LECTURE 14

LOCAL VOLATILITY MODELS

- In a local volatility model, the instantaneous stock volatility $\sigma(S, t)$ is a function of stock price and future time.
- How to build and use a binomial tree with variable local volatility.
- The BSM implied volatility of a standard option in a local volatility model is approximately the average of the local volatilities between the initial stock price and the strike.

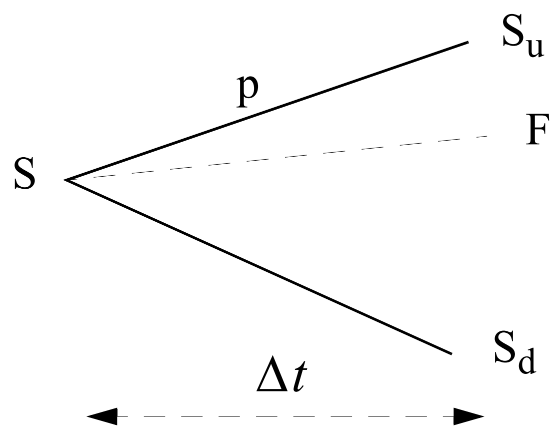
- Building a local volatility tree
- The implied volatility surface that results
- Calibrating a local volatility tree
- The Dupire equation
- The relation between local and implied vol
- Hedge ratios of vanilla options
- Exotics

  etc

Two Midterm Exams include everything up to page 18 of this lecture.

14.1 Binomial Local Volatility Modeling assuming known $\sigma(S, t)$

How do we modify the usual binomial model to build a binomial tree with $\sigma(S, t)$ that closes (in order to avoid computational complexity)? Here we keep Δt constant via another approach:



p is here the **risk-neutral** no-arbitrage probability

This must reduce at $\Delta t \to 0$ to

$$\frac{dS}{S} = (r-d)dt + \sigma(S, t)dZ$$

How do we find p, S_u and S_d?

We know:

Expected value of S is the **forward price** $F = Se^{(r-d)\Delta t}$ or $F = Se^{r\Delta t} - D$

Furthermore, the SDE implies that $\mathrm{var}[S] = (dS)^2 = \sigma^2(S, t)S^2\Delta t$

On the binomial tree:

$$F = pS_u + (1-p)S_d \qquad \text{the mean of } S$$

$$S^2\sigma^2(S,t)\Delta t = p(S_u - F)^2 + (1-p)(S_d - F)^2 \qquad \text{the variance}$$

Solve:

$$p = \frac{F - S_d}{S_u - S_d}$$

$$(F - S_d)(S_u - F) = S^2\sigma^2(S,t)\Delta t$$

$$S_u = F + \frac{S^2\sigma^2(S,t)\Delta t}{F - S_d} \qquad \text{or} \qquad S_d = F - \frac{S^2\sigma^2(S,t)\Delta t}{S_u - F}$$

If you know S_d you can calculate S_u and vice versa.

- Choose the central spine of the tree.

There are many ways to choose the central spine of a binomial tree:

- For every level with an odd number of nodes (1,3,5, etc.) choose the central node to be S. (CRR)
- For every period with even nodes (2,4,6 etc.) choose the two central nodes in those periods to lie above and below the initial stock price *S* exactly as in the CRR tree, given by

$$S_u = Se^{\sigma(S,t)\sqrt{dt}}$$

$$S_d = Se^{-\sigma(S,t)\sqrt{dt}}$$

Once you have the central nodes, you can generate the up and down nodes relative to the central node at each level of the tree by

$$S_u = F + \frac{S^2\sigma^2(S,t)dt}{F - S_d}$$

$$S_d = F - \frac{S^2\sigma^2(S,t)dt}{S_u - F}$$

You could equally well choose a tree whose spine corresponds to the forward price *F* of the stock, growing from level to level. Or anything else.

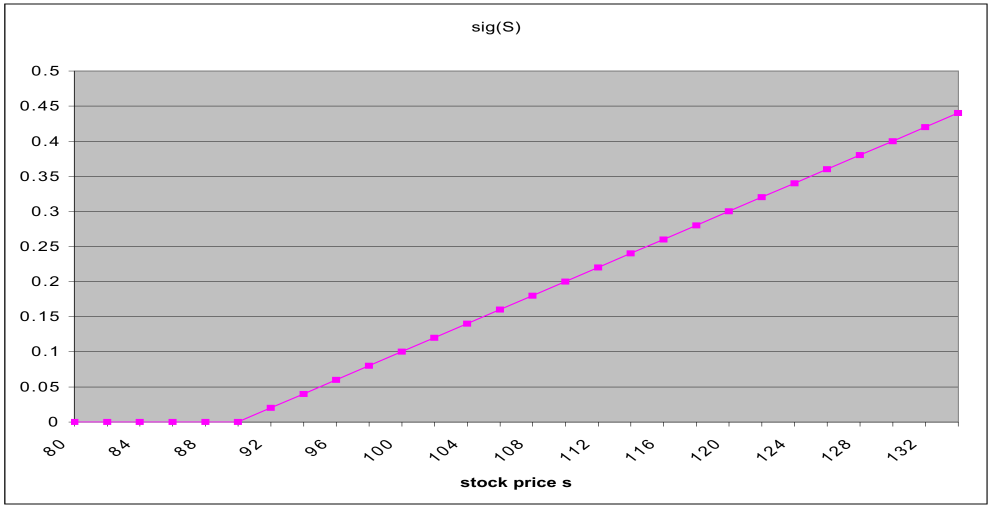
14.2 Investigation of The Relation Between Local and Implied Volatilities.

Calibration: The inverse scattering problem: How to build a local volatility tree that matches the smile?

What is the relation between local volatilities as a function of S and implieds as a function of K?

Intuition: Here is a graph of local volatilities that satisfy a positive skew:

$$\sigma(S) = Max[0.1 + (S/100 - 1), 0]$$



Here is the binomial local-volatility tree for the stock price, assuming $\Delta t = 0.01$, $S = 100$, $r = 0$.

This is a tree with flat volatility 0.1, usual CRR type

stock tree sig = 0.1, p = 0.5

				104.08
			103.05	
		102.02		102.02
	101.01		101.01	
100		100		100
	99		99	
		98.02		98.02
			97.04	
				96.08

$$\sigma^2 = \frac{(F-S_d)(S_u-F)}{S^2\Delta t} = 0.01$$

$$\sigma = 0.1$$

stock tree sigma(s,t)

This is a tree with variable local volatility

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

$$\sigma^2 = \frac{(F-S_d)(S_u-F)}{S^2\Delta t}$$

$$= \frac{(1.28)(1.53)}{(103.5^2)(0.01)} = 0.018$$

$$\sigma = 0.135$$

p-tree

			0.456
		0.488	
	0.453		0.453
0.498		0.498	
	0.441		0.441
		0.509	
			0.434

$$\frac{F-S_d}{S_u-S_d}$$

vol(stock)

			0.135
		0.122	
	0.110		0.110
0.100		0.100	
	0.090		0.090
		0.082	
			0.074

vol ranges from 13.5 to 7.5
as stock ranges from 103.5 to 97.4

14.2.1 An option with strike 102 after 4 periods.

CRR Implied Volatility: What constant volatility CRR tree has the same price for the option? Call with strike 102 has the same value on the *local volatility tree* as it does on a *fixed-volatility CRR tree* with a volatility of 11%. (Analog of BS implied vol for discrete steps)/

