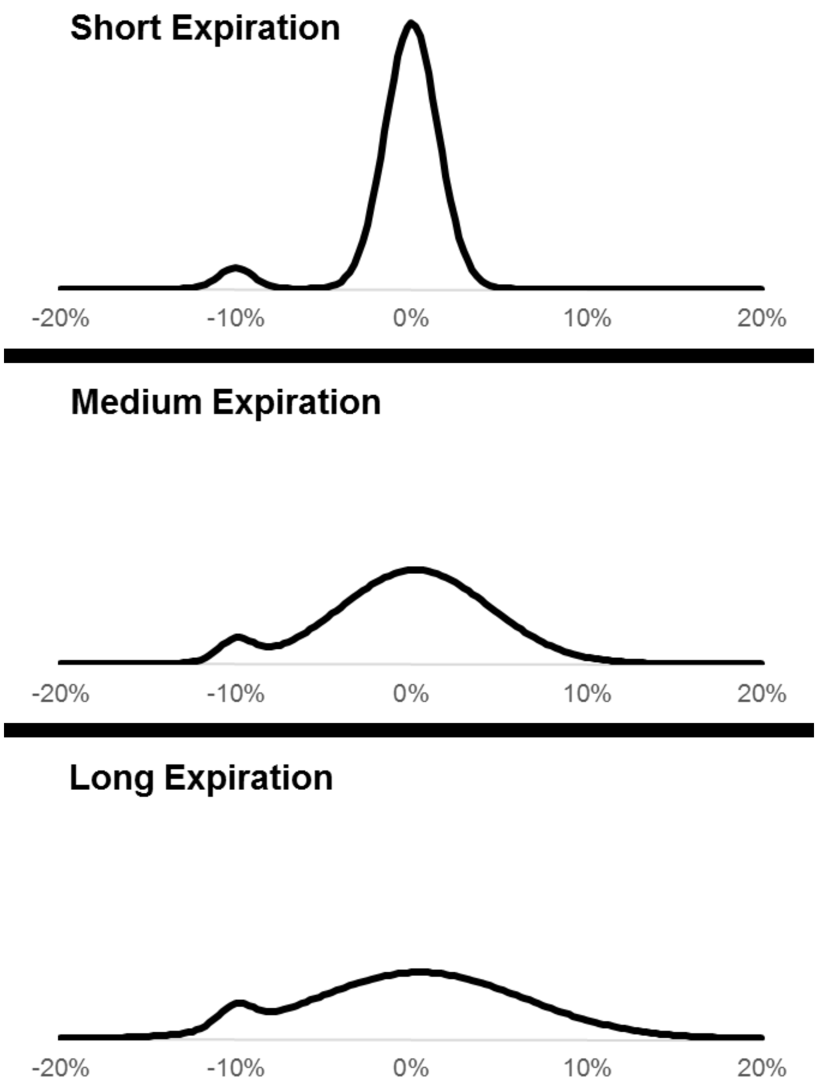
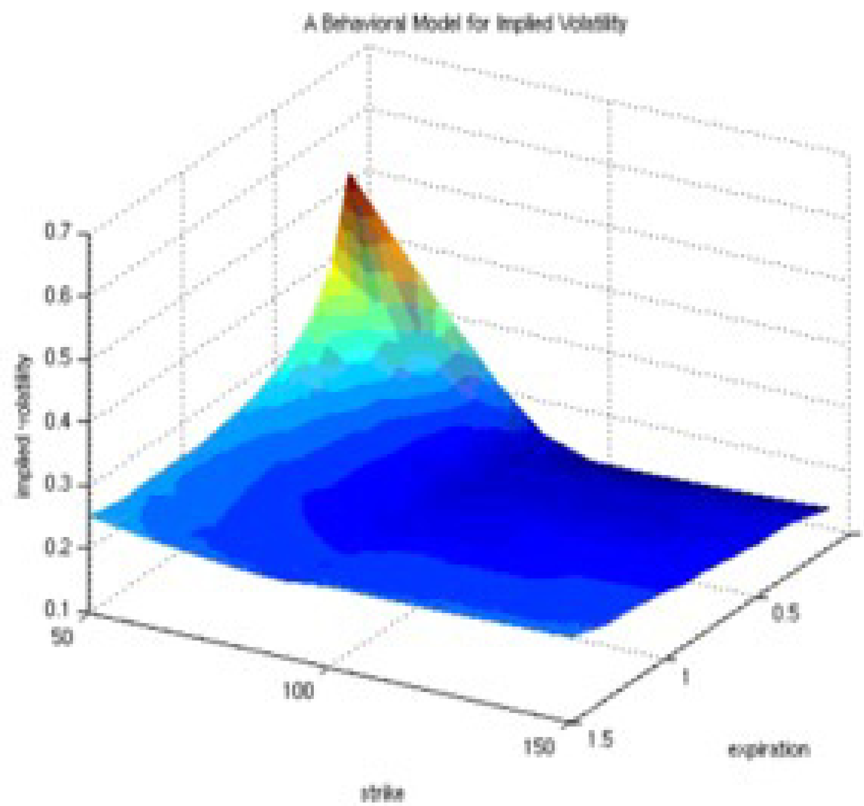
LECTURE 25

JUMP DIFFUSION CONT

25.1 The Effect of Jumps

Just one jump





Although jumps are not diffusive, we want to know what μ and σ their distribution has from a diffusion point of view. How do they modify the diffusion process?

25.2 Modeling Diffusion and Jumps

Discrete binomial approximations over time Δt:

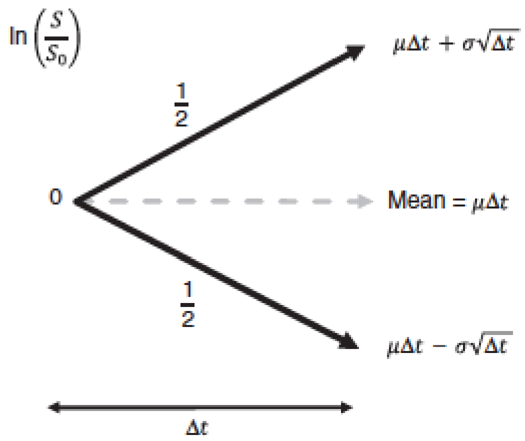


FIGURE 23.3 Binomial Diffusion-Only Model

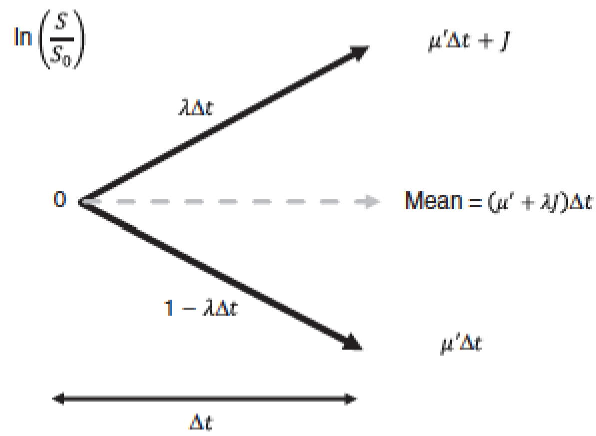


FIGURE 23.4 Binomial Jump Model

For diffusion, **probabilities of both up and down** moves are **finite**, but the **moves** themselves are **small**, of order $\sqrt{\Delta t}$.

