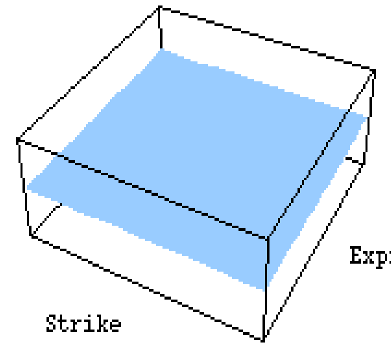
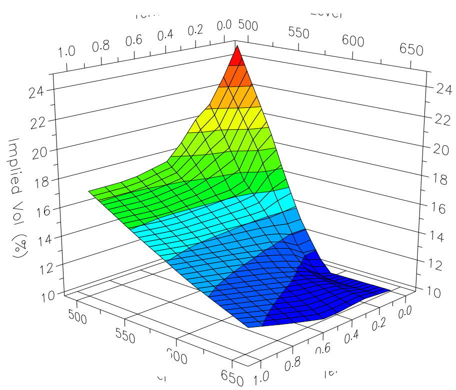
Lecture 1: Introduction to the Smile; Principles of Valuation;

According to classic theory, the Black-Scholes implied volatility of an option should be independent of its strike and expiration. Plotted as a surface, it should be flat, as shown at right.

The volatility surface according to Black-Scholes



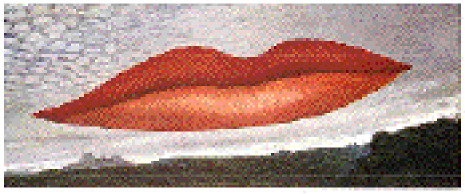
The volatility surface according to S&P options markets



Prior to the stock market crash of October 1987, the volatility surface of index options was indeed fairly flat.

Since the crash, the volatility surface of index options has become skewed. Referred to as the volatility smile, the surface changes over time. Its level at any instant is a varying function of strike and expiration, as shown at left.

The smile phenomenon has spread to stock options, interest-rate options, currency options, and almost ever other volatility market. Since the Black-Scholes model cannot account for the smile, trading desks have begun to use more complex models to value and hedge their options.



After 35 years, there is still no overwhelming consensus as to the correct model. Each market has its own favorite (or two). Despite initial optimism about finding **the** model to replace Black-Scholes, we are still searching.

Recent SPX Implied Volatilities

SPY 25-Delta Put - Call Spread 52-Week Average					
ATM IV	25-Delta Put IV	25-Delta Call IV	Difference Put - Call	High Difference	Low Difference
22.9	25.6	19.6	+6.1	+10.8	+2.6

SPY 25-Delta Put - Call Spread Last 20 Days				
Date	ATM IV	25-Delta Put IV	25-Delta Call IV	Difference Put - Call
12-Jan-2023	17.7	19.1	15.9	+3.2
11-Jan-2023	20.1	21.6	17.9	+3.7
10-Jan-2023	19.6	21.0	17.4	+3.6
9-Jan-2023	20.6	21.9	18.4	+3.5
6-Jan-2023	20.5	21.8	18.2	+3.6
5-Jan-2023	21.9	23.1	19.5	+3.6
4-Jan-2023	21.1	22.4	18.7	+3.7
3-Jan-2023	21.7	22.9	19.2	+3.7
30-Dec-2022	21.5	22.8	19.0	+3.8
29-Dec-2022	21.1	22.4	18.9	+3.5
28-Dec-2022	22.0	23.1	19.8	+3.3
27-Dec-2022	21.3	22.5	19.2	+3.3
23-Dec-2022	21.4	22.8	19.1	+3.7
22-Dec-2022	22.1	23.5	19.9	+3.6
21-Dec-2022	20.3	21.3	18.4	+2.9
20-Dec-2022	21.3	22.4	19.2	+3.2
19-Dec-2022	21.6	22.8	19.5	+3.3
16-Dec-2022	22.0	23.3	19.6	+3.7
15-Dec-2022	21.3	23.0	19.2	+3.8
14-Dec-2022	19.8	21.2	18.1	+3.1

1.1 Introduction

Sell side and buy side: sell side needs a better understanding of derivatives valuation.

But even if you are not going to work in valuing derivatives, options pricing models are the best and most extensible models in finance. It's very important to have a clear understanding of them.

Aim of the Course

1. The principles of financial modeling.
2. Understanding volatility as a quality, a quantity, and an asset.
   Volatility is the propensity to continually change one's current state, irrespective of the direction of the change. Its cause is the fluctuation of opposing desires.
3. Understanding the practical use of the Black-Scholes model. There's more to it than just knowing the equation and its solution. You have to know how to work with it and modify it.
4. Understanding the successes and limitations of the Black-Scholes model.
5. Getting familiar with the behavior of the volatility smile anomaly.
6. The extensions of the Black-Scholes model to accommodate/explain the volatility smile.
7. Understanding the consequences of these extensions. It's easy to make up new and richer models but we want to understand whether they are realistic, whether they are advantageous, and the relation between the model and the phenomenon.
8. How to model the stochastic behavior of volatility.
9. How to make your own models.
10. This is a course about learning how to model and think about models, not necessarily about efficient implementation. So, I'll put a lot of effort into deriving *simple or approximate proofs* of the key model formulas and ideas.

"*I understand what an equation means if I have a way of figuring out the characteristics of its solution without actually solving it.*" Dirac as quoted by Feynman

References (though I’m trying to be self-contained here)

For this class I assume people are familiar with the Black-Scholes equation and solution.

- *The Volatility Smile*, Emanuel Derman and Michael B. Miller, Wiley 2015.
  Developed from this course.
- *Stochastic Volatility Modeling*: L. Bergomi, Chapman & Hall 2016.
  Advanced, difficult but good. Treats stochastic forward volatilities.
- *The Volatility Surface: A Practitioner's Guide* by Jim Gatheral, Wiley 2006.
  This book probably contains material relevant to what I’m doing, though it approaches it somewhat differently and much more formally.
- There are many more ...

Some useful books of a more general nature are listed below:

- *Paul Wilmott on Quantitative Finance*, Wiley, by Paul Wilmott is a very good general book on options theory. He’s not afraid to tell you what he thinks is important and what isn’t, which is valuable.
- *The Concepts and Practice of Mathematical Finance*, CUP 2004, by Mark S. Joshi.
  Good book, particularly on static hedging.
- *Options, Futures and Other Derivatives,* Prentice Hall, by John Hull. The standard comprehensive teaching book.
- *Introduction to Quantitative Finance*, Stephen Blyth, 2013, is a very elegant and compact book with a straightforward approach that tries to make things simple and clear rather than complicated and obscure.

Journals

• Quantitative Finance, lots of econophysics and quant papers

www.ssrn.com has many papers in the FEN (Financial Economics Network) section, and most of the latest papers get posted there before publication.

• Wilmott Magazine

• Risk Magazine

• Journal of Derivatives

Contacting me

ed2114@columbia.edu. If you have questions, please come to my office hours, which I will post. I cannot answer very long questions before or after class.

Grades

Homework assignments weekly. These will count for 10% of the final grade. A further 40% will depend on the midterm and 50% on the final exam.
Homework is due once a week, usually on a Monday at the end of the day, to be submitted on canvas.

Ethics

I don't mind if you discuss homework among yourselves, but then I expect you to think about it and work it out and write it up by yourself afterwards.

Please don't be part of several people handing in identical copies of solutions; don't hand in xeroxes of someone else's solution. If you do, I will consider it cheating

1. Please don't read email or browse the web in class.
NO PHONES DURING CLASS.
If you have some emergency that requires you to keep your phone on, please let me know.

