Homework 7: Monday April 3, 2023

Due Wednesday April 12 2023 **[100 points]**

Problem 1. **[20]**

Assume we look only at options of one-year expiration with any strike K. Suppose that their observed implied volatility at some instant *t* is written as Σ(S,K) for index S and strike K.

(i) Write the at-the-money implied volatility of an option when the index is at level *S* in terms of the function Σ(S,K): i.e. write

$\Sigma_{atm}(S)$ = [your answer]

[10]

(ii) Suppose at time *t* the volatilities Σ(S,K) display a negative skew - i.e. for fixed S, Σ(S,K) decreases as K increases.

If the implied volatilities of all strikes K stay fixed (sticky strike) as S moves to a higher level at a later time, will the at-the-money volatility $\Sigma_{atm}(S)$

(a) increase

(b) decrease

(c) remain unchanged

as the index S rises? [10]

Problem 2: **[20]**

Suppose implied volatility changes according to the sticky delta rule, so that for a fixed expiration T, we have

$$\Sigma(S, K) \equiv f\left(\frac{K}{S}\right)$$

Suppose also that there is a negative skew, so that for fixed S, Σ(S,K) decreases as K increases.

Under these circumstances, what will happen to the volatility of an option with a fixed strike K as S increases – Will it increase/ decrease/ remain unchanged as S increases?

Problem 3: What is the correct delta for an index option? **[60]**

Write the implied volatility of an option with strike K when the index level is at S as Σ(S,K), assuming there is no dependence on time to expiration.

There are two "rules of thumb" that traders use for estimating how the implied Black-Scholes volatility of a European equity index option will vary with index level S. Assuming that the variation of implied volatility with strike is approximately linear, we can write these rules as follows:

$$\text{Sticky Strike Rule: } \Sigma(S, K) = \Sigma_0 - b(K - S_0) \qquad \text{Eq.H8.1}$$

$$\text{Sticky Moneyness Rule: } \Sigma(S, K) = \Sigma_0 - b(K - S) \qquad \text{Eq.H8.2}$$

where S_0 is the current level of the index when the skew is first observed, and S is the value after a move, and Σ_0 is the implied volatility at-the-money when the index $S = S_0$.

The implied tree rule for the variation of implied volatility based on a local volatility picture of the skew is:

$$\text{Implied Tree Rule: } \Sigma(S, K) = \Sigma_0 - b(S + K - 2S_0) \qquad \text{Eq.H8.3}$$

All three rules have the same skew - i.e. the same variation with K - when $S = S_0$, but they vary differently with S.

Using these rules you know how implied volatility varies for fixed strike K when S varies. You therefore know how a call price will vary with S because you know the BS formula for the call price and you know how the implied volatility for fixed K varies with S.

Let $C(S, K, \Sigma(S, K), \tau)$ be the value of a Black-Scholes call with volatility $\Sigma(S, K)$ and time to expiration τ. Define the "correct" delta of a call option in each model to be

$$\Delta(S, K) \equiv \left[\frac{C((S + \varepsilon, K, \Sigma(S + \varepsilon, K)), \tau) - C(S, K, \Sigma(S, K), \tau)}{\varepsilon}\right] \qquad \text{Eq.H8.4}$$

in the limit where ε is small.

Use this formula and a Black-Scholes calculator to find the "correct" deltas for a one-year ($\tau = 1$)at-the-money call option for each of the three cases above (i.e. sticky strike, sticky delta, and implied tree) assuming that for all of them we have $S_0 = 100$, $K = 100$, $\Sigma_0 = 0.2$ (i.e. 20% volatility) and the slope of the skew $b = 0.001$ volatility points per strike point. Assume interest rates and dividend yields are all zero. Choose some small value of ε to evaluate the derivative numerically. In other words, your answer should consist of three different estimated correct deltas, one for each of the rules. Show your calculations for partial credit.

Problem 1. Answer

Answer: (i) $\Sigma_{atm}(S) = \Sigma(S,S)$

Answer: (ii) As S increases the new atm strike K must move to a higher level to keep up with S, by definition. But higher K has lower implied volatility because of the negative skew, and so $\Sigma_{atm}(S)$ must decrease as S increases.