With jumps, the **probability of a jump *J* is small**, of order Δt, but the **jump itself is finite**

For risk neutrality, if the stock grows at r, we need $\mu = r - \frac{1}{2}\sigma^2$ as the **calibrated drift of the log diffusion** in order to get the expected value of the stock to grow at the riskless rate.

25.3 Calibration of the Logarithmic Drift of the Jump Process to the Riskless Growth Rate of the Stock (Risk Neutrality)

How does S evolve? The effect of convexity:

$$
\begin{aligned}
E[S] &= (1-\lambda\Delta t)S\exp(\mu'\Delta t) + \lambda\Delta t S\exp(\mu'\Delta t + J) \\
&= S\exp(\mu'\Delta t)[1 + \lambda\Delta t(e^J - 1)] \\
&\approx S\exp\left[\left\{\mu' + \lambda(e^J - 1)\right\}\Delta t\right]
\end{aligned}
$$

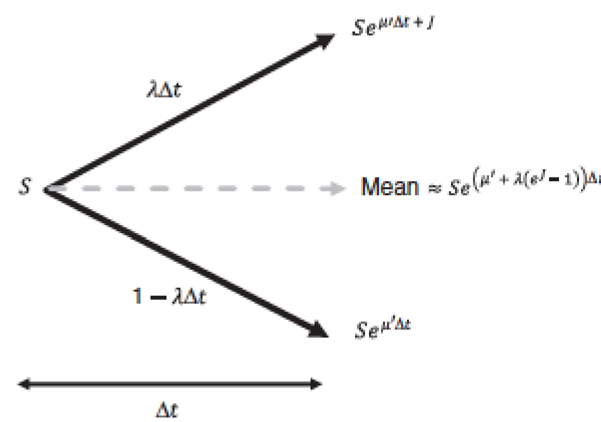


FIGURE 23.5 Binomial Jump Model for Price

A positive jump adds to the drift, a negative jumps lowers the drift.

Thus a calibrated risk-neutral growth would mean

$$
\begin{aligned}
r &= \mu' + \lambda(e^J - 1) \\
\mu' &= r - \lambda(e^J - 1)
\end{aligned}
$$

To maintain risk neutrality and achieve an expected return of r **for the stock**, we have to **compensate the logarithmic drift for the convexity of the jump contribution.**

In continuous-time notation the jump can be written as a Poisson process

$$d\ln S = \mu' dt + J dq$$

Here dq is a jump / Poisson process that is modeled as follows:

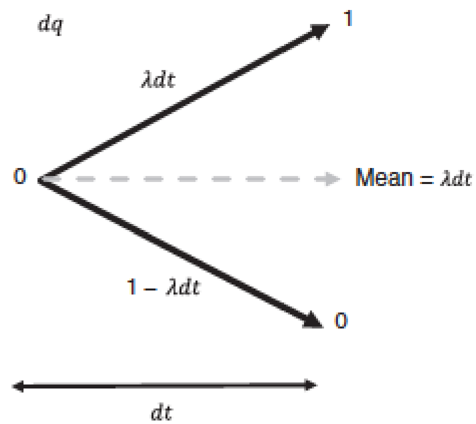


FIGURE 23.6 Binomial Poisson Process

The increment dq takes the values:

1 with probability λdt if a jump occurs
0 with probability $1-\lambda dt$ if no jump occurs
expected value $E[dq] = \lambda dt$

variance $var(dq) = \lambda dt(1-\lambda dt)^2 + (1-\lambda dt)(\lambda dt)^2 = \lambda dt(1-\lambda dt) \to \lambda dt$

25.4 The Poisson Distribution of Jumps

λ = the constant probability per unit time of a jump J occurring in the logarithm of the stock price.

$$P(n, t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

The mean number of jumps $\overline{n(t)}$ during time t is λt.

25.5 Jumps plus Diffusion

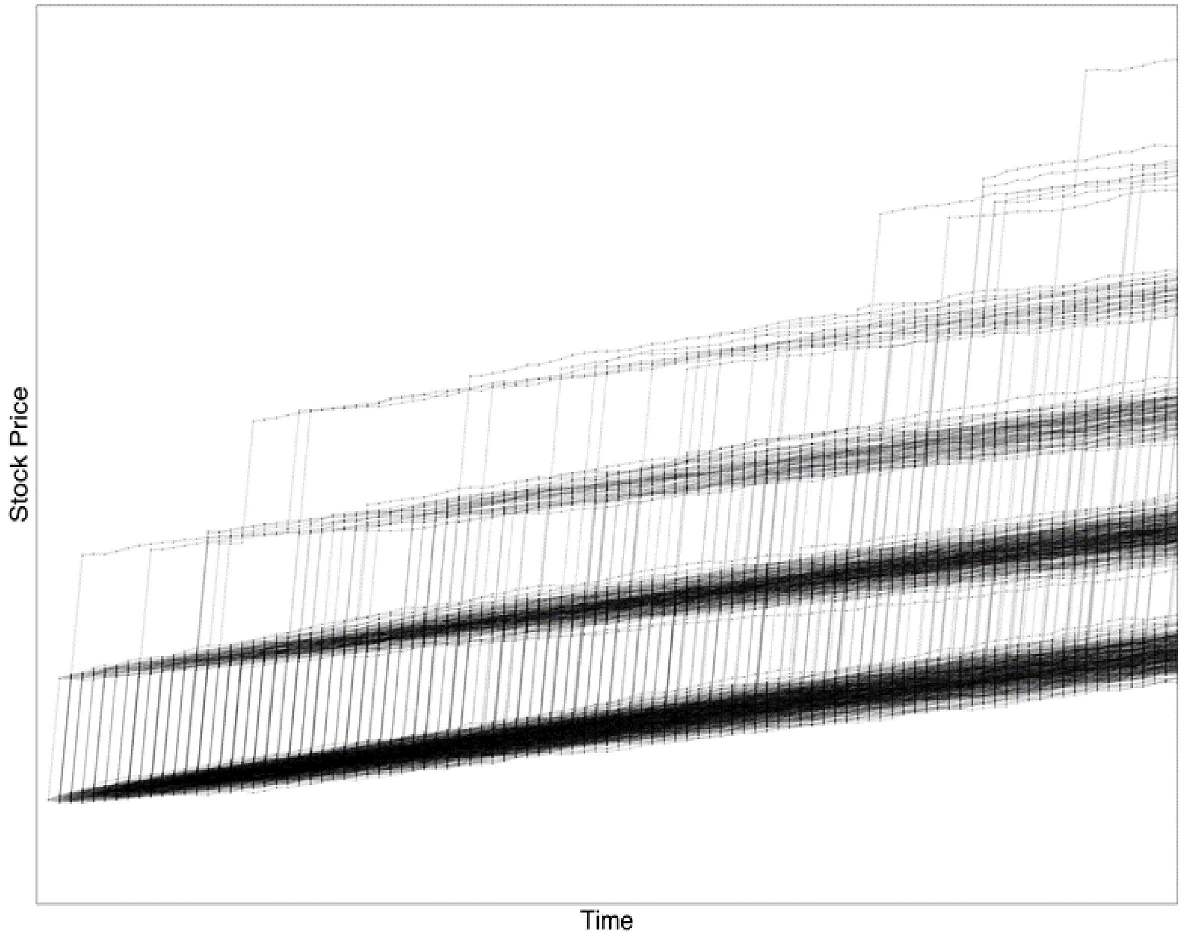
Some comments

- Hedging an option with stochastic volatility requires a stock and another option. Similarly you can hedge an option exactly with stock and a finite number of options for a finite number of jumps of known size,.

