Lecture 8:

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Back To The Smile:
Stylized Facts
Plotting The Skew
Consequences for Trading
Bounds on the Smile

By the way: speculators using options to make a leveraged bet are not trading volatility, but betting on the stock price

Hedgers or market makers are betting on volatility.

8.1 The P&L of Any Hedged Trading Strategy with Div Rate q

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$$(C_0 - \Delta_0 S_0)e^{r\tau} = (C_T - \Delta_T S_T) + \int_0^{\tau} e^{r(\tau - x)} S_x \left\{ [d\Delta_x]_b - \overset{\text{div}}{q\Delta_x \mathrm{dx}} \right\}$$

$$(C_0 - \Delta_0 S_0) = (C_T - \Delta_T S_T)e^{-r\tau} + \int_0^{\tau} e^{-rx} S_x e^{qx} d\left[e^{-qx} \Delta_x \right]_b \qquad \text{Eq 8.1}$$

$$= (C_T - \Delta_T S_T)e^{-r\tau} + \int_0^{\tau} e^{-(r-q)x} S_x d\left[e^{-qx} \Delta_x \right]_b$$

The effect of dividends is to take the current value of the stock using $e^{-(r-q)x}$ value that will produce the future stock and the similarly the current number of shares, which grows like e^{qx} Eq 8.2

You can integrate

$$(C_0 - \Delta_0 S_0)e^{r\tau} = (C_T - \Delta_T S_T) + \int_0^{\tau} e^{r(\tau - x)} S_x \left\{ [d\Delta_x]_b - q\Delta_x \mathrm{dx} \right\}$$

by parts using the relation

$$d\left[e^{r(\tau - x)} S_x \Delta_x \right] = -re^{r(\tau - x)} \Delta_x S_x dx + e^{r(\tau - x)} \Delta_x dS_x + e^{r(\tau - x)} S_x [d\Delta_x]_b$$

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to get

$$(C_0 - \Delta_0 S_0)e^{r\tau} = (C_T - \Delta_T S_T) + \int_0^\tau d\left[e^{r(\tau - x)} S_x \Delta_x\right] + \int_0^\tau (r-q)e^{r(\tau - x)} \Delta_x S_x dx - \int_0^\tau e^{r(\tau - x)} \Delta_x dS_x \qquad \text{Eq 8.3}$$

$$(C_0 - \Delta_0 S_0)e^{r\tau} = (C_T - \Delta_T S_T) + \left[\Delta_T S_T - \Delta_0 S_0 e^{r\tau}\right] + e^{r\tau}\int_0^\tau e^{-rx}\Delta(S_x, x)[dS_x - (r-q)S_x dx]$$

$$C_0 = \underbrace{C_T e^{-r\tau}}_{\text{PV of payoff}} - \underbrace{\int_0^\tau e^{-rx}\Delta(S_x, x)[dS_x - (r-q)S_x dx]}_{\text{PV of change in value of the hedge funded at the interest less dividend rate}} \qquad \text{Eq 8.4}$$

Equation 8.1 and Equation 8.4 provide a way to calculate the value of the option C_0 in terms of its final payoff and the hedging strategy. If you hedge perfectly and continuously to get a riskless position, Black-Scholes tells you that this value will be independent of the path the stock takes to expiration. Else the fair value of C_0 has a spread-out distribution.

8.2 A PDE Model of Transactions Costs

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One can approach transactions costs even more precisely in the framework of Hoggard, Whaley & Wilmott. (There are many other treatments, the first originally tackled by Leland.) In this way you can estimate the effect of transactions costs by simply adjusting the BSM volatility.

Assume zero dividend yield, and

$$dS = \mu S dt + \sigma S Z \sqrt{dt}$$

where Z is drawn from a standard normal distribution. Calculate the change in value of hedged position for non-infinitesimal dt:

$$
\begin{aligned}
d\text{P\&L} &= dV - \Delta dS - \text{cash spent on transactions costs} \\
&\approx \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 dt - \Delta dS - \kappa S|N| \\
&= \frac{\partial V}{\partial t}dt + \left(\frac{\partial V}{\partial S} - \Delta\right)(\mu S dt + \sigma S Z\sqrt{dt}) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 dt - \kappa S|N| \\
&= \cancel{\left(\frac{\partial V}{\partial S} - \Delta\right)}\sigma S Z\sqrt{dt} + \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \mu S \cancel{\left(\frac{\partial V}{\partial S} - \Delta\right)} + \frac{\partial V}{\partial t}\right)dt - \kappa S|N|
\end{aligned}
$$

Choose the continuous hedge ratio $\Delta = \frac{\partial}{\partial S}V(S, t)$ to eliminate the first term.

$$\mathrm{dP\&L} = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \frac{\partial V}{\partial t}\right) dt - \kappa S |N|$$

Then

$$= \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \frac{\partial V}{\partial t}\right) dt - \kappa \sigma S^2 \left|\frac{\partial^2 V}{\partial S^2} Z\right| \sqrt{dt}$$

N itself is stochastic and related to Γ of course.

The P&L is not riskless, unfortunately, but we can calculate its expected value.

The expected value of the change in the P&L is therefore given by

$$E\left[Z^2\right] = 1$$

$$E[|Z|] = \sqrt{\frac{2}{\pi}}$$

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$$E[d\text{P\&L}] = \left[\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \sqrt{\frac{2}{\pi \delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right|\kappa\sigma S^2\right)dt\right]$$

This isn't riskless. **Nevertheless** let's assume we expect to earn the riskless rate on the hedge, on average.

$$E[d\text{P\&L}] = r\left(V - S\frac{\partial V}{\partial S}\right)dt.$$

Combining, we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \sqrt{\frac{2}{\pi \delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right|\kappa\sigma S^2 + rS\frac{\partial V}{\partial S} - rV = 0$$

Modified BS equation with nonlinear extra term proportional to the value of $\Gamma = \frac{\partial^2 V}{\partial S^2}$.

The sum of two solutions to the equation is not necessarily a solution too; you cannot assume that the transactions costs for a portfolio of options is the sum of the transactions costs for hedging each option in isolation.

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For a single long position in a call or a put, $\frac{\partial^2 V}{\partial S^2} \geq 0$, so we can drop the absolute value sign.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\hat{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad \text{Eq.1.1}$$

where

$$\hat{\sigma}^2 = \sigma^2 - 2\kappa\sigma\sqrt{\frac{2}{\pi\delta t}} \qquad \hat{\sigma} \approx \sigma - \kappa\sqrt{\frac{2}{\pi\delta t}}$$

This is the Black-Scholes equation with a modified reduced volatility.For a short position, the effective volatility is enhanced, given by

$\hat{\sigma} \approx \sigma + \kappa\sqrt{\frac{2}{\pi\delta t}}$ and $\kappa \ll \sigma\sqrt{\delta t}$ so that the correction is actually small.

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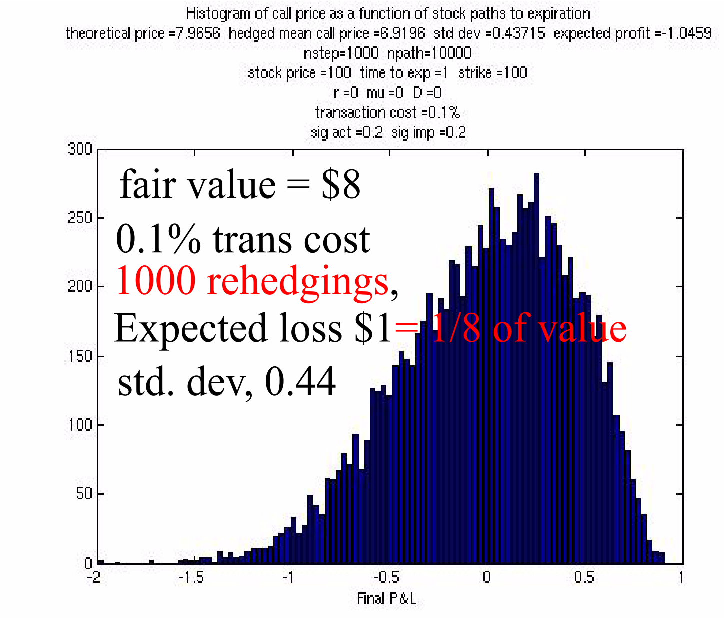
Compare Above Theory with our Earlier Simulations.

cost of initial hedge is k(Δ)S=0.001(0.5)100=$0.05



ATM Call $C \approx \frac{S\sigma\sqrt{\tau}}{\sqrt{2\pi}}$. Percentage change in ATM option is $\frac{\hat{\sigma}-\sigma}{\sigma} = \frac{\kappa}{\sigma}\sqrt{\frac{2}{\pi\delta t}}$

Fractional loss in value of option from transactions costs:

$$\frac{0.001}{0.2}\sqrt{\frac{2}{\pi\frac{1}{1000}}} \approx 0.005(25) = 0.125 \approx \frac{1}{8}$$

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8.3 Stylized Facts about the Smile for Equity Indexes

