Ex 1:

a) $L_n(\theta) = \prod_{i=1}^{n} f(X_i;\theta) = (2\pi)^{-\frac{n}{2}} \prod_{i=1}^{n} X_i^{-\frac{3}{2}} e^{-\frac{1}{2\theta^2}\sum X_i + \frac{n}{\theta} - \frac{1}{2}\sum\frac{1}{X_i}}$ ✓

$\log L_n(\theta) = -\frac{n}{2}\log(2\pi) - \frac{3}{2}\sum \log X_i - \frac{1}{2\theta^2}\sum X_i + \frac{n}{\theta} - \frac{1}{2}\sum \frac{1}{X_i}$ ✓

$\frac{d}{d\theta}\log L_n(\theta) = \frac{1}{\theta^3}\sum X_i - \frac{n}{\theta^2}$ ✓ $= 0$ ✓

$\Longleftrightarrow \quad \theta = \hat{\theta}_n = \frac{1}{n}\sum X_i = \overline{X}_n$ ✓

$\frac{d^2}{d\theta^2}\log L_n(\theta) = -\frac{3}{\theta^4}\sum X_i + \frac{2n}{\theta^3}$ ✓ $\overset{\text{at } \theta = \overline{X}_n}{=} -\frac{3n\overline{X}_n}{\overline{X}_n^4} + \frac{2n}{\overline{X}_n^3}$

$= -\frac{n}{\overline{X}_n^3} < 0$ ✓

$\Longrightarrow \quad \hat{\theta}_n = \overline{X}_n$ is MLE for θ.

b) $\mathbb{E}[\hat{\theta}_n] = \mathbb{E}[X_1] = \theta \Longrightarrow \hat{\theta}_n$ unbiased ✓

$Var(\hat{\theta}_n) = \frac{1}{n}Var(X_1) = \frac{\theta^3}{n}$ ✓

$I_n(\theta) = Var(\dot{\ell}_n(X_n;\theta)) = Var\left(\frac{1}{\theta^3}\sum X_i - \frac{n}{\theta^2}\right) = \frac{1}{\theta^6}\sum Var(X_i)$

$= \frac{n}{\theta^3}$ ✓

Cramér-Rao: Every unbiased estimator T of θ satisfies $Var(T) \geq \frac{1}{I_n(\theta)} = \frac{\theta^3}{n}$ ✓

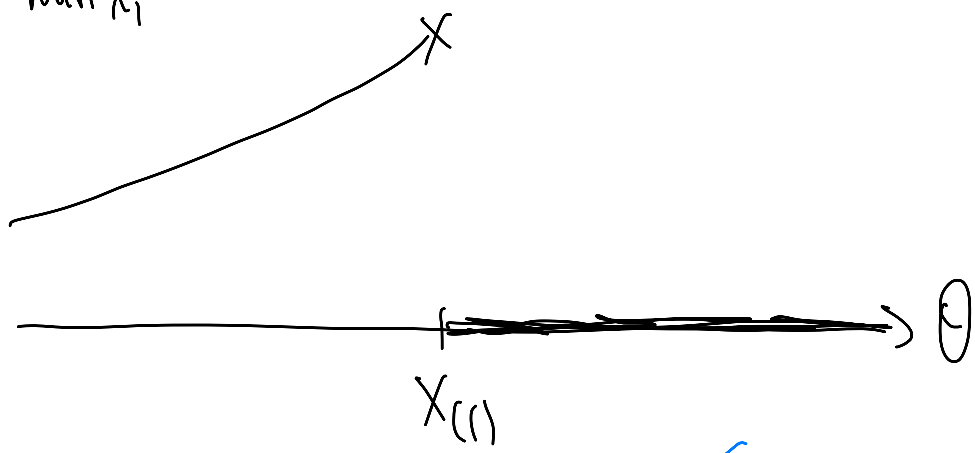
$\Longrightarrow \quad \overline{X}_n$ is MVUE for θ.

