LECTURE 20

STOCHASTIC VOLATILITY

20.1 Revisit Weighted Variance Swaps: Corridor Variance Swaps

We want to create a contract L where hedging it will capture variance only between stock price A and infinity. The weighting function in the weighted variance swap is proptional to $\theta(S-A)$.

That it, we want a contract $L(S)$ whose second derivative is $\frac{\partial^2 L}{\partial S^2} = -\frac{1}{S^2}\theta(S-A)$ so that it's Ito expansion $dL = \frac{\partial L}{\partial S}dS + \frac{1}{2}\frac{\partial^2 L}{\partial S^2}S^2\sigma^2 dt$ will yield $-\frac{1}{S^2}\theta(S-A)S^2\sigma^2 dt = -\frac{1}{2}S^2\sigma^2 dt$ for S > A only.

Let's try to integrate to find L.

$$\frac{\partial^2 L}{\partial S^2} = -\frac{1}{S^2}\theta(S-A)$$

$$\frac{\partial L}{\partial S} = \theta(S-A)\left(\frac{1}{S} - \frac{1}{A}\right)$$

We can check that derivative of 1st derivative is second derivative.

$$L = \theta(S-A)\left[\ln\frac{S}{A} - \frac{1}{A}(S-A)\right]$$

Check that the derivative works:

$$\frac{\partial L}{\partial S} = \cancelto{0}{\delta(S-A)\left[\ln\frac{S}{A} - \frac{1}{A}(S-A)\right]} + \theta(S-A)\left[\frac{1}{S} - \frac{1}{A}\right]$$

Since this is OK, now let's look at the Ito: $L = \theta(S-A)\left[\ln\frac{S}{A} - \frac{1}{A}(S-A)\right]$

$$dL = \frac{\partial L}{\partial S}dS + \frac{1}{2}\frac{\partial^2 L}{\partial S^2}S^2\sigma^2 dt$$

$$= \theta(S-A)\left(\frac{1}{S} - \frac{1}{A}\right)dS - \frac{1}{2}\theta(S-A)\sigma^2 dt$$

Integrate from 0 to T: $L(T) - L(0) = \int_0^T \theta(S-A)\left(\frac{1}{S} - \frac{1}{A}\right)dS - \frac{1}{2}\int_0^T \theta(S-A)\sigma^2 dt$

$$\theta(S_T - A)\left[\ln\frac{S_T}{A} - \frac{1}{A}(S_T - A)\right] = \int_0^T \theta(S-A)\left(\frac{1}{S} - \frac{1}{A}\right)dS - \frac{1}{2}\int \theta(S-A)\sigma^2 dt$$

log payoff at expiration

dynamic rehedging
only if S > A
else gamma = 0
because we don't rehedge

corridor
variance

$$\frac{1}{2}\int \theta(S-A)\sigma^2 dt = \int_0^T \theta(S-A)\left(\frac{1}{S} - \frac{1}{A}\right)dS + \theta(S_T - A)\left[\frac{1}{A}(S_T - A) - \ln\frac{S_T}{A}\right]$$

What replicates the last term on the RHS? Calls with strikes greater than A:

Find the value at expiration of a portfolio π of calls with strike K weighted by K^{-2}, with strike $K > A$, i.e. in the corridor:

$$\pi(S_T) = \int_A^{\infty}\frac{dK}{K^2}(S_T-K)\theta(S_T-K) = \int_A^{S_T}\frac{dK}{K^2}(S_T-K)$$

If $S_T < A$ then this is zero, else $= S_T\left[\frac{1}{A}-\frac{1}{S_T}\right]-\ln\frac{S_T}{A} = \left[\frac{1}{A}(S_T-A)-\ln\frac{S_T}{A}\right]$

Thus we replicate the corridor swap with a dynamic rehedging of a position plus a bunch of calls with strikes from A to infinity, just in the corridor, weighted as usual.

The hedge ratio is $\left(\frac{1}{S}-\frac{1}{A}\right)$ but the position is only rebalanced when $S > A$.

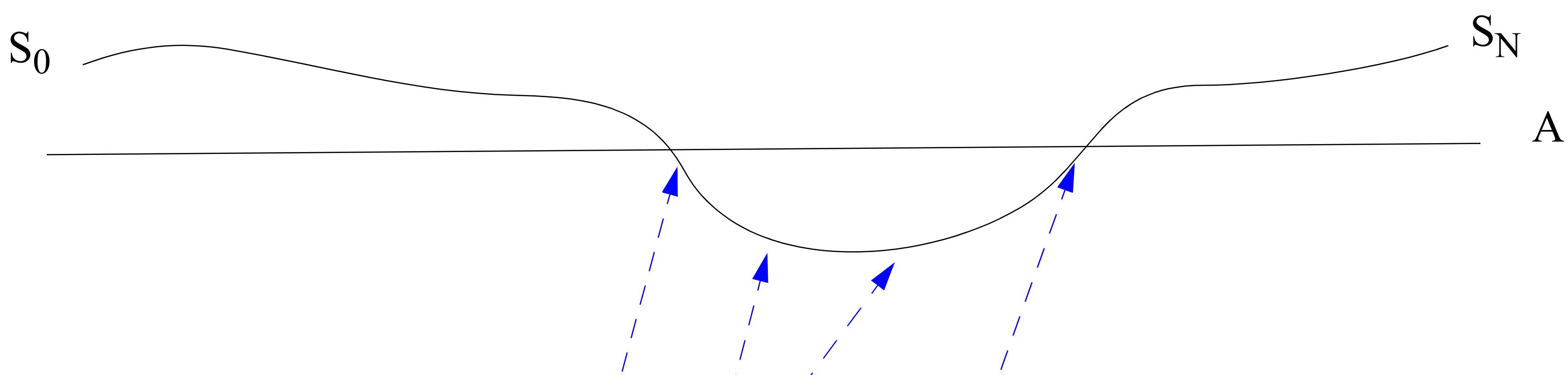
Now recall that we demonstrated in Lecture 4 that the log contract pus the rehedging did indeed generate the total variance. Let’s see how it works here for the corridor variance:

We had in Lecture 4 for variance swap replication after all the rehedgings, starting with \$ in stock and the log contract::

$$
\begin{aligned}
V_N &= 1 - L_N - \sum_{i=0}^{N-1} \frac{S_i - S_{i+1}}{S_i} \\
&= 1 - \ln\left(\frac{S_N}{S_0}\right) + \sum_{i=0}^{N-1} \frac{\Delta S_i}{S_i} \\
&= 1 - \sum_{i=0}^{N-1} \ln\left(\frac{S_{i+1}}{S_i}\right) + \sum_{i=0}^{N-1} \frac{\Delta S_i}{S_i}
\end{aligned}
$$

$$
\begin{aligned}
V_N &= 1 - \sum_{i=0}^{N-1} \left[\frac{\Delta S_i}{S_i} - \frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2\right] + \sum_{i=0}^{N-1} \frac{\Delta S_i}{S_i} \\
&= 1 + \sum_{i=0}^{N-1} \frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2 \\
&= 1 + \sum_{i=0}^{N-1} \frac{\sigma_i^2 \Delta t_i}{2}
\end{aligned}
$$

It works here too: When the stock goes below the corridor bound A, we stop rehedging so there are no more ΔS terms. When it returns, we hedge again. Similarly the



$$\ln S_N/S_0 = \ln S_1/S_0 + \ln S_2/S_1 + \ldots \ln A/S_i + 0 + 0 + \ldots + \ln S_j/A + \ln S_{j+1}/S_j + \ldots$$

so that each ΔS matches each $\ln S_{j+1}/S_j$. So the replication still works.

20.2 Extending Black-Scholes to Stochastic Volatility: The Stochastic Differential Equation for Volatility

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dZ_t$$

$$d\sigma = \text{several possibilities discussed below}$$

The Hull-White (1987) stochastic volatility model with GBM

$$\frac{dV}{V} = \alpha dt + \xi dW \text{ where } V = \sigma^2$$

ξ is the volatility of variance; typical fluctuations of volatility can be very large. How large?

Realized and implied volatilities, like interest rates and credit spreads, are parameters rather than prices **and are range-bound**. For example, between 2005 and 2015 the 30-day realized volatility of the S&P 500 lay between 5% and 82%.

We therefore want to model volatility or variance as a mean-reverting variable, else the range will keep increasing with the square root of the time elapsed.

Stochastic Mean Reversion and its Qualities

Ornstein-Uhlenbeck models:

$$dY = \alpha(m - Y)dt + \beta dW \quad \text{Ornstein Uhlenbeck}$$

$$Y(t) = m + (Y_0 - m)e^{-\alpha t} + \beta\int_0^t e^{-\alpha(t-s)}dW_s \qquad \text{Eq.20.1}$$

The contribution of random previous moves to the long-term value of Y(t) damps out exponentially.

The cross-sectional mean $\overline{Y(t)}$ of Y(t) at time *t*, averaged over all increments dW_s.

$$\overline{Y(t)} = m + (Y_0 - m)e^{-\alpha t}$$

$$E[Y(t) - \overline{Y(t)}]^2 = \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})$$

For small times *t*, variance behaves like $\beta^2 t$, which is like standard Brownian motion.

As $t \to \infty$ the limiting variance is $\lim_{t\to\infty} \mathrm{Var}[Y_t] = \frac{\beta^2}{2\alpha}$.

As α gets larger, the variance gets smaller. Here is a rough sketch of the distribution of the process over time.

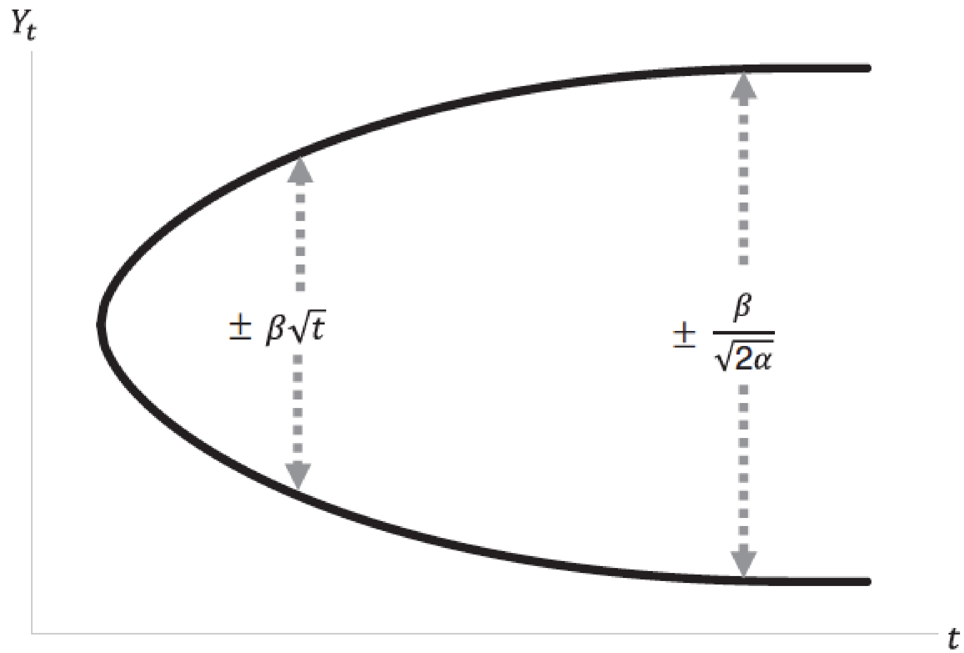


FIGURE 19.4 Schematic Illustration of the Standard Deviation of Y_t

At time $t \approx 1/(2\alpha)$ the variance grows no larger. In contrast, for regular Brownian motion, the linear dependence of the variance on t continues for all time.