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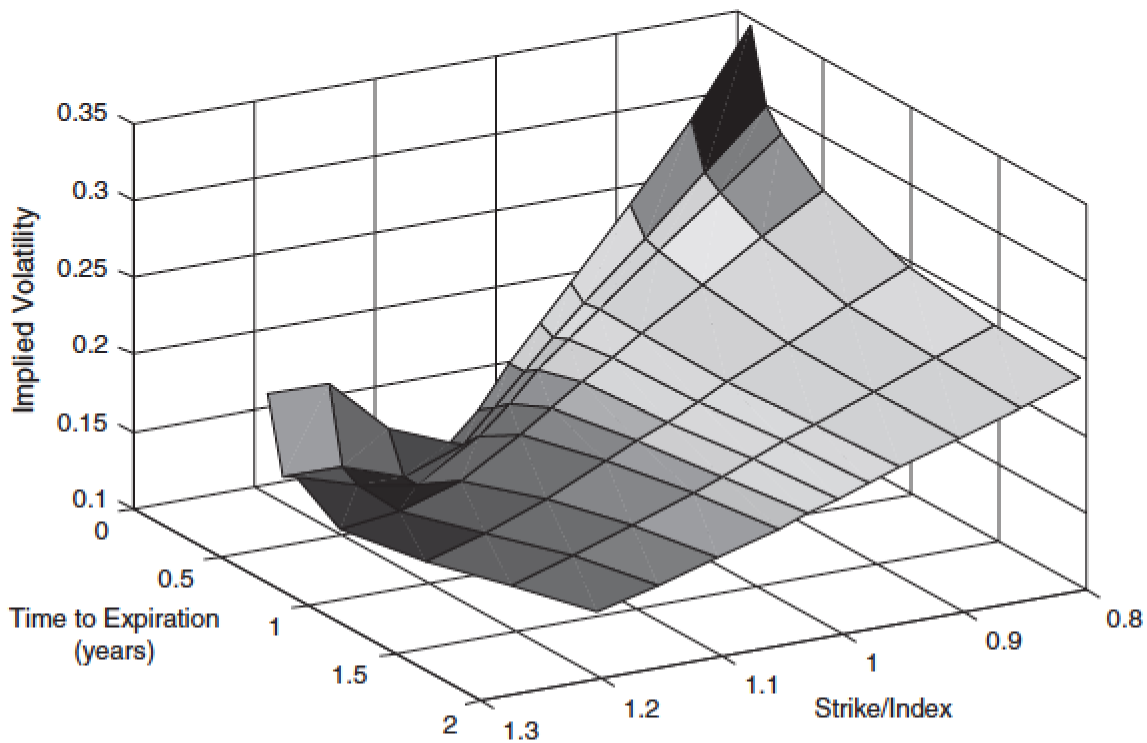

FIGURE 8.4 Volatility Surface, S&P 500, December 31, 2015
Source: Bloomberg.

- Implied volatility tends to be greater than recent realized volatility.
- Almost always negatively skewed. Why? Protection against payoff and volatility spike?
- Wasn't that way before 1987.
- Skew is steeper for short expiration, flatter for longer ones. (In what variable?)

Plotting the Skew

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$m = \frac{K}{S}$ is moneyness. Sometimes we use S_F, the forward price, to get the forward moneyness.

$k = \ln(K/S)$ is log moneyness

$k = \ln(K/S_F)$ is log forward moneyness

$\frac{k}{\sigma\sqrt{\tau}}$ is the standardized log moneyness, the no. of return standard deviations from strike to stock using the average index volatility. This takes account of the increasing width of the distribution in geometric Brownian motion as time passes.

$\psi(\tau) = \left[\frac{\partial}{\partial k}\Sigma(k, \tau)\right]\Big|_{k=0}$ is the slope of the skew at the money.

Slope of Skew in K or ln(K) Flattens for Longer Expirations

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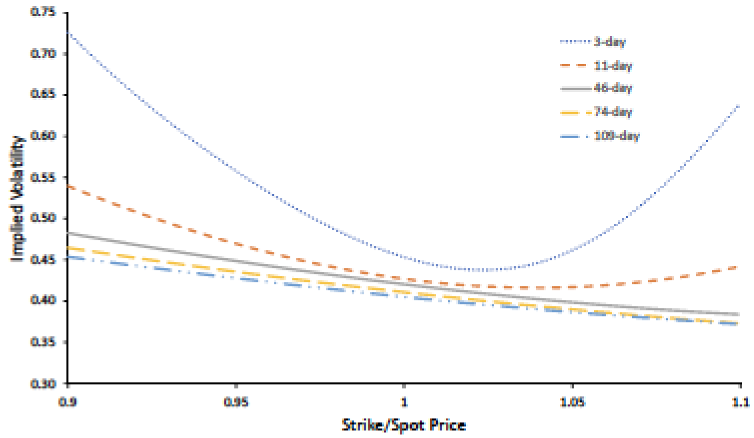

Figure 1: JPM implied volatilities on June 5, 2012. The curves show cubic spline fits at various maturities using raw data from OptionMetrics, plotted against the ratio of the put strike K to the spot price S. The ATM skew is the slope at $K/S = 1$. Its absolute value falls quickly as the maturity increases.

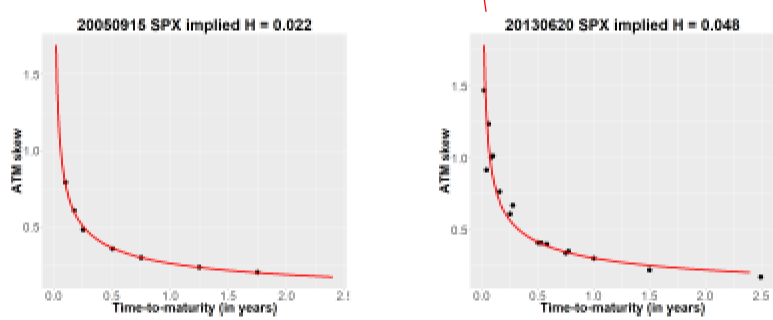

Figure 2: Term structure of the ATM skew for the S&P 500 index, as in similar figures in Gatheral et al. (2018). The charts plot the slope of the ATM skew against option maturity on Sep 15, 2005 (left) and Jun 20, 2013 (right), using OptionMetrics data.

$$\phi(\tau) = \left|\frac{\partial \sigma_{BS}(k,\tau)}{\partial k}\right|_{k=0} \qquad \phi(\tau) \approx \text{constant} \times \tau^{H-1/2}, \quad \text{as } \tau \downarrow 0.$$

$k = \ln(K/S) = \ln(m)$

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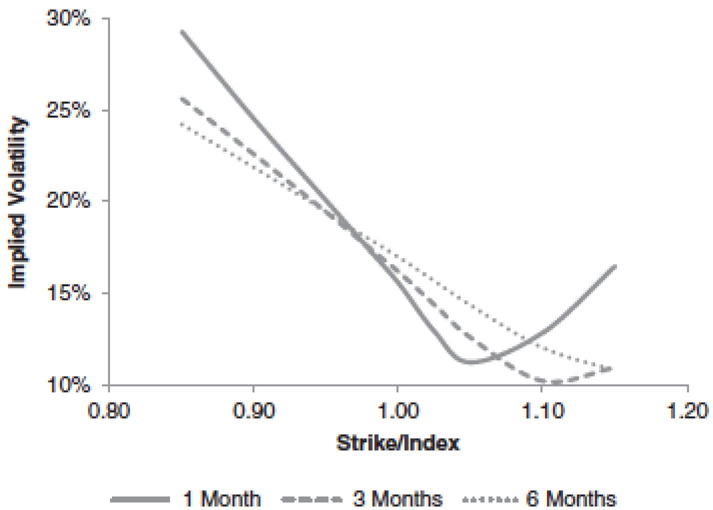
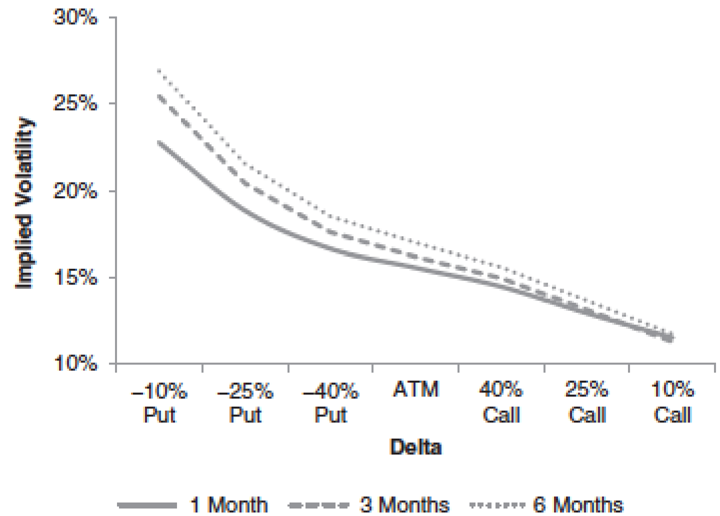
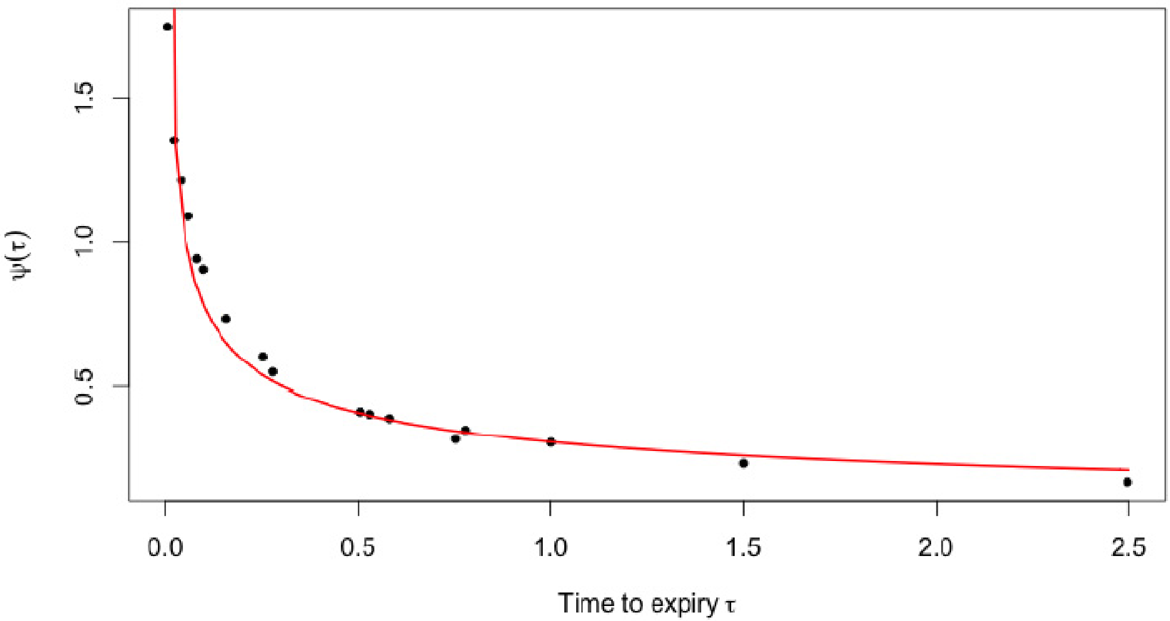
- **For Longer Expirations, Slope Flattens in K, But Steepens Slightly in Delta or** $\left(\log\frac{\text{Strike}}{\text{Spot}}\right) / (\sigma\sqrt{\tau})$





- Term structure of the volatility surface can slope up or down. During a crisis—and a crisis is always characterized by high volatility—the term structure is likely to be downward sloping. The high short-term volatility and lower long-term volatility reflect market participants' belief that uncertainty in the near term will eventually be resolved.

