MATH-365 Week 3 HW - Stochastic Processes

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3.3 Determine which of the following matrices are regular.

$$P = \begin{pmatrix} 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 \\ p & 1-p \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 1 & 0 \\ 0.25 & 0.5 & 0.25 \\ 1 & 0 & 0 \end{pmatrix}.$$

Sol: Using R studio, every element of P^6 & R^3 is >0

Ans: P & R are regular. Q is regular iff $p \in [0,1]$

exploration from textbook

Regular Transition matrix is a matrix A such that for some $n \geq 1$, $(A^n > 0)$.

→ we can notice that each of the matrices P, Q and R have sum of rows = 1 and thus are transition matrices representing Markov chains.

→ Regularity of a transition matrix representing Markov chain corresponds directly to the ability to possess a limiting distribution, which is unique stationary distribution of chain.

→ If for some power n, all zeros in A^n appears at the same place as A^{n+1}, then they will appear at same place for all higher powers (textbook Pg 84)

→ Trying to raise matrices P and R to different powers we can say that they are regular as $(P^n > 0)$ & $(R^n > 0)$ for some $n > 0$. → used this!!

$$Q^2 = \begin{pmatrix} 0 & 1 \\ p & 1-p \end{pmatrix}\begin{pmatrix} 0 & 1 \\ p & 1-p \end{pmatrix} = \begin{pmatrix} p & 1-p \\ p(1-p) & p+(1-p)^2 \end{pmatrix} = \begin{pmatrix} p & 1-p \\ p-p^2 & p^2-p+1 \end{pmatrix}$$

for Q to be regular $(p>0)$ & $(1-p>0)$ & $(p-p^2>0)$ & $(p^2-p+1>0)$

⇒ as p^2-p+1 is always positive, we can say that matrix Q is regular as long as $\boxed{p \in (0,1)}$

3.5 A Markov chain has transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 0 & 1/4 & 0 & 0 & 3/4 \\ 3/4 & 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Describe the set of stationary distributions for the chain.

(b) Use technology to find $\lim_{n\to\infty} \boldsymbol{P}^n$. Explain the long-term behavior of the chain.

(c) Explain why the chain does not have a limiting distribution, and why this does not contradict the existence of a limiting matrix as shown in (b).

Sol (a) stationary distribution (π) must satisfy: $\pi = \pi P$, where $\pi = [\pi_1, \pi_2, \pi_3, \pi_4, \pi_5]$

$$\pi = \pi P = (\pi_1\ \pi_2\ \pi_3\ \pi_4\ \pi_5)\begin{pmatrix} 0 & 1/4 & 0 & 0 & 3/4 \\ 3/4 & 0 & 0 & 0 & 1/4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$\pi_1 = \frac{3}{4}\pi_2 + \frac{1}{4}\pi_5$ --(1)

$\pi_2 = \frac{1}{4}\pi_1 + \frac{3}{4}\pi_5$ --(2)

$\rightarrow 3\times(1) - (2)$

$\Rightarrow 3\pi_1 - \pi_2 = \frac{9\pi_2}{4} - \frac{\pi_1}{4}$

$\Rightarrow \frac{13\pi_1}{4} = \frac{13\pi_2}{4} \Rightarrow \boxed{\pi_1 = \pi_2}$

$\pi_3 = \pi_3$, $\pi_4 = \pi_4$

$\pi_5 = \frac{3}{4}\pi_1 + \frac{1}{4}\pi_2$ --(3) $\Rightarrow \pi_5 = \frac{3}{4}\pi_1 + \frac{1}{4}\pi_1 = \boxed{\pi_1}$

$\Rightarrow \pi = [\pi_1\ \pi_2\ \pi_3\ \pi_4\ \pi_5] = [\pi_1\ \pi_1\ \pi_3\ \pi_4\ \pi_1]$, as $\pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 1$

$\Rightarrow \pi_1 + \pi_1 + \pi_3 + \pi_4 + \pi_1 = 1 \Rightarrow \boxed{3\pi_1 + \pi_3 + \pi_4 = 1}$

All the non-negative vectors of the form $(\pi_1\ \pi_1\ \pi_3\ \pi_4\ \pi_1)$ which satisfy the equation: $\boxed{3\pi_1 + \pi_3 + \pi_4 = 1}$. Ans

(b) By raising P to high power (n), using R,

$$\lim_{n\to\infty} P^n = \begin{array}{c|c} & \begin{matrix} \text{"A"} & \text{"B"} & \text{"C"} & \text{"D"} & \text{"E"} \end{matrix} \\ \begin{matrix} \text{"A"} \\ \text{"B"} \\ \text{"C"} \\ \text{"D"} \\ \text{"E"} \end{matrix} & \begin{pmatrix} 1/3 & 1/3 & 0 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \end{pmatrix} \end{array}$$

- In the long-term, we can say by looking at $\lim_{n\to\infty} P^n$ that if chain starts at C or D, then it stays there forever.
- Moreover, if our chain starts from A, B or E then we have an equal chance to land at any one of A, B or E in long term.