- But with an infinite number of possible jumps, you cannot replicate; you can only minimize the variance of the P&L.

- Merton's model of jump-diffusion regards jumps as "abnormal" market events that have to be superimposed upon "normal" diffusion.
  Mandelbrot, and Eugene Stanley and his econophysics collaborators prefer a single model, rather than a "normal" and "abnormal" model.

A Monte Carlo Simulation of Stock Prices in a Jump-Diffusion Model



25.6 Merton's Jump-diffusion Model And Its PDE

Poisson jumps + GBM diffusion, $\frac{dS}{S} = \mu dt + \sigma dZ + Jdq$ - combination of two processes

$$E[dq] = \lambda dt$$
$$var[dq] = \lambda dt$$

J is like a random percentage dividend that lowers or raises the stock price, but it is **not** paid to the stockholder. Later we'll make J a **normal random** variable.

You can derive a partial differential equation for option valuation: option $C(S, t)$ and usual hedged portfolio $\pi = C - nS$ (bad notation -- here n is number of shares shorted, not number of jumps)

$$\begin{aligned} ndS &= nS(\mu dt + \sigma dZ + Jdq) \\ &= n(\mu S dt + \sigma S dZ) + (nJS)dq \end{aligned}$$

$$\begin{aligned} dC = &\left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right) dt \\ &+ \frac{\partial C}{\partial S}(\mu S dt + \sigma S dZ) + [C(S + JS, t) - C(S, t)]\, dq \end{aligned}$$

if diffuses **if jumps**

Keeping terms up to order dt

$$\begin{aligned} d\pi &= dC - ndS \\ &= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right) dt + \left(\frac{\partial C}{\partial S} - n\right)(\mu S dt + \sigma S dZ) \\ &\quad + [C(S + JS, t) - C(S, t) - nJS]\, dq \end{aligned}$$

Choose number of shares *n* to hedge the diffusion: $n = C_S$. We can't hedge everything with it.

$$d\pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \left[C(S + JS, t) - C(S, t) - \frac{\partial C}{\partial S}JS\right]dq$$

The partially hedged portfolio is still risky because of the possibility of jumps.

$$E[d\pi] = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + E\left[C(S + JS, t) - C(S, t) - \frac{\partial C}{\partial S}JS\right]E[dq]$$

Imagine that we can diversify our portfolio over many different stocks and their options, where the stocks have uncorrelated jumps, so that jump risk becomes diversifiable and can be eliminated. Or suppose simply that even though there is some risk, we expect roughly the riskless return if we average over all jumps:

$$E[d\pi] = r\pi dt = r(C - SC_S)dt \qquad E[dq] = \lambda dt$$

So averaging over all jump sizes gives

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + E\left[C(S+JS,t) - C(S,t) - \frac{\partial C}{\partial S}JS\right]\lambda dt$$
$$= r\left(C - S\frac{\partial C}{\partial S}\right)dt$$

Or

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 + r\left(S\frac{\partial C}{\partial S} - C\right) + E\left[C(S+JS,t) - C(S,t) - \frac{\partial C}{\partial S}JS\right]\lambda = 0 \tag{24.10}$$

This is a mixed difference/partial-differential equation for a standard call with terminal payoff $C_T = max(S_T - K, 0)$. For $\lambda = 0$ it reduces to the Black-Scholes equation. We will solve it a little later by the Feynman-Kaç method as an expected discounted value of the payoffs.

I don’t find the diversification argument very compelling, but we will continue with this logic anyhow.

25.7 Trinomial Jump-Diffusion and Compensation

To add jumps one needs a third, trinomial, leg in the tree:

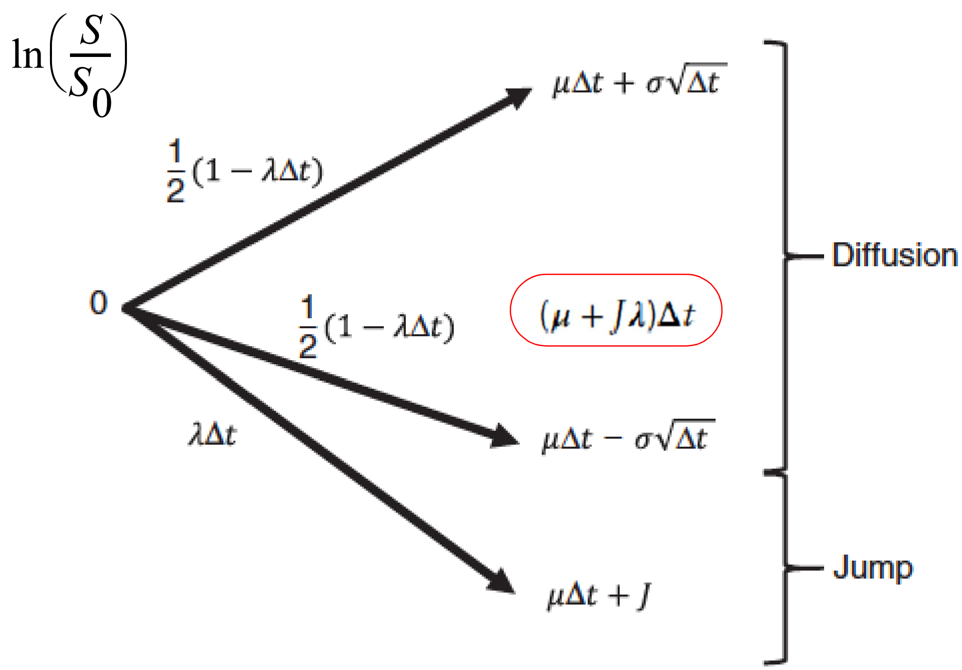


FIGURE 24.2 Trinomial Tree with One Jump

Just as diffusion modifies the drift of the stock price, so do jumps.

μ is the log drift of the diffusion process.

The expected log return after time Δt:

$$E\left[\ln\left(\frac{S}{S_0}\right)\right] = \frac{1}{2}(1-\lambda\Delta t)\left(\mu\Delta t + \sigma\sqrt{\Delta t}\right) + \frac{1}{2}(1-\lambda\Delta t)\left(\mu\Delta t - \sigma\sqrt{\Delta t}\right) + \lambda\Delta t(\mu\Delta t + J) = (\mu + J\lambda)\Delta t \qquad (24.11)$$

Thus the effective **log drift** of the jump-diffusion process will be $\mu_{JD} = \mu + J\lambda$.

