# Mathematics of Finance

## Handout 2

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# Time Series of Daily Stock Prices. Returns.

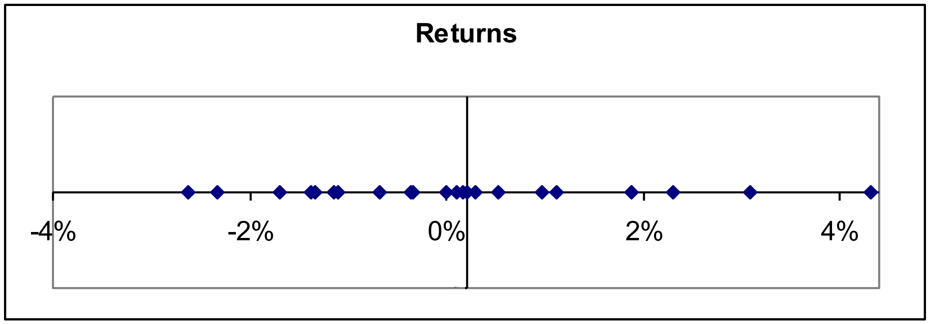
Typically, statistical characteristics of asset prices are calculated based on historical data. Most commonly used are series (also called time series) of daily closing prices. Price changes and returns are calculated usually from close to close.

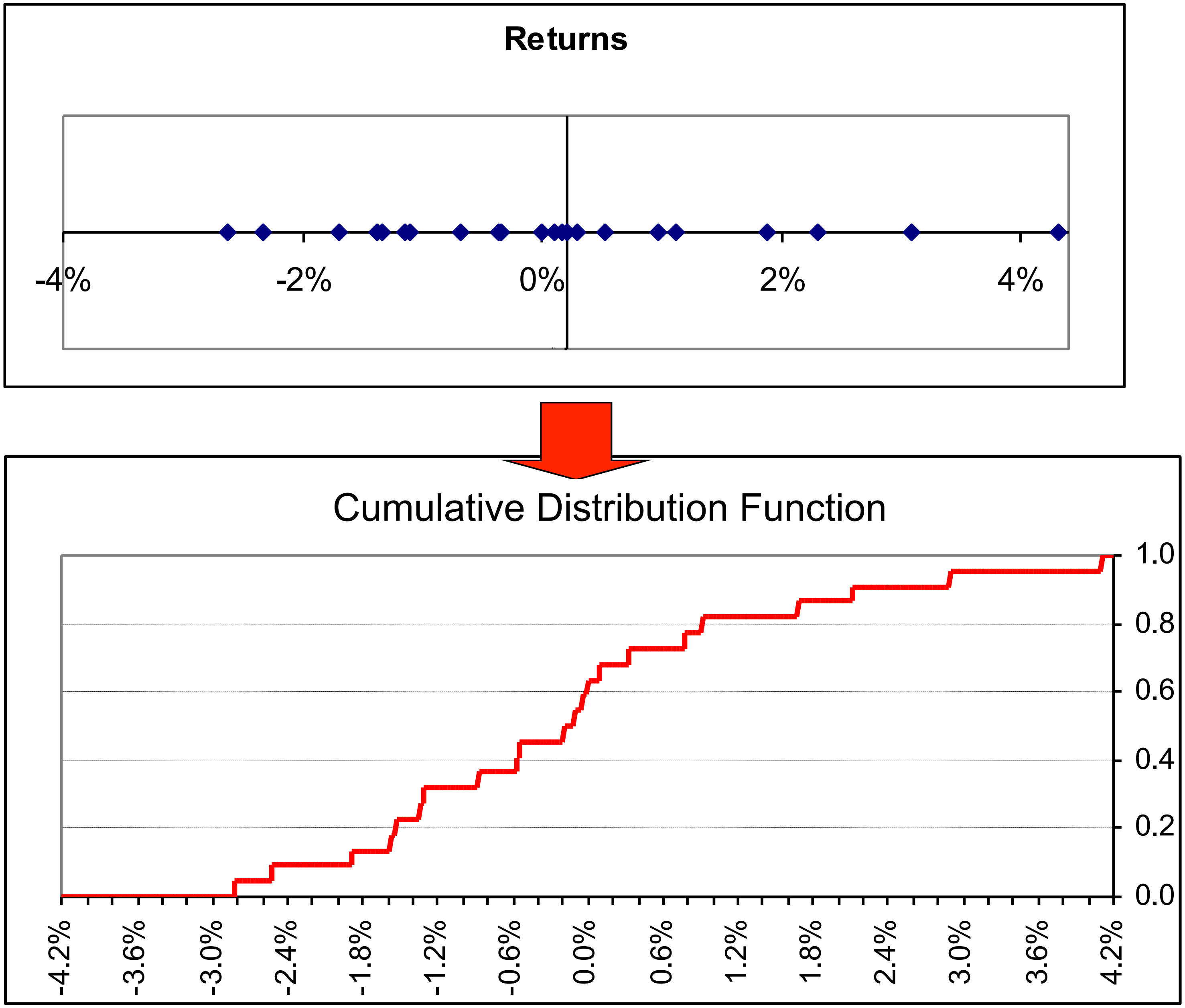
The other popular type of time series of prices are series of daily Open, High, Low, and Close prices and Daily Volume.

| Date | Close Price |
|---|---|
| 30-Dec-20 | 48.81 |
| 29-Dec-20 | 48.86 |
| 28-Dec-20 | 47.85 |
| 24-Dec-20 | 48.77 |
| 23-Dec-20 | 47.40 |
| 22-Dec-20 | 48.14 |
| 21-Dec-20 | 48.91 |
| 18-Dec-20 | 48.75 |
| 17-Dec-20 | 49.18 |
| 16-Dec-20 | 49.45 |
| 15-Dec-20 | 49.00 |
| 14-Dec-20 | 49.00 |
| 11-Dec-20 | 49.02 |
| 10-Dec-20 | 49.67 |
| 9-Dec-20 | 49.95 |
| 8-Dec-20 | 51.40 |
| 7-Dec-20 | 51.35 |
| 4-Dec-20 | 52.68 |
| 3-Dec-20 | 53.40 |
| 2-Dec-20 | 52.99 |
| 1-Dec-20 | 52.11 |
| 30-Nov-20 | 52.21 |
| 27-Nov-20 | 50.15 |

