LECTURE 22

STOCHASTIC VOLATILITY ...

5 classes: 1 on Stochastic vol; 2 on Jump diffusion;
Please Attend For
Mon May 24: Mike Kamal of Citadel;
Wed May 26: Chris Delong of Taconic

Mixing Theorem

The price of an option in a stochastic volatility model with zero correlation is the weighted integral/sum over BS prices over the distribution of path volatilities.

$$V = \sum_{\text{all } \overline{\sigma_T}} f(\overline{\sigma_T}) \times V_{\text{BSM}}(S, t, K, T, r, \overline{\sigma_T})$$

It doesn't matter what order the stochastic volatilities occur in -- as long as the variance along the path is the same, all paths with that variance have the same BS terminal distribution of the stock price.

What is the advantage of this?

The mixing theorem reduces this to a **one-dimensional simulation or integration in the model**.

IF the correlation is different from zero, this doesn't work: then all paths conditional on a definite variance still have a normal distribution of returns, BUT that variance depends on the stock price path, not just on time. The resultant formula is

$$V_t = E\left[V_{\text{BSM}}\left(S_t^*(\overline{\sigma_T}, \rho), K, r, \overline{\sigma_T}^*(\rho), T\right)\right]$$

where the stock price and volatility in the Black-Scholes formula are shifted to "fake" values that differ from actual values by something related to the correlation. Much less useful.

22.1 The zero correlation smile is symmetric

The mixing theorem:

$$C_{SV} = \int_0^{\infty} C_{BSM}(\overline{\sigma_T})\phi(\overline{\sigma_T})d\overline{\sigma_T}$$

Taylor expansion about the average value $\bar{\bar{\sigma}}$ of the path volatility, dropping subscript T.

$$\begin{aligned}
C_{\mathrm{SV}} &= \int_0^{\infty} C_{\mathrm{BSM}}\left(\bar{\bar{\sigma}} + \bar{\sigma} - \bar{\bar{\sigma}}\right) \phi(\bar{\sigma})\, d\bar{\sigma} \\
&\approx \int_0^{\infty} \left[C_{\mathrm{BSM}}(\bar{\bar{\sigma}}) + \left.\frac{\partial C_{\mathrm{BSM}}}{\partial \bar{\sigma}}\right|_{\bar{\bar{\sigma}}} (\bar{\sigma} - \bar{\bar{\sigma}}) + \frac{1}{2} \left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \bar{\sigma}^2}\right|_{\bar{\bar{\sigma}}} (\bar{\sigma} - \bar{\bar{\sigma}})^2 \right] \phi(\bar{\sigma})\, d\bar{\sigma} \qquad (21.13) \\
&\approx C_{\mathrm{BSM}}(\bar{\bar{\sigma}}) + 0 + \frac{1}{2} \left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \bar{\sigma}^2}\right|_{\bar{\bar{\sigma}}} \mathrm{var}[\bar{\sigma}] \\
&\approx C_{\mathrm{BSM}}\left(\bar{\bar{\sigma}}\right) + \frac{1}{2} \left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \bar{\sigma}^2}\right|_{\bar{\bar{\sigma}}} \mathrm{var}\left[\bar{\sigma}\right]
\end{aligned}$$

where $var[\bar{\sigma}]$ is the *variance of the path volatility* $\bar{\sigma}$ *over the life* τ *of the option*.

$$\Sigma \approx \bar{\bar{\sigma}} + \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\left(\frac{1}{\bar{\bar{v}}}\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\bar{\bar{v}}^2}{4}\right] \qquad \bar{\bar{v}} = \bar{\bar{\sigma}}\sqrt{\tau}.$$

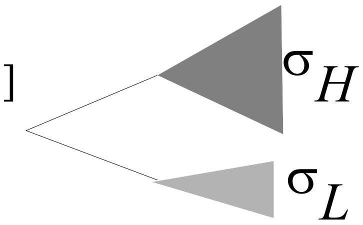
$$\approx \bar{\bar{\sigma}} + \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\frac{1}{\bar{\bar{\sigma}}^2\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\bar{\bar{\sigma}}^2\tau}{4}\right]$$

Quadratic function of $\ln S/K$, parabolically shaped smile that varies as $(\ln S/K)^2$ or $(K-S)^2$.
Sticky moneyness smile, no scale, a function of K/S.

22.2 A Simple Two-State Stochastic Volatility Model

Mixing two path

$$C_{SV} = \frac{1}{2}[C_{BSM}(S, K, \sigma_H) + C_{BSM}(S, K, \sigma_L)]$$



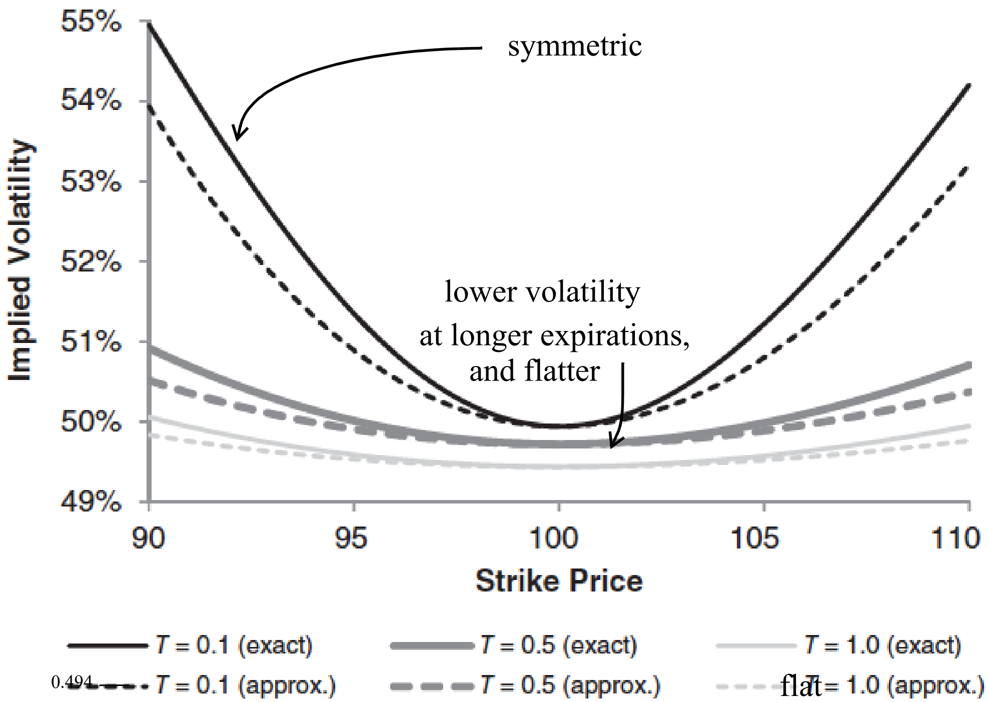
volatilities

Low volatility be 20% and the high volatility 80% with a mean volatility of 50%.

Variance of the volatility is $0.5(0.8-0.5)^2 + 0.5(0.5-0.2)^2 = 0.09$ per year.

In the figure below we show the smile corresponding to the exact mixing formula together with the approximation $\bar{\bar{\sigma}} + \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\frac{1}{\bar{\bar{\sigma}}^2\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\bar{\bar{\sigma}}^2\tau}{4}\right]$

The Volatility Smile in a Discrete Two-Volatility Model With Zero Correlation



- The smile with zero correlation is symmetric;
- the long-expiration smile is relatively flat, while the short expiration skew is more curved (note the τ^{-1} coefficient of $(\ln S/K)^2$ in the formula; and
- at the forward price of the stock, the at-the-money implied volatility decreases monotonically with time to expiration, and lies below the mean volatility of 50%, because of the negative convexity of the Black-Scholes options price at the money.

The approximate solution works quite well.