NUMERICAL ILLUSTRATION OF RELATION BETWEEN LOCAL AND IMPLIED VOL

local vol tree

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

LOCAL VOL TREE CALL STRUCK AT 102

				3.040
			1.510	
		0.790		0.230
	0.386		0.104	
0.204		0.052		0.000
	0.023		0.000	
		0.000		0.000
			0.000	
				0.000

stock tree with 11% vol

				104.50
			103.36	
		102.22		102.22
	101.11		101.11	
100.00		100.00		100.00
	98.91		98.91	
		97.82		97.82
			96.75	
				95.70

CALL TREE FOR STOCK TREE ON RIGHT STRIKE = 102

				2.498
			1.355	
		0.730		0.224
	0.391		0.112	
0.208		0.055		0.000
	0.028		0.000	
		0.000		0.000
			0.000	
				0.000

Note: 11% is the average of the local volatilities between 10% at S=100 and 12% at K=102.

The CRR implied volatility for a given strike is roughly the average of the local volatilities from spot to that strike.

Call with strike 103 on the same tree.

local vol tree

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

LOCAL VOL TREE CALL STRUCK AT 103 (sig = 13%)

				2.040
			0.929	
		0.453		0.000
	0.205		0.000	
0.102		0.000		0.000
	0.000		0.000	
		0.000		0.000
			0.000	
				0.000

stock tree with 11.5% vol

				104.71
			103.51	
		102.33		102.33
	101.16		101.16	
100.00		100.00		100.00
	98.86		98.86	
		97.73		97.73
			96.61	
				95.50

CALL TREE FOR STOCK TREE ON RIGHT STRIKE = 103

				1.707
			0.849	
		0.422		0.000
	0.210		0.000	
0.104		0.000		0.000
	0.000		0.000	
		0.000		0.000
			0.000	
				0.000

Implied volatility is about 11.5%, the average of the local volatilities between S = 100 and K = 103.

What have we learned?

14.2.2 The Rule of 2: Understanding The Relation Between Local and Implied Vols

We see that: Implied volatility $\Sigma(S, K)$ of an option is approximately the average of the expected local volatilities $\sigma(S)$ encountered over the life of the option between spot S and strike K.

Cf: yields to maturity for zero-coupon bonds are an average over future short-term rates over the life of the bond.

Forward short-term rates grow twice as fast with future time as yields to maturity grow with time to maturity.

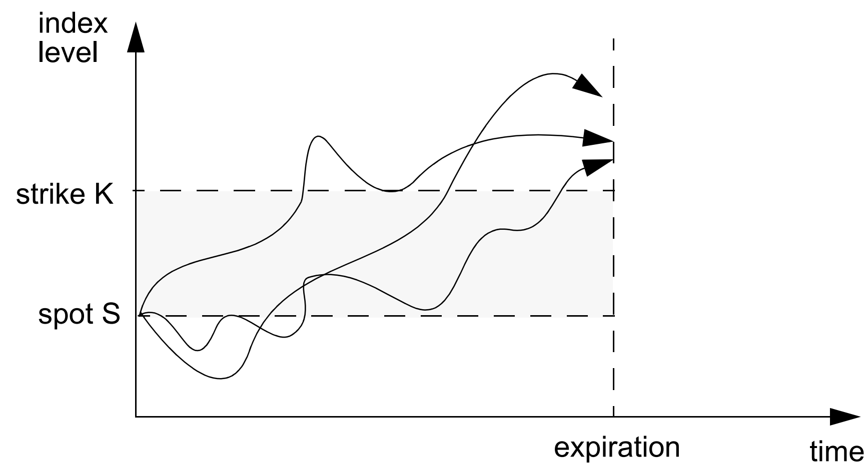
Local volatilities grow approximately twice as fast with stock price as implied volatilities grow with strike.

Illustration/"proof" from *The Local Volatility Surface*. Later we'll prove it much more rigorously.

Simple "sideways" linear vol case: $\sigma(S) = \sigma_0 + \beta S$ for all time t

$\Sigma(S, K)$: Any paths that contribute to the option value must pass between S and K

FIGURE 14.1. Index evolution paths that finish in the money for a call option with strike K when the index is at S. The shaded region is the volatility domain whose local volatilities contribute most to the value of the call option.



Implied volatility for the option of strike K when the index is at S ~ average of the local volatilities

$$\Sigma(S, K) \approx \frac{1}{K-S}\int_S^K \sigma(S')dS'$$

$$\Sigma(S, K) \approx \sigma_0 + \frac{\beta}{2}(S + K)$$

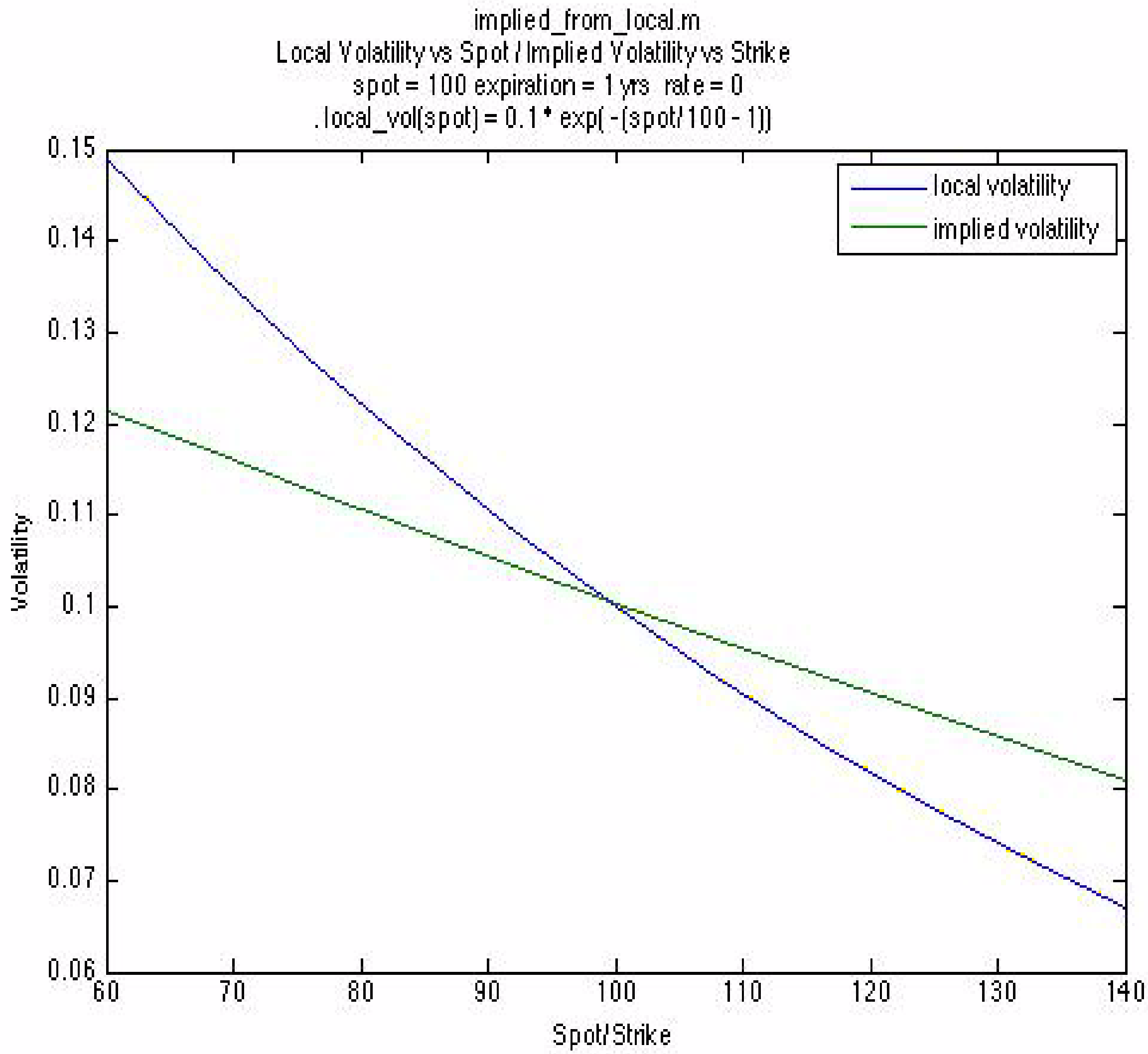
Therefore: If (implied volatility varies linearly with strike K at a fixed market level S) then (it also varies linearly at the same rate with the index level S itself).
Local volatility varies with S at twice that rate.

