# CHAPTER 11

# Managing Vanilla Options Risk

Every book should have a hero. The hero of this book is not a person but an equation: the Black-Scholes formula for pricing European-style options. Like every hero, it has its flaws and no shortage of detractors ready to point them out. But with help from some friends, it can recover to play a vital role in integrating all options risk into a unified, manageable framework. This is the theme of this chapter and the next.

Options risk may be subdivided into two categories: the risk of relatively liquid options, termed *plain-vanilla* or *vanilla options*, and the risk of less liquid options, termed *exotic options*. Managing options risk for vanilla options is quite different from managing options risk for exotic options, so we will discuss them in two separate chapters.

Almost without exception, the only relatively liquid options are European-style calls or puts, involving a single exercise date and a simple payoff function equal to the difference between the final price level of an asset and the strike price. As such, vanilla options can be priced using either the Black-Scholes formula or one of its simple variants (see Hull 2012, Section 14.8, Chapter 16, and Sections 17.8 and 25.13). The only notable exception to the rule that all vanilla options are European style is that some American-style options on futures are exchange traded and liquid. However, the early exercise value of such options—the difference between their value and that of the corresponding European option—is quite small (as discussed in Section 12.5.1). So treating all vanilla options as European-style calls and puts is a reasonable first approximation.

To simplify our discussion of European options, we will utilize the following three conventions:

1. All options are treated as options to exchange one asset for another, which enables us to only consider call options. So, for example, we treat an option to put a share of stock at a fixed price of \$50 as being a call option to exchange \$50 for one share of stock. This is a more natural

way of treating foreign exchange (FX) options than the usual approach, since whether an FX option is a call or a put depends on which currency you use as your base.

2. Options prices and strikes will often be expressed as percentages of the current forward price, so a forward price of 100 (meaning 100 percent) will be assumed.
3. All interest rates and costs of carry are set equal to zero. This means that the volatilities quoted are volatilities of the forward, not the spot; the hedges calculated are for the forward, not the spot; and option payments calculated are for delivery at the option expiry date. Although almost all options traded are paid for at contract date rather than expiry, discount curves derived from market prices, as shown in Section 10.2, can always be used to find the current spot price equivalent to a given forward payment.

With these three conventions, we can use the following formula for Black-Scholes values:

$$BS(K, T, \sigma) = N(d_1) - KN(d_2) \tag{11.1}$$

where $K$ = strike as percentage of current forward to time $T$
$T$ = time to option expiry in years
$N$ = cumulative normal distribution
$\sigma$ = annualized volatility of the forward
$d_1 = [\ln(1/K) + 1/2\, \sigma^2 T]/\sigma \sqrt{T}$
$d_2 = d_1 - \sigma \sqrt{T}$

This is similar to Equation 25.5 in Hull (2012, Section 25.13). Technically, we are using a model in which the zero coupon bond price is the *numeraire* (see Hull 2012, Section 27.4).

Stating the equation in terms of the forward price rather than the spot price is important for reasons other than formula simplification. First, it follows the principle stated and justified in Section 6.2 that all forward risk should be disaggregated from options risk. Second, this has the advantage of not assuming constant interest rates; the volatility of interest rates and their correlation with spot price are all imbedded in the volatility of the forward. The historical volatilities of forwards can often be measured directly. If they cannot be measured directly, they can easily be calculated from the spot volatility, interest rate volatilities, and correlations. Hedges with forwards are often the most liquid hedges available. If a spot hedge is used, then the appropriate interest rate hedges should be used as well, since interest rates and carry costs cannot be assumed to be constant. This combined hedge will be synthetically equivalent to a hedge with a forward.

## 11.1 OVERVIEW OF OPTIONS RISK MANAGEMENT

Even when we limit our discussion to vanilla options, the vast variety of instruments available makes it unlikely that liquidity of any single instrument will be large. For the options on just a single asset, not only do we face the multiplicity of dates we encountered for forward risk products, but each date also has a multiplicity of possible strikes. Once we take into account that options involve an exchange between pairs of assets, the number of possible contracts expands even more rapidly. For example, if a desk trades 10 different currencies, the number of currency pairs of FX options is $10 \times 9 = 90$. In fact, the degree of liquidity available for option products is significantly smaller than that for spot or forward products.

When options market trading first began and, to a more limited extent, as options markets continue to develop for new assets, initial market-maker hedging strategies were often a choice between acting as a broker (attempting to find a structure for which a simultaneous buyer and seller could be found) or relying on an initial static hedge with the underlying instrument until a roughly matching option position could be found. The broker strategy is very limiting for business growth. The static hedge strategy can only convert call positions into put positions, or vice versa; it cannot reduce the nonlinear nature of the option position. As such, it can be used only by trading desks that are willing to severely limit the size of positions (thereby limiting business growth) or to take very large risks on being right about the maximum or minimum levels to which asset prices will move. Static hedging with limited position size remains a viable strategy for a proprietary desk, but not for a market-making desk.