At-the-money, with these parameters, the approximation reduces to

$$\Sigma_{SV}^{ATM} \approx \bar{\bar{\sigma}} + \frac{1}{2} var[\bar{\sigma}] \left[\frac{-\left(\bar{\bar{\sigma}}^4 \tau^2\right)/4}{\bar{\bar{\sigma}}^3 \tau} \right] \approx \bar{\sigma} - \frac{1}{8} var[\bar{\sigma}][\bar{\bar{\sigma}}\tau] \approx 0.5 - \frac{1}{8}(0.09)\bar{\bar{\sigma}}\tau \approx 0.5 - 0.0056\tau$$

For $\tau = 1$, the at-the-money volatility is 0.4944, which agrees well with the figure above.

22.3 Next: The Smile for GBM Continuous Stochastic Volatility with No Mean Reversion and Zero Correlation

A more sophisticated continuous distribution of stochastic volatilities. $d\sigma = a\sigma dt + b\sigma dZ$ $\rho = 0$, an initial volatility of 0.2 and a volatility of volatility of 1.0, straightforward Monte Carlo simulation of stock paths.From the mixing formula:

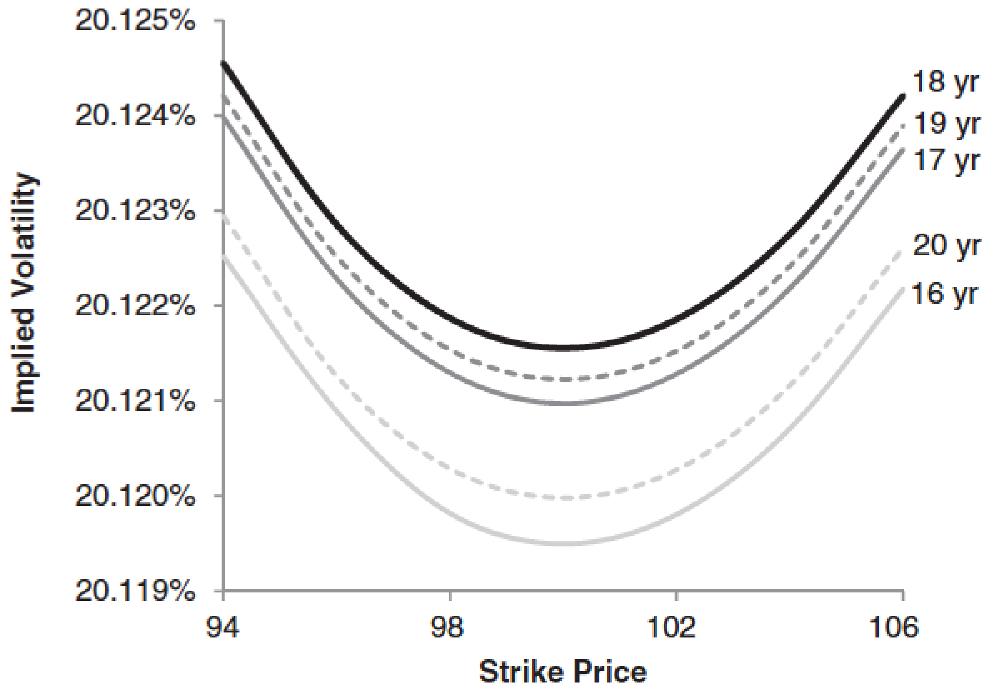


FIGURE 21.4 The Smile for a Variety of Expirations in a GBM Stochastic Volatility Model with Zero Correlation, $a = 0$, $b = 0.1$

Still symmetric but now not monotonic in expiration. $\overset{\Rightarrow}{\sigma}$ is a function of time because of the Brownian motion, which changes the time dependence from the two-state case.

The curvature of the smile in the Figure is approximately independent of time to expiration because of the GBM of stochastic volatility. You can understand this from the formula

$$\Sigma \approx \bar{\bar{\sigma}} + \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\frac{1}{\bar{\bar{\sigma}}^2\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\bar{\bar{\sigma}}^2\tau}{4}\right]$$

$$\approx \bar{\bar{\sigma}} + \frac{1}{2}\frac{\mathrm{var}\,[\bar{\sigma}]}{\bar{\bar{\sigma}}^3\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\mathrm{var}\,[\bar{\sigma}]\,\bar{\bar{\sigma}}\tau}{8}$$

The quadratic skew term is

$$\frac{1}{2}\frac{\mathrm{var}\,[\bar{\sigma}]}{\bar{\bar{\sigma}}^3\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 \approx \frac{1}{2}\frac{\frac{b^2}{3}\sigma^2\tau}{\bar{\bar{\sigma}}^3\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2$$

$$\approx \frac{1}{6}\frac{b^2}{\sigma}\left(\ln\left(\frac{S}{K}\right)\right)^2$$

roughly independent of time to expiration.

This is different from the case where the volatility distribution is two-state discrete.

An Analytic Approximation to Understand Non-monoticity

$$\Sigma_{\text{atm}} \approx \bar{\bar{\sigma}} \left(1 - \frac{1}{8}\text{var}[\bar{\sigma}]\,\tau\right)$$ **Equation GBM** Eq.1.1

Let's estimate the time-to-expiration dependence of these path-volatility quantities in a geometric Brownian motion model for the *instantaneous* volatility σ.

$$d\sigma = a\sigma dt + b\sigma dZ$$

σ^2, therefore, satisfies a similar stochastic differential equation with

$$\text{drift}[\sigma^2] = 2a + b^2$$

$$\text{vol}[\sigma^2] = 2b$$

The extra b^2 term in the drift arises from Ito's Lemma for the square of a Wiener process.

Now let's consider the path variance $\bar{\sigma}^2$ which is relevant to the mixing formula. The path variance is an arithmetic average of the instantaneous variances out to time T, but the variance itself evolves geometrically, and so there is no closed-form expression for its value. Nevertheless, it is well known that the average has approximately 1/2 the drift and $1/\sqrt{3}$ the volatility of the non-averaged variable.

Thus approximately

$$\mathrm{drift}[\bar{\sigma}^2] \approx a + \frac{1}{2}b^2$$

$$\mathrm{vol}[\bar{\sigma}^2] \approx \frac{2b}{\sqrt{3}}$$

But Equations 1.1 involves the square root of $\bar{\sigma}^2$ (i.e., $\bar{\sigma}$), which from Ito's Lemma for the square root of a variable undergoing geometric Brownian motion is

$$\mathrm{drift}[\bar{\sigma}] \approx \frac{1}{2}\left(a + \frac{1}{2}b^2\right) - \frac{1}{8}\left(\frac{2b}{\sqrt{3}}\right)^2$$

$$\approx \frac{a}{2} + \frac{1}{12}b^2$$

$$\mathrm{vol}[\bar{\sigma}] \approx \left(\frac{b}{\sqrt{3}}\right)$$

This drift means that there is a time-dependence to the coefficient $\bar{\bar{\sigma}}$ which grows by GBM in Equations 1.1.

$$\bar{\bar{\sigma}}(\tau) \approx \sigma e^{\left(\frac{a}{2}+\frac{1}{12}b^2\right)\tau}$$
$$\approx \sigma\left[1+\left(\frac{a}{2}+\frac{b^2}{12}\right)\tau+\frac{1}{2}\left(\frac{a}{2}+\frac{b^2}{12}\right)^2\tau^2\right]$$

The volatility of the path volatility is $b/\sqrt{3}$. The variance of the path volatility is therefore

$$\mathrm{var}[\bar{\sigma}] \approx \frac{b^2}{3}\sigma^2\tau$$

Thus

$$\Sigma_{\mathrm{atm}} \approx \bar{\bar{\sigma}}\left(1-\frac{1}{8}\mathrm{var}[\bar{\sigma}]\,\tau\right)$$
$$\approx \sigma\left[1+\left(\frac{a}{2}+\frac{b^2}{12}\right)\tau+\frac{1}{2}\left(\frac{a}{2}+\frac{b^2}{12}\right)^2\tau^2\right]\left(1-\frac{1}{8}\frac{b^2}{3}\sigma^2\tau^2\right)$$
$$\approx \sigma\left[1+\left(\frac{a}{2}+\frac{b^2}{12}\right)\tau+\left(\frac{1}{2}\left(\frac{a}{2}+\frac{b^2}{12}\right)^2-\frac{b^2}{24}\sigma^2\right)\tau^2\right]$$

For a = 0 as in **Figure 21.4** above

$$\Sigma_{\text{atm}} \approx \sigma\left[1+\frac{b^2}{12}\tau+\frac{b^2}{24}\left(\frac{b^2}{12}-\sigma^2\right)\tau^2\right]$$

For b = 0.1 and $\sigma = 0.2$ the τ^2 term has a negative coefficient and explains the non-monotonicity in the Figure. The maximum occurs at $\tau = \dfrac{1}{\sigma^2-\dfrac{b^2}{12}}$, about 25, roughly accounting for the Figure.

