Lecture 2:

1. The Smile
2. Principles of Financial Valuation

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = r\left(C - S\frac{\partial C}{\partial S}\right)$$

Black-Scholes PDE

gain from time passing + gain from stock moving = riskless return on hedged position

Comment on Negative Forward Rates

Assume all zero-coupons have face $1.

If the current price of a (two-year zero coupon) is more than the price of a (one-year zero coupon) (i.e. if the one-year rate one year forward is negative), then:

Short the (two-year zero-coupon)

With the proceeds buy more than one (one-year zero-coupon)

After one year receive more than $1 at maturity from the (one-year zero coupons).

Wait a year with money under your mattress and buy back the two-year at maturity and you are left with some risk-free money.

This is a riskless arbitrage.

2.1 A Quick Look at the Implied Volatility Smile

- **What is implied volatility?** $C_{mkt} = C_{BS}(S, t, K, T, r, \Sigma[S, t, K, T])$. A parameter, not a statistic. A function of S and K and time.
- **Representative S&P 500 implied volatilities for vanilla options prior to 1987**.

This shape is consistent with the Black-Scholes model, which assumes one volatility for the underlying index or stock. (Which options are the most liquid?)

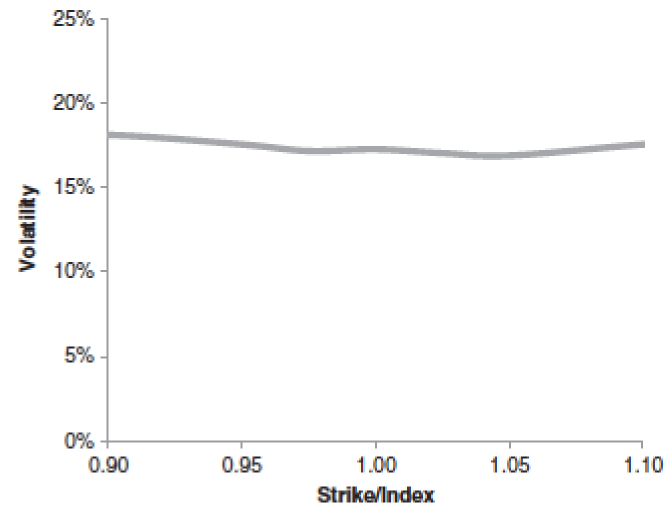


FIGURE 1.1 Representative S&P 500 Implied Volatilities prior to 1987

- **Representative S&P 500 implied volatilities after 1987**

The volatility of a *stock* itself cannot depend upon the option with which you choose to view it. How can one stock have many different volatilities?

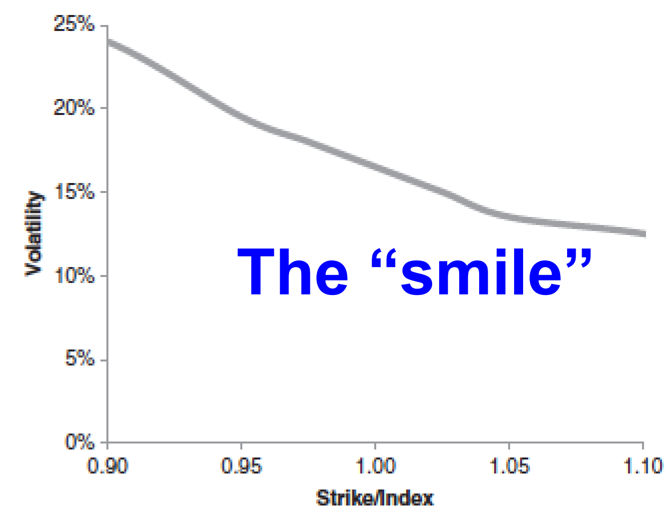


FIGURE 1.2 Representative S&P 500 Implied Volatilities after 1987

2.2 The Challenge: The Black-Scholes model disagrees with the smile.

The Black-Scholes model is being used as **quoting mechanism** to translate from prices to more intuitive implied volatilities.

What's the right replacement??

Why is it important?

(1) **Hedging:** If we **quote** vanilla option prices using Black-Scholes, then

$$C_{mkt} = C_{BS}(S, t, K, T, r, \Sigma[S, t, K, T])$$

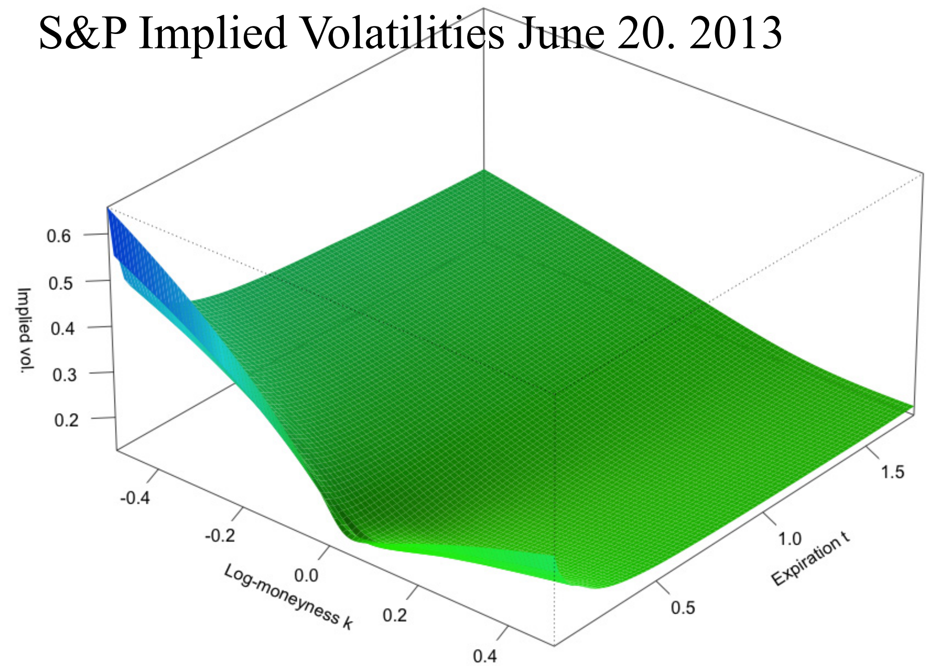
So the hedge ratio is given by

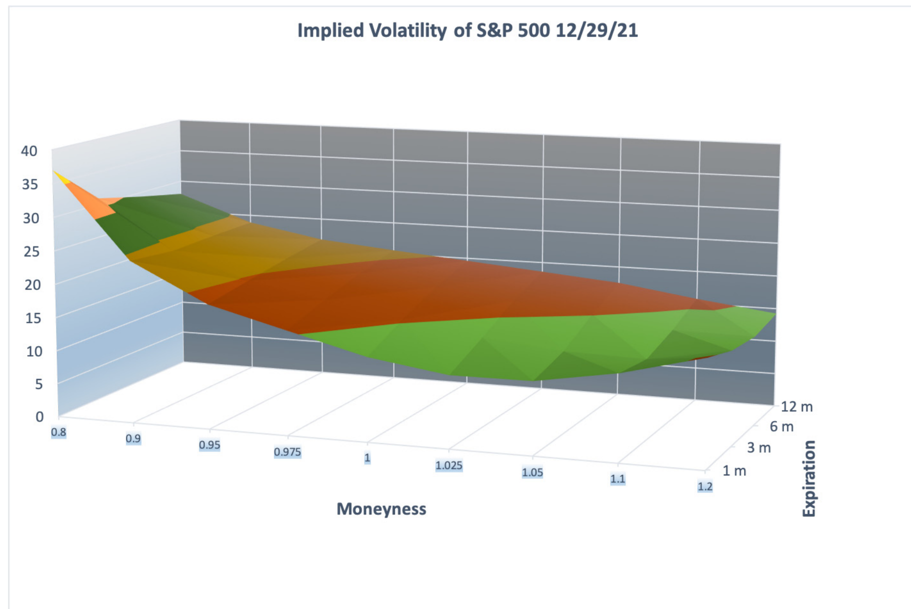
$$\Delta = \frac{\partial C_{\mathrm{mkt}}(S,t,K,T)}{\partial S} = \frac{\partial C_{\mathrm{BSM}}}{\partial S} + \frac{\partial C_{\mathrm{BSM}}}{\partial \Sigma}\frac{\partial \Sigma}{\partial S} = \Delta_{\mathrm{BSM}} + \frac{\partial C_{\mathrm{BSM}}}{\partial \Sigma}\frac{\partial \Sigma}{\partial S}$$

The quoting mechanism doesn't help with that. We need a better model to know how volatility varies with stock price: the *spot-vol dynamics*.