20.3 SABR - A Particular Case of Stochastic Local Vol

Add a stochastic element to a local volatility model.

$$\frac{dS}{S} = \alpha S^{\beta - 1} dW \equiv \sigma(S) dW \qquad \sigma(S) = \frac{\alpha}{S^{1-\beta}}$$

SABR model

$$d\alpha = \xi \alpha dZ$$
$$dZdW = \rho dt$$

Assume $\rho = 0$ since we already have a skew.

One can solve the model analytically or by Monte Carlo but we'll work perturbatively by assuming volatility α is small, vol of vol ξ is small, that skew slope is small so β is close to 1, and that we are close to at the money, just to get an idea of what happens.

Then we'll use our experience with local volatility to estimate the BS implied volatility of this stochastic volatility model.

Estimation Strategy:

1. For Local Volatility, Implied Volatility is Approximately the Average of Local Volatilities
2. For Stochastic Implied Volatility, The Option Value Is Approximate Average Of Option Prices Over The Range Of Volatilities.

So:

For $\xi = 0$, pure local volatility, we know that the implied volatility is roughly the average of the local volatilities from S to K:

$$\Sigma_{\mathrm{LV}}(S,t,K,T,\alpha,\beta) \approx \frac{1}{2}\left(\alpha S^{\beta-1} + \alpha K^{\beta-1}\right)$$
$$\approx \alpha S^{\beta-1}\frac{1}{2}\left[1 + \left(\frac{K}{S}\right)^{\beta-1}\right]$$

Taylor expansion in K for β close to 1, weak skew:

$$\left(\frac{K}{S}\right)^{\beta-1} = e^{(\beta-1)\ln\left(\frac{K}{S}\right)} \approx 1 + (\beta-1)\ln\left(\frac{K}{S}\right)$$

So

$$\Sigma_{\mathrm{LV}}(S,t,K,T,\alpha,\beta) \approx \alpha S^{\beta-1}\frac{1}{2}\left[1 + 1 + \ln\left(\frac{K}{S}\right)(\beta-1)\right]$$
$$\approx \frac{\alpha}{S^{1-\beta}}\left[1 - \frac{(1-\beta)}{2}\ln\left(\frac{K}{S}\right)\right]$$

So for zero volatility of volatility we have a linear skew with negative slope, $\frac{\partial\Sigma}{\partial K} \approx \frac{\partial\Sigma}{\partial S}$ **for at-the-money options K = S**, as is true in local vol.

Now switch on the stochastic volatility $\xi \neq 0$ for small ξ. There is a range of possible α values. The skew becomes stochastic and the implied volatility Σ_{LV} has a range of random values.

Estimate C_{SLV} in this Stochastic Local Vol model as average of the BS prices over the range of α with prob density $f(\alpha)$ and $\int f(\alpha)d\alpha = 1$:

$$C_{\text{SLV}} \approx \int C_{\text{BSM}}(\Sigma_{\text{LV}}(S,t,K,T,\alpha,\beta))f(\alpha)d\alpha$$

Taylor expand this about the mean $\bar{\alpha}$ for small volatility of volatility

$$\begin{aligned} C_{\text{SLV}} &= \int C_{\text{BSM}}(\Sigma_{\text{LV}}(S,t,K,T,\bar{\alpha}+(\alpha-\bar{\alpha}),\beta))f(\alpha)d\alpha \\ &\approx \int \left[C_{\text{BSM}}(\Sigma_{\text{LV}}(S,t,K,T,\bar{\alpha},\beta)) + \left.\frac{\partial C_{\text{BSM}}}{\partial \alpha}\right|_{\bar{\alpha}}(\alpha-\bar{\alpha}) \right. \\ &\quad \left. + \frac{1}{2}\left.\frac{\partial^2 C_{\text{BSM}}}{\partial \alpha^2}\right|_{\bar{\alpha}}(\alpha-\bar{\alpha})^2 \right] f(\alpha)d\alpha \\ &\approx C_{\text{BSM}}(\bar{\alpha}) + \frac{1}{2}\left.\frac{\partial^2 C_{\text{BSM}}}{\partial \alpha^2}\right|_{\bar{\alpha}} \text{var}(\alpha) \end{aligned} \qquad \text{Eq 20.1}$$

The definition of the implied volatility Σ_{SLV} in this model is given by $C_{\mathrm{SLV}} \equiv C_{\mathrm{BSM}}(\Sigma_{\mathrm{SLV}})$.

Because the volatility of volatility has been assumed to be small, α stays close to $\bar{\alpha}$ and thus Σ_{SLV} should not differ by much from $\Sigma_{\mathrm{LV}}(S, t, K, T, \bar{\alpha}, \beta) \equiv \Sigma_{LV}(\bar{\alpha})$ evaluated at $\bar{\alpha}$.