22.4 The Smile in Mean-Reverting Stochastic Volatility Models

Finally, we explore the smile when volatility mean reverts:

$$\frac{dS}{S} = \mu dt + \sigma dZ \qquad d\sigma = \alpha(m-\sigma)dt + \beta\sigma dW \qquad dZdW = \rho dt$$

The following pages show the results of a double Monte Carlo for BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation.

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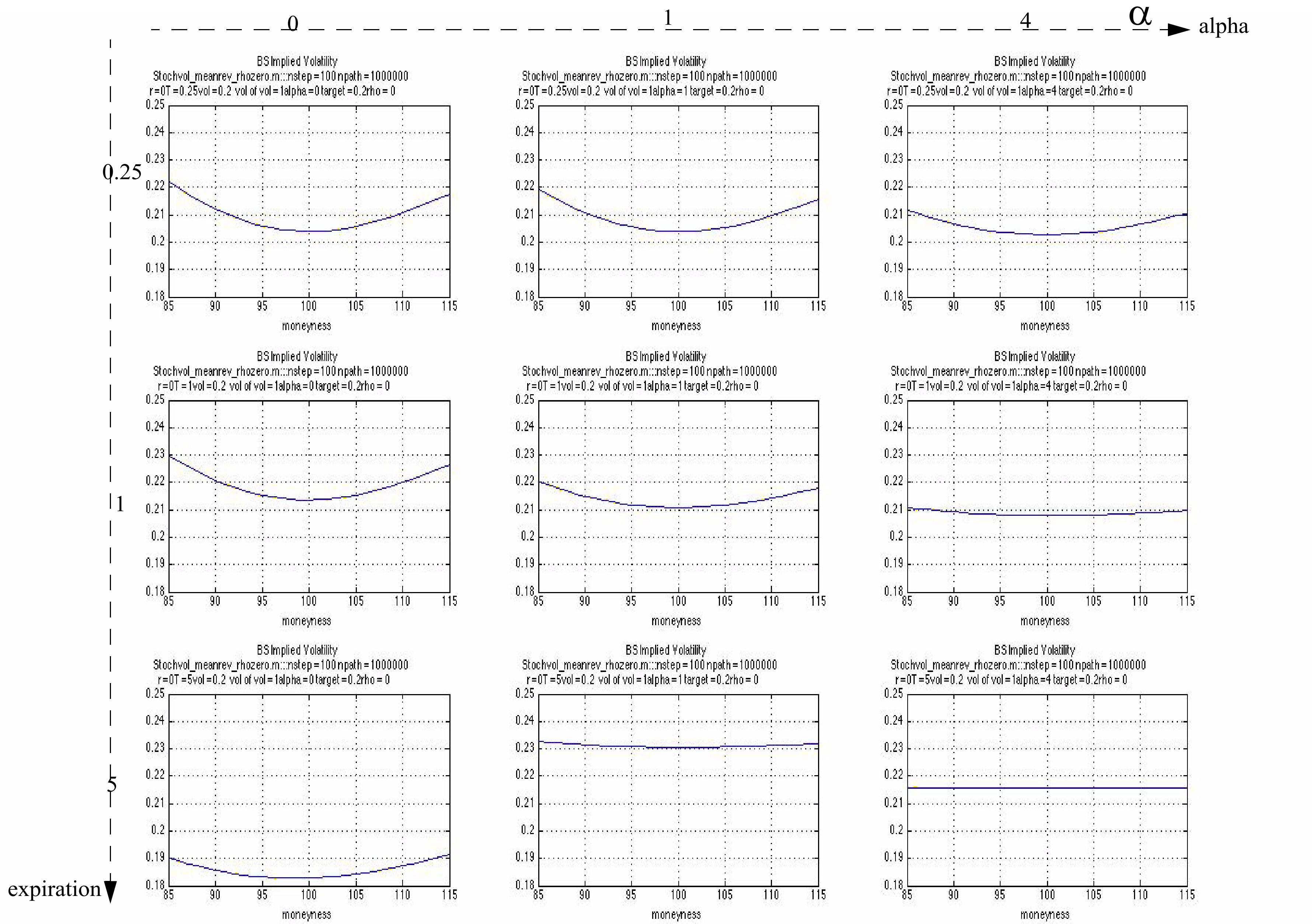
BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation, initial volatility 0.2 and target volatility 0.2.



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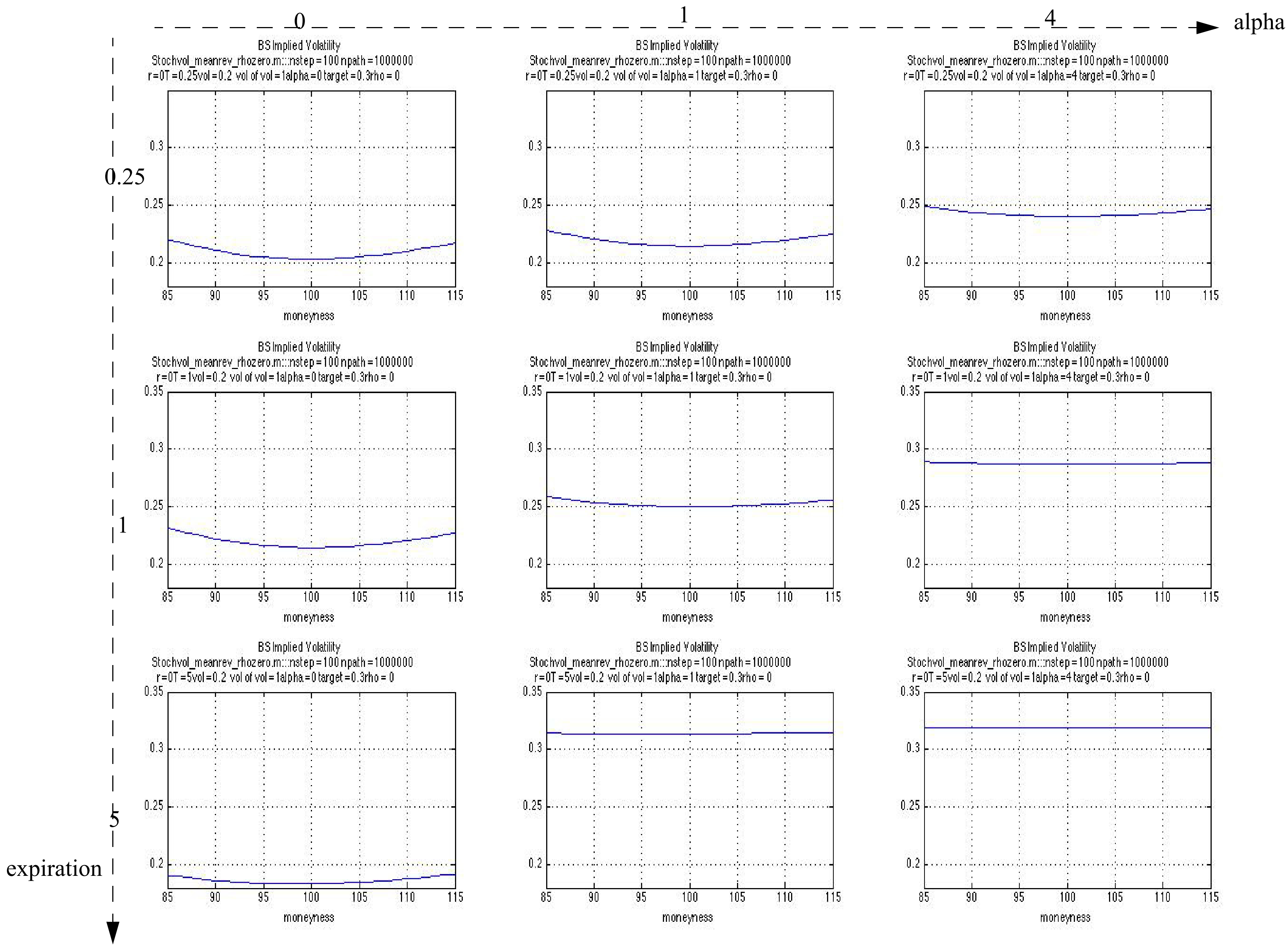
BS Implied Volatility as a function of mean reversion strength and expiration for zero correlation, initial volatility 0.2 and target volatility 0.3.



