Price Impact Models and Applications

Introduction to Algorithmic Trading

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Plan

Last Week
Dynamic liquidity models and no price manipulation results.

For this Week
Applications to optimal execution include four steps:

(a) Communicate a pre-trade cost model to the portfolio team.
(b) Establish a strategic trading schedule based on alpha signals and liquidity models.
(c) Implement microstructure tactics as deviations from the strategic schedule.
(d) Update the schedule when a new order arrives.

Next Week
Back testing and statistical arbitrage.

Last Week's Summary

Liquidity is dynamic
it exhibits time-of-day patterns, as well as stochastic behavior. Trading strategies react strongly to these liquidity fluctations.

Beware of price manipulation
Trading algorithms, especially complicated black-box algorithms, can easily be tricked into thinking there is a price manipulation opportunity.

Round-trip trade manipulation
happens when liquidity increases too fast. For the generalized OW model

$$dI_t = -\beta_t I_t dt + e^{\gamma_t} dQ_t$$

the no-price manipulation condition is

$$2\beta_t + \gamma_t' > 0.$$

Introduction

A Typical Execution Scenario (1/3)

Fundamental events tend to happen after the close
Isichenko (Bloomberg, 2021) details "Event-based predictors":

> *"One can use material company news, mergers and acquisitions, earnings surprises, index rebalancing, or even dividends and splits."*

A portfolio manager has an overnight alpha.
For instance, the portfolio manager takes a view on an earnings announcement after the close.

> *The manager submits a day order considering their overnight alpha.*

A Typical Execution Scenario (2/3)

The portfolio manager engages with an execution team.
This team could be internal or external. Using a pre-trade cost model, the portfolio manager submits an order that best captures the overall trade-off between trading costs and their overnight alpha.

Quantitative Brokers (2019)

> *"Questions such as "how large a trade can we do", or "what time horizon should we set", or even "does this trade have a strong enough alpha to overcome the transaction costs" can all be systematically evaluated using a good pre-trade cost model."*

A Typical Execution Scenario (3/3)

The execution team trades the order.
The execution considers intraday and microstructure signals the portfolio manager lacks. It also tracks a real-time view of liquidity and price impact.

Neuman and Voß (2020)

> *"One of the main challenges in the area of optimal trading with price impact deals with the question of how to incorporate short term predictive signals into a stochastic control framework."*

Pre-Trade Cost Model

What is a Pre-Trade Cost Model?

A model of trading costs for portfolio managers
Prior to submitting an order, the model predicts the costs along the portfolio manager's control variables. For instance, if the portfolio manager decides on

(a) a stock with characteristics (e.g., σ, adv) and
(b) an order size Q,

then, a pre-trade cost model takes the form

$$\text{TC}\left(Q, \sigma, \text{adv}\right).$$

Pre-trade cost models depend on the manager's available actions
For instance, if a trading team offers the manager to pick an algorithm $A \in \{A_1, ...A_n\}$, then,

$$\text{TC}\left(Q, A, \sigma, \text{adv}\right).$$

How do Portfolio Managers use a Pre-Trade Cost Model?

The portfolio manager's control problem
The portfolio manager also solves a control problem. The pre-trade cost model informs the manager's control problem as to their actions' trading costs.

A single-day example
A portfolio manager submitting a day order considering an overnight alpha signal α solves

$$\sup_{Q,A}\{\alpha Q - TC\,(Q, A, \sigma, \mathrm{adv}) + [...]\}$$

where [...] are other portfolio considerations (e.g., risk terms).

Why?

Why do portfolios solve a control problem?
Same reason as trading: reproducibility, automation, and accountability (e.g., is the broker's pre-trade cost model any good?).

Why separate portfolio construction and trading?
A joint control problem mathematically outperforms two separate control problems. However,

(a) The portfolio team and the trading team may not work for the same firm.

(b) Even if they do, the split organizationally may make sense: most professionals are highly specialized.

(c) A joint optimization problem makes it hard to measure individual performance.

