STAT GU4261/GR5261 - Statistical Methods in Finance - Homework #1 Solutions

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Question #1

Problem #4 The solution is written for you in the textbook.

Problems #5 and #6 We produce the following, splitting the code into cases where we are above, below or between the thresholds.

```
niter = 1e5
#initialize simulation result vectors
loss = rep(NA, niter)
profit.100k = rep(NA, niter)
set.seed(1234)
for (i in 1:niter){
  r = rnorm(45, mean = 0.05/253, sd = 0.23/sqrt(253))
  logPrice = log(1e6) + cumsum(r)
  #find min and max, and their indices
  minlogPrice = min(logPrice)
  maxlogPrice = max(logPrice)
  ind.min = which.min(logPrice)
  ind.max = which.max(logPrice)

  #4 cases to consider
  if (minlogPrice > log(950000) & maxlogPrice < log(1100000)){
    profit = exp(logPrice[45]) - 1000000
    if(profit < 0){
      loss[i] = 1
    } else{
      loss[i] = 0
    }
    profit.100k[i] = 0
  }
  if (minlogPrice > log(950000) & maxlogPrice >= log(1100000)){
    loss[i] = 0
    profit.100k[i] = 1
  }
```

```r
  if (minlogPrice <= log(950000) & maxlogPrice < log(1100000)){
    loss[i] = 1
    profit.100k[i] = 0
  }
  if (minlogPrice <= log(950000) & maxlogPrice >= log(1100000)){
    if (ind.min < ind.max){
      loss[i] = 1
      profit.100k[i] = 0
    } else {
      loss[i] = 0
      profit.100k[i] = 1
    }
  }
}

#problem 5 answer
mean(profit.100k)
```

```
## [1] 0.2876
```

```r
#problem 6 answer
mean(loss)
```

```
## [1] 0.56602
```

Question #2

Problem #7 Part (a):

Since the r_t are iid $N(0.06, 0.47)$,

$$r_t(4) \sim N(0.24, 4(0.47))$$

Part (b):

```r
pnorm(2, mean = 4*.06, sd = sqrt(4*.47), lower.tail = TRUE)
```

```
## [1] 0.9003611
```

Part (c):

$$\begin{aligned}
\text{Cov}(r_t(1), r_t(2)) &= \text{Cov}(r_t + r_{t-1}, r_t) \\
&= \text{Cov}(r_t, r_t) + \text{Cov}(r_t, r_{t-1}) \\
&= 0.47
\end{aligned}$$

Part (d):

$$r_t(3) = r_t + r_{t-1} + r_{t-2} \implies r_t(3)|r_{t-2} \sim N(0.6 + 2(0.06), 2(0.47))$$

Problem #8 Part (a):

$$\begin{aligned} P(X_2 > 1.3X_1) &= P(r_1 + r_2 > \log(1.3)) \\ &= P\left(Z > \frac{\log(1.3) - 2\mu}{\sqrt{2}\sigma}\right) \end{aligned}$$

where $Z \sim N(0,1)$

Part (b):

$$\begin{aligned} & F(x) \equiv P(X_1 \le x) = P(r_1 \le \log(x/X_0)) \\ \implies & \text{ density of } X_1 = F'(x) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2}\left(\frac{\log(x/X_1) - \mu}{\sigma}\right)^2\right) \frac{1}{x} \end{aligned}$$

Part (c):

Note that

$$\begin{aligned} P(X_k \le x) &= 0.9 \\ \iff P\left(\sum_{i=1}^{k} r_i \le \log(x/X_0)\right) &= 0.9 \\ \iff P\left(Z \le \frac{\log(x/X_0) - k\mu}{\sigma\sqrt{k}}\right) &= 0.9 \\ \iff \frac{\log(x/X_0) - k\mu}{\sigma\sqrt{k}} &= 1.281552 \end{aligned}$$

Rearrange to solve for x, which is the 0.9 quantile by definition.

Part (d):

$$E(X_k^2) = X_0^2 E(\exp(Y))$$

where $Y \sim N(2k\mu, 4k\sigma^2)$. Using the moment generating function for a Gaussian random variable we immediately see this is equal to

$$X_0^2 \exp\left(2k\mu + 2k\sigma^2\right)$$

Part (e):

Again using the mgf for a Gaussian random variable, we see that $E(X_k) = X_0 E(\exp(W))$ where $W \sim N(k\mu, k\sigma^2)$ so we have, using part (d),

$$\begin{aligned} \text{Var}(X_k) &= E(X_k^2) - (E(X_k))^2 \\ &= X_0^2 \exp(2k\mu + 2k\sigma^2) - X_0^2 \exp(2k\mu + k\sigma^2) \\ &= \exp(2k\mu + k\sigma^2)\Big(\exp(k\sigma^2) - 1\Big) \end{aligned}$$

Question # 3

We proceed by induction. By definition,

$$1 + R_t(1) = \frac{P_t + D_t}{P_{t-1}}$$

Suppose the formula holds for some $k \in \mathbb{N}$. Since the multiple gross return over two periods is equal to the product of the gross returns of each period, the multiple gross return of $k+1$ periods is given by

$$\begin{aligned} 1 + R_t(k+1) = (1 + R_{t-k}(1))(1 + R_t(k)) &= \frac{P_{t-k} + D_{t-k}}{P_{t-k-1}}(1 + R_t(k)) \\ &= \prod_{i=0}^{k} \frac{P_{t-i} + D_{t-i}}{P_{t-i-1}} \end{aligned}$$

Question #4

We compute $G(u) \equiv P(U \leq u)$, the cdf for U. As $U \in [0, 1]$ by the definition of F, $G(u) = 0$ for every $u < 0$ and $G(u) = 1$ for every $u > 1$. Noting that F being strictly increasing implies it has well defined inverse function, we have

$$\begin{aligned} G(u) &= P(X \leq F^{-1}(u)) \\ &= F(F^{-1}(u)) \\ &= u \end{aligned}$$

so that U is a Uniform$[0, 1]$ random variable by definition.

Question #5

Put $k = 1$ in question #2, problem #8.

Question #6

Part (a):
Note that

$$\begin{aligned} E(F_n(x)) &= \frac{1}{n}\sum_{i=1}^{n} E(I(X_i \leq x)) \\ &= \frac{1}{n}\sum_{i=1}^{n} P(X_i \leq x) \\ &= \frac{1}{n}\sum_{i=1}^{n} F(x) \\ &= F(x) \end{aligned}$$

Part (b):

$$
\begin{aligned}
Var(F_n(x)) &= \frac{1}{n^2}\sum_{i=1}^{n} Var(Z_i) \\
&= \frac{1}{n}^2 \sum_{i=1}^{n} F(x)(1 - F(x))
\end{aligned}
$$

Part (c):
$N(0,1)$. This follows immediately by applying the central limit theorem to the sequence of Z_is.