}_{\textit{future alpha}} - \underbrace{\mathbb{E}\left[\int_t^T I_s dQ_s + \frac{1}{2}\int_t^T d[I,Q]_s \middle| \mathcal{F}_t\right]}_{\textit{future transaction costs}}.$$

A Simple Scenario under the Original OW Model

No future trading opportunities
Assume S is a martingale over $[t, T]$ and the trader liquidates Q_t over a horizon of length τ. Then, as $T \to \infty$,

$$\mathbb{E}\left[X_T \,|\, \mathcal{F}_t\right] \to \underbrace{Y_t}_{\text{fundamental P\&L}} - \underbrace{\frac{\lambda}{2+\beta\tau} Q_t^2}_{\text{future liquidation cost}} .$$

Three Takeaways of the Framework

F_t is measurable in real time
We defined the trading footprint as

$$\begin{aligned} F_t &= X_t - Y_t \\ &= Q_t I_t. \end{aligned}$$

Trading systems compute both Q_t and I_t in real time.

F_t captures the bias of accounting P&L

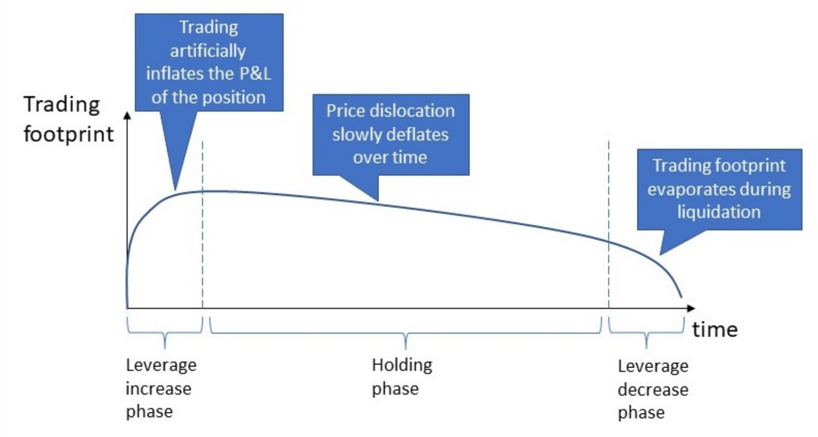
$$\mathbb{E}\left[X_T \mid \mathcal{F}_t\right] \approx X_t - F_t - TC(Q_t)$$

where $TC(Q_t)$ is a standard pre-trade cost model.

F_t has a positive expectation
Under two sets of assumptions, the next section proves *position inflation*:
X_t over-estimates the closing P&L of a position, even ex-liquidation costs.

Applications

Example of Position Inflation for a Roundtrip Trade



Round-Trip Trade Results (1/2)

Proposition
Consider a strategy Q and a generalized OW model I such that

(a) $I_0 = 0$.

(b) $|Q|$ is monotonically increasing over $[0, T]$.

The trading footprint of the portfolio is positive.

The illusion of profit
Even with *martingale* prices and transaction costs, the position entering trade mechanically looks profitable!

Round-Trip Trade Results (2/2)

Proof
Without loss of generality, assume $Q > 0$ and prove that $I_t > 0$. Define Z as

$$dZ_t = \beta_t Z_t dt$$

with $Z_0 = 1$. The solution equals

$$Z_t = e^{\int_0^t \beta_s ds} > 0.$$

Then, one has

$$\begin{aligned} d(I_t Z_t) &= I_t dZ_t + Z_t dI_t \\ &= \beta_t I_t Z_t dt - \beta_t Z_t I_t dt + \lambda_t Z_t dQ_t \\ &= \lambda_t Z_t dQ_t. \end{aligned}$$

Hence, $I_t Z_t$ increases and $I_t Z_t > 0$ for $t \in (0, T]$. It follows that $I_t > 0$ for $t \in (0, T]$.

The Illusion of Profit

For the original OW optimal execution problem
a position entering trade with no alpha has the fundamental P&L

$$\begin{aligned}\mathbb{E}\left[Y_T\right] &= -\mathbb{E}\left[\int_0^T I_t dQ_t + [I, Q]_T\right] \\ &= -\frac{\lambda}{2+\beta T} Q_T^2\end{aligned}$$

and the accounting P&L

$$\begin{aligned}\mathbb{E}\left[X_T\right] &= \mathbb{E}\left[Y_T\right] + I_T Q_T \\ &= \frac{\lambda}{2+\beta T} Q_T^2 \\ &= -\mathbb{E}\left[Y_T\right].\end{aligned}$$

Position Entering and Exiting Trades (1/2)

Consider a trade $Q_T = Q_0 + Q$. Then

$$\mathbb{E}\left[Y_T\right] = -\frac{\lambda}{2+\beta T} Q^2,$$
$$\mathbb{E}\left[X_T\right] = \frac{2\lambda}{2+\beta T} Q_0 Q - \mathbb{E}\left[Y_T\right].$$

Fundamental P&L is identical across orders that trade identically.
Mark-to-market P&L depends on positions.