The last point is a recurring theme in finance (see Module 3).

Example: Brokers and Their Clients

Bacidore (Crédit Suisse, 2020)
Portfolio managers don't share their alpha.

> *"Traders are understandably reluctant to pass their alphas over to a broker due to the potential for lost intellectual property, front-running, etc."*

Alpha Profiling (Wikipedia)
Brokers profile portfolio managers.

> *"Alpha profiling models learn statistically-significant patterns in the execution of orders from a particular trading strategy or portfolio manager and leverages these patterns to associate an optimal execution schedule to new orders. In this sense, it is an application of statistical arbitrage to best execution."*

Information Structure

The portfolio team
knows *why* the trade is happening (e.g. long term alpha).

The execution team
knows *how* the trade is going to happen (e.g. short term liquidity).

In real life, both teams must coordinate without over-sharing their respective information set. This requires a *pre-trade model* to facilitate the communication.

Idealized Execution Problem

Idealized scenario

(a) Broker provides a liquidity model

$$dI_t = -\beta_t I_t dt + e^{\gamma_t} dQ_t.$$

(b) Portfolio manager provides a static overnight alpha

$$\alpha = \mathbb{E}\left[\, S_\tau - S_T \,|\, \mathcal{F}_t\right].$$

Idealized control problem
In the above scenario, one solves

$$\max_Q \mathbb{E}\left[\int_0^T (\alpha - I_t)\, dQ_t - \frac{1}{2}[I, Q]_T\right]$$

to obtain a trading schedule Q and a corresponding pre-trade cost model $TC(Q)$.

Solution

Optimal trading strategy
The target impact state is

$$I_t = \frac{\beta_t + \gamma_t'}{2\beta_t + \gamma_t'}\alpha.$$

The associated order size is

$$Q_T = \frac{\alpha}{\Lambda_T}$$

where Λ_T is

$$\frac{1}{\Lambda_T} = \frac{1}{\lambda_T} + \int_0^T \frac{(\beta_t + \gamma_t')^2}{\lambda_t(2\beta_t + \gamma_t')} dt.$$

Pre-Trade Cost Model for the Client (1/2)

Pre-trade cost model
The pre-trade cost model is

$$
\begin{aligned}
TC(Q) &= -\mathbb{E}\left[\int_0^T I_t dQ_t + \frac{1}{2}[I, Q]_T\right] \\
&= -\mathbb{E}\left[\frac{\Lambda_T}{2} Q_T^2\right].
\end{aligned}
$$

Expected P&L
The corresponding expected P&L is

$$
\begin{aligned}
\mathbb{E}\left[Y_T\right] &= \frac{\alpha}{2}\mathbb{E}\left[Q_T\right] \\
&= \mathbb{E}\left[\frac{\Lambda_T}{2} Q_T^2\right].
\end{aligned}
$$

Pre-Trade Cost Model for the Client (2/2)

Formulas for portfolio managers
Portfolio manager doesn't need intraday liquidity: formulas only depend on *daily* liquidity factor λ_T which the trading team can provide.

For instance, if $\lambda_t = \frac{\sigma}{\mathrm{adv}}$ then

$$\Lambda_T = \frac{\sigma}{(2 + \beta T)\mathrm{adv}}.$$

The optimal order size Q given alpha α is

$$Q = \frac{(2 + \beta T)\mathrm{adv}}{\sigma}\alpha.$$

The pre-trade cost model for this trading schedule is

$$TC(Q) = \frac{\sigma}{(2 + \beta T)\mathrm{adv}} Q^2.$$

Business Takeaways

The portfolio team pays half their alpha in impact costs for an optimally sized order.

> *"As a common feature of linear-quadratic functions, the optimum is reached at the point where the quadratic penalties amount to one-half the linear term.* **This means that it is optimal to give away half of the forecast-driven gross pnl to impact cost***"* *(Isichenko, Bloomberg 2021)*

The trading team can provide a daily pre-trade cost model without sharing their intraday liquidity model.
The client only needs the daily liquidity parameter Λ_T instead of the underlying models β, λ.

