Lecture 3:

Modeling Markets:
Betting on Volatility

3.1 The *Attempted Science* Part of Neoclassical Finance

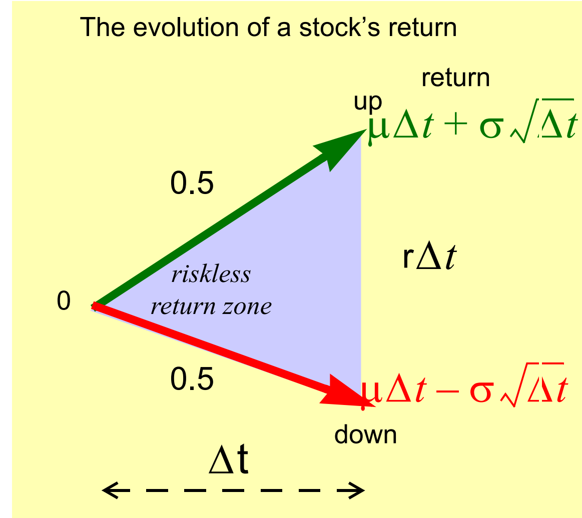
- Brownian motion: Stock returns are assumed to undergo arithmetic Brownian motion, are normally distributed.

- $$\frac{dS}{S} = \mu dt + \sigma dZ$$

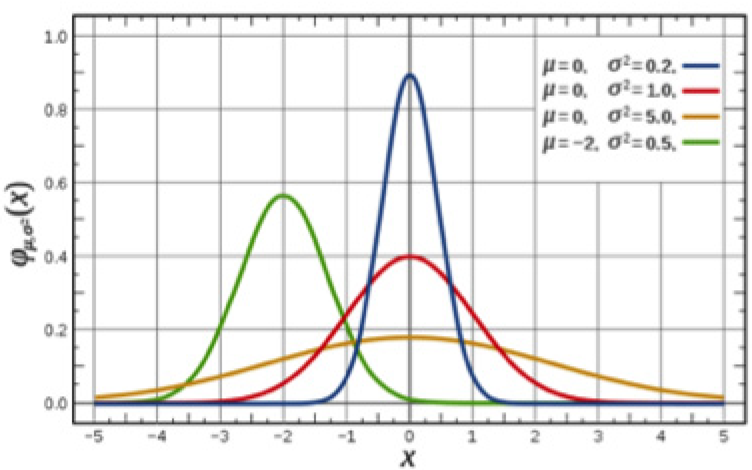
- No Arbitrage means **the riskless return is a convex combination** of up and down returns. If it weren't, you could always make a profit.

- Brownian motion is one model for an efficient market. It is a **good model** for atoms colliding with dust particles, a **not-so-good model** for stocks, as we've seen from the stylized facts.

- **Brownian motion assumes:**
  "financial risk" is a frequentist statistic.
  the stock's motion is completely specified by the volatility σ and expected return μ.



$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$



3.2 Deriving The Relation Between Risk and Return in our Model

In our geometric Brownian motion model for stocks, everything foreseeable about a stock, and in particular the distribution of its future payoffs, is determined by its ***expected return*** μ and the ***volatility*** σ **that represents its risk.**

All that differentiates one stock from another in this model are the values of μ and σ. Two securities with the same μ and σ are in essence identical.

What is the relation between μ and σ?

Note: There is one privileged security, the riskless bond, which has $\sigma = 0$. If there is no risk $\sigma = 0$, you know you will exactly earn the riskless rate and so $\mu = r$ exactly, with no statistical distribution at any given time.

What happens if σ is not zero?

We Will Try to Reduce Risky Securities to Riskless Ones, and Then Use The Law of One Price to Figure Out the Value of a Risky Security.

If you have a risky security whose risk σ you know, but whose return μ you don't know, you can figure out the value of μ via the following strategy:

Embed the risky security into a portfolio such that the portfolio has zero total risk.

Then by the Law of One Price it should have the known return *r* of a riskless bond.

If you know the composition of that riskless portfolio, the proportions that make it riskless, then you can then figure out the expected return of the risky security.

This strategy will lead to both CAPM (the Capital Asset Pricing Model) and the Black-Scholes-Merton Options Valuation Model.

How Can One Reduce Risk?

Hedging: If two securities are positively correlated with each other, if you buy one and short the other, you can reduce the risk of the portfolio.

Diversification: If you take a whole bunch of uncorrelated risky securities together, their volatility decreases because some go up as others go down. If you put enough of them together, their volatility becomes zero asymptotically.

- **Strategy:**

  To find the expected return on a security, try to remove all of its risk by combining it with other securities. If you can do that, then by The Law Of One Price, the combination must earn the riskless rate which is known. Then you can back out the return of the original security.

- **We are going to derive CAPM by combining hedging and then diversification. Option theory will use only hedging.**

3.3 The Relation Between Risk & Return for Stocks: CAPM

Consider a market with an infinite number of stocks S_i, all correlated with the market, which you can think of as the S&P 500 or S&P futures for example.

Let ρ_{iM} be the correlation of the returns between a stock S_i with volatility σ_i and the market M with volatility σ_M. Here is the geometric Brownian motion description:

$$\frac{dM}{M} = \mu_M dt + \sigma_M dZ_M$$

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i\left(\sqrt{1-\rho_{iM}^2}\,dZ_i + \rho_{iM} dZ_M\right)$$

where dZ_i and dZ_M are all uncorrelated with each other. The idiosyncratic risk of the stock is described by dZ_i.You can *hedge away* the M-related risk of the stock S_i by shorting Δ_i shares of the market to create an M-neutral stock $\vec{S}_i$ that has no exposure to movements dZ_M, but only to dZ_i

$$\vec{S}_i = S_i - \Delta_i \times M$$

$d\vec{S}_i = dS_i - \Delta_i \times dM$ has no market risk if $\Delta_i = \rho_{iM}(\sigma_i/\sigma_M)\frac{S_i}{M} \equiv \beta_i \frac{S_i}{M}$.

