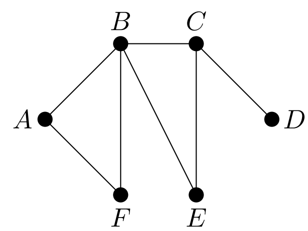
Math 280, Spring 2022

Solutions to Practice Problems 1

1. Consider the following graph G:



(a) Determine the degree sequence of G, as well as $\delta(G)$ and $\Delta(G)$.

(b) Draw the complementary graph to G.

(c) Find a path in G of maximum length, and explain why no longer path is possible.

(d) Find a trail in G of maximum length, and explain why no longer trail is possible.

(e) Write down the adjacency matrix of G.

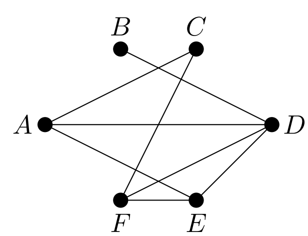
(f) Find the eccentricity of every vertex of G.

(g) Find the radius, diameter, and center of G.

(h) What is the connectivity $\kappa(G)$? Why?

Solutions. (a): The degrees of vertices A–F are, respectively, 2, 4, 3, 1, 2, 2, so the degree sequence is $\boxed{4,3,2,2,2,1}$ with $\boxed{\delta(G) = 1}$ and $\boxed{\Delta(G) = 4}$

(b): Here is $\overline{G}$:



(c): The path $\boxed{D, C, E, B, A, F}$ has length 5. There cannot be a longer path, because this path uses all the vertices, and no path can repeat vertices.
[Side note: There are other paths of length 5.]

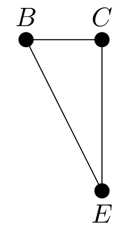
(d): The trail $\boxed{D, C, E, B, A, F, B, C}$ has length 7. There cannot be a longer trail, because this trail uses all of the edges, and no trail can repeat edges.

(e): The adjacency matrix of G is $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

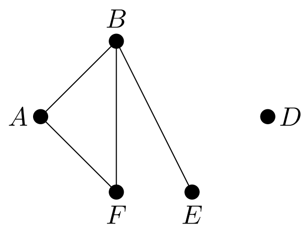
(f): We have $\boxed{\mathrm{ecc}(A) = 3}$ (to get to D), $\boxed{\mathrm{ecc}(B) = 2}$ (to get to D), $\boxed{\mathrm{ecc}(C) = 2}$ (to get to A or F), $\boxed{\mathrm{ecc}(D) = 3}$ (to get to A or F), $\boxed{\mathrm{ecc}(E) = 2}$ (to get to A, D, or F) $\boxed{\mathrm{ecc}(F) = 3}$ (to get to D).

(g): Picking the largest and smallest eccentricities, we have $\boxed{\mathrm{rad}(G) = 2}$ and $\boxed{\mathrm{diam}(G) = 3}$

The center of G is the subgraph spanned by B, C, E, i.e. the center is:



(h): G is connected, so $\kappa(G) \geq 1$. However, removing the vertex C leaves the following disconnected graph $G - C$:



So because it takes removing one vertex to disconnect the graph, $\boxed{\kappa(G) = 1}$

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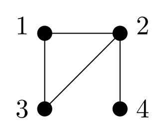
2. Let G be the graph represented by the adjacency matrix $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

(a) Draw the graph G.

(b) Find the number of walks of length 3 from vertex v_2 to vertex v_3 in G.

Solutions. (a): Here is the graph G:



(b): $A^2 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$, so $A^3 = AA^2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 & 1 \\ 4 & 2 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix}$

Since the 2,3-entry of A^3 is 4, there are $\boxed{\text{4 walks}}$ of length 3 on this graph from v_2 to v_3.

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3. Prove that there are no graphs with 10 vertices and 46 edges.

Proof. There are $\binom{10}{2} = \frac{(10)(9)}{2} = 45$ possible places where an edge could go in a graph with 10 vertices, since each edge is determined by an unordered choice of two distinct vertices. Thus, there cannot be such a graph with 46 edges. QED

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4. Let G be a graph with 5 vertices and at least 5 edges. Suppose that G has has no isolated vertices. Prove that G is connected.