Position Entering and Exiting Trades (2/2)

For a position-entering order,

$$Q_0 = 0; \quad Q_T = Q.$$

For a position-exiting order,

$$Q_0 = -Q; \quad Q_T = 0.$$

(a) The optimal trades and arrival slippages are identical for both orders, and the expected fundamental P&L equals

$$\mathbb{E}\left[Y_T\right] = -\frac{\lambda}{2+\beta T} Q^2.$$

(b) The optimal trading strategy has expected accounting P&L

$$\mathbb{E}\left[X_T\right] = -\mathbb{E}\left[Y_T\right] > 0$$

for a position-entering order and

$$\mathbb{E}\left[X_T\right] = \mathbb{E}\left[Y_T\right] < 0$$

for a position-exiting order.

Two Quotes from Caccioli, Bouchaud, and Farmer (2012)

Emphasis mine

When to monitor price dislocations?

> *Whereas mark-to-market valuation only indicates problems with excessive leverage after they have occurred, our method makes them clear* **before positions are entered.**

How to monitor price dislocations?

> *The procedures that we suggest have the key virtue of being extremely easy to implement [...] - anything that can be done with standard risk measures can be easily replicated to take impact into account,* **with little additional effort.**

The Stationary Portfolio Case (1/2)

Proposition
Assume that

(a) I follows the original OW model.

(b) Both I and Q are weakly stationary

$$\forall t > 0, \quad \mathbb{E}\left[I_t^2\right] = I_0^2 \quad \textit{and} \quad \mathbb{E}\left[Q_t^2\right] = Q_0^2.$$

Then the stationary trading footprint is

$$\mathbb{E}\left[F_t\right] = \frac{1}{\lambda} I_0^2.$$

Furthermore, the expected running transaction costs are

$$\mathbb{E}\left[\int_0^T I_t dQ_t + \frac{1}{2}[I, Q]_T\right] = \frac{\beta T}{\lambda} I_0^2.$$

The Stationary Portfolio Case (2/2)

proof
By Itô's formula,

$$
\begin{aligned}
dF_t &= I_t dQ_t + Q_t dI_t + d[I,Q]_t \\
&= \frac{1}{\lambda} I_t \left(\beta I_t dt + dI_t\right) + Q_t \left(-\beta I_t dt + \lambda dQ_t\right) + d[I,Q]_t \\
&= \frac{\beta}{\lambda} I_t^2 dt + \frac{1}{\lambda} I_t dI_t - \beta F_t dt + \lambda Q_t dQ_t + d[I,Q]_t \\
&= -\beta F_t dt + \frac{\beta}{\lambda} I_t^2 dt + \frac{1}{2\lambda} d\left(I_t^2\right) + \frac{\lambda}{2} d\left(Q_t^2\right).
\end{aligned}
$$

The stationarity assumption implies

$$
\frac{d\mathbb{E}\left[F\right]_t}{dt} = -\beta \mathbb{E}\left[F_t\right] + \frac{\beta}{\lambda} I_0^2.
$$

Solving the ODE proves the result for F.

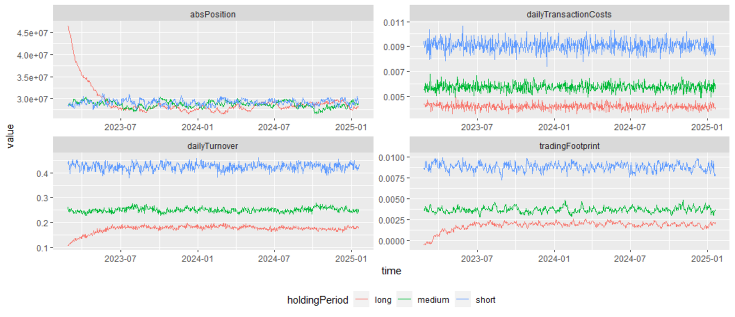
Simulation of a Stationary Portfolio

Simulation settings
Consider the Orstein-Uhlenbeck model for a portfolio of 1000 stocks

$$dQ_t = -\frac{1}{\tau} Q_t dt + \bar{Q} dW_t.$$

τ controls the turnover of a portfolio with fixed size.

