Economics 361
Problem Set #7 (Suggested Solutions)

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Question 1: Returns from Education

Consider the log wage regressions introduced on pages 1495 and 1496. On the top p.1496, the authors write that " ... and ϵ_i is an idiosyncratic error term that is uncorrelated with the other variables on the right hand side of (2)." This is the authors' way of saying that they assume

$$E[lnW_i|SAT_{j*}, X_{1i}, X_{2i}] \;\; = \;\; \beta_0 + \beta_1 SAT_{j*} + \beta_2 X_{1i} + \beta_3 X_{2i}$$

(a) Use the above and the Law of Iterated Expectations to derive $E[lnW_i|SAT_{j*}, X_{1i}]$. Your answer should not be identical to the one proposed by the authors near the bottom of p.1496

ANS: $E[lnW_i|SAT_{j*}, X_{1i}] \; = \; \beta_0 + \beta_1 SAT_{j*} + \beta_2 X_{1i} + \beta_3 E[X_{2i}|SAT_{j*}, X_{1i}]$

On p.1496, the authors argue that

$$E[lnW_i|SAT_{j*}, X_{1i}] \;\; = \;\; \beta_0 + \beta_1 SAT_{j*} + \beta_2 X_{1i} + E[u_i|X_{1i}, \gamma_1 X_{1i} + \gamma_2 X_{2i} + e_{ij*} > C_{j*}]$$

But applying conditional expectation to both sides of equation (3) yields

$$E[lnW_i|SAT_{j*}, X_{1i}] \;\; = \;\; \beta_0 + \beta_1 SAT_{j*} + \beta_2 X_{1i} + E[u_i|SAT_{j*}, X_{1i}]$$

(b) The authors equate $E[u_i|SAT_{j*}, X_{1i}]$ with $E[u_i|X_{1i}, \gamma_1 X_{1i} + \gamma_2 X_{2i} + e_{ij*} > C_j]$. What must be true about the statistical relationship among $\{X_{1i}, X_{2i}, SAT_{j*}, e_{ij}, \epsilon_i\}$ in order for the authors to make this claim? Briefly explain.

ANS: The authors equate conditioning on SAT_{j*} with conditioning on $\gamma_1 X_{1i} + \gamma_2 X_{2i} + e_{ij*} > C_j$. The latter conditioning represents what may be inferred about the "latent quality" of student i (Z_{ij}) from her acceptance by college $j*$. This indicates that the authors believe that SAT_{j*} is statistically informative about $\{X_{1i}, X_{2i}, e_{ij}, \epsilon_i\}$ only through the latent quality variable Z_{ij}, after

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conditioning on X_{1i} (as would be the case when SAT_{j*} is mainly proxying for C_j)

(c) Near the bottom of p.1496, the authors assert that "the coefficient on school-average SAT score is biased upward in this situation." The bias is due to "omitted variables," as discussed in lecture. But why do the authors claim that the bias is **upward**? What, if anything, are the authors assuming about the values of $\{\gamma_1, \gamma_2, \beta_0, \beta_1, \beta_2, \beta_3\}$?

ANS: Combining information from **(a)** and **(b)**

$$E[u_i|X_{1i}, \gamma_1 X_{1i} + \gamma_2 X_{2i} + e_{ij*} > C_j] \quad = \quad \beta_3 E[X_{2i}|SAT_{j*}, X_{1i}]$$

Therefore, $E[u_i|X_{1i}, \gamma_1 X_{1i} + \gamma_2 X_{2i} + e_{ij*} > C_j]$ is higher for "students who were admitted to, and therefore more likely to attend, more selective schools" if one of the following scenario holds

- $\beta_3 > 0$ and $E[X_{2i}|SAT_{j*}, X_{1i}]$ is an increasing function of SAT_{j*}
- $\beta_3 < 0$ and $E[X_{2i}|SAT_{j*}, X_{1i}]$ is a decreasing function of SAT_{j*}

From equation (1) concerning the latent quality Z_{ij} (p.1495), we can see that $E[X_{2i}|SAT_{j*}, X_{1i}]$ is an increasing (decreasing) function of SAT_{j*} when $\gamma_2 > 0$ ($\gamma_2 < 0$). So the authors seem to be assuming that β_3 and γ_2 are both positive or both negative ... although the former is more intuitive than the latter (more able workers are more likely to earn more, more likely to be accepted by selective colleges).