Proof. **(Method 1):** Suppose that G were disconnected, and let H be a connected component. Without loss of generality, H has the fewest vertices of all components of G, i.e., every other component of G has at least as many vertices as H does.

Since G has no isolated vertices, H must contain at least two vertices. If H contains exactly two vertices, call them u_1 and u_2; then H consists of these two vertices and — because H is connected — the edge e between them.
The other three vertices v_1, v_2, v_3 may have edges between them, but because they are in separate components from H_1, there are no edges joining any u_i to any v_j. Thus, the only other edges in G are those between two of v_1, v_2, v_3, of which there are at most $\binom{3}{2} = \dfrac{(3)(2)}{2} = 3$. Thus, in total G has at most $1 + 3 = 4$ edges, contradicting the hypotheses. Thus, G must be connected. QED

(Method 2): Given any edge $e \in E(G)$ with vertices u_1 and u_2, we claim that there must be another edge incident on either u_1 or u_2. To see this, suppose not. Call the other three vertices of the graph v_1, v_2, and v_3. Then all of the other four edges in $E(G) \smallsetminus \{e\}$ must have vertices among v_1, v_2, and v_3. However, there are at most $\binom{3}{2} = 3$ possible edges with vertices among v_1, v_2, and v_3. This contradiction proves our claim.
Now pick any one of the vertices; call it v_1. Since v_1 is not isolated, there is an edge e_1 from v_1 to some other vertex; call this second vertex v_2. By the claim, there is a second edge e_2 incident on either v_1 or v_2. Since G is simple, the other vertex of e_2 must be a third vertex; call it v_3.
Suppose there is no edge from either of the remaining two vertices to any of v_1, v_2, or v_3. Then since those two vertices are not isolated (by hypotheses), there must be an edge between them. By the claim, then, there is yet another edge from one of the the last two to one of the original three, contradicting our supposition.
Thus, there must be an edge from one of v_1, v_2, or v_3 to a fourth vertex, which we call v_4.
Finally, since the fifth vertex v_5 is not isolated, there must be an edge from it to one of the other four. Thus, given any two of the vertices, there must be a walk between them, and hence (by a theorem) a path between them. Thus, G is connected. QED

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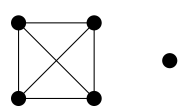
5. Give an example of a graph with 5 vertices and 6 edges that is *not* connected.

Solution. Inspired by the missing "no isolated vertices" hypothesis that was present in the previous problem, here is such a graph:



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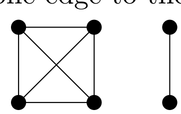
6. Give an example of a simple graph with 6 vertices and 7 edges that has no isolated points and is *not* connected.

Solution. OK, just add one vertex and one edge to the previous answer:



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7. Let G be a graph. Suppose that for any two vertices $u, v \in V(G)$, there is a unique path from u to v in G. Prove that G is a tree.

Proof. Note that G is connected, because for any $u, v \in V(G)$, there is a path from u to v in G, by hypothesis. So it remains to show that G is acyclic.
Suppose (by contradiction) that G has a cycle

$$v_0, v_1, v_2, \ldots, v_k,$$

where $v_k = v_0$, but where all the other $v_i's$ are distinct, and no edges are repeated along the way. (And $k \geq 3$, by definition of cycle.)
The edge e from v_0 to v_1 gives a (one-edge) path from v_1 to v_0. However, the path

$$v_1, v_2, \ldots, v_k$$

is also a path from v_1 to $v_k = v_0$. This second path doesn't use the edge e (since all edges in a cycle are distinct), so it is a different path, violating the uniqueness hypothesis.
By this contradiction, it follows that G is acyclic. Since G was also connected, it is a tree. QED

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8. Let T be a tree, and let $e \in E(T)$ be an edge. Prove that e is a bridge, i.e., that $T - e$ is disconnected.