The variance at time Δt is

$$\begin{aligned} var &\equiv \left(\frac{1-\lambda\Delta t}{2}\right)[\sigma\sqrt{\Delta t} - J\lambda\Delta t]^2 + \left(\frac{1-\lambda\Delta t}{2}\right)[\sigma\sqrt{\Delta t} + J\lambda\Delta t]^2 \\ &\quad + \lambda\Delta t[J(1-\lambda\Delta t])]^2 \\ &= \left(\frac{1-\lambda\Delta t}{2}\right)[2\sigma^2\Delta t + 2J^2\lambda^2(\Delta t)^2] + \lambda\Delta t J^2(1-\lambda\Delta t)^2 \\ &= (1-\lambda\Delta t)[\sigma^2\Delta t] + (1-\lambda\Delta t)J^2\lambda\Delta t(\lambda\Delta t + 1 - \lambda\Delta t) \\ &= (1-\lambda\Delta t)[\sigma^2 + J^2\lambda]\Delta t \end{aligned}$$

so that, as $\Delta t \to 0$, the variance of the log jump diffusion process is $\sigma_{JD}^2 = [\sigma^2 + J^2\lambda]$,

the sum of the diffusion variance plus the expected jump variance. The drift and variance are both affected by the fractional jump J and its probability λ of occurring per unit time.

25.8 The Compensated Logarithmic Drift of the Diffusion Process for Risk Neutrality

How must we choose/calibrate the diffusion log drift μ so that $E[dS] = Srdt$?

First let's compute the stock growth rate under jump diffusion.

$$E\left[\frac{S}{S_0}\right] = \frac{(1-\lambda\Delta t)}{2}e^{\mu\Delta t + \sigma\sqrt{\Delta t}} + \frac{(1-\lambda\Delta t)}{2}e^{\mu\Delta t - \sigma\sqrt{\Delta t}} + \lambda\Delta t e^{\mu\Delta t + J}$$

$$= e^{\mu\Delta t}\left[\frac{(1-\lambda\Delta t)}{2}\left(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}\right) + \lambda\Delta t e^{J}\right]$$

One can show by expanding this to keep terms of order Δt that

$$E\left[\frac{S}{S_0}\right] = \exp\left(\left\{\mu + \frac{\sigma^2}{2} + \lambda(e^J - 1)\right\}\Delta t\right) + \text{higher order terms}$$

so that, if we want the stock to grow risk-neutrally, we must set $r = \mu + \frac{\sigma^2}{2} + \lambda(e^J - 1)$

$$\mu_{JD} = r - \underbrace{\frac{\sigma^2}{2}}_{\text{diffusion compensation}} - \underbrace{\lambda(e^J - 1)}_{\text{jump compensation}}$$

the log drift of the diffusion process so that stock grows at r

The option value is $C_{JD} = e^{-r\tau}\sum_{n=0}^{\infty}\frac{(\lambda\tau)^n}{n!}e^{-\lambda\tau}E\left[\max\left(S_T^n - K, 0\right)\right]$ and we'll show that

$$C_{JD} = e^{-\bar{\lambda}\tau}\sum_{n=0}^{\infty}\frac{(\bar{\lambda}\tau)^n}{n!}C_{BS}\left(S, K, \tau, \sqrt{\sigma^2 + \frac{n\sigma_J^2}{\tau}}, r + \frac{n\left(\bar{J} + \frac{1}{2}\sigma_J^2\right)}{\tau} - \lambda\left(e^{\bar{J} + \frac{1}{2}\sigma_J^2} - 1\right)\right)$$

where $\bar{\lambda} = \lambda e^{\bar{J} + \frac{1}{2}\sigma_J^2}$

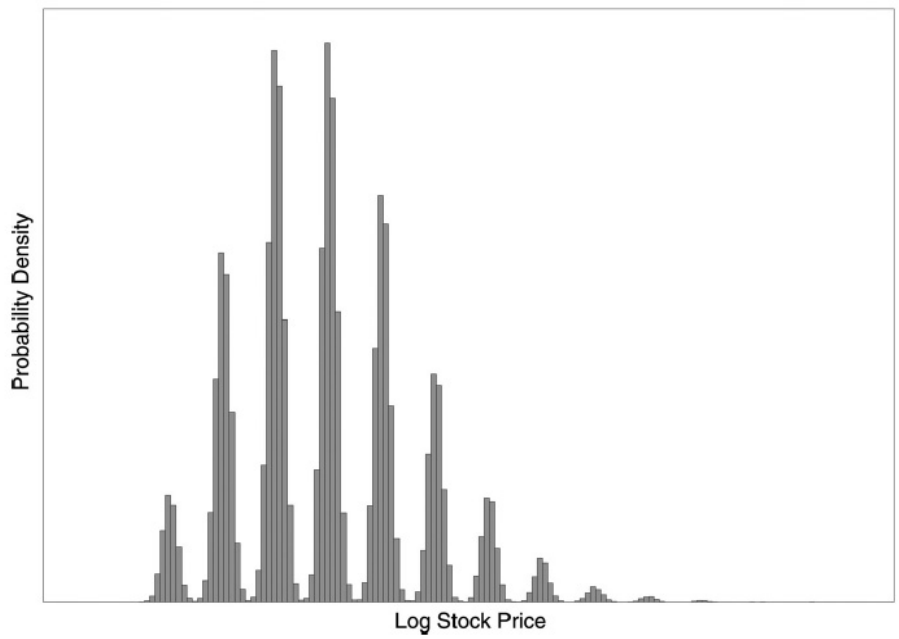


FIGURE 24.4 Multimodal Probability Density Function

25.9 Valuing a Call in the Merton Jump-Diffusion Model

The process we are considering is $\frac{dS}{S} = \mu dt + \sigma dZ + J dq$

where
$$E[dq] = \lambda dt$$
$$var[dq] = \lambda dt$$

J is assumed to be a fixed jump size now, but will later be generalized to a normal random variable.

Risk neutral diffusion drift: $$\mu = r - \frac{\sigma^2}{2} - \lambda(e^J - 1)$$

The value of a standard call in this model is given by

$$C_{JD} = e^{-r\tau} E[(S_T - K, 0)]$$

The risk-neutral terminal value of the stock price is $S_T = Se^{\mu\tau + Jq + \sigma\sqrt{\tau}Z}$

Sum over 0, 1, ... n ... jumps plus the diffusion, where the probability of n jumps $\frac{(\lambda\tau)^n}{n!}e^{-\lambda\tau}$

