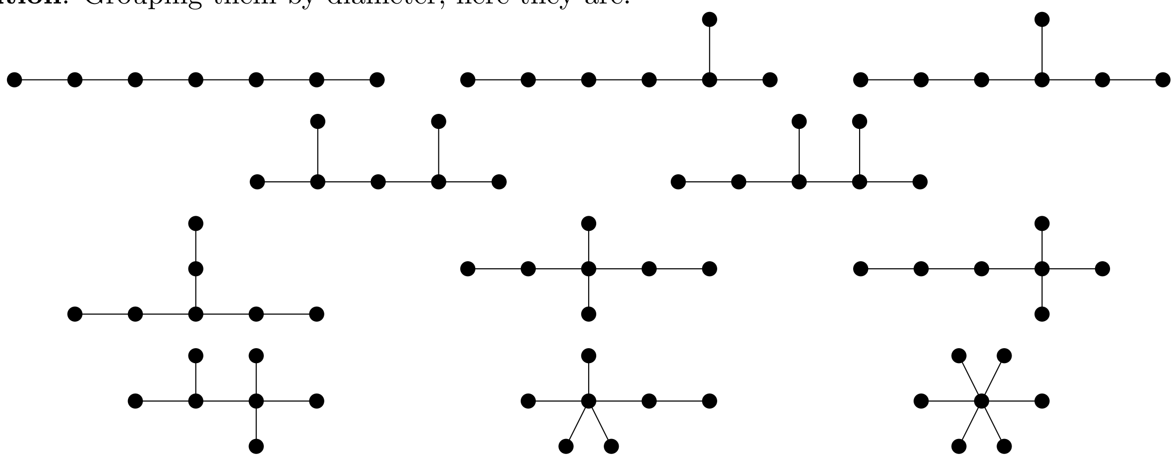
Math 280, Spring 2022

Solutions to Homework #4

1. (20 points) Textbook Section 1.3.1, Problem 1: Draw all unlabeled trees of order 7.

Solution. Grouping them by diameter, here they are:



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2. (6 points) Textbook Section 1.3.1, Problem 3:
Let T be a tree of order $n \geq 2$. Prove that T is bipartite.

Proof. Since T is a tree, it has no cycles, and in particular it has no cycles of odd length. By Theorem 1.3, then, T is bipartite. QED

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3. (6 points) Textbook Section 1.3.2, Problem 2: Let T be a tree that has an even number of edges. Prove that at least one vertex of T has even degree.

Proof. Let $n = |V(T)|$ be the order of T. By Theorem 1.10, T has $n-1$ edges, so our hypothesis says that $n-1$ is even, and hence n is odd.
Recall from Theorem 1.1 that the number of vertices of odd degree is even. Since n is odd, this means there must be at least one vertex $v \in V(T)$ that has even degree. QED

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4. (18 points) Textbook Section 1.3.2, (part of) Problem 5: Let T be a tree, and let $u, v \in V(T)$. Prove that there is *exactly one* path from u to v.

Proof. Because T is connected, there is *at least* one path from u to v, by definition of connectedness. It suffices to show this path is unique.
Suppose there are two different such paths

$$u = x_1, \ldots, x_\ell = v, \qquad u = y_1, \ldots, y_m = v.$$

Let $i \geq 1$ be the smallest integer such that $x_i \neq y_i$. Such i must exist because the paths are different; note that $i \geq 2$ since $x_1 = v = y_1$. In particular, $i - 1 \geq 0$, and we have $x_{i-1} = y_{i-1}$
Let $j \geq i$ be the smallest integer greater than or equal to i such that $x_j = y_k$ for some k. Such j must exist because $x_\ell = v = y_m$. Note that $k \geq i$, because $x_1, \ldots, x_j$ are distinct (by definition of path), so $y_k = x_j$ must be distinct from each of

$$x_1 = y_1,\ x_2 = y_2,\ \ldots\ , x_{i-1} = y_{i-1},$$

and hence $k \geq i$. In addition, we cannot have $k = j = i$, because by definition of i, we have $x_i \neq y_i$.
[However, we might have $j = i$ and $k \geq i+1$, or $k = i$ and $j \geq i+1$.]
Consider the walk W given by

$$x_{i-1}, x_i, \ldots, x_j = y_k, y_{k-1}, \ldots, y_i, y_{i-1},$$

which starts and ends at $x_{i-1} = y_{i-1}$. Note that W consists of $j-(i-1)$ edges from x_{i-1} to x_j, followed by $k-(i-1)$ edges from y_k to y_{i-1}, for a total length of