Proof. Let $v, w \in V(T)$ be the two vertices incident with e. Suppose, towards a contradiction, that $T - e$ is connected. Then there is a path W in $T - e$ from v to w, of the form

$$v = x_1, x_2, \ldots, x_k = w.$$

That is, W is a path in T that does not use the edge e.
We claim that $k \geq 3$. Indeed, if $k = 1$, then $v = w$, which is not possible because $e = vw$. If $k = 2$, then the edge x_1x_2 of $T - e$ is $vw = e$, which is not possible because e is not an edge of $T - e$. Thus, $k \geq 3$ as claimed.
Therefore, appending the edge e to the end of W gives us the closed walk W' given by

$$v = x_1, x_2, \ldots, x_k = w, v$$

which is a walk of length $k \geq 3$ from v to itself, where $x_1, \ldots, x_k$ are all distinct. That is, W' is a cycle in T. But by definition, the tree T is acyclic, so this is a contradiction.
Thus, $T - e$ is disconnected. Since T is connected, this means e is a bridge. QED

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9. Let T be a tree, and let $v \in V(T)$ be a vertex. Define $m = \deg(v)$. Prove that $T - v$ has at least m connected components.

Proof. Let $e_1, \ldots, e_m$ be the edges incident with v, and write $e_i = vw_i$ for each i. It suffices to show that each of the vertices $w_1, \ldots, w_m$ lie in separate components of $T - v$. Given two different such vertices, we may assume without loss that they are w_1 and w_2.
Suppose, towards a contradiction, that w_1 and w_2 are in the same connected component of $T - v$. Then there is a path W in $T - v$ of the form

$$w_1 = x_1, x_2, \ldots, x_k = w_2.$$

Since $w_1 \neq w_2$, we have $k \geq 2$. Define a new walk W' by appending the edges e_2 and then e_1, i.e., given by

$$w_1 = x_1, x_2, \ldots, x_k = w_2, v, w_1.$$

Then W' is a closed walk of length $k + 1 \geq 3$. In addition, since W was a path, the vertices $x_1, \ldots, x_k$ are all distinct; and since W was a path in $T - v$, none of $x_1, \ldots, x_k$ is v. Thus, the only repeated vertex in W' is $w_1 = x_1$ itself, so that W' is a cycle. That is, T has a cycle, which is a contradiction.
Thus, for any $i \neq j$, we have that w_i and w_j lie in separate components of $T - v$, and hence $T - v$ has (at least) m components. QED

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10. Let T be a tree with at least one vertex of degree at least 3. Prove that there is no trail in T that reaches every vertex.

Proof. Let $n = |V(T)|$, and let $v \in V(T)$ be a vertex of degree at least 3. Let $e_1, \ldots, e_m$ be the edges incident with v, so that $m \geq 3$. Write $e_i = vw_i$ for each $i = 1, \ldots, m$.
Suppose there is a trail W in T given by $x_1, \ldots, x_k$ that uses all the edges of T. We will derive a contradiction from the existence of W.
If $x_1 = v$, then without loss of generality, the first edge is $x_1x_2 = e_1$. The rest of the trail W, which is the trail W' given by $w_1 = x_2, \ldots, x_k$, therefore cannot repeat e_1, so it is a trail (and hence a path) in $T - e_1$. However, by problem 8, w_1 lies in a separate component of $T - e_1$ from w_2, and hence the walk W' cannot use edge e_2. Therefore W does not use e_2, a contradiction. We have a similar contradiction if $x_k = v$.
Thus, we must have $x_1, x_k \neq v$. Let $i \geq 2$ be the smallest index such that $v = x_i$; without loss, we have $x_{i-1} = w_1$, and $x_{i+1} = w_2$. Again by problem 8, w_3 lies in a separate component of $T - e_2$ from w_2, and hence the walk W' given by $x_{i+1}, \ldots, x_k$ cannot use edge $e_3 = vw_3$. Therefore W does not use e_3, a contradiction.
In all cases, then, we have a contradiction, as desired. So no such trail W exists. QED

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11. Let T be a tree of order $n \geq 2$. Prove that $\kappa(T) = 1$.

