Class 17: Continuous RVs & Midterm Review

Thursday, November 7, 2019 11:40 AM

- Midterm 2 Tuesday, 11/12, same room assignments

Outline (midterm review):

① Multiple discret random variables (X, Y)

ⓐ Joint PMFs (prob. mass function):

$$p(x,y) = \mathbb{P}(X=x, Y=y) \quad \text{for } x, y \in \mathbb{R}.$$

- Marginal PMFs:

$$p_X(x) = \mathbb{P}(X=x) = \sum_y p(x,y)$$

$$p_Y(y) = \mathbb{P}(Y=y) = \sum_x p(x,y)$$

Use tables!

- Conditional PMFs:

$$p_{X|Y}(x|y) = \mathbb{P}(X=x \mid Y=y) = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)} = \frac{p(x,y)}{p_Y(y)}$$

$$p_{Y|X}(y|x) = \mathbb{P}(Y=y \mid X=x) = \frac{p(x,y)}{p_X(x)}$$

For an event A,

$$p_{X|A}(x) = \mathbb{P}(X=x \mid A).$$

- Conditional expectations

$$\mathbb{E}[X \mid Y=y] = \sum_x x\,\mathbb{P}(X=x \mid Y=y) = \sum_x x\, p_{X|Y}(x|y)$$

$$\mathbb{E}[X \mid A] = \sum_x x\,\mathbb{P}(X=x \mid A) = \sum_x x\, p_{X|A}(x)$$

ⓑ Covariance & correlation (also for continuous RVs)

$$\mathrm{Cov}(X,Y) = \mathbb{E}\big[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])\big]$$
$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Recall: $\mathrm{Var}(X) = \mathrm{Cov}(X,X) = \mathbb{E}\big[(X-\mathbb{E}[X])^2\big]$
$$= \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

$$\rho(X,Y) = \mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$$ always in $[-1, 1]$.

± 1 perfect pos/neg. correlation

0 uncorrelated

ⓒ Independence

Def. X and Y are indep. if

(c) Independence

Def: X and Y are indep. if

$$\mathbb{P}(X=x,\ Y=y) = \mathbb{P}(X=x)\mathbb{P}(Y=y)$$

$$p(x,y) = p_X(x)\ p_Y(y) \qquad \text{for all } x, y.$$

Consequences of independence: $Var(X+Y) = Var(X) + Var(Y)$ and $Cov(X,Y) = 0$

and

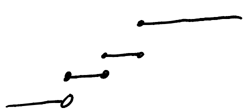
$$\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\,\mathbb{E}[g(Y)]$$

↳ Recall $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$, even without independence!

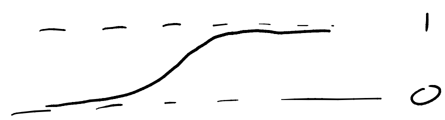
(2) Cumulative Distribution Functions (CDFs)

Def: For any r.v. X (discrete or continuous), the CDF is $F(x) = \mathbb{P}(X \le x)$ for $x \in \mathbb{R}$.

discrete:



continuous:



Note: $0 \le F \le 1$, always increasing as x increases

(If $x \ge y$ then $F(x) \ge F(y)$.)

(3) Continuous RVs:

(a) How to work with PDFs (probability density functions) and CDFs

↳ not the probability of anything!

- $\mathbb{P}(a \le X \le b) = F(b) - F(a) = \int_a^b f(x)\,dx$

  CDF of X   PDF of X

- $F(b) = \int_{-\infty}^{b} f(x)\,dx$ or $F'(x) = \frac{dF}{dx}(x) = f(x)$.

(b) Expectations:

PDF of X

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\,dx \qquad \text{for any function } h$$

(c) Common continuous RVs: (No normal distribution on midterm 2.)

Unif[a, b]   Exponential(λ)

ⓒ Common continuous RVs: (No normal distribution on midterm 2.)

<u>Unif [a, b]</u>

where $a < b$

<u>PDF:</u> $f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{else} \end{cases}$

<u>Exponential(λ)</u>

where $\lambda > 0$

<u>PDF:</u> $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{else} \end{cases}$

Amount of time to wait til first arrival, λ = rate of arrivals.

ⓓ <u>Transformation of distributions</u>:

Given X with some PDF, and given e.g. $Y = X^3$,
how do we find the CDF/PDF of Y?

