Class 4: Conditional Probability

Thursday, September 12, 2019 11:38 AM

- HW 1 due today by 1pm
- HW 2 due Thurs, Sept. 19, 1pm

(Gradescope)

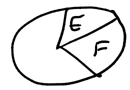
Recap:

P

A probability measure P on a sample space Ω is a function from events to $[0,1]$, satisfying

1. $P(E) \geq 0$ for all E
2. $P(\Omega) = 1$
3. If E and F are disjoint events ($E \cap F = \emptyset$)

$$P(E \cup F) = P(E) + P(F).$$



Properties:

- $P(E^c) = 1 - P(E)$
- For any events E and F,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

- If $E \subseteq F$ (read as "E implies F") then $P(E) \leq P(F)$.

Proof: $F = \underbrace{E}_{\text{disjoint}} \cup \underbrace{(F \cap E^c)}$

$F \cap E^c = F \setminus E$
$= F - E$
"F not E"

$$P(F) = P(E) + P(F \cap E^c) \qquad \text{(axiom 3)}$$
$$\geq P(E) \qquad \text{(axiom 1)}$$

- Discrete Probability: When Ω is finite, we can specify $P(\omega)$ for each outcome $\omega \in \Omega$, as long as $P(\omega) \geq 0$ for all ω and the sum is 1. The prob. of any event $E \subseteq \Omega$ is the sum of probabilities of the outcomes contained in E.

Ex: I have 3 pairs of socks in a drawer, red, blue, & green. I draw 2 socks at random among the 6.

Method 1: $\Omega = \{R, B, G\}^2 = \{RR, RG, RB, GR, GG, GB, BR, BG, BB\}$.
(Keeps track of order of draws.)

Probabilities:

$$P(RR) = \frac{2}{6} \cdot \frac{1}{5} = \frac{1}{15} = P(GG) = P(BB)$$

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$$P(RR) = \underset{\uparrow \text{ 1st draw red}}{\frac{2}{6}} \cdot \frac{1}{5} = \frac{1}{15} = P(GG) = P(BB)$$

out of 5 remaining socks, draw the one red

$$P(RG) = \frac{2}{6} \cdot \frac{2}{5} = \frac{2}{15} = P(RB) = P(GR) = P(GB) = P(BR) = P(BG).$$

- what is the prob. that the two socks match?
  ↳ event $E = \{RR, GG, BB\}$,

$$P(E) = P(RR) + P(GG) + P(BB) = 3 \cdot \frac{1}{15} = \frac{1}{5}.$$

Method 2: $\Omega = \{RR, BB, GG, RG, RB, GB\}$

(Order doesn't matter.)

$P(RR) = P(BB) = P(GG) = \frac{1}{15}$ as above.

Now: $P(RG) = P(\text{one Red \& one Green in any order})$

$$= \underset{\uparrow \text{ red or green}}{\frac{4}{6}} \cdot \frac{2}{5} = \frac{4}{15} = P(RB) = P(GB).$$

Conditional Probability:

Ex 1: Flip 3 coins $\Omega = \{H, T\}^3$.

What is the prob. of getting ≥ 2 heads given that we get ≥ 1 heads?

Outcomes:	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
≥ 1 heads?	✓	✓	✓	✓	✓	✓	✓	✗
≥ 2 heads?	✓	✓	✓	✗	✓	✗	✗	↓ N/A

↳ 4 out of the 7 "≥ 1 heads" outcomes give us "≥ 2 heads"

↳ conditional prob. is $\frac{4}{7}$ ← # favorable outcomes given the extra info / ← # possibilities for extra info

Defn: For two events A and B with $P(B) > 0$,
the conditional probability of A given B, written $P(A \mid B)$,
is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}. \quad \leftarrow \text{renormalize}$$

"Restrict everything to B."

Ex 2: What day of week you do your 3658 HW.

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$\Omega = \{Mon, Tue, \dots, Sat, Sun\}$.

Outcome	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Prob	.1	.2	.5	.01	.04	.07	.08

- If you wait til after the weekend to start, what is the (conditional) prob. that you wait til Wed?