Implied Alpha Formula

Alpha Profiling in the OW model
A trader inverts the pre-trade cost model to define an order's *implied alpha.*

$$\alpha = \Lambda_T Q$$

Use-cases

(a) Frames discussion with client on how much alpha they need to justify an order's size.
"Questions such as [...] "does this trade have a strong enough alpha to overcome the transaction costs" can all be systematically evaluated using a good pre-trade cost model." (Quantitative Brokers, 2019)

(b) Serves as a starting point for alpha profiling.

Non-Linear Price Impact

The AFS model
For sizable orders, a better price impact model is

$$I_t = J_t^c$$

where

$$dJ_t = -\beta J_t dt + \lambda dQ_t.$$

Client pre-trade cost model

$$TC(Q) \propto Q^{1+c}; \quad I = \frac{1}{1+c}\alpha; \quad \mathbb{E}[Y_T] = \frac{c}{1+c}\alpha\mathbb{E}[Q]$$

Implied alpha for profiling

$$\alpha \propto Q^c$$

Scheduling with Intraday Signals

Intraday Alpha Signals

Examples from Almgren (2018)

- *"Prices overshoot and relax*
- *Related assets tend to move together*
- *Presence of imbalance in the market"*

When should an alpha be integrated into the execution schedule?

> *"where the alpha is relatively small over the life of each child order, but meaningful over the life of the parent, the alpha could be incorporated in the core algorithm." (Bacidore, Crédit Suisse 2020)*

On the other hand, low-latency micro-alphas cannot be integrated into the schedule (see next section).

Intraday Alpha Contribution (1/2)

Trader's stochastic control problem
The execution team optimizes

$$\max_{Q} \mathbb{E}\left[\int_0^T \left(\tilde{\alpha} + \alpha_t - I_t\right) dQ_t - \frac{1}{2}[I, Q]_T + [\alpha, Q]_T\right].$$

where

$$\alpha_t = \mathbb{E}\left[\, S_T - S_t \middle|\, \mathcal{F}_t\right]$$

is their intraday alpha and $\tilde{\alpha}$ is the implied alpha from the portfolio team.

Target impact state

$$I_t = \frac{\beta_t + \gamma_t'}{2\beta_t + \gamma_t'}\tilde{\alpha} + \frac{\beta_t + \gamma_t'}{2\beta_t + \gamma_t'}\alpha_t - \frac{1}{2\beta_t + \gamma_t'}\alpha_t'.$$

Intraday Alpha Contribution (2/2)

The optimal order size is

$$Q_T = \frac{\tilde{\alpha}}{\Lambda_T} + Q_T(\alpha)$$

where

$$Q_T(\alpha) = \int_0^T \frac{\beta_t + \gamma'_t}{\lambda_t(2\beta_t + \gamma'_t)} \left((\beta_t + \gamma'_t)\, \alpha_t - \alpha'_t \right) dt$$

is the contribution of the intraday alpha α_t to the order size.

Practical considerations
The formula considers two types of trading signals:

(a) an intraday alpha signal α_t and decay α'_t

(b) dynamic liquidity signals $\lambda_t, \gamma'_t, \beta_t$.

The formula is too complicated to use for communication. However, one implements it in backtest to measure trading signals' effects on the execution schedule.

Must Complete vs Best Effort

Bacidore (Crédit Suisse, 2020)

> *"Note that trades can be submitted as "must complete" orders, meaning the algorithm must aim to complete the entire basket entirely [...]. Others are submitted on a "best efforts" basis, which means that, ideally, the trader would like the algorithm to complete the trade. But leaving residual quantities is permitted."*

Mathematically is Q_T a hard constraint or just an estimation of the order's alpha?

(a) For the *best efforts* basis, $Q(\alpha)$ describes the residual quantities due to the execution team's intraday signals.