Plotted just against log moneyness, we see the slope decreases.



The black dots are non-parametric estimates of the S&P ATM volatility skews as of June 20, 2013; the red curve is the power-law fit $\psi(\tau) = A\,\tau^{-0.4}$.

which is consistent by change of variable with the fact that skew slope increases when plotted against delta or standardized log moneyness.

$$\frac{d\sigma}{d\left(\frac{m}{\sqrt{\tau}}\right)} = \sqrt{\tau}\frac{d\sigma}{d(m)} \sim \sqrt{\tau}\frac{1}{\tau^{0.4}} \sim \tau^{0.1}$$

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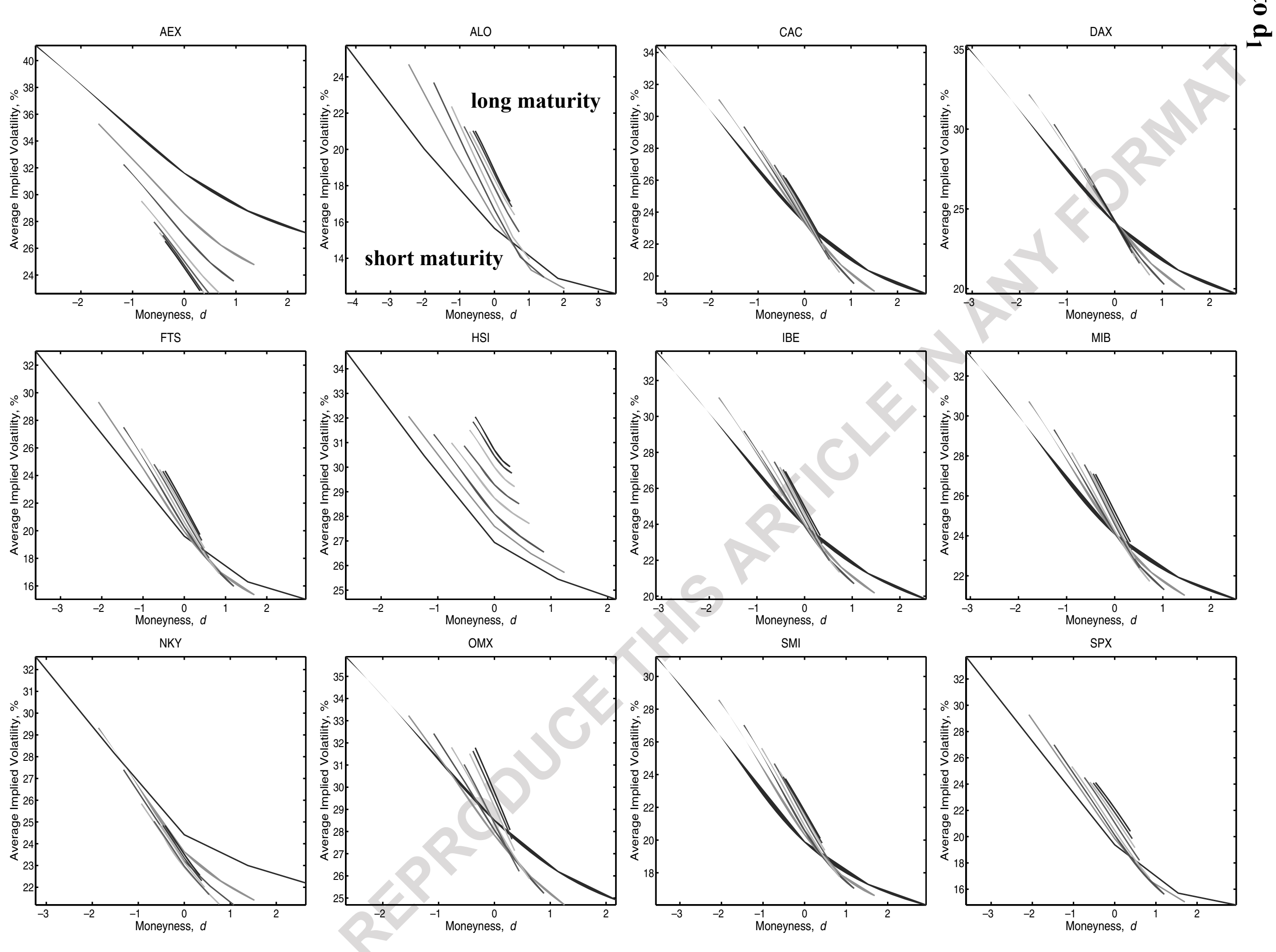
Implied Volatility as a Function of $\left(\log\frac{\text{Strike}}{\text{Spot}}\right)/(\sigma\sqrt{\tau})$ Crash-o-phobia: A Domestic Fear Or A Worldwide Concern? Foresi & Wu JOD Winter 05(

Maturity Pattern of Implied Volatility Smirks



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When plotted against the number of standard deviations between the log of the strike and the log of the spot price for a lognormal process, the slope of the skew actually increases with expiration. Whatever is happening to cause this doesn't fade away with future time.

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Index and implied volatility are negatively correlated.

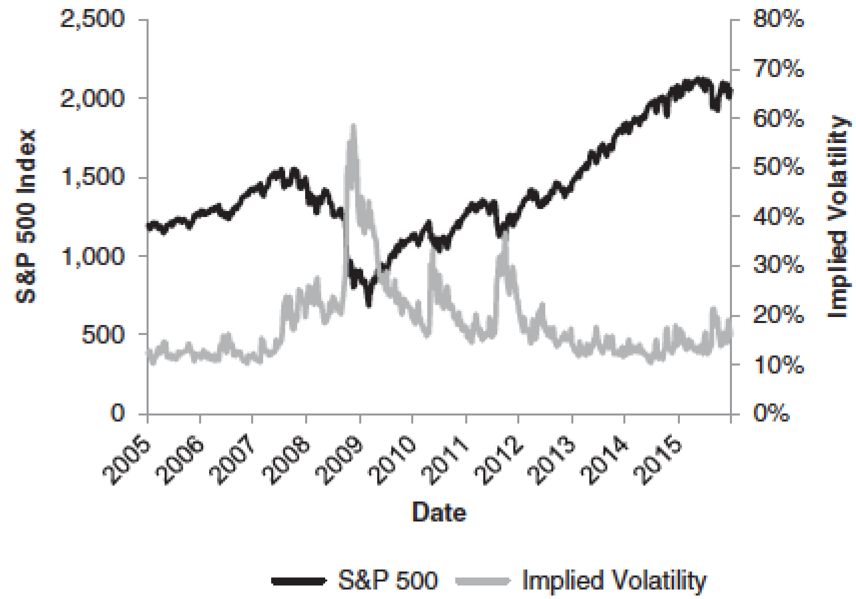

FIGURE 8.9 S&P 500 Level and Three-Month At-the-Money Implied Volatility

- Indexes tend to glide up and crash down. Realized and implied volatility increase in a crash.

We have to be careful not to read too much into Figure 8.9. **What do we mean by an "increase in implied volatility."?** As the market falls, the option that is at-the-money becomes a different option.

The smile shifts **and** we move leftward along the smile at the same time, both of which increase at-the-money volatility in a crash. Why do traders talk most about atm volatility?

ATM volatility is therefore not the volatility of a particular option you own, just like the yield on a 10-year constant maturity treasury is not the yield on something you actually continue to own.

Some of the apparent correlation in the figure above would occur even if $\Sigma(S, t, K, T)$ didn't change with S at all. How much of the correlation is true co-movement and not incidental? When you talk about changes in vol, what are you keeping fixed?

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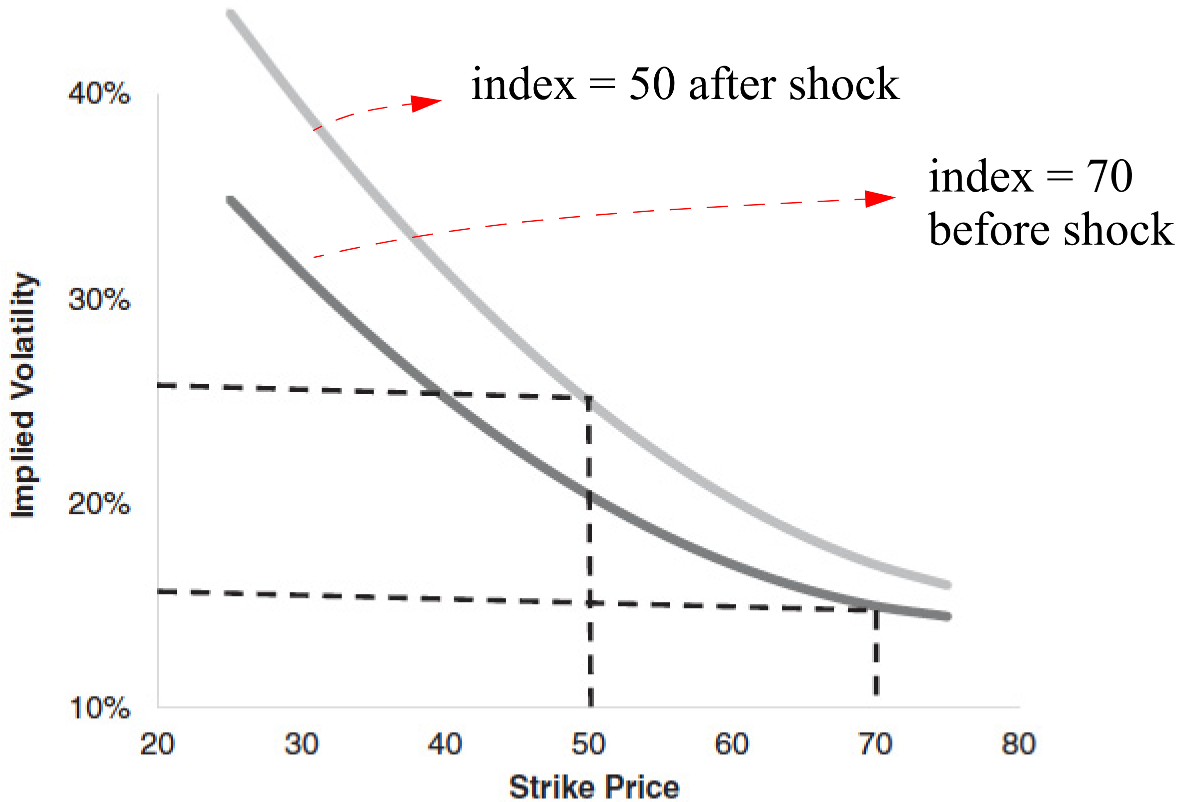

FIGURE 8.10 Effect of a Shock on the Smile

• The volatility of implied volatility and the skew is greatest for short expirations, analogous to the higher volatility of short-term Treasury rates.