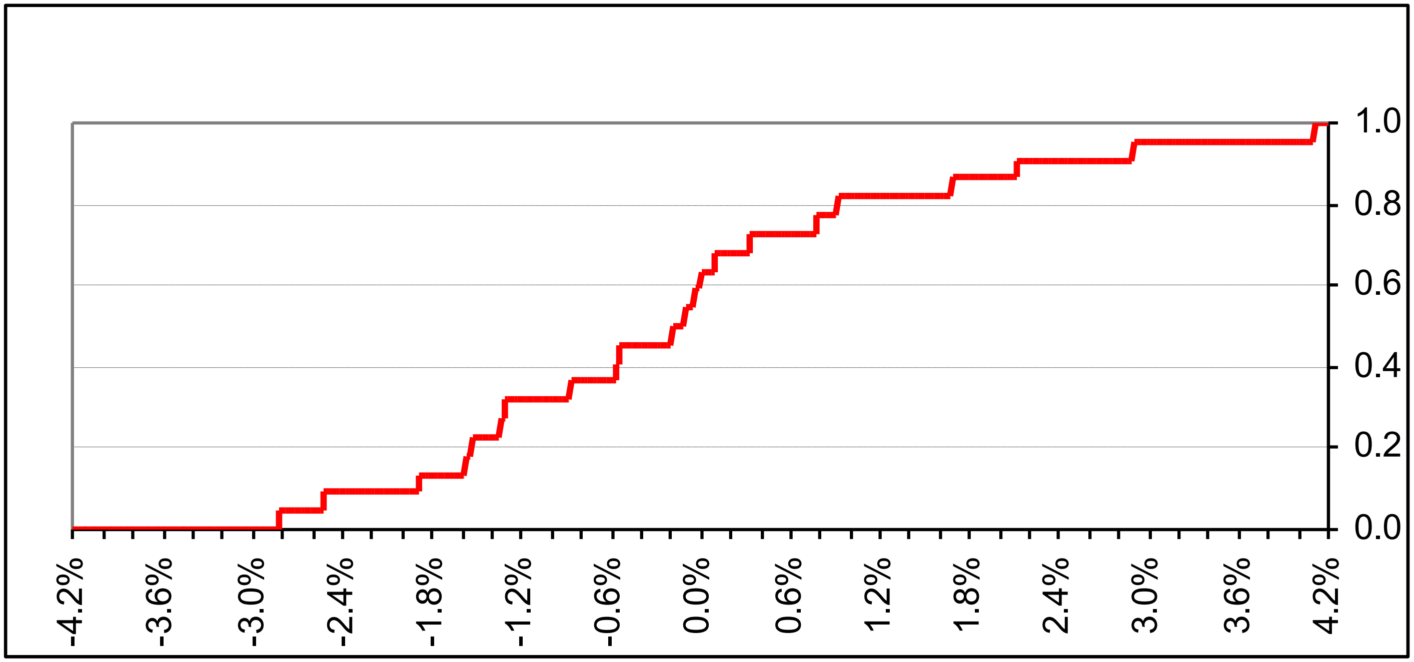
| Date | Return |
|---|---|
| 30-Dec-20 | -0.10% |
| 29-Dec-20 | 2.11% |
| 28-Dec-20 | -1.89% |
| 24-Dec-20 | 2.89% |
| 23-Dec-20 | -1.54% |
| 22-Dec-20 | -1.57% |
| 21-Dec-20 | 0.33% |
| 18-Dec-20 | -0.87% |
| 17-Dec-20 | -0.55% |
| 16-Dec-20 | 0.92% |
| 15-Dec-20 | 0.00% |
| 14-Dec-20 | -0.04% |
| 11-Dec-20 | -1.31% |
| 10-Dec-20 | -0.56% |
| 9-Dec-20 | -2.82% |
| 8-Dec-20 | 0.10% |
| 7-Dec-20 | -2.52% |
| 4-Dec-20 | -1.35% |
| 3-Dec-20 | 0.77% |
| 2-Dec-20 | 1.69% |
| 1-Dec-20 | -0.19% |
| 30-Nov-20 | 4.11% |
| 27-Nov-20 | |

-0.10%=(48.81-48.86)/48.86





Returns
-4%
-2%
0%
2%
4%
Cumulative Distribution Function
1.0
0.8
0.6
0.4
0.2
0.0
-4.2%
-3.6%
-3.0%
-2.4%
-1.8%
-1.2%
-0.6%
0.0%
0.6%
1.2%
1.8%
2.4%
3.0%
3.6%
4.2%




Cumulative distribution function shows what is the fraction of returns equal or below a given number on horizontal axis. For example, from cumulative distribution function graph above, the fraction of returns equal or below –0.6% is approximately 0.38, the fraction of returns equal or below 3.6% is approximately 0.96, below 3.4% also 0.96, below 3.0% already 0.92.

Historical data about prices is available from many data providers, for example Bloomberg, Reuters, CRSP and others.Yahoo website **www.yahoo.com** provides for free historical stock price data in Excel spreadsheet format.For example, for IBM go to **http://finance.yahoo.com/q/hp?s=IBM+Historical+Prices** and select Download Spreadsheet on the bottom of the page.

## Cumulative Distribution Function of Returns. Mean and Standard Deviation of Returns.

In the previous section we took time series of prices , constructed series of returns $r_1 = \frac{p_1}{p_0} - 1, r_2 = \frac{p_2}{p_1} - 1, \ldots, r_n = \frac{p_n}{p_{n-1}} - 1,$ and then defined a **Cumulative Distribution Function** of returns:

$$F(x) = \frac{\text{Number of } r_i \text{ that are less or equal then } x}{\text{Total number of } r_i \text{ (that is } n)},$$

Now we will define several quantities characterizing series of returns.

**Mean** (or average, or expected) return $\bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$

**Variance** of returns $\mathrm{Var}(r_i) = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \bar{r})^2$, where $\bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$

**Standard deviation** of returns

$$\mathrm{Stdev}(r_i) = \sqrt{\mathrm{Var}(r_i)} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(r_i - \bar{r})^2}\ , \text{where } \bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$$

Mean (or average) return shows average level of growth of stock price. Variance and standard deviation of returns show the variability of stock. Stocks with higher variance and standard deviation are riskier.