Suppose we observed $E[u_i|SAT_{j*}, X_{1i}]$ up to some unknown scalar multiple. In other words, suppose we observed $X_{3i} = \alpha\, E[u_i|SAT_{j*}, X_{1i}]$ for each observation i. α is the unknown scalar multiple.

(d) Derive $E[lnW_i|SAT_{j*}, X_{1i}, X_{3i}]$

ANS: Note that $E[u_i|SAT_{j*}, X_{1i}] = \frac{1}{\alpha} X_{3i}$

$$E[lnW_i|SAT_{j*}, X_{1i}, X_{3i}] \quad = \quad \beta_0 + \beta_1 SAT_{j*} + \beta_2 X_{1i} + \frac{1}{\alpha} X_{3i}$$

(e) Briefly explain how one could use this additional piece of data, $\{X_{3i}\}_{i=1}^{N}$, to obtain unbiased estimates of $(\beta_0, \beta_1, \beta_2)$

ANS: Simply regress lnW against (constant, SAT, X_1, X_3). As shown in **(d)**, the relevant linearity condition is satisfied by this regression.

(f) The authors propose to mitigate the omitted variables bias by including "an unrestricted set of dummy variables indicating groups of students who received the same admissions decisions (i.e., the same combination of acceptances and rejections) from the same set of colleges" to wage equation (3). Evaluate this proposal.

ANS: Basically, the authors are using the dummy variables indicating groups of students with the same admissions decisions as a form of fixed effects for the omitted variables X_2. The authors are arguing that students with the same admissions decisions have similar values of $E[u_i|SAT_{j*}, X_{1i}]$. As such, including a dummy variable for each set of students (by admissions decisions) may help mitigate the bias from omitting X_2. But this hinges crucially on the extent to which students with the same admissions decisions have similar values of $E[u_i|SAT_{j*}, X_{1i}]$ ...

Question 2: Measurement Error and Permanent Income Model

(a) Explain briefly why $\begin{pmatrix} x \\ y \end{pmatrix}$ is distributed bivariate normal

ANS: x and y are both linear functions of (z, u, v), which are jointly distributed Normal. As shown in Goldberger Chapter 18 (Proposition 5), linear functions of the same jointly distributed Normal random variables are themselves jointly distributed Normal.

(b) Explain why $E(y \mid x) = \alpha^* + \beta^* x$ where

$$\beta^* = \frac{\text{Cov}(x,y)}{\text{Var}(x)} \qquad \alpha^* = E(Y) - \beta^* E(X)$$

HINT: Goldberger Chapter 18.2

ANS: As shown in Goldberger Chapter 18 (Proposition 3), the conditional distribution $y|x$ for (y, x) jointly Normal is Normal with a conditional mean $E[y|x] = \alpha^* + \beta^* x$. **(a)** established that (y, x) are jointly Normal

(c) Show *explicitly* that $\text{Cov}(x, y) = \beta\sigma_z^2$ and $\text{Var}(x) = \sigma_z^2 + \sigma_u^2$

ANS:

$$\begin{aligned}
\text{Cov}(x,y) &= E[xy] - E[x]\ E[y] \quad = \quad E[(\alpha+\beta z+v)(z+u)] - E[\alpha+\beta z+v]\ E[z+u] \\
&= E[\alpha z + \alpha u + \beta z^2 + \beta z u + zv + uv] - (\alpha+\beta\mu)\ (\mu) \quad = \quad \beta(E[z^2] - (\mu)^2) \quad = \quad \beta\sigma_z^2 \\
\text{Var}(x) &= \text{Var}(z+u) \quad = \quad \text{Var}(z) + \text{Var}(u) + 2\text{Cov}(z,u) \quad = \quad \sigma_z^2 + \sigma_u^2
\end{aligned}$$