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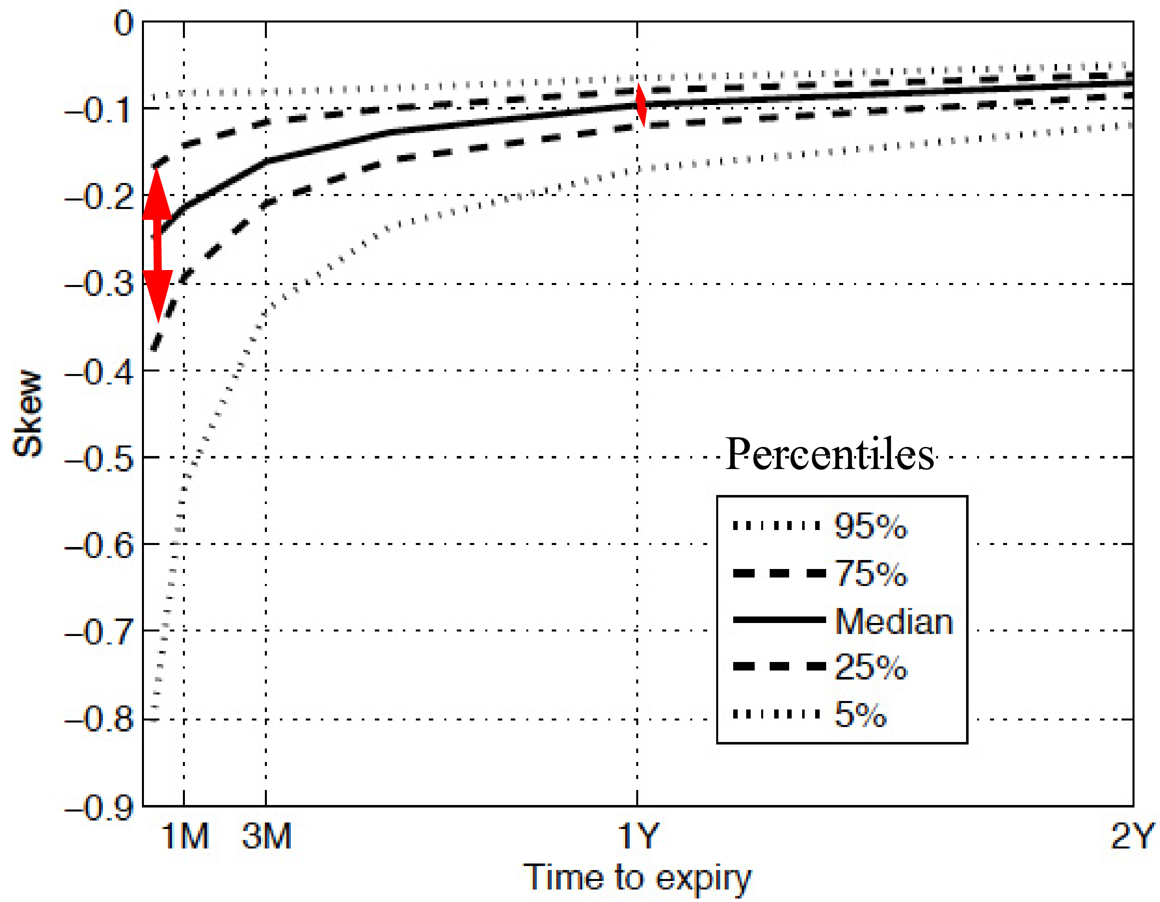

8.4 Different Smiles in Different Markets

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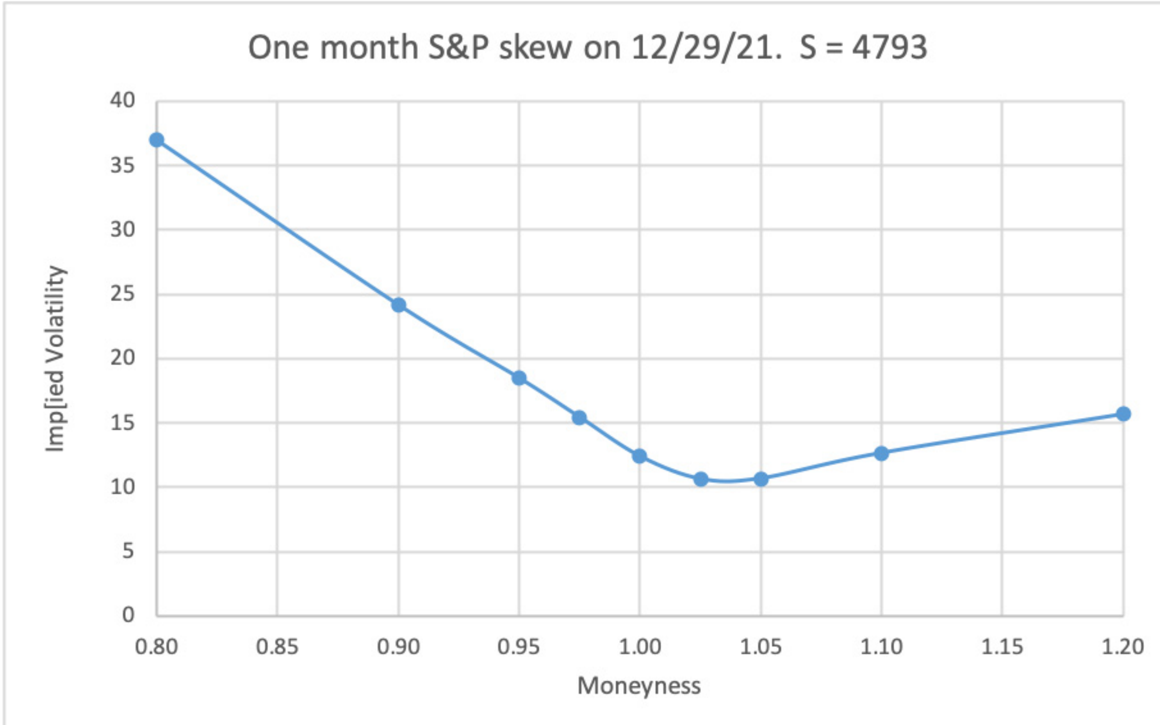

Moneyness	0.80	0.90	0.95	0.98	1.00	1.03	1.05	1.10	1.20
IV	37.0017	24.186	18.4903	15.447	12.4064	10.652	10.6926	12.6682	15.6866

One-month slope at the money is about 6 vol points per 0.05 S&P points = 0.06/(0.05 x 4793) = (0.06)/240 = 0.00025. 6 points change in vol for a 5 point change in moneyness, about 1 vol pt per moneyness pt.

This is the general ballpark. We’ll use this number in making estimates of effect of skew later.

Single stock smiles

A single stock smile is more of an actual smile with both sides turning up.

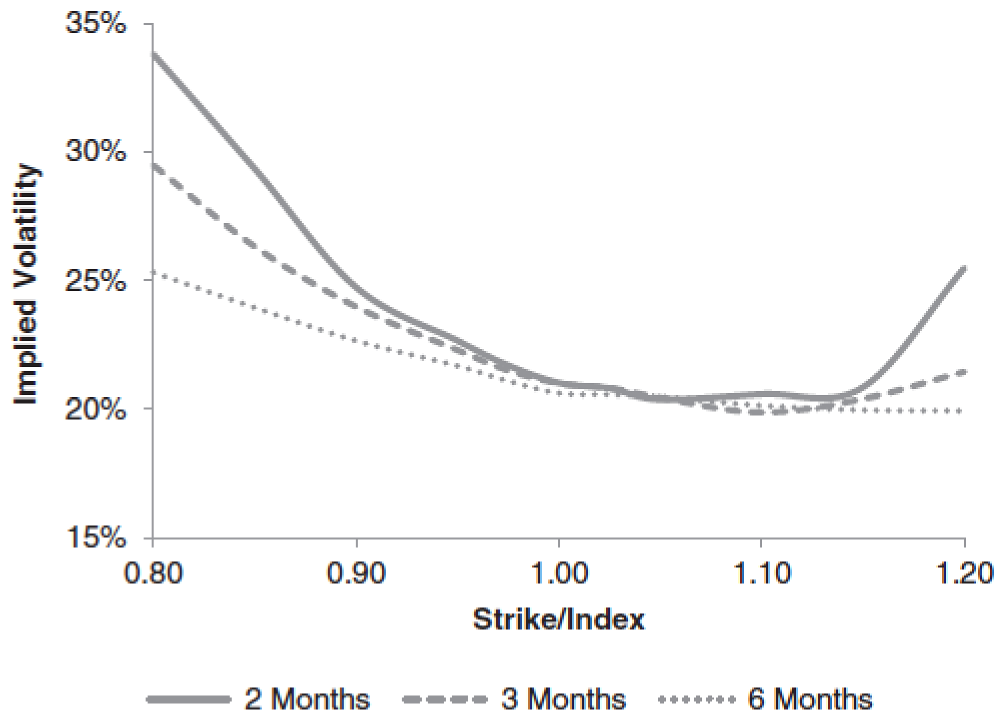

FIGURE 8.11 Volatility Smiles for VOD as of 12/31/2015
Source: Bloomberg.

