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Honors Math, Fall 2018

A Study of the Normality of Transcendental and Irrational Numbers such as π and e

Since the time of the Ancient Greeks and Archimedes, many people have pondered upon irrational numbers, such as π and *e*. As a matter of fact, the approximation of 3.14 for π has been developed by Archimedes himself, and aided him in the design of his inventions. As long as that approximation existed, people wanted to build upon it. Yet, for a very long time, very few tried to continue calculating π, as there were no tools to help with the calculation of it. It was not until the creation of computers that π's calculation began to skyrocket to the millions of digits, as that could not be done by hand. The current world record stands at about 22 trillion digits, which was achieved by Peter Trueb by a special program run by supercomputers. With so many digits calculated, many have tried and failed to find a pattern in pi. So, most mathematicians think that it, among other constants such as *e*, are normal numbers. A number is known as a normal number when any sequence of its digits appears equally often as they would randomly. Yet, the proof of this remains very elusive, and yet remains to be found. Also, with the many strange occurrences in π, such as the Feynman point, which is a first series of six 9's which appears to occur too early in π, one may think that the digits in π are actually not random, and have some sort of an order. This leads one to the question: is π a normal number? And how can that be shown? The way that I will approach this problem is by using statistics to show that these strange patterns' occurrences in π are in fact random, and by depicting π using multiple graphs in order to demonstrate its normality.

After the Babylonians first noted that the ratio of a circumference of a circle was roughly equal to 3 times the radius of it (one tablet gave an even better approximation, 3.125), there were many advancements made in the calculation of π. One very important way that people calculated π was developed by Archimedes, who created the geometric method for doing so. He would inscribe a circle within a perfect 97-gon, and inscribe another 97-gon in the circle. He would then calculate both figure's areas, and π would lay in between the ratios of their perimeters to their radii. Later, a Chinese mathematician known as Zu Chongzhi would use this method, but with a 24,576-gon. Using this method, he had gotten the fraction 355/113, which yields π correct to 7 digits. No major advancements were made until the 15th century, when an Indian mathematician, by the name of Mahadevan, discovered the Madhava-Leibniz series, which was subsequently used to calculate π to 11 digits. Then, at the end of the 16th century, pi's popularity surged. Also, it was at this time that its symbol began to be used. Also, at about this time, the number *e* was being discovered. Its presence was first noted by John Napier, although its discovery is credited to Jacob Bernoulli, who calculated some of its digits when observing compound interest. This led to its calculation by probability by the Bernoulli trials. At about this time, there appeared a new way of calculating π, which was devised by Georges de Buffon. It was simple: if a needle is dropped on a set of lines, then the probability that it will intersect one of the lines is 2 times the length of the needle, over 2 times the distances from each line times π. This allowed some other mathematicians to perform his experiment, to get the

355/113 approximation. While this was going on, mathematicians such as Van Ceulen kept on calculating π using the geometric method, although new methods began to surface. These involved infinite fractions, much like the more modern formula. While this was taking place, Euler began to calculate e, and he used it in many formulas and identities. He used it in some logarithmic growth and decay functions, although there were other uses for it being discovered. He also was the one who gave the number its name, and proved that it is in fact irrational, as its sequence never terminates. The next two major breakthroughs happened with π. π was proven to be an irrational number by Johann Heinrich Lambert. His argument was simple, which was that that if x is a rational number, then tan x cannot be rational; since tan (π/4) = 1, π/4 cannot be rational, therefore π is irrational. Also, it was shown that π is a transcendental number, which means that it cannot be the root of any polynomial with rational coefficients. These discoveries, among some others, had shown that it was in fact impossible to square a circle, or construct a square which has the same area as a given circle. These discoveries did not stop others from continuing to calculate π and e. By the Second World War, π had been calculated to 707 digits by William Shanks, although only 583 were correct. Also, at around this time, calculators were invented. With the aid of these machines, people could accurately calculate π to as many digits as they wanted to. Soon enough, D. F. Ferguson had discovered the error in Shanks's work, and had calculated π to 808 digits. Only a year later, Levi Smith and John Wrench hit the 1000-digit-mark. Yet, a more powerful machine, ENIAC, was just made, a computer which could get 2000 digits of π in only 3 days. With computers being made, the amount of digits found in π grew exponentially, and the one-million-mark was surpassed in 1973. Soon enough, calculating π became a race, and records would rarely stand for a few years.

As computers grew in power, so did their programs, and so did the digits calculated. While all of these things were going on, there was a need for better data processing and statistics, which led to the usage of the chi-squared test. Such a test was first developed by Pierson in the 1900's, and was increasingly used to determine if something followed a normal distribution (a function which shows random values and the probabilities of them occurring, leading to a bell shape in its graph), and if it had significant variation or skewness in its values. This leads one to see that such a test can be used to see if a number's digits are fully random. That number can be π, or e.

In more modern times, π is used increasingly as a decoration, as its chaos and order can be something which is beautiful. One such example is with the Downsview subway station in Toronto. This station has a mural which looks like a spectrograph, but it is just a representation of π. The artist behind this mural, Arlene Stamp, has had a childhood which was deeply entrenched in both mathematics and art. These two talents would compete with each other, yet she eventually connected them. This led to her going to art school. When making her portfolio, she was inspired by Mandelbrot and his fractals. As stated by the article, fractals are, "mathematical curves and shapes which look just as complicated as they do in their original form." (Peterson, 2002) Fractals were not originally accepted that much, yet they soon began to take over. Mandelbrot himself was able to draw these fractals much better, because he could use computers to display them. This is where the story of Arlene Stamp comes back into play. She was fascinated by these works, and liked to display fractals digitally. On the other hand, she wanted to make an entirely new way of expressing fractals, which could even rival computers. She came up with a system which uses π in order to make the station look beautiful, without exceeding a small budget.

In the next section, the article states that, “In other words, π can never be evaluated exactly by any combination of simple operations involving addition, subtraction, multiplication, or division of positive integers, or the extraction of square roots. Nonetheless, π can be expressed by formulas. For example, as the sum of an infinitely long series: $p = 4/1 - 4/3 + 4/5 - 4/7 + 4/9 - \ldots$. Using such a series, it is possible to calculate π to as many decimal places as desired.”(Peterson, 2002) This heavily relates to my claim, as does the next section of the article. Another important point discussed in the article is how it is debated to whether the distribution of digits in the decimal representation of numbers such as π is an even distribution, where the number of each digit (and sequences of digits) are the same. In order to show this in π, Stamp used rectangles with eight different colors in order to show the non-repeating sequence.

The next number after π which I will be focusing on, the number e, is also known as Euler’s number. This number is especially interesting, because of all of its important uses. According to the article by Baraville, all derivatives of the function $y = e^x$ would be equal to $y = e^x$. Using the Maclaurin series expansion, one would get $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$. Such representations of the number e can be applied in many fields, such as harmonic waves in acoustics, mechanics, electricity, and others. The author then talks about doubling in a formation which is reminiscent of Pascal’s triangle, and he defines this for any value of x, which he displays so that it is comparable to the harmonic vibrations equation, and thus, comparable to e. The author then describes compound interest rates, and how they can be used to create a value for e. For example, “In general compounding times per year, we would get from one dollar, at the end of the year $(1 + \frac{1}{n})^n$ and if n approaches infinity, we get $\lim_{1,\infty} (1 + \frac{1}{n})^n$, which is e.”

(Baraville, 1945, 351) This leads to how most examples of natural growth follow this formula. After this, the author continues to prove that there is a relationship between *e* and the sequence of 2, 4, 8, 16, …. He does this by using a compound fraction, which can be derived from the above equations. This is one such that, "$e = 1 + 1 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{1+\dots}}}}}}}}}$" (Baraville, 1945, 351). After this, the author moves on to visual representations of e, which are characterized by logarithmic spirals. His first figure is just one spiral line, with all construction lines. He expands upon this by adding a second figure with twelve spirals. He then adds orthogonal curves to the edges of those 12 spirals to make a third figure. These orthogonal curves travel to the center much faster than the logarithmic ones, yet that does not always hold true. As the spiral's distance gets smaller, the orthogonal curve gets longer. The author then uses vectors and the orthogonal projections in order to deduce the same differential equation as discussed in the very beginning. The author discusses other cases, which he will simplify to get the same equation, which he can link to the number *e.*

My third source (Peterson, 2001) is mostly about π, although it is partially on *e* and on log(2). The article begins with discussing the randomness of the digits in the decimal representation of π. It appears that the digits 0 through 9 appear equally often through the first 30 million digits of pi. Yet, this is only the first 30 million digits, yet π is a transcendental and an irrational number. The only way in which mathematicians thought that this can be proven rigorously is by using a link between number theory and chaotic dynamics. For example, as the author was explaining the conceptual problem which this represented, "However, like many mathematicians before him, Bailey could not prove that such a random distribution would hold beyond the first 30 million decimal digits of pi. Now, there's a shard of hope that mathematicians may yet lay bare the apparent randomness of π's infinite digits. Bailey and Richard E. Crandall of Reed College in Portland, Ore., have identified a potential link between two disparate mathematical fields-number theory and chaotic dynamics-that they suspect could lead to a proof that every digit occurs with the same frequency when pi is written out ad infinitum" (Peterson, 2001, 136) In the next subsection, previous attempts at showing how π is a normal number are described. At the time that it was written, someone had calculated 200 billion digits of it, and someone else found some consecutive 16 digits in π. After numerous statistical tests were conducted on this data, it was shown that in this sample π was a normal number for base 10. In the next author's contribution, it was discussed how π could have some string of digits in the unknown part of π which would skew it. For example, on page 137, it was stated how, "'It is not even known that all digits appear infinitely often,' Wagon remarks. In the case of pi, for example, no one can yet rule out the possibility that at some point beyond the range of current computations of pi's value, its decimal digits revert to a string constrained to, say, only the digits 1 and 0. If this were so, Wagon points out, it would alter the relative frequency of the digit."(Borwein, 2001, 137) Also, it was stated that there were formulas for calculating digits of π for base 2 and 16, and it was proven that it is normal under those bases. The article ends with the idea that though π is far from proven being normal, a large step has been made by connecting chaotic dynamics and number theory. Yet, this paper uses statistics, and not chaotic dynamics.

In order to illustrate the normality of π, I will do two things: apply the Pearson's chi-squared test to the first 1 million of digits in the decimal representation of the number π, and

I will produce a graph of that particular decimal representation of the number π using a JavaScript and HTML program. The chi-squared test is a test which allows one to either state that the null hypothesis (your assumption at the start) is incorrect, or that it can be accepted as true. So, If the chi-squared (χ^2) value for frequencies of digits in π is greater than a certain critical number for a given significance level (typically 0.1% or 1%), known as the critical value, then it will be shown that there exist non-random patterns in the decimal representation, thus illustrating or showing that π is not a normal number. The method for calculating this chi-square is as follows: sum all entries $\frac{(O-E)^2}{E}$, where O is the observed value of frequency of appearance of a certain digit (or patterns) and E is the expected value assuming totally random equal distribution.

For a particular case of the number of appearances of, say, 0's in the first million of digits of a decimal representation of the number π, one needs to calculate the exact count of 0's appearing (O), subtract the E equal to 1,000,000*1/10=100,000 and divide the result by the 100,000. The same formula goes for all other frequencies of double 0's, triple 0's, etc. If this sum is less than 21.666, the critical chi-squared value for a 1% significance level, then the null hypothesis that the difference (O-E) is negligible is confirmed by the data provided. My null hypothesis is: the difference between the sample frequencies of various digits in the decimal representation of π with a million digits is indistinguisable from a perfectly random result. So, the goal is for the chi-squared values to prove it.

Now, in order to get the frequencies for these digits and some of their combinations, I used an HTML-JavaScript program, which reads a plain-text file with decimal representation of π attached to it. Then, if you specify a single digit for it to find the frequency of, it will find and count all instances of that digit. For a two digit pattern (such as 00), it will find all such instances, including the overlapping ones. We have run the program for frequencies for all appearances of digits between 0 and 9 and the frequencies of their repetitive patterns up to length 5 (such as “99999”). All these frequencies can be found in the Table A. Having this, one can calculate the χ^2 value for each frequency. The expected value for the single digit frequencies (such as “0”) would be calculated by multiplying the total by the amount of digits possible, which is 100,000. Yet, the expected value for doubles (such as “00”) should be 9999.99, because one needs to subtract 1 from 1000000 before multiplying by $(1/10)^2$, because the doubles will have one less at the end of π, which will not fit. Yet, this small difference is negligible. This has to be a subtraction by 2 for the triples (such as “000”), by 3 for quadruples, and by 4 for quintuples, etc. One can look at Table B for the calculated χ^2 values, and their respective sums. Now, one needs to get the critical chi-squared value in order to compare the sums to. In order to get this, one needs to know the degrees of freedom. That number, in my case, is found by subtracting 1 from the amount of columns with data (10), is equal to 9. The critical chi-squared value for a 1% significance value, is equal to 21.666. Then, one needs to compare the sum for each row to this value. All of the sums are smaller than this value, which means that under a p-value of 0.01, my null hypothesis is confirmed. As it also turns out, my null hypothesis is even confirmed even under a 5% significance.

Next, one needs to apply the same logic to the frequencies in e. After using my program on e, I got Table C as a result. Then, I calculated the chi-squared values for each frequency, and summed them for each row. These results can be found in Table D. These results prove to be more challenging, as the chi-squared values are larger than the critical value for 1% significance for the “nn” row. This actual chi-squared value is indeed larger than critical for a 1% significance, but it would be smaller than the critical chi-squared value (27.877) for a 0.1% significance value. From this we still conclude that our zero hypothesis, that the statistical difference between the numbers’ frequencies and their totally random predictions is zero, is to be confirmed for the first million of decimal digits of *e*.

With my zero hypothesis confirmed for both π and e, no sufficient evidence to disprove the idea that they are normal numbers was found in this experiment. So we find that decimal digits observed frequencies are the same as those of a totally random selection of decimal digits. So, if the χ^2 test confirms that the frequencies of the digits in 1000000 iterations of π are similar to that of a random frequency, with the difference being irrelevant, then it can be the case that these 1000000 digits come from a normal number.

The next step is to draw π using a concept borrowed from a “random walk”. A random walk is an artificial discrete 1-dimensional process driven by random fair coin flips. A frog starts at a position 0 at time 0. At the next time 1 the frog will hop to either position +1 or -1 depending on a fair coin flip, say Heads is +1 and Tails is -1. Similarly the frog position is determined at time step 2, etc. Here, however, we would replace the coin flip with the digit in the decimal representation of the number π. Starting at a certain point 0 at time 0, at time 1 we would move by a certain distance (say, 1) at an angle corresponding to the next digit of the decimal representation of number π, such as: 0 corresponds to $36^{\circ} = 360^{\circ}/10$, 1 corresponds to $72^{\circ} = 360^{\circ}/10 * 2$, etc. This is a 2-dimensional generalization of the basic 1-diomensional random walk, where the coin flip is replaced with the digits coming from a number π. My walk begins at the center of a 2000 by 2000 pixel screen. The next point can be in 10 directions from the center, all equally spaced between each other. They will each be assigned a digit in clockwise form, from 0 to 9. Then, 10 random colors will be selected, and the line segment from the starting point and the next will be colored in, according to its endpoint's value. Each segment has a constant, already set length. This will be continued until all of the digits have been read and drawn. Because of the way that I had made my program, this can be done for 1000, 10000, 100000, and 1000000 iterations of the digits of pi. After the program has finished drawing each representation, one would notice that the pictures are very different. For example, if one were to look at the first 1000 digits of pi, it would be uncorrelated to 10000 digits, and if one were to look at 10000 digits, it would look nothing like 100000 digits of pi. This lack of a relationship between these random walks would illustrate how pi's digits are, in fact, appearing randomly. Also, when you look at these drawings, you would notice that each color appears almost equally. This can be noticed for almost all such random walks with enough digits. This occurrence would also point to the normality of pi, as the frequencies of its digits should be equal if it is truly random. Yet, one can notice how in some areas, there are randomly more ones, or twos, or other colors. Yet, they would notice that these areas change as more digits are added, which again adds the idea that π is normal. As well as this, my program can allow this to be done one step at a time, so one can watch as the sequence of lines continues on. This sequence would move around a point, giving you the idea of a spiraling pattern, then it would lurch out, then cycle back again, spin, and again move away. Such movements would seem as a pattern, but would quickly turn into something resembling a true random walk, which would again show how π can be a normal number.

The same exact idea can be applied to other irrational numbers, such as e. With e, the random walk would have a different arrangement of focal points and dispersion, but it would follow the same idea that π would. It would spiral around one thing, then move on to another, then spiral back, then jump somewhere, then jump again, and then go back. Also, its distribution of digits would remain relatively constant throughout, and given one part of the random walk, one would not be able to create the next. Also, if one were to select a small piece from e, it would seem to have corresponding parts in pi. For example, one can look at figures 1, 2, and 3. These show the random walk for e after 100000, 10000, and 1000 digits, respectively. One can see how it begins at 1000, with a shape composed of two clusters. Then, after 10000 steps, it has formed many clusters, and had looped between them. Yet, it had not grown that much, in terms of percentage of the previous area. At 100000, the random walk had grown many times over, and seems to have fewer clusters of points, and is shaped more like a curve. From this little experiment with the drawing of π and e, one can see how its appearance of randomness can serve as something to describe its normality.

In conclusion, my analysis of π and e could not show that they are not normal numbers. Based on the chi-squared test, both numbers appear to have the same frequencies at those which a normal number should have. Also, after constructing a random walk, one can notice the randomness of the figures created, and how they can emulate the idea that π and e are normal numbers. Yet, a rigorous proof for the normality of these two numbers has not been constructed, and still remains elusive, but statistically speaking, the first million digits of π and e appear to follow the guidelines of a normal number. With this in mind, it is very possible and probable that these numbers are indeed normal, yet my experiment fails to provide any kind of rigorous proof for this, although my data seems to support this idea. There are many ways that my experiment could be expanded upon. For example, I could have calculated the frequencies and chi-squared values for 1 billion digits of π and *e.* This would have allowed me to gain much more accurate and representative data, as there are 1000 times more values in the samples. Also, I could have statistically compared π and *e*. This would serve as something else to help show that they are normal numbers, because if the statistical difference is insignificant, then their frequencies are similar to each other, which should be the case if they are both normal numbers. Also, many questions have arisen from my research, as this is an entirely new topic for me. I do not know how to actually prove the normality of numbers, such as pi, as I can only show that is is likely for a section of the number. Also, I am wondering how the chi-squared test will react to the frequencies of numbers in a cyclic fraction, such as 23/7, based on different amounts of digits from the fraction. Also, I am wondering what will happen to that fraction if you draw its "random walk," and how it will look like. Finally, I am wondering how will the one billion digit random walk look like for pi? How about a 1 trillion digit random walk? All of these questions can be answered by providing a similar inquiry, as well as improvements in the code, in order to make the calculations less susceptible to human error.

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Appendix

Table A: The frequency of digits and numbers appearing in π

Frequency	0	1	2	3	4	5	6	7	8	9
n	99957	99758	100026	100230	100230	100359	99548	99800	99985	100106
nn	9938	10064	10062	10026	9958	10232	9819	9801	10088	10084
nnn	967	1051	1010	996	949	1040	937	984	983	1003
nnnn	96	120	101	104	81	103	100	103	101	115
nnnnn	6	16	9	10	3	16	15	11	8	10

Table B: Chi-squared values for numbers appearing in π, rounded to the nearest millionth

χ^2	0	1	2	3	4	5	6	7	8	9	Sum
n	0.018490	0.585640	0.006760	0.529000	0.529000	1.288810	2.043040	0.400000	0.002250	0.112360	**5.515350**
nn	0.384276	0.409728	0.384524	0.067652	0.176316	5.382869	3.275741	3.959706	0.774577	0.705769	**15.521160**
nnn	1.088870	2.601209	0.100040	0.015984	2.600801	1.600163	3.968756	0.255937	0.288933	0.009012	**12.529705**
nnnn	0.159976	4.000132	0.010006	0.160024	3.609897	0.090018	0.000000	0.090018	0.010006	2.250097	**10.380175**
nnnnn	1.599974	3.600062	0.099992	0.000000	4.899964	3.600062	2.500050	0.100008	0.399986	0.000000	**16.800099**

Table C: frequencies of digits and numbers in *e*

d'sValue →	0	1	2	3	4	5	6	7	8	9
n	99425	100132	99846	100228	100389	100089	100479	99910	99813	99691
nn	9855	10113	10039	10129	10033	9752	10036	9978	9883	9668
nnn	983	993	1054	1028	970	952	992	980	946	935
nnnn	92	99	111	116	107	73	104	101	105	71
nnnnn	16	9	8	12	11	10	7	8	13	7

Table D: χ^2 values and sums for *e*

χ^2	0	1	2	3	4	5	6	7	8	9	**sum**
n	3.306250	0.174240	0.237160	0.519840	1.513210	0.079210	2.294410	0.081000	0.349690	0.954810	**9.509820**
nn	2.102212	1.277127	0.152178	1.664360	0.108966	6.149910	0.129672	0.048356	1.368667	11.021747	**24.023196**
nnn	0.288933	0.048972	2.916222	0.784114	0.899882	2.303813	0.063968	0.399921	2.915790	4.224748	**14.846362**
nnnn	0.639954	0.009994	1.210070	2.560104	0.490043	7.289860	0.160024	0.010006	0.250031	8.409851	**21.029937**
nnnnn	3.600062	0.099992	0.399986	0.400018	0.100008	0.000000	0.899980	0.399986	0.900028	0.899980	**7.700039**

Figure 1: 100,000 digits of *e*

Figure 2: 10,000 digits of *e*

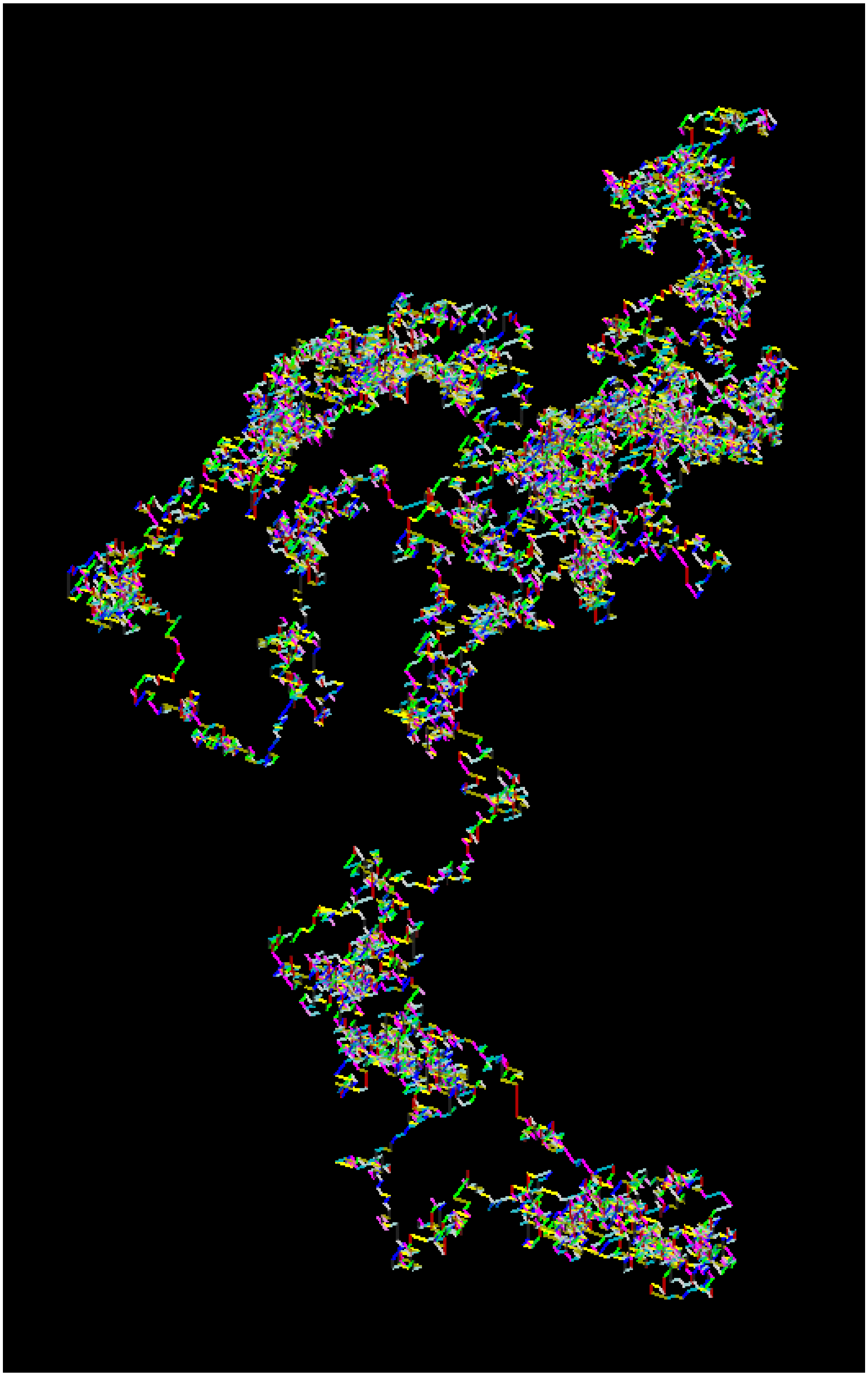

Figure 3: 1,000 digits of *e*

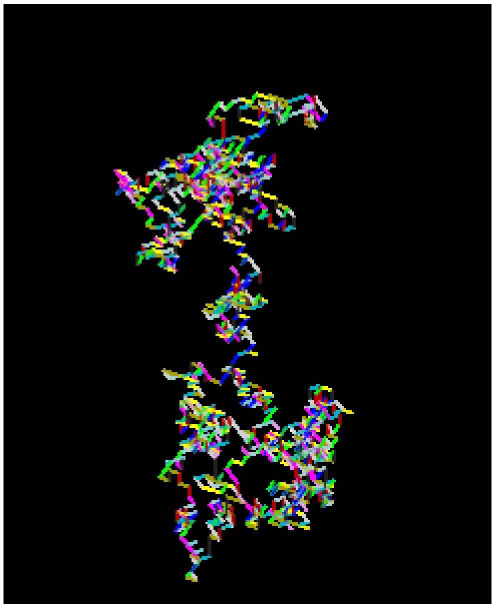

Figure 4: 100,000 digits of π

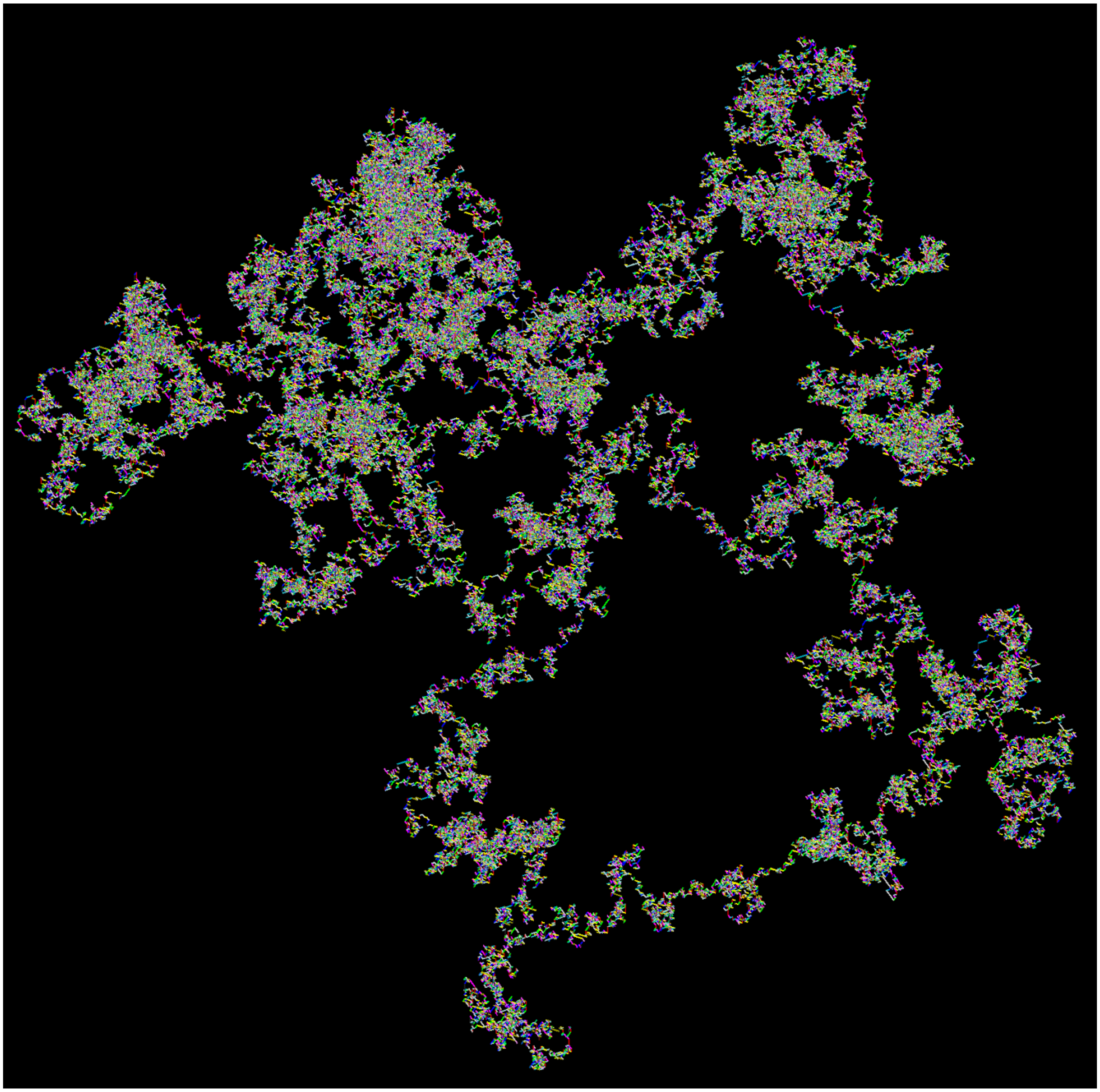

Figure 5: 10,000 digits of pi

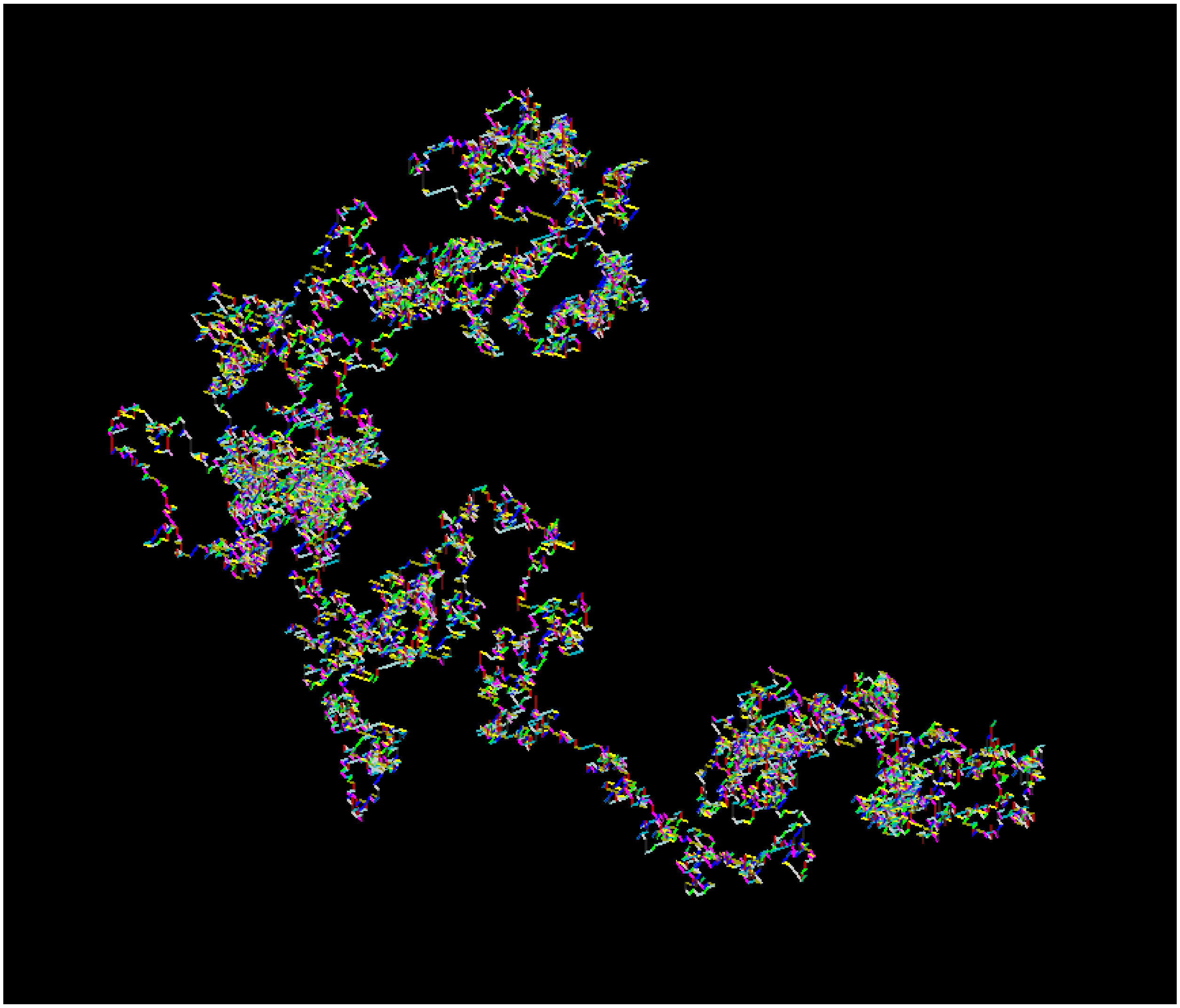

Figure 6: 1,000 digits of π

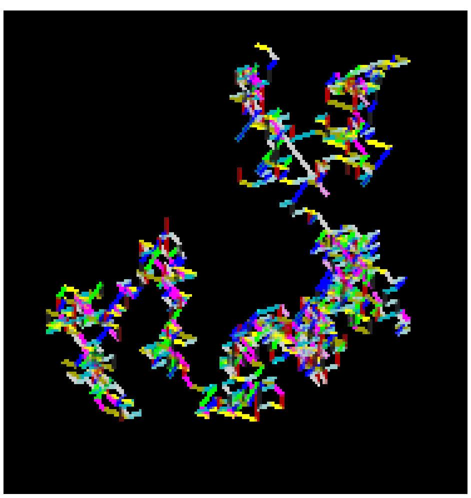

Figure 7: 1,000,000 digits of π