## Calculation of Volatility of a Stock as a Standard Deviation of Percentage Returns. Annualization.

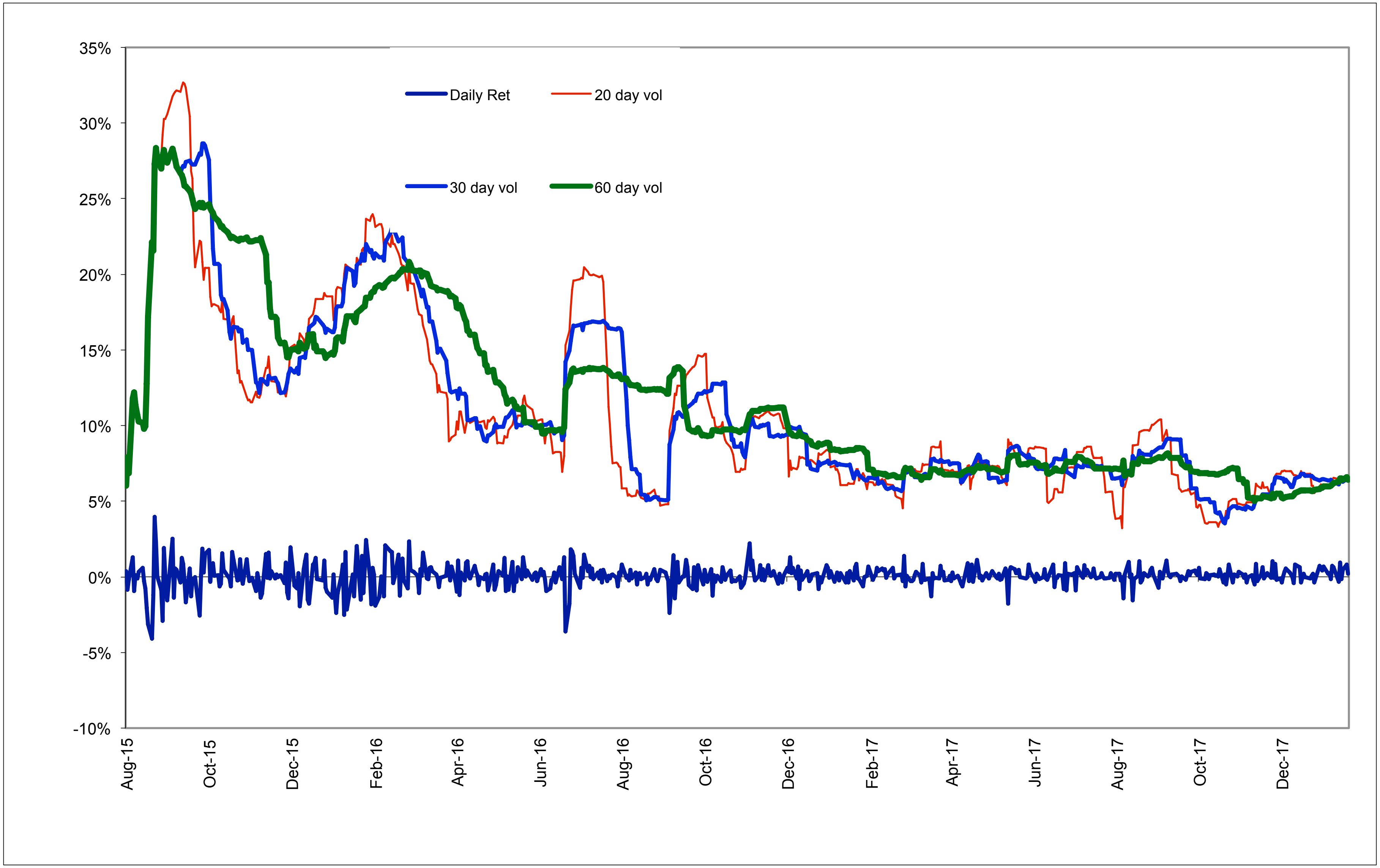
The standard measure of stock risk is Historical Volatility.

Historical Volatility is a standard deviation of daily returns of the stock calculated using last $N$ days multiplied by annualization factor. Typical values of $N$ are 15, 20, 30, 60 days. Sometimes larger $N$ is used.

Annualization factor is typically $\sqrt{\text{number of trading days in a year}}$

For example if a standard deviation of the last 20 days returns of the stock is 1%, number of trading days per year is 255, then annualized 20 day historical volatility of the stock is $0.01 \times \sqrt{255} = 0.159 = 15.9\%$

# SP 500 Volatility



# Random Variables. Cumulative Distribution Function. Probability Density Function.

A random variable is a mathematical model of an outcome of an experiment that is a numerical value. We assume that the experiment can be repeated many times and that the results of a random experiment can assume different values with different probabilities.

We want to characterize analytically the outcome of experiment, in particular we want to know what are the chances that the result of experiment falls between two given numbers $a$ and $b$.

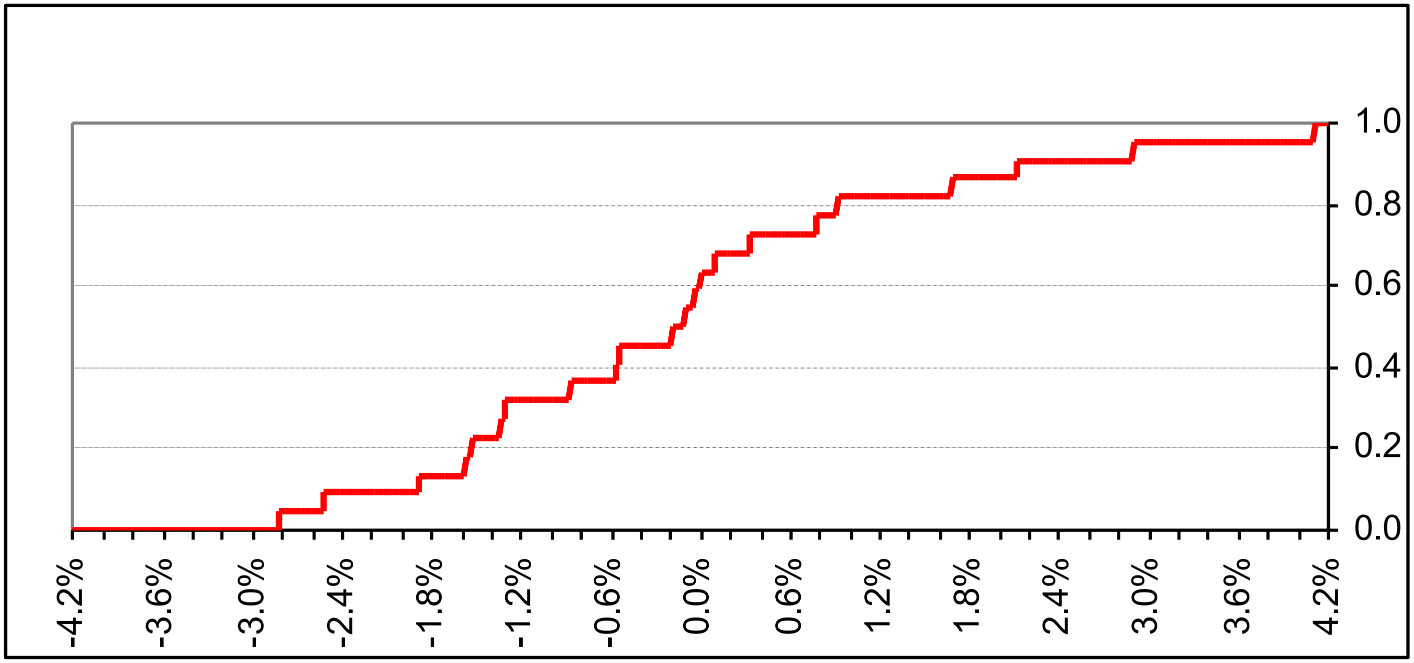
The first numerical characteristic of a random variable that we will use is cumulative distribution function $F(x)$ also called CDF.