The development of dynamic hedging strategies was therefore a major breakthrough for the management of options market making. Consider Table 11.1, which extends an example that Hull (2012, Tables 18.1 and 18.4) presents, using Monte Carlo simulation to evaluate the performance of dynamic hedging strategies.

Table 11.1 shows that even a very naive dynamic hedging strategy, the stop-loss strategy, which calls for a 100 percent hedge of a call whenever the forward price is above the strike and a 0 percent hedge whenever the forward price is below the strike, results in a large reduction in the standard deviation of results—76 percent of option cost relative to 130 percent of option cost for a static hedge. However, an increased frequency of rehedging can only improve stop-loss results up to this point. By contrast, the dynamic

**TABLE 11.1** Performance of Dynamic Hedging Strategies

Price = $49, interest rate = 5 percent, dividend rate = 0, forward price = $50
Strike = $50
Volatility = 20 percent
Time to maturity = 20 weeks (0.3846 years)
Drift rate = 13 percent
Option price = $240,000 for 100,000 shares

| | Performance Measure (Ratio of Standard Deviation to Cost of Option) | | | |
|---|---|---|---|---|
| | | Delta Hedge | | |
| **Frequency of Rehedging** | **Stop Loss** | **No Vol. of Vol.** | **10% Vol. of Vol.** | **33% Vol. of Vol.** |
| 5 weeks | 102% | 43% | 44% | 57% |
| 4 weeks | 93% | 39% | 41% | 52% |
| 2 weeks | 82% | 26% | 29% | 45% |
| 1 week | 77% | 19% | 22% | 47% |
| ½ week | 76% | 14% | 18% | 43% |
| ¼ week | 76% | 9% | 14% | 38% |
| Limit as frequency goes to 0 | 76% | 0% | 11% | 40% |

With no hedging, the performance measure is 130 percent.

hedging strategy corresponding to the Black-Scholes analysis enables the standard deviation to get as close to zero as one wants by a suitable increase in the frequency of rehedging. You can see why the Black-Scholes approach had such an impact on options risk management.

But almost immediately, this was followed by a backlash, focusing on the unrealistic nature of the Black-Scholes assumptions. Principally, these assumptions and the objections are:

- Trading in the underlying asset can take place continuously. (In fact, a practical limit exists on how frequently trading can occur, which places a lower limit on the standard deviation that can be achieved.)
- No transaction costs are involved when trading in the underlying asset. (In practice, transaction costs place an even tighter limit on the frequency of rehedging.)
- The volatility of the underlying asset is a known constant. (If we make the more realistic assumption that volatility is uncertain, with a

standard deviation around a mean, we get results like those in the last two columns of Table 11.1, placing a lower limit on the standard deviation that can be achieved.)
- The underlying asset follows a Brownian motion with no jumps. (In practice, discontinuous jumps in asset prices can occur, even further limiting the degree to which standard deviation can be lowered.)

Trading desks that have tried pure Black-Scholes hedging strategies for large positions have generally found that unacceptably large risks are incurred. A related example is the *portfolio insurance* strategy. Many equity portfolio managers were using this strategy in the mid-1980s to create desired options positions through dynamic hedging. In October 1987, the global stock market crash caused liquidity to dry up in the underlying stocks, leading to trading discontinuities that resulted in large deviations from planned option payoff profiles.

As a result, vanilla options market makers have generally moved in the direction of a paradigm in which they attempt to match the options positions bought and sold reasonably closely, enabling basis risk to be taken both over time while waiting for offsetting trades to be available and with regard to strike and tenor mismatches. The Black-Scholes model is relied on as an interpolation tool to relate observed market prices to prices needed for the residual risk positions left after offsetting closely related buys and sells. Black-Scholes dynamic hedging is used to hedge these residual risk positions.

Three key tools are needed for managing a vanilla options book using this paradigm:

1. A reporting mechanism must be available to measure the amount of basis risk exposure resulting from mismatches in the strike and tenor of options bought and sold. Although summary measures such as *vega* (exposure to a move in implied volatility levels) and *gamma* (the sensitivity of delta to a change in underlying price level) can be useful, the two-dimensional (strike and tenor) nature of the exposure requires a two-dimensional risk measure to be really effective. This measure is the *price-vol matrix* that depicts portfolio valuation sensitivity to the joint distribution of two variables: underlying asset price and implied volatility. It therefore measures exposure to both jumps in underlying asset price and changes in implied volatility. It also measures simultaneous changes in both. We will examine illustrative examples and discuss the use of price-vol matrices in Section 11.4.
2. Dynamic delta hedging of the portfolio of bought and sold options needs to be performed. Guidance for this process comes from the Black-Scholes formula. The targeted hedge for the portfolio is a simple

summation of the targeted hedges of each individual option position, as determined by Black-Scholes. However, given the reality of transaction costs for executing the delta hedges in the underlying, a set of guidelines about how often to hedge is necessary. It has been shown, both by theory and trader experience, that hedging guidelines based on the distance between the current delta hedge and the target delta hedge are more effective than guidelines tied to the frequency of hedging. The degree of tolerance for deviation from the target delta determines a trade-off between higher transaction costs (for lower tolerances) and higher uncertainty of results (for higher tolerances). Section 11.5 discusses these delta-hedging guidelines in more detail along with related issues such as what implied volatility to use to determine the target hedge.