④ <u>Continuous joint distributions</u> (X, Y) with joint PDF f

ⓐ <u>Joint PDFs</u>: $\mathbb{P}((X,Y) \in R) = \iint_R f(x,y)\,dy\,dx = \iint_{R \cap (\text{support of } f)} f(x,y)\,dy\,dx$

joint PDF

$\mathbb{E}[h(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f(x,y)\,dy\,dx$

$\mathbb{P}(1 \le X \le 2,\ 3 \le Y \le 4) = \int_1^2 \int_3^4 f(x,y)\,dy\,dx$

ⓑ <u>Marginal PDFs</u>:

<u>PDF of X</u>: $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ for $x \in \mathbb{R}$

<u>PDF of Y</u>: $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$ for $y \in \mathbb{R}$

ⓒ <u>Independence</u>:

X and Y are indep. if $f(x,y) = f_X(x) f_Y(y)$ for all x, y.

↳ same implications as in discrete case

<u>Analogies</u>:

	<u>Discrete</u>	<u>Continuous</u>
	$P(a \le X \le b) = \sum_{x \text{ between } a \text{ and } b} p_X(x)$	$\mathbb{P}(a \le X \le b) = \int_a^b f_X(x)\,dx$
	$\mathbb{E}[h(X)] = \sum_x h(x) p_X(x)$	$\mathbb{E}[h(X)] = \int_{-\infty}^{\infty} h(x) f_X(x)\,dx$

⋮

$$\mathbb{E}[h(X)] = \sum_x h(x) p_X(x) \qquad \mathbb{E}[h(X)] = \int_{-\infty} h(x) f_X(x)\,dx$$

$\vdots$ $\qquad\qquad$ $\vdots$

Ex 1: (Transformation of RVs)

Q: Let $U \sim \mathrm{Unif}[0,1]$ and $X = 1/U$.

Find the PDF of X.

Solution: PDF of U is $f_U(u) = \begin{cases} 1 & \text{if } 0 \le u \le 1 \\ 0 & \text{else.} \end{cases}$

Start with CDF of X, then differentiate to get PDF.

Since $0 < U < 1$, $X = \frac{1}{U}$ lies between 1 and ∞.

So PDF of X is 0 outside of this range.

For $x > 1$, the CDF of X is

$$\mathbb{P}(X \le x) = \mathbb{P}\left(\frac{1}{U} \le x\right) = \mathbb{P}\left(U \ge \frac{1}{x}\right)$$

$\mathbb{P}(E) = 1 - \mathbb{P}(E^c)$ with $E = \{U \ge \frac{1}{x}\}$.

$0 < \frac{1}{x} < 1$

$$= 1 - \mathbb{P}\left(U < \frac{1}{x}\right)$$

$$= 1 - \int_{-\infty}^{\frac{1}{x}} f_U(u)\,du = 1 - \int_0^{\frac{1}{x}} 1 \cdot du$$

$$= 1 - \frac{1}{x}.$$

CDF of X: $\quad \mathbb{P}(X \le x) = \begin{cases} 1 - \frac{1}{x} & \text{if } x > 1 \\ 0 & \text{if } x \le 1. \end{cases}$

PDF of X:

$$f_X(x) = \frac{d}{dx}\mathbb{P}(X \le x) = \begin{cases} \frac{1}{x^2} & \text{if } x > 1 \\ 0 & \text{if } x \le 1. \end{cases}$$

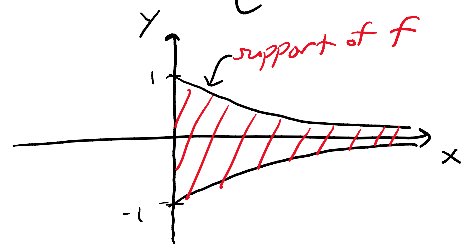
Ex 2: (Joint continuous RVs)

$\frac{1}{2}$ if $x \ge 0, |y| \le e^{-x}$

Ex 2: (Joint continuous RVs)

Suppose (X,Y) have joint PDF $f(x,y) = \begin{cases} \frac{1}{2} & \text{if } x \geq 0,\ |y| \leq e^{-x} \\ 0 & \text{otherwise.} \end{cases}$

$-e^{-x} \leq y \leq e^{-x}$



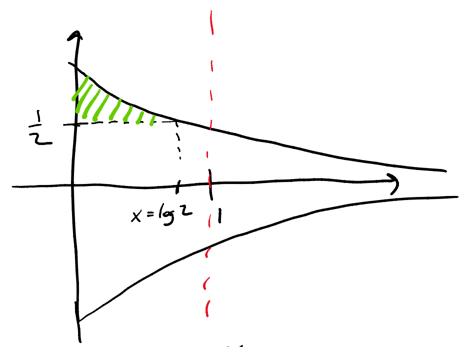
Q1: Check this is a valid joint PDF.