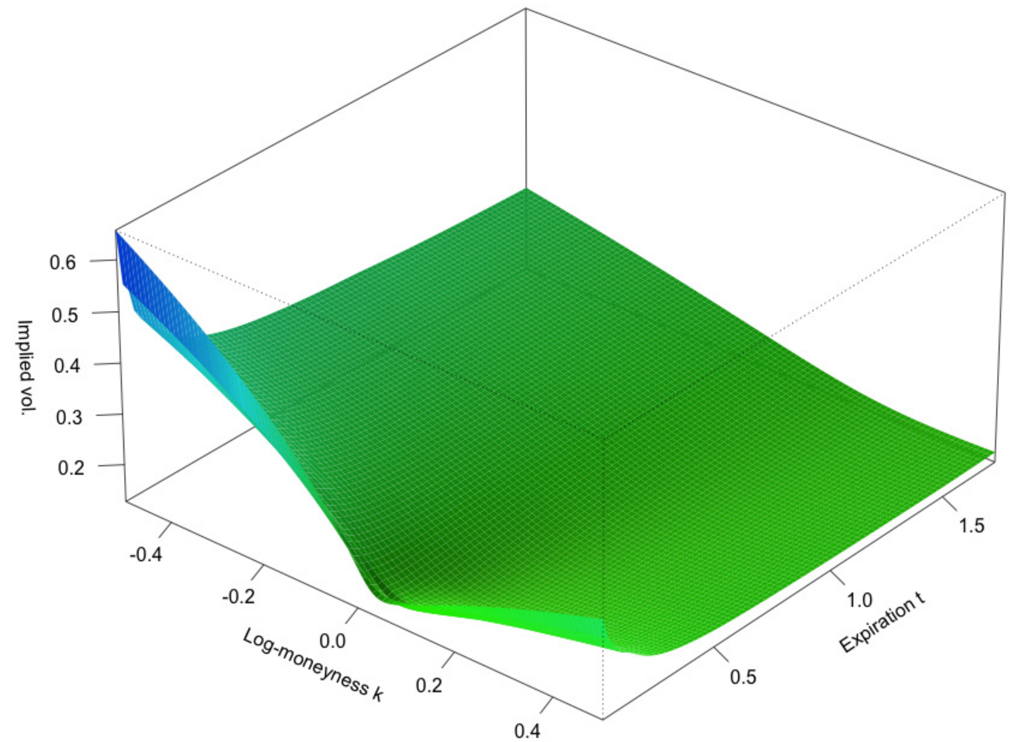
2. Please don't come late to class.

3. Don't post my note or homework on coursehero or other public websites. The copyright is mine. These notes are for you and you alone.

4. There is a lot of cheating in recent years. It's a very serious offence in a university and in the financial world too. In the last two semesters there has been copying of homework or someone's answers in an exam, including copying typos, without understanding what is being written. Also forgery. If someone is found cheating in any way, I will give zero points for that homework, midterm, or entire exam, and send their name to the dean's office and let them take appropriate action.

1.2 Course Outline

- **Overview**
  A quick look at the behavior of stock prices and the behavior of the smile..
  Financial modeling principles:
  From liquid to illiquid values
  Methods:
  Static hedging,
  Dynamic hedging,
  Statistics



S&P Implied Volatilities June 20. 2013

- **The Principle of Replication**
  Replication and the law of one price
  The Capital Asset Pricing Model for stocks
  The theory of dynamic replication for options
  The Black-Scholes equation

- **Trading Volatility**

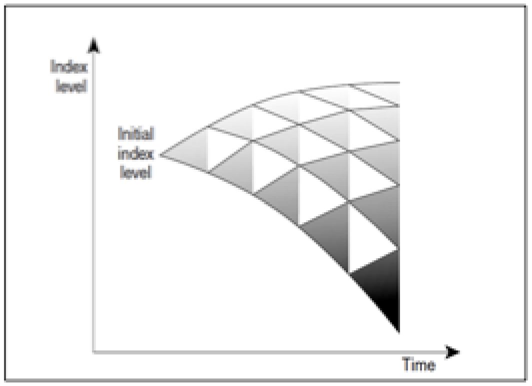
  Replicating Variance Swaps $\left(\sigma_R^2 - K_{var}\right) \times N$ out of calls and puts. The log contract.

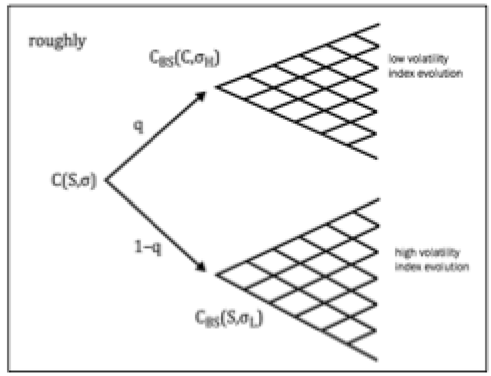
- **The P&L of Hedged Options Strategies in a Black-Scholes World**
  Simulations of discrete hedging with different choices of hedge ratios, transactions costs, …

- **Facts about the Smile**
  The smile in various markets
  The difficulties the smile presents for trading desks and for theorists: pricing & hedging
  Different kinds of volatility: realized, implied, fixed strike, at-the-money, …
  Parameterizing options prices: delta, strike and their relationship

- **Estimating the effects of the smile on hedge ratios and on exotic options**
  Reasons for a smile
  No-riskless-arbitrage bounds on the size of the smile
  Fitting the smile

A survey of smile models.





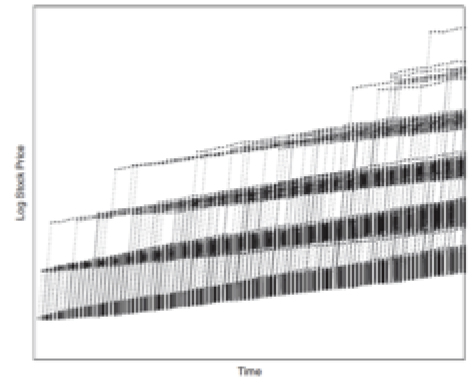


FIGURE 24.3 A Monte Carlo Simulation of the Log Stock Prices in the Jump-Diffusion Model

Static Hedging for European Options in the Presence of a Smile: Implied Distributions
Arrow-Debreu state prices from the Breeden Litzenberger formula
Black-Scholes implied density and its use
Exact static replication of path-independent exotic options using vanilla options

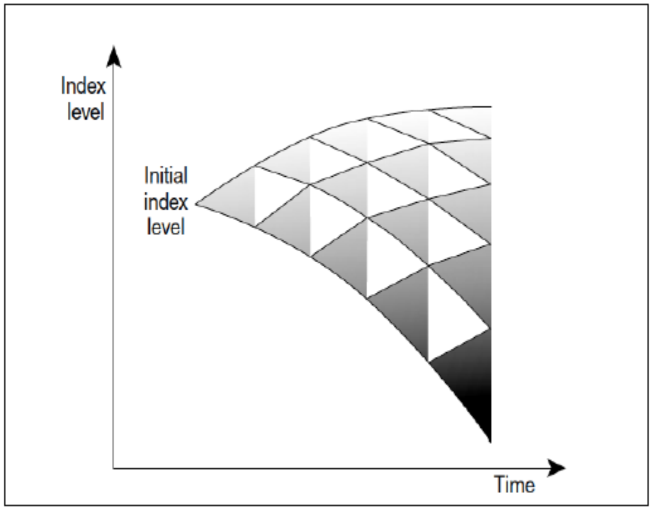
$$V(S,t) = V(A,T)e^{-r(T-t)} + \left.\frac{\partial V(K,T)}{\partial K}\right|_{K=A}\left(S - Ae^{-r(T-t)}\right)$$

$$+ \int_0^A \frac{\partial^2 V(K,T)}{\partial K^2} P(S,K)\,dK + \int_A^\infty \frac{\partial^2 V(K,T)}{\partial K^2} C(S,K)\,dK$$

- **Static Hedging of Path-Dependent Options**
  Approximate static replication of path-dependent exotic options with vanilla options

- **Extending Black-Scholes beyond constant-volatility lognormal stock price evolution**
  Binomial trees
  Time-dependent deterministic rates
  Time-dependent deterministic volatility
  Calibration to rates and volatility

- **Local Volatility Models/ Implied Trees**
  Derman-Kani binomial local volatility trees
  Dupire equation
  Difficulties with the model



- **The Consequences of Local Volatility Models**
  The relationship between local and implied volatility
  Estimating the deltas of vanilla options in the presence of the smile
  Estimating the values of exotic options

- **Model classification**
  Empirical behavior of implied volatility with time and market level
  Sticky strike, sticky delta, sticky implied tree
  Skew stickiness ratio - the possible relation between the current skew and future volatility

- **Stochastic Volatility Models**
  The effect of changes in volatility in the Black-Scholes formula
  The Vanna-Volga heuristic
  Mean reversion of volatility
  The SABR model, the Heston model
  The PDE for option value under stochastic volatility
  The mixing formula for option value under stochastic volatility
  The relationship between local and stochastic volatility
  Stochastic local volatility models -- state of the art

- **Jump-Diffusion Models**
  Are they reasonable, and if so, when?
  Poisson jumps
  The Merton jump-diffusion model and its solution
  Estimating the smile in jump-diffusion models
  Simulations

