Chapter 19: Risk Management

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Need for Risk Management

The financial market has always been risky considering the extremal events such as equity market crash in 1987, failure in bond markets in Q1 1994, Russian financial crisis in 1998, technology bubble burst in 2000, subprime mortgage crisis in 2007 and 2008.

Type of Risk

1. Market risk: the risk of loss arising from changes in the value of tradable or traded assets.
2. Credit risk: the risk of loss due to the failure of the counterparty to pay the promised obligation.
3. Liquidity risk: the risk of loss arising from the inability either to meet payment obligation (funding liquidity risk) or to liquidate positions with little price impact (asset liquidity risk).
4. Operational risk: the risk of loss caused by inadequate or failed internal processes, people and systems, or external events.

Risk metrics

Risk metrics:

1. Variance/volatility (measuring the average risk)
2. Value-at-Risk or VaR (measuring the magnitude of extreme event)
3. Expected-Shortfall or ES (measuring the average loss of extreme event)

Usage:

1. Variance/volatility: for portfolio diversification
2. Value-at-Risk or VaR: for exiting strategies
3. Expected-Shortfall or ES: to control the draw-downs

VaR for single stock

- The Value at Risk (VaR) measures the potential loss in value of a risky asset or portfolio over a defined period for a given confidence interval.
- Example 1: if the VaR on an asset is \$ 100 million at a horizon of one-week and 95% confidence level, there is a only a 5% chance that the value of the asset will drop more than \$ 100 million over any given week.
- Example 2: if the horizon is one week, the confidence coefficient is 99 % (so $\alpha = 0.01$), and the VaR is \$5 million, then there is only a 1% chance of a loss exceeding \$5 million over the next week.
- In this chapter, we will learn how to estimate VaR and use the bootstrap method to construct a CI for it.

Value at Risk (VaR): Introduction

- Value at Risk refers to the maximum loss,which should not be exceeded during a specified period of time with a given probability level.
- VaR uses two parameters,
  - the time horizon denoted by T
  - the confidence level denoted by $1 - \alpha$
- Given these, the VaR is a bound such that the loss over the horizon is less than this bound with probability equal to the confidence coefficient. or equivalently

$$P\{L > VaR(\alpha, T)\} = \alpha$$

I will give a formal definition of loss next

1. For fixed portfolio $\omega = (\omega_1, \omega_2, \ldots, \omega_N)^T$, the loss function L is a random variable that takes positive values when a loss is incurred, and negative ones when a gain occurs.
2. For a stock, If longing, the loss over a horizon $T = 1$ (could be one week, one day, one month ect) is
$$L_t = S_{t-1} - S_t = -R_t \times S_{t-1} = (1 - \exp(r_t)) \times S_{t-1}$$
3. If shorting a stock,
$$L_t = S_t - S_{t-1} = R_t \times S_{t-1} = (\exp(r_t) - 1) \times S_{t-1}$$

- We sometimes use the notation VaR(α) or Var(α, T) to indicate the dependence of VaR on α or on both α and the horizon T .
- Usually, VaR(α) is used with T being understood.
- the time horizon denoted by T
- the confidence level denoted by $1 - \alpha$

Assume that the return R_t on a single stock follows $N(\mu, \sigma^2)$.

The formula of one-step prediction of the VaR if longing a stock, then

$$\begin{aligned}
\alpha = P(L_t > VaR_t(\alpha)) &= P(-R_t S_{t-1} > VaR_t(\alpha)) \\
&= P\left(R_t < \frac{-VaR_t(\alpha)}{S_{t-1}}\right) \\
&= P\left(Z < \frac{\frac{-VaR_t(\alpha)}{S_{t-1}} - \mu}{\sigma}\right)
\end{aligned}$$

This implies that

$$\frac{\frac{-VaR_t(\alpha)}{S_{t-1}} - \mu}{\sigma} = \Phi^{-1}(\alpha) = z_\alpha$$

and

$$VaR_t(\alpha) = -S_{t-1} \times (\mu + z_\alpha \times \sigma)$$

Here z_α is the α lower-quantile of a standard normal r.v. For instance, $z_{0.025} = -1.96$.

VaR for single stock

Assume that the return R_t on a single stock follows $N(\mu, \sigma^2)$.

The formula of one-step prediction of the VaR if shorting a stock, then

$$\begin{aligned}\alpha = P(L_t > VaR_t(\alpha)) &= P(R_t S_{t-1} > VaR_t(\alpha)) \\ &= P\left(R_t > \frac{VaR_t(\alpha)}{S_{t-1}}\right) \\ &= P\left(Z > \frac{\frac{VaR_t(\alpha)}{S_{t-1}} - \mu}{\sigma}\right)\end{aligned}$$

This implies that

$$\frac{\frac{VaR_t(\alpha)}{S_{t-1}} - \mu}{\sigma} = \Phi^{-1}(1-\alpha) = z_{1-\alpha} = -z_\alpha$$

and

$$VaR_t(\alpha) = S_{t-1} \times (\mu - z_\alpha \times \sigma)$$

Here z_α is the α lower-quantile of a standard normal r.v. For instance, $z_{0.025} = -1.96$.