(d) Goldberger claims that using OLS to regress y on x under the above formulation and sample leads to **biased** estimates of (α, β) but **unbiased** estimates of (α^*, β^*). Explain why this is true (do not just repeat Goldberger verbatim – use arguments we have discussed (repeatedly) in class).

ANS: What we want are the estimates of the parameters defining $E[y|z]$ but, as we do not observe z, the best we can do is estimate the parameters of $E[y|x]$. As the parameters differ, regressing y on x will not, in general, provide unbiased estimates of the latter – only unbiased estimates of the former. (α^*, β^*) are the parameters of $E[y|x]$ and (α, β) the parameters of $E[y|z]$.

(e) Goldberger further claims that this “bias” does not depend on (x, y) being distributed bivariate Normal. Explain. (Here, you can use Goldberger’s words – but make sure you understand them!)

ANS: See the last paragraph of p.339

(f) The result in **(d)** is sometimes referred to as the “attenuation bias due to measurement error.” Explain why. **HINT:** See β and β^*

ANS: Note that

$$E[b^{ols}|X] \quad = \quad \beta^* \quad = \quad \beta \underbrace{\left(\frac{\sigma_z^2}{\sigma_z^2 + \sigma_u^2}\right)}_{\theta}$$

But $0 < \theta < 1$ as $\sigma_z^2, \sigma_u^2 > 0$. So $E[b^{ols}|X]$ will be less than β in magnitude – hence “attenuation bias”

(g) Suppose you know the true values of $(\sigma_z^2, \sigma_u^2, \sigma_v^2)$ but not μ. Can you construct unbiased estimates of α and β? If so, state the estimate. If not, explain why not. How does your answer change if you know μ?

HINT: If you know $(\sigma_z^2, \sigma_u^2, \sigma_v^2)$, you also know $\theta = \frac{\sigma_z^2}{\sigma_z^2+\sigma_u^2}$. Also, you will need to use the OLS coefficients from regressing y on x. See **(d)**.

ANS: As you know θ, you can calculate the following $\frac{1}{\theta}b^{ols}$ where b^{ols} is the OLS estimate of β^* from regressing y on x

$$E[\frac{1}{\theta}b^{ols}|x] \quad = \quad \frac{1}{\theta}\beta^* \quad = \quad \frac{1}{\theta}\beta\theta \quad = \quad \beta$$

To estimate α, simply use $\bar{Y} - \frac{1}{\theta}b^{ols}\bar{X}$ where $\bar{Y}$ and $\bar{X}$ are the sample means. If μ were known, you could replace $\bar{X}$ with μ

(h) Suppose it is known that consumption is proportional to permanent income, in the sense that $\alpha = 0$. How does this alter your answers in **(g)**?

ANS: Other than eliminating the need to estimate α, it doesn’t. None of the analysis concerning β is affected by $\alpha = 0$.

(i) Suppose that $\alpha \neq 0$. Additionally, $E[u] = \gamma \neq 0$. So

$$\begin{pmatrix} z \\ u \\ v \end{pmatrix} \sim N\left[\begin{pmatrix} \mu \\ \gamma \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_z^2 & 0 & 0 \\ 0 & \sigma_u^2 & 0 \\ 0 & 0 & \sigma_v^2 \end{pmatrix} \right]$$

How does this unknown γ alter your answers in **(g)**?

ANS: It doesn't affect the estimate of β. It does, however, affect the "bias" of the α estimate, as $E[x]$ no longer equals $E[z]$.