We see $\frac{\partial}{\partial K}\Sigma(S, K, \ldots)$.

We want to know $\frac{\partial}{\partial S}\Sigma(S, K, \ldots)$





(2) **Valuation:** We need a model to **value** exotic options whose price isn't quoted.

2.3 Historical Development of the Smile

- There was always a bit of a smile in currency options markets.
- The equity “smile” is really more of a skew or a smirk.
- 1987 crash: a giant market could drop by 20% or more in a day. Low-strike puts are now perceived to be more likely to end up in the money.
  (What if you hedge and are insensitive to value of index? Increase in volatility might still hurt.)
- The volatility smile has spread to all other options markets.
- Traders and quants in every product area have had to model the smile.
- No area where model risk/uncertainty is more of an issue than in the modeling of the volatility smile. There is no clearly *correct* model.

- You could say:
  Implied volatility reflects the market’s expectation of future volatility, of drama, of fear.
  A bit like the yield curve reflects the market’s expectation of future rates.

Past Black-Scholes Implied Volatilities

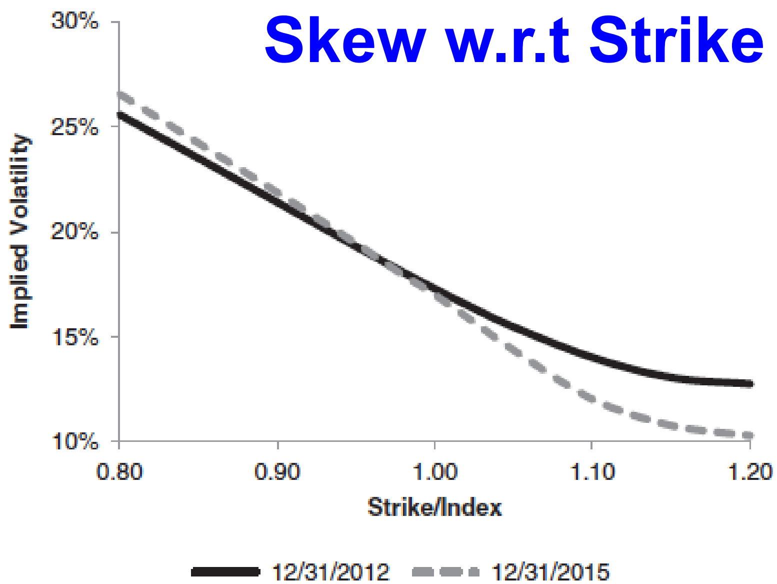


FIGURE 8.2 S&P 500 Six-Month Volatility Smile
Source: Bloomberg.

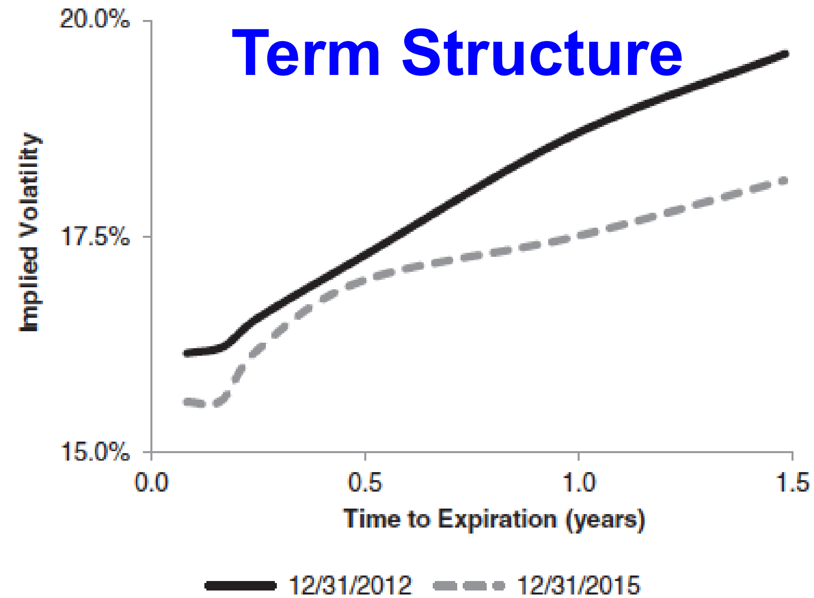


FIGURE 8.3 S&P 500 ATM Volatility Term Structure
Source: Bloomberg.

Surface
or
Smile

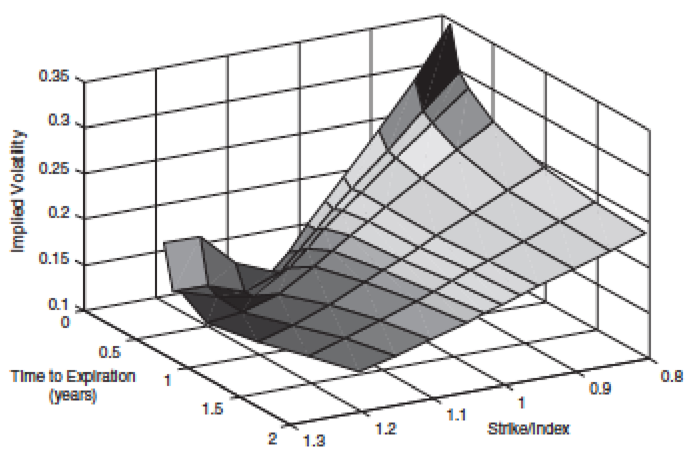
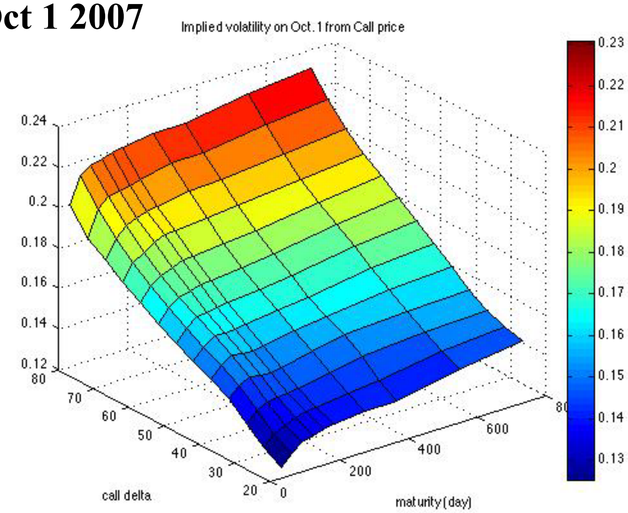


FIGURE 8.4 Volatility Surface, S&P 500, December 31, 2015
Source: Bloomberg.