Single-stock smiles tend to be more symmetric than index smiles. Single stock prices can move dramatically up or down. Indexes like the S&P when they move dramatically, move down.

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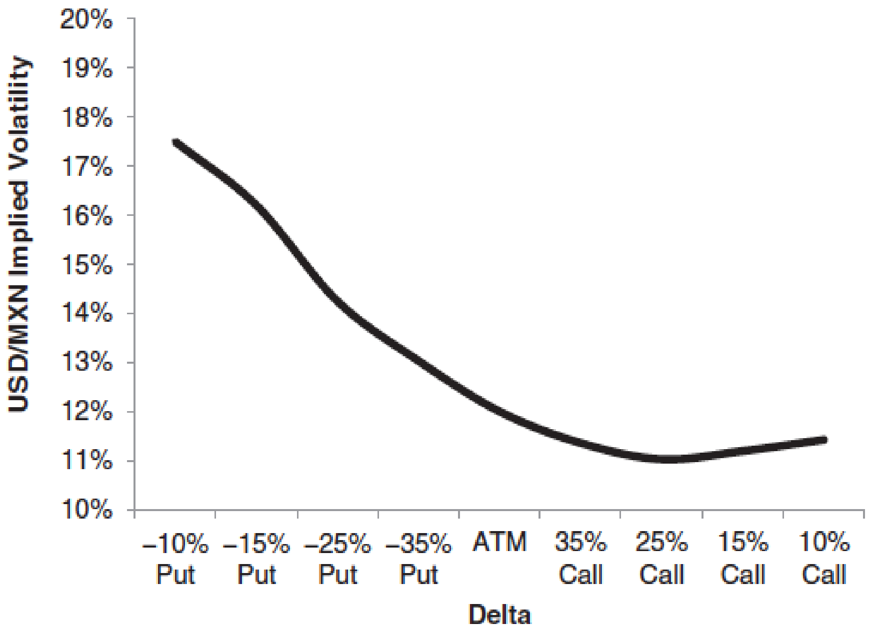
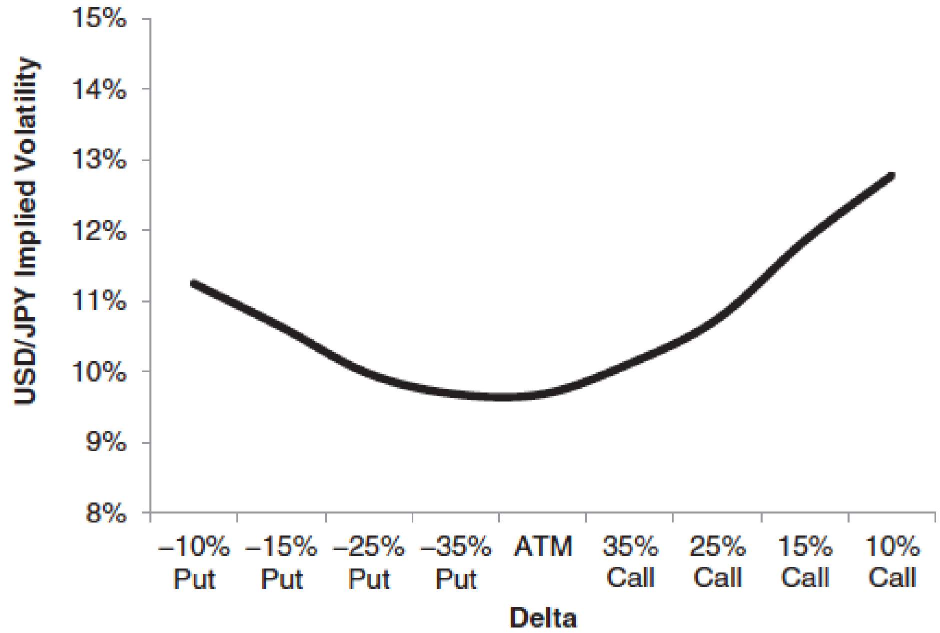
Some currency smiles....





The smiles are more symmetric for “equally powerful” currencies, less so for “unequal” ones. Equally powerful currencies are likely to move up or down.

Equity index smiles tend to be skewed to the downside. The big painful move for an index is a downward move, and needs the most protection. Upward moves hurt almost no-one. An option on index vs. cash is very different and much more asymmetric than an option on JPY vs. USD.

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Interest Rates

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Interest-rate or swaption volatility, which we will not consider much in this course, tend to be more skewed and less symmetric, with higher implied volatilities corresponding to lower interest rate strikes. This can be partially understood as the tendency of interest rates to move normally rather than lognormally as rates get low.

Suppose a rate r evolves under arithmetic Brownian motion, with normal increments:

$$dr = \sigma_a dZ,$$

If you insist on viewing this as GBM with a variable volatility, then you must write

$$\frac{dr}{r} = \frac{\sigma_a}{r}dZ \equiv \sigma_g dZ$$

which corresponds to a negative skew $\sigma_g = \dfrac{\sigma_a}{r}$

This is a locally varying volatility from the point of view of GBM.

Expectations of changes in asset volatility as the market approaches certain significant levels can also give rise to skew structure. For example, investors' perceptions of support or resistance levels in currencies and in interest rates suggest that realized volatility and hence, presumably, lognormal BS implied volatility will both decrease as those levels are approached.

8.5 Graphing the Smile:

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You see the yield curve at one instant and wonder what will happen to it later. Similarly with the smile.

The snapshot $\Sigma(S_0, t_0, K, T)$. What is $\Sigma(S, t; K, T)$?

Plot $\Sigma(\ )$ vs. strike K, moneyness K/S, forward moneyness K/S_F, $(\ln K/S_F)/(\sigma\sqrt{\tau})$, or even more generally against $\Delta = N(d_1)$, which depends on S, K, t and *implied volatility* Σ itself.

Some traders like to plot the smile against Δ because they believe it's more invariant, as we showed in the plot against $(\ln S/K)/(\sigma\sqrt{\tau})$. There are advantages:

- Plotting implied volatilities against Δ immediately indicates the hedge for the option.
- Since Δ depends on both strike and expiration, you can compare the implied volatilities of differing expirations and strikes as a function of single variable.
- Finally, d_2 is roughly the number of standard deviations the stock price must move to expire in the money and $N(d_2)$ is the risk-neutral probability of this happening. An "actuarial" measure.

Using the wrong quoting convention can distort the simplicity of the underlying dynamics. Perhaps the Black-Scholes model uses the wrong dynamics for stocks and therefore the smile looks peculiar in that quoting convention: cf: ABM vs GBM for interest-rate implied volatilities.

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The meaning of delta

Suppose that

$$\frac{dS}{S} = rdt + \sigma dZ$$

$$d\ln(S) = \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dZ$$

$$\ln\frac{S_t}{S_0} = \left(r - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z$$

The risk-neutral probability of $S_t > K$ is $P(S_t > K)$ given by

$$P(\ln S_t > \ln K) = P\left(\ln\frac{S_t}{S_0} > \ln\frac{K}{S_0}\right) = P\left[\left(r - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z > \ln\frac{K}{S_0}\right]$$

$$= P\left[Z > \frac{\ln K/S_0 - \left(r - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right] = P[Z > -d_2] = P[Z < d_2] = N(d_2) \approx N(d_1)$$

$$\approx \Delta$$

Delta is approximately the risk-neutral probability of the option finishing in the money at expiration.

The Relationship between Δ and Strike

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As the stock moves you more likely quote a certain moneyness or delta rather than a strike.

A standard measure of the skew is the *risk reversal*: difference in volatility between an out-of-the-money call option with a 25% Δ and an out-of-the-money put with a –25% Δ.

Moneyness rather than strike because it's more general to talk about moneyness as markets move.

What percentage of moneyness corresponds to a given Δ?

For simplicity set $r = 0$ and assume small volatility.

$$\Delta = N(d_1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_1} \exp\left(-\frac{y^2}{2}\right)dy$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} \exp\left(-\frac{y^2}{2}\right)dy + \int_{0}^{d_1} \exp\left(-\frac{y^2}{2}\right)dy\right] \approx \frac{1}{2} + \frac{d_1}{\sqrt{2\pi}}$$

$$d_{1,2} = \frac{\ln\frac{S}{K}}{\Sigma\sqrt{\tau}} \pm \frac{\Sigma\sqrt{\tau}}{2} \text{ and } \tau = T - t$$

At the money $S = K$ $\qquad \Delta \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\frac{\Sigma\sqrt{\tau}}{2} \approx 0.5 + (0.4)(0.5)\Sigma\sqrt{\tau}$

For 20% volatility 1 year expiration $\qquad \Delta \approx 0.5 + 0.04 = 0.54$.

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Slightly out of the money: $K = S + \delta S$ $\quad \ln\left(\frac{S}{S+\delta S}\right) = -\ln(1 + \delta S/S) \approx -\frac{\delta S}{S}$

$$d_1 = \frac{\ln\frac{S}{K}}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2} \approx -\frac{(\delta S)/S}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2}$$

Then for a slightly out-of-the-money option,

$$\Delta \approx \frac{1}{2} + \frac{d_1}{\sqrt{2\pi}} \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left(\frac{\Sigma\sqrt{\tau}}{2} - \frac{(\delta S)/S}{\Sigma\sqrt{\tau}}\right)$$

percent move of

total variance

Suppose $(\delta S)/S = 0.01$, $T = 1$ year $\Sigma = 0.2$.

Then $\Delta \approx 0.54 - \frac{(0.4)(0.01)}{0.2} = 0.54 - 0.02 = 0.52$

Thus, Δ decreases by two basis points for every 1% that the strike moves out of the money.

The difference between a 50-delta and a 25-delta option therefore corresponds to about a 12% or 13% move in the strike price.

