E4718 Midterm Examination 1
March 20, 2023

70 minutes, 75 points total
Formula sheet is on the last page.

Write your name: ______________________________

Write your UNI: _______________________________

Please sign the honor pledge:

I pledge that I have neither given nor received unauthorized aid during this examination.

Signature:______________________________________

Write your name and UNI on this examination sheet and on the blue book. Write your answers in the blue books and then hand back this entire examination with your blue books.

Note: Even if you cannot prove some result required in the first part of some problem, you can still use those results in subsequent parts of the problem.

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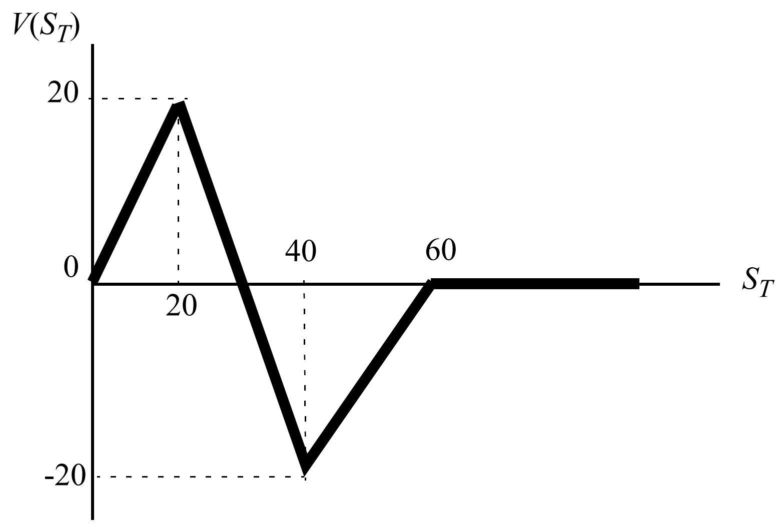
Problem 1: **[25 points]**

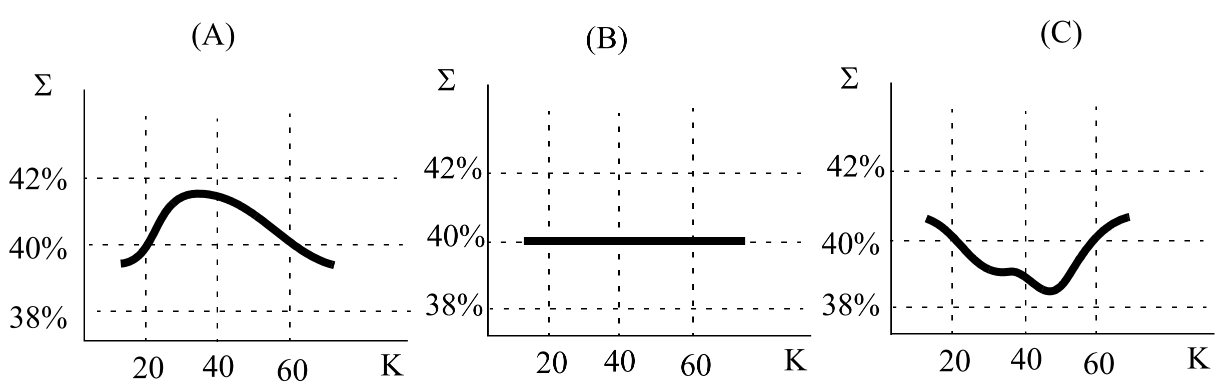
You own a one-year European-style exotic option V with payoff $V(S_T)$ at expiration, as given by the heavy line in the following diagram, where S_T is the terminal stock price in dollars.



(i) Write down a portfolio consisting **only** of standard European-style options with **non-zero strike** that has the same payoff as this exotic option. [15 points]

(ii) The implied volatility skew $\Sigma(K)$ as a function of the strike K for standard options with a one-year expiration can take three different shapes, A, B, or C as shown below.