$f \geq 0$ ✓

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^{\infty}\int_{-e^{-x}}^{e^{-x}} \frac{1}{2}\,dy\,dx$$

$$= \int_0^{\infty} e^{-x}\,dx = 1.$$ ✓

Q2: Find $\mathbb{P}(Y \geq \frac{1}{2},\ X \leq 1)$.



$$\mathbb{P}(Y \geq \tfrac{1}{2},\ X \leq 1) = \int_{-\infty}^{1}\int_{\frac{1}{2}}^{\infty} f(x,y)\,dy\,dx$$

$$= \int_0^{\log 2}\int_{\frac{1}{2}}^{e^{-x}} \frac{1}{2}\,dy\,dx$$

$$= \int_0^{\log 2} \frac{1}{2}\left(e^{-x} - \frac{1}{2}\right)dx$$

$$= \ldots$$

Line $y = e^{-x}$ crosses $y = \frac{1}{2}$ when $\frac{1}{2} = e^{-x}$, $x = -\log\frac{1}{2} = \log 2$.

So $0 \leq x \leq 1$ and $y \geq \frac{1}{2}$ and $-e^{-x} \leq y \leq e^{-x}$ happens

when $0 \leq x \leq \log 2$ and $\frac{1}{2} \leq y \leq e^{-x}$.

Q3: Find the marginal PDF of X.

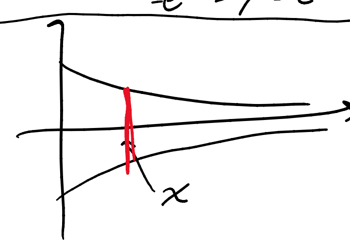
$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{-e^{-x}}^{e^{-x}} \frac{1}{2}\,dy = \frac{1}{2}\cdot 2e^{-x}$$

$$= e^{-x}, \quad \text{for } x \geq 0.$$

And $f_X(x) = 0$ for $x < 0$.

$\boxed{\text{Support of } f\text{: } x \geq 0,\ -e^{-x} \leq y \leq e^{-x}}$

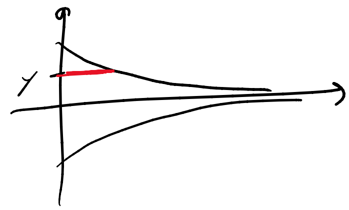


Q4: Find marginal PDF of Y.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$x \ge 0$

$-1 \le -e^{-x} \le y \le e^{-x} \le 1$



Easy case: $y \ge 1$ or $y \le -1$, $f_Y(y) = 0$.

For $0 < y < 1$, x ranges from 0 to $e^{-x} = y$, or $x = -\log y$.

So $f_Y(y) = \int_0^{-\log y} \frac{1}{2}\,dx = -\frac{1}{2}\log y$.

For $-1 < y < 0$, x ranges from 0 to $-e^{-x} = y$, or $x = -\log(-y)$.

so $f_Y(y) = \int_0^{-\log(-y)} \frac{1}{2}\,dx = -\frac{1}{2}\log(-y)$.

Full answer:
$$f_Y(y) = \begin{cases} -\frac{1}{2}\log y & \text{if } 0 < y < 1 \\ -\frac{1}{2}\log(-y) & \text{if } -1 < y < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Q4: Find $\mathbb{E}[Ye^{-x}]$. (Use $\mathbb{E}[h(X,Y)] = \int\int h(x,y) f(x,y)\,dy\,dx$.)

$$\boxed{\begin{matrix} x \ge 0 \\ -e^{-x} \le y \le e^{-x} \end{matrix}}$$

$$\mathbb{E}[Ye^{-x}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y e^{-x} f(x,y)\,dy\,dx$$

$$= \int_0^{\infty}\int_{-e^{-x}}^{e^{-x}} y e^{-x} \cdot \frac{1}{2}\,dy\,dx$$

$$= \int_0^{\infty} \left[\frac{1}{4} y^2 e^{-x}\right]_{y=-e^{-x}}^{e^{-x}} dx$$

$$= \frac{1}{4}\int_0^{\infty} e^{-x}\left(e^{-2x} + e^{-2x}\right) dx = \ldots$$