Math 280, Spring 2022

## Solutions to Homework #3

1. (8 points) Suppose $G$ is a graph that has 10 edges and 6 vertices, and suppose that the degrees of five of those vertices are $2, 2, 3, 4, 4$, and the sixth has some degree $n$.
(a) Find the integer $n$, i.e., the degree of the sixth vertex.
(b) Is $G$ connected? (Yes, no, or maybe?) If "yes" or "no", prove it; if "maybe", draw two examples of such a graph $G$: one that is connected and one that is not.

**Solution/Proof**. (a): By an early theorem, the sum of the degrees of all the vertices is twice the number of edges, and hence

$$2 + 2 + 3 + 4 + 4 + n = 2 \cdot 10, \qquad \text{i.e.,} \qquad n = 5.$$

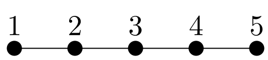
---

(b): Yes, $G$ is connected. By part (a), there is a vertex $v$ with $\deg(v) = 5$, and hence $v$ must be adjacent to each one of the other five vertices. Therefore, the connected component of $G$ that contains $v$ also contains all six vertices.

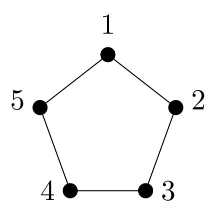
---

2. (15 points) For each of the graphs $P_5$, $C_5$, and $K_5$:
(a) draw the graph
(b) find the eccentricity of each vertex
(c) find the radius and diameter of the graph
(d) find its adjacency matrix.

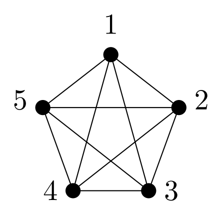
**Proof**. (a): Here are the three graphs:



$P_5$



$C_5$



$K_5$

---

(b,c): For $P_5$, by inspection (e.g. the shortest path from vertex 1 to vertex 5 has 4 edges), we see

$$\mathrm{ecc}(1) = \mathrm{ecc}(5) = 4, \qquad \mathrm{ecc}(2) = \mathrm{ecc}(4) = 3, \qquad \mathrm{ecc}(3) = 2,$$

and hence, picking the smallest and largest of these numbers, we see

$$\mathrm{rad}(P_5) = 2 \qquad \text{and} \qquad \mathrm{diam}(P_5) = 4.$$

For $C_5$, by inspection (e.g. the shortest path from vertex 1 to vertex 3 or 4 has 2 edges), we see

$$\mathrm{ecc}(1) = \mathrm{ecc}(2) = \mathrm{ecc}(3) = \mathrm{ecc}(4) = \mathrm{ecc}(5) = 2,$$

and hence, picking the smallest and largest of these numbers, we see

$$\mathrm{rad}(P_5) = 2 \qquad \text{and} \qquad \mathrm{diam}(P_5) = 2.$$

For $K_5$, by inspection (the shortest path between any two distinct vertices is a single edge), we see

$$\mathrm{ecc}(1) = \mathrm{ecc}(2) = \mathrm{ecc}(3) = \mathrm{ecc}(4) = \mathrm{ecc}(5) = 1,$$

and hence, picking the smallest and largest of these numbers, we see

$$\mathrm{rad}(P_5) = 2 \qquad \text{and} \qquad \mathrm{diam}(P_5) = 1.$$

---

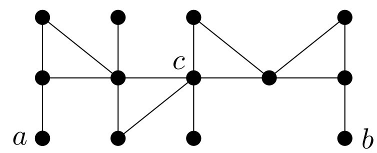
(d): By inspection, the adjacency matrices are:

$$P_5: \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad C_5: \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad K_5: \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

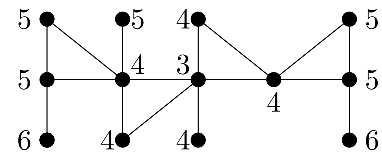
---

3. (8 points) Textbook, Section 1.2.1, Problem 1: Find the radius, diameter, and center of the graph.