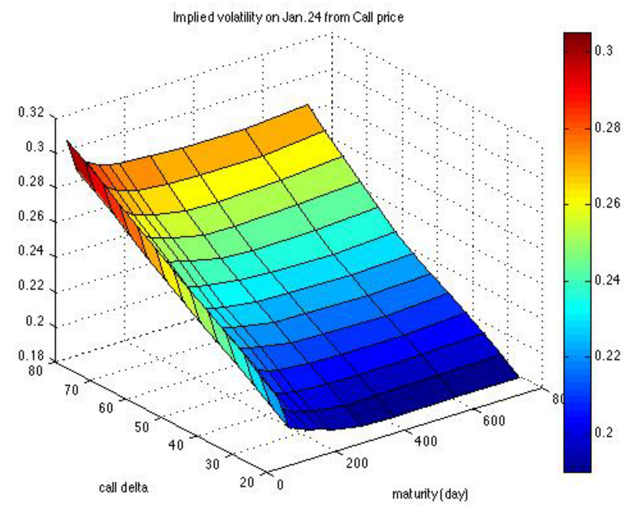
Some Historical S&P 500 Smiles

1. S&P 500

Oct 1 2007



Jan 24 2008



2.4 SVI Parameterization (*Stochastic Volatility Inspired*)

• A commonly used parameterization of the implied volatility smile for a fixed expiration T is

$$\Sigma^2(m, T) = a + b[\rho(m-c) + \sqrt{(m-c)^2 + \theta^2}]$$

where $m = \ln\frac{K}{F}$ is the **log forward moneyness**, $F = Se^{rT}$ is the forward price of the underlying stock S, and T is the time to expiration of the option.

What features does this have? Homework 1 Problem 3.

a - level
b -
ρ -
c -
θ -

• This is an empirical formula. It is a parameter in a model in which it should be constant, i.e.independent of S. It cannot be 100% correct theoretically because we must worry about arbitrage violations induced by the parameter. See Homework 1 Problem 2.

(Cf. parameterizing interest rates: you cannot write down a steeply negative yield curve because it implies negative forward rates which violate the principal of no riskless arbitrage.)

2.5 Stylized Facts about the Smile for Equity Indexes

- Current implied volatility tends to be greater than recent realized volatility.
- Almost always negatively skewed. Why? Protection against payoff and volatility spike in a crash?
- Wasn't that way before 1987.
- Skew is steeper for short expiration, flatter for longer ones. (Steeper in what variable?)
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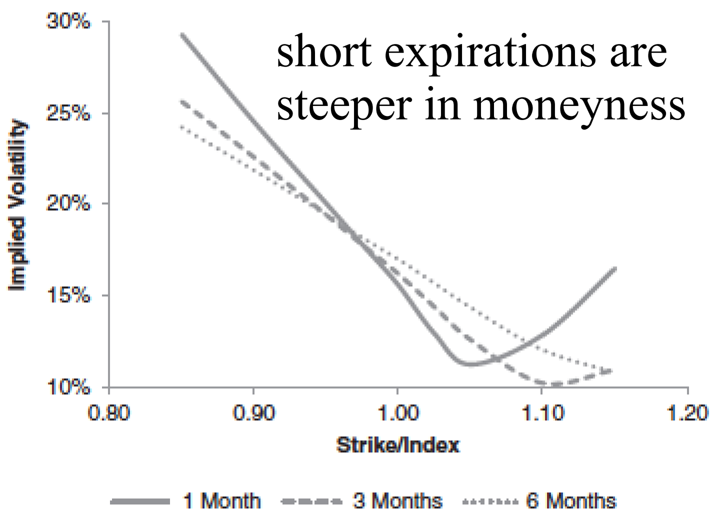
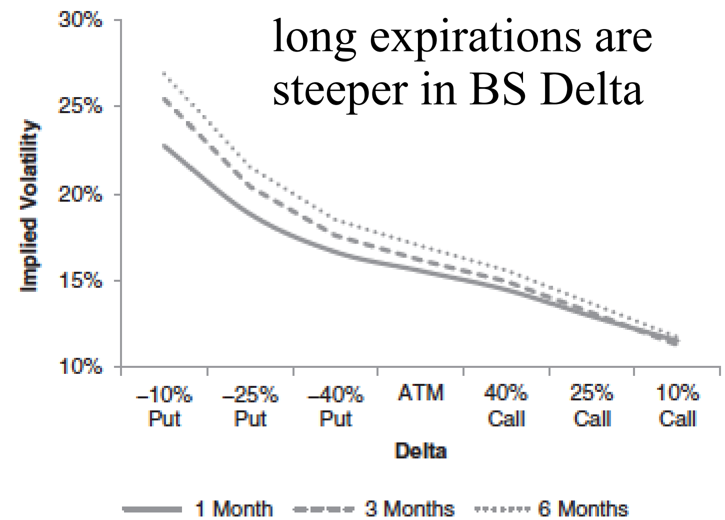




- Term structure of the volatility surface can slope up or down. During a crisis—and a crisis is always characterized by high volatility—the term structure is likely to be downward sloping. The high short-term volatility and lower long-term volatility reflect market participants' belief that uncertainty in the near term will eventually be resolved.

• Index and implied volatility are negatively correlated. Indexes tend to glide up and crash down.

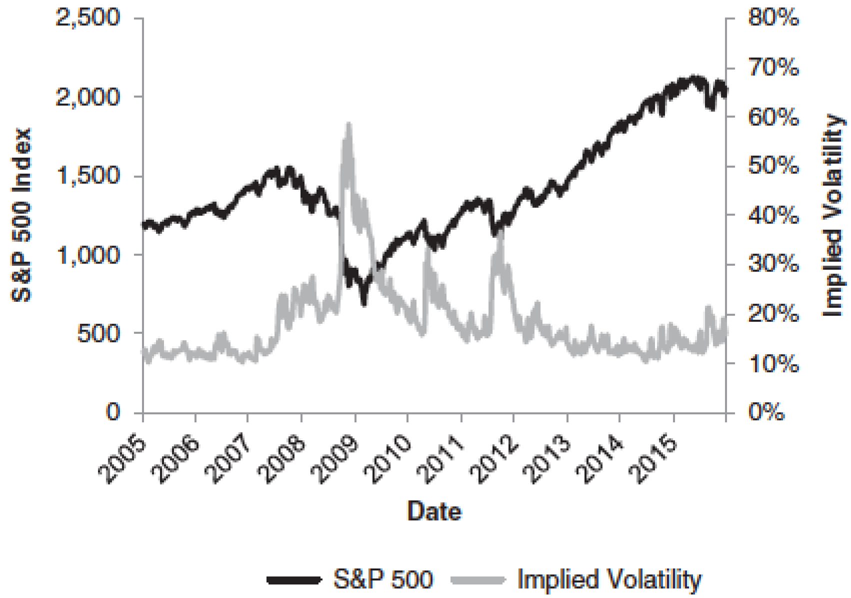


FIGURE 8.9 S&P 500 Level and Three-Month At-the-Money Implied Volatility

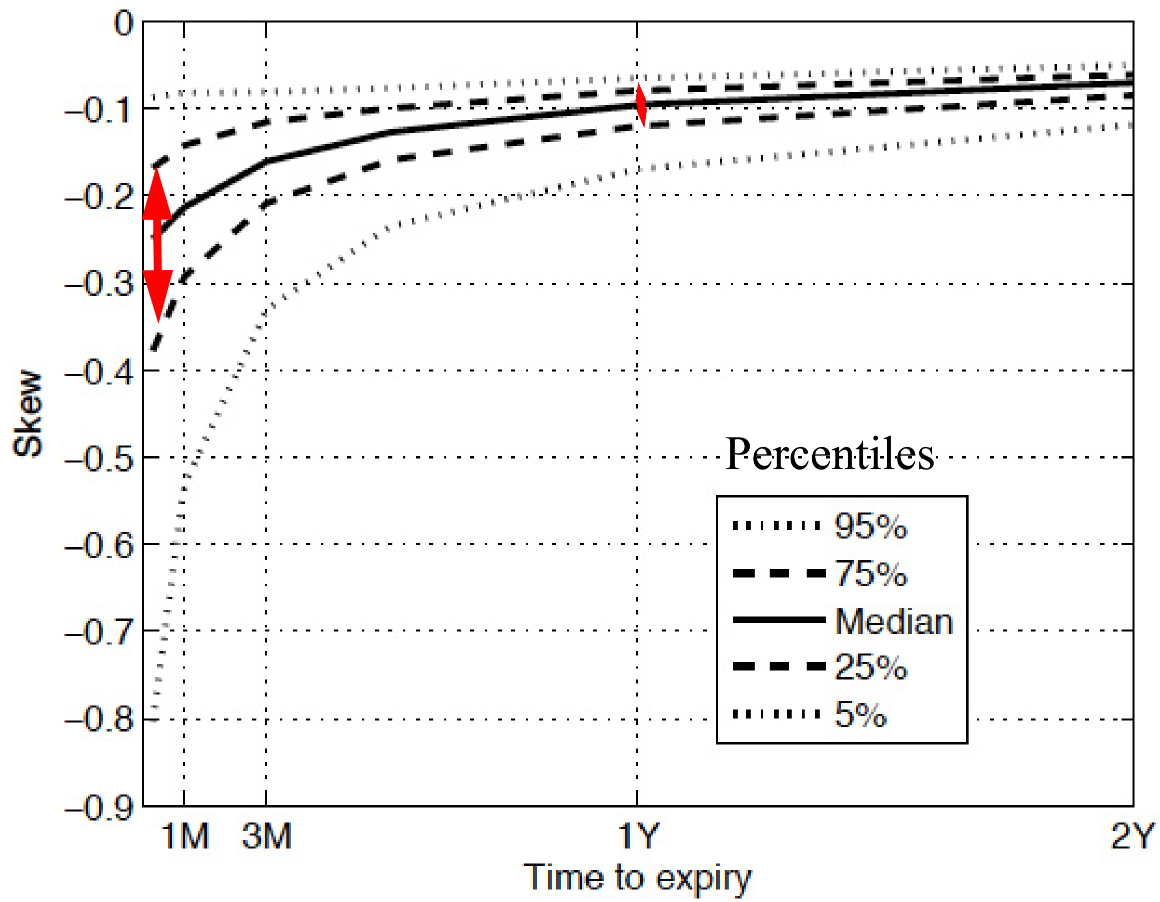
• Realized **and** implied volatility increase in a crash.