Each M-neutral stock $\vec{S}_i$ is now uncorrelated with the market and uncorrelated with all other M-neutral stocks.

The M-neutral stock has expected return per unit time

$$\frac{1}{dt}\left\{\frac{E\left[d\vec{S}_i\right]}{\vec{S}_i}\right\} = \frac{1}{dt}\left\{\frac{E[dS_i - \Delta_i \times dM]}{S_i - \Delta_i \times M}\right\} = \frac{\mu_i S_i - \Delta_i \mu_M M}{S_i - \Delta_i \times M} = \frac{\mu_i S_i - \beta_i \mu_M S_i}{S_i(1-\beta_i)} = \frac{\mu_i - \beta_i \mu_M}{(1-\beta_i)}$$

We can do this for each stock S_i in the market. If we diversify over all M-neutral stocks we can create a portfolio with asymptotically zero volatility, and so by the Law of One Price it must earn the riskless return r, so that[1] each stock in the portfolio can earn no more than the riskless return. So

$$\frac{\mu_i - \beta_i \mu_M}{(1-\beta_i)} = r$$

$$(\mu_i - r) = \beta_i(\mu_M - r)$$ CAPM in “Efficient Markets”

$$(\underbrace{\mu - r}_{\text{stock's excess return}}) = \underbrace{\beta}_{\text{stock's beta}}(\underbrace{\mu_M - r}_{\text{market's excess return}})$$ Capital Asset Pricing Model

What this relation is really saying is that if you can hedge away all market risk, and then diversify over all idiosyncratic risk, no risk is left, and so you must earn the riskless rate.

1. Spelled out in more detail in Section 2 of *The Perception of Time, Risk and Return During Periods of Speculation*, Quantitative Finance Vol 2 (2002) 282–296, or http://emanuelderman.com/perception-time/

[Finance For Future Generations]

- Feynman: One sentence about physics to guide future civilizations

- One sentence about finance to guide future civilizations:

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  If

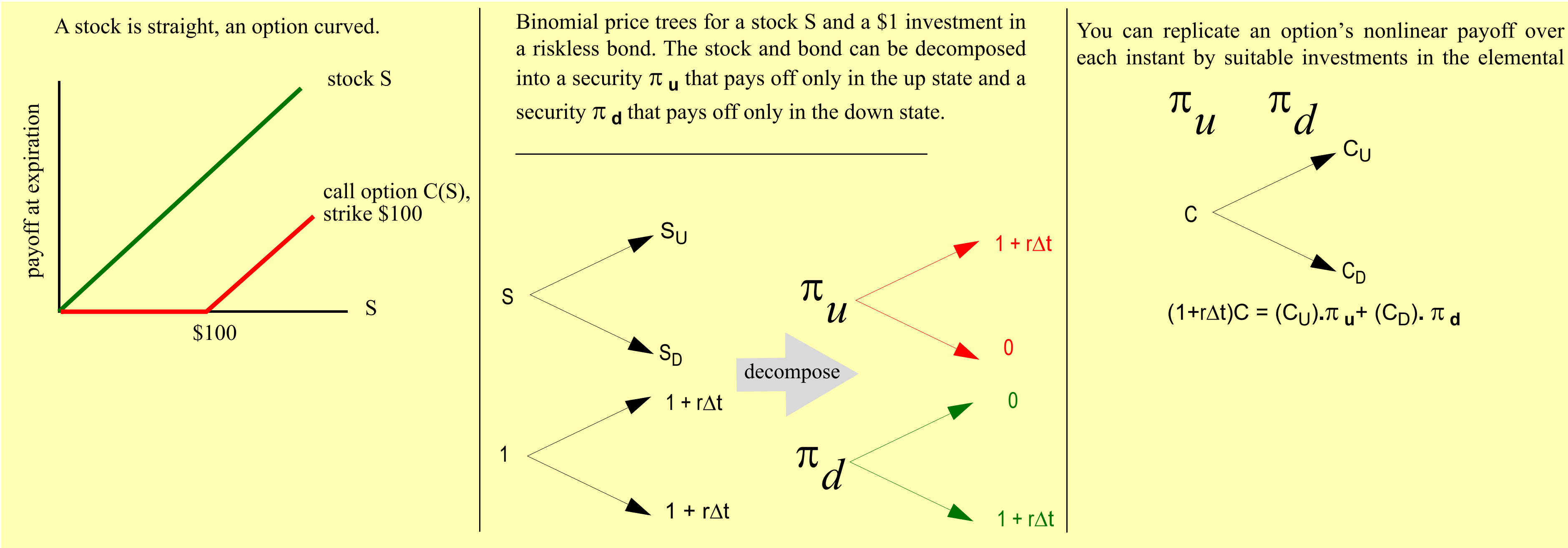
  you can hedge away all correlated risk, and
  then diversify over all uncorrelated risk,

  you should expect to earn the riskless return.

This is a sensible principle.

The difficulty is that correlation and diversification can't really be carried out because risk is not really purely statistical. It can't be specified for all time by a stochastic pde or Monte Carlo.

3.4 Binomial Derivative Valuation by Replication



A derivative is a contract whose payoff depends on the price of a "simpler" *underlier*. The most relevant characteristic is the *curvature* of its payoff *C(S)*, as illustrated for a simple call option. **What is the financial value of owning curvature?**

You can use linear algebra to decompose the stock and bond into a basis of two more elemental securities π_u and π_d, each respectively paying \$(1+rΔt) in only one of the final states.

Then you can replicate the payoff of any non-linear function *C(S)* over the next instant of time Δt, no matter into which state the stock evolves. Note that the portfolio consisting of both π_u and π_d is riskless and is therefore worth \$1. This is a homework problem.