(j) The above difficulties stem from the desire to estimate (α, β) rather than (α^*, β^*). For which types of statistical inference may an estimate of (α^*, β^*) suffice? For which do you need an estimate of (α, β)?

ANS: See pp.344-346, starting with "Why is knowledge of α^* and β^* in the CEF $E[y|x) = \alpha^* + \beta^* x$ not adequate?"

Question 3: Simultaneity and the Keynesian Model

(a) Under what *economic* condition do you indirectly observe z as well?

ANS: if the national income identity (equilibrium condition) holds, as then $z = x - y$ and we observe both x and y.

(b) Show *explicitly* how the *reduced form* equations

$$\begin{aligned} y &= (\alpha + \beta z + u)/(1 - \beta) \\ x &= (\alpha + z + u)/(1 - \beta) \end{aligned}$$

can be derived from the *structural* equations given above (the consumption and national income identity equations).

ANS: Substitute the 2 Keynesian equations into each other.

(c) Briefly explain why **(b)** implies that (y, x) is joint distributed multivariate normal. **HINT:** Goldberger Chapter 18.2

ANS: See **Q2(a)**

(d) Show *explicitly* the following results concerning the moments of x and y

$$\begin{aligned} \text{Cov}(x,y) &\equiv \sigma_{xy} = \frac{\beta\sigma_z^2+\sigma_u^2}{(1-\beta)^2} & \text{Var}(x) &\equiv \sigma_x^2 = \frac{\sigma_z^2+\sigma_u^2}{(1-\beta)^2} \\ E[y] &\equiv \mu_y = \frac{\alpha+\beta\mu}{1-\beta} & E[x] &\equiv \mu_x = \frac{\alpha+\mu}{1-\beta} \end{aligned}$$

ANS: Use the Reduced Form Equations ...

$$\begin{aligned} \text{Cov}(x,y) &= E[\left(\frac{\alpha+z+u}{1-\beta}\right)\left(\frac{\alpha+\beta z+u}{1-\beta}\right)] - E[\frac{\alpha+z+u}{1-\beta}]\ E[\frac{\alpha+\beta Z+u}{1-\beta}] \\ &= \frac{\beta^2E[z^2]-\beta(E[z])^2+E[u^2]-(E[u])^2}{(1-\beta)^2} = \frac{\beta\sigma_z^2+\sigma_u^2}{(1-\beta)^2} \\ \text{Var}(z) &= \text{Var}(\frac{\alpha+z+u}{1-\beta}) = \frac{\text{Var}(z)+\text{Var}(u)+2\text{Cov}(z,u)}{(1-\beta)^2} = \frac{\sigma_z^2+\sigma_u^2}{(1-\beta)^2} \\ E[y] &= E[\frac{\alpha+\beta z+u}{1-\beta}] = \frac{\alpha+\beta\mu}{1-\beta} \\ E[x] &= E[\frac{\alpha+z+u}{1-\beta}] = \frac{\alpha+\mu}{1-\beta} \end{aligned}$$

(e) Show *explicitly* that the bias from "estimating" β using OLS on (y,x) is

$$E[b^{ols}-\beta \mid x] = (1-\theta)(1-\beta)$$

where $\theta = \sigma_z^2/(\sigma_z^2+\sigma_u^2)$.

HINT: See **(3b)** and **(3d)**

ANS:

$$\begin{aligned} E[b^{ols}-\beta \mid x] &= E[b^{ols}|x]-\beta = \beta^*-\beta = \frac{\text{Cov}(x,y)}{\text{Var}(x)}-\beta = \left(\frac{\frac{\beta\sigma_z^2+\sigma_u^2}{(1-\beta)^2}}{\frac{\sigma_z^2+\sigma_u^2}{(1-\beta)^2}}\right)-\beta \\ &= \frac{\beta\sigma_z^2+\sigma_u^2}{\sigma_z^2+\sigma_u^2}-\beta = \frac{(1-\beta)\sigma_u^2}{\sigma_z^2+\sigma_u^2} = (1-\beta)(1-\theta) \end{aligned}$$

(f) How does the bias resulting from "estimating" β via OLS on (y, x) change with [1] an increase in the variance of z [2] an increase in the variance of u? Which variance do you prefer to be bigger, σ_z^2 or σ_u^2?