• When the index moves sharply down,

All option implied volatilities increase;
short-term implied volatility increases more sharply;
the short-term negative skew steepens.
long-term volatility and the long-term skew increase too, but less so.

• Implied volatility is a model parameter, like interest rates. It seems to be mean-reverting. It can jump up quickly and decline less rapidly, though nowadays it declines more rapidly than before, perhaps because people expect the Fed to save the market.

• The volatility of implied volatility and of the skew is greatest for short expirations, analogous to the higher volatility of short-term Treasury rates.



PRINCIPLES OF FINANCIAL VALUATION

The Principles of Financial Valuation

Don't get misled by mathematics, theorems, lemmas. Understand them, but this is real world.

- What is financial engineering?
  Cf. Mechanical engineering, Electrical engineering, Bio-engineering.

  Science seeks to discover the fundamental principles that describe the world, and is usually reductive.
  Engineering is about using those principles, constructively, for a purpose.

  Financial engineering, layered above financial science, would be the study of how to create functional financial devices – convertible bonds, warrants, default swaps, etc. – that perform in desired ways.
- What is financial science?
  Our elegant scientific theories don't describe the behavior of assets very well.
  (People have emotions.)
  Stock evolution isn't Brownian.
  So let's use as little modeling as possible.
- Extreme axiomatization is therefore not that useful.Concentrate on concepts, then use math.

2.6 Terminology: Price & Value

- Price = what you have to pay to acquire a security.
  Value is what it is worth. The price is fair when it is equal to the value.
- Can you name one security whose price is equal to its value?
- Judging value, in even the simplest way, involves the construction of a model or theory.
- Fischer Black: markets are efficient when prices are 1/2 to 2 times value!

The Purpose of Models in Finance

- Example: Valuing a unique (“exotic”) 57th Street Manhattan apartment.
- Models are used to rank securities by value.
- Models are used to interpolate or extrapolate from liquid prices to illiquid prices.
- Models allow you to quantify intuition, to turn linear quantities you can have intuition about into nonlinear dollar values.

Styles of Modeling: Absolute and Relative

- Absolute vs. Relative Value Models.

- Absolute Value/Description: Newtonian or Quantum Mechanics or Maxwell's Equations are
  theories of the world.
  Geometric Brownian motion is a *model* of valuation, an analogy, comparing stock prices to the diffusion of particles of smoke, not an accurate description.

- Relative Value:
  Used for going from liquid prices --> illiquid values.
  Relative valuation is less ambitious.
  We regard derivatives as molecules made out of simpler atoms.
  Black-Scholes tells you the price of an option in terms of the price of a stock and a bond.

- In this course we adopt the view point of an options trading desk, as manufacturers or wholesalers of options. We are relativists.
  Derivatives can be constructed or deconstructed.
  Stocks to Options
  Vanilla to Exotic Options
  Fruit salad: What is *the implied price* of pears?

2.7 The One Commandment of Quantitative Finance

Andy Lo.
You might think there are many laws, but …

You can derive everything in neo-classical finance from one principle:

- **The Law of One Price**: If you want to know the value of a security, use the price of another security that's as similar to it as possible.

- **The Principle of No Riskless Arbitrage**: Any two securities with identical future payoffs, *no matter how the future turns out*, should have identical current prices.

- How to use the law:
  Target security
  Replicating portfolio: a collection of liquid securities that collectively have the same future payoffs as the target, *no matter how the future turns out.*

- ***The Science Part of Finance***:

  describe the stochastic behavior of all future payoffs.
  usually a model and at best partially correct.

The Engineering Part of Finance:

construct the replicating portfolio and prove similarity of payoffs.

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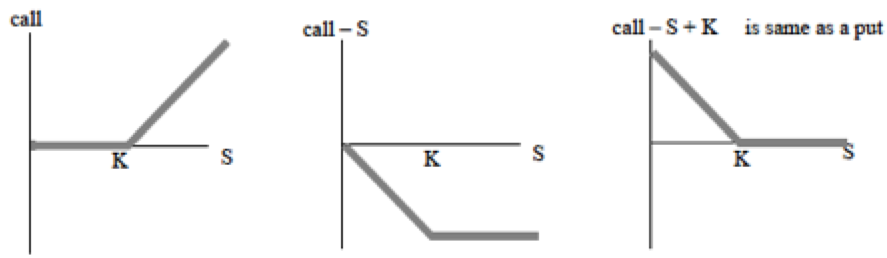
Engineering: Styles of Replication

- **Static:**
  The best method when you can do it;
  Example: replicating a put from a call: P = C - S + K



Once you replicate statically, you have only credit risk. Even if you can do it approximately, you eliminate a lot of model risk and rehedging costs.

- **Dynamic**
  Underlying positions needs to be adjusted many times
  Black-Scholes-Merton theory of options valuation

  The risks:
    - Transactions costs
    - Liquidity
    - Systems and IT issues
    - Exposure to volatility and models. (Volatility has dynamics too, later).

Always try static first, then dynamic.

- So, in this course, *first try use static replication* for valuing new securities. If we cannot, *then we will use dynamic replication*.

Models are unreliable guides to the world of finance, and because you don't know which is the right one, *it's best to use as little modeling as possible. And, if you have to use a model, it's always good to use more than one so you understand the model-dependence of your result.*

Implied Variables and Realized Variables

Physics models and theories start from today and **predict the future**.

Financial models, because they are not quite right, often rely on *implied variables*. Implied variables are opinions or views about the future extracted from current prices and a model. They are then used **predict the value of another security today**. (The apartment price from other apartments; the value of a mortgage from Treasuries; an exotic option value from vanilla options.)

Financial models **calibrate** the future to current known prices of liquid securities whose prices we trust to produce the current value of **implied variables** that match known prices today:

Yield to maturity, forward rates, implied volatility.

Implied variables describe what people think will happen filtered through a model.

Realized variables describe what actually happens.

What matters is not only what will happen, but what people *think* will happen.

But: what people think will happen affects what happens today.

Realized volatility is a statistic.

Implied volatility is NOT a statistic, but a parameter.

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MODELING MARKETS:

2.8 The Efficient Market Hypothesis

Experience shows that it is difficult or impossible to successfully and consistently predict what's going to happen to the stock market tomorrow based on all the information you have today.

We have already seen in the stylized facts that there is little autocorrelation between returns.

The EMH formalizes this experience by stating that it is impossible to beat the market, because current prices reflect all current economic and market information.