22.1 Mean-Reverting Stochastic Volatility and the Asymptotic Behavior of the Smile.

$$\Sigma \approx \bar{\bar{\sigma}} + \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}}\left[\frac{1}{\bar{\bar{\sigma}}^2\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\bar{\bar{\sigma}}^2\tau}{4}\right]$$

From p4, approximately for small vol of vol Eq.22.2

Now employ intuition about mean reversion for σ.

Short Expirations, Zero Correlation

In the limit that $\tau \to 0$

$$\lim_{\tau\to 0}\Sigma \approx \bar{\bar{\sigma}} + \frac{1}{2}\mathrm{var}[\bar{\sigma}]\frac{1}{\bar{\bar{\sigma}}^3\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2$$

For small times, $var[\sigma] = \beta'\tau$. Substituting this relation into Equation leads to the expression

$$\lim_{\tau\to 0}\Sigma \approx \bar{\bar{\sigma}} + \frac{1}{2}\beta'\frac{1}{\bar{\bar{\sigma}}^3}\left(\ln\left(\frac{S}{K}\right)\right)^2$$

$\tau \to 0$ limit Eq.22.3

Smile is quadratic and finite as $\tau \to 0$ for short expirations, and depends on volatility of volatility.

Long Expirations

As $\tau \to \infty$ $$\Sigma_{SV} \approx \bar{\bar{\sigma}} - \frac{1}{8} var[\bar{\sigma}][\bar{\bar{\sigma}}\tau]$$

where $\bar{\sigma}$ is the path volatility over the life of the option and is itself a function of the time to expiration due to the stochastic nature of the instantaneous volatility.

For Ornstein-Uhlenbeck processes the path volatility to expiration $\bar{\sigma}$ converges to a constant along all paths as $\tau \to \infty$, and so $\bar{\sigma}$ has zero variance as $\tau \to \infty$, $var[\bar{\sigma}] \to \text{const}/\tau$.

$$\Sigma_{SV} \approx \bar{\bar{\sigma}} - \frac{\text{const}}{8} \bar{\bar{\sigma}} \tag{Eq.22.4}$$

NO smile at large expirations.

Why is the correction term negative? The option price $C_{BS}(\sigma)$ has negative convexity, and for a concave function $f(x)$, the average of the function $\overline{f(x)}$ is less than the function $f(\bar{x})$ of the average.

Thus, for zero correlation, we expect to see stochastic volatility smiles that look like this:

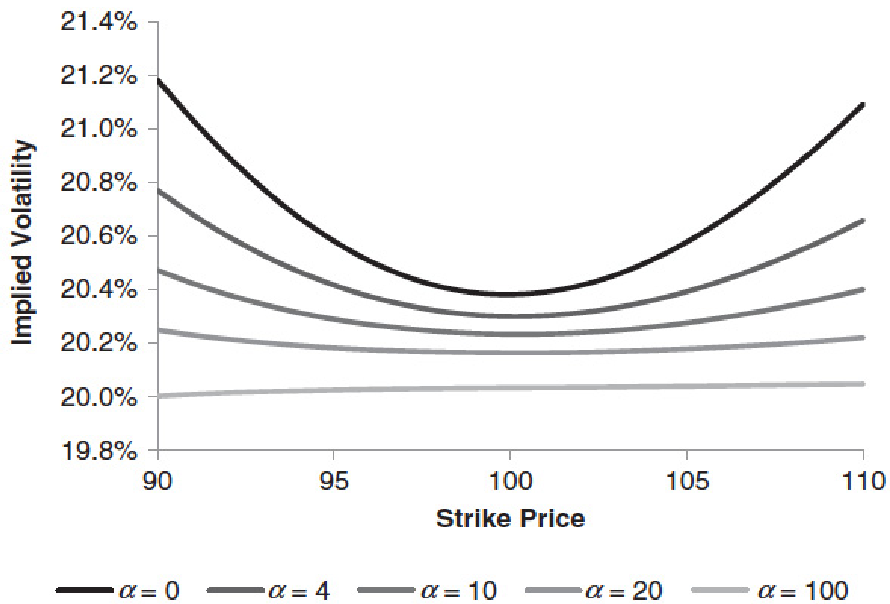


FIGURE 22.2 The Smile for a Mean-Reverting Stochastic Volatility Model with $\rho = 0$ and Varying Mean Reversion Strength

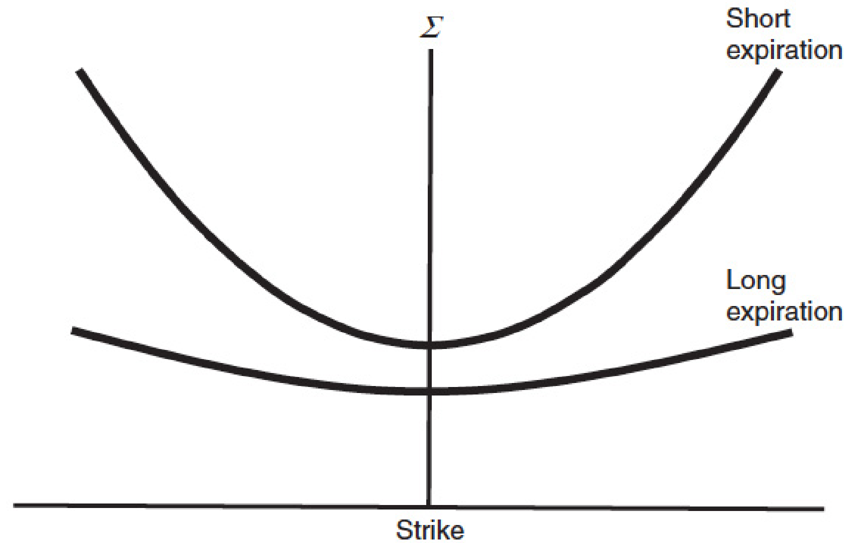


FIGURE 22.1 The Smile for Stochastic Volatility Model with $\rho = 0$

We can understand this intuitively as follows. In the long run, all paths will have the same volatility if volatility mean reverts, and so the long-term skew is flat. In the short run, bursts of high volatility act almost like jumps, and induce fat tails

22.2 Non-zero correlation ρ in stochastic volatility models

No correlation lead to a symmetric smile.

With correlation the smile still depends on (K/S_F) but the dependence is not quadratic.

FIGURE 22.3 The Smile as a Function of Correlation in a Stochastic Volatility Model with Zero Mean Reversion

A very steep short-term skew is difficult in these models; since volatility diffuses continuously in these models, at short expirations volatility cannot have diffused too far. A very high volatility of volatility and very high mean reversion are needed to account for steep short-expiration smiles.