Because the volatility of volatility is small, we write

$$\Sigma_{\mathrm{SLV}} \equiv \Sigma_{\mathrm{LV}}(\bar{\alpha}) + (\Sigma_{\mathrm{SLV}} - \Sigma_{\mathrm{LV}}(\bar{\alpha})).$$

and then

$$\begin{aligned} C_{\mathrm{SLV}} &= C_{\mathrm{BSM}}(\Sigma_{\mathrm{LV}}(\bar{\alpha}) + (\Sigma_{\mathrm{SLV}} - \Sigma_{\mathrm{LV}}(\bar{\alpha}))) \\ &\approx C_{\mathrm{BSM}}(\bar{\alpha}) + \frac{\partial C_{\mathrm{BSM}}}{\partial \Sigma_{\mathrm{LV}}}(\Sigma_{\mathrm{SLV}} - \Sigma_{\mathrm{LV}}(\bar{\alpha})) \end{aligned}$$

Eq 20.2

change in vega as stock moves

Comparing the above two equations for C_{SLV}, we obtain

$$\Sigma_{\mathrm{SLV}} \approx \Sigma_{\mathrm{LV}}(\bar{\alpha}) + \frac{\frac{1}{2}\left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \alpha^2}\right|_{\bar{\alpha}} \mathrm{var}(\alpha)}{\frac{\partial C_{\mathrm{BSM}}}{\partial \Sigma_{\mathrm{LV}}}} \qquad \text{(A)}$$

Evaluate the BS derivatives above **for small total variance** $\Sigma^2\tau$ **and close to at-the-money** S = K

$$\Sigma_{\mathrm{LV}}(\bar{\alpha}) \approx \bar{\alpha}/S^{1-\beta}.$$

Then from the chain rule, since $\sigma \approx \alpha S^{\beta - 1}$

$$\begin{aligned}\left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \alpha^2}\right|_{\bar{\alpha}} &\approx \left(\frac{1}{S^{1-\beta}}\right)^2 \left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2}\right|_{\sigma=\Sigma_{\mathrm{LV}}} \\ &\approx \left(\frac{\Sigma_{\mathrm{LV}}}{\bar{\alpha}}\right)^2 \left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2}\right|_{\sigma=\Sigma_{\mathrm{LV}}}\end{aligned}$$

Furthermore in the SABR model α undergoes GBM with variance $\mathrm{var}(\alpha) \approx \bar{\alpha}^2\xi^2\tau$. So

$$\frac{\frac{1}{2}\left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial\alpha^2}\right|_{\bar{\alpha}} \mathrm{var}(\alpha)}{\frac{\partial C_{\mathrm{BSM}}}{\partial\Sigma_{\mathrm{LV}}}} \approx \frac{1}{2}\left[\left(\frac{\Sigma_{\mathrm{LV}}}{\bar{\alpha}}\right)^2 \left.\frac{\frac{\partial^2 C_{\mathrm{BSM}}}{\partial\sigma^2}}{\frac{\partial C_{\mathrm{BSM}}}{\partial\sigma}}\right|_{\sigma=\Sigma_{\mathrm{LV}}} (\bar{\alpha}\xi)^2\tau\right]$$

$$\approx \frac{1}{2}\Sigma_{\mathrm{LV}}^2 \boxed{\left.\frac{\frac{\partial^2 C_{\mathrm{BSM}}}{\partial\sigma^2}}{\frac{\partial C_{\mathrm{BSM}}}{\partial\sigma}}\right|_{\sigma=\Sigma_{\mathrm{LV}}}} \xi^2\tau \qquad \text{(B)}$$

$$V = \frac{\partial C_{\mathrm{BSM}}}{\partial\sigma} = \frac{\sqrt{\tau}}{\sqrt{2\pi}} S e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right)^2} \qquad \text{vega}$$

$$\frac{\partial^2 C_{\mathrm{BSM}}}{\partial\sigma^2} = \frac{V}{\sigma}\left[\frac{\ln^2\left(\frac{S}{K}\right)}{\sigma^2\tau} - \frac{\sigma^2\tau}{4}\right] \qquad \text{volga}$$

Using our volga and vanna formulas for small total variance $\sigma^2\tau$

$$\boxed{\frac{\frac{\partial^2 C_{\mathrm{BSM}}}{\partial\sigma^2}}{\frac{\partial C_{\mathrm{BSM}}}{\partial\sigma}}} = \frac{1}{\sigma}\left[\frac{1}{\sigma^2\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\sigma^2\tau}{4}\right] \approx \frac{1}{\sigma^3\tau}\left(\ln\left(\frac{S}{K}\right)\right)^2$$

So from (B) above, setting $\sigma = \Sigma_{LV}$

$$\frac{1}{2}\frac{\left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial\alpha^2}\right|_{\bar{\alpha}}\mathrm{var}(\alpha)}{\frac{\partial C_{\mathrm{BSM}}}{\partial\Sigma_{\mathrm{LV}}}} \approx \frac{1}{2}\frac{\xi^2}{\Sigma_{\mathrm{LV}}(\bar{\alpha})}\left(\ln\left(\frac{S}{K}\right)\right)^2$$

and substituting this into Equation (A) we get

$$\Sigma_{SLV} \approx \Sigma_{LV}(\bar{\alpha})\left\{1+\frac{1}{2}\left[\frac{\xi}{\Sigma_{LV}(\bar{\alpha})}\right]^2\left[\ln\frac{S}{K}\right]^2\right\} \qquad \text{Eq.20.2}$$

The local volatility smile $\Sigma_{LV}(\bar{\alpha})$ is altered by the addition of a quadratic term in $\ln\frac{S}{K}$