Jiu-jitsu approach, turn weakness into strength: *I can't figure out how things work, so I'll make the inability to do that a principle.*

2.9 Uncertainty, Risk & Return

Quantifiable Uncertainty or Risk

What do you mean when you say there's a 1/8 chance of throwing 3 heads in succession?

Frequentist Probabilities

Tossing a coin: **history doesn't matter**.

Unquantifiable Uncertainty:

What do you mean when you say that there's a small probability of a revolution overthrowing the United States government in the next year?

Consider: The likelihood of a revolution in some country or the probability of a terrorist attack. The chance that an earthquake of magnitude 6.7 or greater will occur before the year 2030 in the San Francisco Bay Area.

No way of honestly estimating probabilities.

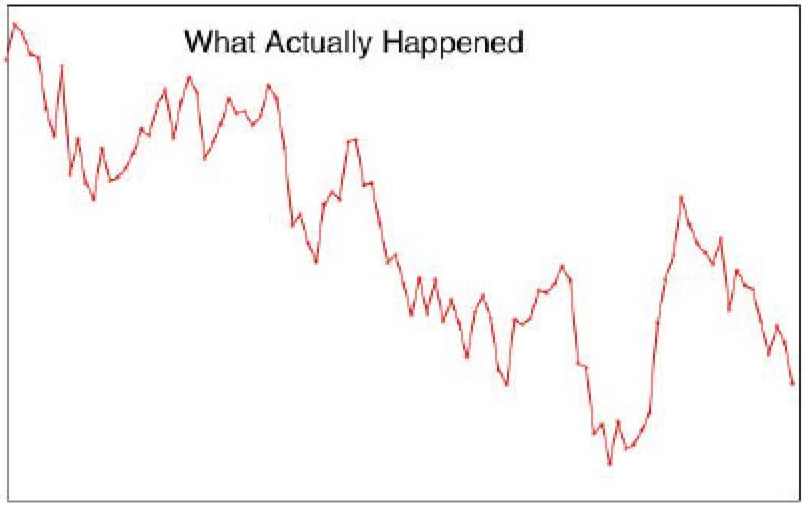
In human affairs frequentist probabilities are not known and **history matters**. People have memories. And we don't have enough statistical information for unique events.

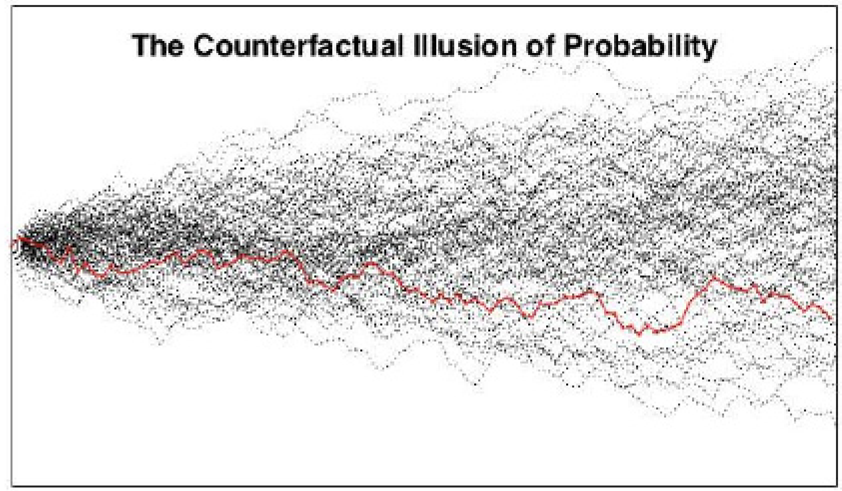
9/11 wasn't a part of the probability distribution for the US until after it happened.

Most events of interest in life have unquantifiable probabilities.

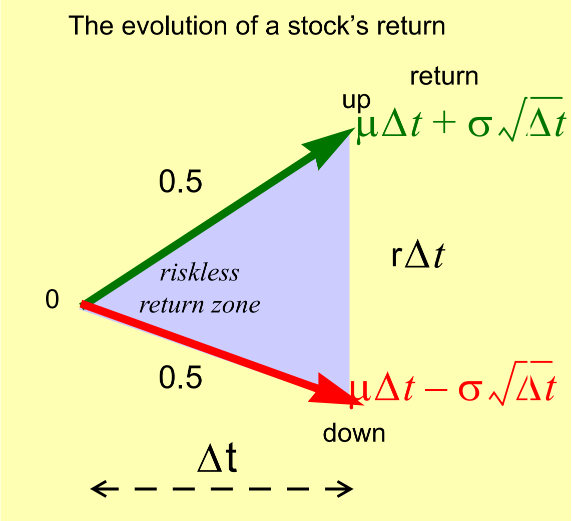
Most financial models assume that unquantifiable uncertainties are actually frequentist. We are mostly going to do this too, but we need to always remember the assumptions we make.

Markets are not exactly like flipping coins. There isn't a well-defined *a priori* probability of a market crash. Probability is a bit of an illusion, a fantasy about what might have happened.



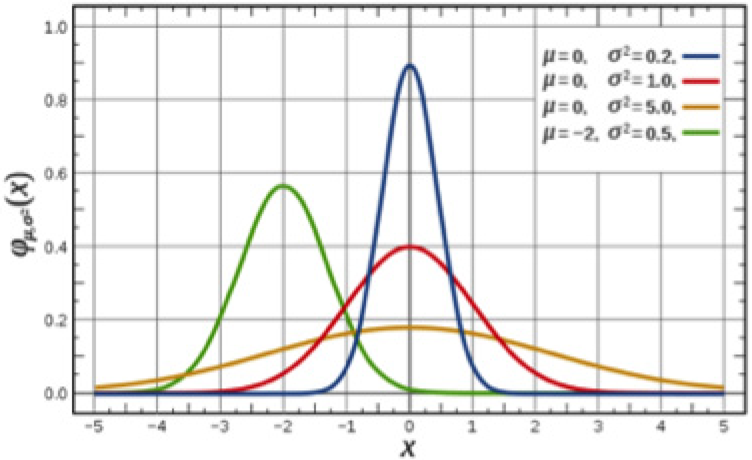


2.10 The *Attempted Science* Part of Neoclassical Finance



- Brownian motion: Stock returns are assumed to undergo arithmetic Brownian motion, are normally distributed.
- $$\frac{dS}{S} = \mu dt + \sigma dZ$$
- No Arbitrage means **the riskless return is a convex combination** of up and down returns. If it weren't, you could always make a profit.
- Brownian motion is one model for an efficient market. It is a **good model** for atoms colliding with dust particles, a **not-so-good model** for stocks, as we've seen from the stylized facts.
- **Brownian motion assumes:**
  "financial risk" is a frequentist statistic.
  the stock's motion is completely specified by the volatility σ and expected return μ.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$



2.11 Deriving The Relation Between Risk and Return in our Model

In our geometric Brownian motion model for stocks, everything foreseeable about a stock, and in particular the distribution of its future payoffs, is determined by its ***expected return*** μ and the ***volatility*** σ **that represents its risk.**

All that differentiates one stock from another in this model are the values of μ and σ. Two securities with the same μ and σ are in essence identical.

What is the relation between μ and σ?

Note: There is one privileged security, the riskless bond, which has $\sigma = 0$. If there is no risk $\sigma = 0$, you know you will exactly earn the riskless rate and so $\mu = r$ exactly, with no statistical distribution at any given time.

What happens if σ is not zero?

We Will Try to Reduce Risky Securities to Riskless Ones, and Then Use The Law of One Price to Figure Out the Value of a Risky Security.

If you have a risky security whose risk σ you know, but whose return μ you don't know, you can figure out the value of μ via the following strategy:

Embed the risky security into a portfolio such that the portfolio has zero total risk.

Then by the Law of One Price it should have the known return r of a riskless bond.

If you know the composition of that riskless portfolio, the proportions that make it riskless, then you can then figure out the expected return of the risky security.

This strategy will lead to both CAPM (the Capital Asset Pricing Model) and the Black-Scholes-Merton Options Valuation Model.

