Selected Levy PDF Calculus

$$\mathcal{P}_{\alpha,\gamma}(x,\tau) = \frac{1}{\pi}\int_0^{+\infty} \underbrace{e^{-\gamma\tau q^{\alpha}}}_{\chi(q),\ \text{Levy characteristic fn}} \cdot \cos qx \cdot dq \;=$$

$\alpha \to 2$

$\gamma \to \sigma$

Levy Gauss

$x \to \Delta p$

$\tau \to \Delta t$

in new variables $\begin{cases} \tilde{q} = q\cdot(\gamma\tau)^{1/\alpha} \\ \tilde{x} = \dfrac{x}{(\gamma\tau)^{1/\alpha}} \end{cases}$ - self-similar variable

$$= \frac{1}{(\gamma\tau)^{1/\alpha}} \cdot F_\alpha(\tilde{x}), \quad \text{where } F_\alpha(\tilde{x}) \equiv \frac{1}{\pi}\int e^{-\tilde{q}^{\alpha}} \cdot \cos(\tilde{q}\tilde{x}) \times d\tilde{q}.$$

Taylor series expansion:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

$$\int_0^{+\infty} q^{2n} e^{-q^{\alpha}} dq = \frac{1}{\alpha}\cdot\int_0^{+\infty} p^{\frac{2n+1}{\alpha}-1} \cdot e^{-p} \cdot dp =$$

$$= \frac{1}{\alpha}\cdot\Gamma\left(\frac{2n+1}{\alpha}\right), \quad \begin{pmatrix} q^{\alpha} = p \\ q = p^{1/\alpha} \end{pmatrix} \quad \text{where: } \Gamma(z) \equiv \int_0^{\infty} t^{z-1} \cdot e^{-t} \cdot dt$$

- gamma function

We obtained:

$$\int_0^{+\infty} q^{2n}\cdot e^{-q^{\alpha}}\cdot dq = \frac{1}{\alpha}\cdot\Gamma\left(\frac{2n+1}{\alpha}\right).$$