(c) As the long-term behaviour of chain depends on the initial state (as per part (b)), therefore it does not have a limiting distribution although it has a set of possible stationary distributions. As we can see that the matrix $(\lim_{n\to\infty} P^n)$ doesn't have equal rows, we can say that this doesn't contradict the fact that limiting matrix exists.

3.10 A Markov chain has transition matrix $\boldsymbol{P}$ and limiting distribution $\boldsymbol{\pi}$. Further assume that $\boldsymbol{\pi}$ is the initial distribution of the chain. That is, the chain is in stationarity. Find the following:

(a) $\lim_{n\to\infty} P(X_n = j | X_{n-1} = i)$

(b) $\lim_{n\to\infty} P(X_n = j | X_0 = i)$

(c) $\lim_{n\to\infty} P(X_{n+1} = k, X_n = j | X_0 = i)$

(d) $\lim_{n\to\infty} P(X_0 = j | X_n = i)$

Sol

(a) $\lim_{n\to\infty} P(X_n = j \mid X_{n-1} = i) = \lim_{n\to\infty} P(X_1 = j \mid X_0 = i) = \boxed{P_{ij}}$ Ans

(b) $\lim_{n\to\infty} P(X_n = j \mid X_0 = i) = (P^n)_{ij} = \boxed{\pi_j}$ Ans

(c) $\lim_{n\to\infty} P(X_{n+1} = k, X_n = j \mid X_0 = i)$

$= \lim_{n\to\infty} P(X_{n+1} = k \mid X_n = j)\, P(X_n = j \mid X_0 = i)$

$= \lim_{n\to\infty} P(X_1 = k \mid X_0 = j)\, P(X_n = j \mid X_0 = i) = \boxed{P_{jk}\pi_j}$ Ans

(d) $\lim_{n\to\infty} P(X_0 = j \mid X_n = i) = \lim_{n\to\infty} \dfrac{P(X_0 = j, X_n = i)}{P(X_n = i)}$

$= \lim_{n\to\infty} \dfrac{\overbrace{P(X_n = i \mid X_0 = j)}^{\pi_i}\, P(X_0 = j)}{\underbrace{P(X_n = i)}_{\pi_i}} = \dfrac{\pi_j \cancel{\pi_i}}{\cancel{\pi_i}} = \boxed{\pi_j}$ Ans

[here $P(X_0 = j)$ is π_j because we are told to assume that the initial distribution and limiting distribution are both equal to π]

3.14 The California Air Resources Board warns the public when smog levels are above certain thresholds. Days when the board issues warnings are called *episode* days. Lin (1981) models the daily sequence of episode and nonepisode days as a Markov chain with transition matrix

$$P = \begin{array}{c c} & \begin{array}{cc} \text{Nonepisode} & \text{Episode} \end{array} \\ \begin{array}{c} \text{Nonepisode} \\ \text{Episode} \end{array} & \begin{pmatrix} 0.77 & 0.23 \\ 0.24 & 0.76 \end{pmatrix} \end{array}.$$

(a) What is the long-term probability that a given day will be an episode day?

(b) Over a year's time about how many days are expected to be episode days?

(c) In the long-term, what is the average number of days that will transpire between episode days?

Sol

(a) Longterm probability of an episode day $= \left(\lim_{n\to\infty} P^n\right)_{12 \text{ or } 22}$

$= \boxed{0.489\ldots}$ Ans [from calculator]

(b) The expected number of episode days over a year

$= 365 \times$ longterm probability of an episode day.

$= 365 \times 0.489362\ldots \approx \boxed{178.6 \text{ days}}$ Ans

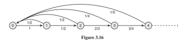
(c) In the long-term, average number of days that will transpire between episode days $= [1/(\text{longterm prob. of ep. day})]$

$= (1/0.489362\ldots) \approx \boxed{2}$ Ans

3.27 Sinclair (2005). Consider the infinite Markov chain on the non-negative integers described by Figure 3.16.

(a) Show that the chain is irreducible and aperiodic.

(b) Show that the chain is recurrent by computing the first return time to 0 for the chain started at 0.



(c) Show that the chain is null recurrent.

Sol

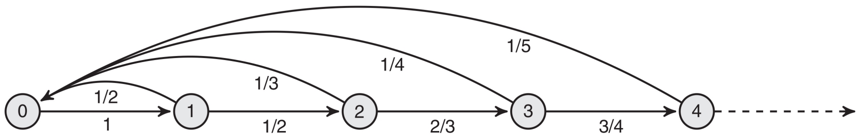


Figure 3.16

(a) Observations: — state 0 can be reached from any other non-zero state.
— we can go from 0 to whichever state we want with a non-zero probability (if we observe $0 \to 1 \to 2 \to 3 \ldots$)

$\Rightarrow$ chain is $\boxed{\text{IRREDUCIBLE}}$

Observations: — the loops like $0 \to 1 \to 0$, $0 \to 1 \to 2 \to 0$, ... exist for $\forall\, x$ in $(0 \to 1 \to 2 \ldots x \to 0)$ where $x \geq 1$

$\Rightarrow$ state 0 is accessible in more than or equal to 2 steps.