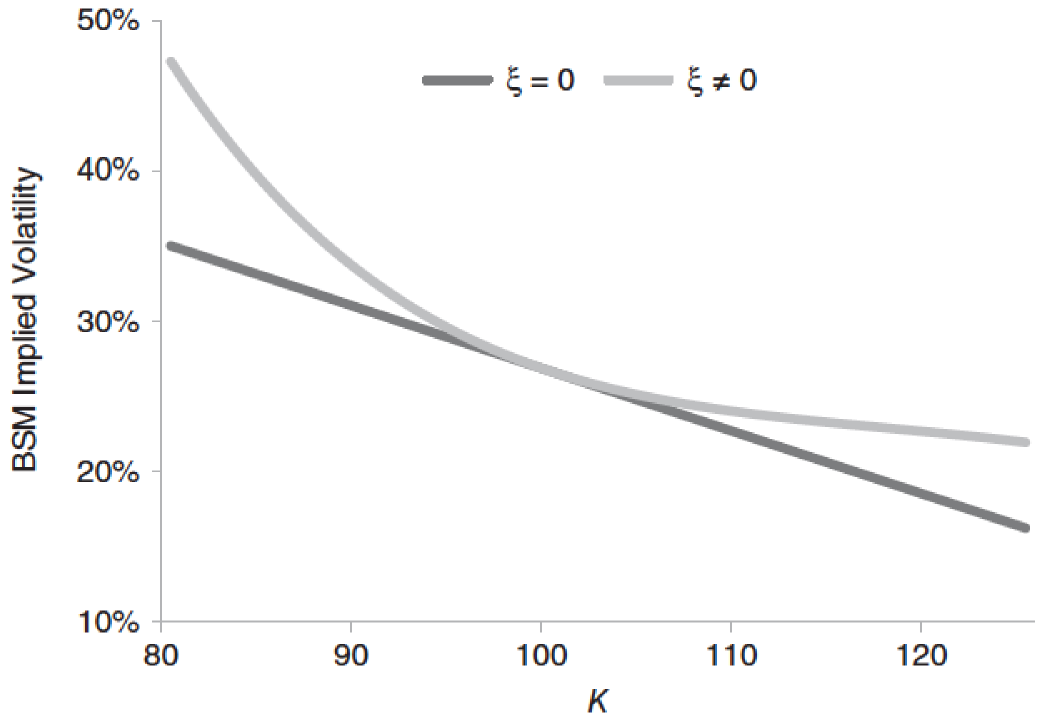


FIGURE 20.1 The Impact of Stochastic Volatility of Volatility on the Smile in the SABR Model

The relative size of the stochastic contribution is $\left[\frac{\xi}{\Sigma_{LV}(\bar{\alpha})}\right]^2$.

No need for correlation between volatility and stock price in order to obtain a smile if we start from local volatility.

We just covered the SABR model which starts from local volatility and a skew and adds stochasticity to the skew to get the stochastic volatility model. Now we go back to starting from Black-Scholes with no skew and adding volatility of volatility, the alternative approach.

20.4 The PDE for Valuing Options With Stochastic Volatility

Extending the Black-Scholes riskless-hedging argument.

$$\begin{aligned} dS &= \mu S dt + \sigma S dW \\ d\sigma &= p(S, \sigma, t)dt + q(S, \sigma, t)dZ \\ dW dZ &= \rho dt \end{aligned} \qquad \text{Eq.20.3}$$

Now consider two options $V(S, \sigma, t)$ and $U(S, \sigma, t)$

$\Pi = V - \Delta S - \delta U$, short Δ shares of S and short δ options U to hedge V.

$$\begin{aligned} d\Pi = dt &\left[\begin{array}{l} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}q^2 + \dfrac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho \\ -\delta\left(\dfrac{\partial U}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}q^2 + \dfrac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho\right) \end{array} \right] \\ &+ dS\left(\frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} - \Delta\right) + d\sigma\left(\frac{\partial V}{\partial \sigma} - \delta\frac{\partial U}{\partial \sigma}\right) \end{aligned}$$

We can eliminate all risk by choosing Δ and δ to satisfy

$$\frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} - \Delta = 0 \qquad \left(\frac{\partial V}{\partial \sigma} - \delta\frac{\partial U}{\partial \sigma} = 0\right)$$

$$\Delta = \frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S} \qquad \delta = \frac{\partial V}{\partial \sigma}\Big/\frac{\partial U}{\partial \sigma} \tag{Eq.20.4}$$

$$\text{Then } d\Pi = dt\left[\begin{array}{l} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}q^2 + \dfrac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho \\ -\delta\left(\dfrac{\partial U}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}q^2 + \dfrac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho\right) \end{array}\right] \tag{Eq.20.5}$$

No riskless arbitrage: $$d\Pi = r\Pi dt = r[V - \Delta S - \delta U]dt$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho - rV$$

$$-\delta\left(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}q^2 + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho - rU\right)$$

$$+ r\Delta S = 0$$

But $\Delta = \frac{\partial V}{\partial S} - \delta\frac{\partial U}{\partial S}$ and so

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} - rV$$

$$= \delta\left(\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 U}{\partial \sigma^2}q^2 + \frac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial U}{\partial S} - rU\right)$$

and $\delta = \frac{\partial V}{\partial \sigma} \Big/ \frac{\partial U}{\partial \sigma}$

$$\frac{\dfrac{\partial V}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 V}{\partial \sigma^2}q^2 + \dfrac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\dfrac{\partial V}{\partial S} - rV}{\dfrac{\partial V}{\partial \sigma}}$$

$$= \frac{\dfrac{\partial U}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial S^2}\sigma^2 S^2 + \dfrac{1}{2}\dfrac{\partial^2 U}{\partial \sigma^2}q^2 + \dfrac{\partial^2 U}{\partial S \partial \sigma}\sigma q S \rho + rS\dfrac{\partial U}{\partial S} - rU}{\dfrac{\partial U}{\partial \sigma}} \qquad \text{Eq.20.6}$$

$$= \phi(S, \sigma, t) \qquad \text{separation of variables}$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} + \phi(S, \sigma, t)\frac{\partial V}{\partial \sigma} - rV = 0 \qquad \text{Valuation PDE}$$

This is the PDE for the value of an option with stochastic volatility σ.

