Math 280, Spring 2022

Solutions to Homework #1

1. (8 points) Textbook, Section 1.1.2, Problem 1: If G is a graph of order n, what is the maximum possible size of G?

Solution. Since there are n vertices in G, there are $\binom{n}{2} = \frac{n(n-1)}{2}$ two-element subsets of $V(G)$.
Since $E(G)$ is a subset of the set of *all* two-element subsets of $V(G)$, the size of G, i.e., $E(G)$, is at most this number.
That is, the maximum possible size is $\binom{n}{2} = \frac{n(n-1)}{2}$.

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2. (12 points) Textbook, Section 1.1.2, Problem 2: Let G be a graph of order $n \geq 2$. Prove that the degree sequence of G has at least one pair of repeated entries.

Proof. For any vertex $v \in V(G)$, there are only $n-1$ other vertices (i.e., elements of $V(G) \smallsetminus \{v\}$), so there can be at most $n-1$ edges incident with v. Thus,

$$0 \leq \deg(v) \leq n-1 \quad \text{for all } v \in V(G).$$

Suppose, towards a contradiction, that all n vertices have different degrees. Since there are n vertices and n integers between 0 and $n-1$, by the pigeonhole principle, there must be one vertex of each degree from 0 to $n-1$.
In particular, there must be vertices $a, b \in V(G)$ such that $\deg(a) = 0$ and $\deg(b) = n-1$. Since $n \geq 2$, these degrees are different, so $a \neq b$. Because $\deg(b) = n-1$, and there are only $n-1$ vertices in $V(G) \smallsetminus \{b\}$, there must be an edge from b to every other vertex. In particular, there must be an edge from b to a. But then there is at least one edge incident to a, so $0 = \deg(a) \geq 1$, which is a contradiction.
Thus, our supposition must be false. That is, there are at least two vertices with the same degree. Hence, there is at least one pair of repeated entries in the degree sequence of G. QED

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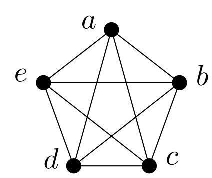
3. (15 points) Textbook, Section 1.1.2, Problem 3(c,d):
For this graph:



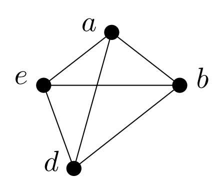
(c) What is the maximum length of a circuit in this graph? Give an example of such a circuit.
(d) What is the maximum length of a circuit that does not include vertex c? Give an example of such a circuit.

Solution. (c): There are 10 edges in this graph, so certainly no trail, and hence no circuit, can have length more than 10. And there are circuits of length 10, such as:

$$a, b, c, d, e, a, c, e, b, d, a$$

(i.e., trace the outer pentagon all the way around, and then the star all the way around). So this circuit has maximum length, because it uses all the edges, so there cannot be a longer one.

(d): A circuit not including c means a circuit on the subgraph $G - c$, i.e., on this graph:



which is a complete graph on 4 vertices. In particular, any one vertex among $\{a, b, d, e\}$ is like any of the others, so we can relabel the four vertices and it doesn't change anything about the graph.
We claim that no circuit has length more than 4. To prove this, given an arbitrary circuit, by relabeling the vertices, we may assume without loss of generality that the first vertex is a. By relabeling the remaining three vertices if necessary, we may similarly assume the second vertex in the circuit is b. Relabeling the other two vertices, we may assume the third vertex is d. (It cannot be a again, because that would reuse the edge ab, which is not allowed for a circuit. It also cannot be b, because the third edge must leave from b and land somewhere else.)
The fourth vertex in the circuit cannot be d (because the third edge starts from d), and it cannot be b (because that would reuse edge bd). If the fourth vertex is a, then because the only remaining unused edge incident to a is ae; and after using that, there is no way to return to a to complete the circuit. That is, if the fourth vertex is a, then the circuit must end there, with length 3.
The only other possibility is that the fourth vertex is e; that is, our path so far is a, b, c, e. If the fifth vertex is b, then we will be stuck at b, having used all three edges ab, bd, and be incident with b. Thus, the fifth vertex cannot be b, as this would make it impossible to leave b to complete the circuit at a. The fifth vertex also cannot be e (since it must be different from the fourth vertex), and it cannot be d (because edge de has already been used). Thus, the fifth vertex must be a.
The only remaining edge incident with a is ad. But if we use it, then we cannot leave d to return to a to complete the circuit, as we will have used all three edges incident with d. So our circuit must stop at the fifth vertex, meaning that it has length at most $5 - 1 = 4$, as claimed.
Finally, as illustrated in the claim, the path a, b, d, e, a is a circuit of length 4. So 4 is the maximum length of a circuit omitting vertex c.