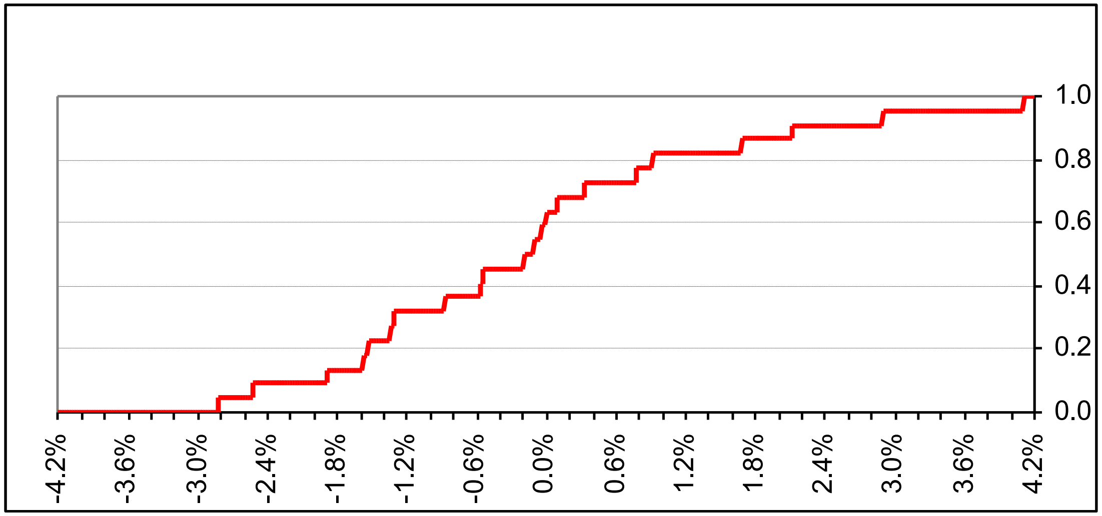
***F(x)=Probability that the value outcome of experiment is less or equal then x.***

If the outcomes of experiment are independent of each other and we make a large number of experiments then

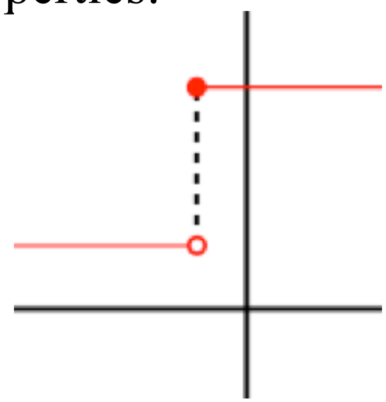
***F(x)~(Number of outcomes less or equal then x)/(Total Number of outcomes).***

A typical CDF is shown on the picture below.






Cumulative Distribution Function (CDF) has several important properties:



*1*. *F* is non-decreasing.
*2*. *F* is continuous from the right. Function is right-continuous if no jump occurs when the limit point is approached from the right.
3. *F* approaches 1 as *x* approaches plus infinity, and approaches 0 as *x* approaches minus infinity. So *F* is between 0 and 1.

Every function with these three properties is a CDF: for every such function *F* it is possible to find a random variable so that the *F* is the cumulative distribution function of that random variable.
In the future we are going to denote random variables by capital letters. For example *X*. We can now write

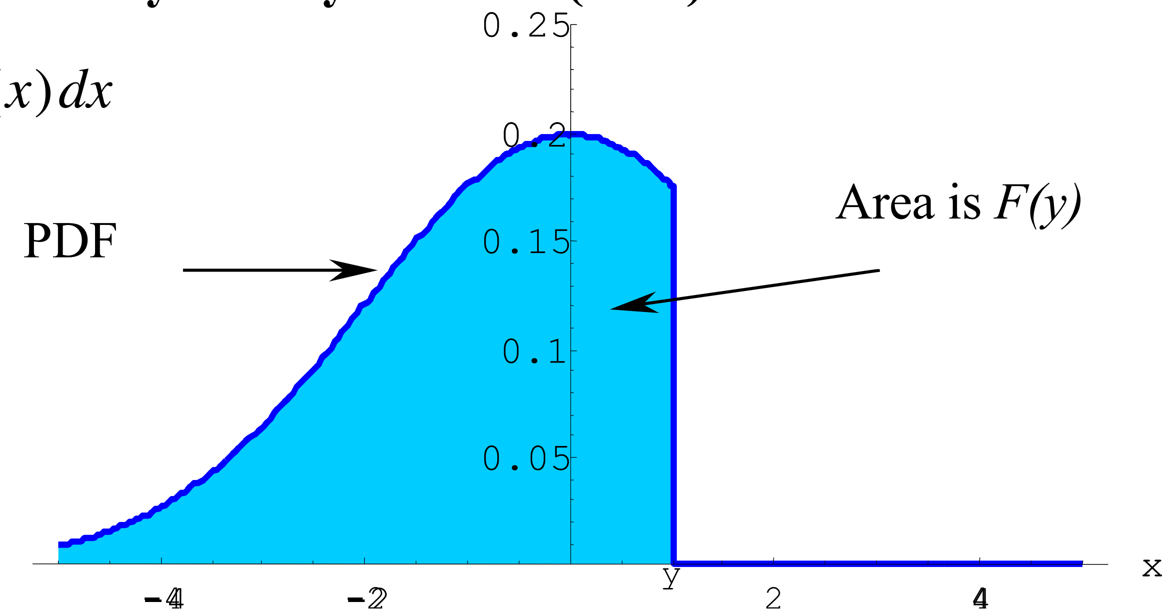
$$F(x)=Probability\ (X <= x)$$

If the CDF $F$ of a random variable $X$ is continuous, then $X$ called a continuous random variable. If $F$ is absolutely continuous, then there exists an integrable function $P(x)$ such that for all numbers a, b

$$F(b) - F(a) = \int_a^b P(x)\,dx = \Pr(a < X <= b)$$

The function $P$ is equal to the derivative of $F$ almost everywhere, and it is called the **probability density function (PDF)** of the random variable $X$.

$$F(y) = \int_{-\infty}^{y} P(x)\,dx$$




Intuitively, if we make a large number of experiments generating random variable $X$, the numbers thus generated would be spread along the axis with some density. This density is described by the probability density function.

Intuitively, if we make a large number of experiments generating random variable *X,* the numbers thus generated would be spread along the axis with some density. This density is described by the Probability Density Function.

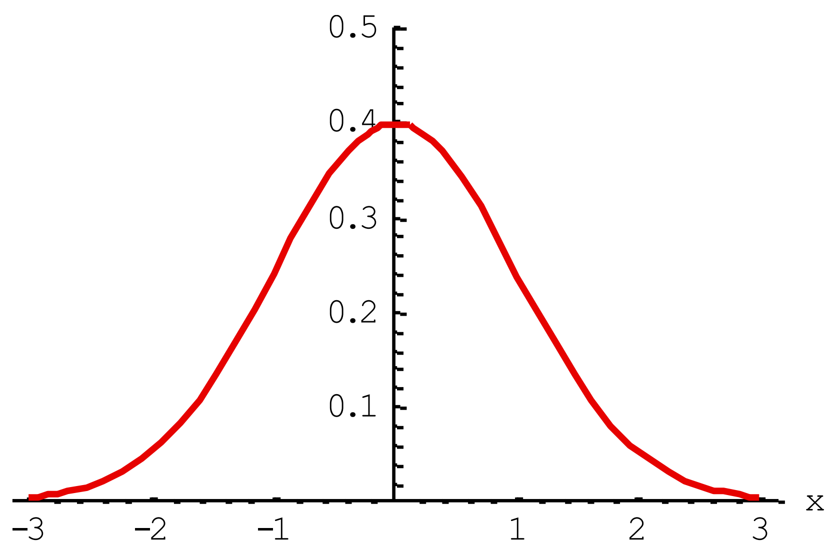
Note that it is not always true that for a random variable there exist a "good" Probability Density Function. For example mass ½ at points 0 and 1 for random variable taking values 0 and 1 with probability 1/2.