Notice: we don’t know the value of the function ϕ!

20.5 The Sharpe-ratio meaning of $\phi(S, \sigma, t)$

See what the PDE says about expected risk and return of the option V using Ito's lemma:

$$
\begin{aligned}
dS &= \mu S dt + \sigma S dW \\
d\sigma &= p(S, \sigma, t)dt + q(S, \sigma, t)dZ \\
dWdZ &= \rho dt
\end{aligned}
$$

$$
\begin{aligned}
dV &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 dt + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho dt \\
&= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}\mu S dt + \frac{\partial V}{\partial \sigma}p dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 dt + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho dt \\
&\quad + \frac{\partial V}{\partial S}\sigma S dW + \frac{\partial V}{\partial \sigma}q dZ \\
&\equiv \mu_V V dt + V\sigma_{V_S}dW + V\sigma_{V_\sigma}dZ
\end{aligned}
\qquad \text{Eq.20.7}
$$

$$
\mu_V = \frac{1}{V}\left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial \sigma}p + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho\right] \qquad \text{Eq.20.8}
$$

$$
\sigma_{V_S} = \frac{S}{V}\frac{\partial V}{\partial S}\sigma \qquad \sigma_{V_\sigma} = \frac{1}{V}\frac{\partial V}{\partial \sigma}q \qquad \sigma_V \equiv \sqrt{\sigma_{V_S}^2 + \sigma_{V_\sigma}^2 + 2\rho\sigma_{V_S}\sigma_{V_\sigma}}
$$

where σ_{V_S} and σ_{V_σ} are the partial volatilities of option V with total volatility σ_V.

The stochastic volatility PDE was

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho + rS\frac{\partial V}{\partial S} + \phi(S, \sigma, t)\frac{\partial V}{\partial \sigma} - rV = 0 \text{ and}$$

$$\mu_V = \frac{1}{V}\left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial \sigma}p + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho\right]$$

Thus the red terms in the LHS of the stochastic volatility PDE for the value of the option are

$$\frac{1}{V}\left[\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}q^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\sigma q S \rho\right] \equiv \mu_V - \mu\left(\frac{\partial V}{\partial S}\frac{S}{V}\right) - p\left(\frac{\partial V}{\partial \sigma}\frac{1}{V}\right)$$

Substituting this into the stochastic vol PDE

$$\mu_V - \mu\left(\frac{\partial V}{\partial S}\frac{S}{V}\right) - p\left(\frac{\partial V}{\partial \sigma}\frac{1}{V}\right) + r\frac{S}{V}\frac{\partial V}{\partial S} + \phi(S, \sigma, t)\frac{1}{V}\frac{\partial V}{\partial \sigma} - r = 0$$

or $$(\mu_V - r) = \frac{S}{V}\frac{\partial V}{\partial S}(\mu - r) + \frac{1}{V}\frac{\partial V}{\partial \sigma}(p - \phi)$$

or $$(\mu_V - r) = \sigma_{V_S}\frac{(\mu - r)}{\sigma} + \sigma_{V_\sigma}\frac{(p - \phi)}{q}$$

or $$\frac{(\mu_V - r)}{\sigma_V} = \frac{\sigma_{V_S}}{\sigma_V}\frac{(\mu - r)}{\sigma} + \frac{\sigma_{V_\sigma}}{\sigma_V}\frac{(p - \phi)}{q}$$

Excess return per unit of risk for the option = the excess return per unit of risk for the stock and the excess return per unit of risk for volatility, weighted by each one's contribution to volatility of V.

ϕ plays the role for stochastic volatility that the riskless rate r plays for a stochastic stock price.

In the Black-Scholes case, r is the rate at which the stock price must grow in order that option payoffs can be discounted at the riskless rate.

Similarly, ϕ is the drift p that volatility must undergo in order that option prices with stochastic volatility be discounted at the riskless rate (or values can grow risk-neutrally).

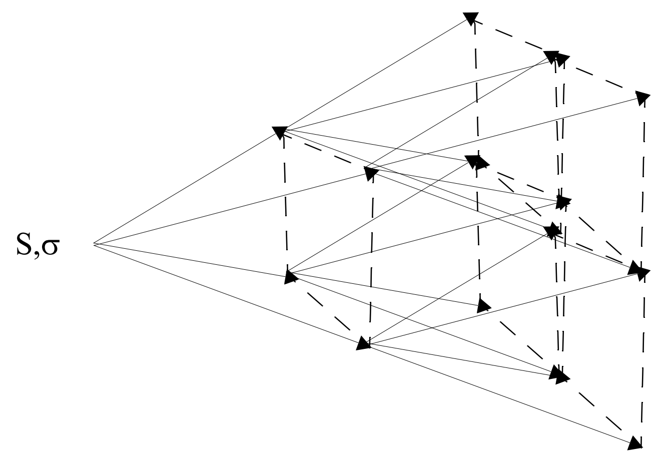
ϕ is not equal to r because σ is not traded. ϕ is the rate at which volatility σ must grow in order that the price of the option V grows at the rate r when you can hedge away all risk.

From a calibration point of view, ϕ or p must be chosen to make option prices grow at the riskless rate.