The move δS to decrease the delta from atm 0.54 to 0.25 is approximately given by

$$\frac{1}{\sqrt{2\pi}} \frac{(\delta S)/S}{\Sigma\sqrt{\tau}} \approx 0.29 \text{ or } (\delta S)/S = 0.29\sqrt{2\pi}\Sigma\sqrt{\tau} \approx 0.29 \times 2.5 \times 0.2 \approx 0.15$$

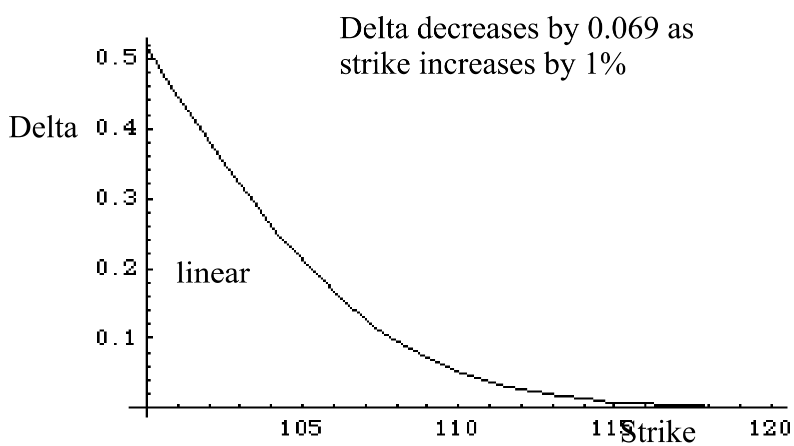
Thus the strike of the 25-delta call is about 115. Actually it's about 117 if you use the exact Black-Scholes formula to compute deltas.

More generally $$\text{change in Delta} \approx \frac{1}{\sqrt{2\pi}}\left(-\frac{(\delta S)/S}{\Sigma\sqrt{\tau}}\right)$$

and a one-basis point change in Δ corresponds to a change in $(\delta S)/S$ of about $0.025\Sigma\sqrt{\tau}$.

Delta depends upon the percent move in stock price divided by the square root of the total variance. For a greater volatility or time to expiration and you need a bigger move in the strike to get to the same Δ.

A 1-month call with S = 100 and with zero interest rates, 20% volatility, a 1% move in strike produces $(\sqrt{12})2 = 6.9$ b.p., much bigger because the total variance is smaller.



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8.6 No-Arbitrage Bounds on the Smile

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An analogy:

Yield to maturity: the quoted parameter that determines bond prices: $B_T = 100\exp(-y_T T)$

Implied volatility Σ: the quoted parameter that determines options prices in Black-Scholes.

When you deal with parameters in models, you have to be careful that they don't produce arbitrageable prices.

There are no-arbitrage bounds on bond yields.

For example suppose $B_1 = 90$ and $B_2 = 91$. Two-year bond worth more than one-year.

$\pi = \frac{91}{90}B_1 - B_2$ has zero cost.

After one year the long position is worth more than \$100, so if you wait for B_2 to mature and pay off the face value you have a riskless profit so there is something wrong with these yields. They produce negative forward prices.

There are similar constraints on option implied volatilities

Some of the Merton Inequalities for Strike

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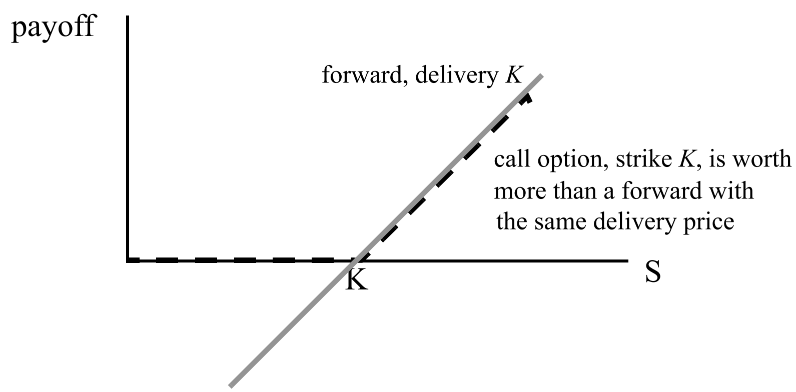
Assume zero dividends, European calls.

1. A call is always worth more than a forward: $C \ge S - Ke^{-r(T-t)}$:

   Proof: An option is always worth more than a forward, because it has the same payoff when $S_T > K$, and is worth more when $S_T < K$.

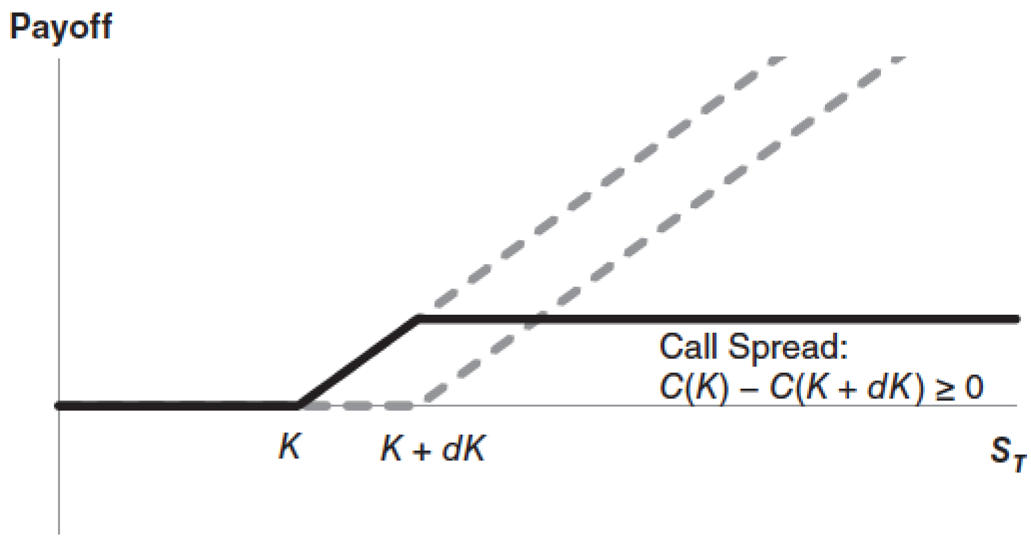
   Diagrammatically:



2. For the same expiration, options prices satisfy two constraints on their derivatives:

$$\frac{\partial C}{\partial K} \le 0 \text{ and } \frac{\partial^2 C}{\partial K^2} \ge 0,$$

Proof 1: Look at payoff of a *call spread* at expiration:



In the limit,

$$\frac{\partial C}{\partial K} \le 0$$

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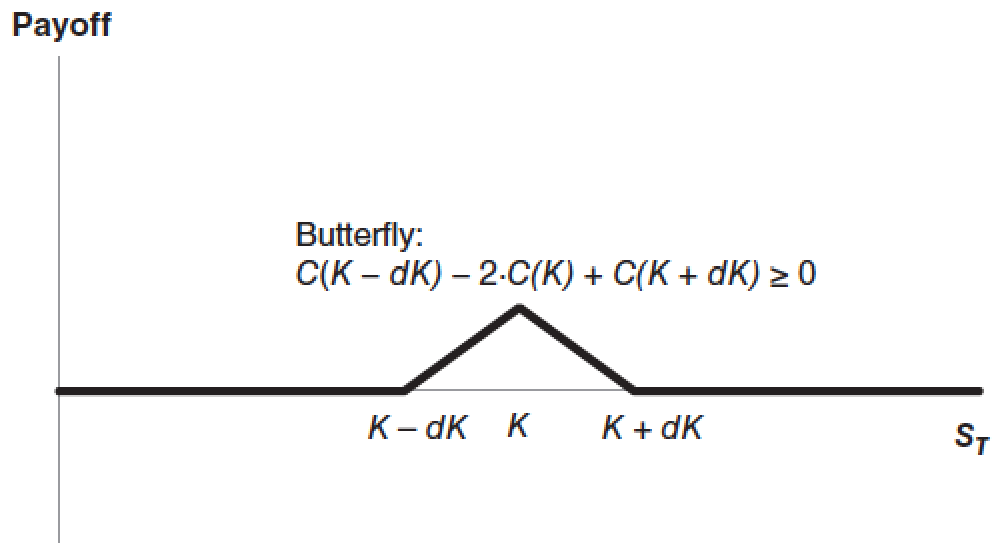
3. Look at the payoff of a *butterfly spread* at expiration:



$$\pi_B = C(K - dK) - 2C(K) + C(K + dK)$$
$$= [C(K + dK) - C(K)] - [C(K) - C(K - dK)]$$

$$\lim_{dK \to 0} \frac{\pi_B}{dK^2} = \frac{\partial^2 C}{\partial K^2}$$

From the law of no-riskless-arbitrage,

$$\frac{\partial^2 C}{\partial K^2} \geq 0$$

Similarly for puts, $\frac{\partial P}{\partial K} \geq 0$ $\qquad$ $\frac{\partial^2 P}{\partial K^2} \geq 0$

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Inequalities for the slope of the smile

