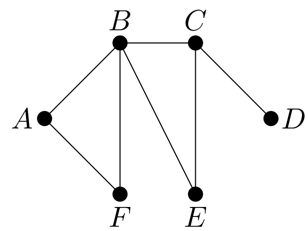
Math 280, Spring 2022

Practice Problems for Midterm Exam 1

(A little more difficult, and much longer, than the real exam)

1. Consider the following graph G:



(a) Determine the degree sequence of G, as well as $\delta(G)$ and $\Delta(G)$.

(b) Draw the complementary graph to G.

(c) Find a path in G of maximum length, and explain why no longer path is possible.

(d) Find a trail in G of maximum length, and explain why no longer trail is possible.

(e) Write down the adjacency matrix of G.

(f) Find the eccentricity of every vertex of G.

(g) Find the radius, diameter, and center of G.

(h) What is the connectivity $\kappa(G)$? Why?

2. Let G be the graph represented by the adjacency matrix $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

(a) Draw the graph G.

(b) Find the number of walks of length 3 from vertex v_2 to vertex v_3 in G.

3. Prove that there are no graphs with 10 vertices and 46 edges.

4. Let G be a graph with 5 vertices and at least 5 edges. Suppose that G has has no isolated vertices. Prove that G is connected.

5. Give an example of a graph with 5 vertices and 6 edges that is *not* connected.

6. Give an example of a simple graph with 6 vertices and 7 edges that has no isolated points and is *not* connected.

7. Let G be a graph. Suppose that for any two vertices $u, v \in V(G)$, there is a unique path from u to v in G. Prove that G is a tree.

8. Let T be a tree, and let $e \in E(T)$ be an edge. Prove that e is a bridge, i.e., that $T - e$ is disconnected.

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9. Let T be a tree, and let $v \in V(T)$ be a vertex. Define $m = \deg(v)$. Prove that $T - v$ has at least m connected components.

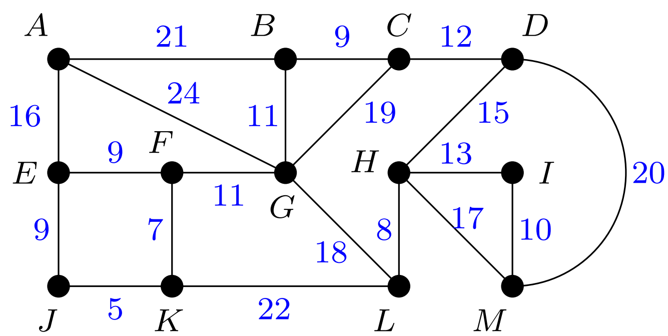
[In fact, it has *exactly* m, but I'm only asking you to prove "at least" here.]

10. Let T be a tree with at least one vertex of degree at least 3. Prove that there is no trail in T that reaches every vertex.

(*Suggestion*: use one of the previous two problems.)

11. Let T be a tree of order $n \geq 2$. Prove that $\kappa(T) = 1$.

12. Use Kruskal's Algorithm to find a minimal spanning tree of the following weighted graph:



13. Find an example of a graph G that has a vertex $v \in V(G)$ such that v is a cut vertex of G, but also, v lies on a cycle of G.

14. Let G be a graph, and let $e \in E(G)$ be an edge. Suppose that e is not a bridge of G. Prove that e lies on some cycle of G.

15. Let G be a graph of order $n \geq 1$ and of size m. Prove that

$$\delta(G) \leq \frac{2m}{n} \leq \Delta(G).$$

16. Let G be a graph with adjacency matrix A.

(a) Suppose A is 7×7. What does this say about G?

(b) Suppose exactly 26 of the entries of A are 1's. What does this say about G?

(c) Suppose that the $(2,5)$ entry of A^4 is 6. What does this say about G?

(d) Suppose that the $(3,4)$ entry of $I+A+A^2$ is 0, but the entire third row of $I+A+A^2+A^3$ is nonzero. What does this say about G?

17. Let P_{50} be the path graph with 50 vertices, numbered 1 to 50 from one end to the other. Let A be the associated adjacency matrix.

(a) What is the $(3,42)$ entry of A^{20}? Why?

(b) What is the smallest integer $k \geq 0$ such that the $(6,38)$ entry of A^k is nonzero? Why?

(c) For k as in part (b), what is the $(6,38)$ entry of A^k? Why?