Let's see how well it works with actual local volatilities and implied volatilities in the model.

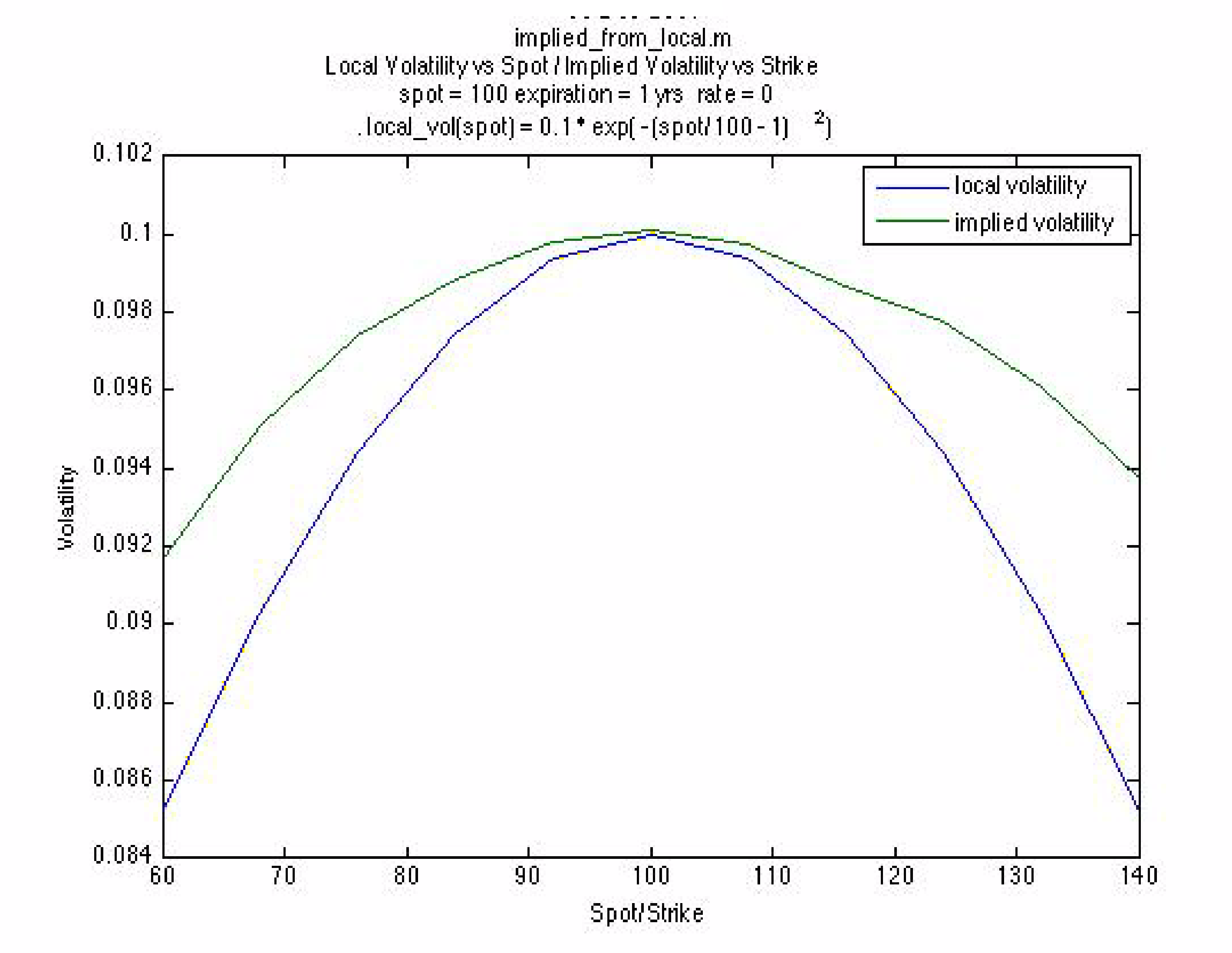
14.2.3 Testing Some Examples of Local and Implied Volatilities.

$\sigma(S, t) = 0.1\exp(-[S/100 - 1])$ inserted into a binomial model **with many periods**. Calculate options prices from the binomial local vol model, and then convert the prices to BS implied vols. Note the slopes.