If we know the market price of just one option U, and **we assume an evolution process for volatility**, $d\sigma = \phi(S, \sigma, t)dt + q(S, \sigma, t)dZ$, then we can choose/calibrate the effective drift ϕ of volatility so that the calculated discounted expected value of U matches its market price.

Then we can value all other options from the same pde.

In a quadrinomial picture of stock prices where volatility and stock prices are stochastic, as illustrated in the figure below, we much calibrate the drift of volatility ϕ so that the value of an option U is given by the expected risklessly discounted value of its payoffs.



Once we've chosen ϕ to match that one option price, then, assuming we have the correct model for volatility, all other options can be valued risk-neutrally by discounting their expected payoffs.

Of course, **it may be naive** to assume that just one option can calibrate the entire volatility evolution process, just as one bond cannot determine the whole yield curve.)
Note that even though the payoffs of the option are the same as in the Black-Scholes world, the evolution process of the stock is different, and so the option price will be different too.

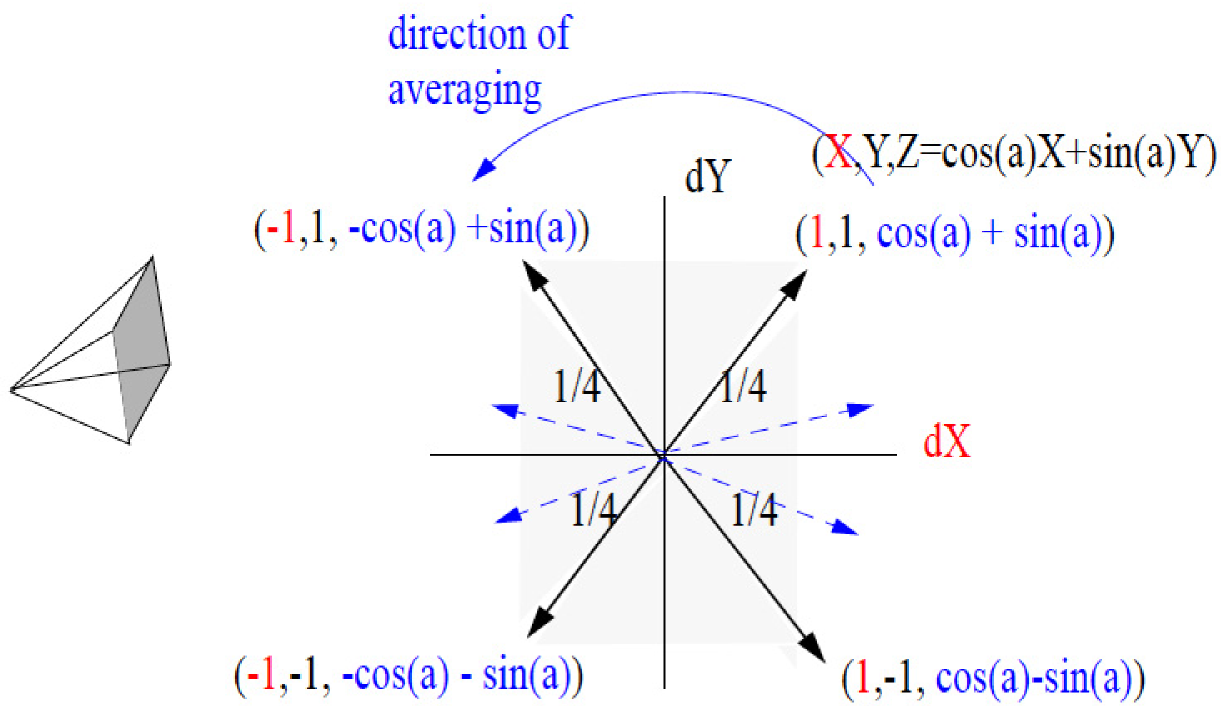
The homework problems give you a stochastic process, an *r* and a drift and then you have to calculate the expected discounted value of the payoff assuming it grows risk-neutrally.

20.6 A Method for Simulating Two Correlated Stochastic Variables Step by Step

random increments X and Y are uncorrelated and Z is correlated with X

Correlated Normal Distributions: Method 1:Equal Probabilities

$$Z = \rho X + \sqrt{1-\rho^2}\,Y \equiv X\cos a + Y\sin a \text{ where } \cos a = \rho$$



The correlation between dZ and dX is cos(a).

Simulating Stochastic Vol Step by Step, An Example


```
function [call, implied] = stochastic_vol_option(vol, mu_vol, vol_vol, rho, r, strike, s, t, npaths,
nsteps, rseed);


dt = t/nsteps;
randn('seed', rseed);
call_vec = zeros(npaths,1);

for j=1:npaths
   s_value=s;
   vol_value = vol;
   for i=1:nsteps
      % generate the random increment for the stock
      z_s = randn(1,1);
      % generate the random increment for the volatility
      z_vol = sqrt(1-rho*rho) * randn(1,1) + rho * z_s;
      % generate the next stock price dt later
      s_value = s_value*exp((r-0.5*vol_value*vol_value)*dt + vol_value*sqrt(dt)*z_s);
       % generate the next volatility dt later
      vol_value = vol_value*exp((mu_vol-0.5*vol_vol*vol_vol)*dt + vol_vol*sqrt(dt)*z_vol);
   end
   call_vec(j) = max(0,s_value-strike); % option payoff on this path
end

% Discounted expected value for call
call_vec = exp(-r*t)*call_vec;
call     = mean(call_vec);
implied  = blsimpv(s, strike, r, t, call, 2);
```

