STAT GU4261/GR5261 - Statistical Methods in Finance - Homework #2 Solutions

February 5th, 2023

Question 1

Load data and perform Kolmogorov-Smirnov tests using `lillie.test` from `nortest` package.

```r
asset_returns <- read.csv("HW2-S22-DATA.csv")
my_ks_test <- function(x) {
  nortest::lillie.test(x)
}
lapply(asset_returns, my_ks_test)
```

```
## $MSOFT
##
##  Lilliefors (Kolmogorov-Smirnov) normality test
##
## data:  x
## D = 0.084955, p-value = 0.03317
##
##
## $GE
##
##  Lilliefors (Kolmogorov-Smirnov) normality test
##
## data:  x
## D = 0.053323, p-value = 0.5508
##
##
## $GM
##
##  Lilliefors (Kolmogorov-Smirnov) normality test
##
## data:  x
## D = 0.078251, p-value = 0.06817
```

The KS statistics for MSOFT, GE and GM are 0.085, 0.053 and 0.078 with corresponding p-values 0.03317, 0.5508 and 0.06817 respectively. We reject the null hypothesis for normal data for MSOFT but we do no have the enough evidence to reject the null for GE and GM. Note: `ks.test` does not give accurate p-values for the composite hypothesis of normality (mean and variance unknown).

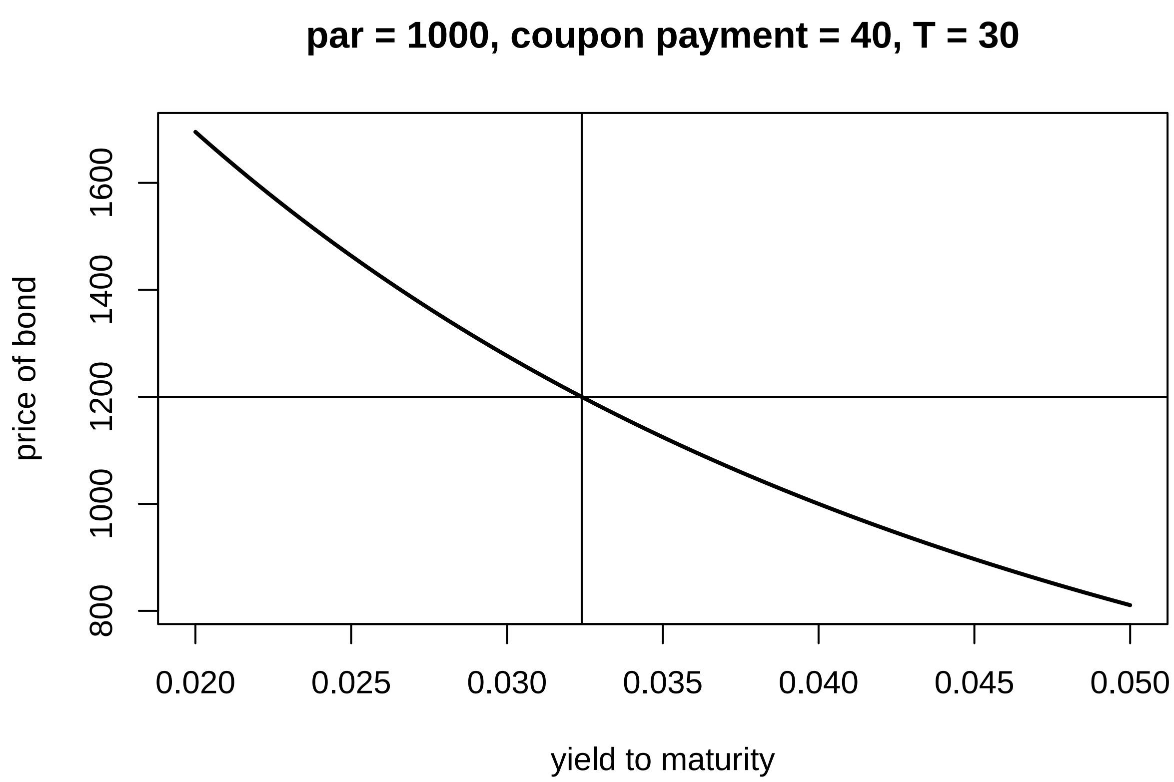
Question 2

Problem 1 pp. 38

```r
bondvalue <- function(c, T, r, par) {
  # Computes bv = bond values (current prices) corresponding
  # to all values of yield to maturity in the
  # input vector r
  #
  # INPUT
  # c = coupon payment (semiannual)
  # T = time to maturity (in years)
  # r = vector of yields to maturity (semiannual rates)
  # par = par value
  #
  bv <- c / r + (par - c / r) * (1 + r)^(-2 * T)
  bv
}

price <- 1200 # current price of the bond
C <- 40 # coupon payment
T <- 30 # time to maturity
par <- 1000 # par value of the bond
r <- seq(0.02, 0.05, length = 300)
value <- bondvalue(C, T, r, par)
yield2M <- spline(value, r, xout = price) # spline interpolation

plot(
  r, value,
  xlab = "yield to maturity", ylab = "price of bond", type = "l",
  main = "par = 1000, coupon payment = 40, T = 30", lwd = 2
)
abline(h = 1200)
abline(v = yield2M)
```



From the above plot we can see that the graphical estimate is in very close agreement to the spine interpolation estimate for yield to maturity. Graphically the yield to maturity is near 0.0325 (midway between 0.030 and 0.035) and the value obtained from spline interpolation is 0.0324. Note that the above graph is exactly Figure 3.1, pp. 24.

Problem 2 pp. 38

From the help page of `uniroot` we see that this function finds the solution (or root) of the equation $r^2 - 0.5 = 0$ within the interval $(0.7, 0.8)$. Indeed a value of 0.707107 is obtained form the following code.

```r
uniroot(function(r) r^2 - 0.5, c(0.7, 0.8))$root
```

```
## [1] 0.707107
```

Problem 3 pp. 38

```r
uniroot(function(r) bondvalue(C, T, r, par) - price, c(0.02, 0.05))$root
```

```
## [1] 0.03238059
```

A yield to maturity of 0.03281 semi-annually or 0.064761 annually is obtained.

Problem 4 pp. 38

```r
get_yield2M <- function(C, T, par, price, left = 0.02, right = 0.05) {
  # Function that calls `uniroot` to solve for yield to maturity
  uniroot(function(r) bondvalue(C, T, r, par) - price, c(left, right))$root
```

```r
}
get_yield2M(C = 280, T = 8, par = 10000, price = 9800)
```

```
## [1] 0.02958708
```

A yield to maturity of 0.029587 semi-annually or 0.059174 annually is obtained.

Problem 5 pp. 38

```r
get_yield2M(C = 35, T = 20, par = 1000, price = 1050)
```

```
## [1] 0.03275297
```

A yield to maturity of 0.032753 semi-annually or 0.065506 annually is obtained.

Question 3

Problem 1 pp. 40

(a)
Yield to maturity in 20 years from Equation (3.25) is

$$y_{20} = \frac{1}{20}\int_0^{20} r(t)dt = \frac{1}{20}\int_0^{20}(0.028 + 0.00042t)dt = 0.028 + 0.0042 = 0.0322.$$

(b)
Using (3.22) and (3.23), the price of this bond is

$$P(15) = 1000 \cdot D(15) = 1000\exp(-\int_0^{15}(0.028 + 0.00042t)dt) = 1000\exp(-0.46725) = 626.72.$$

Problem 3 pp. 40

(a)
Bond will be selling above par. If bond were selling below par, then we know coupon rate will be below current yield - which is not the case.

(b)
When the bond is selling above par, the yield to maturity is below current yield. Thus yield to maturity will be below 2.8%.

Remark.
We prove this claim mathematically below. For a yield to maturity y, the bond price is

$$P = \sum_{t=1}^{T}\frac{C}{(1+y)^t} + \frac{\text{PAR}}{(1+y)^T} = \frac{C}{y}\times\left(1 - \frac{1}{(1+y)^T}\right) + \frac{\text{PAR}}{(1+y)^T}.$$

Dividing by P we get the expression for current yield

$$\frac{C}{P} = y \times \frac{1 - \frac{\text{PAR}}{P}\frac{1}{(1+y)^T}}{1 - \frac{1}{(1+y)^T}}.$$

It follows from the above expression that current yield > yield to maturity if and only if $\frac{1-\frac{\text{PAR}}{P}\frac{1}{(1+y)^T}}{1-\frac{1}{(1+y)^T}} > 1$. That is, current yield > yield to maturity if and only if price > PAR.

Problem 4 pp. 40

(a)
5 year continuously compounded rate is

$$y_5 = \frac{1}{5}\int_0^5 r(t)dt = \frac{1}{5}\int_0^5 (0.032 + 0.001t + 0.0002t^2)dt = 0.032 + 0.0025 + 0.001667 = 0.0362.$$

(b)
Price of this bond is

$$PAR \cdot \exp(-5 \cdot y_5) = 834.57 \times PAR.$$

Problem 7 pp. 40

(a)
$C = \frac{0.085}{2} \times 1000 = 42.5$ The coupon payments would be \$42.5 every 6 months.

(b)
The bond is currently worth

$$\sum_{t=1}^{38} \frac{42.5}{(1.038)^t} + \frac{1000}{(1.038)^{38}} = \frac{42.5}{0.038} + \left\{1000 - \frac{42.5}{0.038}\right\}(1.038)^{-38} = 1089.717908.$$

Question 4

Problem 15 pp. 42

$$\begin{aligned}
\frac{d}{d\delta}\sum_{i=1}^{N} C_i \exp\{-T_i(y_{T_i} + \delta)\}\bigg|_{\delta=0} &= -\sum_{i=1}^{N} C_i T_i \exp\{-T_i y_{T_i}\} \\
&= -\sum_{i=1}^{N}\left(\sum_{j=1}^{N} C_j \exp\{-T_j y_{T_j}\}\right)\omega_i T_i \\
&= -\text{DUR}\sum_{j=1}^{N} C_j \exp\{-T_j y_{T_j}\}.
\end{aligned}$$

From the definition of derivative, for some small δ, we have

$$\frac{\sum_{i=1}^{N} C_i \exp\{-T_i(y_{T_i}+\delta)\} - \sum_{i=1}^{N} C_i \exp\{-T_i y_{T_i}\}}{\delta} \approx -\text{DUR} \sum_{j=1}^{N} C_j \exp\{-T_j y_{T_j}\}.$$

So the change in bond price is approximately

$$\text{change bond price} \approx -\delta \text{DUR} \sum_{j=1}^{N} C_j \exp\{-T_j y_{T_j}\}.$$

Equivalently,

$$\frac{\text{change bond price}}{\text{bond price}} \approx -\text{DUR} \times \delta.$$

Question 5

Problem 21 pp. 43

We need to solve the following equation:

$$1015 = \frac{25}{r} + (1000 - \frac{25}{r})(1+r)^{-8}.$$

Using the code from Question 2

```r
C <- 25
T <- 4
par <- 1000
price <- 1015
get_yield2M(C = 25, T = 4, par = 1000, price = 1015)
```

```
## [1] 0.02293077
```

The semiannual yield to maturity of this bond is 2.2931%.