- The expected shortfall and the abbreviation ES is defined as

$$
\begin{aligned}
ES_t(\alpha) &= E(L_t | L_t \geq VaR_t(\alpha)) \\
&= \frac{\int_0^{\alpha} VaR_t(u)du}{\alpha}
\end{aligned}
$$

- $ES_t(\alpha)$ is the average of $VaR_t(u)$ over all u that are less than or equal to α.

Proof: We make use of the following: If X has pdf f(x) then, given $X > A$, X has pdf

$$g(x) = \begin{cases} \frac{f(x)}{P(X>A)}, & x > A \\ 0, & \text{otherwise} \end{cases}$$

Let F_{L_t} be the distribution of L_t. We have

$$\begin{aligned} ES_t(\alpha) = E(L_t|L_t \geq VaR_t(\alpha)) &= \frac{\int_{VaR(\alpha)}^{\infty} \ell dF_{L_t}(\ell)}{P(L_t \geq VaR_t(\alpha))} \\ &= -\frac{\int_{VaR(\alpha)}^{\infty} \ell d(1 - F_{L_t}(\ell))}{P(L_t \geq VaR_t(\alpha))} \end{aligned}$$

let $u = (1 - F_{L_t}(\ell))$ so that $du = d(1 - F_{L_t}(\ell))$ and $\ell = P(L_t > u) = VaR_t(u)$

$$\int_{VaR_t(\alpha)}^{\infty} \ell d(1 - F_{L_t}(\ell)) = \int_{\alpha}^{0} VaR_t(u)du$$

and as a result

$$ES_t(\alpha) = -\frac{\int_{VaR_t(\alpha)}^{\infty} \ell d(1 - F_{L_t}(\ell))}{P(L_t \geq VaR_t(\alpha))} = \frac{\int_0^{\alpha} VaR_t(u)du}{\alpha}$$

- If $R \sim N(0.04, (0.18)^2)$, and in a long position at $t = 0$ is $S_0 = \$100,000$ for T=one year, what is the value of $VaR_1(0.05)$.
- Answer:

$$VaR_1(\alpha) = -S_0(\mu + z_\alpha \sigma) = -100000(0.04 - 1.645(0.18)) = 25610.$$

- VaR depends heavily on α
- In applications, risk measures will rarely, if ever, be known exactly as in this simple example. Instead, risk measures are estimated, and estimation error is another source of uncertainty. This uncertainty can be quantified using a confidence interval for the risk measure. We turn next to these topics.

- We assume $t = 0$ in the rest and use $VaR(\alpha)$ to denote $VaR_1(\alpha)$.
- In this case, the loss distribution is not assumed to be in a parametric family such as the normal or t-distributions.
- Suppose we want a confidence coefficient of $1 - \alpha$ for the risk measures. Therefore, we estimate the α-quantile of the return distribution, which is the α -upper quantile of the loss distribution.
- In the nonparametric method, this quantile is estimated as the α-quantile of a sample of historic returns, which we will call $\hat{q}(\alpha)$.
- If S_0 is the size of the current position, then the nonparametric estimate of VaR is

$$VaR^{np}(\alpha) = -S_0 \times \hat{q}(\alpha)$$

- (superscripts and subscripts will sometimes be placed on VaR and ES to provide information. Here, the superscript np means nonparametrically estimated)

Nonparametric Estimation of VaR and ES

To estimate ES, let $R_1, R_2, \ldots, R_n$ be the historic returns and define $L_i = -S_0 \times R_i, i = 1, 2, \ldots, n$. Then

$$\widehat{ES}^{np}(\alpha) = \frac{\sum_{i=1}^{n} L_i I[L_i > \widehat{VaR}(\alpha)]}{\sum_{i=1}^{n} I[L_i > \widehat{VaR}(\alpha)]}$$

which is the average of all L_i exceeding $\widehat{VaR}(\alpha)$. Here

$$I[L_i > \widehat{VaR}(\alpha)] = \begin{cases} 1 & \text{if } L_i > \widehat{VaR}(\alpha) \\ 0 & \text{otherwise} \end{cases}$$

Example

- Suppose you hold S=$20,000 position in as an S&P 500 index fund (your returns are those of this index) and you want a 24 hour VaR.
- We use the 2783 daily returns of the S&P 500 the data set SP500 in R's Ecdat package.
- Using quantile(r500, probs=c(0.05)), the 5th percentile is -0.01513775
- and the $\widehat{VaR}^{np}(0.05) = 20000(0.01513775) = 302.755$ and $\widehat{ES}^{np}(0.05) = 391.23$.
- The command I used in R to compute it is -20000*sum(r500[r500< -0.01513775])/length(r500[r500<-0.01513775])