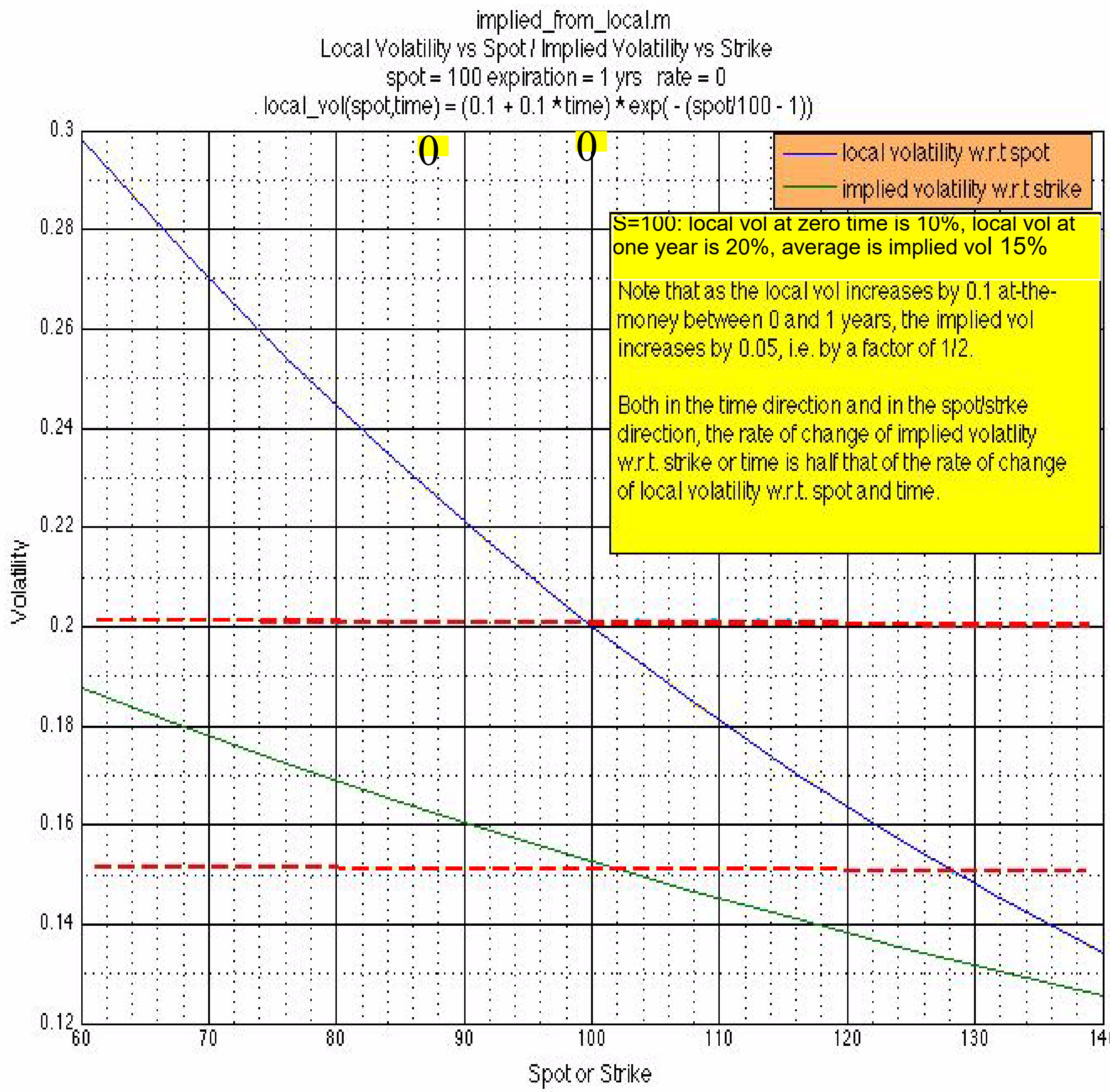
There can be no arbitrage violations of implied vol when you choose positive local vols.



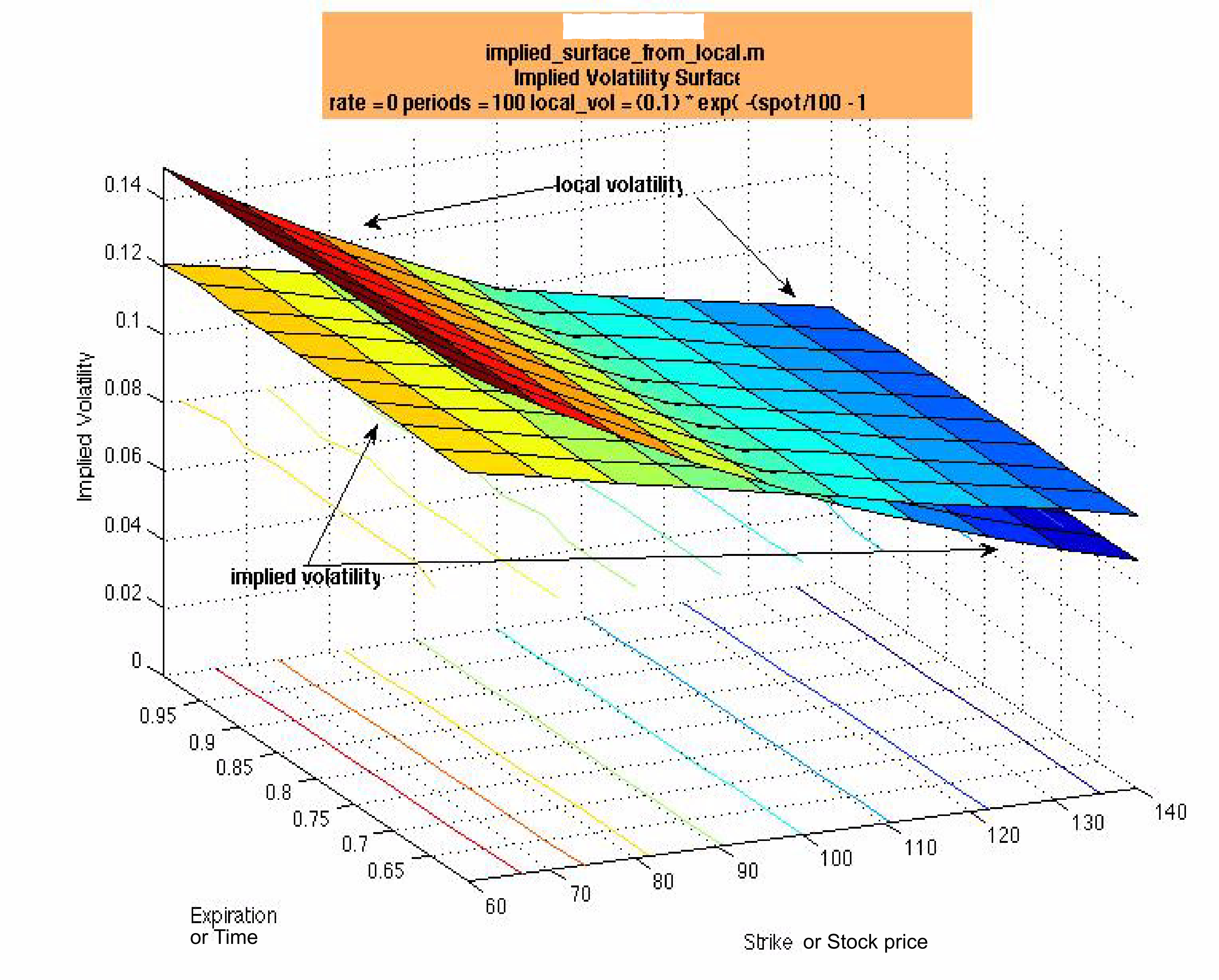
$$\sigma(S, t) = 0.1\exp(-[S/100 - 1]^2)$$