3. Options for which liquid market prices are not available are valued based on interpolation from options that do have liquid market prices available. The interpolation methodology translates prices of liquid options into implied volatilities using the Black-Scholes formula, interpolates these implied volatilities to implied volatilities for less liquid options (interpolation is based on both strike and tenor), and then translates implied volatilities to prices of the less liquid options, again using the Black-Scholes formula. Limits and reserves are needed to control uncertainty in the interpolation process. Section 11.6 gives a detailed account of this interpolation method.

Note how closely bound together the three operative legs of this paradigm are. The Black-Scholes formula serves as the glue that binds them together:

- The price-vol matrix shows how the portfolio valuation will change based on a joint distribution of changes in underlying asset price and implied volatility. However, many (probably most) of the options in the portfolio lack liquid market prices, so their valuation depends on the interpolation step. Furthermore, the calculation of the change in option value for a change of asset price and implied volatility is calculated using the Black-Scholes formula.
- As will be seen in the detailed discussion of the price-vol matrix, all calculations are done under the assumption that exposure to small changes in underlying asset price have been delta hedged with a position in the underlying asset, so the validity of the price-vol matrix depends on the execution of this dynamic delta hedging.
- The need for this approach to options risk management is based on the flat rejection of the key assumptions of the Black-Scholes model: continuous rehedging, no transaction costs, no price jumps, and known and

> constant volatility. How, then, can we continue to rely on the Black-Scholes model to calculate the impact of changes in underlying asset price, calculate the target delta hedges, and play a critical role in value interpolation? The answer is that position limits based on the price-vol matrix are being counted on to keep risk exposures low enough that deviations from the Black-Scholes assumptions will not have that large an effect. Small risk exposures mean that the size of required delta hedges will be small enough that transaction costs will not be that significant. Small risk exposures mean that the differences between the Black-Scholes model and the presumably much more complex true model (whatever that may be) are small enough to hold down the errors due to valuing and hedging based on a model that is only an approximation to reality.

It is important to be aware of the degree to which this paradigm depends on the availability of market liquidity for hedging instruments. The paradigm works best when reasonable liquidity in vanilla options is available for at least some combinations of strike and tenor. This enables risks to be hedged by actively pursuing the purchase and sale of options to lower exposures as measured by the price-vol matrix. As we will see in Exercise 11.1, price-vol matrix exposures can be held reasonably flat even if only a small number of strike-tenor combinations provide significant liquidity. The valuation of options with other strike-tenor combinations can be interpolated from the liquid set.

If a particular options market does not have liquidity, the paradigm can still work reasonably well as long as the underlying asset has liquidity. The price-vol matrix now serves primarily as a measure of position imbalance. It can serve as a signal to marketers to encourage customer business at some strike-tenor combinations and discourage it at others. It can be used to place limits on new customer business when this would cause risk to exceed management guidelines. It can be used as input to setting limits and determination of reserves against illiquid concentrations of risk. It can also be used as input to calculations of portfolio risks such as value at risk (VaR) and stress tests. Price interpolation, in the absence of liquid market quotations, becomes primarily a mechanism to enforce the consistency of valuations. Delta hedge calculations continue to serve the function of directing dynamic hedging and ensuring the proper representation of options positions in firmwide reports of spot and forward risk.

It is far more questionable to employ this paradigm in the absence of liquidity in the underlying asset. In this case, it is doubtful that dynamic delta hedging can be carried out in any systematic way, and it probably becomes preferable to analyze positions based primarily on how they will behave under longer-term scenarios, with limits and reserves calculated from this scenario analysis. An example where this may apply is for options written on hedge fund results where there are restrictions on the ability to buy and sell the underlying, which is an investment in the hedge fund. A specific case to illustrate this point is the option Union Bank of Switzerland (UBS) wrote on Long-Term Capital Management (LTCM) performance (see Section 4.1.5).

How well does this paradigm work? Trading desks that have years of experience using it have generally been satisfied with the results. But this is insider knowledge and may be specific to conditions in particular markets. How can outsiders get comfortable with these assumptions, and how can these assumptions be tested in new options markets to which they might be applied? The best tool available is Monte Carlo simulation, in which all of the Black-Scholes assumptions can be replaced with more realistic assumptions, including limits on hedge frequency, transaction costs, uncertain volatility, nonlognormal changes in the underlying price, and price jumps. In Section 11.3, we examine the results of a typical Monte Carlo simulation to see what it indicates about the feasibility of this risk management paradigm.