Ex 2:

a) $L_n(\theta) = 3^n \theta^{3n} \prod_{i=1}^{n} x_i^{-4} \prod_{i=1}^{n} I_{[\theta,\infty)}(x_i)$

$x_i \geq \theta \Leftrightarrow \theta \leq x_i$ $\rightarrow = 3^n \theta^{3n} \prod_{i=1}^{n} x_i^{-4} \prod_{i=1}^{n} I_{[0,x_i]}(\theta)$

$\theta \leq x_i$ for all i $\Leftrightarrow \theta \leq \min x_i$ $\rightarrow = u(\underline{x}_n)\, \theta^{3n}\, I_{[0,x_{(1)}]}(\theta)$



$\Rightarrow$ MLE is $\hat{\theta}_n = X_{(1)}$.

b) $E[X] = \int_{\theta}^{\infty} x \times 3\theta^3 x^{-4}\, dx = 3\theta^3 \int_{\theta}^{\infty} x^{-3}\, dx$

$= 3\theta^3 \frac{\theta^{-2}}{2} = \frac{3}{2}\theta$

$\Rightarrow \theta = \frac{2}{3}\mu_1 \Rightarrow \hat{\theta}_{MOM} = \frac{2}{3}\overline{X}_n$

c) $L_n(\theta) = \underbrace{u(\underline{x}_n)}_{u(\underline{x}_n)} \times \underbrace{\theta^{3n} I_{[0,\overset{r(\underline{x}_n)}{x_{(1)}}]}(\theta)}_{v(r(\underline{x}_n);\,\theta)}$

$\Rightarrow$ $X_{(1)}$ is a sufficient statistic ✓

$\hat{\theta}_{MOM}$ is not a function of $X_{(1)}$ ✓

$\xRightarrow[\text{Blackwell}]{\text{Rao-}}$ There is an estimator that makes $\hat{\theta}_{MOM}$ inadmissible. ✓

Ex 3:

a) $X_1, \dots, X_n \overset{iid}{\sim} \text{Gamma}(2, \beta)$

$$\Rightarrow \sum X_i \sim \text{Gamma}(2n, \beta) \checkmark$$

$$\Rightarrow \underbrace{2\beta \sum X_i}_{\text{pivot!}} \sim \text{Gamma}\left(2n, \tfrac{1}{2}\right) \checkmark = \chi^2_{4n} \checkmark$$

Thus:

$$\mathbb{P}\left(q(\chi^2_{4n}; \alpha) \leq 2\beta \sum X_i \right) = 1 - \alpha \checkmark$$

$$\Leftrightarrow \mathbb{P}\left(\beta \geq \frac{q(\chi^2_{4n}; \alpha)}{2 \sum X_i} \right) = 1 - \alpha \checkmark$$

Hence: $\left(\frac{q(\chi^2_{4n}; \alpha)}{2 \sum X_i}, \infty \right)$ is a $(1-\alpha)$-CI for β. ✓

b) $$\frac{q(\chi^2_{4n}; \alpha)}{2 \sum X_i} = \frac{q(\chi^2_{40}; 0.05)}{2n \overline{X}_n} = \frac{26.51}{20 \times 4.2} \checkmark = 0.3156 \checkmark$$

Ex 4:

a) $\mathbb{E}[X_1] = \frac{2+\theta}{2} = 1+\frac{\theta}{2}$, $\overline{X}_n \xrightarrow{\mathbb{P}} \mathbb{E}[X_1]$ (LLN)

$\Rightarrow \log \overline{X}_n \xrightarrow{\mathbb{P}} \log \mathbb{E}[X_1] = \log\left(1+\frac{\theta}{2}\right) = g(\theta)$

b) CLT: $\sqrt{n}\left(\overline{X}_n - \left(1+\frac{\theta}{2}\right)\right) \xrightarrow{d} N(0, \underbrace{Var(X_1)}_{=\frac{\theta^2}{12}})$

δ-method $\Rightarrow$

$g(x) = \log x$

$g'(x) = \frac{1}{x}$

$$\sqrt{n}\,\frac{g(\overline{X}_n) - g\left(1+\frac{\theta}{2}\right)}{g'\left(1+\frac{\theta}{2}\right)} \xrightarrow{d} N\left(0, \frac{\theta^2}{12}\right)$$

$$\Rightarrow g(\overline{X}_n) = \log \overline{X}_n \overset{approx}{\sim} N\left(g\left(1+\frac{\theta}{2}\right), \frac{\theta^2\left(g'\left(1+\frac{\theta}{2}\right)\right)^2}{12n}\right)$$

$$= N\left(\log\left(1+\frac{\theta}{2}\right), \frac{\theta^2}{12n\left(1+\frac{\theta}{2}\right)^2}\right)$$