Proof. Since T is connected, we have $\kappa(T) \geq 1$.
We have $\kappa(T) \leq n - 1$ by definition of κ, and hence if $n = 2$ we are done. Thus, we may assume for the remainder of the proof that $n \geq 3$.
Since T is a tree, there are $n-1$ edges. If all vertices have degree at most 1, then Theorem 1.1 says

$$2n - 2 = 2(n-1) = 2|E(T)| = \sum_{v \in V(T)} \deg(v) \leq |V(T)| = n,$$

whence $n \leq 2$, a contradiction. Therefore, there is some vertex v with $\deg(v) \geq 2$. By problem 9, $T-v$ has at least 2 components and hence is disconnected. Therefore $\kappa(T) \leq 1$, and hence $\kappa(T) = 1$. QED

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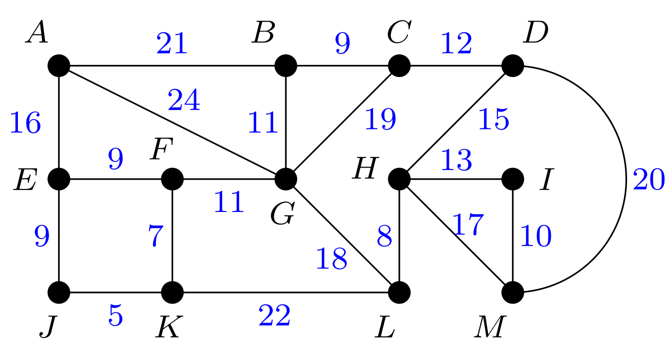
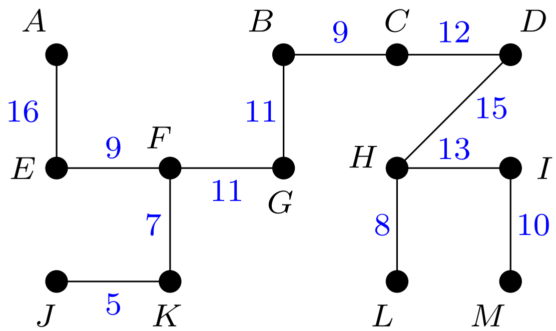
12. Use Kruskal's Algorithm to find a minimal spanning tree of the following weighted graph:



Solution.
Step 1: Add the shortest edge, JK (length 5).
Step 2: Add the next shortest edge, FK (length 7).
Step 3: Add the next shortest edge, HL (length 8).
Step 4: Add the next shortest edge, BC (length 9).
Step 5: Add the next shortest edge, EF (length 9).
[Alternatively, add EJ, which has the same length.]

Step 6: The next shortest edge, EJ, would form a cycle; instead add the *next* shortest edge, IM (length 10).
Step 7: Add the next shortest edge, BG (length 11).
Step 8: Add the next shortest edge, FG (length 11).
Step 9: Add the next shortest edge, CD (length 12).
Step 10: Add the next shortest edge, HI (length 13).
Step 11: Add the next shortest edge, DH (length 15).
Step 12: Add the next shortest edge, AE (length 16).
Halt; we have added 12 edges, which attains a spanning tree on our 13 vertices:



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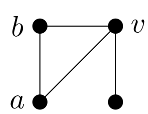
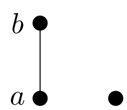
13. Find an example of a graph G that has a vertex $v \in V(G)$ such that v is a cut vertex of G, but also, v lies on a cycle of G.

Solution. There are many ways to do this, but here is one example of such a graph G:



Note that v is part of the cycle v, a, b, v, but the subgraph $G - v$ is:



which has two components, whereas G had only one, so v is indeed a cut vertex of G.

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14. Let G be a graph, and let $e \in E(G)$ be an edge. Suppose that e is not a bridge of G. Prove that e lies on some cycle of G.