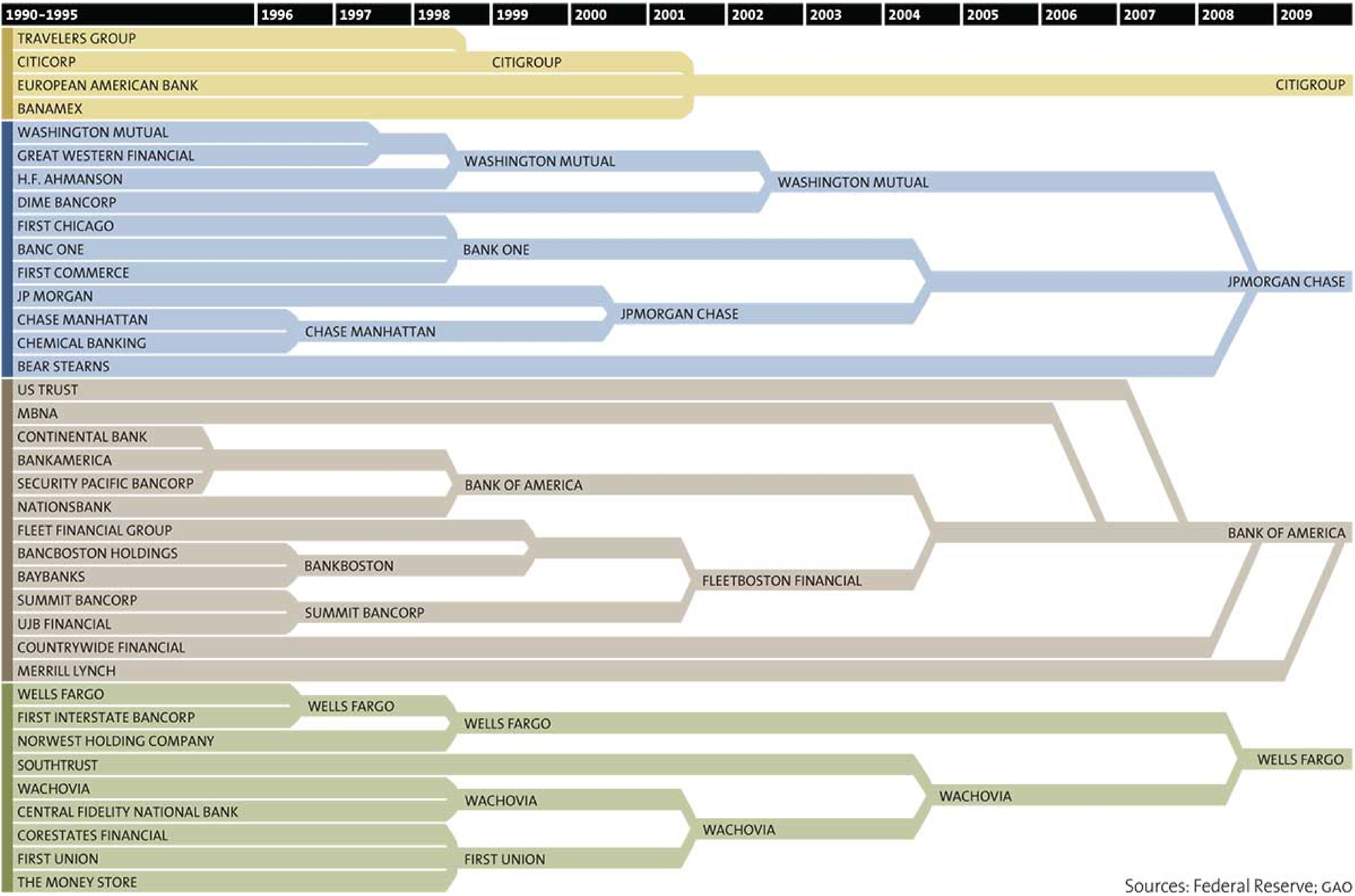
- **Other models of volatility evolution: Fractional Brownian Motion (with Autocorrelation)**
-
  Some Guest Speakers from Industry

Today and the Next Class:

The Stylized Facts about stock and index prices, and stylized facts about the smile

Principles of Financial Valuation

The Sell Side World



Stylized Facts About Stock Prices

Stylized Facts are common properties across a wide range of instruments, markets, time periods

1.3 Stylized Statistical Properties of Asset Returns

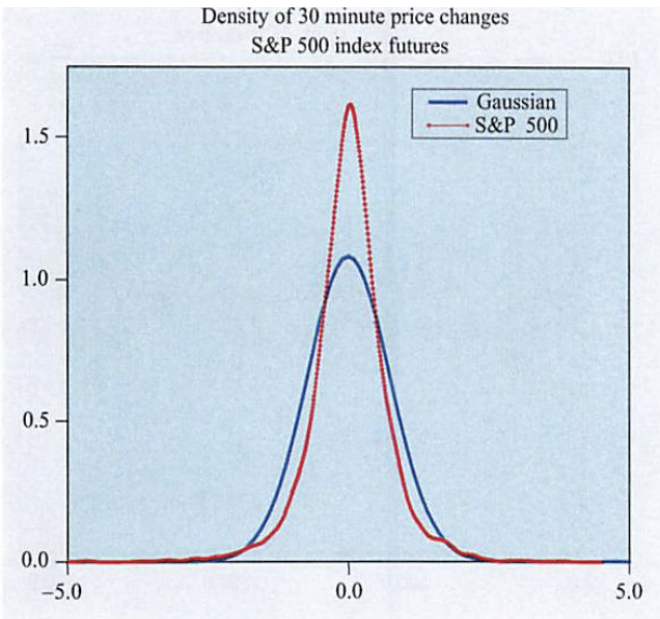
http://rama.cont.perso.math.cnrs.fr/pdf/empirical.pdf

$$X(t) = \ln S(t) \qquad r(t, \Delta t) = X(t+\Delta t) - X(t) = \ln\frac{S(t+\Delta t)}{S(t)} \approx \frac{\Delta S(t)}{S(t)}$$

Return at time t over the subsequent period Δt. Often $\Delta t = 1\,\text{day}$.

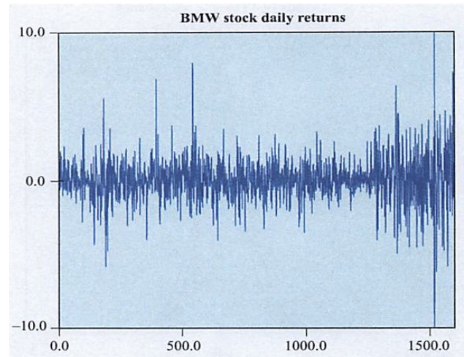
$\text{corr}[r(t+\tau, \Delta t), r(t, \Delta t)]$ is autocorrelation between daily returns at time t and time τ later.

- Returns are uncorrelated.
- Returns have heavy tails -- non-Gaussian. Power laws for the tail



- More large down moves than up moves for stocks, but not for exchange rates (why?). “Indexes glide up, crash down.”

- As Δt increases, distribution of $r(t, \Delta t)$ becomes more Gaussian for long time periods.
- Returns are highly variable. Volatility is high.
- Volatility clusters.



- There seems to be an autocorrelation of **absolute returns** $\mathrm{corr}[|r(t+\tau, \Delta t)|^2, |r(t, \Delta t)|^2]$ as a function of τ, falling off like $\tau^{-0.3}$ roughly (when the paper was written).
- **Leverage effect**: the volatility of an asset is negatively correlated with the returns of an asset.

  $L(\tau) = \mathrm{corr}[|r(t+\tau, \Delta t)|^2, r(t, \Delta t)]$ is negative for small τ, then decays to zero.

  In words: Return goes up, then volatility tends to get lower; return down, then volatility higher.

  $L(\tau) \neq L(-\tau)$

  If you ran the movie backwards you would be able to tell is wasn't the real world, because in the backward world first volatility would go up and then the return would go down. But in the real world, a negative return is followed by higher volatility.

If we wanted to model the distribution really well, we'd need a mean, a volatility, a tail, and asymmetry of tails, and $L(\tau) \neq L(-\tau)$.

A note about assumptions we make in doing statistical analysis. We assume:

1. **Stationarity**. That the distribution remains the same over time. (Seasonality? Can try to remove)

2. **Ergodicity**. That the time average of returns is the same as the expected value of returns. Because when we calculate expected value we are using returns at different times.