- Let $F(y|\theta)$ be a parametric family of distributions used to model the return distribution and suppose that $\hat{\theta}$ is an estimate of θ, such as, the MLE computed from historic returns. Then $F^{-1}(\alpha|\hat{\theta})$ is an estimate of the $\alpha-$quantile of the return distribution and
- A a parametric estimate of $VaR(\alpha)$ is

$$\widehat{VaR}^{par}(\alpha) = -S_0 \times F^{-1}(\alpha|\hat{\theta}).$$

As before, S_0 is the size of the current position

- Let $f(y|\theta)$ be the density of $F(y|\theta)$. Then the estimate of expected shortfall

$$\widehat{ES}^{par}(\alpha) = -\frac{S_0}{\alpha}\int_0^{F^{-1}(\alpha|\hat{\theta})} uf(u|\hat{\theta})du$$

- Suppose the return has a t-distribution with mean equal to μ, scale parameter equal to λ, and ν degrees of freedom. Let f_ν and F_ν be, respectively, the t-density and t-distribution function with ν degrees of freedom. Then

$$VaR^t(\alpha) = -S_0 \times \{\mu + \lambda F_\nu^{-1}(\alpha)\}.$$

- The expected shortfall is

$$ES^t(\alpha) = S_0 \times \left\{-\mu + \lambda \left(\frac{f_\nu(F_\nu^{-1}(\alpha))}{\alpha} \left[\frac{\nu + \{F_\nu^{-1}(\alpha)\}^2}{\nu - 1} \right] \right) \right\}$$

- For the normal distribution

$$ES^{norm}(\alpha) = S_0 \times \left\{-\mu + \sigma \left(\frac{\phi((\Phi^{-1}(\alpha)}{\alpha} \right) \right\}$$

- In practice, we estimate the parameter (using say the maximum likelihood estimation technique) and the use the estimates in the formulas above to get estimates for VaR and ES.

Confidence Intervals for VaR and ES Using the Bootstrap

- The estimates of VaR and ES are precisely that, just estimates. If we had used a different sample of historic data, then we would have gotten different estimates of these risk measures.
- A confidence interval for VaR or ES is easily obtained by bootstrapping.
- Suppose we have a large number, B, of resamples of the returns data. Then a $VaR(\alpha)$ or $ES(\alpha)$ estimate is computed from each resample and for the original sample.
- The confidence interval can be based upon either a parametric or nonparametric estimator of $VaR(\alpha)$ or $ES(\alpha)$. Suppose that we want the confidence coefficient of the interval to be $1-\gamma$ The intervals confidence coefficient should not be confused with the confidence coefficient of VaR, which we denote by $1-\alpha$
- The $\gamma/2$-lower and -upper quantiles of the bootstrap estimates of VaR and $ES(\alpha)$ are the limits of the basic confidence interval.

- When VaR is estimated for a portfolio of assets rather than a single asset, parametric estimation based on the assumption of multivariate normal or t- distributed returns is very convenient, because the portfolios return will have a univariate normal or t-distributed return.
- VaR can be estimated, assuming normally distributed returns on the portfolio by

$$\widehat{VaR}_P(\alpha) = -S_0(\hat{\mu}_P + \hat{\sigma}_P \Phi^{-1}(\alpha))$$

- Moreover

$$\widehat{ES}_P^{norm} = S_0 \times \left\{ -\hat{\mu}_P + \hat{\sigma}_P \left(\frac{\phi((\Phi^{-1}(\alpha)}{\alpha} \right) \right\}$$

- If the returns have a t-distribution, then

$$VaR_P^t(\alpha) = -S_0 \times \{\hat{\mu}_P + \hat{\lambda}_P F_{\hat{\nu}}^{-1}(\alpha)\}.$$

- The expected shortfall is

$$ES_P^t(\alpha) = S_0 \times \left\{ -\hat{\mu}_P + \hat{\lambda}_P \left(\frac{f_{\hat{\nu}}(F_{\hat{\nu}}^{-1}(\alpha))}{\alpha} \left[\frac{\hat{\nu} + \{F_{\nu}^{-1}(\alpha)\}^2}{\hat{\nu} - 1} \right] \right) \right\}$$

- Suppose that the daily returns (R_A, R_B) on Stocks A and B have a bivariate normal distribution with mean vector

$$\mu = \begin{pmatrix} 0.08 \\ 0.10 \end{pmatrix}$$

and variance-covariance matrix

$$\Sigma = \begin{pmatrix} 0.0289 & 0.0213 \\ 0.0213 & 0.0625 \end{pmatrix}.$$

1. Suppose that you hold a \$100000 position in each stock (i.e $S_0 = 100000$ for each one of them), compute $VaR_A(0.05)$ and $VaR_B(0.05)$.
2. What is VaR(0.05) of a portfolio holding 50000 is Stock A and 50000 in Stock B?
3. Compute the expected short fall for each asset and the portfolio in the previous questions using $\alpha = 0.05$.