## 11.2 THE PATH DEPENDENCE OF DYNAMIC HEDGING

To understand options pricing, an important distinction must be made between path-independent and path-dependent options. A path-independent option's payout depends only on what the price of some underlying asset will be at one particular point in time and does not depend on the actual path of price evolution between the current date and that future date. All European-style options are path independent. Exotic options are divided between path-independent and path-dependent options. In Chapter 12 on managing exotic options risk, we will see that path-independent options are generally much easier to risk manage than are path-dependent options.

Although, when considered in isolation, European-style options are path independent, once we start to evaluate the impact of dynamic hedging, we find that dynamic hedging makes "every option become path dependent." (This is quoted from Taleb [1997, Chapter 16]. I strongly recommend reading Taleb's Chapter 16 along with this chapter.) This is a direct consequence of the limitations of the Black-Scholes assumptions, since continuous hedging at a known constant volatility would result in a definite value with no variation (hence, you would achieve not just path independence, but independence of the final underlying asset value as well). Sporadic dynamic hedging and stochastic volatility make the realized value of a dynamic hedging strategy dependent on the full price history of the underlying asset. Let's illustrate this with a few examples.

The first example is based on one presented in Taleb (1997, 270). It is an out-of-the-money call on $100 million par value of a stock with 30 days to expiration that is purchased for $19,000. If no dynamic hedging is attempted, then the option will expire either out-of-the-money for a total loss of the $19,000 premium or in-the-money with upside potential. The amount of return will be completely dependent on where the underlying asset price finishes in 30 days. Suppose a trader wanting to reduce the uncertainty of this payoff attempts to dynamically hedge her position. Taleb demonstrates a plausible price path for the underlying asset that results in a loss of $439,000, not even counting any transaction costs. The **NastyPath** spreadsheet provided on the course website enables you to see the details of this path and experiment with the impact of other possible paths. What is it about the path that leads to a loss that is so large relative to the option's cost? Try to reach your own conclusion. I will provide my answer at the end of Section 11.5.

The second example is drawn from my own experience. In early 1987, I was part of a team at Chase Manhattan that introduced a new product—a term deposit for consumers that would guarantee a return of principal plus a small interest payment, but could make higher interest payments based on a formula tied to the closing price of the Standard & Poor's (S&P) stock index on the maturity date of the deposit. Although the stock market had been showing very good returns in the mid-1980s, stock market participation among smaller investors was still not well developed. Therefore, a product that would be Federal Deposit Insurance Corporation (FDIC) insured, would guarantee against loss, and would provide some upside stock participation quickly attracted a sizable amount of investment.

Our hedging strategy for this product was to invest part of the proceeds in standard deposit products, ensuring the ability to return principal plus guaranteed minimum interest, and use the remainder to fund an S&P index call position. As might be anticipated by those who remember the financial events of 1987, this product suffered an untimely demise in the autumn of that year. After the stock market crash of October 19, consumer interest in possible stock market participation sharply diminished, so new funds stopped coming in. We also experienced severe losses on our hedging of the existing product, and the postmortem we conducted to determine the reason for these losses produced some interesting results.

The equities options markets were at a very early stage of development in 1987, so there was virtually no liquidity for options with tenors beyond a few months. Since our market research had determined that there would be little interest in a deposit product with tenors shorter than a year or two, we had decided to initially rely entirely on a dynamic hedging strategy, using a Black-Scholes–determined delta hedge. We were certainly aware of the vulnerability of this approach to high volatility, but we had done extensive research on the historical patterns of stock market volatility and concluded that we could price the product at an implied volatility that allowed a margin for error that would result in hedging losses only in extremely rare cases.

Not surprisingly, our postmortem showed significant losses due to our inability to carry out the delta-hedging strategy during the period of October 19 and the following few days. The cash and futures equities markets during that period were highly illiquid in the face of panicky selling, and there were even some short periods in which the markets were closed in an attempt to restore stability to chaotic trading. Illiquid markets in the underlying during large price moves result in gapping losses to options sellers employing dynamic hedging strategies. We were not alone in this vulnerability. In October 1987, a substantial number of asset managers following portfolio insurance strategies in which they attempted to achieve the payoff profiles of an option through delta hedging experienced heavy losses as a result of this gapping.

What was less expected, though, was our finding that a considerable part of our loss would have been experienced even if the markets had not gapped. Our loss was due to higher-than-anticipated volatility. This was despite the fact that when we looked over the tenor of our deposit product the average realized volatility was well within the range we had anticipated in pricing the product. Here's where path dependence comes in. The average realized volatility consisted of very high volatility during a short period when the market was plunging sharply, which was preceded and followed by periods of much lower volatility. However, exposure to volatility depends on the relationship between the price level and strike. The higher-than-average volatility during the period when prices were falling sharply cost us much more than we saved from the lower-than-average volatility during the other periods.