THE SMILE

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$ Black-Scholes PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = r\left(C - S\frac{\partial C}{\partial S}\right)$$

gain from time passing + gain from stock moving = riskless return on hedged position

1.4 A Quick Look at the Implied Volatility Smile

- **What is implied volatility?** $C_{mkt} = C_{BS}(S, t, K, T, r, \Sigma[S, t, K, T])$. A parameter, not a statistic. A function of S and K and time.
- **Representative S&P 500 implied volatilities prior to 1987**.

This shape is consistent with the Black-Scholes model, which assumes one volatility for the underlying index or stock. (Which options are the most liquid?)

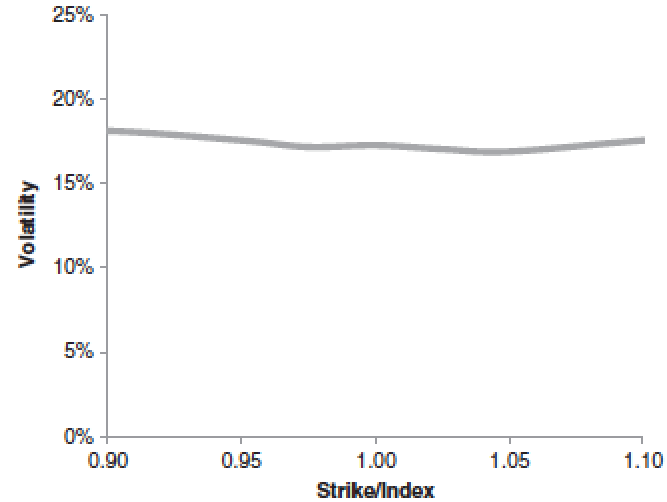


FIGURE 1.1 Representative S&P 500 Implied Volatilities prior to 1987

- **Representative S&P 500 implied volatilities after 1987**

The volatility of a *stock* itself cannot depend upon the option with which you choose to view it. How can one stock have many different volatilities?

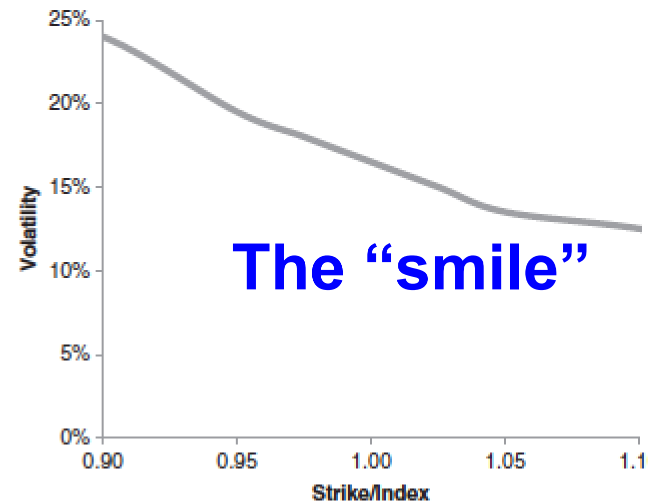


FIGURE 1.2 Representative S&P 500 Implied Volatilities after 1987

1.5 The Challenge: The Black-Scholes model disagrees with the smile.

The Black-Scholes model is being used as **quoting mechanism** to translate from prices to more intuitive implied volatilities.

What's the right replacement??

Why is it important?

(1) **Hedging:** If we **quote** vanilla option prices using Black-Scholes, then

$$C_{mkt} = C_{BS}(S, t, K, T, r, \Sigma[S, t, K, T])$$

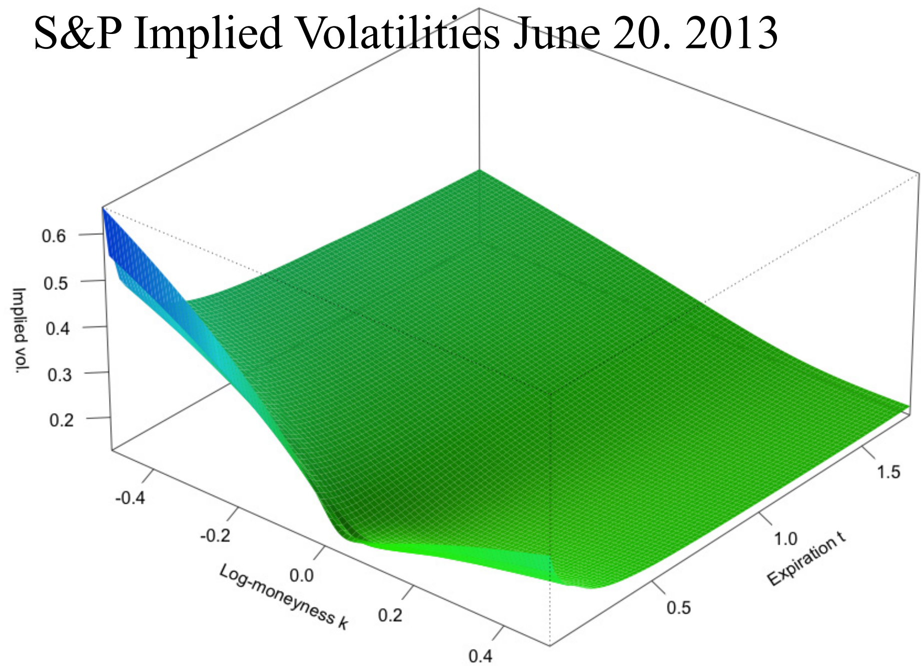
So the hedge ratio is given by

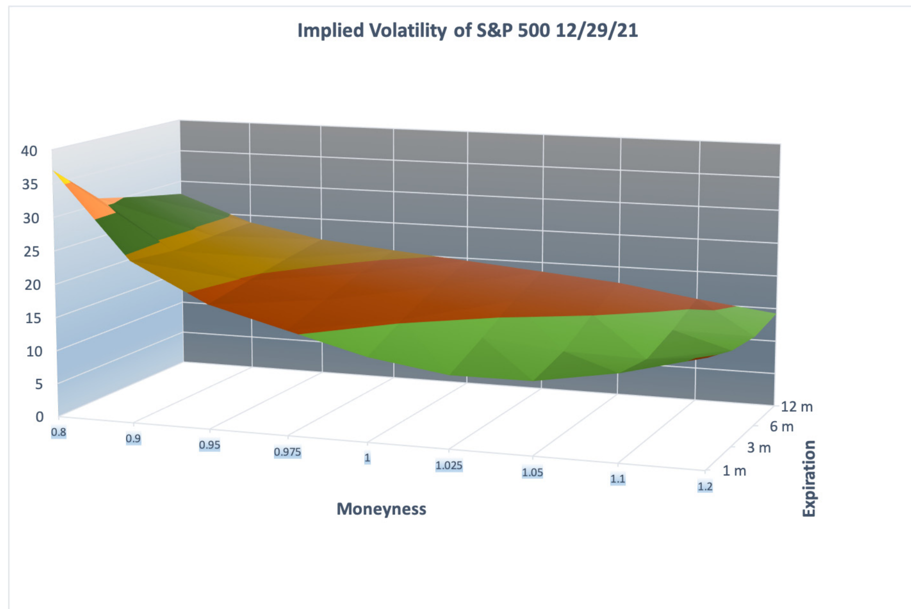
$$\Delta = \frac{\partial C_{\text{mkt}}(S,t,K,T)}{\partial S} = \frac{\partial C_{\text{BSM}}}{\partial S} + \frac{\partial C_{\text{BSM}}}{\partial \Sigma}\frac{\partial \Sigma}{\partial S} = \Delta_{\text{BSM}} + \frac{\partial C_{\text{BSM}}}{\partial \Sigma}\frac{\partial \Sigma}{\partial S}$$

The quoting mechanism doesn't help with that. We need a better model to know how volatility varies with stock price: the *spot-vol dynamics*. We see

$\frac{\partial}{\partial K}\Sigma(S, K, \ldots)$. We want to know

$$\frac{\partial}{\partial S}\Sigma(S, K, \ldots)$$





(2) **Valuation:** We need a model to **value** exotic options whose price isn't quoted.

1.6 Historical Development of the Smile

- There was always a bit of a smile in currency options markets.
- The equity "smile" is really more of a skew or a smirk.
- 1987 crash: a giant market could drop by 20% or more in a day. Low-strike puts are now perceived to be more likely to end up in the money.
  (What if you hedge and are insensitive to value of index? Increase in volatility might still hurt.)
- The volatility smile has spread to all other options markets.
- Traders and quants in every product area have had to model the smile.
- No area where model risk/uncertainty is more of an issue than in the modeling of the volatility smile. There is no clearly *correct* model.

- You could say:
  Implied volatility reflects the market's expectation of future volatility, of drama, of fear.
  A bit like the yield curve reflects the market's expectation of future rates.