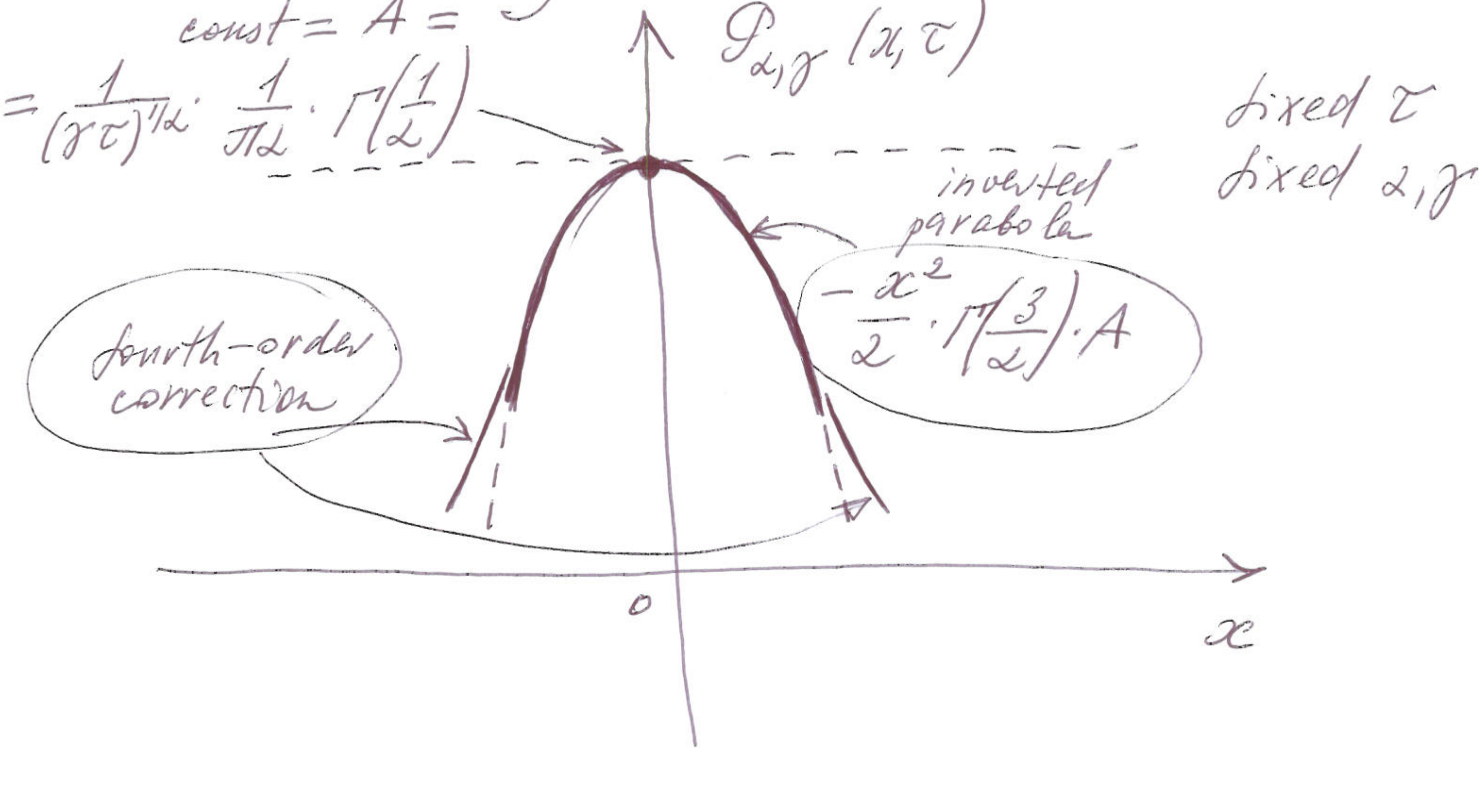
Therefore,

$$\boxed{P_{\alpha,\gamma}(x,\tau) = \frac{1}{(\gamma\tau)^{1/\alpha}}\cdot\frac{1}{\pi}\sum_{n=0}^{+\infty} x^{2n}\cdot\frac{(-1)^n}{(2n)!}\cdot\left(\int_0^{\infty} e^{-q^{\alpha}}\cdot q^{2n}\cdot dq\right) = \frac{1}{(\gamma\tau)^{1/\alpha}}\cdot\frac{1}{\pi\alpha}\cdot\sum_{n=0}^{\infty} x^{2n}\cdot\Gamma\left(\frac{2n+1}{\alpha}\right)\cdot\frac{(-1)^n}{(2n)!}}$$

This is asymptotic series expansion for Levy P.D.F.

$$P_{\alpha,\gamma}(x,\tau) = \frac{1}{(\gamma\tau)^{1/\alpha}}\cdot\frac{1}{\pi\alpha}\cdot\Big\{\underbrace{\Gamma\left(\frac{1}{\alpha}\right)}_{n=0} - \underbrace{\frac{x^2}{2}\cdot\Gamma\left(\frac{3}{\alpha}\right)}_{n=1} + \frac{x^4}{24}\cdot\Gamma\left(\frac{5}{\alpha}\right) - \ldots\Big\}.$$

const $= A = \frac{1}{(\gamma\tau)^{1/\alpha}}\cdot\frac{1}{\pi\alpha}\cdot\Gamma\left(\frac{1}{\alpha}\right)$



