- HW7: Due today
- HW8: Due Dec 6^{th}.
- Today: MGF (continued)

Recap:

Def: (MGF)

let X be some RV. the moment generating function of X is a mapping $\mathbb{R} \to \mathbb{R}$

$$M_X(s) = E[e^{sX}]$$

(Sometimes $M(s)$)

Examples (last week): Bernoulli, Exponential, —

Side note: Expectations do not need to be $< \infty$

Fun example: St Petersburg Paradox

Bet 2\$ to start, If I win, I net 2\$

If I lose, I repeat the bet with 4\$

If I lose again, " " " 8\$

" " " " " 16\$

Until I win for the first time

- let X be the necessary nb of bets to win
- Suppose the bet is fair

$X \sim \text{Geom}(1/2)$

At round X, I bet 2^X

So the previous rounds, I lost

$$2 + 4 + 8 + \dots + 2^{X-1} = \sum_{k=1}^{X-1} 2^k = 2^X - 2$$

The net gain is $\underbrace{2^X}_{\substack{\text{net gain} \\ \text{at round } X}} - \underbrace{(2^X - 2)}_{\text{lost before}} = 2$

How much bankroll do I need?

$$E\left[2^X - 2\right] = E[2^X] - 2$$

$$E[2^X] = \sum_{k=1}^{\infty} 2^k \, \mathbb{P}(X = k) = \sum_{k=1}^{\infty} 2^k \left(\frac{1}{2}\right)^{k-1} \frac{1}{2}$$

$$= \sum_{k=1}^{\infty} \frac{2^k}{2^k} = \sum_{k=1}^{\infty} 1 = \infty$$

Another example of infinite expectation: Recitation 6 (Q2)

Example of MGF computation: $X \sim \mathcal{N}(0,1)$

Standard normal

PDF $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$M_X(s) = \mathbb{E}[e^{sX}] = \int_{-\infty}^{\infty} e^{sx} f(x)\, dx$$
$$= \int_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx - \frac{x^2}{2}}$$

Trick: Completing the squares

$$sx - \frac{x^2}{2} = \frac{2sx - x^2}{2} = \frac{2sx - x^2 - s^2 + s^2}{2}$$
$$= \frac{-(x-s)^2 + s^2}{2}$$

$$M_X(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{sx - \frac{x^2}{2}}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2 + s^2}{2}}\, dx$$

$$= e^{s^2/2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{(x-s)^2}{2}}}_{} dx$$

PDF of $\mathcal{N}(s,1)$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-s)^2}{2}} dx = 1$$

$\boxed{M_X(s) = e^{s^2/2}}$ when $X \sim \mathcal{N}(0,1)$

Example 5 $\quad Y \sim \mathcal{N}(\mu, \sigma^2)$ for μ, $\sigma^2 > 0$

Y has the same distribution as $\mu + \sigma X$ where $X \sim \mathcal{N}(0,1)$

$$M_Y(s) = \mathbb{E}[e^{sY}] = \mathbb{E}[e^{s(\mu+\sigma X)}]$$

$$= \mathbb{E}[e^{s\mu + s\sigma X}]$$

$$= e^{s\mu}\mathbb{E}[e^{s\sigma X}]$$

$$= e^{s\mu} M_X(s\sigma)$$

$$= e^{s\mu + \frac{(\sigma s)^2}{2}}$$

Def: For integer $n \geq 1$, we define the n^{th} moment of X as the number $\mathbb{E}[X^n]$

1st moment $E[X]$

2nd moment $E[X^2]$

Fact $\mathbb{E}[X^n] = \frac{d^n M_X}{ds^n}(0) = \left.\frac{d^n M_X(s)}{ds^n}\right|_{s=0}$

Proof Case X discrete (X takes finitely many values)

$$\frac{dM_X}{ds}(s) = \frac{d}{ds} E[e^{sX}]$$

$$= \frac{d}{ds} \sum_k e^{sk} \underbrace{\mathbb{P}(X=k)}_{\text{does not depend on } s}$$