This phenomenon can be easily illustrated with some simple Black-Scholes calculations. Suppose you have written a one-year call option with a strike equal to the current forward price. You intend to delta hedge and expect volatility to average 20% over the year. If you are wrong and volatility averages 30%, your expected losses will be $BS(100\%, 1, 30\%) - BS(100\%, 1, 20\%) = 11.923\% - 7.966\% = 3.957\%$. Suppose one-tenth of a year goes by and the forward price is at the same level as when you wrote the option. Your remaining exposure to volatility averaging 30% is $BS(100\%, 0.9, 30\%) - BS(100\%, 0.9, 20\%) = 11.315\% - 7.558\% = 3.757\%$. So $3.757\%/3.957\% = 94.9\%$ of your volatility exposure comes in the last 90% of the option's life and only 5.1% comes in the first 10% of the option's life (a consequence of the fact that $\sqrt{.9} / \sqrt{1} = .949$). However, if the price at the end of one-tenth of a year has fallen by 30%, the remaining exposure to volatility averaging

30% is $BS(70\%, 0.9, 30\%) - BS(70\%, 0.9, 20\%) = 1.188 - 0.184 = 1.004$. So $(1.004\%/3.957\%) = 25.4\%$ of your volatility exposure comes in the last 90% of the option's life and 74.6% comes in the first 10% of the option's life. A very similar effect will be seen for a large rise in underlying price.

With the benefit of experience, we concluded that we had badly underestimated the risk of the product. First, we had not taken into account the potential losses from pricing gaps. Second, the chances of volatility being very high during a short time period are much larger than the chances of it being very high during a long time period, so we had not properly calculated our vulnerability to a short period of high volatility combined with a large price move. Third, we had not looked at the impact of other market participants pursuing strategies similar to ours, thereby decreasing liquidity by competing with us for hedges in the underlying when we most needed them.

What would have been a more prudent way of managing this risk? We had been considering, but had not implemented, a proposal from a broker in exchange-traded, shorter-term S&P options for a hedge of our longer-term options with these shorter-term options. See Section 11.6.3 for a discussion of the risk characteristics of this hedge.

## 11.3 A SIMULATION OF DYNAMIC HEDGING

In the immediately preceding section, we established that, under realistic economic assumptions, dynamically hedged options are path dependent. In the section before that, we observed the need for testing how well the paradigm of managing options risk using Black-Scholes theory works. Both sections point toward using Monte Carlo simulation to see what the probability distribution of results can be for dynamically hedging an options portfolio.

Using Monte Carlo simulation for dynamic hedging options is an invaluable tool for understanding how the management of an options trading book works in practice. When new options products or hedging strategies are proposed, traders and risk managers alike will want to look at simulation results to assess potential pitfalls. This is an example of the use of simulation in model testing recommended in Section 8.4.3. Simulation gives the flexibility to take into account the impact on hedging results of real-life constraints such as liquidity constraints on the size of changes in hedges that can be performed in a given time period (or the impact of larger changes on the price at which the hedge can be executed).

Simulation also provides a vital learning tool for people who are unfamiliar with the workings of options markets. Theoretical demonstrations of the power of dynamic hedging rarely carry the conviction that can be provided by observing hundreds of simulation paths that, despite wild gyrations in underlying prices, produce almost identical hedging results. Nothing short of actually suffering through a losing options strategy can convey the pain of an unsuccessful hedge as will observing the losses pile up on a simulation path.

In the course I teach, on which this book is based, I have always insisted that each student personally program and run simulations of a dynamic hedge. I lack a comparable power of persuasion over readers of this book, but I urge each of you to do as much of Exercise 11.2 as you can. Even if you lack the time to program your own simulation, you should at least do parts 4 and 5 of this exercise using the provided spreadsheets.

What features do we want a Monte Carlo simulation of dynamic hedging to contain?

- The simulation must be over a sufficiently large number of possible price paths to produce stable statistics. Prices for the underlying variable must be sampled at enough points on each path to allow for rehedging.
- Since volatility of the underlying price is not constant, but is a stochastic variable, a random process should drive it. Data to determine reasonable values of volatility can be obtained by looking at historical distributions of realized volatility for separate time periods. A separate volatility should be chosen for each path generated.
- The distribution of the underlying price does not necessarily need to be lognormal. Different mixtures of normal and lognormal processes should be tried.
- Rehedges should be allowed only at periodic intervals, and transaction costs of the hedge should be calculated explicitly. Different rules for determining hedge amounts, as discussed in Section 11.5, should be considered.
- When calculating Black-Scholes deltas for rehedging, you generally do not want to take advantage of knowing what volatility is being used for the path, since this would not be available in making actual hedging decisions. Either you want to use the same implied volatility to calculate rehedges on all paths or you want to use some adaptive rule tying volatility used to the history of price moves on the path up to the time of the rehedge.
- A random process of significant price jumps, where no rehedging is permitted until after the jump is completed, can be used as a simulation of periods of illiquidity.
- When simulating a portfolio of options for one particular expiry date, it is usually convenient to assume that all hedges are performed with

a forward with the same expiry to avoid needing to keep track of discounting rates. When simulating options with different expiry dates, some assumptions about discounting rates must be used to arrive at relative prices between forwards.