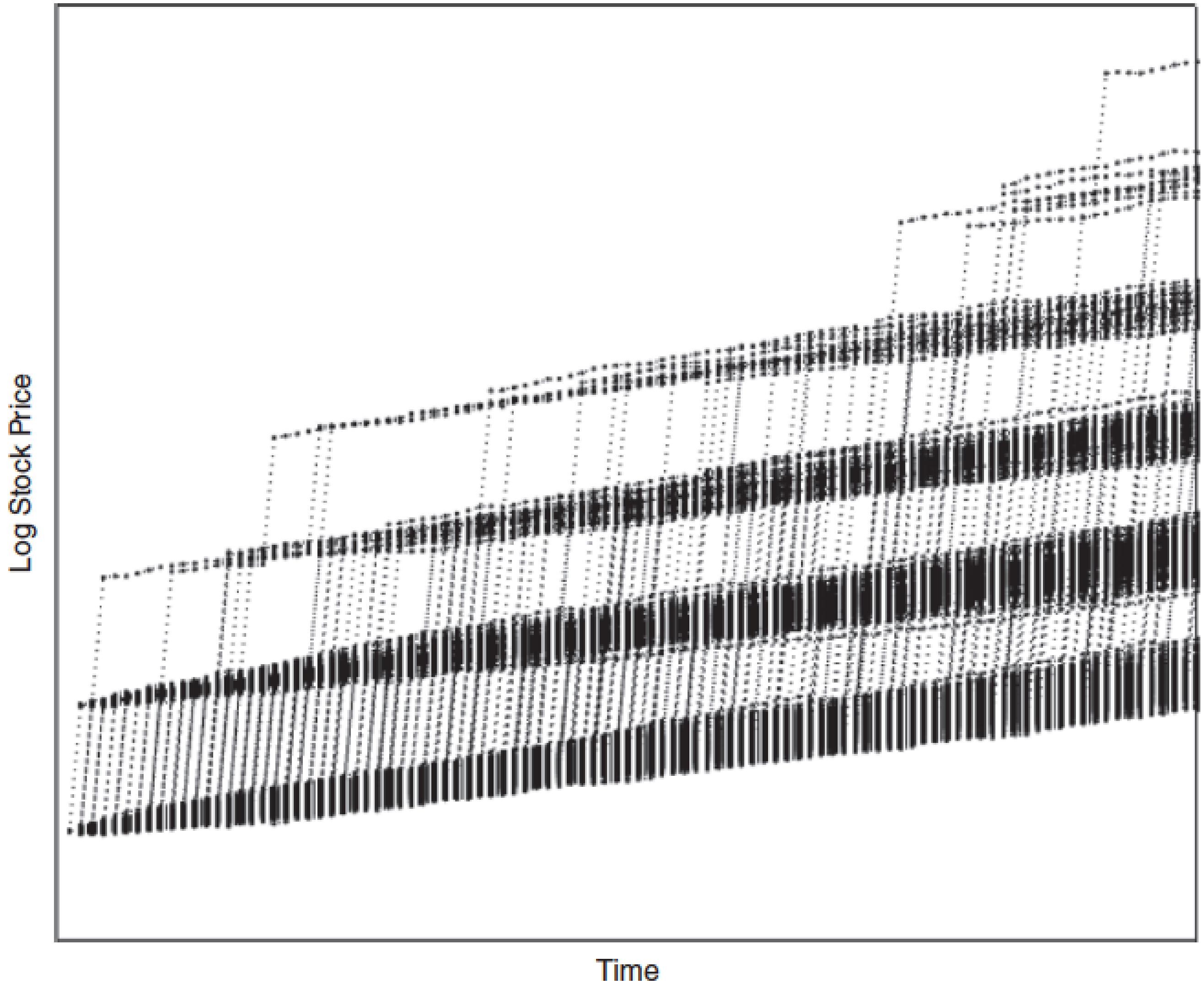


FIGURE 24.3 A Monte Carlo Simulation of the Log Stock Prices in the Jump-Diffusion Model

Thus,

$$C_{JD} = e^{-r\tau} \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} E\left[\max\left(S_T^n - K, 0\right)\right]$$

where S_T^n is the terminal lognormal distribution of the stock price that started with initial price S and diffused with log drift $\mu = r - \frac{\sigma^2}{2} - \lambda(e^J - 1)$ and underwent n jumps as well as the diffusion.

Each term is an expectation over a lognormal stock price that, after time τ, has undergone n jumps in addition to the diffusion log drift μ, and therefore effectively diffused at a rate μ_n given by the logarithmic drift μ plus the n jumps added to it

$$\mu_n = \mu + \frac{nJ}{\tau} = r - \frac{\sigma^2}{2} - \lambda(e^J - 1) + \frac{nJ}{\tau}$$

where the last term is divided by τ because the jumps are not τ-dependent diffusions drifts.

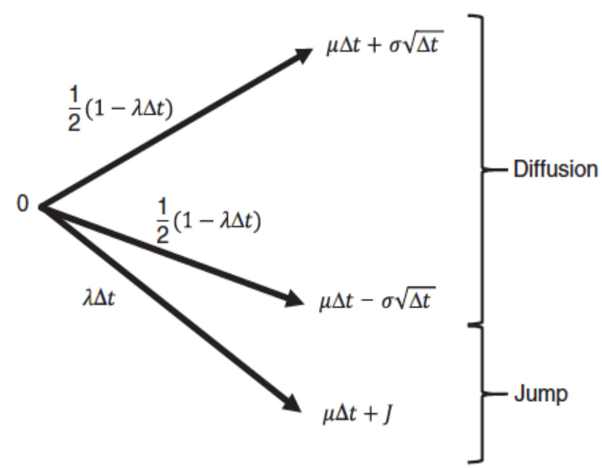


FIGURE 24.2 Trinomial Tree with One Jump

We can write $$\mu_n \equiv r_n - \frac{\sigma^2}{2}$$

where $$r_n = r - \lambda(e^J - 1) + \frac{nJ}{\tau}$$

$r_n = \mu_n + \frac{\sigma^2}{2}$ is the log drift **plus** the $\frac{\sigma^2}{2}$ Ito term that together correspond to the **expected growth rate of the stock price** (not the log of the stock) under diffusion plus exactly n jumps.

$r_n = \mu_n + \frac{\sigma^2}{2}$ is the growth rate of the stock price with diffusion and exactly n jumps, and plays the role of the risk-neutral rate for each n-jump diffusion, which is like a n-**shifted BS diffusion**.

Now recall $$C_{\mathrm{JD}} = e^{-r\tau} \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} E\left[\max\left(S_T^n - K, 0\right)\right]$$

Recall from BSM that if the log grows at $r - \sigma^2/2$ then the stock grows at r and we simply enter r into the BSM formula for the option; the formula itself puts in the $\sigma^2/2$ in $d_{1,2}$.

Analogous to BSM, for the n-**shifted BS diffusion**,

$$C_{BSM}(S, K, \tau, \sigma, r_n) \equiv e^{-r_n\tau} E\left[max\left(S_T^n - K, 0\right)\right] \qquad E\left[\max\left(S_T^n - K, 0\right)\right] = e^{r_n\tau} C_{\mathrm{BSM}}(S, K, \tau, \sigma, r_n)$$

So

$$C_{JD} = e^{-r\tau} \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} e^{r_n\tau} C_{BS}(S, K, \tau, \sigma, r_n)$$

$$= e^{\cancel{-r\tau}} \sum_{n=0}^{\infty} \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} e^{\left(\cancel{r} - \lambda(e^J - 1) + \frac{nJ}{\tau}\right)\tau} C_{BS}(S, K, \tau, \sigma, r_n)$$

$$= e^{-(\lambda e^J \tau)} \sum_{n=0}^{\infty} \frac{(\lambda e^J \tau)^n}{n!} C_{BS}\left(S, K, \tau, \sigma, r + \frac{nJ}{\tau} - \lambda(e^J - 1)\right)$$

Writing $\bar{\lambda} = \lambda e^J$ as the "effective" probability of jumps, we obtain

$$C_{\mathrm{JD}} = e^{-\bar{\lambda}\tau} \sum_{n=0}^{\infty} \frac{(\bar{\lambda}\tau)^n}{n!} C_{BSM}\left(S, K, \tau, \sigma, r - \lambda(e^J - 1) + \frac{nJ}{\tau}\right)$$

. This is a **mixing formula.** Now generalize from one to a **normal distribution of jumps** with

$$J \sim N(\mu_J, \sigma_J^2) \qquad E[J] = \bar{J} \qquad var[J] = \sigma_J^2$$

Then
$$E[e^J] = e^{\bar{J} + \frac{1}{2}\sigma_J^2}$$

Incorporating the expectation over this distribution of jumps has two effects:

- J gets replaced by $\bar{J} + \frac{1}{2}\sigma_J^2$
- second, the variance of the jump process adds to the variance of the entire distribution in the Black-Scholes formula by blurring the mean of each n-jump subdistribution, so that we must replace σ^2 by $\sigma^2 + \frac{n\sigma_J^2}{\tau}$ because n jumps adds $\frac{n\sigma_J^2}{\tau}$ amount of variance. (The division by τ is necessary because variance is defined in terms of geometric Brownian motion and grows with time, but the variance of normally distributed J is independent of time.)