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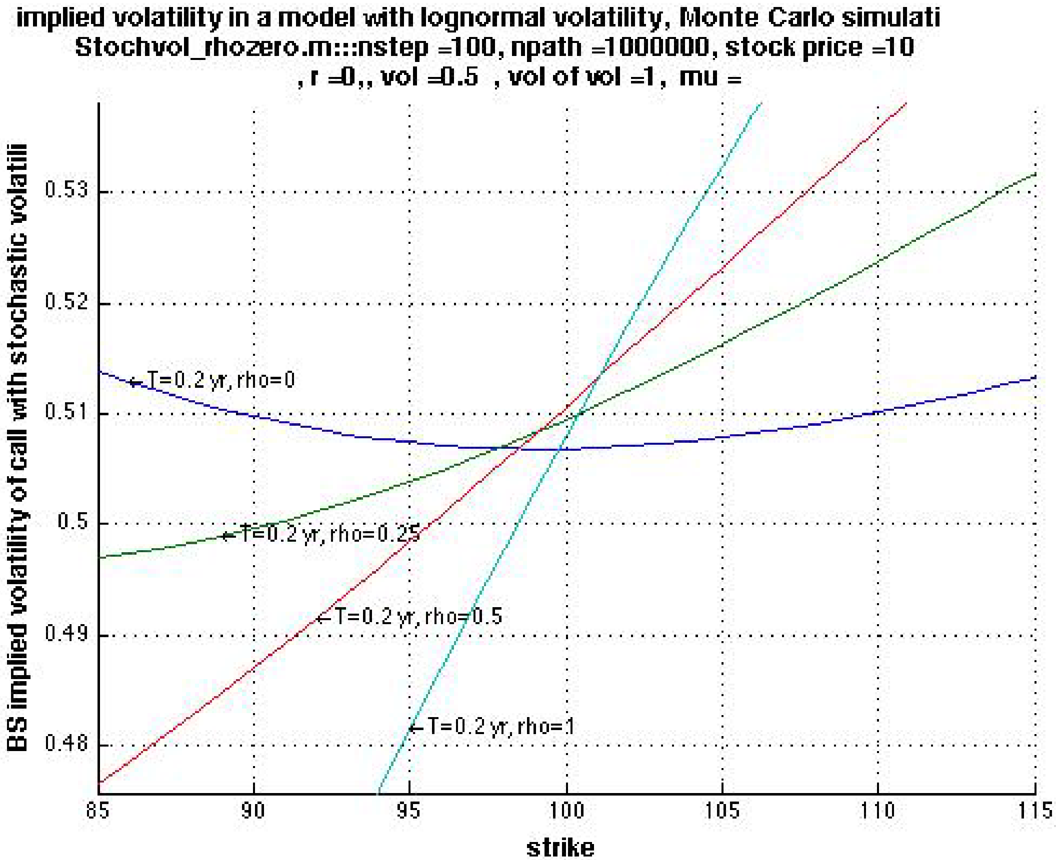
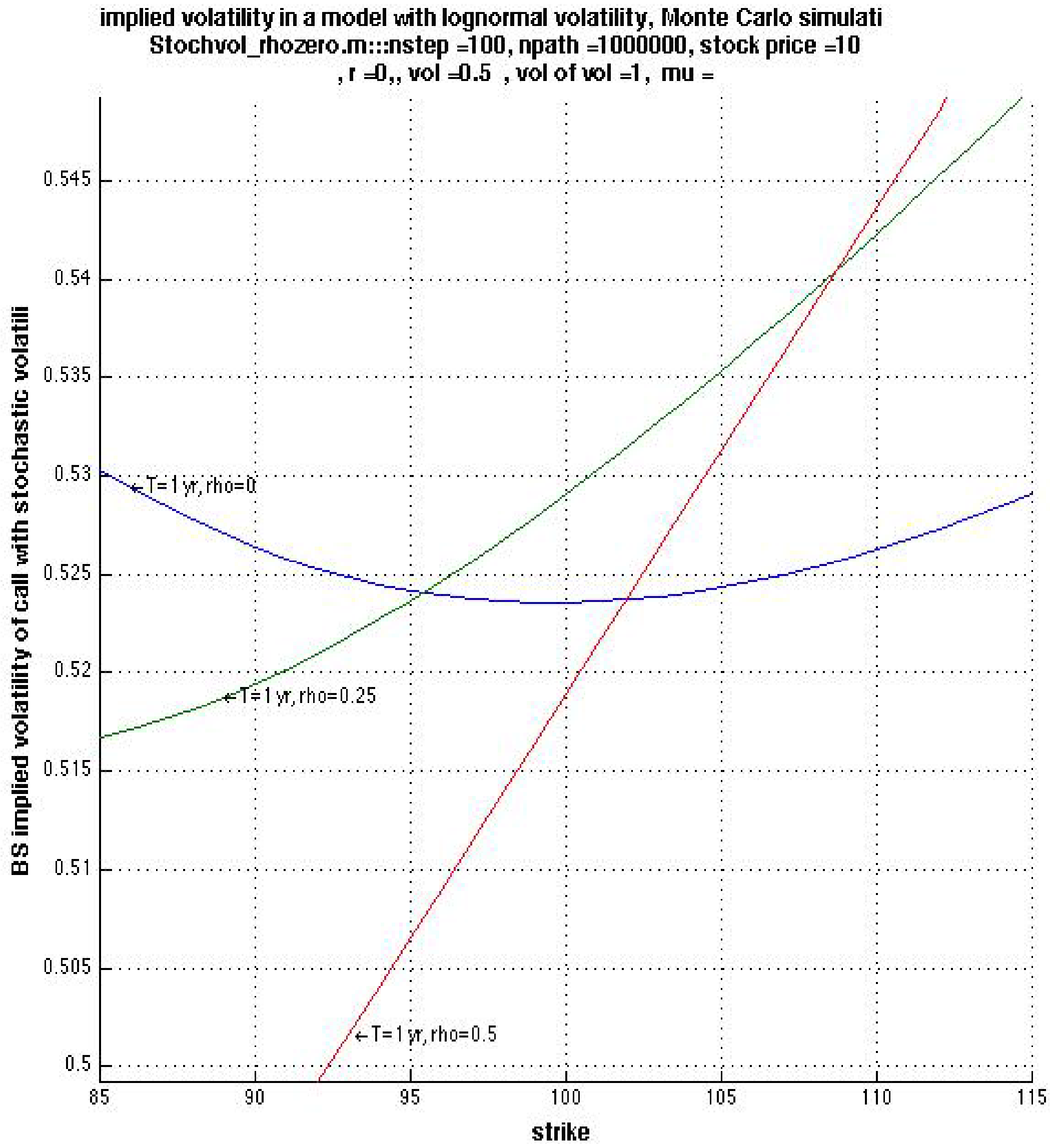
22.3 Simulation of Non-Zero Correlation, No Mean Reversion

Monte Carlo simulation for $\tau = 0.2\text{yrs}$ with non-zero ρ. You can see that increasing the value of the correlation steepens the slope of the smile.



$\tau = 1yr.$



implied volatility in a model with lognormal volatility, Monte Carlo simulati
Stochvol_rhozero.m:::nstep =100, npath =1000000, stock price =10
, r =0,, vol =0.5 , vol of vol =1, mu =
BS implied volatility of call with stochastic volatili
strike
T=1 yr, rho=0
T=1 yr, rho=0.25
T=1 yr, rho=0.5

22.4 Simulation with Mean Reversion and Correlation

volatility is 20%, the long-term volatility m is 20%, the volatility of volatility is 50%, and the correlation between the stock and its volatility is –30%.