In effect, we are testing the performance of the Black-Scholes model as a hedging tool by running a Monte Carlo simulation based on a more complex, and presumably more accurate, model of underlying price behavior than Black-Scholes utilizes. Why not just value and hedge options by directly using this more complex and complete model? For two reasons:

1. **Computational complexity.** The speed of the computation of the Black-Scholes model for valuation and the fast and direct computation of the target underlying hedge are enormous advantages in providing timely risk information on portfolios of options that may have many thousands of deals outstanding at any given time. By contrast, more complex models can be orders of magnitude slower when computing valuations and often lack a direct computation of target hedges, requiring multiple runs of the valuation algorithm to determine the appropriate hedge. This advantage can particularly be seen in Monte Carlo testing of hedge effectiveness. At each potential rehedge point, the Black-Scholes target hedge is a simple equation; a more complex model may require full recalibration to compute each hedge (see Section 12.3.2 for a discussion of this point in conjunction with hedging barrier options).
2. **Validity.** We don't necessarily know what the correct model is. For testing hedge performance with Monte Carlo, we can make different runs with alternative candidates for the correct model.

As a first example of a simulation, let's look at a comparison between hedging an option using a pure Black-Scholes hedge and hedging using a combination of Black-Scholes delta hedging and hedging with other options. We may suppose that an option has been sold at a strike for which no liquidity is readily available. We can either utilize a dynamic hedging strategy or buy some options at strikes for which liquidity is available and then utilize a dynamic hedging strategy for the residual risk.

For this example, we will assume that a one-year option has been sold at a strike 5 percent in-the-money and that one-year options are available for purchase at strikes at-the-money and 10 percent in-the-money. For the second case, we will consider purchasing the same notional amount of options as has been sold, but split 50–50 between the at-the-money option and 10 percent in-the-money option. The reason for thinking that this might be a good hedge will be shown in Section 11.4. There we will see that the

**TABLE 11.2** Monte Carlo Simulation Comparing Pure Dynamic Delta Hedging with Combined Static Option and Dynamic Delta Hedging

| | Standard Deviation of P&L | | Standard Deviation of P&L | | | |
|---|---|---|---|---|---|---|
| | Given 0% Standard Deviation of Volatility | | Given 33% Standard Deviation of Volatility | | Transaction Costs | |
| Number of Rebalancing | Unhedged | Two-Sided Hedge | Unhedged | Two-Sided Hedge | Unhedged | Two-Sided Hedge |
| 10 | 25.7% | 6.4% | 50.6% | 6.3% | 1.5% | 0.1% |
| 20 | 19.8% | 5.6% | 41.5% | 6.7% | 2.2% | 0.2% |
| 50 | 12.4% | 4.6% | 40.9% | 5.5% | 3.5% | 0.4% |
| 100 | 8.5% | 3.6% | 42.6% | 4.9% | 5.0% | 0.6% |
| 200 | 6.3% | 2.5% | 41.6% | 4.8% | 7.1% | 0.9% |
| 300 | 5.1% | 1.9% | 39.9% | 3.8% | 8.5% | 1.1% |
| 400 | 4.3% | 1.8% | 40.1% | 4.1% | 9.9% | 1.2% |
| 500 | 3.9% | 1.6% | 38.4% | 3.9% | 11.2% | 1.4% |
| 600 | 3.5% | 1.4% | 35.9% | 3.4% | 12.0% | 1.5% |
| 700 | 3.3% | 1.3% | 41.0% | 3.5% | 13.3% | 1.6% |
| 800 | 3.2% | 1.3% | 39.2% | 3.7% | 14.4% | 1.7% |
| 900 | 2.9% | 1.5% | 40.0% | 3.8% | 15.0% | 1.9% |

All results are shown as a percentage of the cost of the option to be hedged.
The option is a one-year call struck 5 percent in-the-money.
The expected volatility is 20 percent, and all hedges are calculated based on a 20 percent implied volatility.
The two-sided hedge has half a call struck at-the-money and half a call struck 10 percent in-the-money.
Transaction costs are based on a bid-ask spread of one-fourth point per $100.

price-vol matrix for this portfolio (Table 11.9) shows very little sensitivity to changes in either the price level or implied volatility. This does not, by itself, prove that the hedge will work well over the life of the option, since it only shows a snapshot at one point in time. In fact, you will be able to see from Tables 11.10 and 11.11 in Section 11.4 that although this portfolio does continue to show low sensitivity to price on volatility shifts for a substantial time period, this sensitivity increases at some point in its evolution. So we need the Monte Carlo simulation to get a statistical measure of the sensitivity. Table 11.2 shows the results of the simulation.