The value of the option is the price of the mixture of stock and bond that replicates it. The coefficients depend on the difference between the up-return and the down-return at each node, that is, on the stock's volatility σ.

The choice-of-currency/numeraire trick in modeling

You can use any currency to value a security if markets are efficient.

A convenient choice of currency can greatly simplify thinking about a problem, and sometimes reduce its dimensionality.

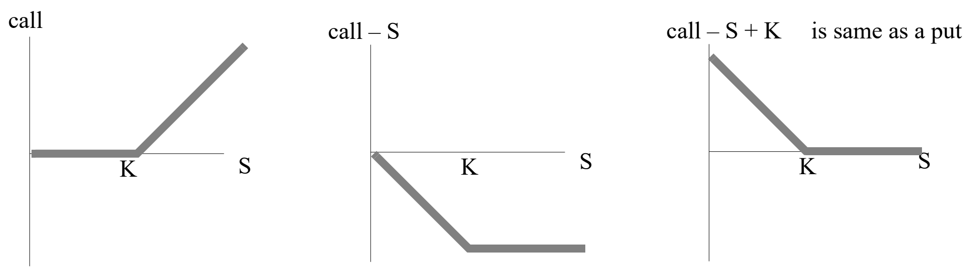
Convertible bonds, for example, which involve an option to exchange a bond for stock, can sometimes be fruitfully modeled by choosing a bond itself as the natural valuation currency.

METHODS OF REPLICATION

3.5 Exact Static Replication for European Payoffs With Valuation Independent of Smile

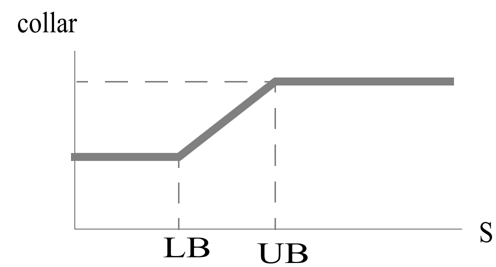
If you can create a static replicating portfolio for your payoff, and you know the prices of the ingredients in the static replication, **you have very little model risk.**

European put from a call: Put-Call Parity



Thus price of put = price of call - price of stock + PV(K).

A collar is a very popular instrument for portfolio managers who have made some gains during the year and now want to make sure they keep some upside but don't lose too much downside.



You can write the payoff as

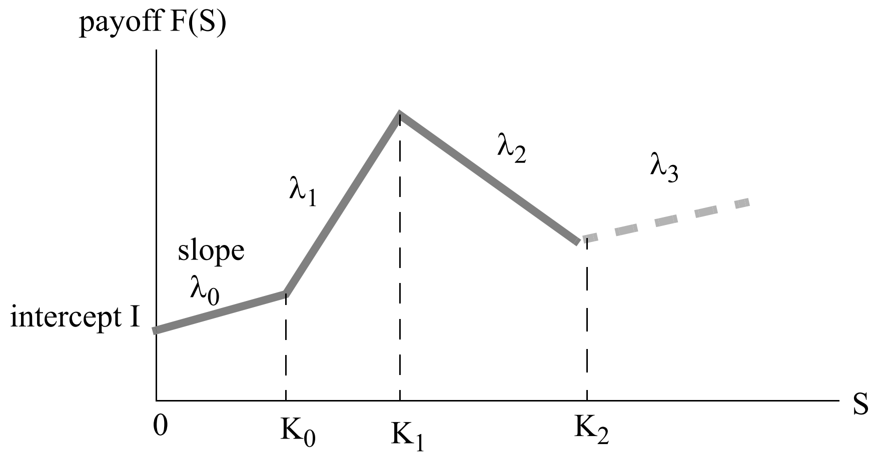
$$LB + call(S, LB) - call(S, UB)$$

Using put-call parity: $S + put(S, LB) - call(S, UB)$.

Its popularity forces dealers to be short puts and long calls.

Generalized European payoffs:

Piecewise-linear function of the terminal stock price S



Replicating portfolio, starting from the left, consists of a zero-coupon bond ZCB(I) plus a series of calls $C(K_i)$:

$$ZCB(I) + \lambda_0 S + (\lambda_1 - \lambda_0)C(K_0) + (\lambda_2 - \lambda_1)C(K_1) + \dots$$

whose value can be determined from market prices.

(You can start from the right using puts rather than from the left using calls.)

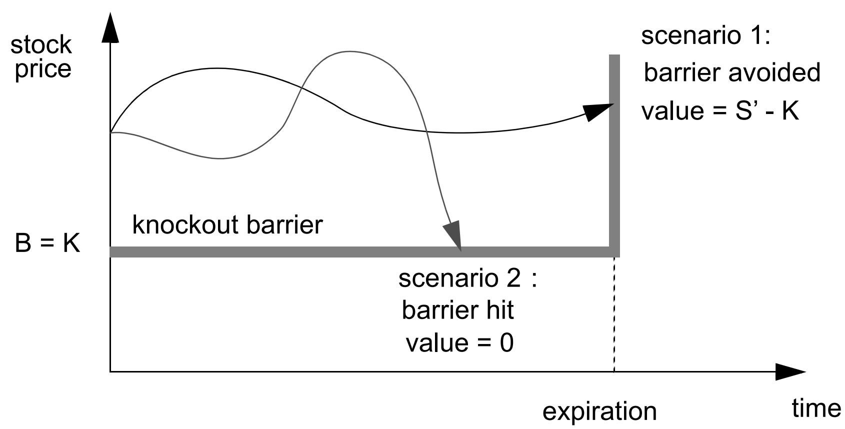
3.6 Approximate Static Replication for a Down-and-Out Call with Strike = Barrier, Only Approximately Smile-Independent

This option has a time-dependent boundary. It is worth zero if the stock ever touches B.

Stock price S and dividend yield d, strike K and out-barrier $B = K$.