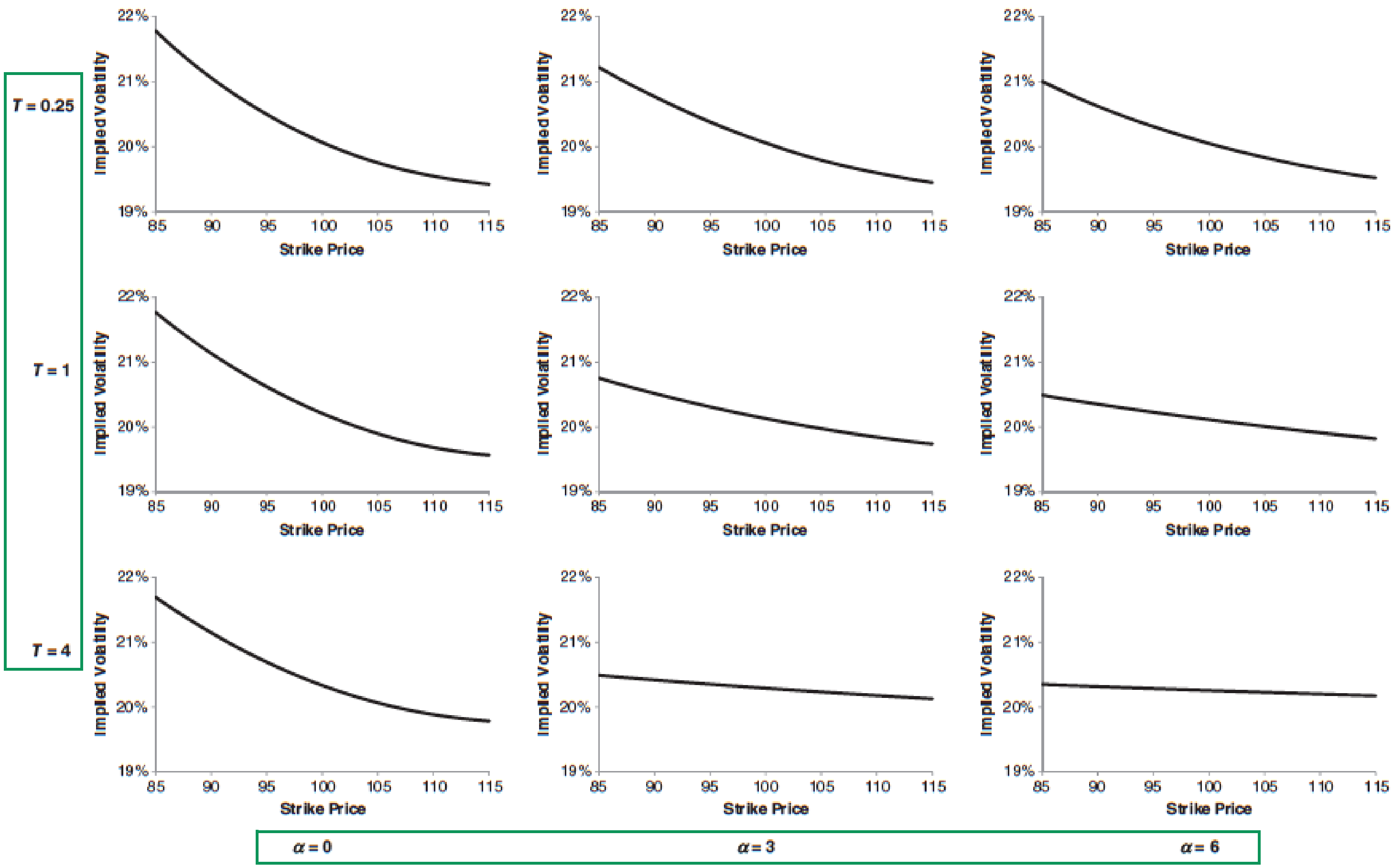


FIGURE 22.4 The Smile in a Mean-Reverting Stochastic Volatility Model and Its Variation with Time to Expiration and Mean Reversion Strength (Long-Term Volatility = 20%)

volatility is 20%, the long-term volatility m is **40%**, the volatility of volatility is 50%, and the correlation between the stock and its volatility is –30%.