In the context of the discussion of model risk in Section 8.4, the 50–50 mixture of at-the-money option and 10 percent in-the-money option constitutes the liquid proxy that would be used to represent the 5 percent in-the-money option in standard risk reports, such as VaR and stress tests. The Monte Carlo simulation would be used to generate a probability distribution of how much extra risk there is in holding the 5 percent in-the-money option than there is in holding the liquid proxy. The assumption that the 50–50 mixture will constitute a good hedge all the way to the expiration of the option is a simplifying assumption that makes the Monte Carlo simulation easier. In reality, a trading desk would change this mixture through time, particularly as time to option expiry was close. But while a Monte Carlo simulation that included changes in the mixture would be more realistic, it would also be far more difficult to perform. Changes in the volatility surface would need to be simulated, since changes in the mixture will require purchases and sales of options at future dates; transaction costs for purchases and sales of options would need to be included; behavioral rules for trading decisions would be needed on the trade-off between these transaction costs and the desirability of changing the mixture.

What conclusions can we reach?

- If the standard deviation of volatility is zero, then both the pure dynamic hedging and the mixed-option/dynamic hedging strategies can achieve as low a standard deviation of results as you like by increasing the frequency of rebalancing the dynamic hedge, although the mixed strategy achieves a given level of standard deviation with far fewer rebalancings than the pure strategy. For either strategy, there is a trade-off between higher expected transaction costs with more frequent rebalancing and lower standard deviations of results. (Standard deviations of total results, including transaction costs, don't differ significantly from the standard deviations without transaction costs, which are shown in Table 11.2.) However, the mixed strategy can achieve a desired level of standard deviation at a far lower transaction cost level than the pure strategy. For example, achieving a 3% standard deviation with the pure strategy requires about 900 rebalancings with an associated transaction cost of 15.0%. Achieving a 3% standard deviation with the mixed strategy requires about 150 rebalancings with an associated transaction cost of about 0.8%.
- If the standard deviation of volatility is 33%, then there is a lower bound on how much the standard deviation of results can be decreased. For both the pure and mixed strategies, this lower bound is reached at about 250 rebalancings. The lowest level of standard deviation of

results that can be achieved by the mixed strategy is about one-tenth of what can be achieved by the pure strategy, roughly 4% compared to roughly 40%.

- The inability to reduce the standard deviation of results below a lower bound is due to both the uncertainty of volatility and the use of incorrect volatility inputs in forming hedge ratios. However, the first effect is many times larger than the second. A Monte Carlo run with 33% standard deviation of volatility, but with hedge ratios on each Monte Carlo path based on the actual volatility of that path, results in a lower bound on the standard deviation of results that is only reduced from 40 to 36%

Please note that although we are using standard deviation as a convenient summary statistic to give a rough feel for relative levels of uncertainty, both in this example and others in this book, more detailed analysis would be needed before arriving at any precise conclusions. For example, if a measure was being developed for a risk versus return trade-off as input to a decision on a trading strategy, a more complete set of measures of the probability distribution of returns should be used. The discussion of measures of portfolio risk in Section 7.1.2 gives more of a flavor for these considerations.

These results will not be surprising when we examine the price-vol matrix in Table 11.9 in Section 11.4. From the relative insensitivity of portfolio value to a shift in implied volatility we will see there, you would expect low sensitivity to the standard deviation of volatility. The small size of the portfolio's convexity translates into small changes in the delta when prices move, so transaction costs should be low. A reasonable inference, which is supported by experience with Monte Carlo simulations, is that a trading desk can estimate its vulnerability to uncertain volatility and transaction costs by forecasting how large its price-vol matrix positions are likely to be given the anticipated flows of customer business and the availability of hedges with liquid options. Management can keep these vulnerabilities under control by placing limits on the size of price-vol matrix positions.

It is important to recognize the distinction between the two aspects of dynamic hedging costs—transaction costs that arise from bid-ask spreads and gamma hedging costs from buying high and selling low that would be present even if all trades were at midmarket. Transaction costs are a direct function of the frequency of rehedging, and a trade-off occurs between higher transaction costs and lower variability of profit and loss (P&L) with less frequent rehedging. By contrast, there is no a priori reason to believe that the level of gamma hedging costs will vary in any systematic way with the frequency of rehedging.

A good way to see this latter point is to look at how P&L is related to the gap between actual hedges held and the theoretical hedge called for by the Black-Scholes formula. The expected value of this P&L under the standard Black-Scholes assumption is given by the formula:

$$\sum_{\substack{small\ time\\ periods}} (\Delta_{actually\ held} - \Delta_{theoretical}) \times \text{ expected price change of underlying forward} \tag{11.2}$$

A full mathematical derivation of this formula can be found in Gupta (1997). I will give an alternative derivation using a simple financial argument. In the presence of the Black-Scholes assumptions, use of the theoretical delta will lead to an expected return of zero, so any holdings above or below the theoretical delta can be regarded as proprietary positions that will lead to the same expected return as an outright position in the underlying forward.

The consequence of this formula for the relationship between gamma hedging costs and the frequency of rehedging is that as rehedging becomes less frequent, it widens the gap between $\Delta_{actually\ held}$ and $\Delta_{theoretical}$. However, unless a correlation between the sign of this gap and the sign of the expected price change in the underlying forward is expected for some reason, the expected value of the incremental P&L should be zero. (Although this formula is strictly correct only in the case where the Black-Scholes assumptions hold, Monte Carlo simulation with stochastic volatility shows similar results.)