$\Rightarrow$ 0 is $\boxed{\text{APERIODIC}}$ $\Rightarrow$ chain is $\boxed{\text{APERIODIC}}$

(b) The probability of returning at zero after n steps is by moving straight $(n-1)$ steps & then looping round to 0 state.

$$\Rightarrow \left(1 \times \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{4}} \times \cdots \times \frac{\cancel{n-3}}{\cancel{n-2}} \times \frac{\cancel{n-2}}{n-1}\right) \times \frac{1}{n} = \frac{1}{n(n-1)} \quad (\forall\, n \geq 2)$$

$$\Rightarrow P_{\text{total of returning to } 0} = \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right) = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3} + \cdots\cdots \infty = \boxed{1}$$

$\Rightarrow$ state 0 is $\boxed{\text{RECURRENT}}$ $\Rightarrow$ chain is $\boxed{\text{RECURRENT}}$

(c) on next page...

3.27(c) Let Expected time to return to k state be $E_{time}(k)$...

gauged by number of steps

$$E_{time}(0) = \sum_i (\text{time})_i (p_{0i}) = \sum_{n=2}^{\infty} (\cancel{n}) \left(\frac{1}{\cancel{n}(n-1)}\right)$$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{n-1}\right) = \infty \; .$$ (doesn't converge)

$\Rightarrow$ diverges

$\Rightarrow$ Hence, 0 is $\boxed{\text{NULL RECURRENT}}$ $\Rightarrow$ chain is $\boxed{\text{NULL RECURRENT}}$

Next Problem Starts on next page...

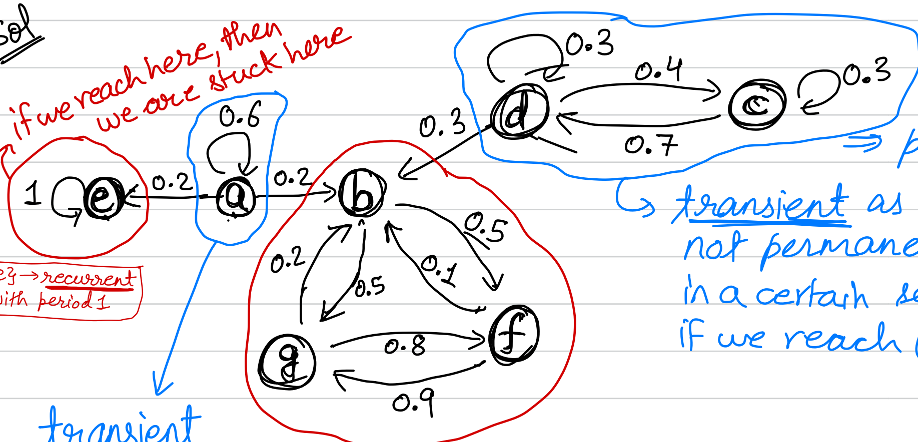
3.29 Consider a Markov chain with transition matrix

$$
P = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array}
\begin{array}{c}
\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\
\begin{pmatrix}
0.6 & 0.2 & 0 & 0 & 0.2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\
0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 \\
0 & 0.3 & 0.4 & 0.3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0.1 & 0 & 0 & 0 & 0 & 0.9 \\
0 & 0.2 & 0 & 0 & 0 & 0.8 & 0
\end{pmatrix}
\end{array}.
$$

Identify the communication classes. Classify the states as recurrent or transient, and determine the period of each state.

Sol

if we reach here, then we are stuck here



$\Rightarrow$ {e} → recurrent with period 1

$\Rightarrow$ period 1.

↳ transient as we are not permanently stuck in a certain section even if we reach ⓓ or ⓒ

transient as we are not stuck there (with period 1)

↳ if we reach ⓑ then we are stuck forever in the cycle of ⓑ, ⓕ, ⓖ.

↳ Recurrent with period 1.

$\Rightarrow$ {e} is recurrent with period 1
$\Rightarrow$ {a} is transient ~~with period 1~~
$\Rightarrow$ {b, f, g} is recurrent with period 1
$\Rightarrow$ {c, d} is transient ~~with period 1.~~

Ans

{e} : period = gcd(1) = 1 (aperiodic)
{a} : period = gcd(1) = 1 (aperiodic) → transient is always aperiodic
{b, f, g} : period = gcd(2, 3, 4, 5, ...) = 1 (aperiodic) ⇒ all diagonal elements of P^2, P^3 are > 0.
{c, d} : period = gcd(1, 2, 3...) = 1 (aperiodic) → transient is always aperiodic