

Important:

You are allowed to bring one letter-sized page written on one side of your own notes.

1. True or False:

- a. There exist r.v.'s X and Y such that $\text{Var}(Y|X) > \text{Var}(Y)$ with positive probability.
- b. The Monte Carlo method in Bayesian inference is useful for approximating posterior expectations $E[g(\theta)|\mathbf{x}]$ but cannot be used to compute approximate Bayesian confidence intervals.
- c. If $X_1, \dots, X_n | \theta$ are iid $\text{Poisson}(\theta)$ and $\theta \sim \Gamma(a, b)$, then the posterior predictive distribution of X_{new} is Poisson distributed as well.
- d. In the Binomial model with uniform prior on θ , the mean of the posterior predictive distribution can be written as weighted average of the population mean θ and the sample mean \bar{x} .
- e. Let X, Y, Z be random variables with joint density $p(x, y, z)$ such that there exist functions f, g , and h with the property that $p(x, y, z) \propto f(x)g(y)h(z)$. Then X, Y and Z are mutually independent.
- f. If X_1, X_2, \dots, X_n are exchangeable then they are also iid.
- g. If $\theta^{(1)}, \dots, \theta^{(S)}$ are iid from the posterior $f(\theta|x_1, \dots, x_n)$, then a Monte Carlo approximation of $E(g(\theta)|x_1, \dots, x_n)$ is given by $\frac{1}{S} \sum_{i=1}^S g(\theta^{(i)})$.
- h. A probability density function that satisfies $f(x) \propto e^{ax^2 + bx}$ for any real x , and some $a < 0$, is necessarily that of a normal distribution.
- i. The R command `pnorm(-4.5, -3.4, sqrt(10))` gives the probability $P(X > -4.5)$, where $X \sim N(\mu = -3.4, \sigma^2 = 10)$.
- j. Suppose we have one observation X such that:
$$X | \theta \sim N(\theta, \sigma^2)$$
$$\theta \sim N(\mu, \tau^2)$$
Then the marginal distribution of X is normal.
- k. Bonus: A 95% Bayesian confidence interval for θ is a random interval $[l(X), u(X)]$ such that, for any value of $\theta \in \Theta$, $P[l(X) < \theta < u(X)|\theta] = 0.95$.

2. Let $X_1 \mid \theta \sim N(\theta, 1)$, where $\theta \sim N(0, 1)$.

a. Derive the posterior distribution of $\theta \mid x_1$

b. Derive the posterior predictive distribution of $X_{\text{new}} \mid x_1$.

3. Suppose we have independent $X_1, \dots, X_n \sim \text{Bernoulli}(\theta)$ and $\theta \sim \text{Beta}(a, b)$. Use without proof the fact that the posterior distribution is: $\theta \mid x \sim \text{Beta}(a + x, b + n - x)$, where $x = \sum_{i=1}^n x_i$.

- Suppose $a = b = 1$. For which case will the 95% posterior interval be shorter: $(n = 10, x = 4)$ or $(n = 20, y = 8)$? Explain why, but no need to prove it rigorously.
- Suppose the observed data are $n = 20$ and $x = 8$. For which prior distribution will the 95% posterior interval be shorter, $\text{Beta}(2, 3)$ or $\text{Beta}(4, 6)$? Again, just explain why.
- Derive the general formula for the posterior predictive distribution of X_{new} .
- If $n = 2$, $x = 0$, $a = 4$, and $b = 4$, which one is smaller, the posterior predictive variance or the prior predictive variance?

4. Based on Paul's symptoms, he believes it is equally likely that he has either Covid or the flu. Thus, each disease has a prior probability of 0.5; assume there is zero probability that he has both, and zero probability he has neither.

It is known that:

- A rapid Covid test is 70% sensitive and 97% specific. Meaning, if Paul has Covid, the probability of a negative test is 0.3; if he has the flu, the probability of a negative test is 0.97.
- An influenza test is 60% sensitive and 80% specific. So, if Paul has Covid, the probability of a negative flu test is 0.8; if Paul has the flu, the probability of a negative test is 0.4.

a) Suppose Paul takes a rapid Covid test and gets a negative result. What is his updated belief about the probability he has Covid versus the flu?

b) Suppose that in addition to the negative Covid test, Paul takes an independent flu test and gets a negative result there as well. What is his updated belief about the probability he has Covid versus the flu?

c) Which result changed Paul's belief more dramatically, the negative Covid test, the negative flu test, or did they both have an equal effect? Does it make intuitive sense?