The general formula is therefore:

$$C_{JD} = e^{-\bar{\lambda}\tau} \sum_{n=0}^{\infty} \frac{(\bar{\lambda}\tau)^n}{n!} C_{BS}\left(S, K, \tau, \sqrt{\sigma^2 + \frac{n\sigma_J^2}{\tau}}, r + \frac{n\left(\bar{J} + \frac{1}{2}\sigma_J^2\right)}{\tau} - \lambda\left(e^{\bar{J} + \frac{1}{2}\sigma_J^2} - 1\right)\right)$$

where $\bar{\lambda} = \lambda e^{\bar{J} + \frac{1}{2}\sigma_J^2}$

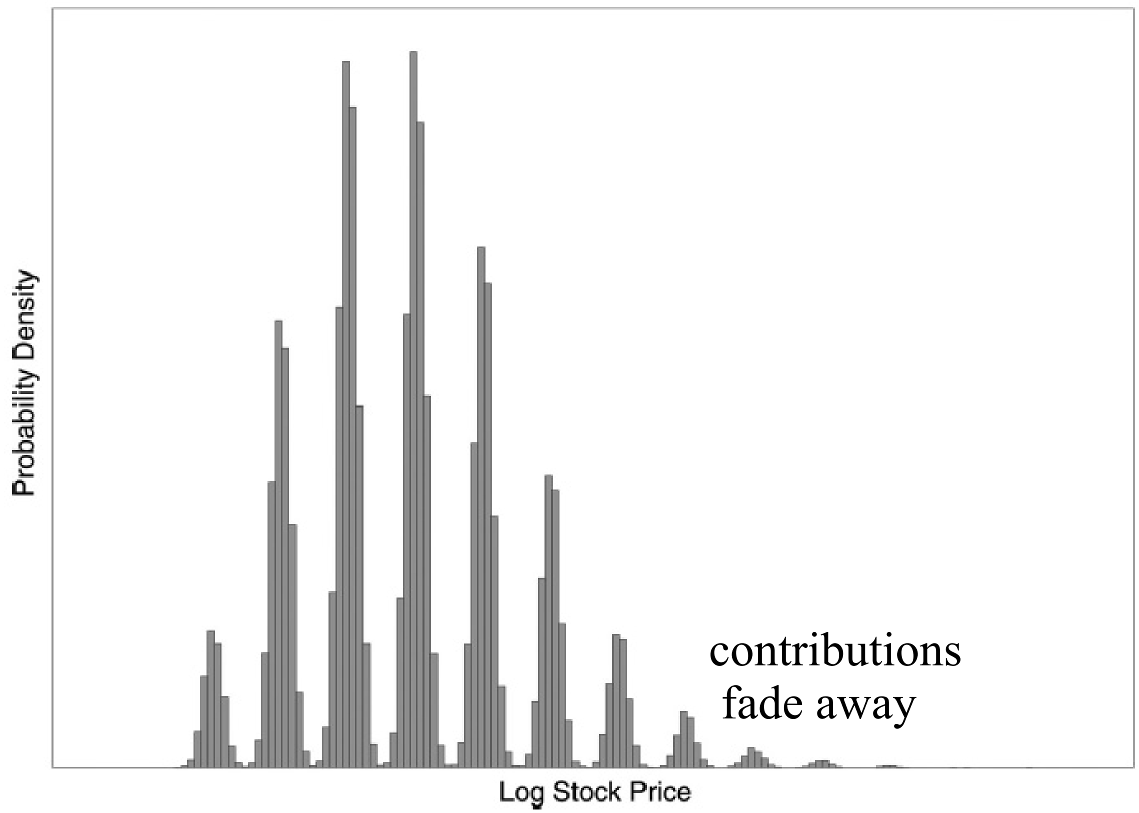
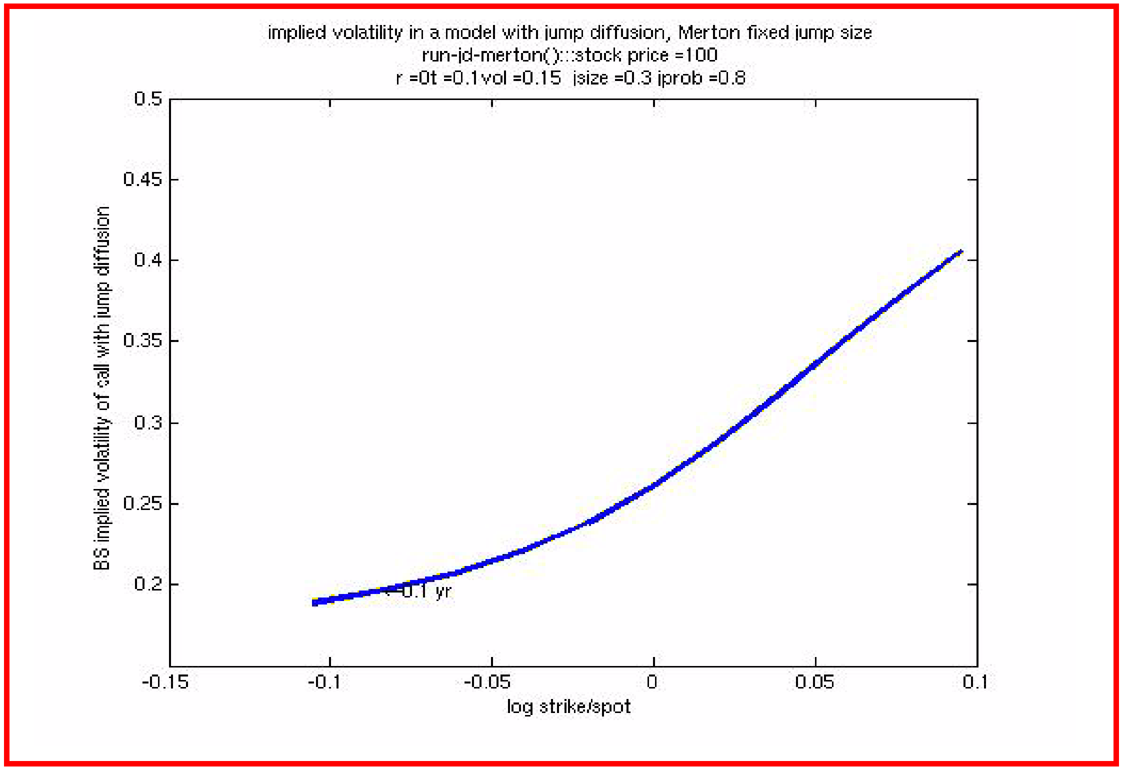


FIGURE 24.4 Multimodal Probability Density Function

Example

- Calculate the value of an option in the Merton model. Consider a stock with current price 100 that has a diffusion volatility of 15% and a probability of 0.8 jumps per year of making a jump of fixed size +30%. Assume interest rates and dividends are zero. Assume risk-neutral pricing.