(b) For the *must complete* basis, $\tilde{\alpha}$ must be changed to guarantee the order's completion.

Tactical Microstructure Deviations

What are Micro-Alphas? (1/2)

Isichenko (Bloomberg, 2021)

> *"Given the extreme time sensitivity of micro alpha [...], a competitive advantage is gained by using low-latency, or high-frequency, strategies and co-located access infrastructure."*

Isichenko provides a practical implicit definition of micro-alphas: signals that require co-location to be profitable.

Crucial challenge

(a) High-frequency signals require a decentralized approach to take advantage of low-latency infrastructure, such as co-located servers.

(b) However, strategic execution schedules are, by nature, centralized algorithms.

(c) Therefore, traders implement low-latency signals separately and incorporate them differently into the core execution problem.

What are Micro-Alphas? (2/2)

Primary tool: tactical deviations
When central decision making is impossible, traders implement decentralized decisions as tactical deviations form the centralized schedule.

Nehren et al. (ADIA, 2020)

> "**Systematic Deviation:** *[...] If a prediction of short term price move is possible, it can leverage that information by making a systematic decision to get ahead or stay behind the schedule and take advantage of that price information. This has to be done carefully to avoid incurring any additional market impact or risk of catching up that will cancel out any possible benefit of the move. The schedule also needs to have a component that will move the strategy back on schedule".*

Micro-Alpha Examples (1/2)

Zovko (2017)

"The method we describe in this paper colours the dark liquidity by dynamically detecting evidence of causality between dark and lit fills and provides reactive information for a trader to adjust their trading."

Kolm, Turiel, and Westray (Alliance & Bernstein, 2021)

"achieve state-of-the-art predictive accuracy by training simpler "off-the-shelf" artificial neural networks on stationary inputs derived from the order book"

Micro-Alpha Examples (2/2)

Cont, Cucuringu, and Zhang (2021)

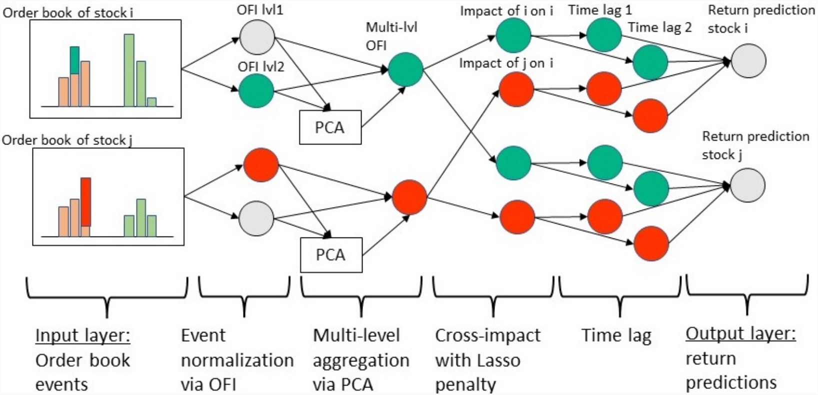

Figure 1: Model architecture in Cont, Cucuringu, and Zhang (2021) containing an input layer, four computational layers, and an output layer.

Major US Equity Trading Venues

Examples
NASDAQ: *Carteret, NJ*; **NYSE:** *Mahwah, NJ*; **BATS:** *Secaucus, NJ*; **Futures:** *Aurora, IL*

Implications for Optimal Execution (1/3)

Definition (Trade and impact deviation)
If Q is the trading schedule and Q^r the realized trading process, then define the *trade deviation* as

$$\delta Q = Q^r - Q.$$

Define the *price impact deviation* as

$$\delta I = I\left(Q^r\right) - I\left(Q\right).$$

Implications for Optimal Execution (2/3)

The deviation formula
Then the price impact deviation satisfies the SDE

$$d\delta I_t = -\beta_t \delta I_t dt + \lambda_t d\delta Q_t$$

with $\delta I_0 = 0$.