Moments or Structure Functions:

$$S_\nu(\tau) = \overline{|x|^\nu} = \int_{-\infty}^{+\infty} |x|^\nu \cdot \mathcal{P}_{\alpha,\mu}(x,\tau) \cdot dx =$$

$$= \frac{2}{\pi} \int_0^{+\infty} dx \int_0^{+\infty} dq \cdot x^\nu \cdot e^{-\gamma \tau q^\alpha} \cdot \cos qx =$$

$$q = \frac{\tilde{q}}{(\gamma\tau)^{1/\alpha}} \; ; \; \tilde{q} = (\gamma\tau)^{1/\alpha} \cdot q$$

$$= \frac{2}{\pi} \cdot \int_0^{+\infty} dx \cdot \int_0^{+\infty} \frac{d\tilde{q}}{(\gamma\tau)^{1/\alpha}} \cdot x^\nu \cdot e^{\tilde{q}^\alpha} \cdot \cos\left[\tilde{q} \cdot \frac{x}{(\gamma\tau)^{1/\alpha}}\right] =$$

$$\tilde{x} = \frac{x}{(\gamma\tau)^{1/\alpha}} \; ; \; x = (\gamma\tau)^{1/\alpha} \cdot \tilde{x}$$

$$= \frac{2}{\pi} \cdot \int_0^{+\infty} d\tilde{x} \cdot \cancel{(\gamma\tau)^{1/\alpha}} \cdot \int_0^{+\infty} \frac{d\tilde{q}}{\cancel{(\gamma\tau)^{1/\alpha}}} \cdot (\gamma\tau)^{\nu/\alpha} \cdot \tilde{x}^\nu \cdot e^{\tilde{q}^\alpha} \cdot \cos(\tilde{q} \cdot \tilde{x})$$

$$= \frac{2}{\pi} \cdot (\gamma\tau)^{\nu/\alpha} \cdot \underbrace{\int_0^{+\infty} d\tilde{x} \cdot \int_0^{+\infty} d\tilde{q} \cdot \tilde{x}^\nu \cdot e^{\tilde{q}^\alpha} \cdot \cos(\tilde{q} \cdot \tilde{x})}_{\text{Const},\ B}$$

$$\boxed{S_\nu(\tau) = \overline{|x|^\nu} = \frac{2}{\pi} (\gamma\tau)^{\nu/\alpha} \cdot B}$$ if B is finite

for example, $S_2(\tau) \sim \tau^{2/\alpha} \sim \tau^{\frac{1}{1-\epsilon/2}} \simeq \tau^{1+\epsilon/2}$

for $\alpha = 2-\epsilon$

Power-law tails of Levy P.D.F.

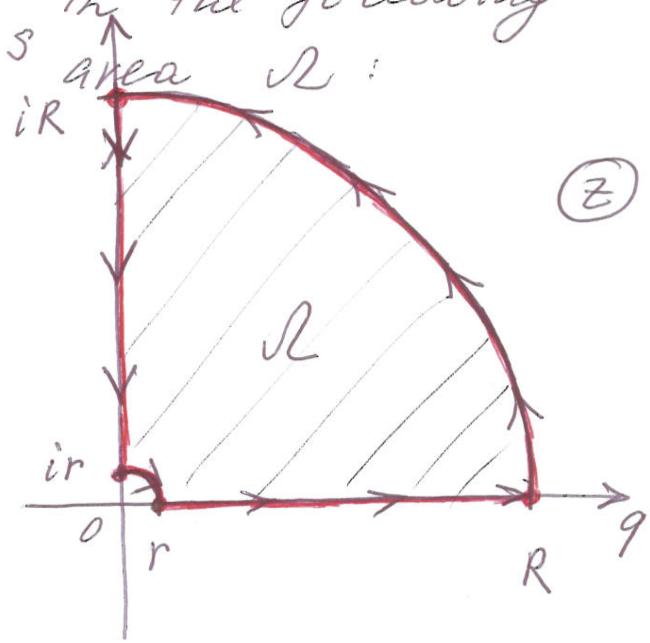
References: Article 124, Book 1 (pg. 25-26).

$$\frac{1}{\pi}\int_0^\infty \underbrace{e^{-\gamma\tau q^2}\cos qx}_{F(q) \text{ for } q\text{-real}}\, dq$$