  Plot the implied volatility Black-Scholes smile for a call option with 0.1 year to expiration, for all strikes between 90 and 110. [30 points]



25.10 The Jump-Diffusion Smile (Qualitatively)

- Jump-diffusion tends to produce a steep realistic very short-term smile in strike or delta, because the jump happens instantaneously and moves the stock price by a large amount.
- Stochastic volatility models, in contrast, have difficulty producing a very steep short-term smile unless volatility of volatility is very large. (Local vol can always do it.)
- The long-term smile in a jump-diffusion model tends to be flat, because at large times the effect on the distribution of the diffusion of the stock price, whose variance grows like $\sigma^2\tau$, tends to overwhelm the diminishing Poisson probability of large moves via many jumps. Thus jumps produce steep short-term smiles and flat long-term smiles.
- Recall that mean-reverting stochastic volatility models also produce flat long-term smiles.
- Jumps of a fixed size tend to produce multi-modal densities centered around the jump size. Jumps of a higher frequency tend to wash out the multi modal density and produce a smoother distribution of multiply overlaid jumps at longer expirations.
- A higher jump frequency λ produces a steeper smile at longer expiration, because jumps are more probable and therefore are more likely to occur into the future as well.
- Andersen and Andreasen claimed that a jump-diffusion model can be fitted to the S&P 500 skew with a diffusion volatility of about 17.7%, a jump probability of $\lambda = 8.9\%$, an expected jump size of 45% and a variance of the jump size of 4.7%. A jump this size and with this probability seems excessive when compared to real markets, and suggests that the options market is paying a greater risk premium for protection against crashes.

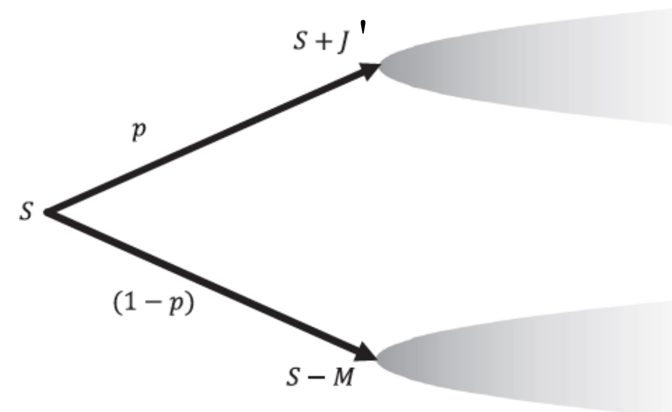
25.11 A Simplified Perturbative Treatment Of The Atm Jump-diffusion Skew With A Small Probability Of A Large Single Jump Up (Assume r = 0)

Simple mixing: Note different definition of J' below.

J' - here, a big immediate **jump up** in the stock price, (not the log of the stock price as earlier), with a small probability p

M - a small move down with a large probability (1-p).

Thereafter by diffusion with volatility σ.



We assume that p is small, one jump or zero only.

Risk-neutrality $S = p(S+J') + (1-p)(S-M)$

$M = \frac{p}{1-p}J' \approx pJ'$ to leading order in p, $M \ll J'$

The mixing formula

$$\begin{aligned} C_{JD} &= p \times C_{BSM}(S+J',\sigma) + (1-p)C_{BSM}(S-M,\sigma) \\ &\approx p \times C_{BSM}(S+J',\sigma) + (1-p)C_{BSM}(S-pJ',\sigma) \end{aligned}$$

Mixing

to leading order in p, where $C_{BSM}(S, \sigma)$ is the Black-Scholes option price for strike K and volatility σ

Assume a regime of the following dimensionless numbers

$$p \ll \sigma\sqrt{\tau} \ll \frac{J'}{S}$$

Small probability *p* of a **large jump** *J'*, where **"small" and "large" mean relative to diffusion** standard deviation $\sigma\sqrt{\tau}$. Also assume strike is close to the at-money, $K \sim S$. Keep largest terms.

Since $J'/S \gg \sigma\sqrt{\tau}$, the **positive jump** J' takes the call deep into the money, so that the first call $C(S+J', \sigma)$ in the mixing equation becomes equal to a forward whose value is

$$C(S+J', \sigma) \to (S+J') - Ke^{-r\tau}$$

For simplicity from now on we will also assume that $r = 0$.

$$C_{JD} = p \times (S + J' - K) + (1 - \cancel{p})C_{BSM}(S - pJ', \sigma)$$

Because *pJ* is small in the second term, the option is close to at-the-money, with

$$C_{BSM}(S, \sigma) \sim S\sigma\sqrt{\tau}$$

so $pC_{BSM}(S, \sigma)$ is of order $p\sigma\sqrt{\tau}S$ much smaller than *pS* in the first term.

Therefore

$$
\begin{aligned}
C_{\mathrm{JD}} &\approx p \times (S + J' - K) + C_{\mathrm{BSM}}(S - pJ', \sigma) \\
&\approx p \times (S - K + J') + C_{\mathrm{BSM}}(S, \sigma) - pJ' \frac{\partial C_{\mathrm{BSM}}}{\partial S} \\
&\approx C_{\mathrm{BSM}}(S, \sigma) + p \times \left[S - K + J' \left(1 - \frac{\partial C_{\mathrm{BSM}}}{\partial S} \right) \right] \\
&\approx C_{\mathrm{BSM}}(S, \sigma) + p \times [S - K + J'(1 - N(d_1))]
\end{aligned}
$$

Now if K is close to at-the-money, we know that $N(d_1) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \frac{\ln S/K}{\sigma\sqrt{\tau}}$

Therefore

$$
C_{\mathrm{JD}} \approx C_{\mathrm{BSM}}(S, \sigma) + p \times \left[(\cancel{S -} K) + J' \left(\frac{1}{2} - \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma\sqrt{\tau}} \ln\left(\frac{S}{K}\right) \right) \right]
$$

Now **close to at-the-money**, the $S - K$ term in the above equation is negligible compared with both J' and the $J' \ln S/K$ if $\sigma\sqrt{\tau}$ is small, because

$$J'\frac{\ln S/K}{\sigma\sqrt{\tau}} = J'\frac{\ln\left(1+\frac{S-K}{K}\right)}{\sigma\sqrt{\tau}} \approx \frac{J'}{K}\left\{\frac{S-K}{(\sigma\sqrt{\tau})}\right\} \approx O\left(\frac{S-K}{(\sigma\sqrt{\tau})}\right) » S-K[$$

Therefore

$$C_{JD} \approx C_{BSM}(S,\sigma) + pJ'\left(\frac{1}{2} - \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{\tau}}\ln\left(\frac{S}{K}\right)\right)$$

This is the approximate formula for the jump-diffusion call price close to at the money in the case where we consider only one jump under the conditions $p « \sigma\sqrt{\tau} « (J'/S)$, i.e. a small probability (relative to volatility) of a large one-sided **positive** jump. Recall that $C(S, \sigma)$ is the Black-Scholes option price.