Furthermore, the expected fundamental P&L of the realized trades satisfies

$$\mathbb{E}\left[Y_T(Q^r)\right] - \mathbb{E}\left[Y_T(Q^*)\right] = -\mathbb{E}\left[\int_0^T \frac{2\beta_t + \gamma_t'}{2\lambda_t}\left(\delta I_t\right)^2 dt + \frac{1}{2\lambda_T}\left(\delta I_T\right)^2\right].$$

Implications for Optimal Execution (3/3)

Co-located algos don't have time to confer with the central schedule or other algos.
They act immediately considering the last schedule update and their venue-level signals.

(a) Algo knows the expected gain from its micro-alpha.

(b) Algo knows current impact target.

(c) Algo acts on micro-alpha if it's larger than the impact deviation cost.

(d) Central scheduler registers the impact deviation and updates the schedule to return on track in impact space.

Special Case: Block Fills

What is a block fill?
Most fills happen on a limit order book and are *tiny.* This leads to near-continuous trading strategies. However, sometimes the trading strategy jumps with a big fill. E.g.,

(a) the open and closing auction.
(b) dark pools.
(c) over-the-counter (OTC) trading.
(d) indication of interest (IOI).

These allow trading strategies to immediately “jump” to the correct impact state. If such a block opportunity is unavailable, then the strategy uses a limit order book model to smoothly trade towards the target.

Reacting to New Orders

Scenario Examples

What if an order is submitted shortly after another order finishes?
The second order must consider the impact of the first order.

What if a portfolio manager updates the order's size?
How to update the trading strategy to reflect the novel information?

What two portfolio managers submit correlated orders?
If the trading team can combine information from multiple sources, how do they deal with crowding (order correlation)?

Myopia in Trade and Impact space

Harvey et al. (Man Group, 2022)
In trade space, myopic strategies are sub-optimal.

> *"Common approaches suffer from a type of myopia: impact is only measured for the current transaction. In many cases, orders are correlated and the impact of the first order will affect the execution of future orders."*

Myopia in impact space
The myopic formula

$$I_t = \frac{\beta_t + \gamma_t'}{2\beta_t + \gamma_t'}\tilde{\alpha} + \frac{\beta_t + \gamma_t'}{2\beta_t + \gamma_t'}\alpha_t - \frac{1}{2\beta_t + \gamma_t'}\alpha_t'.$$

as long as one updates the variables to reflect all information.

General Principle

Trading is a myopic relationship between alpha, impact, and liquidity

New order information updates alpha signals to reflect new information.

Traders update their trading targets accordingly.

$$I_t = f(\tilde{\alpha}, \cdots)$$

changes to

$$I_t = f(\tilde{\alpha} + \delta\tilde{\alpha}, \cdots).$$

A Useful Heuristic

Slow-moving liquidity
Often, one observes that $\gamma_t' \ll \beta_t$. This heuristic

(a) simplifies the target impact state to

$$I_t = \frac{1}{2}\left(\tilde{\alpha} + \alpha_t - \beta_t^{-1}\alpha_t'\right).$$

(b) rules out price manipulation.

(c) allows for rapid back-of-the-envelope computations.

It is verified under normal market conditions.

Follow Up Order Example (1/2)

A first order finishes and a second order is placed shortly after.
Consider an order Q over $[T_1, T_2]$ with implied alpha $\tilde{\alpha}$. Assume that a previous order Q^0 with implied alpha $\tilde{\alpha}^0$ finished trading at $T_0 < T_1$. Then, by myopia in impact space,

$$I_t(Q + Q^0) = \frac{1}{2}\left(\tilde{\alpha} + \alpha_t - \beta_t^{-1}\alpha_t'\right).$$

A straightforward computation
Shift the past order's impact $I(Q^0)$ to single out the new order's impact,

$$\begin{aligned} I_t\left(Q\right) &= \frac{1}{2}\left(\tilde{\alpha} + \alpha_t - \beta_t^{-1}\alpha_t'\right) - I_t(Q^0) \\ &= \frac{1}{2}\left(\tilde{\alpha} + \alpha_t - \beta_t^{-1}\alpha_t'\right) - e^{-\int_{T_0}^{t} \beta_s ds}\tilde{\alpha}^0. \end{aligned}$$