Proof. Let a, b be the two vertices incident with e, i.e., $e = ab$. Then a and b are in the same component of G, since there is a path a, b along the edge e between them.
By assumption, $G - e$ has the same number of components as G does, and hence a and b must still be in the same component of $G - e$. Thus, there is a path in $G - e$ from a to b, i.e. a path W given by

$$a = x_1, x_2, \ldots, x_k = b$$

in G that does not use the edge e. Thus we must have $k \geq 3$, or else the above path would be the path a, b along e. Thus, the following closed walk W' in G given by

$$a = x_1, x_2, \ldots, x_k = b, a,$$

which is formed by appending e to the end of W, has length $k \geq 3$. Note that $x_1, \ldots, x_k$ are all distinct (because W is a path), and hence W' is a cycle in G, and it uses the edge e, as desired. QED

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15. Let G be a graph of order $n \geq 1$ and of size m. Prove that

$$\delta(G) \leq \frac{2m}{n} \leq \Delta(G).$$

Proof. We have $\deg(v) \geq \delta(G)$ for every $v \in V(G)$, and hence

$$2m = 2|E(G)| = \sum_{v \in V(G)} \deg(v) \geq \sum_{v \in V(G)} \delta(G) = n\delta(G).$$

Dividing by $n \geq 1$ gives $\delta(G) \leq 2m/n$, the first inequality.
Similarly, $\deg(v) \leq \Delta(G)$ for every $v \in V(G)$, and hence

$$2m = 2|E(G)| = \sum_{v \in V(G)} \deg(v) \leq \sum_{v \in V(G)} \Delta(G) = n\Delta(G).$$

Dividing by $n \geq 1$ gives $\Delta(G) \geq 2m/n$, the second inequality. QED

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16. Let G be a graph with adjacency matrix A.
   (a) Suppose A is 7×7. What does this say about G?
   (b) Suppose exactly 26 of the entries of A are 1's. What does this say about G?
   (c) Suppose that the $(2, 5)$ entry of A^4 is 6. What does this say about G?
   (d) Suppose that the $(3, 4)$ entry of $I + A + A^2$ is 0, but the entire third row of $I + A + A^2 + A^3$ is nonzero. What does this say about G?

Solution. (a): The fact that A is 7×7 means $\boxed{G \text{ has order } 7}$

(b): A 1 shows up in entry (i, j) and in entry (j, i) if and only if ij is an edge of G. Thus, each edge of G produces exactly two 1's in the matrix A. Since $26/2 = 13$, this means $\boxed{G \text{ has size } 13}$ (i.e., G has 13 edges).

(c): By a theorem, this means
$\boxed{\text{there are exactly 6 different walks of length 4 from vertex 2 of } G \text{ to vertex 5}}$

(d): Since the $(3, 4)$ entry of S_2 is 0, this means there are no paths of length 2 or less between vertices 3 and 4, so that in particular, $d(3, 4) > 2$, and hence $\text{ecc}(3) > 2$. Since there are no 0's in the third row of S_3, this means that $d(3, i) \leq 3$ for every vertex i, and hence $\text{ecc}(3) \leq 3$.
Combining these two facts means that $\boxed{\text{ecc}(3) = 3}$

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17. Let P_{50} be the path graph with 50 vertices, numbered 1 to 50 from one end to the other. Let A be the associated adjacency matrix.
   (a) What is the $(3, 42)$ entry of A^{20}? Why?
   (b) What is the smallest integer $k \geq 0$ such that the $(6, 38)$ entry of A^k is nonzero? Why?
   (c) For k as in part (b), what is the $(6, 38)$ entry of A^k? Why?

Solution. (a): To get from vertex 3 to vertex 42, we must use all of the vertices $4, 5, \ldots, 41$ in between, and hence all of the $42 - 3 = 39$ edges between them. So there is no walk shorter than length 39 between vertices 3 and 42, and in particular no such walk of length 20. So the $(3, 42)$ entry of A^{20} is $\boxed{0}$

(b and c): To get from vertex 6 to 38, we must use all of the vertices $7, 8, \ldots, 37$ in between, and hence all of the $38 - 6 = 32$ edges between them. So there is no walk shorter than length 32 between them, and there is *one* of length 32, namely

$$6, 7, 8, \ldots, 37, 38.$$

Thus, $\boxed{k = 32}$ is the smallest integer such that the $(6, 38)$ entry of A^k is nonzero, and this entry is $\boxed{1}$ because there is only one such walk.