20.7 The Simpler Characteristic Solution to the Stochastic Volatility Model with Zero Correlation

Before Stochastic Vol, First Recall: The continuous-time treatment of a time-dependent volatility:

$$dS = rSdt + \sigma(t)SdW_t \text{ in the risk-neutral world.}$$

$$d\ln S = \frac{dS}{S} - \frac{1}{2}\frac{1}{S^2}\sigma^2(t)S^2dt = rdt + \sigma(t)dW_t - \frac{1}{2}\sigma^2(t)dt$$

$$\ln S(t) = \ln S(0) + rt - \frac{1}{2}\int_0^t \sigma^2(s)ds + \int_0^t \sigma(s)dW_s$$

The distribution is given by a sum of normals, which itself is normal, with total variance given by

$$E\int_0^t\int_0^t \sigma(u)\sigma(s)dW_u dW_s = \int_0^t\int_0^t \sigma(u)\sigma(s)E[dW_u dW_s]$$

$$= \int_0^t\int_0^t \sigma(u)\sigma(s)duds\delta(u-s) = \int_0^t \sigma^2(u)du = t\bar{\sigma}^2(t)$$

Thus at time t the mean of the distribution of $\ln S$ is at $\left(r - \frac{\bar{\sigma}^2}{2}\right)t$

where $\bar{\sigma}^2 = \frac{1}{t}\int_0^t \sigma^2(s)ds$ is the **path variance, known and deterministic.**

Thus $\log S(t)$ is distributed normally too, as follows.

$$\log S_t/S_0 \sim \mathcal{N}\left(\left(r - \frac{\bar{\sigma}^2}{2}\right)t, \bar{\sigma}^2 t\right)$$

Thus, calculating the value of the option as an expected value of the payoff in a risk-neutral world, we find

$$C = C_{BSM}(S, t, K, T, r, \bar{\sigma}^2(t))$$

The Black-Scholes-Merton value of the option is the discounted expected value of the payoff after a time-dependent volatility, discounted at the riskless rate.

$$V = V_{BSM}(S, t, K, T, r, \bar{\sigma}^2(t)) = \exp(-r\tau)E_Q[payoff]$$

Now let's look at stochastic volatility rather than deterministic volatility

The price of the hedged European option is given by the expected risk-neutral value of the terminal payoff where the stock price $S(t)$ and the variance $v(t) = \sigma^2(t)$ are both stochastic:

$$V = e^{-r(T-t)} \sum_{\text{all paths}} p(\text{path}) \times \text{payoff}|_{\text{path}}$$

V is the integral over the payoff conditional on the two diffusions.

Hull and White (1987) characterize **each path** by its terminal stock price S_T and the average variance along that path

$$\overline{\sigma_T^2} = \frac{1}{T}\int_0^T \sigma_t^2 dt$$

We refer to $\overline{\sigma}_T$ as the path volatility -- the square root of the average path variance.

Then

$$V = e^{-r(T-t)} \sum_{\text{all } \overline{\sigma_T}} \sum_{\substack{\text{paths of } S_T \\ \text{given } \overline{\sigma_T}}} p(\overline{\sigma_T}, S_T) \times \text{payoff}|_{\text{path}}$$

where $p(\overline{\sigma}_T, S_T)$ is the probability of one of these paths.

If the stock and its volatility are uncorrelated ($\rho = 0$), then the probability p can factor such that

$$p(\overline{\sigma_T}, S_T) = f(\overline{\sigma_T}) \times g(S_T)$$

and so

$$V = e^{-r(T-t)} \sum_{\text{all } \overline{\sigma_T}} f(\overline{\sigma_T}) \sum_{\substack{\text{paths of } S_T \\ \text{given } \overline{\sigma_T}}} g(S_T) \times \text{payoff}|_{\text{path}}$$

But because $g(\ )$ is the usual lognormal distribution for a known path volatility.t

$$V_{\text{BSM}}(S, t, K, T, r, \overline{\sigma_T}) = e^{-r(T-t)} \sum_{\substack{\text{paths of } S_T \\ \text{given } \overline{\sigma_T}}} g(S_T) \times \text{payoff}|_{\text{path}}$$

Combining these previous two equations we find that

$$V = \sum_{\text{all } \overline{\sigma_T}} f(\overline{\sigma_T}) \times V_{\text{BSM}}(S, t, K, T, r, \overline{\sigma_T})$$

When the correlation is zero, the stochastic volatility solution for a standard European option is the weighted sum over the BSM solutions with different path volatilities.

Mixing Theorem

The price of an option in a stochastic volatility model with zero correlation is the weighted integral/sum over BS prices over the distribution of path volatilities.

$$V = \sum_{\text{all } \overline{\sigma_T}} f(\overline{\sigma_T}) \times V_{\text{BSM}}(S, t, K, T, r, \overline{\sigma_T})$$

It doesn't matter what order the stochastic volatilities occur in -- as long as the variance along the path is the same, all paths with that variance have the same BS terminal distribution of the stock price.

What is the advantage of this?

The mixing theorem reduces this to a **one-dimensional simulation or integration in the model**.

IF the correlation is different from zero, this doesn't work: then all paths conditional on a definite variance still have a normal distribution of returns, BUT that variance depends on the stock price path, not just on time. The resultant formula is

$$V_t = E\left[V_{\text{BSM}}\left(S_t^*(\overline{\sigma_T}, \rho), K, r, \overline{\sigma_T}^*(\rho), T\right)\right]$$

where the stock price and volatility in the Black-Scholes formula are shifted to "fake" values that differ from actual values by something related to the correlation. Much less useful.

[Refs: Fouque, Papanicolaou and Sircar book, and Roger Lee, *Implied and Local Volatilities under Stochastic Volatility*, International Journal of Theoretical and Applied Finance, 4(1), 45-89 (2001).]