Scenario 1 in which the barrier is avoided and the option finishes in-the-money.

Scenario 2 in which the barrier is hit before expiration and the option expires worthless.



In scenario 1 the call pays out $S'-K$, the payoff of a forward contract with delivery price K worth

$$F = Se^{-dt} - Ke^{-rt}$$

For paths in scenario 2, the down-and-out call immediately expires with zero value. In that case, the above forward F that replicates the barrier-avoiding scenarios of type 1 is worth $Ke^{-dt'} - Ke^{-rt'}$. This is close to zero. When the stock hits the barrier you must sell the forward to end the trade.

DYNAMIC REPLICATION

3.7 Quick Derivation of the Black-Scholes PDE

Assume GBM with zero rates for simplicity.

In time Δt, $\Delta S \approx \sigma S \sqrt{\Delta t}$.

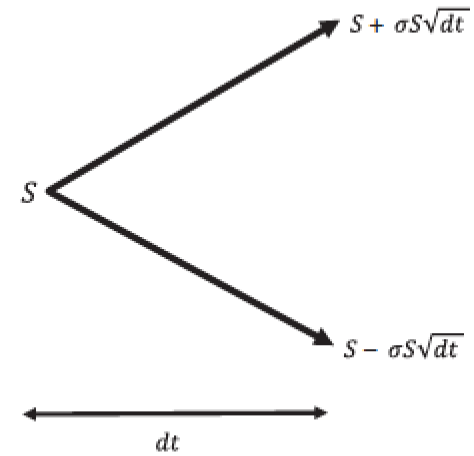


FIGURE 3.5 Binomial Model of Underlying Stock Price, $\mu = 0$

The stock S is a primitive, linear underlying security that provides a linear position in ΔS.

If you are long an option, you get a positive payoff whether the stock goes up or down! The call has curvature, or convexity.

$$\Gamma = \frac{\partial^2 C}{\partial S^2} \neq 0$$

What is the fair price for $C(S,t,K,T)$?

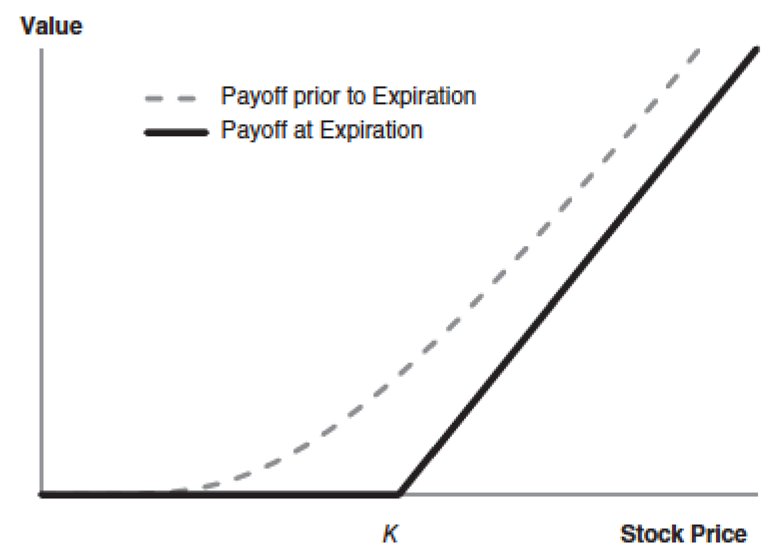


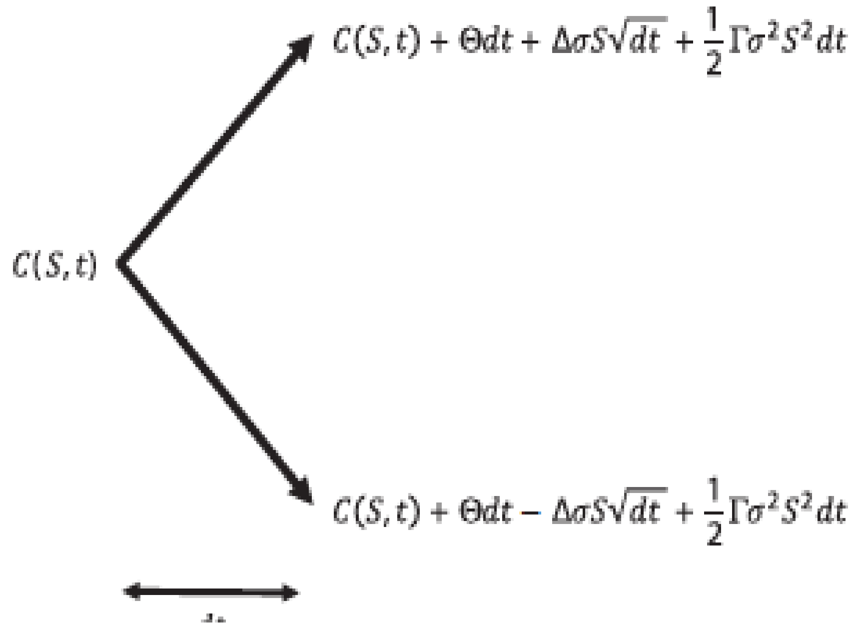
FIGURE 3.6 The Payoff of a Vanilla Call Option at Expiration

We can do a Taylor series expansion on the unknown price $C(\;)$ and examine how its value changes as time Δt passes and the stock moves by an amount ΔS:

$$C(S + \Delta S, t + \Delta t) = C(S, t) + \left.\frac{\partial C}{\partial t}\right|_{S,\,t} \Delta t + \left.\frac{\partial C}{\partial S}\right|_{S,\,t} \Delta S + \left.\frac{\partial^2 C}{\partial S^2}\right|_{S,\,t} \frac{(\Delta S)^2}{2} + \ldots$$

This is a quadratic function of ΔS. The linear term behaves like the stock price itself, the quadratic terms increases no matter what the sign of the move in S.