$$(j-i) + (k-i) + 2 \geq 3.$$

In addition, all of the vertices in W are distinct from each other. The x's are distinct because they are in a path, and similarly the y's are distinct; and by definition of j, none of $x_i, \ldots, x_{j-1}$ is equal to any of the $y's$.
Thus, W is a closed walk of length at least 3 with no repeated vertices (other than the first and last); i.e., W is a cycle. But T is a tree and hence has no cycles, so this is a contradiction. Hence, our assumption that there are two different paths is impossible, and there is only one such path. QED

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5. (14 points) Textbook Section 1.3.2, (part of) Problem 6:
Let T be a tree, and let $u, v \in V(T)$ be two distinct vertices that are *not* adjacent. Define a new graph G with the same vertex set $V(G) = V(T)$ but with one extra edge $e = uv$. That is, $E(G) = E(T) \cup \{e\}$, where the new edge e runs between u and v.
Prove that the new graph G has exactly one cycle.

Proof. Since T is connected, there is a path P given by $u = x_1, \ldots, x_k = v$ in T from u to v. Moreover, because u and v are not adjacent in T, we have $k \geq 3$, i.e., this path has length at least 2.
Since $uv = e$ is an edge in G, we may form a walk W in G given by

$$u = x_1, \ldots, x_k = v, u.$$

Then W is a cycle, since it is a closed walk of length at least 3 with no repeated vertices other than the first and last. So G contains at least one cycle.
To show uniqueness, consider an arbitrary cycle C in W. We consider two cases.
Case 1. C does not use the edge e. Then C is a walk in T, and hence a cycle in T. But T is a tree and so has no cycles, so this is impossible.
Case 2. Otherwise, C uses the edge e. Since we may start a cycle at any one of its vertices and move in either direction along it, we may start C at the vertex u and end with the edge e, so that C is:

$$u = y_1, \ldots, y_m = v, u.$$

Then removing the edge e from the end, we have that the walk Q given by $u = y_1, \ldots, y_m = v$ is a path from u to v in T. By the previous problem, there is only one path from u to v in T, so $Q = P$. That is, we have $k = m$ and $x_1 = y_1, \ldots, x_k = y_k$. Therefore, the cycle C is precisely the same as the cycle W. QED

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6. (10 points) Let T be a tree of order $n \geq 2$, and suppose that none of the vertices of T have degree 2. Prove that T has more than $n/2$ leaves.
Proof. Let $X \subseteq V(T)$ be the set of leaves (i.e., vertices of T of degree 1), and let $Y = V(T) \smallsetminus X$ be the set of vertices that are not leaves. By the hypothesis that $\deg(v) \neq 2$ for all $v \in V(T)$, we have

$$\deg(y) \geq 3 \quad \text{for all } y \in Y.$$

Let $m = |X|$ be the number of leaves. We have $|X| + |Y| = n$, since there are n total vertices, and therefore $|Y| = n - m$. Moreover, by Theorem 1.10, we have $|E(T)| = n - 1$. Thus, by Theorem 1.1, we have

$$\begin{aligned} 2(n-1) = 2|E(T)| &= \sum_{v \in V(T)} \deg(v) = \sum_{x \in X} \deg(x) + \sum_{y \in Y} \deg(y) \\ &= m \cdot 1 + \sum_{y \in Y} \deg(y) \geq m + 3|Y| = m + 3(n-m) = 3n - 2m. \end{aligned}$$

Rearranging, then, we have

$$2m \geq n+2, \qquad \text{i.e.,} \qquad m \geq \frac{n}{2}+1 > \frac{n}{2},$$

i.e., there are (strictly) more than $n/2$ leaves. QED

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7. (14 points) Textbook Section 1.3.3, Problem 1:
Let G be a connected graph. Prove that G contains at least one spanning tree.

Proof. **Method 1**: Let $n = |V(G)|$, and let T be a subtree of G (i.e., a subgraph that is a tree) with the maximum possible number of vertices m among all subtrees of G. Note that $m \geq 1$, because a one-point graph is a tree; so we may choose a vertex $v \in V(T)$. It suffices to show that $m = n$, i.e., that T uses all the vertices of G.