Someone using the Black-Scholes model will quote the jump-diffusion price as $C_{BSM}(S, \Sigma)$ where

$$\begin{aligned} C_{JD} &= C_{BSM}(S,\Sigma) \\ &= C_{BSM}(S,\sigma+\Sigma-\sigma) \\ &\approx C_{BSM}(S,\sigma) + \frac{\partial C_{BSM}}{\partial\sigma}(\Sigma-\sigma) \end{aligned}$$

Comparing equations we obtain

$$\Sigma \approx \sigma + \frac{pJ'\left(\frac{1}{2} - \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{\tau}}\ln\left(\frac{S}{K}\right)\right)}{\frac{\partial C_{\mathrm{BSM}}}{\partial \sigma}}$$

For options close to at the money, $\frac{\partial C_{\mathrm{BSM}}}{\partial \sigma} = S\sqrt{\tau}N'(d_1) \approx \frac{S\sqrt{\tau}}{\sqrt{2\pi}}$, so that

$$\begin{aligned}\Sigma &\approx \sigma + pJ'\frac{\sqrt{2\pi}}{S\sqrt{\tau}}\left(\frac{1}{2} - \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{\tau}}\ln\left(\frac{S}{K}\right)\right) \\ &\approx \sigma + \frac{pJ'}{S\sqrt{\tau}}\left(\sqrt{\frac{\pi}{2}} + \frac{1}{\sigma\sqrt{\tau}}\ln\left(\frac{K}{S}\right)\right)\end{aligned}$$

(Recall p for one jump is proportional to time in Merton model, so skew doesn't really diverge for short times.

We see that the jump-diffusion smile **is linear in** $\ln K/S$ when the strike is close to being at-the money, and the implied volatility increases when the strike increases, as we would have expected for a positive jump J'.

Improvement: We want to do a bit better and use the mixing formula from the Merton jump-diffusion model that we derived earlier. We have been using here J' as the jump in S, but in our previous derivation of the general jump-diffusion formula, we used J as the jump in $\ln(S)$. Fixing notation:

So $(\ln S) + J = \ln(S + J')$ or $J = \ln(1 + J'/S)$ or $J' = S(e^{J} - 1)$.

In the Merton model $p = \bar{\lambda}\tau e^{-\bar{\lambda}\tau}$ where $\bar{\lambda} = \lambda e^{J} = \lambda e^{\ln\left(1 + \frac{J'}{S}\right)} = \lambda\left(1 + \frac{J'}{S}\right)$ so

$$\Sigma \approx \sigma + \frac{\bar{\lambda}\sqrt{\tau}e^{-\bar{\lambda}\tau}J'}{S}\left(\sqrt{\frac{\pi}{2}} + \frac{1}{\sigma\sqrt{\tau}}\ln\left(\frac{K}{S}\right)\right)$$
$$\approx \sigma + \bar{\lambda}e^{-\bar{\lambda}\tau}\frac{J'}{S}\left(\sqrt{\frac{\pi\tau}{2}} + \frac{1}{\sigma}\ln\left(\frac{K}{S}\right)\right)$$

As $\tau \to 0$ for short expirations, the implied volatility smile becomes

$$\Sigma(K, S) \approx \sigma + \bar{\lambda}\frac{J'}{S}\frac{1}{\sigma}\ln\left(\frac{K}{S}\right)$$

This is a finite smile at small times proportional to the percentage jump and its probability, and linear in $\ln K/S$. The greater the expected jump, the greater the skew. This is a model suitable for explaining the short-term equity index skew.

For larger expirations the skew flattens proportional to $e^{-\overline{\lambda\tau}}$. Asymmetric jumps produce a steep short-term skew and a flatter long-term skew, Here is the skew for a jump probability of 0.1 per year and a percentage jump size of 40%, with a diffusion volatility of 10%, for options with 0.1 years to expiration.

$$\Sigma \approx \sigma + \frac{\bar{\lambda}\sqrt{\tau}e^{-\bar{\lambda}\tau}J'}{S}\left(\sqrt{\frac{\pi}{2}} + \frac{1}{\sigma\sqrt{\tau}}\ln\left(\frac{K}{S}\right)\right)$$

$$\approx 0.1\ \ 2 + 0.56 \times \ln\left(\frac{K}{S}\right)$$

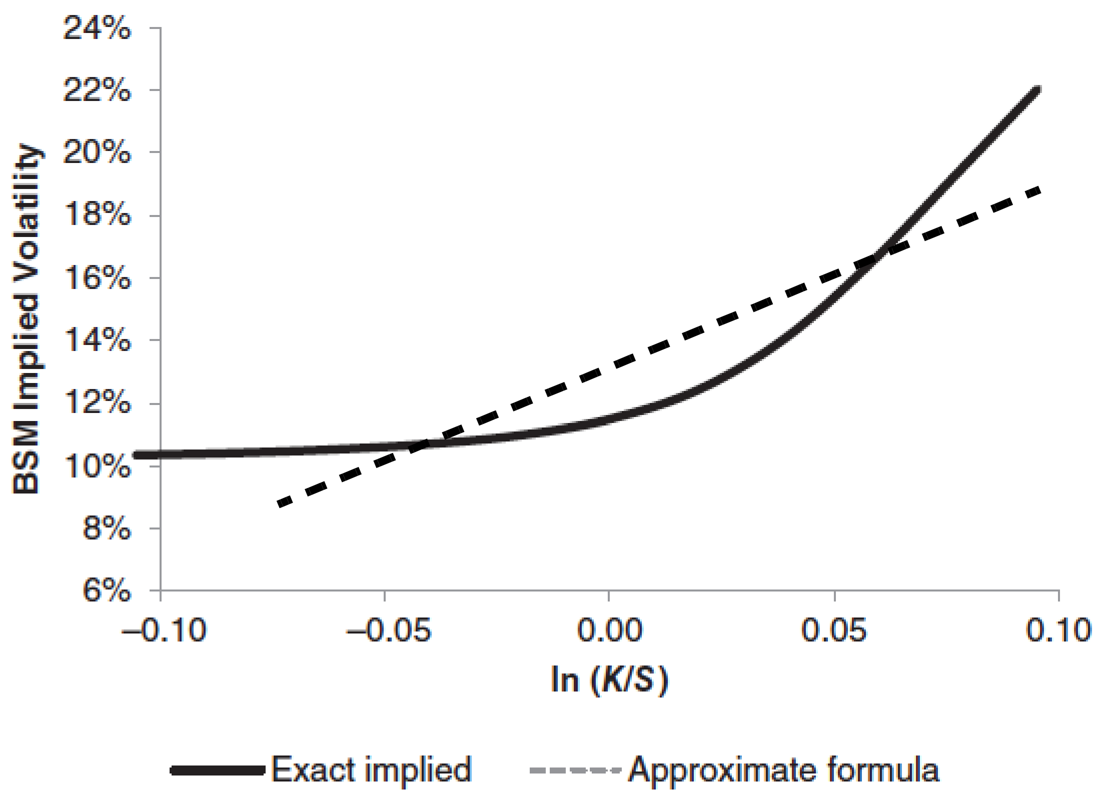


FIGURE 24.6 Jump-Diffusion Smile with a Positive Jump

Conclusion

First understand how realized and implied volatility varies in your market.

Understand the effect of these variations on Black-Scholes values.

Look for static hedges.

Then study dynamic hedging.

Stochastic + local volatility is perhaps most reasonable combination.

Try to build the simplest model with the least number of moving parts that can match most of the features of your market.

There is no easy solution that avoids thinking.