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4. (7 points) Let G be a graph of odd order. Suppose that all the vertices of G have the same degree r. Prove that r is an even number.

Proof. Suppose (towards contradiction) that r is odd. Let $n = |V(G)|$ be the number of vertices, which is odd, and let $m = |E(G)|$ be the number of edges. Then by the degree theorem, we have

$$nr = \sum_{i=1}^{n} r = \sum_{v \in V(G)} \deg(v) = 2|E(G)| = 2m.$$

But $2m$ is even, while nr is odd, because n and r are both odd. This is a contradiction, so our supposition must have been wrong. That is, r is even. QED

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5. (12 points) Textbook, Section 1.1.2, Problem 6: Prove that every closed walk of odd length in a graph contains a cycle of odd length.

Proof. Call our graph G. We prove this result for closed walks of length $2k+1$ by induction on $k \geq 0$.

Base case: For $k = 0$, let W be a closed walk of length 1. We may write W as w_0, w_1 with $w_0 = w_1$ (because W is closed), but this contradicts the definition of walk, since $w_0 w_1$ must be an edge, and in particular $w_0 \neq w_1$. That is, there are *no* closed walks of length 1, so the desired statement is vacuously true for the case $k = 1$.

Inductive step: For any $m \geq 2$, assume the statement is true for all $0 \leq k \leq m - 1$; we will prove it for m. Let W be a closed walk of length $2m + 1$. Write W as

$$w_0, w_1, \ldots, w_{2m+1}$$

with $w_0 = w_{2m+1}$. If all of $w_0, \ldots, w_{2m}$ are distinct, then W is already a cycle of odd length, and we are done.
Otherwise, there are integers i, j with $0 \leq i < j \leq 2m$ such that $w_i = w_j$. Let $s = j - i$. Note that $1 \leq s \leq 2m - 1$. Define the following walks:

$$W_1 \text{ is } w_i, w_{i+1}, \ldots, w_j, \quad \text{and} \quad W_2 \text{ is } w_0, w_1, \ldots, w_i, w_{j+1}, w_{j+2}, \ldots, w_{2m+1}.$$

(That is, W_1 consists of the $s+1$ vertices of W from w_i to w_j; and W_2 is formed from W by deleting the $s-1$ vertices from w_{i+1} to w_{j-1}.)
Observe that W_1 is a closed walk (since it starts at w_i and ends at $w_j = w_i$. In addition, W_2 is also a walk, because $w_i = w_j$ is adjacent to w_{j+1}; and W_2 is also closed, since it starts and ends at $w_0 = w_{2m+1}$. In addition, W_1 lists $s+1$ vertices and hence has length s, while W_2 lists $(2m+1)-(s-1) = 2m+2-s$ vertices, and hence has length $2m+1-s$.
If s is odd, then W_1 is a closed walk of odd length $s < 2m+1$, so by the inductive hypothesis, W_1 and hence W contains an odd-length cycle.
Otherwise, s is even, in which case $2m+1-s$ is odd (and strictly smaller than $2m+1$). Thus, W_2 is a closed walk of odd length $2m+1-s < 2m+1$, so by the inductive hypothesis, W_2 and hence W contains an odd-length cycle. QED

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6. (8 points) Let G be a graph of order n and size t. Let $\overline{G}$ be the complement graph of G. (See the textbook, Section 1.1.3, item 3.) Find the order and size of $\overline{G}$.

Solution. Since $V(\overline{G}) = V(G)$ by definition, the order of $\overline{G}$ is also n.
As we saw in problem 1 of this assignment, there are $\binom{n}{2}$ possible edges between vertices of $V(G)$. Together, $E(G)$ and $E(\overline{G})$ consist of all the possible edges, with no repeats, and hence

$$|E(G)| + |E(\overline{G})| = \binom{n}{2} = \frac{n(n-1)}{2}.$$

Since $|E(G)| = t$, it follows that the size of $\overline{G}$ is

$$|E(\overline{G})| = \frac{n(n-1)}{2} - t.$$