For which skew is V worth the least amount today, and why? [10 points]



Solution to Problem 1:

(i) Short 1 put with strike 60, long 3 puts with strike 40, short 3 puts with strike 20.

(ii) The implied volatilities are 20 and 60 are always the same. Only the implied volatility at 40 varies. Since we are long 3 puts with strike 40, V is worth the least when the 40-strike put is worth the least, i.e. in case (C).

Problem 2. **[35 points]**

(i) You trade options on a stock which pays no dividends. The annually compounded riskless rate is r. When the market opens at 10 a.m. today you notice that the prices of two-year put options on the stock for any strike K satisfy

$$PutPrice(K) = \frac{30}{31}K + 30\left[\exp\left(\frac{-K}{31}\right) - 1\right]$$

Derive the expression in terms of K and r that describes the risk-neutral probability of moving from a stock price S today at $t = 10$ a.m. to a stock price S_T exactly two years later?

[10 points]

(ii) Show that r is 1.65%.

[10 points]

(iii) What is the fair numerical value of the underlying stock at 10 a.m. today? **[8 points]**

(iv) What is the formula for the fair value of a two-year call with strike K on the stock at 10 a.m. today?

[7 points]

Solution:

(i) $P(K) = \frac{30}{31}K + 30\left[e^{-K/31} - 1\right]$

$\frac{\partial P}{\partial K} = \frac{30}{31} - \frac{30}{31}e^{-K/31}$

$\frac{\partial^2 P}{\partial K^2} = \frac{30}{(31)^2}e^{-K/31}$

$\rho(S,t,K,T) = \frac{30}{(31)^2} e^{-K/31}(1+r)^2$

(ii) ~~A zero coupon bond pays ...~~
The integral over all K must equal 1.

$\int_0^\infty \rho(S,t,K,T)\,dK = 1$

$\frac{30}{(31)^2}\int_0^\infty e^{-K/31}dK\,(1+r)^2 = 1$ $\qquad \frac{1}{31}\int_0^\infty e^{-K/31}dK = \int_0^\infty e^{-x}dx = 1$

$(x = K/31)$

$\frac{30}{31}(1+r)^2 = 1$

$(1+r)^2 = \frac{31}{30}$ $\qquad r = 1.65\%$ $\qquad \rho(S,t,K,T) = \frac{1}{31}e^{-K/31}$

(iii) The value of the stock is the PV of getting K in all states:

$S = \frac{1}{(1+r)^2}\int_0^\infty K\rho(S,t,K,T)\,dK = \frac{30}{31}\frac{1}{31}\int_0^\infty Ke^{-K/31}\,dK$

let $x = K/31$ $\qquad = 30\int_0^\infty xe^{-x}dx$ integrate by parts

$= 30$

(iv) $C - P = S - K/(1+r)^2$ PC parity

$C = P + S - K\frac{30}{31} = \cancel{\frac{30K}{31}} + 30\left[e^{-K/31}\cancel{-1}\right] \cancel{+30} \cancel{-30K/31}$

$= 30e^{-K/31}$

Problem 3. Multiple Choice Questions: Just Write the Right Choice in Your Blue Book **[15 points]**

(3.1) A recently invented Product Model (PM) for the volatility smile produces call prices C_{PM} that take the form

$$C_{PM}(S, t, K, T, a) = C_{BS}\left(S, t, K, T, \Sigma\left(\frac{SK}{a^2}\right)\right)$$

where the Black-Scholes implied volatility Σ in the above equation is a function of the single variable $\frac{SK}{a^2}$, and a is a constant.

The at-the-money slope of the skew with respect to K is observed to be positive.

The appropriate delta of an at-the-money call option in this model is:

(a) less than the corresponding Black-Scholes delta;
(b) greater than the corresponding Black-Scholes delta;
(c) cannot say.