$$C(S + dS,\ t + dt) = C(S,\ t) + \Theta dt + \Delta dS + \frac{1}{2}\Gamma dS^2$$



If you hedge away the linear term in ΔS by shorting $\Delta = \dfrac{\partial C}{\partial S}$ shares the profit and loss of the hedged option position looks like this:

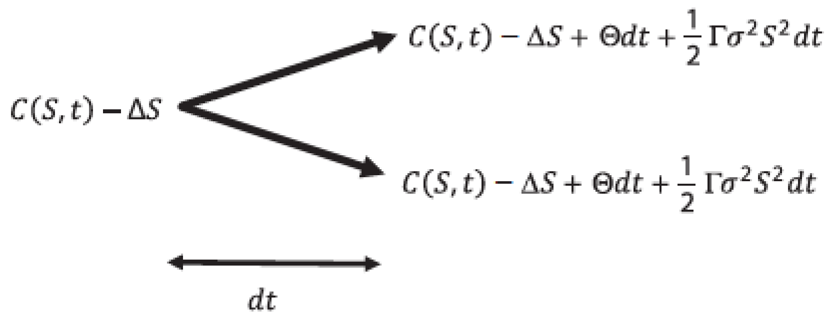


FIGURE 3.8 Delta-Hedged Call Option, $\mu = 0$

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Positive convexity generates a profit or loss on the hedged position that is quadratic in (ΔS).

3.8 What Should You Pay for Convexity? Replication Tells Us

$$V = C(S, t) - \Delta S.$$

Positive convexity:

$$dV(S,t) = \Theta dt + \frac{1}{2}\Gamma\sigma^2 S^2 dt \quad = \Theta dt + \frac{1}{2}\Gamma dS^2$$

Suppose we think we know the future volatility of the stock, Σ.

Total change in value of the hedged position is $d\text{P\&L} = dV = \frac{1}{2}\Gamma(\Sigma^2 S^2 dt) + \Theta(dt)$

If we know Σ, the P&L is completely deterministic, irrespective of the direction of the move.

Therefore it replicates a riskless bond and must earn zero interest: $\Theta + \frac{1}{2}\Gamma S^2 \Sigma^2 = 0$

This is the Black-Scholes equation for zero interest rates:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0$$

time decay and curvature are linked

$$C_{BS}(S, t, K, T, \Sigma) = SN(d_1) - KN(d_2)$$

$$d_{1,2} = \frac{\ln(S/K) \pm 0.5\Sigma^2(T-t)}{\Sigma\sqrt{T-t}}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$$

By differentiation,

$$\Delta_{BS} \equiv \frac{\partial C_{BS}}{\partial S} = N(d_1)$$

The option's Δ tells you how many shares to short of the stock so as to remove the linear exposure of the option so you can trade its quadratic part.

When the riskless rate r is non-zero,

$$\frac{\partial C}{\partial t} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = r\left(C - \frac{\partial C}{\partial S} S\right)$$

profit per unit time = riskless interest on capital

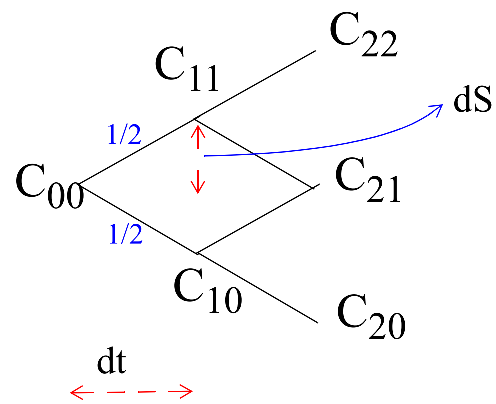
$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

$$C(S, K, \tau, \sigma, r) = SN(d_1) - Ke^{-r\tau}N(d_2)$$

$$d_{1,2} = \frac{\ln\left(\frac{S}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}y^2}\,dy$$

3.9 What Does This PDE Mean Discretely?



$$\frac{\partial C}{\partial t} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0$$ say for $r = 0$.

$$\frac{\partial C}{\partial t} = \frac{C_{21} - C_{00}}{2dt} \qquad \frac{\partial C}{\partial S} = \frac{C_{11} - C_{10}}{2dS}$$

$$\frac{\partial^2 C}{\partial S^2} = \left[\frac{\frac{C_{22} - C_{21}}{2dS} - \frac{C_{21} - C_{20}}{2dS}}{2dS}\right] = \frac{C_{22} - 2C_{21} + C_{20}}{4(dS)^2}$$

$$\text{pde} = \frac{C_{21} - C_{00}}{2dt} + \frac{1}{2}(\Sigma^2 S^2)\frac{C_{22} - 2C_{21} + C_{20}}{4(dS)^2} = 0$$

$$C_{00} = \frac{(\Sigma^2 S^2 dt)}{4(dS)^2}[C_{22} - 2C_{21} + C_{20}] + C_{21}$$ and suppose GBM $(dS)^2 = \Sigma^2 S^2 dt$

$$C_{00} = \frac{[C_{22} + 2C_{21} + C_{20}]}{4}$$

The current option price is the average of its future possible values; it's a martingale!

3.10 The Second-Most Important Formula in Options Theory

How much profit should you expect to make when you hedge an option at the implied volatility? We can calculate it. To keep things simple, assume zero dividends and interest rates.

At time *t*: Buy the call for price C_i corresponding to a Black-Scholes implied volatility Σ, so

$$\frac{\partial C_i}{\partial t} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C_i}{\partial S^2} = 0 \text{ or } \frac{\partial C_i}{\partial t} = -\frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C_i}{\partial S^2}$$

Now hedge it by shorting $\Delta_\Sigma = \frac{\partial C_i}{\partial S}$ shares of stock, so that $\pi(S, t) = C_i - \Delta_\Sigma S$ is riskless if the future volatility is Σ.