How Can One Reduce Risk?

Hedging: If two securities are positively correlated with each other, if you buy one and short the other, you can reduce the risk of the portfolio.

Diversification: If you take a whole bunch of uncorrelated risky securities together, their volatility decreases because some go up as others go down. If you put enough of them together, their volatility becomes zero asymptotically.

- **Strategy:**

To find the expected return on a security, try to remove all of its risk by combining it with other securities. If you can do that, then by The Law Of One Price, the combination must earn the riskless rate which is known. Then you can back out the return of the original security.

2.12 The Relation Between Risk & Return for Stocks: CAPM

Consider a market with an infinite number of stocks S_i, all correlated with the market, which you can think of as the S&P 500 or S&P futures for example.

Let ρ_{iM} be the correlation of the returns between a stock S_i with volatility σ_i and the market M with volatility σ_M. Here is the geometric Brownian motion description:

$$\frac{dM}{M} = \mu_M dt + \sigma_M dZ_M$$

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i\left(\sqrt{1-\rho_{iM}^2}\,dZ_i + \rho_{iM} dZ_M\right)$$

where dZ_i and dZ_M are all uncorrelated with each other. The idiosyncratic risk of the stock is described by dZ_i.You can *hedge away* the M-related risk of the stock S_i by shorting Δ_i shares of the market to create an M-neutral stock $\vec{S}_i$ that has no exposure to movements dZ_M, but only to dZ_i

$$\vec{S}_i = S_i - \Delta_i \times M$$

$d\vec{S}_i = dS_i - \Delta_i \times dM$ has no market risk if $\Delta_i = \rho_{iM}(\sigma_i/\sigma_M)\frac{S_i}{M} \equiv \beta_i \frac{S_i}{M}$.

Each M-neutral stock $\vec{S}_i$ is now uncorrelated with the market and uncorrelated with all other M-neutral stocks.

The M-neutral stock has expected return per unit time

$$\frac{1}{dt}\left\{\frac{E\left[d\vec{S}_i\right]}{\vec{S}_i}\right\} = \frac{1}{dt}\left\{\frac{E[dS_i - \Delta_i \times dM]}{S_i - \Delta_i \times M}\right\} = \frac{\mu_i S_i - \Delta_i \mu_M M}{S_i - \Delta_i \times M} = \frac{\mu_i S_i - \beta_i \mu_M S_i}{S_i(1-\beta_i)} = \frac{\mu_i - \beta_i \mu_M}{(1-\beta_i)}$$

We can do this for each stock S_i in the market. If we diversify over all M-neutral stocks we can create a portfolio with asymptotically zero volatility, and so by the Law of One Price it must earn the riskless return r, so that[1] each stock in the portfolio can earn no more than the riskless return. So

$$\frac{\mu_i - \beta_i \mu_M}{(1-\beta_i)} = r$$

$$(\mu_i - r) = \beta_i(\mu_M - r) \qquad \text{CAPM in “Efficient Markets”}$$

$$(\mu - r) = \beta(\mu_M - r) \qquad \text{Capital Asset Pricing Model}$$

stock’s excess return   stock’s beta   market’s excess return

What this relation is really saying is that if you can hedge away all market risk, and then diversify over all idiosyncratic risk, no risk is left, and so you must earn the riskless rate.

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1. Spelled out in more detail in Section 2 of *The Perception of Time, Risk and Return During Periods of Speculation*, Quantitative Finance Vol 2 (2002) 282–296, or http://emanuelderman.com/perception-time/

[Finance For Future Generations]

- Feynman: One sentence about physics to guide future civilizations

- One sentence about finance to guide future civilizations:

  .

  If

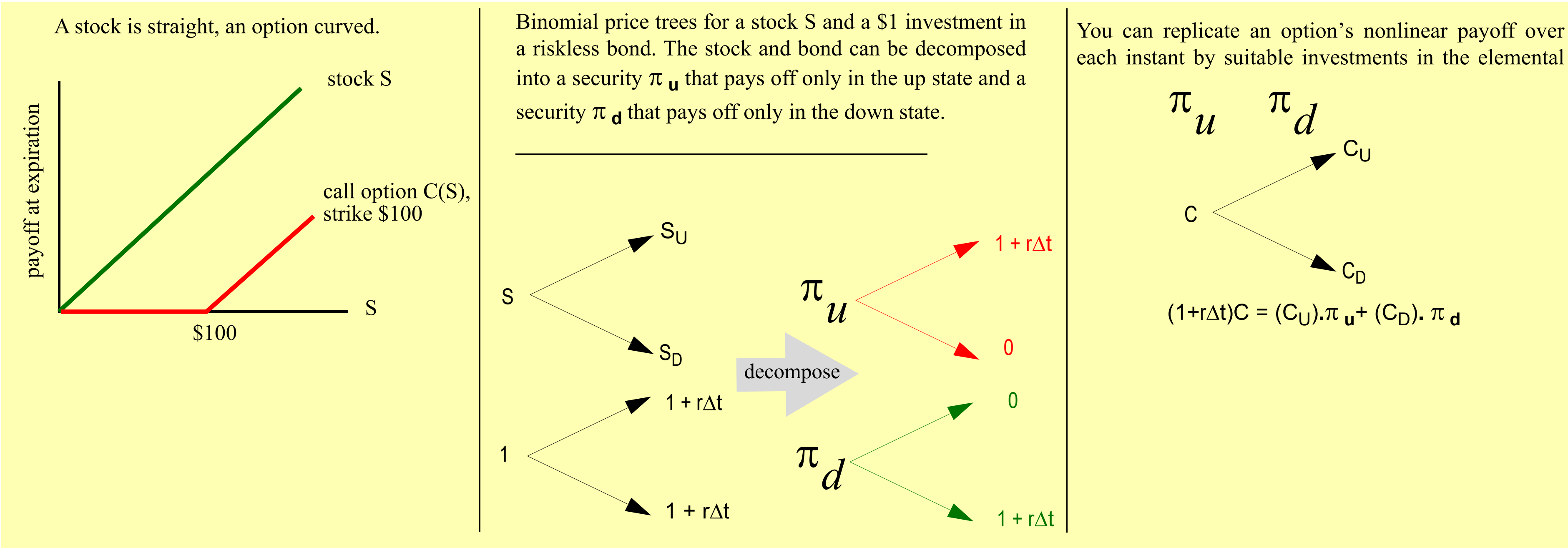
  you can hedge away all correlated risk, and
  then diversify over all uncorrelated risk,

  you should expect to earn the riskless return.

This is a sensible principle.

The difficulty is that correlation and diversification can't really be carried out because risk is not really purely statistical. It can't be specified for all time by a stochastic pde or Monte Carlo.

2.13 Binomial Derivative Valuation by Replication



A derivative is a contract whose payoff depends on the price of a "simpler" *underlier*. The most relevant characteristic is the *curvature* of its payoff *C(S)*, as illustrated for a simple call option. **What is the financial value of owning curvature?**

You can use linear algebra to decompose the stock and bond into a basis of two more elemental securities π_u and π_d, each respectively paying $(1+r\Delta t)$ in only one of the final states.

Then you can replicate the payoff of any non-linear function *C(S)* over the next instant of time Δt, no matter into which state the stock evolves. Note that the portfolio consisting of both π_u and π_d is riskless and is therefore worth $1. This is a homework problem.

The value of the option is the price of the mixture of stock and bond that replicates it. The coefficients depend on the difference between the up-return and the down-return at each node, that is, on the stock's volatility σ.

The choice-of-currency/numeraire trick in modeling

You can use any currency to value a security if markets are efficient.

A convenient choice of currency can greatly simplify thinking about a problem, and sometimes reduce its dimensionality.