Suppose not, i.e., that there is some vertex w of G that is not in T. Since G is connected, there is a path

$$v = x_1, x_2, \ldots, x_k = w$$

from v to w in G. Let $i \geq 1$ be the smallest positive integer such that $x_i \notin V(T)$; note that i exists because $x_k = w \notin V(T)$. We also have $i \geq 2$ because $x_1 = v \in V(T)$.

Let $e_i = x_{i-1}x_i$, which is an edge of G, because it is in the above path. Let T' be the subgraph of G given by

$$V(T') = V(T) \cup \{x_i\} \subseteq V(G) \qquad \text{and} \qquad E(T') = E(T) \cup \{e_i\} \subseteq E(G).$$

Note that T' is indeed a graph, because for any $e \in E(T')$, we have either $e \in E(T)$ (in which case both vertices of e are in $V(T) \subseteq V(T')$), or else $e = e_i$, in which case the two vertices of e are $x_{i-1}, x_i \in V(T')$. Since $V(T) \subseteq V(T') \subseteq V(G)$ and $E(T) \subseteq E(T') \subseteq E(G)$, it follows that T is a subgraph of T', and that T' is a subgraph of G.

We claim that T' is connected. Indeed, given any $y, z \in V(T')$, if both are in $V(T)$, then there is a path between them in T and hence in T'. If $y = z = x_i$, then there is a (trivial) path between them. The only other possibility is that one of them is x_i and the other is in $V(T)$; without loss, $y = x_i$ and $z \in V(T)$. In that case, there is a path from z to x_{i-1} in T, and we can append the edge e_i to the end of that path to produce a path in T' from z to $x_i = y$, proving our claim.

Finally, because T is a tree with $|V(T)| = m$, we have $|E(T)| = m-1$ by Theorem 1.10. Therefore, since T' is formed by adding one vertex and one edge, we have $|V(T')| = m+1$ and $|E(T')| = m = |V(T')|-1$. By Theorem 1.12, since T' is connected and has one more vertex than it has edges, we have that T' is a tree.

Thus, G has a subtree T' with $m+1$ vertices, contradicting the maximality of T. Therefore, our assumption that T did not use all the vertices must have been false. That is, G has a subtree T that uses all n vertices, and hence is a spanning tree. QED

Method 2: Consider the set S of all subgraphs H of G that are connected and have $V(H) = V(G)$. Note that S is finite (because G has only finitely many subgraphs at all) and nonempty (because G itself is one of the graphs in the set S). So it makes sense to define T to be an element of S (that is, a connected subgraph of G with $V(T) = V(G)$) with the fewest possible number of cycles among all the graphs in S.

We claim that T is acyclic. To prove this claim, suppose there is a cycle C in T, given by

$$x_1, x_2, \ldots, x_k = x_1.$$

That is, each of $x_1, \ldots, x_{k-1}$ is distinct, we have x_i adjacent to x_{i+1} for each $i = 1, \ldots, k-1$, and $k \geq 4$. Let e be the edge $e = x_1x_2 \in E(T)$, and let $K = T - e$, so that K is a subgraph with $V(K) = V(T) = V(G)$.

In addition, K is connected, for the following reason. For any vertices $v, w \in V(K)$, there is a path in T from v to w, given by

$$v = y_1, y_2, \ldots, y_\ell = w.$$

If none of the edges in this path is e, then it is a path in K as well. Otherwise, we have $y_i y_{i+1} = e$ for some i, and hence either $y_i = x_1$ and $y_{i+1} = x_2$, or vice versa; without loss, assume $y_i = x_1$ and $y_{i+1} = x_2$. Then the following is a walk in K from v to w:

$$v = y_1, y_2, \ldots, y_i = x_1 = x_k, x_{k-1}, \ldots, x_3, x_2 = y_{i+1}, y_{i+2}, \ldots, y_\ell = w.$$

Since there is a walk in K from v to w, there is also a path in K from v to w. Thus, K is indeed connected.

That is, K is a connected subgraph of G with $V(K) = V(G)$, and hence K is one of the graphs in the set S. However, K has fewer cycles than T does, since every cycle of K must be a cycle in T (since K is a subgraph of T), but T has the cycle C while K does not. This contradicts the fact that T has the fewest cycles of all graphs in S. This contradiction proves our claim.

Thus, T is acyclic (by the claim) and connected (since it belongs to S), so it is a tree. It also has $V(T) = V(G)$ (since it belongs to S), so it is a spanning tree. QED