Past Black-Scholes Implied Volatilities

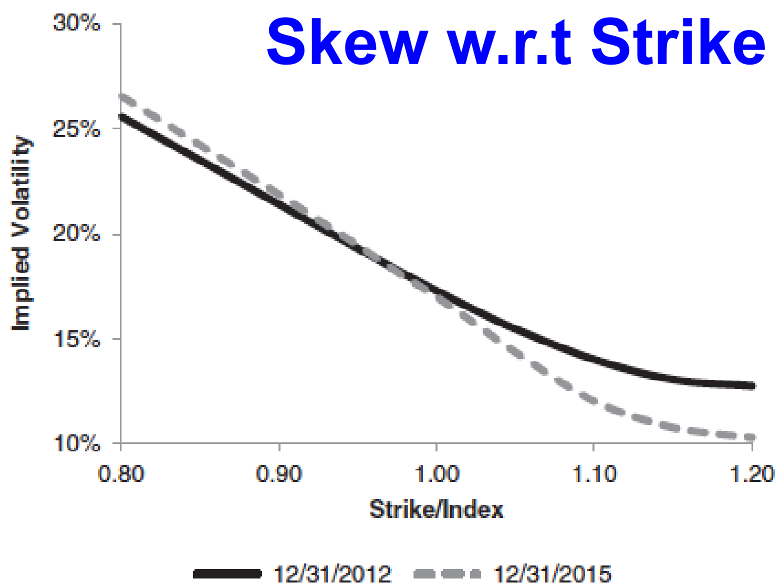


FIGURE 8.2 S&P 500 Six-Month Volatility Smile
Source: Bloomberg.

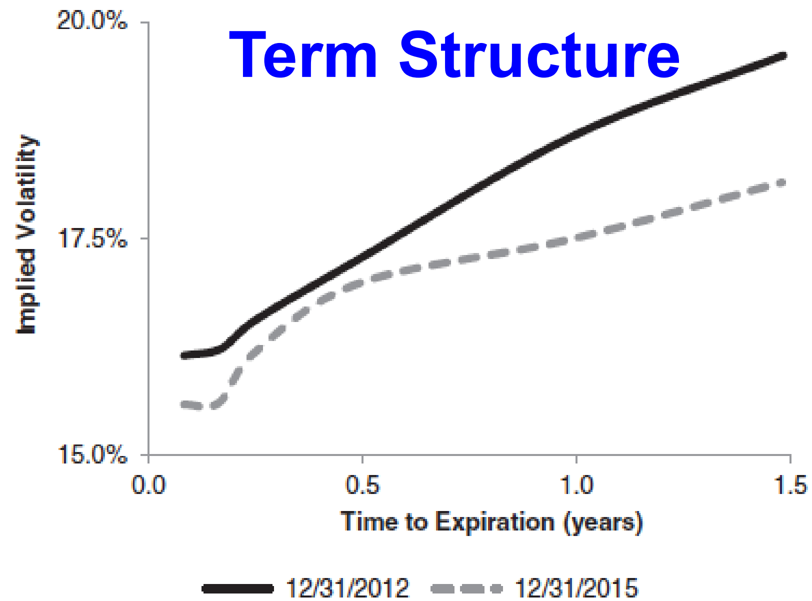


FIGURE 8.3 S&P 500 ATM Volatility Term Structure
Source: Bloomberg.

Surface or Smile

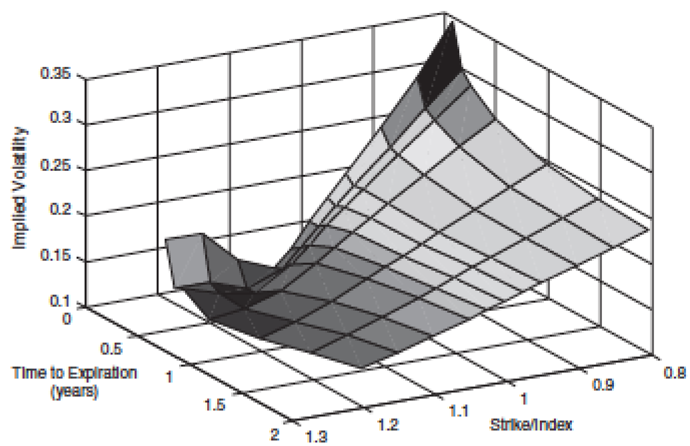
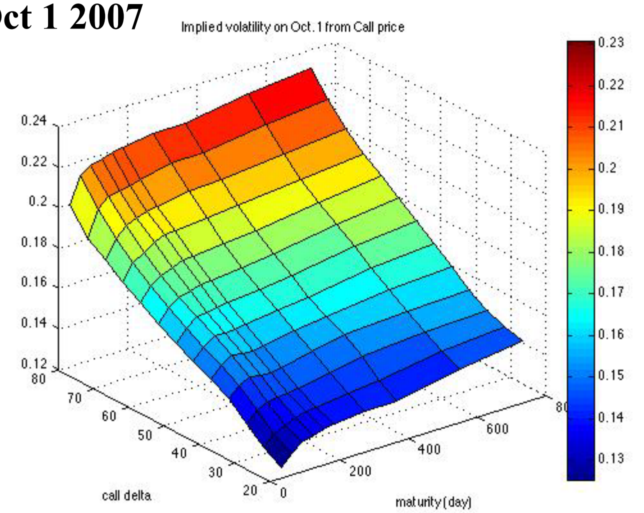


FIGURE 8.4 Volatility Surface, S&P 500, December 31, 2015
Source: Bloomberg.

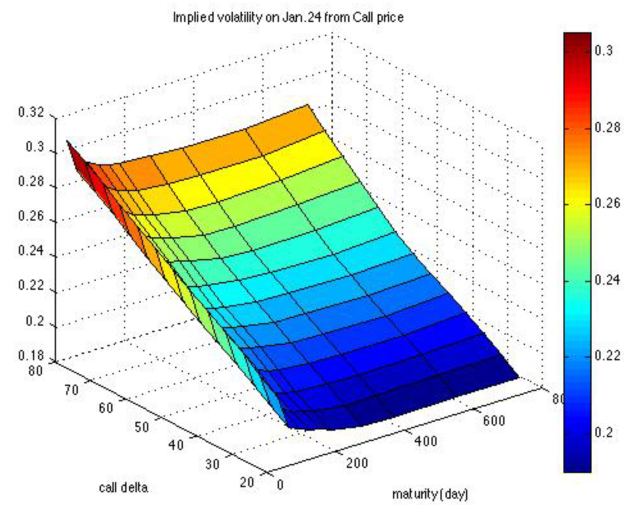
Some Historical S&P 500 Smiles

1. S&P 500

Oct 1 2007



Jan 24 2008



1.7 SVI Parameterization (*Stochastic Volatility Inspired*)

- A commonly used parameterization of the implied volatility smile for a fixed expiration T is

$$\Sigma^2(m, T) = a + b[\rho(m-c) + \sqrt{(m-c)^2 + \theta^2}]$$

where $m = \ln\frac{K}{F}$ is the **log forward moneyness**, $F = Se^{rT}$ is the forward price of the underlying stock S, and T is the time to expiration of the option.

What features does this have? Homework 1 Problem 3.