ANS: Just take the appropriate derivatives:

$$\begin{aligned}
\frac{\partial E[b^{ols} - \beta \mid x]}{\partial \sigma_z^2} &= -(1-\beta)\frac{\partial \theta}{\partial \sigma_z^2} = -(1-\beta)\left(\frac{1}{\sigma_z^2 + \sigma_u^2}\right)\left(1 - \frac{\sigma_z^2}{\sigma_z^2 + \sigma_u^2}\right) < 0 \\
\frac{\partial E[b^{ols} - \beta \mid x]}{\partial \sigma_u^2} &= -(1-\beta)\frac{\partial \theta}{\partial \sigma_u^2} = -(1-\beta)\left(\frac{1}{\sigma_z^2 + \sigma_u^2}\right)\left(\frac{-1}{(\sigma_z^2 + \sigma_u^2)^2}\right) > 0
\end{aligned}$$

We want σ_z^2 to be bigger as that reduces the bias.

(g) Derive the variance of y in terms of $\{\alpha, \beta, \sigma_z^2, \sigma_u^2, \sigma_{xy}\}$

HINT: See **(b)**

ANS: Use the Reduced Form Equations

$$\text{Var}(y) = \text{Var}(\frac{\alpha + \beta z + u}{1 - \beta}) = \frac{\beta^2 \sigma_z^2 + \sigma_u^2}{(1-\beta)^2}$$

(h) Use **(f)** and **(g)** to evaluate the following statement

> "The simultaneity bias of the naïve OLS estimate of β falls as the variation in z accounts for a larger share of the variation in y. In other words, the simultaneity bias is mitigated to the extent that private investment shocks drive consumption fluctuation more than consumption shocks."

ANS: From **(g)**, we see that there are two ways in which the variation in z can account for a larger share of the variation in y: (1) σ_z^2 can increase (ceteris paribus) and (2) β can increase (ceteris paribus). We know from **(f)** that an increase in σ_z^2 decreases the bias. We know from

$$\frac{\partial}{\partial \beta}(1-\beta)(1-\theta) = -(1-\theta) < 0$$

that an increase in β decreases the bias as well.

This makes intuitive sense as the result simply states that as more of the variation in y is driven by what we observe (z) than what we do not observe (u), bias falls. Remember, simultaneity bias is a "variant" of omitted variables bias (i.e. the unobserved shock u).

Question 4: Supply & Demand

(a) Given the assumptions above, do the **structural** equations individually satisfy the classical OLS assumptions? Given the assumptions above, do the **reduced form** equations satisfy the classical OLS assumptions? Explain *briefly.*

ANS: The structural equations do not satisfy the CRM assumptions. Specifically, they violate the second assumption as $E(P|Q,C) \neq \alpha_0 + \alpha_1 Q_t + \delta C_t$ and $E(Q|P,I) \neq \beta_0 + \beta_1 P_t + \gamma I_t$. This is because we are give market **equilibrium** data: $Q_t^s = Q_d^t = Q^t$. More details can be found on first page of the "Simultaneous Equations Model: Supply & Demand" (SEM) Handout.