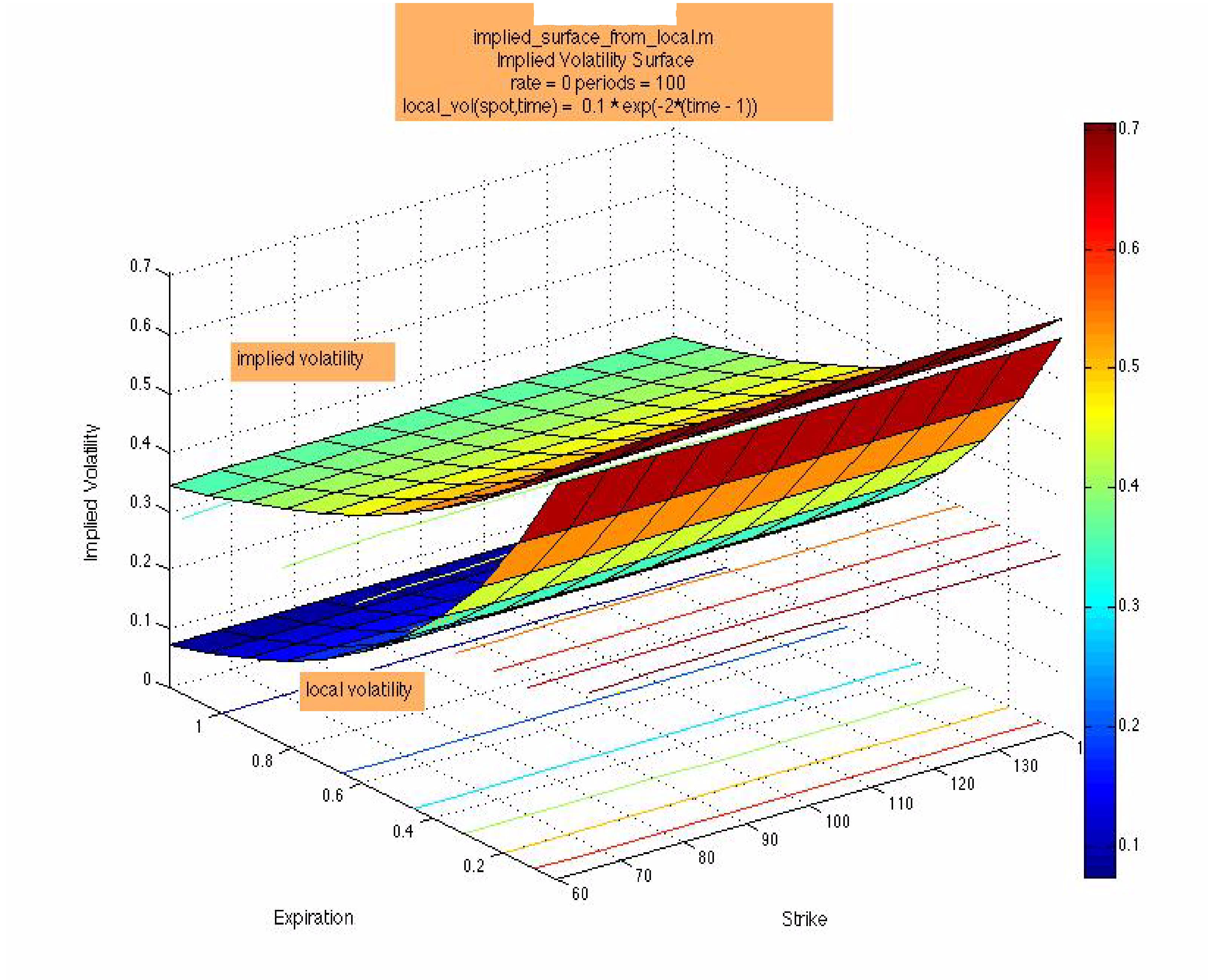
$\sigma(S, t) = (0.1 + 0.1t)\exp(-[S/100 - 1])$ Look at one-year implied volatility skew:



Dependent only on *S*: $\sigma(S, t) = 0.1\exp(-[S/100 - 1])$: Plot surface

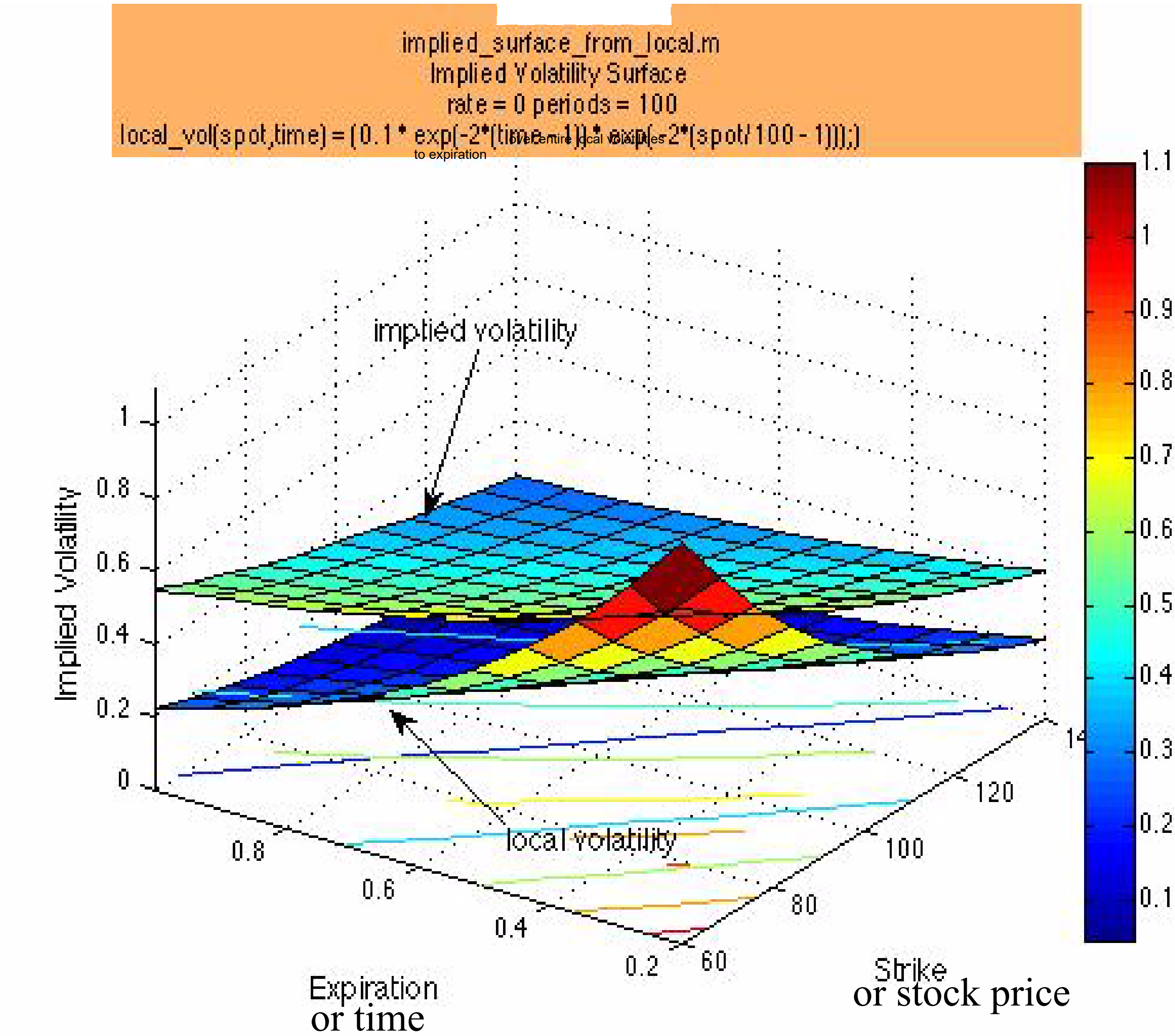


Dependent only on *t*: $\sigma(S, t) = 0.1\exp(-2[t-1])$: Plot Surface



Dependent on S and t: $\sigma(S, t) = 0.1\exp(-2[t-1])\exp(-2[S/100-1])$**. Local vol is 10% at t = 1 and S = 100.**