Follow Up Order Example (2/2)

Past order's impact is like negative alpha.
Two points of view on follow up orders:

(a) The past order is part of the same strategy. Therefore, the orders' combined impact targets

$$I_t(Q + Q^0) = \frac{1}{2}\left(\tilde{\alpha} + \alpha_t - \beta_t^{-1}\alpha_t'\right).$$

(b) The past order is external to the strategy. Therefore, the second order's impact targets

$$I_t(Q) = \frac{1}{2}\left(\tilde{\alpha} + \alpha_t - \beta_t^{-1}\alpha_t'\right) - e^{-\int_{T_0}^{t}\beta_s ds}\tilde{\alpha}^0.$$

Both lead to trivial implementations in a systematic trading system.

Combining A Broker's Simple Orders

A broker's client sends an update to an active order
A simple model treats the second order as a new order with implied alpha $\tilde{\alpha}$ to be *added* to the previous implied alpha $\tilde{\alpha}^0$. Therefore, the new impact target is

$$I_t = \frac{1}{2}\left(\tilde{\alpha} + \tilde{\alpha}^0 + \alpha_t - \beta_t^{-1}\alpha_t'\right).$$

Why additive implied alphas?
Presumably, the client is updating orders using a coherent strategy: the update only reflects new information and has no correlation with the order's previous alpha.

Independent alphas are additive.

The next slide covers the *correlated* alpha case.

Combining Portfolio Managers' Overlapping Orders (1/2)

Waelbroeck et al. (2012)

> *"Different portfolio managers are reasonably likely to think coherently, either because a particular investment style is currently in fashion, or because their reasoning is influenced by analysts that publish reports"*

Two extreme cases

(a) If portfolio managers use identical signals, trade only one order to avoid double counting the shared alpha.

(b) If portfolio managers use independent signals, trade a combined order adding the two implied alphas (previous slide).

Combining Portfolio Managers' Overlapping Orders (2/2)

If the trading team can combine information sets,
Criscuolo and Waelbroeck's (2012) alpha profiling method combines the two implied alphas' predictive power considering historical correlations. Otherwise, they must treat the orders as independent.

General outline
Correlation makes the combined alpha sub-additive. Unlike in the independent case, the formula depends on historical trading data. Standard statistical learning methods estimate this sub-additivity from data.

Weekly Summary (1/2)

Communicate a pre-trade cost model
E.g., under the OW model,

$$TC(Q) = -\frac{\Lambda_T}{2} Q^2; \quad \mathbb{E}[Y_T] = \frac{\alpha}{2} Q$$

where

$$\Lambda_T = \frac{\sigma}{(2 + \beta T)\text{adv}}.$$

Consider intraday signals for the trading schedule
Under the OW model, an intraday alpha signal α_t changes the order by

$$Q_T(\alpha) = \int_0^T \frac{\text{adv}}{2\sigma} \left(\beta \alpha_t - \alpha_t'\right) dt.$$

Weekly Summary (2/2)

Micro-alphas are de-centralized signals.
They lead to tactical deviations form the trading schedule. For the OW model,

$$\mathbb{E}\left[Y_T(Q^r)\right] - \mathbb{E}\left[Y_T(Q^*)\right] = -\mathbb{E}\left[\beta \int_0^T \frac{\mathsf{adv}}{\sigma} \left(\delta I_t\right)^2 dt + \frac{\mathsf{adv}}{2\sigma} \left(\delta I_T\right)^2\right].$$

For multiple orders, update the implied alpha.
The myopic relationship updates the target impact state.

Questions?

Next week
Waelbroeck's simulation engine.

(a) Detrending historical prices.

(b) Backtest new strategies.

(c) Application to statistical arbitrage.