In contrast, the reduced form equations, individually, *will* satisfy the Gauss-Markov assumptions. The first assumption can be verified by just checking the data to see that the "Q" matrix formed from the constant, income, and cost is invertible. The second assumption holds as you can show that

$$\begin{aligned} E(\epsilon_{1t}|C,I) &= E(\ \frac{\alpha_1 \nu_t + \eta_t}{1 - \alpha_1 \beta_1}\ |C,I) = 0 \\ E(\epsilon_{2t}|C,I) &= E(\ \frac{\beta_1 \eta_t + \nu_t}{1 - \alpha_1 \beta_1}\ |C,I) = 0 \end{aligned}$$

Zero conditional mean of the reduced form "error term" implies that $E(P|C,I) = \pi_{11} + \pi_{12} I + \pi_{13} C$ and $E(Q|C,I) = \pi_{21} + \pi_{22} I + \pi_{23} C$. Similarly, the third assumption holds as you can show that

$$\begin{aligned} V(\epsilon_{1t}|C,I) &= \frac{1}{(1 - \alpha_1 \beta_1)^2} [\ \alpha_1^2 \sigma_\nu^2 + \sigma_\eta^2\ ] \\ V(\epsilon_{2t}|C,I) &= \frac{1}{(1 - \alpha_1 \beta_1)^2} [\ \beta_1^2 \sigma_\eta^2 + \sigma_\nu^2\ ] \end{aligned}$$

Note: η_t, ν_t are independent of each other - hence, their covariance is zero. The above shows that homoskedasticity is satisfied. To show that there are no autocorrelation, note that in addition to the fact that η, ν are independent of each other, each η_t and ν_t are $i.i.d.$ draws. So η_s, η_t for $s \neq t$ are uncorrelated – meaning that $\epsilon_{1s}, \epsilon_{1t}$ are uncorrelated (same deal for $\epsilon_{2s}, \epsilon_{2t}$). For more details, see the SEM Handout.

(b) Use OLS to calculate unbiased estimates of the reduced form parameters:
$(\Pi_{11}, \Pi_{12}, \Pi_{13}, \Pi_{21}, \Pi_{22}, \Pi_{23})$

ANS: Regress P on a constant, I, C and regress Q on a constant, I, C.

Variable	Coef.	Std. Err.
Reduced Form Equation for Price		
Constant (Π_{11})	.7501654	.2076297
Cost (Π_{13})	.6544147	.1011581
Income (Π_{12})	.2849983	.0335826
Reduced Form Equation for Quantity		
Constant (Π_{21})	.2441900	.1977304
Cost (Π_{23})	-.1473084	.0963351
Income (Π_{22})	.5793507	.1977304

(c) Derive Var($\epsilon_{1t} \mid I_t, C_t$),Var($\epsilon_{2t} \mid I_t, C_t$), and Cov($\epsilon_{1t}, \epsilon_{2t} \mid I_t, C_t$) in terms of the structural parameters $(\alpha_0, \alpha_1, \delta, \beta_0, \beta_1, \gamma)$, σ_η^2, and σ_ν^2. Note: $(\epsilon_{1t}, \epsilon_{2t})$ are the implied reduced form disturbance/error terms. (See lecture notes).

ANS: For the variances, see **(2a)**. Covariance is given below:

$$\text{Cov}(\epsilon_{1t}, \epsilon_{2t}|I, C) \quad = \quad \frac{1}{(1-\alpha_1\beta_1)^2}[\ \alpha_1\sigma_\nu^2 + \beta_1\sigma_\eta^2\ ]$$

(d) In light of the answer in **(c)**, propose a way of estimating the reduced form parameters that might achieve lower MSE than the method used in **(b)**. **HINT:** Think SUR Model

ANS: The above covariance result implies that there exists **cross-equation correlation** between the two reduced form equations! Note that the reduced form equations are equations concerning the relationship between the market *equilibrium* price and quantity. Hence, given that market equilibrium is determined by "supply = demand," it should come as no surprise that the two reduced form "error terms" should be correlated.