Local volatilities stay constant along lines where S increases and t decreases. Similarly for implieds



Implementation Difficulties With Binomial Trees

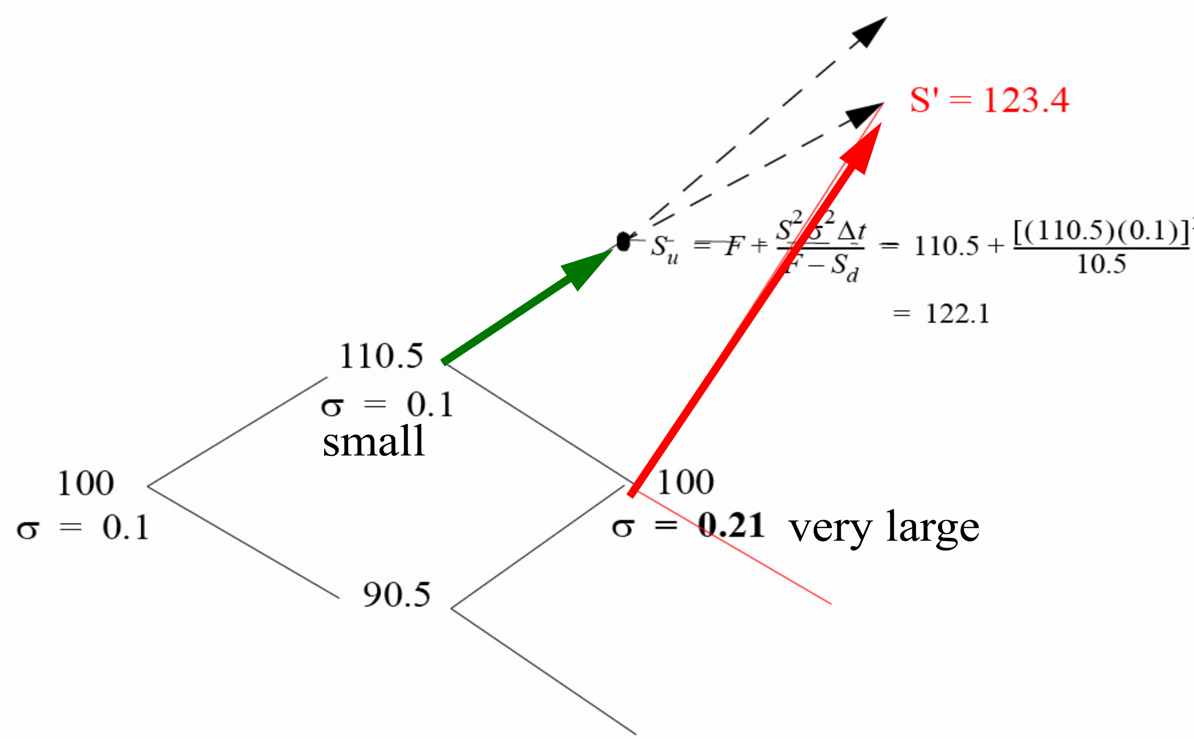
The stock prices at the nodes and the transition probabilities we discussed are uniquely determined by forward rates and the local volatility function we specify.

If $\sigma(S, t)$ varies too rapidly with stock price or time, then, for finite Δt, you can get binomial transition probabilities greater than 1 or less than zero.

Here is an example with $\Delta t = 1$ and $r = 0$.

The local volatility at level 3 at S = 100 is 0.21.

The S' node, S's up node in level 4 should be the down node from S_u in level 3, it should lie below S_u, but in fact lies above it, and so violates the no-arbitrage condition



You can remedy this with smaller time steps Δt, but then you are trying to extract to extract more information that is available from implied volatilities that have coarser Δt-spacing. Therefore, it's sometimes easier to use other methods, e.g. trinomial trees. They provide greater flexibility in placing the nodes to avoid arbitrage situations.

14.3 Implied Trees and Calibration

We went from $\sigma(S, t)$ to $\Sigma(S, t, K, T)$ and found the Rule of 2.

In actual implementation, one observes **discretely spaced** implied volatilities $\Sigma(S, t, K_i, T_i)$ for discrete strikes K_i and expirations T_i, and one wants to calibrate a **continuous** local volatility surface $\sigma(S, t)$. This is a problem. One needs to smooth or parametrize the local vol surface, but then one needs to make assumptions about the tails.

"Implied trees" are a generalization of implied volatility. Implied volatility is a single variable defining a Black-Scholes tree; the implied tree is a representation of local volatility defining an implied tree of the stock price consistent with options prices.

Binomial framework in the paper *The Volatility Smile and Its Implied Tree* Derman, E. & I. Kani. "*Riding on a Smile.*" RISK, 7(2) Feb.1994, pp. 139-145. Also *The Local Volatility Surface* Continuous-time derivation of the same results: Dupire, *Pricing With A Smile*, RISK, January 1994, pages 18-20.

The key point of these papers is that there is a unique stochastic stock evolution process with a variable continuous volatility $\sigma(S, t)$ that can fit all options prices and their *continuous* implied volatilities $\Sigma(S, t, K, T)$. The tree representation of this stochastic process is called the *implied tree*.

- *The Volatility Smile and Its Implied Tree* (see my textbook) shows that, when the stock price is S at time t, the local volatility $\sigma(K, T)$ at a future stock price K and time T, is determined numerically from options prices $C(S, t, K, T)$ in the vicinity of K, T.
- This is an *inverse problem*: going backwards from **output** $C(S, t, K, T)$ to **input** $\sigma(S, t)$ to the stochastic process $\frac{dS}{S} = \mu dt + \sigma(S, t)dZ$
- Analogous to finding a potential in physics from viewing the way particles move under its influence.
- Theoretically it's straightforward, but it is rather difficult in practice. It's an "ill-posed" problem. Beginning with sparse implied volatilities and interpolating them into a surface, small changes in the interpolated input can cause dramatic changes in the output.
- In practice, it may be better to assume some parametric form for the local volatility function and then find the parameters for local volatility that make the tree's option prices match as closely as possible the market's option prices.

14.3.1 Dupire Equation for Local Volatility in Terms of Option Prices

This equation describes the mathematical relationship between **continuous** implied and **continuous** local volatility.

Recall forward rates from **two** zero-coupon bonds:

$$B_2 = B_1 e^{-y_{12}} \qquad y_{12} = \ln B_1 - \ln B_2$$

Similarly, assuming continuity, you can derive the local volatility from **three** option prices (or their corresponding BS implied volatilities) in a simple way. Once you find the local volatilities from the implied volatilities surface, you can use them to build a tree or MC process, and then use it to value options.

Recall Breeden-Litzenberger formula: The risk-neutral probability density of making a transition is given by:

$$p(S, t, K, T) = e^{r(T-t)} \frac{\partial^2}{\partial K^2}(C_{mkt}(S, t, K, T)) \text{ -- continuous version of butterfly spread}$$

$p(S, t, K, T)dK$ is the density or risk-neutral probability function $p(\ )$ that tells you the no-arbitrage price you have to pay at time t for earning \$1 if the future stock price at time T lies between K and $K + dK$, and zero if it's anywhere else. It is determined by the second derivative of the **market options prices**. It depends only on the market prices of options, C, and their derivatives.