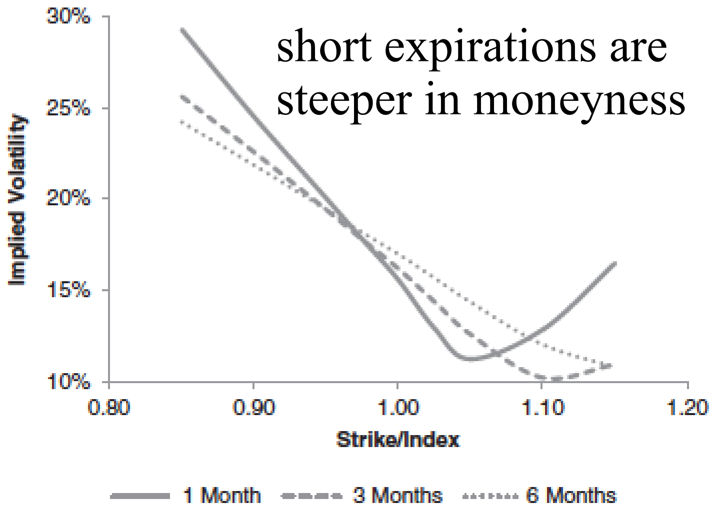
a - level
b -
ρ -
c -
θ -

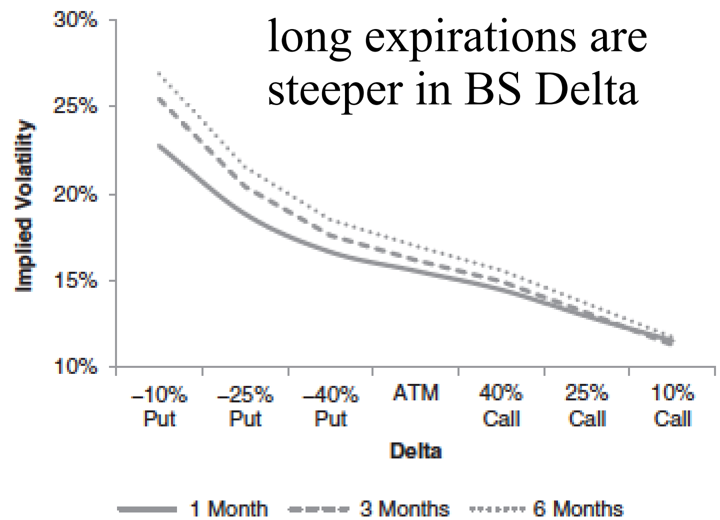
- This is an empirical formula. It is a parameter in a model in which it should be constant, i.e.independent of S. It cannot be 100% correct theoretically because we must worry about arbitrage violations induced by the parameter. See Homework 1 Problem 2.

  (Cf. parameterizing interest rates: you cannot write down a steeply negative yield curve because it implies negative forward rates which violate the principal of no riskless arbitrage.)

1.8 Stylized Facts about the Smile for Equity Indexes

- Current implied volatility tends to be greater than recent realized volatility.
- Almost always negatively skewed. Why? Protection against payoff and volatility spike in a crash?
- Wasn't that way before 1987.
- Skew is steeper for short expiration, flatter for longer ones. (Steeper in what variable?)
-





- Term structure of the volatility surface can slope up or down. During a crisis—and a crisis is always characterized by high volatility—the term structure is likely to be downward sloping. The high short-term volatility and lower long-term volatility reflect market participants' belief that uncertainty in the near term will eventually be resolved.

• Index and implied volatility are negatively correlated. Indexes tend to glide up and crash down.

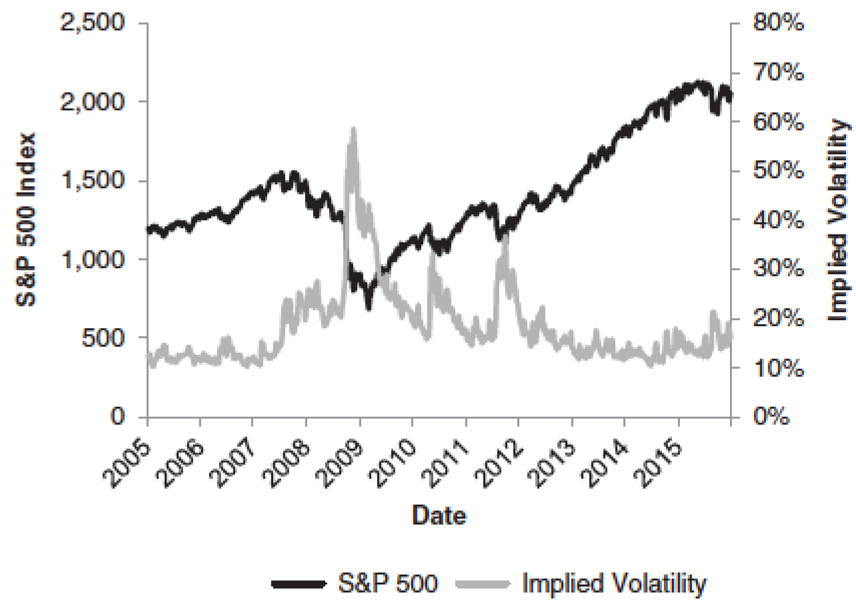


FIGURE 8.9 S&P 500 Level and Three-Month At-the-Money Implied Volatility

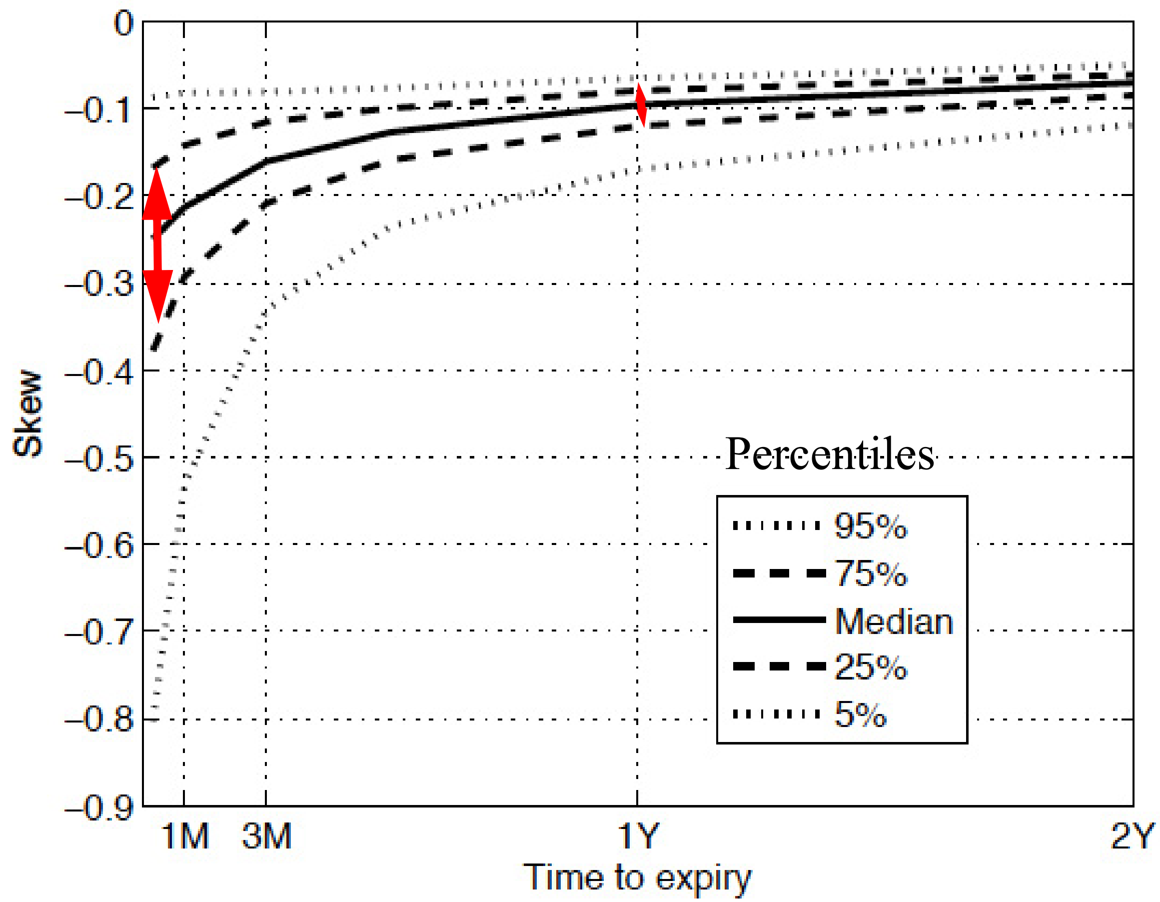
• Realized **and** implied volatility increase in a crash.

• When the index moves sharply down,

All option implied volatilities increase;
short-term implied volatility increases more sharply;
the short-term negative skew steepens.
long-term volatility and the long-term skew increase too, but less so.

• Implied volatility is a model parameter, like interest rates. It seems to be mean-reverting. It can jump up quickly and decline less rapidly, though nowadays it declines more rapidly than before, perhaps because people expect the Fed to save the market.

• The volatility of implied volatility and of the skew is greatest for short expirations, analogous to the higher volatility of short-term Treasury rates.



PRINCIPLES OF FINANCIAL VALUATION

The Principles of Financial Valuation

Don't get misled by mathematics, theorems, lemmas. Understand them, but this is real world.

- What is financial engineering?
  Cf. Mechanical engineering, Electrical engineering, Bio-engineering.

  Science seeks to discover the fundamental principles that describe the world, and is usually reductive.
  Engineering is about using those principles, constructively, for a purpose.