**Solution**. Let $a$ and $b$ be the vertices at the lower left and lower right, respectively; and call the one in the middle of the picture $c$, like so:




It is not hard to see that for any of vertices $v$ in the graph, the furthest vertex away is either $a$ or $b$ (or a tie between the two). Using this thought, here are the eccentricities of each vertex:




Taking the smallest and largest eccentricity, we have

$$\mathrm{rad}(G) = 3 \quad \text{and} \quad \mathrm{diam}(G) = 6.$$

Since $c$ is the only vertex $v$ with $\mathrm{ecc}(v) = \mathrm{rad}(G)$, the center of $G$ is the one-point graph $c$, i.e. with vertex set $\{c\}$ and edge set $\varnothing$.

---

4. (10 points) Textbook, Section 1.2.1, Problem 5:
Let $G$ be a graph, and let $u, v \in V(G)$ be adjacent vertices. Prove that their eccentricities $\mathrm{ecc}(u)$ and $\mathrm{ecc}(v)$ differ by at most 1.

**Proof**. The shortest path from $u$ to $v$ is the length 1 path $u, v$, because of the edge joining them. (Note that they must be distinct, again because of the edge joining them, so their distance apart is not zero.) That is, $d(u, v) = d(v, u) = 1$.
Let $m = \mathrm{ecc}(u)$, so that there is some vertex $a \in V(G)$ such that $d(u, a) = m$. By the triangle inequality, then we have

$$\mathrm{ecc}(u) = m = d(u, a) \leq d(u, v) + d(v, a) = 1 + d(v, a),$$

and hence, by definition of the eccentricity of $v$,

$$\mathrm{ecc}(v) \geq d(v, a) \geq m - 1 = \mathrm{ecc}(u) - 1.$$

Reversing the roles of $u$ and $v$, the same argument also yields

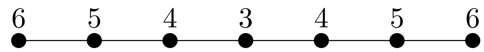
$$\mathrm{ecc}(u) \geq \mathrm{ecc}(v) - 1, \quad \text{i.e.,} \quad \mathrm{ecc}(v) \leq \mathrm{ecc}(u) + 1.$$

Combining these two inequalities gives $\mathrm{ecc}(u) - 1 \leq \mathrm{ecc}(v) \leq \mathrm{ecc}(u) + 1$, as desired. QED

---

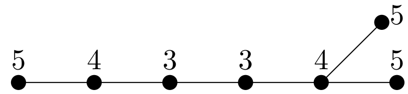
5. (12 points) Textbook, Section 1.2.1, Problem 8(a,b,c):
(a) Draw a graph of order 7 that has radius 3 and diameter 6.
(b) Draw a graph of order 7 that has radius 3 and diameter 5.
(c) Draw a graph of order 7 that has radius 3 and diameter 4.

**Solution**. (a): $P_7$ does the trick here, where I've marked each vertex with its eccentricity:



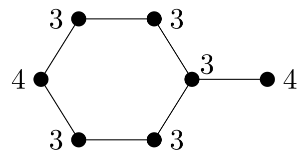

Noting the min and max eccentricities shows $\text{rad}(P_7) = 3$ and $\text{diam}(P_7) = 6$, as desired.

(b): Let $G$ be the following graph, with eccentricities marked in:




Noting the min and max eccentricities shows $\text{rad}(G) = 3$ and $\text{diam}(G) = 5$, as desired.

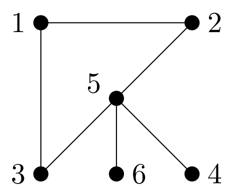
(c): Let $H$ be the following graph, with eccentricities marked in:




Noting the min and max eccentricities shows $\text{rad}(H) = 3$ and $\text{diam}(H) = 4$, as desired.

---

6. (18 points) Let $G$ be the following graph:




(a) Find the adjacency matrix $A$ of $G$.
(b) Find all the walks of length 3 from vertex 1 to vertex 4. What is the total number of such walks, and (without computing $A^3$) what does this say about the matrix $A^3$?
(c) How many closed walks of length 3 are there in $G$? Without computing $A^3$, how would this number be related to the matrix $A^3$?
(d) Find the eccentricities of all the vertices of $G$.