So, the reduced form equations individually satisfy the Gauss-Markov assumptions **and** has cross-equation correlation – SUR Model. Stack the equation as follows (using the same definition of X as above – for Σ I use the Kronecker product (Goldberger, p.325) for notational simplicity):

$$\underbrace{\begin{pmatrix} P \\ Q \end{pmatrix}}_{Y} = \underbrace{\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}}_{X_{stack}} \begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} V(\epsilon_{1t}|C, I) & \text{Cov}(\epsilon_{1t}, \epsilon_{2t}|C, I) \\ \text{Cov}(\epsilon_{1t}, \epsilon_{2t}|C, I) & V(\epsilon_{2t}|C, I) \end{pmatrix} \otimes I_{(N\times N)}$$

Apply GLS to this stacked regression equation

$$\begin{pmatrix} \hat{\Pi}_1 \\ \hat{\Pi}_2 \end{pmatrix} = (X'_{stack}\ \Sigma^{-1}\ X_{stack})^{-1}\ X'_{stack}\ \Sigma^{-1}\ Y$$

Note that this requires you to *know* Σ or at least Σ up to some proportionality (Ω). Moreover, knowing σ_η^2 and σ_ν^2 are insufficient: see **(c)**.

So, actually, this is an example where *feasible* GLS would work but not GLS (as you do not know enough about Σ). While you may not know Σ, you can calculate a consistent estimate of Σ, $\hat{\Sigma}$, as follows (OK, this part is not necessary to get full credit for this question):

1. To estimate $V(\epsilon_{1t}|C, I)$, run OLS on the first reduced form equation and get the OLS residuals e_1. Calculate the estimate of the variance in the usual manner: $s_1^2 = \frac{e_1' e_1}{N-k}$.

2. Use the analogous procedure to get e_2 and $s_2^2 = \frac{e_2' e_2}{N-k}$. Note that $k = 3$ as we have three variables (constant, C, I)

3. To estimate the $\text{Cov}(\epsilon_{1t}, \epsilon_{2t}|C, I)$, find the sample covariance between the two series of OLS residuals from previous steps: $s_{12}^2 = \frac{e_1' e_2}{N-k}$

4. So $\hat{\Sigma} = \begin{pmatrix} s_1^2 & s_{12}^2 \\ s_{12}^2 & s_2^2 \end{pmatrix} \otimes I_{N \times N}$

$\hat{\Sigma}$ is a consistent estimate of Σ following much of the same intuition as what I showed you for White's Heteroskedasticity Consistent Estimator (of the OLS variance).

Keep in mind that while GLS is (weakly) better than OLS under non-spherical errors, we do not know if FGLS is better than OLS as FGLS has mostly known *asymptotic* properties. Aitken's Theorem holds for the GLS estimator, not the FGLS estimator.

OPTIONAL: Estimate the reduced form parameters using the method proposed in (d). Good practice to see if you understand GLS and SUR.

(e) Use your estimates from **(b)** to calculate consistent estimates of the structural parameters, via **indirect least squares** (ILS).

ANS: Just combine your answer from (b) with the algebra on the SEM Handout.
For example, $\hat{\alpha}_1 = \frac{\hat{\Pi}_{12}}{\hat{\Pi}_{22}}$ and $\hat{\beta}_1 = \frac{\hat{\Pi}_{23}}{\hat{\Pi}_{13}}$

(f) Suppose you ran OLS on the structural equations, ignoring the simultaneity "bias" problem. Although these OLS "estimates" are neither unbiased nor consistent for the structural parameters, these "estimates" are consistent for (BLANK) ?

HINT: Recall the Keynesian Model and (α, β) versus (α^*, β^*)

ANS: They still (under fairly general conditions) provide consistent estimates of the coefficients for the best linear predictor (under MSE) of P given Q and C and similarly the best linear predictor (under MSE) of Q given P and I.