Similarly: $\sigma(S, t)$ in the model can be found from **market prices of options** and their derivatives. Assuming zero interest rates and dividends, the local volatility at the stock price K is given by:

$$\frac{\sigma^2(K,T)}{2} = \frac{\dfrac{\partial C(S,t,K,T)}{\partial T}}{K^2\dfrac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

We'll see that this is the continuous version of the procedure we used to construct a local volatility binomial tree.

If interest rates are non-zero, but for zero dividend yield,

,

$$\frac{\sigma^2(K,T)}{2} = \frac{\dfrac{\partial C(S,t,K,T)}{\partial T} + rK\dfrac{\partial C(S,t,K,T)}{\partial K}}{K^2\dfrac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

Recall $T_1\sigma_1^2 + (T_2 - T_1)\sigma_2^2 = T_2\sigma_2^2$. This is the mathematically correct generalization of the notion of forward stock volatilities $\sigma(T)$ to local volatilities.

14.4 Understanding the Equation

$$\frac{\sigma^2(K,T)}{2} = \frac{\frac{\partial C(S,t,K,T)}{\partial T}}{K^2 \frac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

We can interpret the numerator of the equation in economic terms.

$$\frac{\partial C(S,t,K,T)}{\partial T} = \lim_{dT \to 0} \frac{C(S,t,K,T+dT) - C(S,t,K,T)}{dT}$$

It is proportional to $1/dT$ infinitesimal calendar spreads for standard calls with strike K.

At expiration time T, the calendar spread has significant value only for $S_T \sim K$.

Far below $S_T \sim K$, both calls are worthless. Far above $S_T \sim K$, both calls are forward contracts with equal value.

At $S_T \sim K$ the relevant variance that determines the non-zero value of the calendar spread is $\sigma^2(K, T)$. This volatility determines whether the longer option is worth more than the shorter one.

Similarly the denominator:

$$\frac{\partial^2 C}{\partial K^2} = \frac{C(S, t, K + dK, T) - 2C(S, t, K, T) + C(S, t, K - dK, T)}{(dK)^2}$$

is proportional to an infinitesimal butterfly spread for standard calls with strike K.

Therefore the local variance $\sigma^2(K, T)$ at stock price K and time T is proportional to the ratio of the price of a calendar spread to a butterfly spread.

A calendar spread and a butterfly spread are **combinations of tradeable options**, and so the local volatility can be extracted from traded options prices (if they are available)!

14.4.1 More Intuition

The price at time t of a calendar spread

$$C(S, t, K, T+dT) - C(S, t, K, T)$$

measures the risk-neutral probability $p(S,t,K,T)$ of the stock moving from S at time t to K at time T, times the variance $\sigma^2(K, T)$ at K and T that is responsible for the adding option value. Roughly,

$$\text{calendar spread} \sim p(S, t, K, T)\sigma^2(K, T).$$

But, according to the Breeden-Litzenberger, the probability

$$p(S, t, K, T) \sim \frac{\partial^2}{\partial K^2} C(S, t, K, T) \sim \text{butterfly spread}$$

So, roughly speaking, combing the two equations above, we have

$$\text{calendar spread} \approx \text{butterfly spread} \times \sigma^2(K, T)$$

or

$$\sigma^2(K, T) \approx \frac{\text{calendar spread}}{\text{butterfly spread}}$$

14.4.2 About The Deceptive Appearance of the Equation Which We Will Derive

$$\frac{\sigma^2(K,T)}{2} = \frac{\frac{\partial}{\partial T}C(S,t,K,T) + rK\frac{\partial}{\partial K}C(S,t,K,T)}{K^2\frac{\partial^2 C}{\partial K^2}}$$

can be rewritten as

$$\frac{\partial C(S,t,K,T)}{\partial T} + rK\frac{\partial C(S,t,K,T)}{\partial K} - \frac{\sigma^2(K,T)}{2}K^2\frac{\partial C^2(S,t,K,T)}{\partial K^2} = 0$$

This resembles Black-Scholes equation with t replaced by T and S replaced by K. But …

- **Black-Scholes equation holds for any contingent claim on S**, relating the value of *any option* at S, t to the value of that option locally at $S + dS$, $t + dt$, **keeping strike and expiration fixed**. It was derived from no-arbitrage, by hedging the claim and then setting the value of the hedged portfolio equal to the riskless rate.
- **Dupire equation holds only for standard calls (or puts)** and relates the value of a *standard* option with strike and expiration at K, T **to the same option with a different strike and expiration** $K + dK$, $T + dT$ when S, t is kept fixed.

14.4.3 Note: The Local Variance is Always Positive, As It Should Be

Dupire Denominator: $\frac{\partial^2 C}{\partial K^2}$ is the positive price of a positive payoff.

Dupire Numerator for option with strike $y = Ke^{rT}$ and expiration T, price $C(y, T)$:

$$\frac{d}{dT}C(y, T) = \frac{\partial C}{\partial y}\frac{\partial y}{\partial T} + \frac{\partial C}{\partial T} = \frac{\partial C}{\partial T} + ry\frac{\partial C}{\partial y}$$

We now show that this is also positive:

Go to discrete case: $C(S, t, Ke^{r(T+dT)}, T+dT) - C(S, t, Ke^{rT}, T)$

Evaluate this calendar spread at $t = T$ and $S_T > Ke^{rT}$:

Then first leg is in the money, worth $S_T - Ke^{rT}$ -- simply spot minus strike.

Second leg is $C\left(S_T, T, Ke^{r(T+dT)}, T+dT\right)$ always worth more than a forward by no arbitrage:

$$C\left(S_T, T, Ke^{r(T+dT)}, T+dT\right) \geq S_T - (Ke^{r(T+dT)})e^{-rdT} = S_T - Ke^{rT}$$

So the second leg is worth more than first, so the numerator at any earlier time t is always positive.