However most of the random variables that we shall see here will have a "good" Probability Density Function *P(x)*.

An important type of a random variable is a Standard Normal (also called Gaussian) random variable. Its probability density function is a "bell shaped curve".

**Example.** A **Standard Normal (or Gaussian)** random variable has the probability density function

$$P(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$



# Expectation, Variance, and Standard Deviation of Random Variables.

**Definition.** Let $X$ be a random variable with probability density function *P(x)*. Then the number

$$\mathrm{E}(X) = \int_{-\infty}^{+\infty} xP(x)dx$$

is called the **mean or expected value** of the random variable $X$. Mean shows the average value that you would get if you repeat experiment of getting a random variable with probability density function *P(x)* many many times.

The number

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} (x - \mathrm{E}(X))^2 P(x)dx$$

is called the **variance**.

**Definition.** The number

$$\sigma(X) = \sqrt{\mathrm{Var}(X)}$$

is called the **standard deviation** of the random variable $X$.

Notions of mean, variance and standard deviation defined here correspond to the same notions for time series defined before. If we consider larger and larger time series that are results of sampling a variable with probability density function *P(x)* then mean, variance and standard deviation in time series sense will converge to mean, variance and standard deviation defined here through integrals.

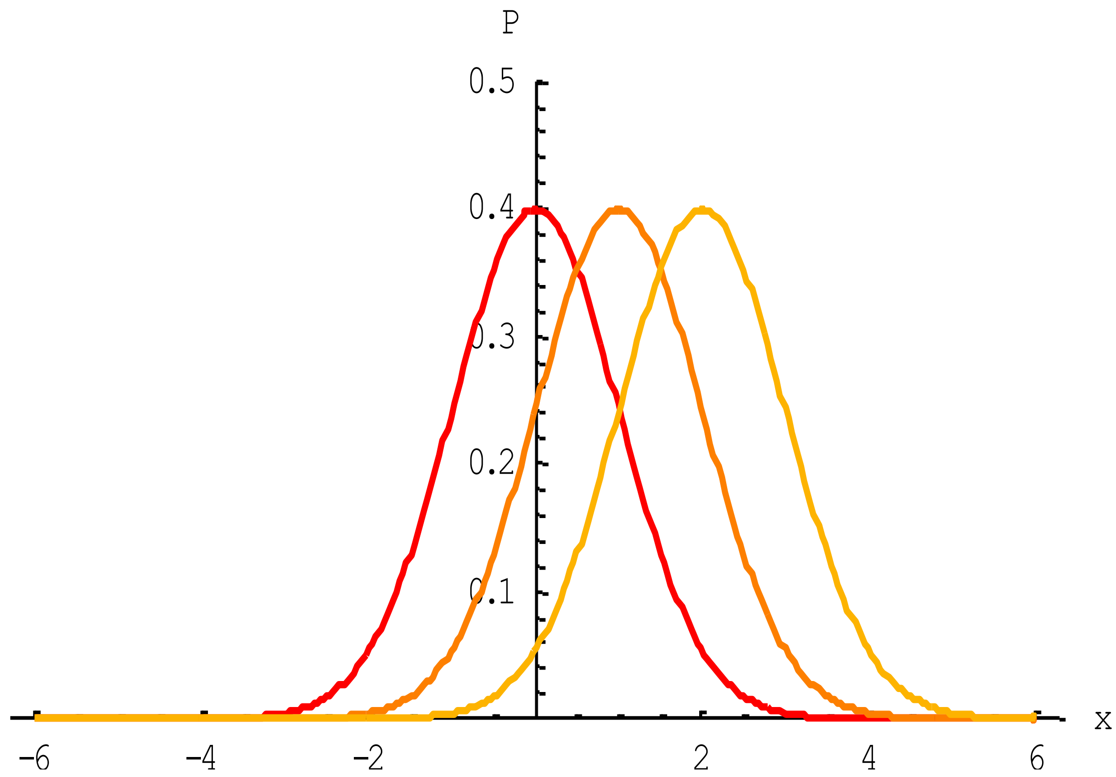
# Normal Random Variables.

**Definition.** A **Normal random variable** with mean μ and standard deviation σ has a probability density function

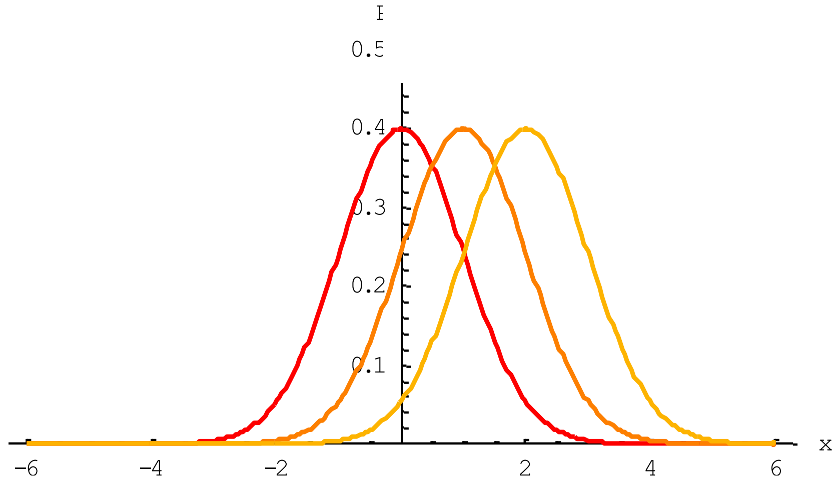
$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Here μ can be any real number and σ can be any positive real number. The standard normal random variable has mean μ=0 and standard deviation σ =1 (and variance =1).

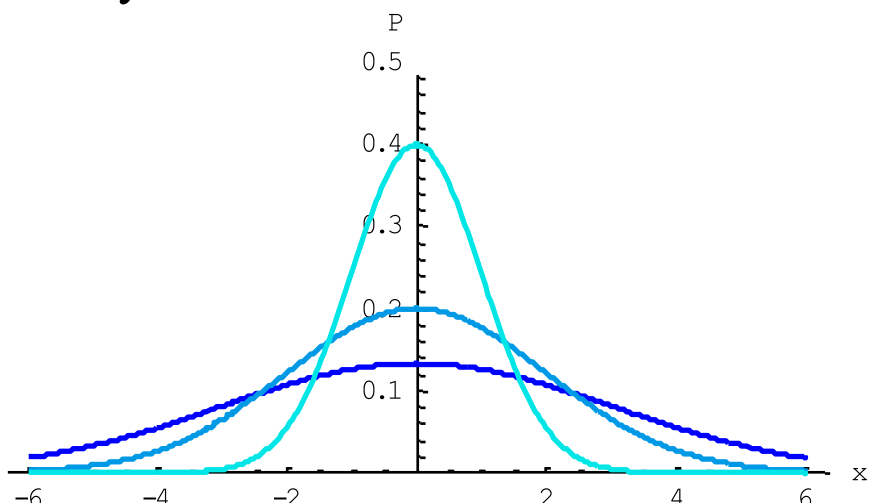
Probability density functions of normal random variables with σ =1 and μ=0, μ=1, μ=2.