We can expand q in complex plane: $F(z)$

$$z = q + i\cdot s$$

$$F(z) = \frac{1}{2\pi} e^{-\gamma\tau z^2} \cdot \cancel{\cos} e^{i(z\cdot x)}$$ – analytical fn in the following area Ω:



$$\int_{\partial\Omega} F(z)\,dz = 0$$

$$\parallel$$

$$\underbrace{\int_{C_r}}_{\substack{\downarrow\\0\\r\to 0+}} + \underbrace{\int_{C_R}}_{\substack{\downarrow\\0\\R\to+\infty}} + \int_r^R + \int_{iR}^{ir} = 0$$

$$\Downarrow$$

$$\int_r^R F(z)\,dz = -\int_{iR}^{ir} F(z)\,dz = \int_{ir}^{iR} F(z)\,dz$$

$$\mathrm{Re}\left[\frac{1}{\cancel{2}\pi}\int_{ir}^{iR} e^{-\gamma\tau z^{\alpha}} \cdot e^{izx} \cdot dz\right]_{z=is} = \frac{1}{\cancel{2}\pi}\, i \int_{r}^{R} \underbrace{e^{-\gamma\tau i^{\alpha} s^{\alpha}}}_{\text{Taylor series expansion}} \cdot e^{-s\cdot x}\, ds =$$

$$= \frac{1}{\cancel{2}\pi}\cdot i \cdot \sum_{k=0}^{\infty} \int_{r}^{R} ds\, \frac{(-1)^k (\gamma\tau)^k i^{k\alpha} \cdot s^{k\alpha}}{k!} \cdot e^{-s\cdot x}$$

$$s\cdot x = t \;;\quad s = \frac{t}{x}$$

$$= \frac{1}{\cancel{2}\pi}\cdot i \cdot \sum_{k=0}^{\infty} \int_{r\cdot x}^{R\cdot x} dt \cdot \frac{(-1)^k \cdot (\gamma\tau)^k \cdot i^{k\alpha} \cdot t^{k\alpha}}{x^{k\alpha+1}} \cdot e^{-t} =$$

$$= \frac{1}{\cancel{2}\pi}\cdot i \cdot \sum_{k=0}^{\infty} \frac{(-1)^k (\gamma\tau)^k \cdot e^{i\frac{\pi}{2}k\alpha}}{x^{k\alpha+1}} \cdot \underbrace{\int_{r\cdot x}^{R\cdot x} t^{k\alpha} \cdot e^{-t} \cdot dt}$$

as $r \to 0^+$, $R \to \infty$

$$\to \mathrm{Re}\left\{\frac{1}{\cancel{2}\pi}\, i \sum_{k=0}^{\infty} \frac{(-1)^k (\gamma\tau)^k e^{i\frac{\pi}{2}k\alpha}}{x^{k\alpha+1}} \times \Gamma(k\alpha+1)\right\}$$

$$\int_{0}^{+\infty} t^{(k\alpha+1)-1} e^{-t}\, dt = \Gamma(k\alpha+1)$$

$$\mathrm{Re}\left(i\, e^{i\frac{\pi}{2}k\alpha}\right) = \mathrm{Re}\left\{ i\cos\left(\frac{\pi}{2}k\alpha\right) + i\sin\left(\frac{\pi}{2}k\alpha\right)\right\} =$$

$$= -\sin\left(\frac{\pi}{2}k\alpha\right).$$

$$\frac{1}{\pi}\int_0^{\infty} e^{-\gamma\tau q^{\alpha}} \cos(qx) \underset{\substack{\text{as} \\ x\to+\infty}}{\longrightarrow} -\frac{1}{\pi}\cdot\sum_{k=1}^{\infty} \frac{(-1)^k(\gamma\tau)^k \sin\left(\frac{\pi k\alpha}{2}\right)}{|x|^{k\alpha+1}} \times \Gamma(k\alpha+1)$$

$$= -\frac{1}{\pi}\cdot\left\{ -\frac{\gamma\tau\sin\left(\frac{\pi\alpha}{2}\right)}{|x|^{\alpha+1}} \times \Gamma(\alpha+1) + h.o.t.\right\} =$$

$$= \frac{1}{\pi}\cdot\gamma\tau\cdot\sin\left(\frac{\pi\alpha}{2}\right)\cdot\Gamma(\alpha+1)\cdot\frac{1}{|x|^{\alpha+1}} + h.o.t.$$