  Financial engineering, layered above financial science, would be the study of how to create functional financial devices – convertible bonds, warrants, default swaps, etc. – that perform in desired ways.
- What is financial science?
  Our scientific theories don't describe the behavior of assets very well. (People have emotions.)
  Stock evolution isn't Brownian.
  So let's use as little modeling as possible.
- Extreme axiomatization is therefore not that useful.Concentrate on concepts, then use math.

1.9 Terminology: Price & Value

- Price = what you have to pay to acquire a security.
  Value is what it is worth. The price is fair when it is equal to the value.
- Can you name one security whose price is equal to its value?
- Judging value, in even the simplest way, involves the construction of a model or theory.
- Fischer Black: markets are efficient when prices are 1/2 to 2 times value!

The Purpose of Models in Finance

- Example: Valuing a unique ("exotic") 57th Street Manhattan apartment.
- Models are used to rank securities by value.
- Models are used to interpolate or extrapolate from liquid prices to illiquid prices.
- Models allow you to quantify intuition, to turn linear quantities you can have intuition about into nonlinear dollar values.

Styles of Modeling: Absolute and Relative

- Absolute vs. Relative Value Models.

- Absolute Value/Description: Newtonian or Quantum Mechanics or Maxwell's Equations are
  theories of the world.
  Geometric Brownian motion is a *model* of valuation, an analogy, comparing stock prices to the diffusion of particles of smoke, not an accurate description.

- Relative Value:
  Used for going from liquid prices --> illiquid values.
  Relative valuation is less ambitious.
  We regard derivatives are molecules made out of simpler atoms.
  Black-Scholes tells you the price of an option in terms of the price of a stock and a bond.

- In this course we adopt the view point of an options trading desk, as manufacturers or wholesalers of options. We are relativists.
  Derivatives can be constructed or deconstructed
  Stocks to Options
  Vanilla to Exotic Options
  Fruit salad: What is *the implied price* of pears?

1.10 The One Commandment of Quantitative Finance

Andy Lo.
You might think there are many laws, but …

You can derive everything in neo-classical finance from one principle:

- **The Law of One Price**: If you want to know the value of a security, use the price of another security that's as similar to it as possible.

- **The Principle of No Riskless Arbitrage**: Any two securities with identical future payoffs, *no matter how the future turns out*, should have identical current prices.

- How to use the law:
  Target security
  Replicating portfolio: a collection of liquid securities that collectively have the same future payoffs as the target, *no matter how the future turns out.*

- ***The Science Part of Finance***:

  describe the stochastic behavior of all future payoffs.
  usually a model and at best partially correct.

The Engineering Part of Finance:

construct the replicating portfolio and prove similarity of payoffs.

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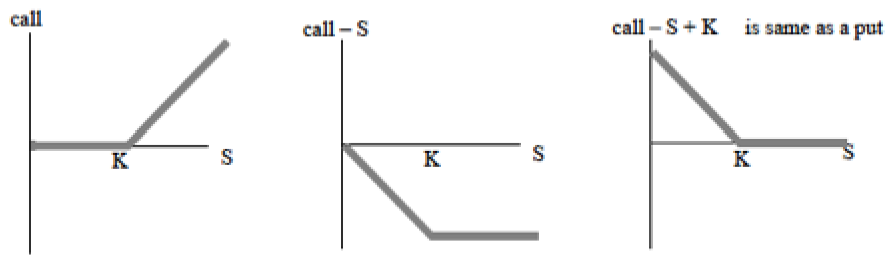
Engineering: Styles of Replication

- **Static:**
  The best method when you can do it;
  Example: replicating a put from a call: P = C - S + K



  Once you replicate statically, you have only credit risk. Even if you can do it approximately, you eliminate a lot of model risk and rehedging costs.

- **Dynamic**
  Underlying positions needs to be adjusted many times
  Black-Scholes-Merton theory of options valuation

  The risks:
    Transactions costs
    Liquidity
    Systems and IT issues
    Exposure to volatility and models. (Volatility has dynamics too, later).

Always try static first, then dynamic.

- So, in this course, *first try use static replication* for valuing new securities. If we cannot, *then we will use dynamic replication*.

Models are unreliable guides to the world of finance, and because you don't know which is the right one, *it's best to use as little modeling as possible. And, if you have to use a model, it's always good to use more than one so you understand the model-dependence of your result.*

Implied Variables and Realized Variables

Physics models and theories start from today and **predict the future**.

Financial models are mostly about finding the current value of a security. They extract opinions or views about the future from current prices and **predict the value of another security today**. (The apartment price; the value of a mortgage; an exotic option value).

Financial models **calibrate** the future to current known prices of liquid securities whose prices we trust to produce the current value of **implied variables** that match known prices today:

Yield to maturity, forward rates, implied volatility.

Implied variables describe what people think will happen filtered through a model.

Realized variables describe what actually happens.

What matters is not only what will happen, but what people *think* will happen.

But: what people think will happen affects what happens today.

Realized volatility is a statistic.

Implied volatility is NOT a statistic, but a parameter.

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MODELING MARKETS:

1.11 The Efficient Market Hypothesis

Experience shows that it is difficult or impossible to successfully and consistently predict what's going to happen to the stock market tomorrow based on all the information you have today.

We have already seen in the stylized facts that there is little autocorrelation between returns.

The EMH formalizes this experience by stating that it is impossible to beat the market, because current prices reflect all current economic and market information.

Jiu-jitsu approach: *I can't figure out how things work, so I'll make the inability to do that a principle.*

1.12 Uncertainty, Risk & Return

Quantifiable Uncertainty or Risk

What do you mean when you say there's a 1/8 chance of throwing 3 heads in succession?

Frequentist Probabilities

Tossing a coin: **history doesn't matter**.

Unquantifiable Uncertainty:

What do you mean when you say that there's a small probability of a revolution overthrowing the United States government in the next year?

Consider: The likelihood of a revolution in some country or the probability of a terrorist attack. The chance that an earthquake of magnitude 6.7 or greater will occur before the year 2030 in the San Francisco Bay Area.

No way of honestly estimating probabilities.

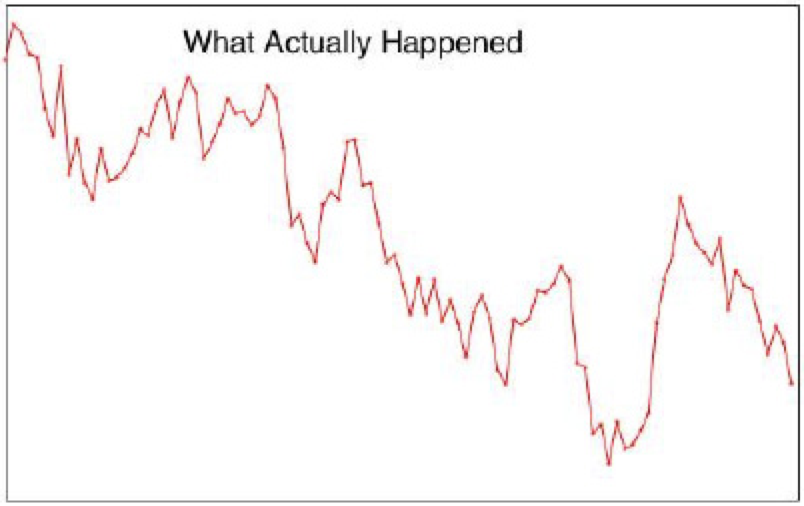
In human affairs frequentist probabilities are not known and **history matters**. People have memories.

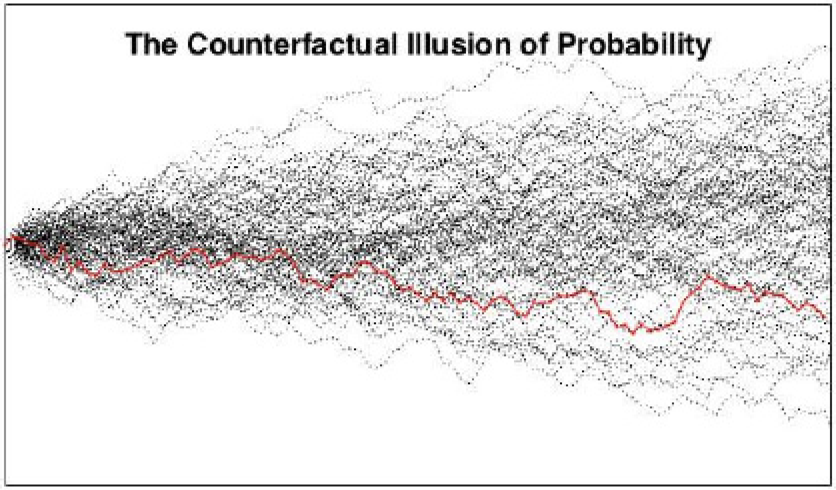
9/11 wasn't a part of the probability distribution for the US until after it happened.