Are there cases where we might expect a relationship between the sign of the delta gap and the sign of expected price changes in the underlying forward? Let's consider a case that will cast an interesting light on a long-standing debate among practitioners. The debate is over what options pricing is appropriate for a market in which the underlying process shows *mean reversion*, resulting in a narrower dispersion of future price levels than would be implied by a pure random walk with the short-term volatility of the underlying process. One group argues that delta-hedging costs are completely a function of short-term volatility, so mean reversion is irrelevant to pricing. The opposing group argues that risk-neutral valuation principles should result in the same pricing of options as would be implied by the probability distribution of final prices; compare the discussion here to Rebonato (2004, Sections 4.7 and 4.8).

Some of this dispute reflects a failure to distinguish between the short-term volatility of spot prices and forward prices. If the market is pricing

**TABLE 11.3** Impact of Drift and Mean Reversion on Dynamic Hedging Results

| | All Paths | Upward Drift | Downward Drift | Mean Reversion |
|---|---|---|---|---|
| 20 rehedges | –0.07% | –0.33% | –0.45% | +0.57% |
| 100 rehedges | +0.01% | –0.06% | –0.10% | +0.20% |
| 1,000 rehedges | –0.01% | 0% | 0% | –0.02% |

the mean reversion process into the forward prices, we should expect to see a lower historical short-term volatility of forward prices than a historical short-term volatility of spot prices. Equivalently, this can be viewed as a correlation between changes in spot prices and changes in the discount rate of the forwards, a pattern that can be seen in the market for seasonal commodities. When seasonal demand is high or seasonal supply is low, spot prices rise, but so does the discount rate, dampening the rise in forward prices. When seasonal demand is low or seasonal supply is high, spot prices fall, but so does the discount rate, dampening the fall in forward prices. Since the option can be delta hedged with the forward, replication costs will be tied to the volatility of the forward, so we should expect implied option volatilities to reflect the impact of mean reversion relative to the volatility of the spot price.

Suppose that a trader believes that the market has not adequately priced in mean reversion, so he expects that forward prices will show mean reversion. In this case, we cannot resolve the controversy between the two differing views on options pricing by an appeal to the difference between short-term volatility of spot and forward prices. Let us look at the results of a Monte Carlo simulation in which we ignore transaction costs and study the impact of rehedging at a fixed number of evenly spaced intervals. We will calculate statistics for the whole sample of paths, but also for three subsamples:

1. The third of paths having the highest finishing forward prices, which we can take as representing upward drift of the forward.
2. The third of paths having the lowest finishing forward prices, which we can take as representing downward drift.
3. The remaining third of the cases, which we can take as representing mean reversion.

Table 11.3 shows the resulting expected values of a delta-hedging strategy for a written (sold) option (for a purchased option, the signs would be reversed).

What conclusions can we draw?

- As you increase the frequency of rehedging, you get the same expected results regardless of drift or mean reversion. This is consistent with the theoretical result that, under the Black-Scholes assumptions, standard deviation of results goes to zero as the frequency of rehedging increases, so the P&L will be the same on every path. It is also consistent with Equation 11.2, since frequent rehedging drives the difference between the $\Delta_{actually\ held}$ and $\Delta_{theoretical}$ terms to zero.
- As you decrease the frequency of rehedging, you increase the losses from a sold option with drift or a purchased option with mean reversion, and you increase the gains from a sold option with mean reversion on a purchased option with drift. All of these results are consistent with Equation 11.2. For example, here's the reasoning for mean reversion on a sold option: It is likely that one period's up move will be followed by the next period's down move, and vice versa. After an up move, the $\Delta_{theoretical}$ on the sold option will increase, but if no rehedge is performed, due to the infrequency of rehedging, this will make the $\Delta_{actually\ held} - \Delta_{theoretical}$ for the next period be negative. Since the expected price change in the next period is negative, the expected P&L is the product of two negatives, and hence positive.
- The consequence of the last point for hedging strategies is that if you anticipate mean reversion, you should try to decrease hedging frequency for a sold option (which also saves transaction costs, but increases the uncertainty of return) and try to increase hedging frequency for a bought option (but this needs to be balanced against the increase in hedging costs and uncertainty of return). This is intuitively correct. As the option seller, you want to hold off on rehedging since you expect the market to rebound; as the option buyer, you want to take advantage of the market move with a rehedge prior to the expected rebound. Conversely, if you anticipate a drifting market, whether up or down, you should try to decrease hedging frequency for a bought option and increase hedging frequency for a sold option.
- If you cannot anticipate either drift or mean reversion, there is no difference in gamma hedging costs based on the frequency of rehedging, so the decision rests purely on the trade-off between transaction costs and the uncertainty of return.

## 11.4 RISK REPORTING AND LIMITS

The best tool for managing residual options risk on a trading desk is the *price-vol matrix*, which depicts valuation sensitivity to joint