What happens when the stock moves to $S + dS$ at time $t + dt$ with a volatility $\sigma \neq \Sigma$, so that $(dS)^2 = \sigma^2 S^2 dt$ actually?

$$d\pi = \frac{\partial C_i}{\partial t} dt + \cancel{\frac{\partial C_i}{\partial S} dS} + \frac{1}{2}\frac{\partial^2 C_i}{\partial S^2}(dS)^2 - \cancel{\Delta_\Sigma dS} = \frac{\partial C_i}{\partial t} dt + \frac{1}{2}\frac{\partial^2 C_i}{\partial S^2}(dS)^2$$

Then

$$= \frac{\partial C_i}{\partial t} dt + \frac{1}{2}\frac{\partial^2 C_i}{\partial S^2}\sigma^2 S^2 dt = -\frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C_i}{\partial S^2} + \frac{1}{2}\frac{\partial^2 C_i}{\partial S^2}\sigma^2 S^2 dt = \frac{1}{2}\frac{\partial^2 C_i}{\partial S^2} S^2 \left(\sigma^2 - \Sigma^2\right) dt$$

3.11 Hedging an Option Means Betting On Volatility

$$d\pi = \frac{1}{2}\frac{\partial^2 C_i}{\partial S^2}S^2(\sigma^2 - \Sigma^2)dt = \frac{1}{2}\Gamma_i S^2(\sigma^2 - \Sigma^2)dt$$

To profit, you need the realized volatility to be greater than the implied volatility.

A short position profits when the opposite is true.

Note: Black-Scholes uses a single unique volatility for all strikes K and expirations T, because the volatility, real or implied, is the volatility of the stock, not the option. If Black-Scholes is correct, then Σ is independent of K, t, T and S. But we know there is a smile …

Here is an illustration of the contributions to the P&L:

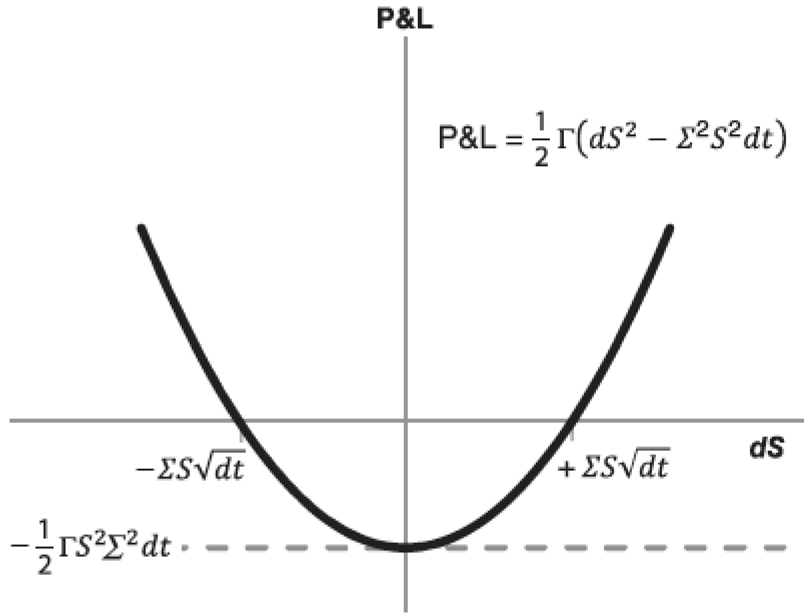


FIGURE 3.10 P&L from Implied versus Realized Volatility

To profit, you need the realized volatility to be greater than the implied volatility. A short position profits when the opposite is true.

Recall: Black-Scholes uses a single unique volatility for all strikes K and expirations T, because the volatility, real or implied, is the volatility of the stock, not the option. If Black-Scholes is correct, then Σ is independent of K, t, T and S.

One Vanilla Option is Not a Pure Bet on Volatility

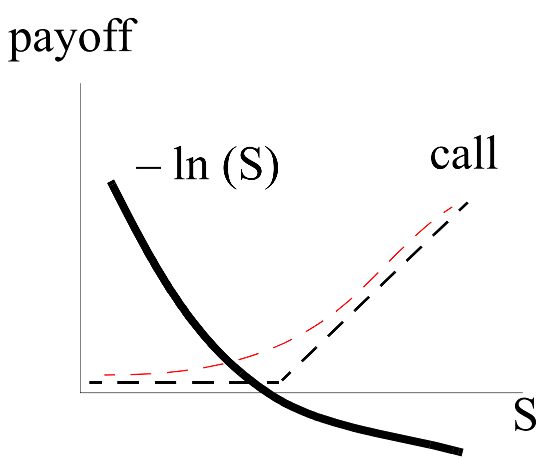
Net P&L from hedging an option $= \frac{1}{2}\int \Gamma S^2(\sigma^2 - \Sigma^2)dt$: sensitive to S and σ

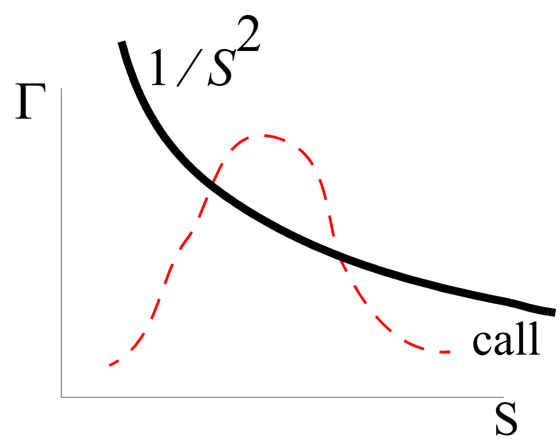
In a BS world, you can capture pure volatility if you own a derivative O whose curvature satisfies

$$\Gamma_O = 1/S^2 \quad \text{P\&L(O)} = \int \frac{1}{2}(\sigma^2 - \Sigma^2)\Delta t \text{ sensitive only to } \sigma$$

A security with this gamma is the "log contract" with BS value $O = -\ln S$ and $\Delta = -1/S$, **independent** of volatility, unlike an ordinary call! You delta-hedge it by owning $1 worth of stock always.