Probability density functions of normal random variables with $\sigma = 1$ and $\mu = 0$, $\mu = 1$, $\mu = 2$.



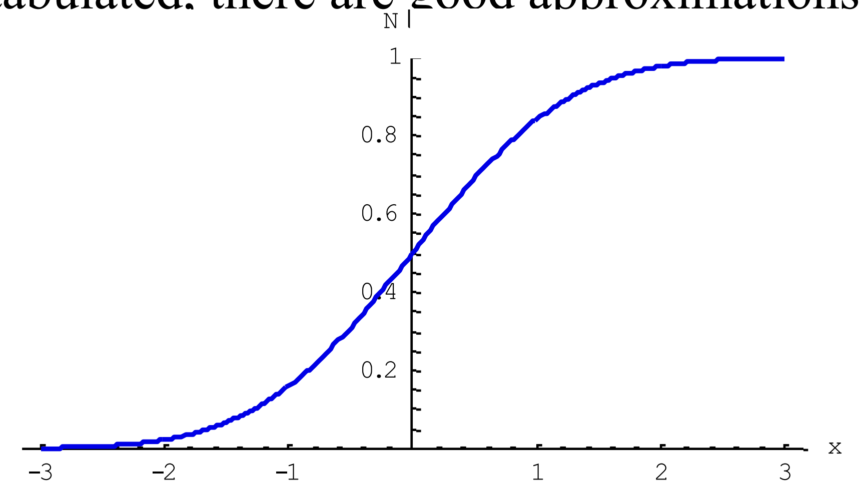

Probability density functions of normal random variables with $\sigma = 1$, $\sigma = 2$, $\sigma = 3$, and $\mu = 0$.



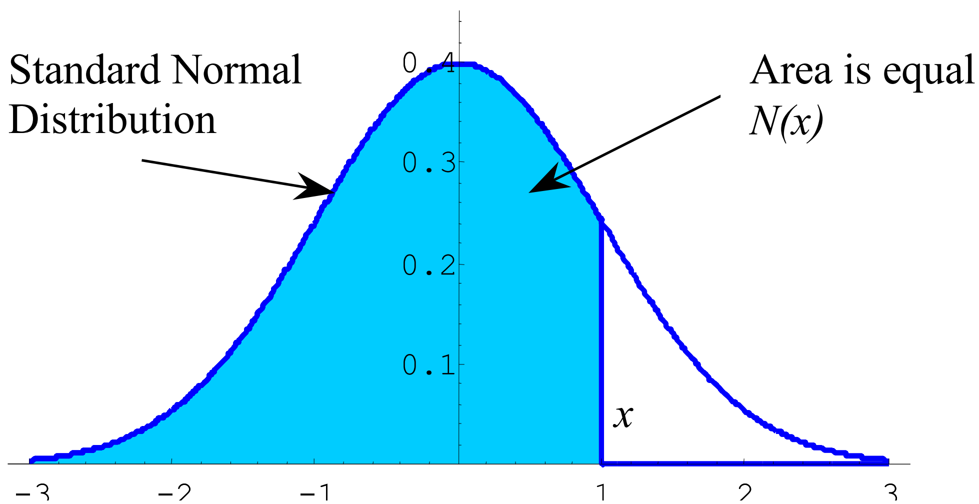

The larger is the $\sigma$ the wider is the graph of probability density function. Dark blue has $\sigma = 3$. The normal random variable is particularly important in probability theory and its applications because of the Central Limit Theorem. Qualitatively that result can be formulated as follows: if a random variable $X$ is the sum of a large number of independent random variables and each of these independent variables variable is small compared to the sum, then the distribution of $X$ can be well approximated by the normal distribution.

Definition. **Cumulative normal distribution** function $$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$$

This function is well tabulated, there are good approximations of it, and fast computer algorithms




Graph of the cumulative normal distribution function.

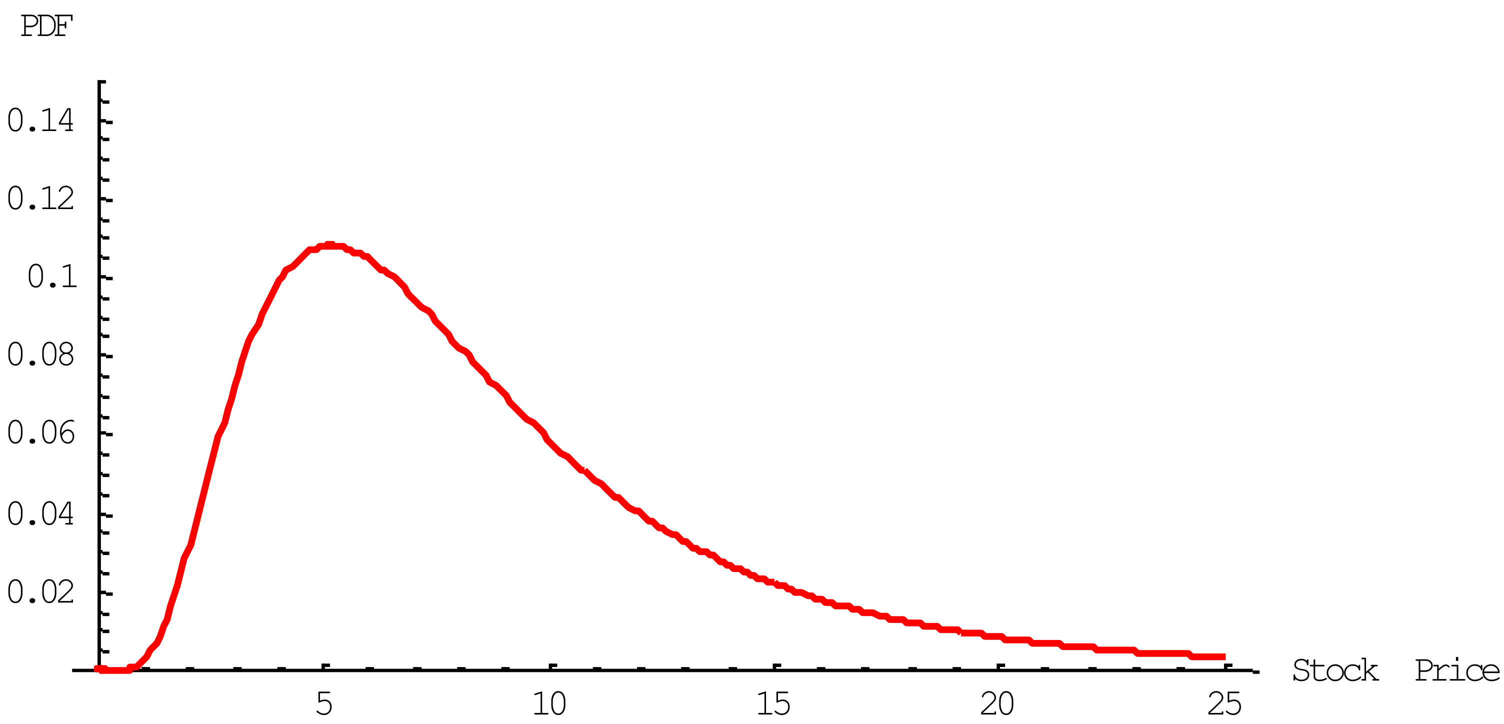


## Log-Normal Random Variables.

Definition: A random variable $X$ is called **log-normal** if $Y=\ln X$ is a normal random variable. If $Y$ is a normal variable with mean $\mu$ and standard deviation $\sigma$ then the variable $X=\exp(Y)$ is log-normal with probability density function