$B = \{Mon, Tue, Wed\}$

$A = \{Wed\}$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{Wed\})}{P(\{Mon, Tue, Wed\})}$$

$$= \frac{.5}{.1 + .2 + .5} = \frac{5}{8}.$$

Rules:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Multiplication Rule: $P(A \cap B) = P(A \mid B) P(B)$

Iterate to get

$$P(A \cap B \cap C) = P(A \cap (B \cap C)) = P(A \mid B \cap C) \underline{P(B \cap C)}$$

$$= P(A \mid B \cap C) P(B \mid C) P(C)$$

Note: If $P(B) > 0$, then $P(\cdot \mid B)$ is a probability measure!

i.e. it satisfies the axioms from before.

Ex. $P(\Omega \mid B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$

Ex. If E and F are disjoint,

then $P(E \cup F \mid B) = P(E \mid B) + P(F \mid B)$.

Bayes' Rule: Relates $P(A \mid B)$ to $P(B \mid A)$. (Not equal in general!)

Ex. $\Omega = \{H, T\}^2$ 2 coin flips, $A = \{HH\} =$ "2 heads"

$B = \{HH, TH, HT\} =$ "≥ 1 heads".

$$P(A \mid B) = \frac{P(A \cap B)}{\;} = \frac{P(HH)}{\;} = \frac{1/4}{\;} = 1/$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(HH)}{P(\{HH, TH, HT\})} = \frac{1/4}{3/4} = 1/3.$$

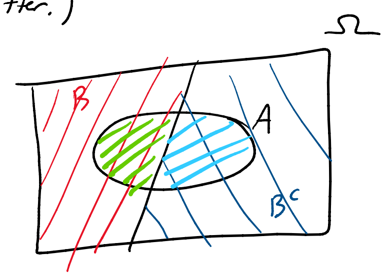
$$P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{P(HH)}{P(HH)} = 1.$$

Assume $P(A) > 0$, $P(B) > 0$. Then

$$\underbrace{P(A|B)}_{\text{definition}} = \frac{P(A\cap B)}{P(B)} \underset{A\cap B = B\cap A}{=} \frac{P(B\cap A)}{P(B)} \underset{\text{multiplication rule}}{=} \underbrace{\frac{P(B|A)P(A)}{P(B)}}_{\text{Bayes Rule}}.$$

Law of Total Probability:

Lets you relate unconditional probabilities to conditional probabilities.
(In many applications, you know only the latter.)



$A\cap B$ $\quad$ $A\cap B^c$ $\leftarrow$ or $A\setminus B$

Let A and B are events.

Then $A = \underbrace{(A\cap B)}_{\text{disjoint}} \cup \underbrace{(A\cap B^c)}$.

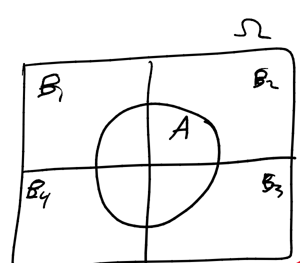
So

$$P(A) = P(A\cap B) + P(A\cap B^c)$$

$$= \underbrace{P(A|B)P(B) + P(A|B^c)P(B^c)}_{\text{multiplication rule}}.$$

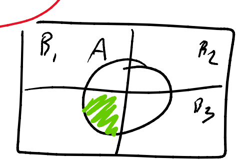
$\leftarrow$ Law of Total Prob.

More general version for multiple events:



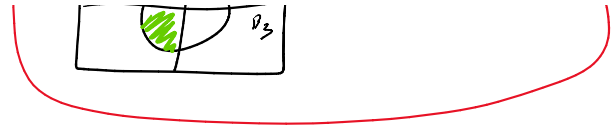
Suppose $B_1, \ldots, B_n$ are mutually exclusive
and $B_1 \cup B_2 \cup \cdots \cup B_n = \Omega$. (Partition.)

Otherwise we might miss part of A.



Then $A = \underbrace{(A\cap B_1)}_{\text{piece of } A \text{ in } B_1, \ldots} \cup (A\cap B_2) \cup \ldots \cup (A\cap B_n)$.

So $P(A) = P(A\cap B_1) + P(A\cap B_2) + \cdots + P(A\cap B_n)$



So
$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \dots + P(A \cap B_n)$$

$$= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \dots + P(A|B_n)P(B_n)$$

multiplication rule

$$= \sum_{i=1}^{n} P(A|B_i)P(B_i).$$