A log contract, hedged, will capture realized variance.





VARIANCE SWAPS
HOW TO TRADE PURE VOLATILITY
(A LESSON IN REPLICATION)

3.12 Volatility and Variance Swap Contracts

A Volatility swap is a forward contract on realized volatility. At expiration it pays the difference in dollars between the actual volatility realized by the index over the lifetime of the contract σ_R and some previously agreed upon “delivery” volatility K_{vol}:

$$(\sigma_R - K_{vol}) \times N$$ where N is the notional amount.

Similarly, a **variance swap** is a forward contract on realized variance. It pays

$$\left(\sigma_R^2 - K_{var}\right) \times N$$

Variance Swaps 63

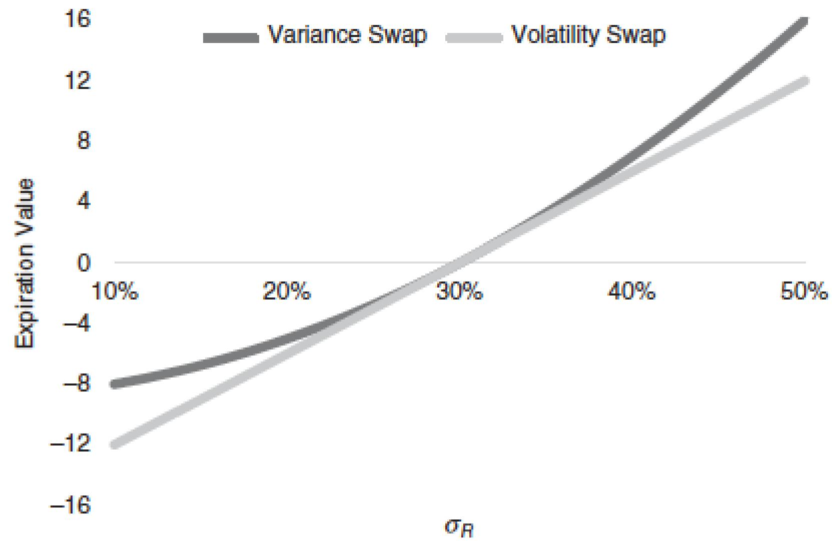


FIGURE 4.3 Comparison of a Volatility Swap with a Variance Swap

The variance swap is a derivative of the volatility swap. In theory, we could replicate the variance swap out of the volatility swap -- if we knew the evolution of volatility.

Common Sense: The delivery price of the volatility swap should be lower than the square root of the delivery price of variance, else the variance swap always does better than the volatility swap.

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3.13 Engineering Approach to Variance Replication in a BS World Using Vanilla Options

Zero interest rates and dividend yields for simplicity, so $C = C(S, K, v)$ where $v = \sigma\sqrt{\tau}$.

$$C_{BS} = SN(d_1) - KN(d_2) \qquad d_{1,2} = \frac{\ln S/K \pm v^2/2}{v}$$

Then the exposure to variance is given by

$$\kappa \equiv \frac{\partial C_{BS}}{\partial \sigma^2} = \frac{S\sqrt{\tau}}{2\sigma\sqrt{2\pi}} e^{-d_1^2/2}$$

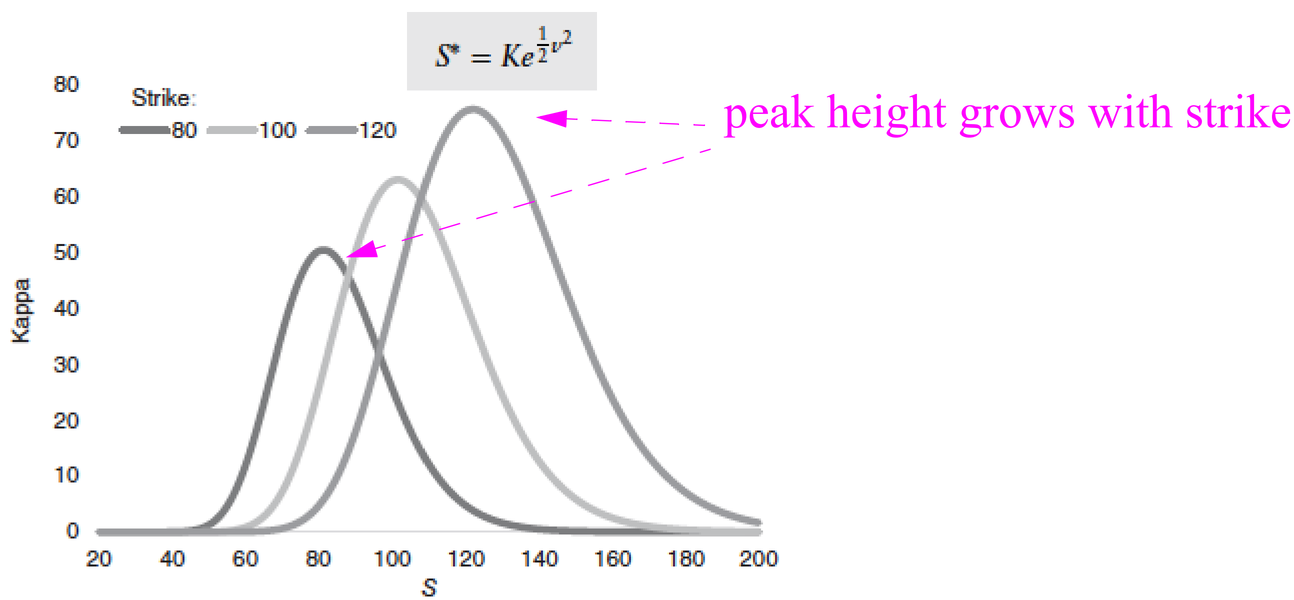

FIGURE 4.1 The Variance Vega for Three Strike Prices

You can see that the option has sensitivity to S and σ, and is therefore not a good way to make a clean bet on volatility. It peaks roughly at $S = K$, and so increases as the strike increases. What we want is a portfolio whose exposure κ to variance is independent of the stock price S, so that we can be exposed to volatility alone no matter what the stock price does.