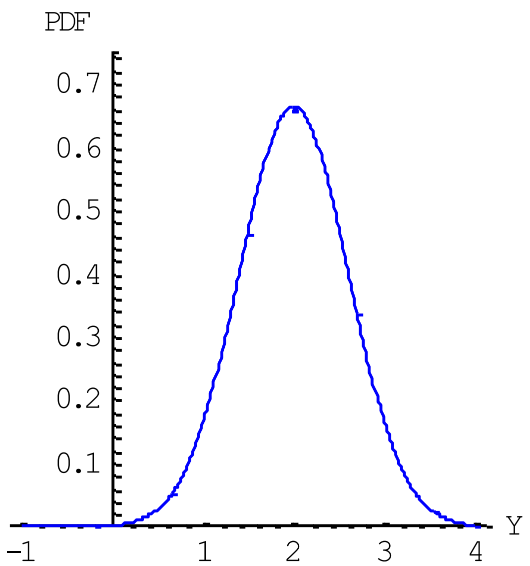
$$p(x) = \frac{1}{x} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-(\ln x - \mu)^2}{2\sigma^2}}$$

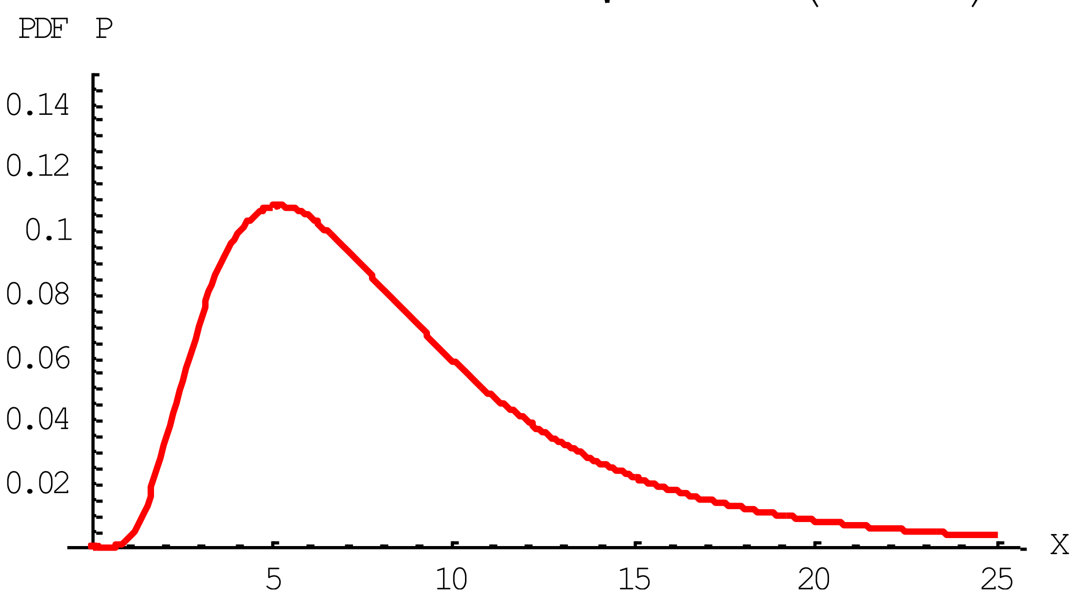
Probability density function of log-normal distribution with $\mu=2$ and $\sigma=0.6$



The parameters μ and σ in the formula for probability density function of the log-normal variable are not mean and standard deviation of the log-normal variable *X* but those of *Y*=ln *X*.

The mean of *X* is $e^{(\mu+\frac{\sigma^2}{2})}$ and standard deviation $\sqrt{e^{(2\mu+\sigma^2)}\left(e^{\sigma^2}-1\right)}$




| | |
|---|---|
| Y | exp(Y)=X |
| Y is Normal | X is log-normal |
| Y=ln(X) | X |

So if Y is normal $P(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(y-\mu)^2}{2\sigma^2}}$ with mean μ stdev σ

Then $X = e^Y$ is lognormal $p(x) = \frac{1}{x} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{\frac{-(\ln x-\mu)^2}{2\sigma^2}}$

Where mean of X is $e^{(\mu+\frac{\sigma^2}{2})}$ and stdev $\sqrt{e^{(2\mu+\sigma^2)}\left(e^{\sigma^2}-1\right)}$

## Stock Returns and Stock prices.

A commonly accepted model in finance is when stock returns (and most other assets returns) are normally distributed. It is possible to show that if returns are normally distributed then prices are Log-normally distributed. There is a reasonably good agreement of this model with observable data for returns in the "regular" state of the market. However the extremal event returns do not fit into this normality assumption.

Example:

If the asset has annual percentage returns Y with mean $\mu$=10% and $\sigma$ =25%

Then $X=exp(Y)$ has mean 1.140 and standard deviation 0.290

The cumulative distribution function (CDF) at y is given by the integral of the probability density function (PDF) P(x) up to y.

$$F(y) = \int_{-\infty}^{y} P(x)dx$$

The PDF can therefore be obtained by differentiating the cdf (perhaps in a generalized sense).

The quantile is effectively the inverse of the CDF. It gives the value of y at which CDF F(y)= q. The median is given by value of y where F(y)=1/2, quartiles, deciles and percentiles can also be expressed as quantiles. Quantiles are used in constructing confidence intervals for statistical parameters.