$$= \sum_k \frac{d}{ds} e^{sk} \mathbb{P}(X=k)$$

$$= \sum_k k e^{sk} \mathbb{P}(X=k)$$

So at $s=0$ $\frac{dM_X}{ds}(0) = \sum k \mathbb{P}(X=k) = E[X]$

$$\frac{d^n M_X}{ds^n}(s) = \sum_k \frac{d^n}{ds^n} e^{sk} \mathbb{P}(X=k)$$

$$= \sum_k k^n e^{sk} \, \mathbb{P}(X = k)$$

at $s = 0$ $\quad \dfrac{d^n M_X(s)}{ds^n}(0) = \sum_k k^n \, \mathbb{P}(X = k)$

$$= \mathbb{E}[X^n]$$

<u>Ex 3 revisited</u> $\quad X \sim \text{Exp}(\lambda)$
$\lambda > 0$

$$M_X(s) = \begin{cases} \dfrac{\lambda}{\lambda - s} & \text{if } s < \lambda \\ \infty & s \geq \lambda \end{cases}$$

let $s < \lambda$

$$\boxed{\frac{d}{dx}\left(\frac{1}{u(x)}\right) = \frac{-u'}{u^2}}$$

$$M_X'(s) = \frac{d}{ds}\left(\frac{\lambda}{\lambda - s}\right)$$

$$= \frac{-\lambda \cdot (-1)}{(\lambda - s)^2} = \frac{\lambda}{(\lambda - s)^2}$$

$$M_X''(s) = \frac{d^2}{ds}\left(\frac{\lambda}{\lambda - s}\right) = \frac{d}{ds}\left(\frac{\lambda}{(\lambda - s)^2}\right)$$

$$= \frac{2\lambda}{(\lambda - s)^3}$$

$$M'''_X(s) = \frac{6\lambda}{(\lambda - s)^4} \qquad \frac{d^k M_X}{ds^k}(s) = \frac{k!\,\lambda}{(\lambda - s)^{k+1}}$$

By taking $s = 0$

$$E[X] = M'_X(0) = \frac{1}{\lambda}$$

$$E[X^2] = M''_X(0) = \frac{2}{\lambda^2}$$

$$E[X^n] = \frac{n!}{\lambda^n}$$

Inversion Theorem

If X and Y have the same moment generating function M, and if $M(s) < \infty$ for all $s \in [-a, a]$ for some $a > 0$, Then X and Y have the same distribution

Ex: If X and Y are discrete with MGFs

$$M_X(s) = \sum_k e^{sk}\,\mathbb{P}(X = k), \quad M_Y(s) = \sum_k e^{sk}\,\mathbb{P}(Y = k)$$

If $M_X = M_Y$ (same MGF)

Then $\sum_k e^{sk}\,\mathbb{P}(X = k) = \sum_k e^{sk}\,\mathbb{P}(Y = k)$

for all $s \in \mathbb{R}$

So the coefficients $\mathbb{P}(X=k)$ and $\mathbb{P}(Y=k)$ must match

$\Longrightarrow \quad X \underset{\text{distribution}}{\sim} Y$

Ex: $X \sim \mathrm{Ber}(p)$ $\qquad$ $Y \sim \mathrm{Ber}(q)$

$$M_X(s) = E[e^{sX}] = pe^s + (1-p)$$
$$M_Y(s) = E[e^{sY}] = qe^s + (1-q)$$

Suppose $M_X(s) = M_Y(s) \Longrightarrow pe^s + 1 - p = qe^s + (1-q)$ for all $s \in \mathbb{R}$

$\Longrightarrow \quad p = q$

Ex: X has MGF

$$M(s) = \frac{1}{2} e^{3s} + \frac{1}{4} e^{s} + \frac{1}{4}$$
$$= \sum_k e^{sk}\, \mathbb{P}(X=k)$$

Then X is discrete with PMF

$$\mathbb{P}(X=k) = \begin{cases} \frac{1}{2} & \text{if } k=3 \\ \frac{1}{4} & \text{if } k=1 \\ \frac{1}{4} & \text{if } k=0 \end{cases}$$

Independance:

Suppose X and Y are independent

Then

$$M_{X+Y}(s) = \mathbb{E}\left[e^{s(X+Y)}\right]$$

$$= \mathbb{E}\left[e^{sX} e^{sY}\right]$$

$$= \mathbb{E}\left[e^{sX}\right] \mathbb{E}\left[e^{sY}\right]$$

$$= M_X(s) \; M_Y(s)$$

Summation of RVs $\Longrightarrow$ Multiplication of MGFs

Convolution:

- Discrete Suppose X and Y are independent with values in $\{0, 1, 2, \ldots\}$

Then $\mathbb{P}(X+Y=n) = \sum_{k=0}^{n} \mathbb{P}(X=k, Y=n-k)$

by independance $\longleftarrow$ $= \sum_{k=0}^{n} \mathbb{P}(X=k)\, \mathbb{P}(Y=n-k)$

$$\boxed{p_{X+Y}(n) = \sum_{k=0}^{n} p_X(k)\, p_Y(n-k)}$$

This is called the convolution of p_X and p_Y

Example: $X \sim \text{Poisson}(\lambda)$ $\quad Y \sim \text{Poisson}(\mu)$

X and Y independent.

$$\mathbb{P}(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad \mathbb{P}(Y=k) = e^{-\mu}\frac{\mu^k}{k!}$$

$$\mathbb{P}(X+Y=n) \overset{\text{Convolution}}{=} \sum_{k=0}^{n} \mathbb{P}(X=k)\,\mathbb{P}(Y=n-k)$$

$$= \sum_{k=0}^{n} e^{-\lambda}\frac{\lambda^k}{k!}\, e^{-\mu}\frac{\mu^{n-k}}{(n-k)!}$$

$$= e^{-(\lambda+\mu)} \sum_{k=0}^{n} \frac{\lambda^k \mu^{n-k}}{k!\,(n-k)!}$$

$$\boxed{\begin{aligned}(a+b)^n &= \sum_{k=0}^{n}\binom{n}{k} a^k b^{n-k} \\ &= \sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!} a^k b^{n-k}\end{aligned}}$$

$$= \frac{e^{-(\lambda+\mu)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\lambda^k \mu^{n-k}$$

$$\boxed{\mathbb{P}(X+Y=n) = \frac{e^{-(\lambda+\mu)}}{n!}(\lambda+\mu)^n}$$

So $X+Y \sim \text{Poisson}(\lambda+\mu)$

More general: The sum of n independent poisson with parameters $\lambda_1, \ldots \lambda_n$ is Poisson with parameter $\lambda_1 + \lambda_2 + \cdots + \lambda_n$

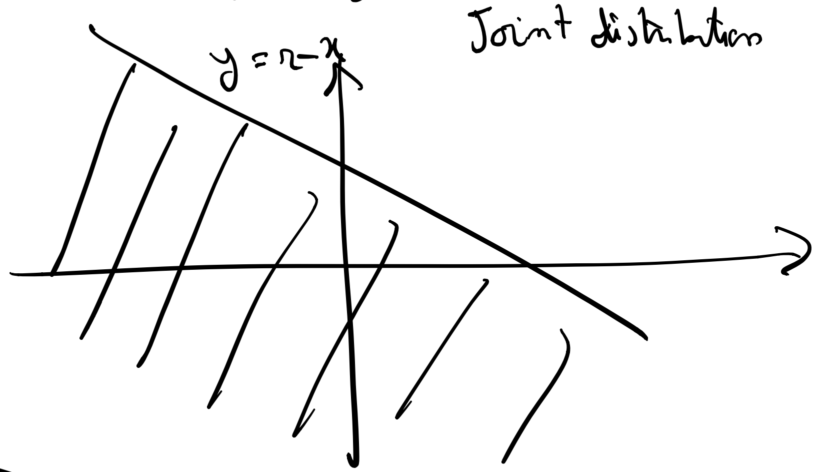
Convolution with continuous

Say X and Y are independent and continuous

CDF of $X+Y$

$$F_{X+Y}(r) = \mathbb{P}(X+Y \leqslant r)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{r-x} \underbrace{f_X(x) f_Y(y)}_{\text{Joint distribution}} \, dy \, dx$$



$$= \int_{-\infty}^{\infty} f_X(x) \underbrace{\int_{-\infty}^{r-x} f_Y(y) \, dy}\, dx$$

$$\mathbb{P}(X+Y \leqslant r) = \int_{-\infty}^{\infty} f_X(x) \, F_Y(r-x) \, dx$$

PDF of $X+Y$

$$f_{X+Y}(z) = \frac{d}{dz} \mathbb{P}(X+Y \leq z)$$

$$= \frac{d}{dz} \int_{-\infty}^{\infty} f_X(x) \, F_Y(z-x) \, dx$$

$$= \int_{-\infty}^{\infty} f_X(x) \, \frac{d}{dz} F_Y(z-x) \, dx$$

$$\boxed{f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) \, f_Y(z-x) \, dx}$$

$$\overset{u=z-x}{=} \int_{-\infty}^{\infty} f_X(z-u) \, f_Y(u) \, du$$

This is called convolution of f_X and f_Y

Indep RVs

- Convolution at level of PDF / PMF
- Multiplication at level of MGF
- Addition at level of RVs themselves