[5 points]

3.1 The answer is (b)

$$\Delta = \frac{\partial C_{PM}}{\partial S} = \frac{\partial C_{BS}}{\partial S} + \frac{\partial C_{BS}}{\partial \Sigma}\frac{\partial \Sigma}{\partial S} = \Delta_{BS} + \frac{\partial C_{BS}}{\partial \Sigma}\frac{\partial \Sigma}{\partial S}$$

Now the last term on the RHS of the equation is positive, because $\frac{\partial C_{BS}}{\partial \Sigma}$ is positive in BS, and $\frac{\partial \Sigma}{\partial S}$ is positive because Σ increases with K (the skew), but since Σ is a function of $\frac{SK}{a^2}$, it also therefore increases with S. Thus the delta is greater than BS.

(3.2) You buy a one-year call struck at-the-money at an implied volatility of 5%, and from then the option always trades at an implied volatility of 5%. You always hedge the option to expiration at an implied volatility of 5%. Interest rates and dividend yields are zero, and the current stock price is 100.

Over the next year the realized volatility is always 10%. Which of the following two scenarios for the stock price produces the largest P&L for the hedged portfolio?

(a) the stock stays close to 100 all the way between purchase of the option and expiration.

(b) the stock diffuses up to 130 in the first month and then stays close to 130 between the first month and expiration.

Solution. (a) has the largest P&L,

Explanation, though just the answer is enough:

From the formula sheet, for a long position hedged at implied volatility,

$$\boxed{d\text{P\&L} = \frac{1}{2}\Gamma_I S^2(\sigma^2 - \Sigma^2)dt}$$

The term $\boxed{(\sigma^2 - \Sigma^2)}$ is always positive so you make a profit at every instant. You gain more when $\boxed{\Gamma S^2}$ is large, which occurs when the stock stays close to at the money, i.e. 100. Hence (b) makes less profit.

More carefully $\boxed{\Gamma S^2 = \frac{Se^{-d_1^2/2}}{\sigma\sqrt{T-t}}}$

In (a) $\boxed{S = 100}$ and $\boxed{d_1 = \frac{\ln\frac{S}{K}}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2} = \frac{\Sigma\sqrt{\tau}}{2} = 0.025}$ and $\boxed{Se^{-d_1^2/2} \approx 100}$

In (b) $\boxed{S = 130}$ and $\boxed{d_1 = \frac{\ln\frac{S}{K}}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2} = \frac{\ln\frac{130}{100}}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2} = \frac{0.26}{0.05} + 0.025 \approx 5.27}$ and

$\boxed{Se^{-d_1^2/2} \approx 130 \times 0.005}$ which is much smaller and so $\boxed{\Gamma S^2}$ is much smaller.

Formulas

The **Black-Scholes formula** for a European call option at time t with strike K expiring at time T on a non-dividend-paying stock of price S with future volatility σ and a continuously compounded riskless rate r is given by

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

$$C_{BS}(S, t, K, T, r, \sigma) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$d_{1,2} = \frac{\ln(S_F/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$S_F = e^{r(T-t)}S$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \exp\left(\frac{-y^2}{2}\right) dy$$

You may find these formulas useful too:

$$S_F N'(d_1) = KN'(d_2) \text{ where } N'(x) \equiv \frac{dN}{dx}.$$

$$\frac{\partial C_{BS}}{\partial K} = -e^{-r(T-t)}N(d_2) \qquad \frac{\partial C_{BS}}{\partial S} = N(d_1)$$

$$\frac{\partial C_{BS}}{\partial \sigma} = SN'(d_1)\sqrt{T-t}$$

When you hedge a long position in an option V at implied volatility Σ, the instantaneous P&L during time dt is given by

$$d\text{P\&L} = \frac{1}{2}\Gamma S^2(\sigma^2 - \Sigma^2)dt$$

where σ is the realized volatility of the stock S and $\Gamma = \frac{\partial^2 V}{\partial S^2}$ evaluated at the implied volatility.