Simulation results



A Fire-Sale Model

A Liquidation Model Without Fire-Sale

Assumptions
Let Q_0, I_0 be the starting position to a liquidation. Assume that

(a) the OW model holds.

(b) the unperturbed price S is a martingale.

(c) a fraction $\eta \in (0,1]$ of the position is liquidated using the OW optimal execution strategy over $[0, T]$.

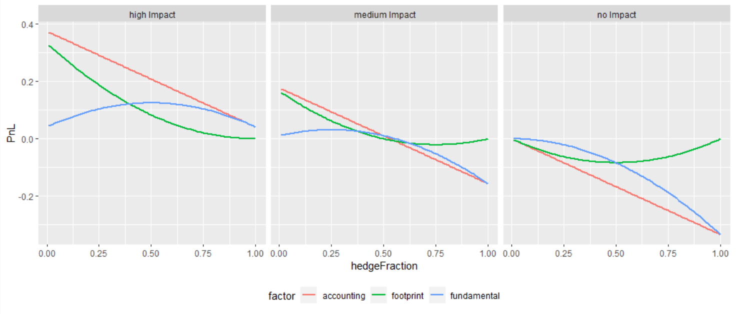
Proposition
The expected fundamental P&L is

$$\mathbb{E}\left[Y_T - Y_0\right] = \frac{1}{2\lambda} I_0^2 - \frac{1}{(2+\beta T)\,\lambda}\left(I_0 - \lambda\eta Q_0\right)^2$$

and the trading footprint is

$$F_T = \frac{2(1-\eta)}{2+\beta T}\left(F_0 - \lambda\eta Q_0^2\right).$$

Sensitivity to Initial Conditions



Observations

(a) With a high initial impact state, most losses stem from temporary dislocations. They show up in the trading footprint.

(b) With a low initial impact state, most losses stem from permanent transaction costs. They show up in the fundamental P&L.

A Liquidation Model with Fire-Sale

Additional assumptions

(a) the rest of the market is also hedging a quantity $-\bar{Q}$ over $[0, T]$.

(b) their impact follows the same OW model, with initial condition $\bar{I}_0$.

(c) both sets of executions optimize unaware of each other's impact.

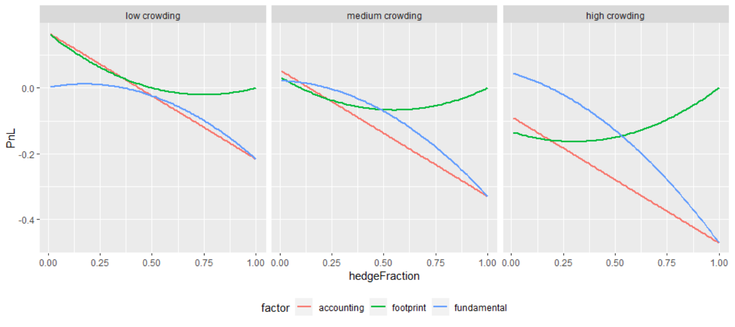
Proposition
The expected fundamental P&L is

$$\mathbb{E}\left[Y_T - Y_0\right] = (\ldots) - \frac{1}{(2+\beta T)\,\lambda}\left(I_0 - \lambda\eta Q_0\right)\left(\bar{I}_0 - \lambda\bar{Q}\right)$$

and the trading footprint is

$$F_T = (\ldots) + \frac{2(1-\eta)}{2+\beta T}\left(Q_0\bar{I}_0 - \lambda Q_0\bar{Q}\right).$$

Sensitivity to Crowding



Observations

(a) Crowding affects the fundamental P&L most in the full hedging case.

(b) Crowding affects the trading footprint across all hedging fractions.

Weekly Summary

Accounting P&L over-estimates a position's closing P&L
We prove position inflation for simple scenarios based on the OW model.

Price impact matters for liquidity risk management
Price impact causes both position inflation and liquidation costs.

Managing liquidity risk starts before leveraging the position up
Portfolio managers, risk managers and regulators can measure price impact in real time and predict mechanical P&L movements ahead of time.

Questions?

Next week
Module 3: measuring price impact considering common trading biases.

(a) Bouchaud's list of four trading biases.
(b) An informal primer on causal graphs.
(c) Introduction to live trading experiments.