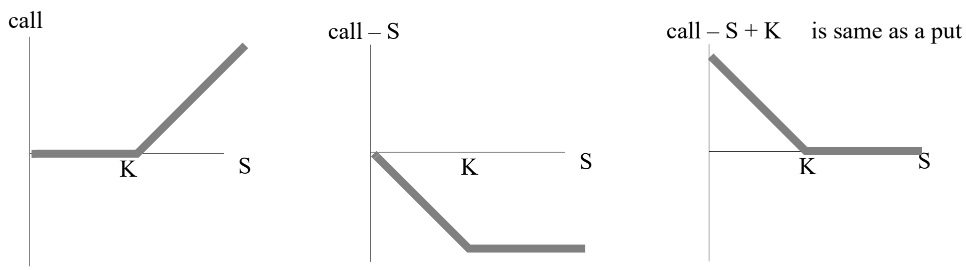
Convertible bonds, for example, which involve an option to exchange a bond for stock, can sometimes be fruitfully modeled by choosing a bond itself as the natural valuation currency.

Methods of Replication

2.14 Exact Static Replication for European Payoffs With Valuation Independent of Smile

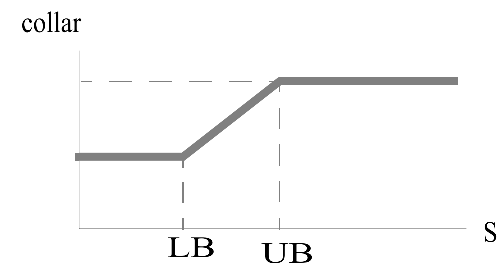
If you can create a static replicating portfolio for your payoff, and you know the prices of the ingredients in the static replication, **you have very little model risk.**

European put from a call: Put-Call Parity



Thus price of put = price of call - price of stock + PV(K).

A collar is a very popular instrument for portfolio managers who have made some gains during the year and now want to make sure they keep some upside but don't lose too much downside.



You can write the payoff as

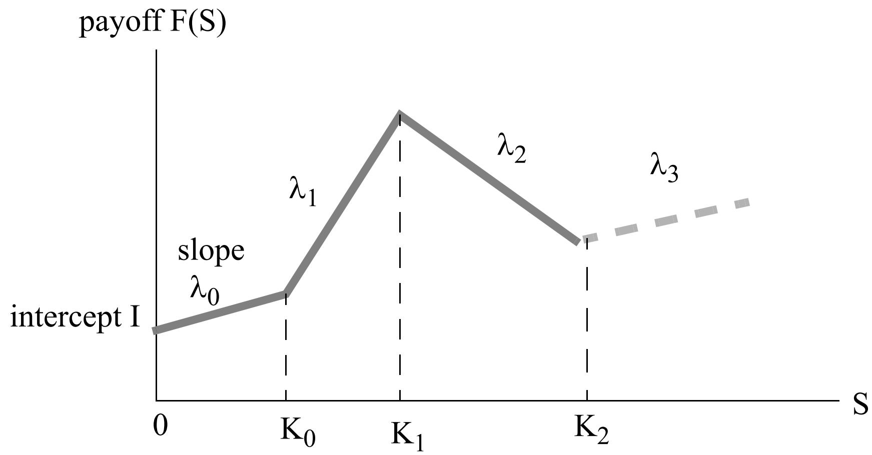
$$LB + call(S, LB) - call(S, UB)$$

Using put-call parity: $S + put(S, LB) - call(S, UB)$.

Its popularity forces dealers to be short puts and long calls.

Generalized European payoffs:

Piecewise-linear function of the terminal stock price S



Replicating portfolio, starting from the left, consists of a zero-coupon bond ZCB(I) plus a series of calls $C(K_i)$:

$$ZCB(I) + \lambda_0 S + (\lambda_1 - \lambda_0)C(K_0) + (\lambda_2 - \lambda_1)C(K_1) + \ldots$$

whose value can be determined from market prices.

(You can start from the right using puts rather than from the left using calls.)

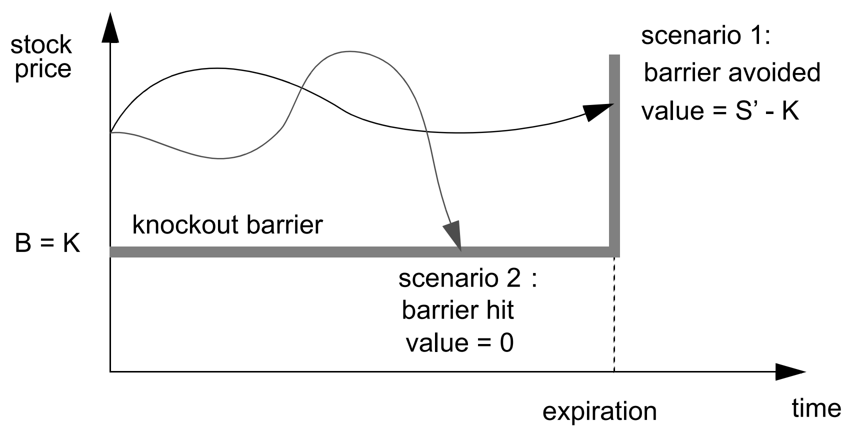
2.15 Approximate Static Replication for a Down-and-Out Call with Strike = Barrier, Only Approximately Smile-Independent

This option has a time-dependent boundary. It is worth zero if the stock ever touches B.

Stock price S and dividend yield d, strike K and out-barrier $B = K$.

Scenario 1 in which the barrier is avoided and the option finishes in-the-money.

Scenario 2 in which the barrier is hit before expiration and the option expires worthless.



In scenario 1 the call pays out $S' - K$, the payoff of a forward contract with delivery price K worth

$$F = Se^{-dt} - Ke^{-rt}$$

For paths in scenario 2, the down-and-out call immediately expires with zero value. In that case, the above forward F that replicates the barrier-avoiding scenarios of type 1 is worth $Ke^{-dt'} - Ke^{-rt'}$. This is close to zero. When the stock hits the barrier you must sell the forward to end the trade.

Dynamic Replication

2.16 Quick Derivation of the Black-Scholes PDE

Assume GBM with zero rates for simplicity.

In time Δt, $\Delta S \approx \sigma S\sqrt{\Delta t}$.

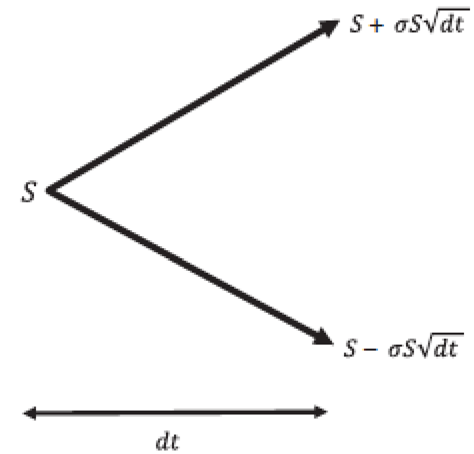


FIGURE 3.5 Binomial Model of Underlying Stock Price, $\mu = 0$

The stock S is a primitive, linear underlying security that provides a linear position in ΔS.

If you are long an option, you get a positive payoff whether the stock goes up or down! The call has curvature, or convexity.

$$\Gamma = \frac{\partial^2 C}{\partial S^2} \neq 0$$

What is the fair price for $C(S,t,K,T)$?