Problem 2. Answer:

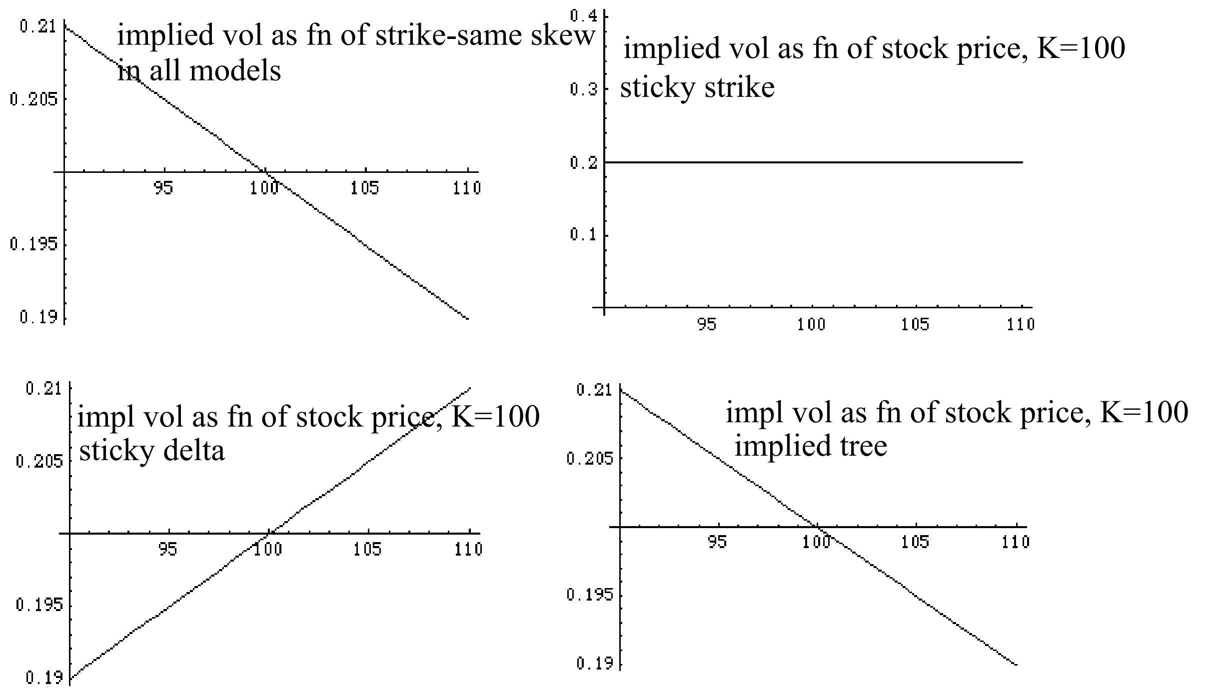
Since $\boxed{\Sigma(S, K) \equiv f\left(\frac{K}{S}\right)}$

S increasing is equivalent to K decreasing, which causes volatility to increase in a negatively skewed environment. Therefore, fixed-strike volatility will increase as S increases in order that ATM volatility remains unchanged.

Problem 3: Answer: (using Mathematica)

slope= 0.001
VolSS[s_,k_,slope_] := 0.2 - slope(k-100) (* sticky strike *)
VolSD[s_,k_,slope_] := 0.2 - slope(k- s) (* sticky delta *)
VolIT[s_,k_,slope_] := 0.2 - slope(k+s- 200) (* implied tree *)

Variation of Implied Volatility with Strike and with Stock Price in Three Different Models



Use ε = 0.1 to evaluate the derivative numerically.

BSDeltaSS[s_, k_, r_, q_, t_] := (BlackScholesCall[s + 0.1, k, VolSS[s + 0.1, k, slope], r, q, t] -
BlackScholesCall[s, k, VolSS[s, k, slope], r, q, t])/0.1
= 0.54

BSDeltaSD[s_,k_,r_,q_,t_] := (BlackScholesCall[s+0.1,k,VolSD[s+0.1,k,slope],r,q,t]-
BlackScholesCall[s,k,VolSD[s,k,slope],r,q,t])/0.1
= 0.58

BSDeltaIT[s_, k_, r_, q_, t_] := (BlackScholesCall[s + 0.1, k, VolIT[s + 0.1, k, slope], r, q, t] -
BlackScholesCall[s, k, VolIT[s, k, slope], r, q, t])/0.1

:= 0.50 approximately