Lecture 16: Analysis of Skiplists

- Basic Probability
- Height and Max. Height
- Expected Running Time

⇒ Basic Probability

Goal: formalize vocabulary and concepts to reason about random processes

Concrete Examples — $\boxed{\text{SAMPLE SPACE}}$ (Ω = Omega)

1) flipping coin ⟶ $\{H, T\}$
2) rolling dice ⟶ $\{1, 2, 3, 4, 5, 6\}$
3) Shuffling/picking card ⟶ $\{1\diamondsuit, 2\heartsuit, \ldots, A\diamondsuit\}$

Also, each element in sample space has an associated probability.

$$\boxed{\underbrace{P(\omega)}_{\text{outcome in } \Omega} = \text{likelihood that } \omega \text{ happens}}$$

Properties

(1) $P(\omega) \geq 0$

(2) $P(\omega_1) + P(\omega_2) + \ldots + P(\omega_n) = 1 \Rightarrow \boxed{\sum_{i=1}^{n} P(\omega_i) = 1}$

A random variable associates a real value/number to each possible outcome.

eg Head → win \$1
Tail → lost \$1

⤳ Random Var X: $X(H) = 1$, $X(T) = -1$.

Intuitively: expect to win as much as lose, so "expected winnings" = 0.

$$\frac{1}{2}\cdot(1) + \frac{1}{2}\cdot(-1) = 0$$

Expected value of Random Variable

$$\begin{cases} \Omega = \{\omega_1, \omega_2, \dots, \omega_n\} \\ P = \text{probability for each outcome} \\ X = \text{rand. var assigns value to each outcome} \end{cases}$$

$E(X)$ = Expected value or avg of x. = $\boxed{\sum X P(X)}$

⇒ Probability of event A is |SUM| of probabilities of all outcomes in A

⊛ Computing Expected Height & Size of Skiplist

Recall: To generate height of a node:

- flip coin until we see first heads
- heights = # tails before heads.

$\Omega = \{H, TH, TTH, TTTH, \dots\}$

P : $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$ $\frac{1}{16}$ ... $\frac{1}{2^k}$

H : 0 1 2 3 $(k-1)$

random variable for height

Define r.v. ~~x~~ as total number of coins flipped = $H+1$

X : 1 2 3 4 K

$P(X=1) = \frac{1}{2}$

$P(X=2) = \frac{1}{4}$

⋮

$P(X=K) = \frac{1}{2^K}$

$E(X) = \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 + \dots + \frac{1}{2^k}\cdot K + \dots$

⇒ Trick to compute $E(X)$

Define $k = 1, 2, 3, \ldots$ $I_k = \begin{cases} 1, \text{if at least } k \text{ flips required} \\ 0, \text{otherwise} \end{cases}$

$$E(I_k) = P(I_k = 1)\cdot 1 + P(I_k = 0)\cdot 0$$
$$= P(I_k = 1) = P(\text{first } (k-1) \text{ flips are all tails})$$
$$= \frac{1}{2^{k-1}}.$$

⇒ How are X related to I_ks?

$$\begin{cases} \bullet \text{ if } X=1, \text{ then } I_1 = 1 \,\|\, I_2, I_3, I_4 \ldots = 0 \\ \bullet \text{ if } X=2, \text{ then } I_1, I_2 = 1 \,\|\, I_3, I_4, \ldots = 0 \\ \bullet \text{ if } X=3, \text{ then } I_1, I_2, I_3 = 1 \,\|\, I_4, I_5, \ldots = 0. \end{cases}$$

In all cases,

$$X = I_1 + I_2 + \ldots$$

So,

$$E(X) = E(I_1 + I_2 + I_3 + I_4 + \ldots)$$
$$= E(I_1) + E(I_2) + \ldots$$
$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^k} \approx \boxed{2}$$

Conclusion

$$E(H) = E(X) - 1 = 2 - 1 = \boxed{1} ✓$$