Using the Equation

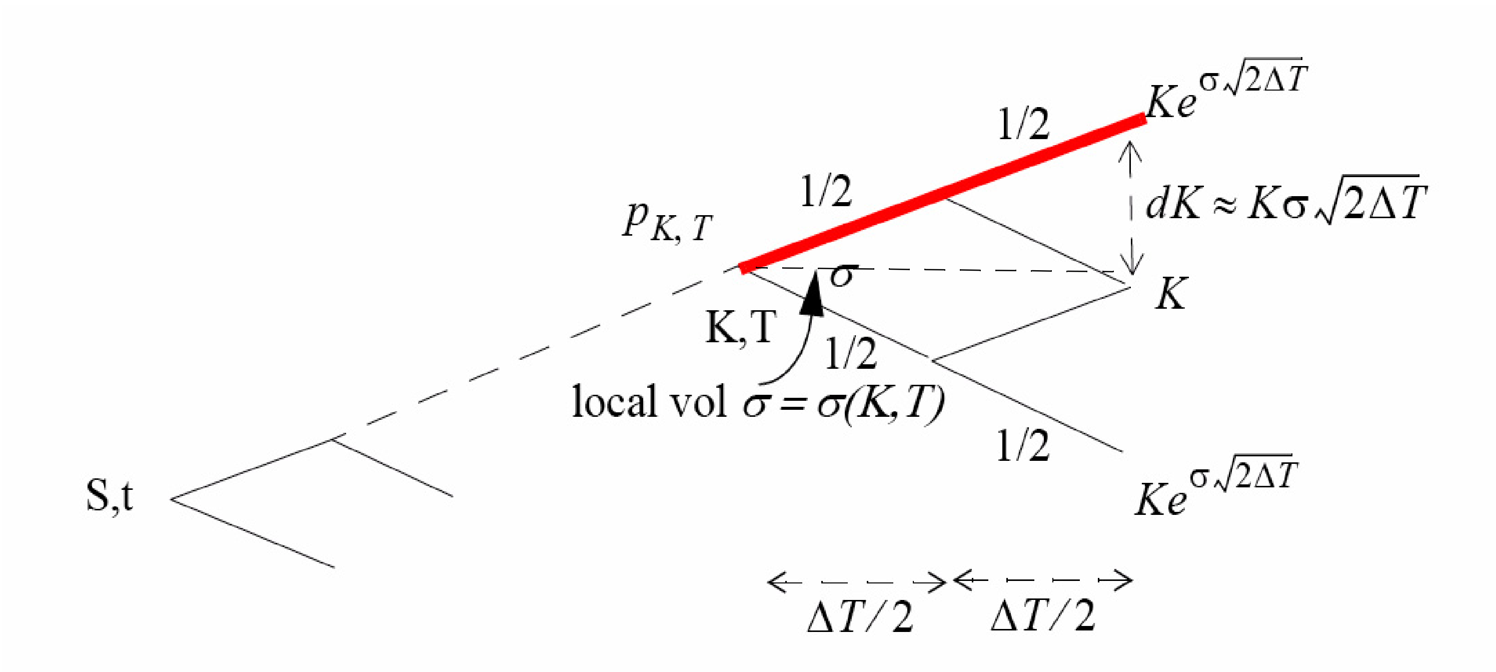
$$\frac{\sigma^2(K,T)}{2} = \frac{\frac{\partial}{\partial T}C(S,t,K,T) + rK\frac{\partial}{\partial K}C(S,t,K,T)}{K^2\frac{\partial^2 C}{\partial K^2}}$$

- Find $\sigma(K, T)$ from options prices and hence build local volatility tree from any options prices and their derivatives.
- Or you can do Monte Carlo over the stochastic process with the local volatility evolution.
- **This provides one consistent model** that values all standard options correctly rather than having to use several different inconsistent Black-Scholes models with different underlying volatilities.
- Volatility arbitrage trading. You can calculate the future local volatilities implied by options prices and then see if they seem reasonable. If some of them look too low or too high in the future, you can think about buying or selling future butterfly and calendar spreads to make a bet on future volatility.

14.5 A Poor Man's Derivation of the Dupire Equation in a Binomial Framework. (Zero Rates and Dividends)

$$\frac{\sigma^2(K,T)}{2} = \frac{\dfrac{\partial C(S,t,K,T)}{\partial T}}{K^2 \dfrac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

Let's use a Jarrow-Rudd tree that goes from (S,t) to (K,T) through **two half-periods of time** $\Delta T/2$, keeping interest rates zero for pedagogical simplicity.



We now show that the calendar spread at time t obtains all its optionality from future nodes at $S_T = K$ that move up the heavy red line to a nonzero positive payoff, so that:

$$C(S, t, K, T+dT) - C(S, t, K, T) \equiv \frac{\partial C}{\partial T}\Delta T = \overset{\text{probability}}{p_{K,T}}\frac{1}{4} \times \overset{\text{payoff}}{dK}$$

Proof: Nodes S_T below (K,T) at expiration time T contribute zero value to $C(S, t, K, T)$ and produce transitions to nodes at time $T + dT$ that have zero payoff for $C(S, t, K, T+dT)$. Thus all nodes below (K,T) contribute zero to the calendar spread.

Nodes S_T above (K,T) at expiration time T produce a payoff $S_T - K$. Each of these nodes transition to three nodes S_{T+dT} above K at time $T + dT$ that have the same risk-neutral expected value $S_T - K$ at time T, (assuming interest rates are zero). So these nodes above (K,T) also contribute zero to the calendar spread.

So, only the node at (K,T) matters.

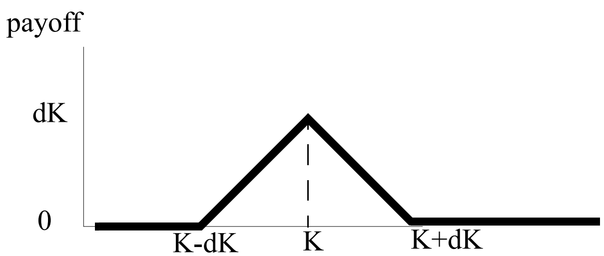
The value of the calendar spread per unit time is $$\frac{\partial C}{\partial T} = \frac{1}{4}p_{K,T} \times \frac{dK}{dT}$$

where $p_{K,T}$ is the risk-neutral probability of getting to (K, T) and $dK \approx \Delta K = K\sigma\sqrt{2\Delta T}$.

We can get $p_{K,\,T}$ from a **butterfly spread portfolio**

$p_{K,\,T}$ is the value of a portfolio that pays \$1 if the stock price is at node K, and zero for all other nodes at time T.

The butterfly spread $C(S, t, K + dK, T) - 2C(S, t, K, T) + C(S, t, K-dK, T)$ pays dK rather than \$1:



Dividing by dK produces a payoff which is \$1 at the node K and zero at adjacent nodes.

$$
\begin{aligned}
p_{K,T} &= \frac{C(S,t,K-dK,T) - 2C(S,t,K,T) + C(S,t,K+dK,T)}{dK} \\
&= \frac{C(S,t,K+dK,T) - C(S,t,K,T)}{dK} - \frac{C(S,t,K,T) - C(S,t,K-dK,T)}{dK} \\
&\approx \frac{\partial C(S,t,K,T)}{\partial K} - \frac{\partial C(S,t,K-dK,T)}{\partial K} \\
&\approx \frac{\partial^2 C(S,t,K,T)}{\partial K^2} dK
\end{aligned}
$$

Combining the expression for $p_{K,\,T}$ with $\frac{\partial C}{\partial T} = \frac{1}{4}p_{K,\,T} \times \frac{dK}{dT}$

$$\frac{\partial C}{\partial T} = \frac{1}{4}p_{K,T}\frac{dK}{dT}$$
$$= \frac{1}{4}\frac{\partial^2 C(S,t,K,T)}{\partial K^2}\frac{dK^2}{dT}$$

$$dK = K\sigma\sqrt{2dT},$$

So

$$\frac{\partial C(S,t,K,T)}{\partial T} = \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C(S,t,K,T)}{\partial K^2}$$

so that the local volatility $\sigma(K, T)$ is given by

$$\frac{\sigma^2(K,T)}{2} = \frac{\dfrac{\partial C(S,t,K,T)}{\partial T}}{K^2\dfrac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

You can regard this as a *definition* of the effective local volatility from options prices and has meaning beyond the model, even with stochastic volatility, as we will see later.