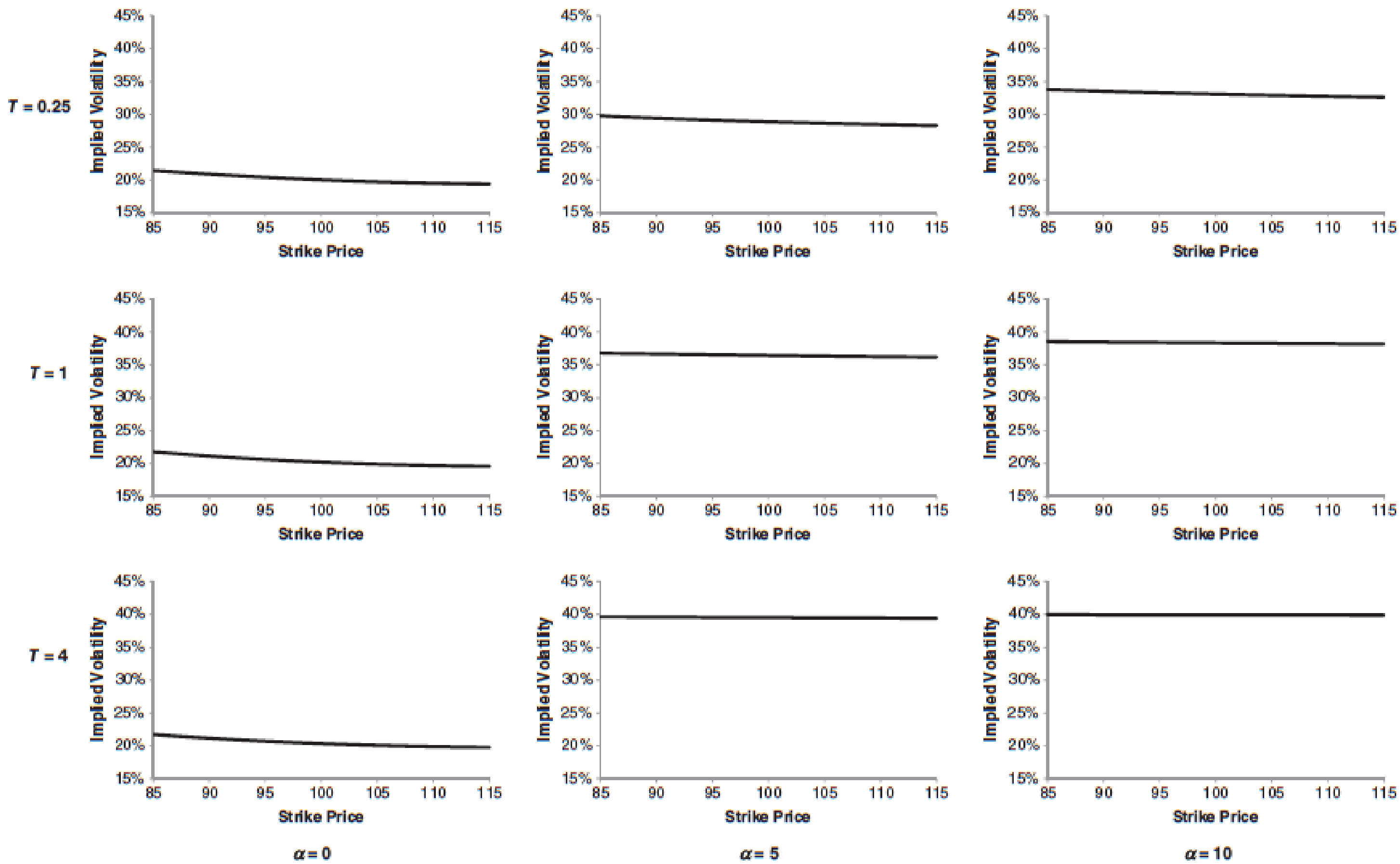


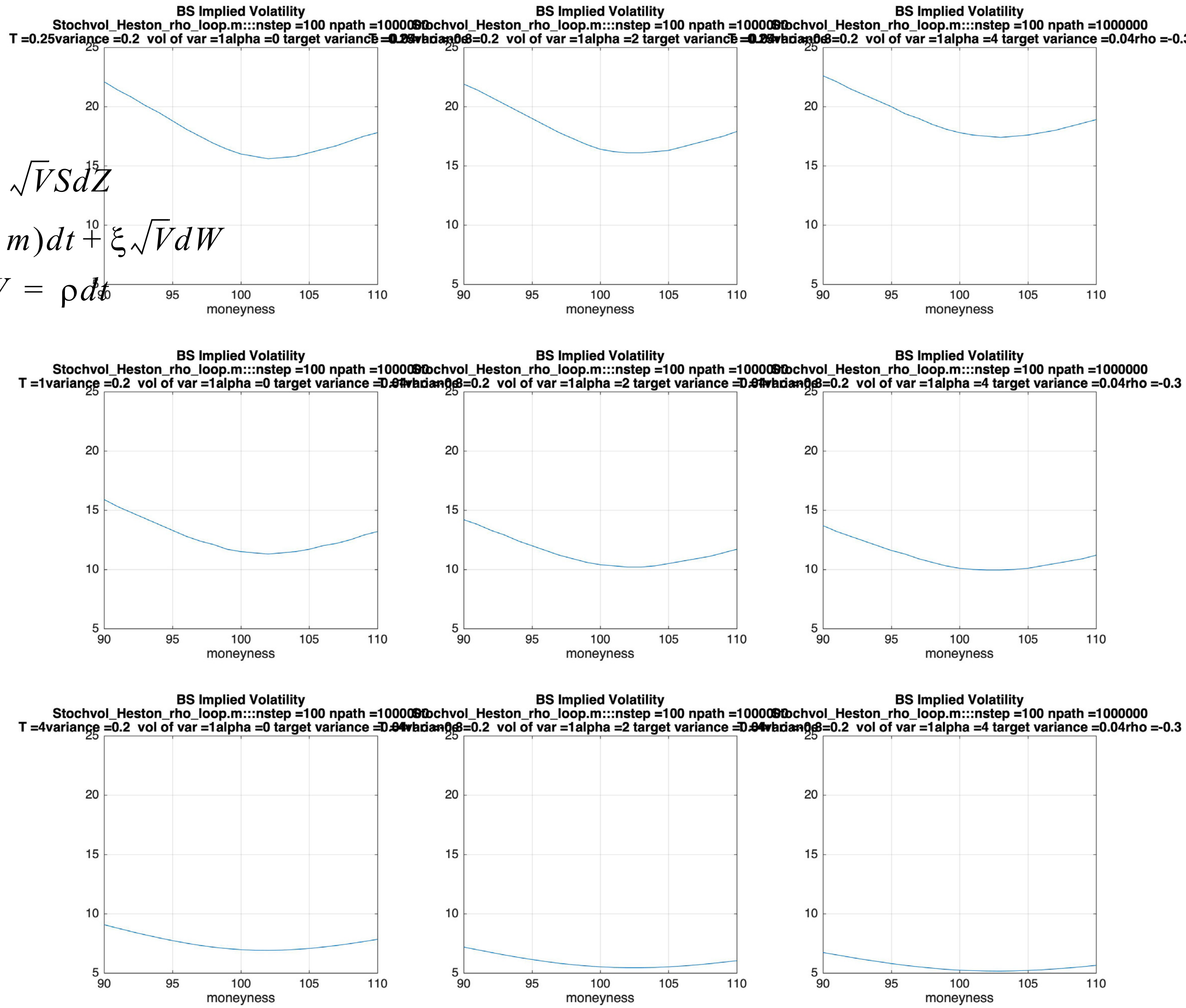
FIGURE 22.5 The Smile in a Mean-Reverting Stochastic Volatility Model and Its Variation with Time to Expiration and Mean Reversion Strength (Long-Term Volatility = 40%)

Heston with correlation -30% and mean reversion -- note flattening and downward drift of skew

$$dS = \sqrt{V}SdZ$$

$$dV = -\kappa(V - m)dt + \xi\sqrt{V}dW$$

$$dZdW = \rho dt$$



Comparison of vanilla hedge ratios under Black-Scholes, Local Volatility and Stochastic Volatility models when all are calibrated to the same negative skew

Calibrated to the same current negative skew for the S&P, different models have different evolutions of volatility, different hedge ratios, different deltas, different forward skews, different exotic options values.

Black-Scholes: Implied volatility is independent of stock price. The correct delta is the Black-Scholes delta.

Local Volatility: Local volatility goes down as market goes up, so the correct delta is smaller than Black-Scholes.

Stochastic Volatility:

Implied volatility is a function of K/S,
Negative skew means that implied volatility goes up as K goes down
Then implied volatility must go up as S goes up, contingent on the stochastic volatility not changing.

Therefore, the st**ock-only hedge ratio will be greater than Black-Scholes, contingent on the level of the stochastic volatility remaining the same.** But, remember, in a stochastic volatility model there are two hedge ratios, a delta for the stock and another hedge ratio for the volatility, so just knowing how one hedge ratio behaves doesn't tell the whole story anymore.

22.5 Best stock-only hedge in a stochastic volatility model is like a local vol model

Although stochastic volatility models suggest a hedge ratio greater than Black-Scholes in a negative skew environment, that hedge ratio is only the partial hedge ratio w.r.t. the stock degree of risk, and doesn't mitigate the volatility risk.

What is the best stock-only hedge, best in the sense that you don't hedge the volatility but try to hedge away as much risk as possible with the stock alone, by minimizing the P&L volatility?

Best stock-only hedge is a lot like a local volatility hedge ratio, and is indeed smaller than hedge ratio in a Black-Scholes model.

Simplistic **stochastic implied volatility** model

$$\begin{aligned} \frac{dS}{S} &= \mu dt + \Sigma dZ \\ d\Sigma &= p dt + q dW \\ dZ dW &= \rho dt \end{aligned}$$

We have for simplicity assumed that the stock evolves with a realized volatility equal to the implied volatility of the particular option itself. Then for an option $C_{BS}(S, \Sigma)$ where both S and Σ are stochastic, we can find **the hedge that minimizes the instantaneous variance of the hedged portfolio.** That's as good as we can do with stock alone.

This partially hedged portfolio is $\pi = C_{BS} - \Delta S$

Then in the next instant $d\pi = \left(\frac{\partial C_{BS}}{\partial S} - \Delta\right) dS + \frac{\partial C_{BS}}{\partial \Sigma} d\Sigma = (\Delta_{BS} - \Delta) dS + \kappa d\Sigma$

The instantaneous variance of this portfolio is defined by $(d\pi)^2 = var[\pi]dt$ where

$$var[\pi] = (\Delta_{BS} - \Delta)^2 (\Sigma S)^2 + \kappa^2 q^2 + 2(\Delta_{BS} - \Delta)\kappa S \Sigma q \rho$$

The value of Δ that minimizes the residual variance of this portfolio is given by

$$\frac{\partial}{\partial \Delta} var[\pi] = -2(\Delta_{BS} - \Delta)(\Sigma S)^2 - 2\kappa S \Sigma q \rho = 0 \qquad \Delta = \Delta_{BS} + \rho\left(\frac{\kappa q}{\Sigma S}\right)$$

The second derivative $\frac{\partial^2}{\partial \Delta} var[\pi]$ is positive, so that this hedge produces a minimum variance.

The hedge ratio Δ is less than Δ_{BS} when ρ is negative. The best stock-only hedge in a stochastic volatility model tends to resemble the local volatility hedge ratio.

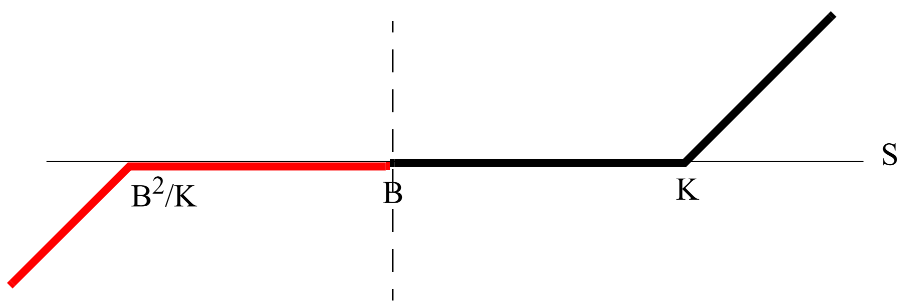
22.6 The Value of Barrier Options in Stochastic Volatility Models ... is often independent of the stochastic volatility model!