Most events of interest in life have unquantifiable probabilities.

Most financial models assume that unquantifiable uncertainties are actually frequentist. We are mostly going to do this too, but we need to always remember the assumptions we make.

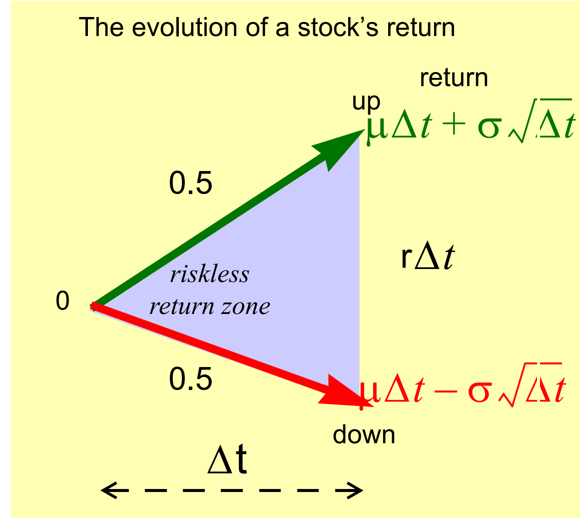
Markets are not exactly like flipping coins. There isn't a well-defined *a priori* probability of a market crash. Probability is a bit of an illusion, a fantasy about what might have happened.





1.13 The *Attempted Science* Part of Neoclassical Finance

- Brownian motion: Stock returns are assumed to undergo arithmetic Brownian motion, are normally distributed.



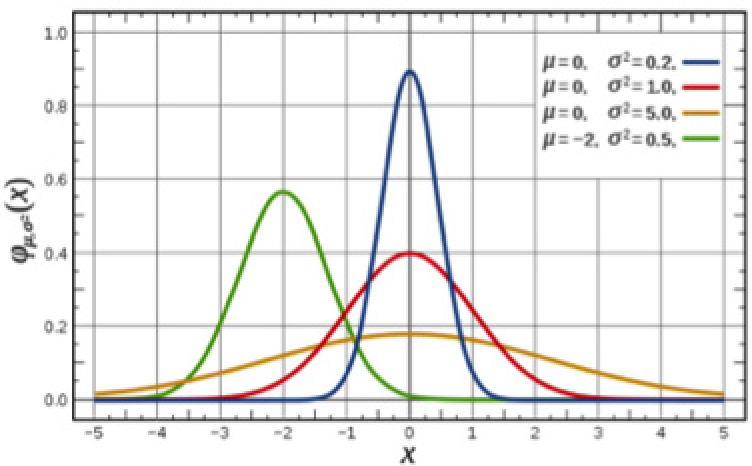
- $$\frac{dS}{S} = \mu dt + \sigma dZ$$

- No Arbitrage means **the riskless return is a convex combination** of up and down returns. If it weren't, you could always make a profit.

- Brownian motion is one model for an efficient market. It is a **good model** for atoms, a **not-so-good model** for stocks, as we've seen from stylized facts.

- **Brownian motion assumes:**
  "financial risk" is a frequentist statistic.
  the stock's motion is completely specified by the volatility σ and expected return μ.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$



1.14 Deriving The Relation Between Risk and Return in our Model

In our geometric Brownian motion model for stocks, everything foreseeable about a stock, and in particular the distribution of its future payoffs, is determined by its ***expected return*** μ and the ***volatility*** σ **that represents its risk.**

All that differentiates one stock from another in this model are the values of μ and σ. Two securities with the same μ and σ are in essence identical.

What is the relation between μ and σ?

Note: There is one privileged security, the riskless bond, which has $\sigma = 0$. If there is no risk $\sigma = 0$, you know you will exactly earn the riskless rate and so $\mu = r$ exactly, with no statistical distribution at any given time.

What happens if σ is not zero?

Then We Will Use The Law of One Price to Reduce Risky Securities to Riskless Ones.

If you have a risky security whose risk σ you know, but whose return μ you don't know, you can figure out the value of μ via the following strategy:

Embed the risky security into a portfolio such that the portfolio has zero total risk.

Then by the Law of One Price it should have the known return r of a riskless bond.

If you know the composition of that riskless portfolio, the proportions that make it riskless, then you can then figure out the expected return of the risky security.

This strategy will lead to both CAPM (the Capital Asset Pricing Model) and the Black-Scholes-Merton Options Valuation Model.

How Can One Reduce Risk?

Hedging: If two securities are positively correlated with each other, if you buy one and short the other, you can reduce the risk of the portfolio.

Diversification: If you take a whole bunch of uncorrelated risky securities together, their volatility decreases because some go up as others go down. If you put enough of them together, their volatility becomes zero asymptotically.

- **Strategy:**

To find the expected return on a security, try to remove all of its risk by combining it with other securities. If you can do that, then by The Law Of One Price, the combination must earn the riskless rate which is known. Then you can back out the return of the original security.

1.15 The Relation Between Risk & Return for Stocks: CAPM

Consider a market with an infinite number of stocks S_i, all correlated with the market, which you can think of as the S&P 500 or S&P futures for example.

Let ρ_{iM} be the correlation of the returns between a stock S_i with volatility σ_i and the market M with volatility σ_M. Here is the geometric Brownian motion description:

$$\frac{dM}{M} = \mu_M dt + \sigma_M dZ_M$$

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i\left(\sqrt{1-\rho_{iM}^2}\,dZ_i + \rho_{iM} dZ_M\right)$$

where dZ_i and dZ_M are all uncorrelated with each other. The idiosyncratic risk of the stock is described by dZ_i.You can *hedge away* the M-related risk of the stock S_i by shorting Δ_i shares of the market to create an M-neutral stock $\vec{S}_i$ that has no exposure to movements dZ_M, but only to dZ_i

$$\vec{S}_i = S_i - \Delta_i \times M$$

$d\vec{S}_i = dS_i - \Delta_i \times dM$ has no market risk if $\Delta_i = \rho_{iM}(\sigma_i/\sigma_M)\frac{S_i}{M} \equiv \beta_i \frac{S_i}{M}$.

Each M-neutral stock $\vec{S}_i$ is now uncorrelated with the market and uncorrelated with all other M-neutral stocks.

The M-neutral stock has expected return per unit time

$$\frac{1}{dt}\left\{\frac{E\left[d\vec{S}_i\right]}{\vec{S}_i}\right\} = \frac{1}{dt}\left\{\frac{E[dS_i - \Delta_i \times dM]}{S_i - \Delta_i \times M}\right\} = \frac{\mu_i S_i - \Delta_i \mu_M M}{S_i - \Delta_i \times M} = \frac{\mu_i S_i - \beta_i \mu_M S_i}{S_i(1-\beta_i)} = \frac{\mu_i - \beta_i \mu_M}{(1-\beta_i)}$$

We can do this for each stock S_i in the market. If we diversify over all M-neutral stocks we can create a portfolio with asymptotically zero volatility, and so by the Law of One Price it must earn the riskless return r, so that[1] each stock in the portfolio can earn no more than the riskless return. So

$$\frac{\mu_i - \beta_i \mu_M}{(1-\beta_i)} = r$$

$$(\mu_i - r) = \beta_i(\mu_M - r) \qquad \text{CAPM in “Efficient Markets”}$$

$$(\mu - r) = \beta(\mu_M - r) \qquad \text{Capital Asset Pricing Model}$$

stock’s excess return   stock’s beta   market’s excess return

What this relation is really saying is that if you can hedge away all market risk, and then diversify over all idiosyncratic risk, no risk is left, and so you must earn the riskless rate.

1. Spelled out in more detail in Section 2 of *The Perception of Time, Risk and Return During Periods of Speculation*, Quantitative Finance Vol 2 (2002) 282–296, or http://emanuelderman.com/perception-time/

[Finance For Future Generations]

- Feynman: One sentence about physics to guide future civilizations
- Feynman: One sentence about biology to guide future civilizations:
- One sentence about finance to guide future civilizations:

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  If

  you can hedge away all correlated risk, and
  then diversify over all uncorrelated risk,

  you should expect to earn the riskless return.

This is a sensible principle.

The difficulty is that correlation and diversification can't really be carried out because risk is not really purely statistical. It can't be specified for all time by a stochastic pde or Monte Carlo.