Mean (average value) $$\mu = \int_{-\infty}^{+\infty} xP(x)dx$$

Median $$\int_{-\infty}^{m} P(x)dx = \int_{m}^{+\infty} P(x)dx$$

Mode (local maximums of PDF) M such that *P(M)*=local max *P(x)*

Mean abs deviation (preferred) $$MAD = \int_{-\infty}^{+\infty} |x - m| P(x)dx$$

Mean abs deviation (alternative) $$MAD = \int_{-\infty}^{+\infty} |x - \mu| P(x)dx$$

Mean

$$\mu = \int_{-\infty}^{+\infty} x P(x) dx$$

Mean Abs Deviation

$$MAD = \int_{-\infty}^{+\infty} |x - \mu| P(x) dx$$

Variance

$$\mathrm{Var} = \int_{-\infty}^{+\infty} (x - \mu)^2 P(x) dx$$

Standard Deviation

$$\sigma(X) = \sqrt{\mathrm{Var}(X)}$$

Skewness

$$\mathrm{Skewness} = \int_{-\infty}^{+\infty} \frac{(x - \mu)^3}{\sigma^3} P(x) dx$$

Kurtosis

$$\mathrm{Kurtosis} = \int_{-\infty}^{+\infty} \frac{(x - \mu)^4}{\sigma^4} P(x) dx$$

N-th central moment

$$\mathrm{M_n} = \int_{-\infty}^{+\infty} (x - \mu)^n P(x) dx$$

Characteristic function
(Fourier transform)

$$\phi(t) = \int_{-\infty}^{+\infty} P(x) \exp(itx) dx$$

Characteristic function (Fourier transform)

$$\phi(t) = \int_{-\infty}^{+\infty} P(x) \exp(itx) dx$$

Each probablility distribution that satisfies additional technical conditions has a unique characteristic function which is sometimes used instead of the PDF to define a distribution.

Skewness summarizes the asymmetry

$$\text{Skewness} = \int_{-\infty}^{+\infty} \frac{(x-\mu)^3}{\sigma^3} P(x)dx$$

Kurtosis summarizes the peakedness

$$\text{Kurtosis} = \int_{-\infty}^{+\infty} \frac{(x-\mu)^4}{\sigma^4} P(x)dx$$

ExcessKurtosis=Kurtosis-3

The kurtosis for any normal distribution N(μ,σ), is three.
For this reason, excess kurtosis is defined as Kurtosis-3
so that the normal distribution has an excess kurtosis of zero.

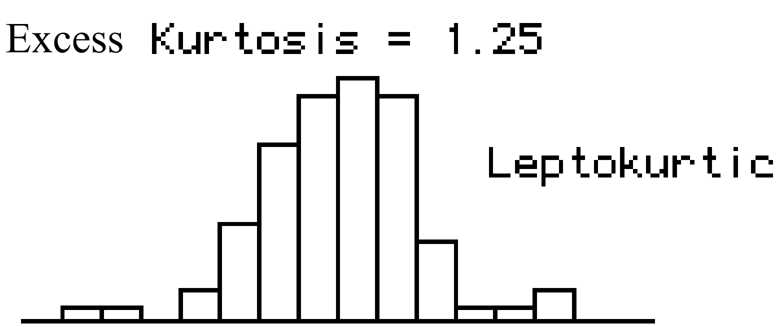
Kurtosis describes the **movement of probability mass from the shoulders of a distribution into its center and tails.**

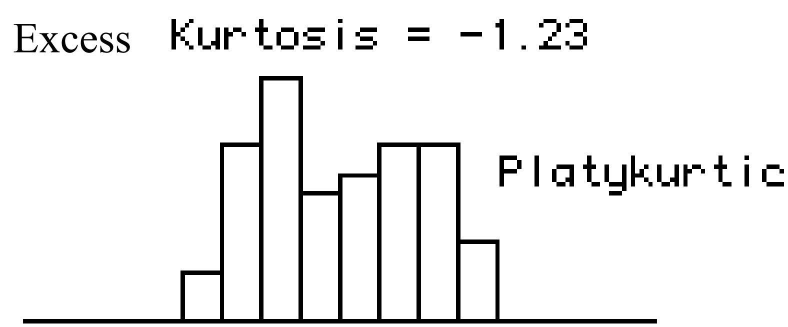
Distributions with **positive** excess kurtosis are called "**leptokurtic**"; with **negative** excess kurtosis are called "**platykurtic**."
Distributions with the same excess kurtosis as the normal distribution (**zero** excess kurtosis) are called "**mesokurtic**."

**Positive** excess kurtosis = relatively large tails:
higher peaks around the mean compared to normal distributions, thicker tails on both sides.

**Negative** excess kurtosis = smaller tails:
flatter peak around  mean, thinner tails.

Heavy tails have much more influence on excess kurtosis than does the peakedness of the distribution near the mean.



*Skewness* for $Y_1$, $Y_2$, ..., $Y_N$,
the formula for skewness is:

$$\text{Skewness} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \frac{(Y_i - m)^3}{(N-1)s^3}$$

where m is the mean, s is the standard deviation, and *N* is the number of data points. The skewness for a normal distribution is zero, and any symmetric data should have a skewness near zero.

Negative values for the skewness indicate data that are skewed left and positive values for the skewness indicate data that are skewed right.

Skewed left means that the left tail is heavier than the right tail.

For $Y_1$, $Y_2$, ..., $Y_N$, the formula for kurtosis is:

$$\text{Kurtosis} = \sum_{\text{i=1}}^{\text{n}} \frac{(Y_i - m)^4}{(N-1)s^4}$$

where m is the mean, s is the standard deviation, and $N$ is the number of data points.

The kurtosis for any normal distribution $N(\mu,\sigma)$, is three. For this reason, excess kurtosis is defined as

$$\text{ExcessKurtosis} = \sum_{\text{i=1}}^{\text{n}} \frac{(Y_i - m)^4}{(N-1)s^4} - 3$$

so that the normal distribution has an excess kurtosis of zero. Positive excess kurtosis indicates a "peaked" and “fat tail” distribution and negative excess kurtosis indicates a "flat" and “thin tail” distribution.

Normal Distribution

Notation $N(\mu, \sigma^2)$

| | |
|---|---|
| **Parameters** | $\mu \in \mathbb{R}$ = mean (location)<br>$\sigma^2 > 0$ = variance (squared scale) |
| **Support** | $x \in \mathbb{R}$ |
| **PDF** | $\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ |

CDF

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right]$$

| | |
|---|---|
| **Mean** | $\mu$ |
| **Median** | $\mu$ |
| **Mode** | $\mu$ |
| **Variance** | $\sigma^2$ |
| **Skewness** | $0$ |
| **Ex. kurtosis** | $0$ |

| | |
|---|---|
| **Notation** | $\mathrm{Lognormal}(\mu,\,\sigma^2)$ |
| **Parameters** | $\mu \in (-\infty, +\infty)$,<br><br>$\sigma > 0$ |
| **Support** | $x \in (0, +\infty)$ |
| **PDF** | $\dfrac{1}{x\sigma\sqrt{2\pi}}\ e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}$ |
| **CDF** | $\dfrac{1}{2}+\dfrac{1}{2}\operatorname{erf}\left[\dfrac{\ln x-\mu}{\sqrt{2}\sigma}\right]$ |
| **Mean** | $\exp\left(\mu+\dfrac{\sigma^2}{2}\right)$ |
| **Median** | $\exp(\mu)$ |
| **Mode** | $\exp(\mu-\sigma^2)$ |
| **Variance** | $[\exp(\sigma^2)-1]\exp(2\mu+\sigma^2)$ |
| **Skewness** | $(e^{\sigma^2}+2)\sqrt{e^{\sigma^2}-1}$ |
| **Ex. kurtosis** | $\exp(4\sigma^2)+2\exp(3\sigma^2)+3\exp(2\sigma^2)-6$ |