**Solution**. (a): The adjacency matrix is $A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

(b): Any walk from 1 to 4 has to start by going from 1 to either 2 or 3, and end by going from 5 to 4. To get length 3, only one extra edge is allowed, so there are 2 such walks:

$$1, 2, 5, 4 \qquad \text{and} \qquad 1, 3, 5, 4.$$

Thus, both the $(1, 4)$ and $(4, 1)$ entries of $A^3$ must be 2, the number of walks of length 3 from vertex 1 to vertex 4, or from vertex 4 to vertex 1.
[I would also accept the weaker statement that just the $(1, 4)$ entry is 2, or just the $(4, 1)$ entry.]

(c): There are no closed walks of length 3 in $G$, because as no two consecutive vertices can coincide, such a walk would be of the form $a, b, c, a$ with $a, b, c$ being three distinct vertices. This excludes vertices 4 and 6, and the remaining four vertices form a 4-cycle but no shorter closed walks.
Thus, the number of closed walks of length 3 is 0. This means that there are 0 walks of length 3 from vertex $i$ to vertex $i$, for each of $i = 1, \ldots 6$. That is, every entry on the diagonal of $A^3$ is 0.
[I would also accept the weaker statement that the trace of $A^3$ is 0.]

(d) The furthest vertex from either vertex 4 or 6 is vertex 1, which is distance 3 away. The vertices 2,3,5 are all distance at most 2 from any other vertices. Thus, the eccentricities are:

$$\mathrm{ecc}(1) = \mathrm{ecc}(4) = \mathrm{ecc}(6) = 3, \qquad \mathrm{ecc}(2) = \mathrm{ecc}(3) = \mathrm{ecc}(5) = 2.$$

---

7. (10 points) Textbook, Section 1.2.2, Problem 3:
Let $G$ be a graph with $V(G) = \{v_1, \ldots, v_n\}$ and with adjacency matrix $A$. For each $j = 1, \ldots, n$, prove that the $(j, j)$ entry of $A^2$ is $\deg(v_j)$.
**Proof**. Given any $j = 1, \ldots, n$, the $(j, j)$ entry $a_{j,j}$ of $A^2$ is the number of walks of length 2 from $v_j$ to $v_j$. Such walks are precisely those of the form

$$v_j, w, v_j$$

where $w$ is a vertex adjacent to $v_j$. Thus, each such walk gives a unique vertex $w \in N(v_j)$ adjacent to $v_j$; and conversely any $w \in N(v_j)$ gives the walk above. Hence, the $(j, j)$ entry of $A^2$ is

$$a_{j,j} = |\{\text{walks of length 2 from } v_j \text{ to } v_j\}| = |N(v_j)| = \deg(v_j)$$ QED

---

8. (15 points) Let $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, and let $G$ be the graph with adjacency matrix $A$.

(a) Compute $A^2$ and $A^3$.
(b) How many walks are there in $G$ from vertex 1 to vertex 2 of length exactly 3?
(c) Find the radius and the diameter of $G$.
(d) Draw the graph $G$.

**Solution**. (a): Direct computation gives $A^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix}$

and

$$A^3 = AA^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 & 4 \\ 3 & 2 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 4 & 4 & 3 & 2 \end{bmatrix}$$

(b): The number of walks from $v_1$ to $v_2$ of length 3 is the $(1, 2)$ entry of $A^3$, which is 3.

(c): Define $S_k = I + A + \cdots + A^k$, and we have $S_0 = I$, and

$$S_1 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 2 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ 2 & 2 & 1 & 4 \end{bmatrix}$$

By Theorem 1.9, the radius is the smallest positive integer $r$ such that at least one row of $S_r$ has all nonzero entries. This happens for the fourth row of $S_1$, so $\boxed{\mathrm{rad}(G) = 1}$
Also by Theorem 1.9, the diameter is the smallest positive integer $m$ such that all entries of $S_m$ are nonzero. This happens for $S_2$, so $\boxed{\mathrm{diam}(G) = 2}$

(d): Based on $A$, here is $G$: