# Distribution

Econ 361 Week 2

# Joint, Marginal, Conditional

- (Joint distribution, marginal distribution, conditional distribution) are all types of probability mass function (pmf) or probability density function (pdf) – depending on whether the underlying random variables are continuous or discrete
- When the random variables are all discrete, these "distributions" can be interpreted as probabilities
  - e.g. the joint distribution of (X, Y) when (X, Y) are both discrete random variables provides the probability of the joint event (X = x, Y = y)

    $= P_{XY}(x, y) = P(X = x, Y = y)$
  - e.g. similarly, the conditional distribution of Y given X = 2 provides the probability of the conditional event (X = 2, Y = y)

    $= P(Y = y \mid X = 2) = P(X = 2, Y = y) / P_X(2) = P(X = 2, Y = y) / P(X = 2)$

# Two Dice Roll Experiment

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | **Second Die Roll (Y)** | | | | | | | |
| First Die Roll (**X**) | | "Roll 1" | "Roll 2" | "Roll 3" | "Roll 4" | "Roll 5" | "Roll 6" | |
| | "Roll 1" | | | | | | | |
| | "Roll 2" | | | | | | | |
| | "Roll 3" | | | | | | | |
| | "Roll 4" | | | | | | | |
| | "Roll 5" | | | | | | | |
| | "Roll 6" | | | | | | | |
| | | | | | | | | |

36 Atomistic, mutually exclusive events ("simple" events) that span the Sample Space

Introduce two random variables, (**X**, **Y**), to characterize possible outcomes of this random experiment

**X** maps each of the 36 events in the Sample Space to the number rolled by first die

**Y** maps each of the 36 events in the Sample Space to the number rolled by second die

# Two (*Fair*) Dice Roll Experiment
# Joint Distribution of (X, Y)

| | | Second Die Roll (Y) | | | | | | |
|---|---|---|---|---|---|---|---|---|
| First Die Roll (**X**) | | "Roll 1" | "Roll 2" | "Roll 3" | "Roll 4" | "Roll 5" | "Roll 6" | |
| | "Roll 1" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 2" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 3" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 4" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 5" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 6" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | | | | | | | | |

36 Atomistic, mutually exclusive events ("simple" events) that span the Sample Space

Introduce two random variables, (**X**, **Y**), to characterize possible outcomes of this random experiment

- **X** maps each of the 36 events in the Sample Space to the number rolled by first die
- **Y** maps each of the 36 events in the Sample Space to the number rolled by second die

# Two (*Fair*) Dice Roll Experiment Marginal Distribution of Y

| | | Second Die Roll (Y) | | | | | | |
|---|---|---|---|---|---|---|---|---|
| First Die Roll (**X**) | | "Roll 1" | "Roll 2" | "Roll 3" | "Roll 4" | "Roll 5" | "Roll 6" | |
| | "Roll 1" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 2" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 3" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 4" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 5" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 6" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | | **1/6** Y = 1 | **1/6** Y = 2 | **1/6** Y = 3 | **1/6** Y = 4 | **1/6** Y = 5 | **1/6** Y = 6 | |

Marginal distribution ***"sums across"*** the relevant evaluations of the joint distribution

For continuous random variables, often refer to ***"integrating out"*** the other random variables in the relevant joint distribution

# Two (*Fair*) Dice Roll Experiment
# Marginal Distribution of X

| | Second Die Roll (Y) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | "Roll 1" | "Roll 2" | "Roll 3" | "Roll 4" | "Roll 5" | "Roll 6" | |
| First Die Roll (**X**) | "Roll 1" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | **1/6 X = 1** |
| | "Roll 2" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | **1/6 X = 2** |
| | "Roll 3" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | **1/6 X = 3** |
| | "Roll 4" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | **1/6 X = 4** |
| | "Roll 5" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | **1/6 X = 5** |
| | "Roll 6" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | **1/6 X = 6** |
| | | | | | | | | |

Marginal distribution ***"sums across"*** the relevant evaluations of the joint distribution

For continuous random variables, often refer to ***"integrating out"*** the other random variables in the relevant joint distribution

# Two (*Fair*) Dice Roll Experiment Conditional Distribution (1/2)

| | | Second Die Roll (Y) | | | | | | |
|---|---|---|---|---|---|---|---|---|
| First Die Roll (**X**) | | "Roll 1" | "Roll 2" | "Roll 3" | "Roll 4" | "Roll 5" | "Roll 6" | |
| | "Roll 1" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | **1/6 X = 1** |
| | "Roll 2" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | **1/6 X = 2** |
| | "Roll 3" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | **1/6 X = 3** |
| | "Roll 4" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | **1/6 X = 4** |
| | "Roll 5" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | **1/6 X = 5** |
| | "Roll 6" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | **1/6 X = 6** |
| | | **1/6 Y = 1** | **1/6 Y = 2** | **1/6 Y = 3** | **1/6 Y = 4** | **1/6 Y = 5** | **1/6 Y = 6** | |

Conditional probability restricts the relevant Sample Space and then reconsider the relative frequency within this restricted Sample Space

Example: P(X | Y = 3) "Conditional distribution of X given Y = 3"

# Conditional Probability “Definition”

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \qquad \text{for } P(B) > 0$$

For our example: B is event corresponding to Y = 3

A is event corresponding to X = x

# Two (*Fair*) Dice Roll Experiment Conditional Distribution (2/2)

| | Second Die Roll (Y) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| First Die Roll (**X**) | | “Roll 1” | “Roll 2” | “Roll 3” | “Roll 4” | “Roll 5” | “Roll 6” | |
| | “Roll 1” | | | (1/36) / (1/6) = 1/6 | | | | **1/6 X = 1** |
| | “Roll 2” | | | = 1/6 | | | | **1/6 X = 2** |
| | “Roll 3” | | | = 1/6 | | | | **1/6 X = 3** |
| | “Roll 4” | | | = 1/6 | | | | **1/6 X = 4** |
| | “Roll 5” | | | = 1/6 | | | | **1/6 X = 5** |
| | “Roll 6” | | | = 1/6 | | | | **1/6 X = 6** |
| | | **1/6 Y = 1** | **1/6 Y = 2** | **1/6 Y = 3** | **1/6 Y = 4** | **1/6 Y = 5** | **1/6 Y = 6** | |

Conditional probability restricts the relevant Sample Space and then reconsider the relative frequency within this restricted Sample Space

Example: $P(X = x \mid Y = 3) = P(X = x, Y = 3) / P(Y = 3) = (1/36) / (1/6) = 1/6$

# More than 2 Random Variables

- Suppose the random experiment consisted of ***three*** die rolls
  - Let Z be the die roll that maps the simple event in the Sample Space to the number rolled by the third die
- The concepts of joint/marginal/conditional distributions generalize to the case of N random variables (where N is some positive integer > 1)
  - To get joint distribution of (X, Y), sum across the relevant realizations of the joint distribution (X, Y, Z) – i.e. “integrate out” Z
  - The conditional distribution of (X, Y) given Z = 2 can be obtained from taking the ratio of the appropriate evaluations of the joint distribution (X, Y, Z) and the marginal distribution of Z evaluated at Z = 2

# Transformations of Random Variables

- Let’s go back to the two die roll random experiment
- Instead of a third die roll, consider a third random variable that is a transformation of the other two random variables (X, Y*)*
- Say W = X + Y
  - W is the random variable that maps the simple events in the Sample Space to the sum of the two die rolls
- Knowing the joint distribution of (X, Y), can we arrive at the marginal distribution of W?

# Two (*Fair*) Dice Roll Experiment Marginal Distribution of W = X + Y (1/2)

| | | **Second Die Roll (Y)** | | | | | | |
|---|---|---|---|---|---|---|---|---|
| First Die Roll (**X**) | | "Roll 1" | "Roll 2" | "Roll 3" | "Roll 4" | "Roll 5" | "Roll 6" | |
| | "Roll 1" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 2" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 3" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 4" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 5" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 6" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | | | | | | | | |

Consider the event W = 7

As W is a transformation of (X, Y), consider the joint events (X = x, Y = y) that correspond to (W = 7)

- $P_W(7)$ = P(W = 7) = sum of all the P(X = x, Y = y) such that x + y = 7
- $P_W(7)$ = P(W = 7) = sum of all the P(X = x, Y = y) such that x + y = 7 = 1/36 * 6 = 1/6

Note: (X, Y, Z) all have the same domain (same Sample Space as same random experiment)

# Two (*Fair*) Dice Roll Experiment Marginal Distribution of W = X + Y (2/2)

| | | **Second Die Roll (Y)** | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | "Roll 1" | "Roll 2" | "Roll 3" | "Roll 4" | "Roll 5" | "Roll 6" | |
| First Die Roll (**X**) | "Roll 1" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 2" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 3" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 4" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 5" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | "Roll 6" | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | |
| | | | | | | | | |

Consider the event W = 4

As W is a transformation of (X, Y), consider the joint events (X = x, Y = y) that correspond to (W = 4)

- $P_W(4) = P(W = 4)$ = sum of all the $P(X = x, Y = y)$ such that $x + y = 4$
- $P_W(4) = P(W = 4)$ = sum of all the $P(X = x, Y = y)$ such that $x + y = 4 = 1/36 * 3 = 1/12$

Note: There are some realizations of X for which W = 4 is impossible (similarly some realizations of Y)

# What about continuous random variables?

- (Joint, marginal, conditional) distributions may no longer be interpreted as probabilities; they must be interpreted as densities
- For continuous random variables, the respective values of some (joint, marginal, conditional) distribution for two events that are "adjacent" to each other, say $X = x$ and $X = x + \varepsilon$ , are usually connected ,
- This allows us to "integrate" where we had previous summed
  - Think "Riemann sum" interpretation of integrals

# Continuous Random Variables

- For continuous random variables, helps to “graph” the (joint, marginal, conditional) distributions

- Most famous example: the “Bell Curve” for the marginal distribution of X when X is distributed Normal

- Shape of the marginal distribution provides some insights
  - Where is the distribution centered?  Symmetric?  Modality?  Fat or think tails?