We have just derived that:

$$\boxed{\mathcal{G}_{\alpha,\gamma}(x,\tau) \Longrightarrow \frac{1}{\pi}\cdot\gamma\tau\cdot\sin\left(\frac{\pi\alpha}{2}\right)\cdot\Gamma(\alpha+1)\cdot\underbrace{\frac{1}{|x|^{\alpha+1}}}_{\text{power law}} + h.o.t. \quad \text{as } |x|\to+\infty}$$

1) $\int_R^{+\infty} \frac{dx}{x^{\alpha+1}}$ $\alpha = 0$ – diverges; $\alpha > 0$ – converges $\alpha+1 > 1$

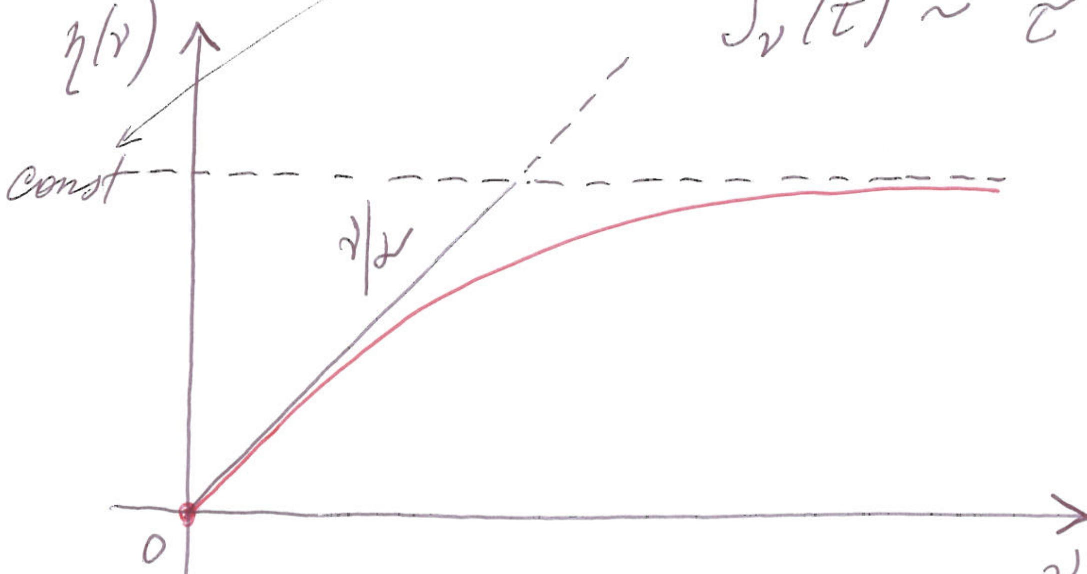
2) $S_\nu(\tau) \simeq \int_R^{+\infty} \frac{dx}{x^{-\nu+\alpha+1}}$ $-\nu+\alpha+1 > 1$ converges for $\nu < \alpha$

a particular case $\alpha = 2 - \varepsilon$, $\varepsilon > 0$ is small
then $S_2(\tau)$ is <u>formally divergent</u>.

<u>Intermittency of bi-scaling behavior:</u>

$$S_\nu(\tau) \sim \begin{cases} \frac{2}{\pi}(\gamma\tau)^{\nu/\alpha} \cdot B & \text{if } \nu < \alpha \\ \text{const (tail-dependent)} & \text{if } \nu \geq \alpha \end{cases}$$

$S_\nu(\tau) \sim \tau^{\eta(\nu)}$ ← critical exponent



Critical diagram for Structure Functions.