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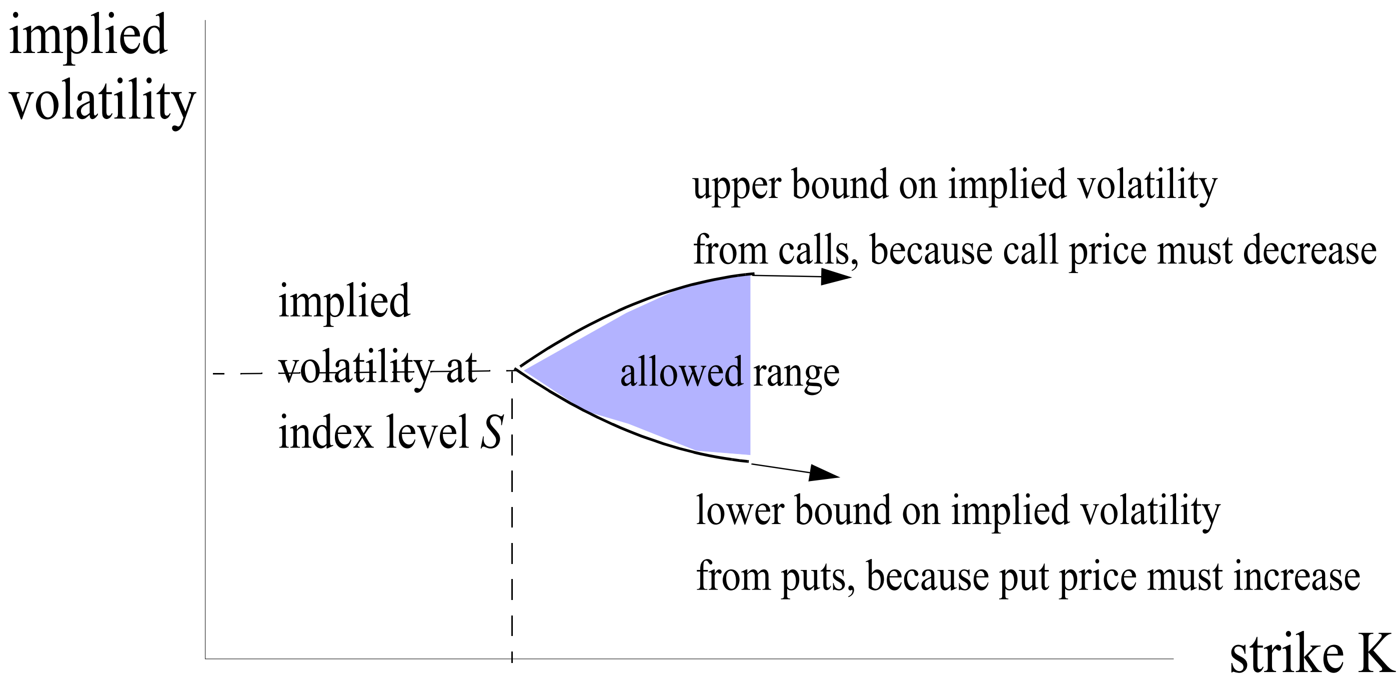
The constraints on $\frac{\partial C}{\partial K} < 0$ and $\frac{\partial P}{\partial K} \geq 0$ put limits on the slope of the smile independent of model.

Therefore they put constraints on the implied volatility parameters in the BS formula as a function of strike.



Now let's develop this idea more quantitatively.

$$C = C_{BS}(S, t, K, T, r, \Sigma)$$

$$\frac{\partial C}{\partial K} = \frac{\partial C_{BS}}{\partial K} + \frac{\partial C_{BS}}{\partial \Sigma}\frac{\partial \Sigma}{\partial K} \leq 0$$

$$\frac{\partial C_{BS}}{\partial \Sigma} = S\sqrt{\tau}N'(d_1) \equiv Ke^{-r\tau}\sqrt{\tau}N'(d_2) \qquad \text{Eq.8.2}$$

$$\frac{\partial \Sigma}{\partial K} \leq -\frac{\frac{\partial C_{BS}}{\partial K}}{\frac{\partial C_{BS}}{\partial \Sigma}} = \frac{e^{-r\tau}N(d_2)}{Ke^{-r\tau}\sqrt{\tau}N'(d_2)} = \frac{N(d_2)}{K\sqrt{\tau}N'(d_2)}$$

For small volatility, at the money: $d_2 \approx 0$, $N(d_2) \approx 0.5$ and $N'(d_2) \approx \frac{1}{\sqrt{2\pi}}$:

$$\frac{\partial \Sigma}{\partial K} \leq \sqrt{\frac{\pi}{2}}\frac{1}{K\sqrt{\tau}} \approx \frac{1.25}{K\sqrt{\tau}} \qquad \text{Eq.8.3}$$

For 1-month options on the S&P with $S = K = 4000$ $\qquad K\frac{\partial \Sigma}{\partial K} \leq 4.3$

For a 5% change in moneyness $\Delta K = 200$) volatility must change less than 0.22 or 22 vol points. Recall: the S&P skew slope for one-month options on page 22 changed by 6 vol points for a 5% change in moneyness, so it's not too far from the no-arbitrage limit.

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Asymptotically short-term expiration

$$\frac{\partial \Sigma}{\partial K} \le \frac{N(d_2)}{K\sqrt{\tau}N'(d_2)}$$

At-the-money forward, as $\tau \to 0$

$$d_2 \to -\frac{\Sigma\sqrt{\tau}}{2} \to 0$$

$$N(d_2) \to \frac{1}{2}$$

$$N'(d_2) \to \frac{1}{\sqrt{2\pi}}$$

and so

$$\frac{\partial \Sigma}{\partial K} \le O(\tau^{-1/2}) \qquad \text{as } \tau \to 0.$$

As the time to expiration $\tau \to 0$, the slope steepness can increase no faster than $O(\tau^{-1/2})$.

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Asymptotically long expiration

At the other extreme, as $\tau \to \infty$, $d_2 \to -\infty$, and therefore

$$\frac{\partial \Sigma}{\partial K} \le \frac{1}{K\sqrt{\tau}} \frac{N(d_2)}{N'(d_2)} \sim O\left(\frac{1}{\sqrt{\tau}} \frac{1}{\sqrt{\tau}}\right) \sim O\left(\frac{1}{\tau}\right)$$

To prove the line above we have made use of the asymptotic relation

$N(d_2)/N'(d_2) \sim O(\tau^{-0.5})$ as $\tau \to \infty$.

The area under the tail gets smaller faster than the height of the tail.

Thus, the slope of the smile can decrease with time to expiration no more slowly than $O(\tau^{-1})$.

Reference: *Arbitrage Bounds on the Implied Volatility Strike and Term Structures of European-Style Options*. Hardy M. Hodges, Journal of Derivatives, Summer 1996, pp. 23-35.

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8.7 Some Behavioral Reasons for an Implied Volatility Skew

There are two approaches to options trading:

1. Consumers buying protection from or seeking exposure to the underlier.
2. Sophisticated people trading volatility as an asset.

- Knowledge of past behavior in options markets suggests a skew in options would be wise. (How much, though? What's the fair value?) Implied and realized volatilities go up after a crash.

- Expectation of future changes in volatility naturally gives rise to a term structure.

- Expectation of changes in volatility as support or resistance levels in currencies and interest rates suggests that realized volatility will decrease as those levels are approached.

- Expectation of an increase in the cross-sectional correlation between the returns of constituent stocks in the index as the market drops can cause an increase in the volatility of the entire index.

$$r = \sum w_i r_i \qquad \sigma^2 = \sum w_i w_j \sigma_{ij}$$

- Dealers' tend to be short options because they sell zero-cost collars (short otm call-long otm put) to investors who want protection against a decline.

8.8 An Overview of Smile-Consistent Models

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Three choices:

(i) Model the stochastic evolution of the underlying asset *S* and its realized volatility, and then deduce $\Sigma(S, t, K, T)$: fundamental approach, avoid arbitrage violations; but it's hard to get the right equation;

(ii) directly model the dynamics of the parametric surface $\Sigma(S, t, K, T)$.
more intuitive but we are modeling a parameter in a bad model, not a price, and hard to avoid arbitrage violations.

(iii) Pragmatic vanna-volga models to value an exotic option by replicating it with a portfolio of vanilla options in the BS world, by making the exotic and the vanillas have the same vanna and volga. Then adjust the value for a skewed world by how much the portfolio of vanillas change as the skew is turned on. It's analogous to using the Black-Scholes world as a control variate.

In the end, different markets have different smiles and it is unlikely that one grand replacement for Black-Scholes will cover all smiles in all markets.

Local Volatility Models -- the first smile models.

Black-Scholes: $\Sigma(S, t, K, T) = \sigma$ is independent of strike and expiration.

Local volatility models: $$\frac{dS}{S} = \mu(S, t)dt + \sigma(S, t)dZ$$

$\sigma(S, t)$ is a *deterministic* function of a *stochastic variable* S.

This is a one-factor model so our replication strategy and risk-neutral valuation still works.
But is it true?

The issue: Calibration: how to choose $\sigma(S, t)$ to match market values of $\Sigma(S, t, K, T)$?
The model provides great intuition.

People make use of it for trading and as a proxy for other models, and extend it with some stochastic volatility.

What might account for local volatility being a function $\sigma(S, t)$?

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***The leverage effect*:** leverage makes volatility increase as the stock price moves lower:

$$S = A - B \qquad \text{assets - liabilities}$$

$$\frac{dA}{A} = \sigma dZ$$

$$\frac{dS}{S} = \frac{dA}{S} = \frac{A\sigma dZ}{S} = \frac{(S+B)}{S}\sigma dZ$$

$$\sigma_S = \sigma(1 + B/S)$$

***Constant Elasticity of Variance (CEV) models (Cox and Ross)*:**

$$\frac{dS}{S} = \mu(S,t)dt + \sigma S^{\beta-1}\, dZ$$

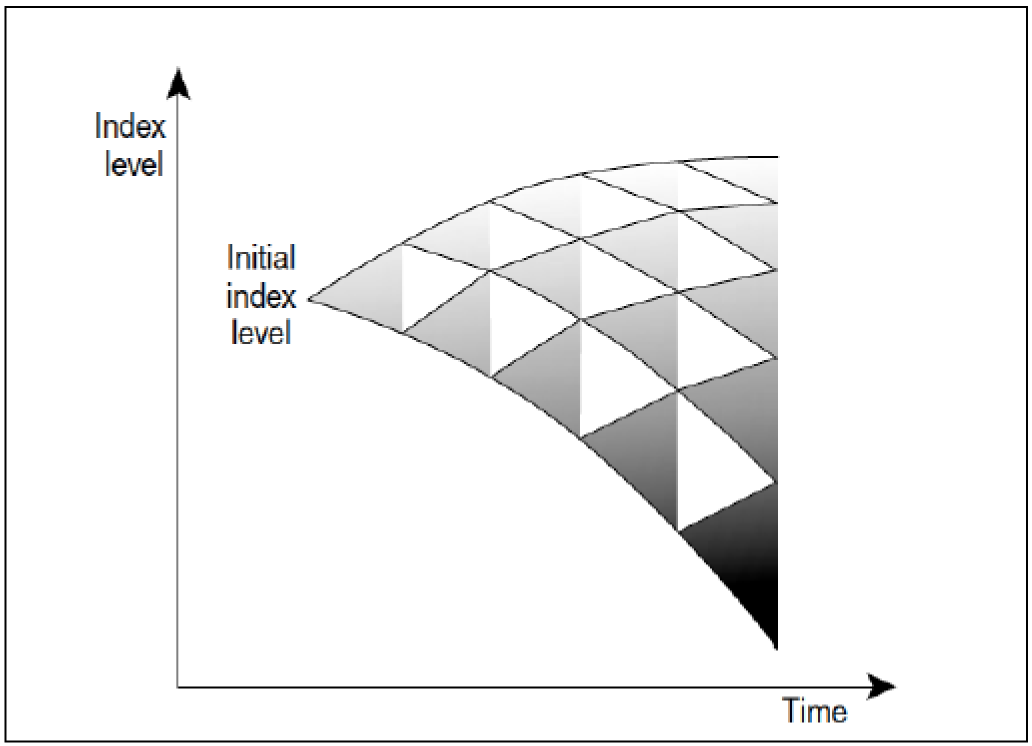
$$\text{Vol} = \sigma S^{\beta - 1}$$

$$\frac{S}{V}\frac{\partial V}{\partial S} = \text{constant}$$

$\beta = 1$ lognormal; $\beta = 0$ normal evolution. β needs to be large and negative, but then model has mathematical problems.

