Columbia University
MATH GR5260 Spring 2023
Programming for Quant and Computational Finance
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Homework Assignment 2
Release date: Feb 16^{th} 2023 (Thu)
Due date: Mar 2^{nd} 2023 (Thu) 11:00pm

HOMEWORK GUIDELINE

Submit your solution file(s) onto CourseWorks by the specified due date and time.

A. Theory parts

Your solution to theory questions can be (i) hand-written and scanned as a pdf file or (ii) prepared using Word and be converted into a pdf file or (iii) be included in the solution for practice parts.

B. Practice parts

Your Python source code and outputs shall be prepared and submitted in a format (eg. pdf, html, .ipynb, .py, etc.) that the graders can understand. Before submitting your solution, make sure that you have run your program successfully to generate all required outputs. Points may be deducted if some outputs are missing.
You are free to use standard python packages distributed by Anaconda.
Do not submit additional packages that can be downloaded from the web. Just provide instructions in the solution file.

C. Other files

Include files that may be requested by the homework assignment. Make proper references in the solution file.

Note: If you suspect there are typos in this homework, or some questions are wrong, please first discuss with your TAs.

SIMULATION-BASED PRICING

Let the present time be 0. Let $t \geq 0$ be the present or a future time, and S_t the underlying asset value at time t. Let r_s be the short rate process. Consider a European style derivative having a single payoff at its maturity time T. Let X be the its payoff of at time T. For example, for a European call option with strike K on a stock, then S_T is the terminal stock price and $X = max(S_T - \mathrm{K}, 0)$. In theory, the value of the option is given by $V = E\left[exp\left(-\int_0^T r_s ds\right) X\right]$ where the expectation E is taken under the risk-neutral measure.

Let $Y = exp\left(-\int_0^T r_s ds\right) X$ be the discounted payoff. Then we have $V = E[Y]$. Under simulation-based pricing, one would simulate $Y_1, Y_2, \ldots Y_N$. The value under N simulations is given by $V_N = \frac{1}{N}(Y_1 + Y_2 + \cdots + Y_N)$, and a 95% confidence interval of this price is given by $\left[V_N - \frac{1.96*sd}{\sqrt{N}}, V_N + \frac{1.96*sd}{\sqrt{N}}\right]$ where sd is the sample standard deviation (ie. square root of the sample variance) computed from $Y_1, Y_2, \ldots Y_N$.

QUESTION 1

Consider the Black-Scholes model settings on a stock S_t where:

- the continuously compounded dividend rate is assumed to be constant q.
- the BS volatility is assumed to be constant σ.
- the short rate is deterministic and also a constant, $r_s = r$.

In this case, $S_T = S_0 * exp\left(\int_0^T \left(r_s - q - \frac{\sigma^2}{2}\right) ds + \int_0^T \sigma dW_s\right)$ where W_s is a standard Brownian motion.

Thus, we can generate samples of terminal stock price via $S_T = S_0 * exp\left(\left(r - q - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T}z\right)$

where z is the standard normal random variate.

a) Write a function to compute and return the theoretical BS price and delta of a European call option. The function inputs shall include: current stock price S_0, strike K, time to option expiry (in years) T, short rate r.

b) Write a function to compute the simulated price V_N of a European call option. Besides the inputs as described part (a), the number of simulations N shall also be an input. The function shall return V_N and the 95% confidence interval $\left[V_N - \frac{1.96*sd}{\sqrt{N}}, V_N + \frac{1.96*sd}{\sqrt{N}}\right]$.

Convergence and variance reduction

Now take the following example data: $S_0 = 180, r = 1\%, q = 1.5\%, \mathrm{K} = 160, \sigma = 20\%$

c) Use the example data to plot the simulated price V_N against the number of simulations N where $1 \leq N \leq 500000$. The figure shall also include a plot for (i) the upper bound (ie. $V_N + \frac{1.96*sd}{\sqrt{N}}$) (ii) lower bound (ie. $V_N - \frac{1.96*sd}{\sqrt{N}}$) with 95% confidence (iii) a flat line for the theoretical BS price. Label each of the lines properly.

d) Enhance your function in part (b) to include an optional input flag for antithetical draw. If the flag is True, the simulated price $\widetilde{V_N}$ will be computed with antithetical draws using N random samples. If the flag is False, the simulated price V_N is computed as in part (b) with N random samples.

Now let's compare the empirical distribution for V_{2N} and $\widetilde{V_N}$.

e) Use the example data and your function in part (d) to generate 1000 simulated prices for V_{2N} and $\widetilde{V_N}$. Here set $N = 500000$. Chart the distribution of V_{2N} and $\widetilde{V_N}$ in the same figure. Compute and compare the sample variance of the 1000 simulated prices for V_{2N} and $\widetilde{V_N}$.

Greek calculations

There are two potential ways to compute the numerical delta of the option using MC simulation.

First the simulated price is a function of the current stock price. One can view V_N as $V_N(S_0)$. With a small shift (say, $\varepsilon = 0.01S_0$) of the current stock price, one can compute the one-sided delta as:

$$\Delta_1 = \frac{V_N(S_0+\varepsilon) - V_N(S_0)}{\varepsilon}$$

Another approach is that since random samples are already generated in computing $V_N(S_0)$, we may as well re-use these samples to estimate $V_N(S_0+\varepsilon)$. Suppose the standard normal random samples are z_1 ,..., z_N. Let's denote the simulated price computed from these samples by $V_N(S_0\,; z_1\,, \dots, z_N)$. Then one can estimate the one-sided delta as:

$$\Delta_2 = \frac{V_N(S_0+\varepsilon\,; z_1\,, \dots, z_N) - V_N(S_0\,; z_1\,, \dots, z_N)}{\varepsilon}$$

f) Write a function to compute and return Δ_1. Use the example data to call your function several times with different shifts. Comment if you get "good" Deltas. Does it converge to the theoretical delta?

g) Write a function to compute and return Δ_2. Use the example data to call your function several times with different shifts. Comment if you get "good" Deltas. Does it converge to the theoretical delta?

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