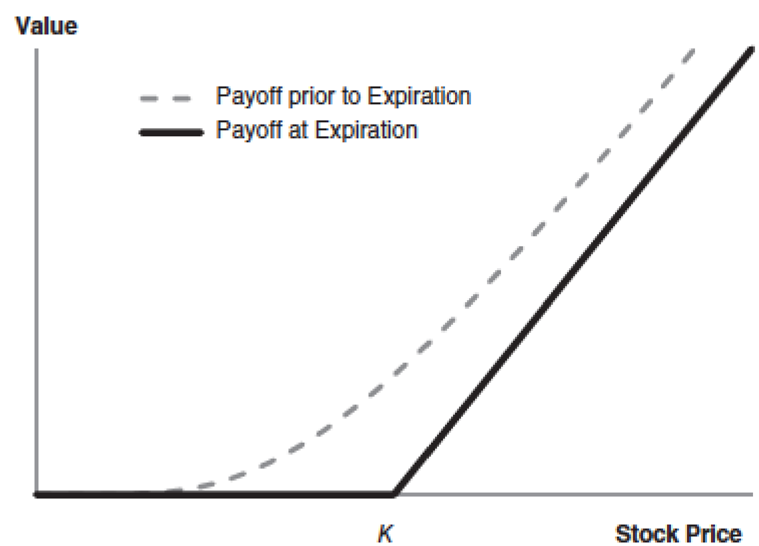


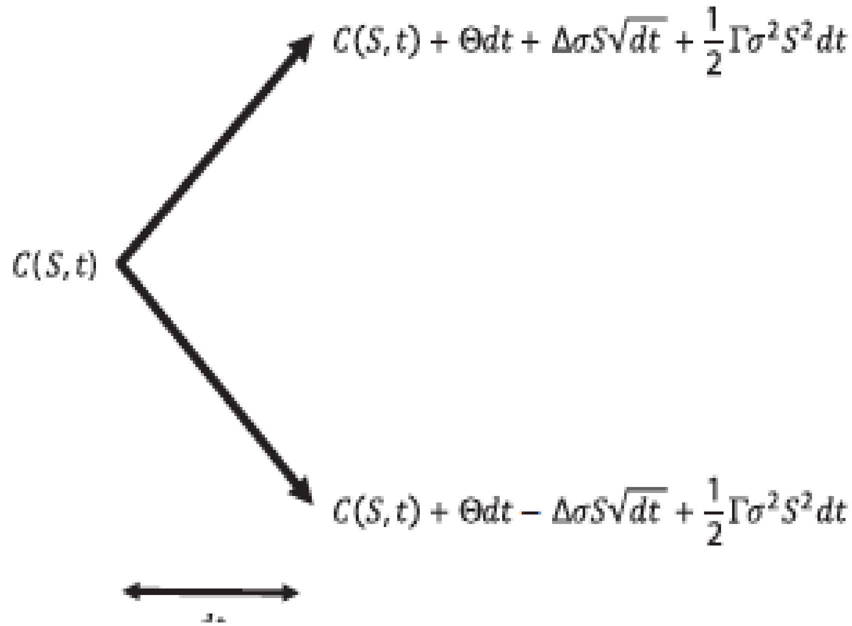
FIGURE 3.6 The Payoff of a Vanilla Call Option at Expiration

We can do a Taylor series expansion on the unknown price $C(\ )$ and examine how its value changes as time Δt passes and the stock moves by an amount ΔS:

$$C(S+\Delta S, t+\Delta t) = C(S, t) + \left.\frac{\partial C}{\partial t}\right|_{S,\,t}\Delta t + \left.\frac{\partial C}{\partial S}\right|_{S,\,t}\Delta S + \left.\frac{\partial^2 C}{\partial S^2}\right|_{S,\,t}\frac{(\Delta S)^2}{2} + \ldots$$

This is a quadratic function of ΔS. The linear term behaves like the stock price itself, the quadratic terms increases no matter what the sign of the move in S.

$$C(S+dS,\ t+dt) = C(S,\ t) + \Theta dt + \Delta dS + \frac{1}{2}\Gamma dS^2$$



If you hedge away the linear term in ΔS by shorting $\Delta = \dfrac{\partial C}{\partial S}$ shares the profit and loss of the hedged option position looks like this:

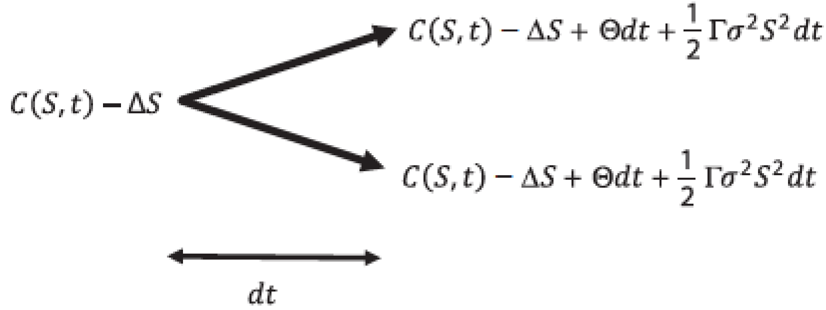


FIGURE 3.8 Delta-Hedged Call Option, $\mu = 0$

:

Positive convexity generates a profit or loss on the hedged position that is quadratic in (ΔS).

2.17 What Should You Pay for Convexity? Replication Tells Us

$$V = C(S, t) - \Delta S.$$

Positive convexity:

$$dV(S,t) = \Theta dt + \frac{1}{2}\Gamma\sigma^2 S^2 dt \quad = \Theta dt + \frac{1}{2}\Gamma dS^2$$

Suppose we think we know the future volatility of the stock, Σ.

Total change in value of the hedged position is $d\text{P\&L} = dV = \frac{1}{2}\Gamma(\Sigma^2 S^2 dt) + \Theta(dt)$

If we know Σ, the P&L is completely deterministic, irrespective of the direction of the move.

Therefore it replicates a riskless bond and must earn zero interest: $\Theta + \frac{1}{2}\Gamma S^2\Sigma^2 = 0$

This is the Black-Scholes equation for zero interest rates:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0 \qquad \text{time decay and curvature are linked}$$

$$C_{BS}(S, t, K, T, \Sigma) = SN(d_1) - KN(d_2)$$

$$d_{1,2} = \frac{\ln(S/K) \pm 0.5\Sigma^2(T-t)}{\Sigma\sqrt{T-t}}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$$

By differentiation,

$$\Delta_{BS} \equiv \frac{\partial C_{BS}}{\partial S} = N(d_1)$$

The option’s Δ tells you how many shares to short of the stock so as to remove the linear exposure of the option so you can trade its quadratic part.

When the riskless rate *r* is non-zero,

$$\frac{\partial C}{\partial t} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = r\left(C - \frac{\partial C}{\partial S}S\right)$$

profit per unit time = riskless interest on capital

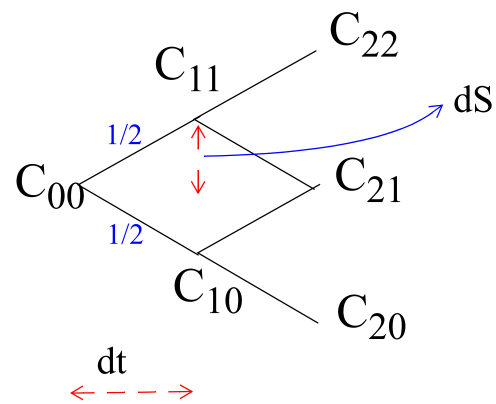
$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

$$C(S, K, \tau, \sigma, r) = SN(d_1) - Ke^{-r\tau}N(d_2)$$

$$d_{1,2} = \frac{\ln\left(\frac{S}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}y^2}\,dy$$

2.18 What Does This PDE Mean Discretely?



$$\frac{\partial C}{\partial t} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0 \qquad \text{say for } r = 0.$$

$$\frac{\partial C}{\partial t} = \frac{C_{21} - C_{00}}{2dt} \qquad \frac{\partial C}{\partial S} = \frac{C_{11} - C_{10}}{2dS}$$

$$\frac{\partial^2 C}{\partial S^2} = \left[\frac{\frac{C_{22} - C_{21}}{2dS} - \frac{C_{21} - C_{20}}{2dS}}{2dS}\right] = \frac{C_{22} - 2C_{21} + C_{20}}{4(dS)^2}$$

$$\text{pde} = \frac{C_{21} - C_{00}}{2dt} + \frac{1}{2}(\Sigma^2 S^2)\frac{C_{22} - 2C_{21} + C_{20}}{4(dS)^2} = 0$$

$$C_{00} = \frac{(\Sigma^2 S^2 dt)}{4(dS)^2}[C_{22} - 2C_{21} + C_{20}] + C_{21} \qquad \text{and suppose GBM} \quad (dS)^2 = \Sigma^2 S^2 dt$$

$$C_{00} = \frac{[C_{22} + 2C_{21} + C_{20}]}{4}$$

The current option price is the average of its future possible values; it's a martingale!