Recall a down-and-out option:

We showed in Lecture 11 that, in a Black-Scholes world with zero rates, the fair value for a down-and-out call with strike K and barrier B is given by a combination of two European options:

$$C_{DO}(S, K) = C_{BS}(S, K) - \frac{S}{B} C_{BS}\left(\frac{B^2}{S}, K\right) = C_{BS}(S, K) - \frac{K}{B} P_{BS}\left(S, \frac{B^2}{K}\right) \qquad \text{Eq.22.5}$$

.Its value vanishes at any time vanishes on the barrier.



If you can replicate a barrier option by a combination of European options (approximately), then its value is the value of the European options. But the value of the European options are determined by the smile independent of the model for volatility.

As long as the stochastic volatility model is calibrated to the smile, it will therefore approximately produce the same value for the barrier option independent of the model, roughly.

22.5 Some Problems of Heston Stochastic Volatility

$$\begin{aligned} dS &= \sqrt{V} S dZ \\ dV &= -\kappa(V-m)dt + \xi\sqrt{V}dW \\ dZdW &= \rho dt \end{aligned}$$

Heston with zero drift, dividends, rates

Let's see what the dynamics of volatility is.

The smile of volatility of volatility:

$$\begin{aligned} d\sqrt{V}^2 &= (*)dt + \xi\sqrt{V}dW \\ 2\sqrt{V}d\sqrt{V} &= (*)dt + \xi\sqrt{V}dW \\ d\sqrt{V} &= (*)dt + \xi dW \end{aligned}$$

So volatility, the square root of variance, varies normally, like arithmetic Brownian motion, but the evidence from data is that volatility is more lognormal, with changes proportional to the current level of volatility. There are other problems to do with the Skew Stickiness Ratio, with Forward Variances, all not unconstrained enough. See Bergomi's book.

Nevertheless Heston is very popular because it can be solved analytically.

22.6 Calibration Of Stochastic Volatility In General

With only a few parameters to choose, you can fit the implied volatilities of a few vanilla options, but then you will have a mismatch between the model and the other vanilla options.

In general:

- the initial level of volatility and the mean reversion control the overall level of implied volatility;
- the volatility of volatility (and the mean reversion) control the convexity of the smile $\left(\ln\frac{S}{K}\right)^2$ term that we saw from the volga term;
- the correlation ρ controls the skew, the $\ln\frac{S}{K}$ term in its description that's related to vanna.

For risk management, it's hard to understand really well the connection between changes in implied volatility we see and changes in the stochastic parameters.

Steepness of the skew problem in stochastic volatility

Hard to get a steep short-term skew because it is the volatility of the volatility that feeds the correlation ρ into producing a volatility skew, and for short expirations the volatility of volatility hasn't had enough time to act.

If you make the volatility of volatility very high, that may work, but then you need very strong mean reversion to prevent volatility from getting very high or low.

22.7 Stochastic Local Volatility Briefly

22.6.1 Recall Problems and Benefits of Local Volatility

The good thing about local vol is that it matches all the hedging instruments of an exotic option correctly - i.e. it can fit the smile.

But we'd like to be free to specify the future **dynamics** of the stock price **and** future implied volatilities, and local vol constrains the dynamics of forward vols and implied vols.

The volatility of local volatility is not independent, is too small, too determined, to model stochastic volatility realistically. Related to the short-term forward skew being too flat.

$$\frac{dS}{S} = \mu dt + \sigma(S, t)dZ$$

$$d\sigma = \frac{\partial\sigma}{\partial t}dt + \frac{\partial\sigma}{\partial S}dS + \frac{1}{2}\frac{\partial^2\sigma}{\partial S^2}\sigma^2 S^2 dt$$

The dS terms contains dZ and thus indicates the lognormal volatility of the volatility ξ.

$$d\sigma = \frac{\partial\sigma}{\partial S}\sigma S dZ + \ldots = \xi\sigma dZ + \ldots$$

$$\xi = S\frac{\partial\sigma}{\partial S} = \frac{\partial}{\partial \ln S}\sigma(S, t)$$

Forward vol of vol is determined by the slope of the local vol, which is determined by the skew, which isn't realistic and underprices forward starting options and the forward skew.

22.6.2 Adding Stochastic Volatility

So how can one overcome the difficulty that

- local volatility can fit any smile and any steepness of the current skew but has a low volatility of volatility
- stochastic volatility provides inadequate calibration and, with Heston, insufficient volatility of volatility as well.

A popular method in practice is to combine the two models to get **stochastic local volatility**.

The SABR model is a version of this, but not general enough because it's parametric.

The idea is as follows:

- combine local volatility and stochastic volatility into one model, like SABR, but with a more general local volatility function $A(S, t)$
- stochastic volatility provides the forward volatility of volatility and forward skew;
- local volatility provides the short-term skew and the calibration to any smile.

Example

$$\begin{aligned} \frac{dS}{S} &= \mu dt + A(S, t)\sigma_t dW \\ d\sigma_t &= (*)dt + \xi\sigma_t dZ \\ dXdZ &= \rho dt \end{aligned}$$

We need to find $A(S, t)$ given some volatility of volatility ξ.

To calibrate:

- Recall that for stochastic volatility we proved that the Dupire one-factor local volatility is given by averaging over all the stochastic factors for the volatility:

definition

known from market

$$\sigma^2_{local}(S,t) \equiv \left.\frac{\dfrac{\partial C_{market}}{\partial t}}{\dfrac{K^2}{2}\dfrac{\partial^2 C_{market}}{\partial K^2}}\right|_{K=S} = E\left[\{A(S,t)\sigma_t\}^2 \middle| S\right]$$

average over all stochastic σ_t

from theory we did earlier

$$A^2(S,t)E\left[\sigma_t^2 \middle| S\right] = \sigma^2_{local}(S,t)$$ known from the market

$$A^2(S,t) = \frac{\sigma^2_{local}(S,t)}{E\left[\sigma_t^2 \middle| S\right]}$$

the average at S,t depends upon earlier A(S,t) that led to that S,t

Can solve this one step at a time to find the $A(S,t)$ moving forward in time for the $A(S,t)$.

Detailed but simple in principle.

Thus you can choose a volatility evolution ξ and then adjust the local vol $A(S,t)$ to fit the smile.

General Comments on Economics and Skew Models

Economic effects causing an Equity Skew: many obvious local effects

• Markets **jump** down, realized vol increases, implied vol then increases too.

• Leverage-like asymmetry. Equity is assets minus debt. Getting closer to bankruptcy increases the skew..

• Supply and demand asymmetry. It is more natural for equity to be held long than short, which makes downwards protection more important, puts more valuable.

Economic effects causing an FX Skew / Smile: seem level related

• Anticipated government and central bank intervention to stabilize FX rates.

• Foreign investors buy FX rate protection with options.

• Stochastic volatility may be important in approaching levels.

Economic effects causing an Interest Rate skew and smile seem local:

• Interest rate levels are absolutely important -- 1% is very different from 10% -- and volatilities are connected to absolute levels of interest rates rather than stock prices.

• Local volatility is therefore important

• Anticipated central bank actions tend to be normally distributed.

Does Stochastic Volatility match the facts?

None of these economic effects are very well described by simply a strong correlation between the asset and the uncertainty in volatility.

In contrast most stochastic volatility models incorporate a skew by virtue of strong correlation of volatility and stock.

But you need a lot of volatility of volatility and correlation to get a steep short-term smile, and then a lot of mean reversion to prevent wild levels of future volatility.

Perhaps better to start with SLV: a local vol skew and then add a stochastic perturbation.

Conclusion

Stochastic volatility models produce a rich structure of smiles from only a few stochastic variables. There is some element of stochastic volatility in all options markets.

SV models seem to provide a reasonable description of currency options markets where the dominant features of the smile are consistent with fluctuations in volatility. But SLV models seem more suitable for markets that have a pronounced permanent skew. and can be calibrated better.

Remember though: the stochastic evolution of volatility is not really yet well understood and involves many at presently unverifiable assumptions.

One of the interesting new approaches is modeling volatility as a fractional Brownian motion which produces a rougher volatility.