CEV is a parametric model and cannot fit an arbitrary smile; local volatility models are non-parametric and $\sigma(S, t)$ can be calibrated numerically.

Schematic view of a local volatility model



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Stochastic Volatility Models

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Volatility is actually random too.

$$dS = \mu S dt + \sigma S dZ_t$$

$$d\sigma = p\sigma dt + q\sigma dW_t$$

$$E[dWdZ] = \rho dt$$

In that case perfect replication is impossible if you allow yourself to hedge only with the stock.

Analogy: to hedge a long position in a bond that is exposed to interest rates, you have to short another bond. You cannot short an interest rate, only a security. The hedged portfolio, long one bond, short another, must earn the riskless rate. Hence one can derive the PDE for interest-rate sensitive securities and their prices. (Vasiçek)

Similarly, to hedge an option's exposure to volatility, you have to short another option. You cannot short σ. ***Assuming (!?)*** you know the stochastic process for volatility, then you can hedge one option's exposure to volatility with another option and derive an arbitrage-free formula for options values. Stochastic volatility models assume that the correlation ρ is constant but that is stochastic too.

Schematic view of stochastic volatility model

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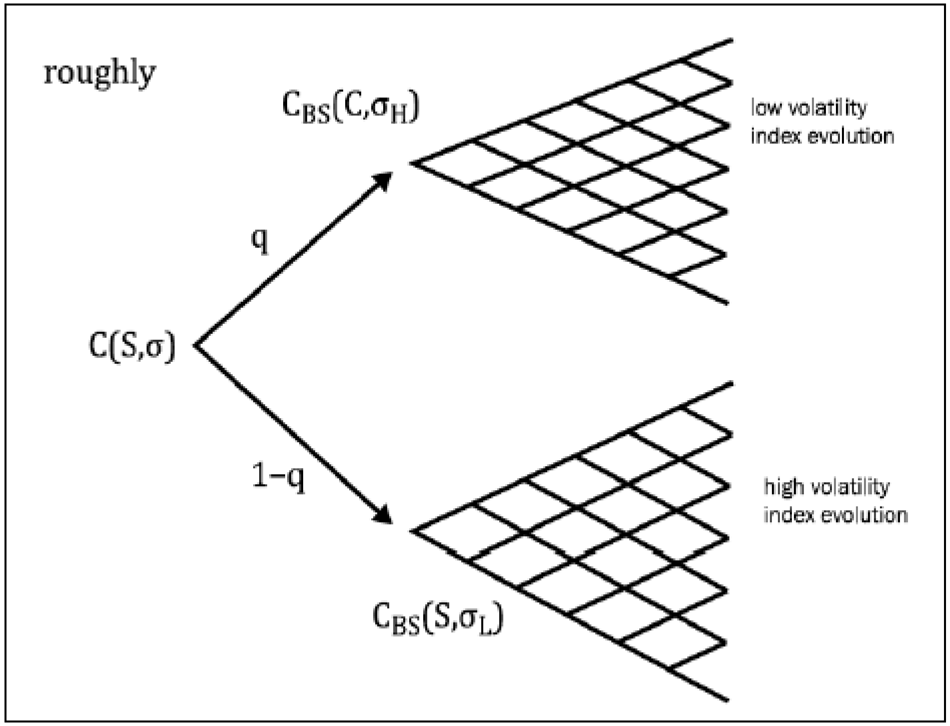

Jump-Diffusion Models

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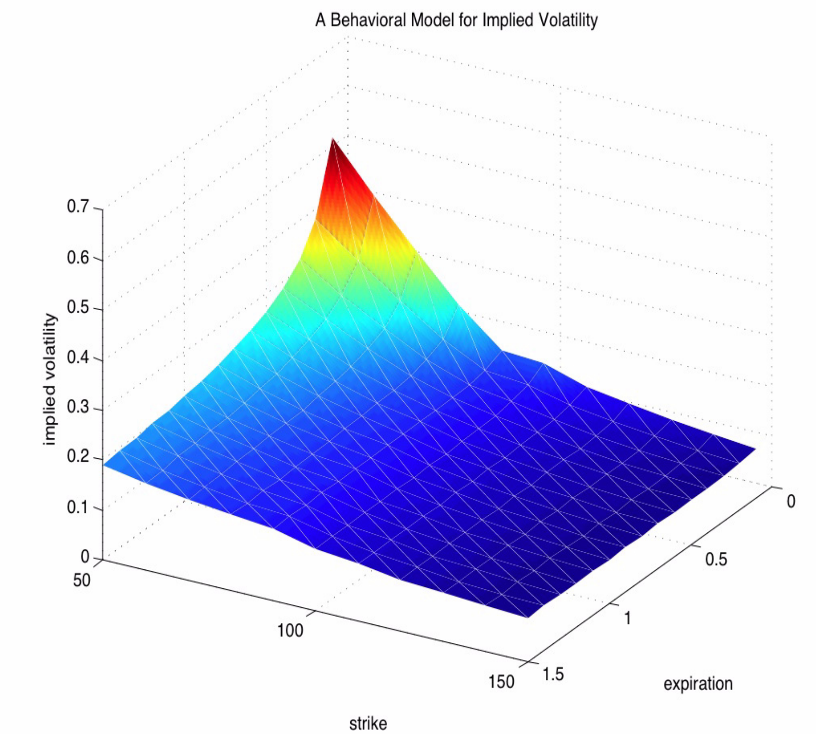
Black-Scholes ignores discontinuous jumps. Merton model allows an arbitrary number of jumps plus diffusion.



With a finite number of jumps of known size in the model, one can hedge the jumps perfectly by dynamic trading in a finite number of options, the stock and the bond, and so achieve risk-neutral pricing.

With an arbitrary or infinite possible number of jumps, one cannot, but people use risk-neutral pricing anyway.

Jumps affect the short-expiration part of the skew more than the long expiration part of the skew.

Schematic view of a jump-diffusion model

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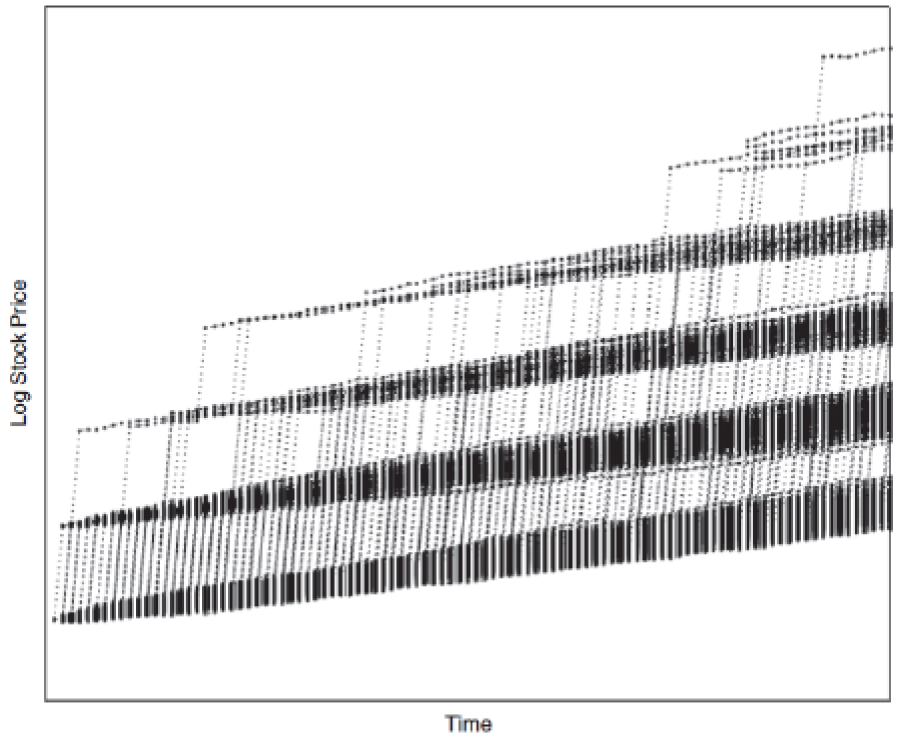

FIGURE 24.3 A Monte Carlo Simulation of the Log Stock Prices in the Jump-Diffusion Model

A Plenitude of Other Models

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Local stochastic volatility: use local volatility to match the skew, stochastic volatility to add realism.

There are many other smile models too, which we may discuss later: fractional Brownian motion for volatility, forward variance models, mixing models, variance gamma models, stochastic volatility models of other types, stochastic implied volatility models …

In practice, one has to see which model best describes the market one is working in.

In the real world there is indeed diffusion, jumps and stochastic volatility!

There are too many different ways of fitting the observed smile that the model is non-parsimonious and offers too many choices.

In the end, you want to model the market with reasonable (but not perfect) accuracy via a fairly simple model that captures most of the important behavior of the asset.

A model is only a model, not the real thing.