(g) Calculate consistent estimates of the structural parameters, via **two stage least squares** (2SLS) *without* using ivreg or ivregress. i.e. do it "old school" and run both regressions. Show your steps, clearly

ANS: Save the fitted values from previous reduced form equation $(\hat{P}, \hat{Q})$ and use them as regressors in the second stage regression

Variable	Coef.	Std. Err.
Supply		
Constant	.6300418	.2191001
$\hat{Q}$	.4919270	.0579659
Cost	.7268797	.1011415
Demand		
Constant	.4130518	.2700567
$\hat{P}$	-.2250994	.1472080
Income	.6435037	.0538648

(h) Now compare your answers in **(g)** with the results from running the following commands:

- ivregress 2sls p (q = i) c
- ivregress 2sls q (p = c) i

ANS: Results from using ivregress

Variable	Coef.	Std. Err.
Supply		
Constant	.6300418	.2168133
Q	.4919270	.0573609
Cost	.7268797	.1000859
Demand		
Constant	.4130518	.2929935
P	-.2250994	.1597109
Income	.6435037	.0594397

The coefficient estimates are the same, for all cases. But the standard errors reported in **(h)** differ from the standard errors in the second stage regression of **(g)** for reasons noted earlier (different estimate of σ^2 – recall the discussion about the asymptotic equivalency of using either $\frac{e'e}{N-k}$ or $\frac{e'e}{N}$ to estimate σ^2). The reported standard errors are estimates of the standard errors for the *asymptotic* distribution of the 2SLS estimates.

(i) Use your answer in **(h)** to test the following economic hypotheses:

- Cost Pass-through: Producers pass along the entire burden of cost changes to the consumers $(\delta = 1)$
- Demand for the first observation is unit price elastic $(\frac{\partial Q_1^d}{\partial P_1}\frac{P_1}{Q_1^d} = -1)$

Keep in mind that you are using a consistent estimate and asymptotic standard errors. So your hypothesis tests should be asymptotically valid tests: i.e. you should be looking up critical values in the standard Normal table (not t-table)

For the unit price elastic hypothesis, note the following

- You can use the Data Browser (under Top Menu, Data) to find the values of P_1 and Q_1^d. Note that $Q_1^d = Q_1^s = Q_1$.
- You should be able to tell me why $\frac{\partial Q_1^d}{\partial P_1} = \beta_1$ (Think exogenous variables and earlier discussion on partial derivatives and interpreting regression coefficients)

ANS: Start with Cost-Pass through.

- $H_0 : \delta = 1$
- Under H_0: $\frac{\hat{\delta}_{2SLS}-1}{\sqrt{Var(\hat{\delta}_{2SLS})}} \overset{a}{\sim} N(0,1)$
- Z-test statistic $= \frac{.7268797-1}{.1000859} = -2.72886$
- Critical Value (two-sided, $\alpha = 0.05$) = 1.96 (and -1.96)

  Could have used different α or even one-sided ... just be able to explain why
- Conclusion: reject H_0 as $-2.72886 < -1.96$

Now consider unit price elasticity.

- Using data browser, $P_1 = 3.569332$ and $Q_1^d = Q_1 = 4.851808$
- Own price elasticity for $i = 1$ is $\frac{\partial Q_1^d}{\partial P_1}\frac{P_1}{Q_1^d} = \beta_1 \cdot \frac{3.569332}{4.851808}$
- $H_0 : \beta_1 \cdot \frac{3.569332}{4.851808} = -1$
- Above can be re-written as $H_0 : \beta_1 = -\frac{4.851808}{3.569332} = -1.395304$
- Under H_0: $\frac{\hat{\beta}_{2SLS}-(-1.395304)}{\sqrt{Var(\hat{\beta}_{2SLS})}} \overset{a}{\sim} N(0,1)$
- Z-test statistic $= \frac{-0.2250994+1.395304}{0.1597109} = 7.101611$
- Critical Value (two-sided, $\alpha = 0.05$) = 1.96 (and -1.96)

  Could have used different α or even one-sided ... just be able to explain why
- Conclusion: reject H_0 as $7.101611 > 1.96$

  Note that at $\beta_1 = \hat{\beta}_1 = -0.2250994$, the own price elasticity is $-0.2250994 \cdot \frac{3.569332}{4.851808} > -1$

  So your estimates suggest that demand is inelastic