Construct a portfolio $\pi(S) = \int_0^{\infty} \rho(K) C(S, K, v) dK$ such that $\kappa = \frac{\partial \pi}{\partial \sigma^2}$ is independent of S.

$$\frac{\partial \pi}{\partial \sigma^2} = \int_0^{\infty} \rho(K) \frac{S\sqrt{\tau}}{2\sigma} \frac{e^{-d_1^2/2}}{\sqrt{2\pi}} dK \sim \int_0^{\infty} \rho(K) S f\left(\frac{K}{S}, v\right) dK$$

We can make the S-dependence of $\rho(\ \ )$ explicit by changing variable to $x = K/S$ so that

$$\frac{\partial \pi}{\partial \sigma^2} = \int_0^{\infty} \rho(xS) S^2 f(x, v) dx$$

In order for this to be independent of S, we require that $\rho(K) \sim 1/K^2$

A density of options whose weights decrease as K^{-2} will give the correct volatility dependence.

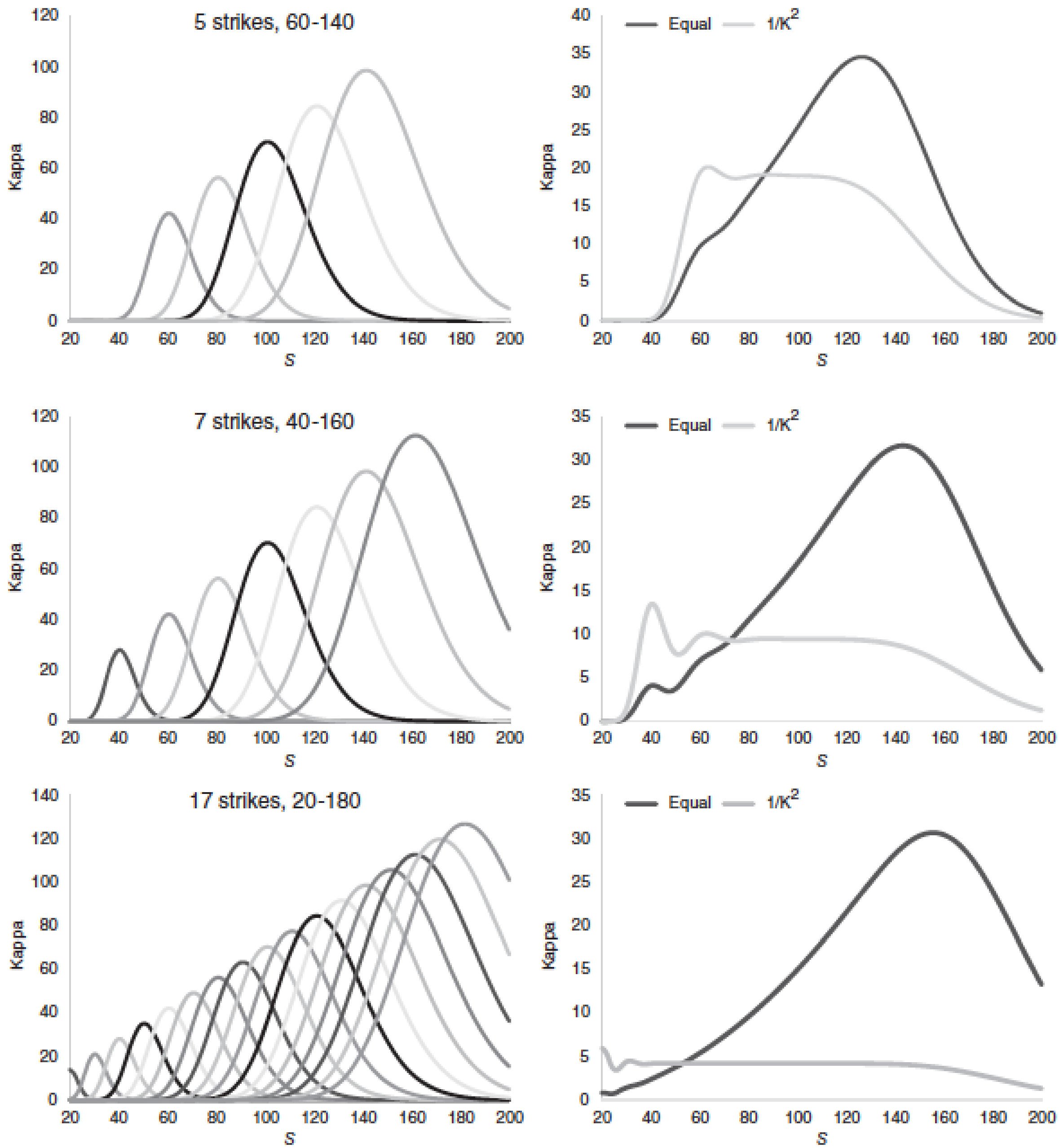

FIGURE 4.4 The Variance Vega of a Portfolio of Vanilla Options: Replicating a Variance Swap with Options Using Two Weighting Schemes

$1/K^2$ Calls and-or Puts Replicate a Log Payoff and a Forward

Use liquid puts below some strike S^* and use calls with strikes above S^*: $\pi(S, S^*, v)$.

The payoff at expiration at stock value S_T is $\pi\left(S_T, S^*, v\right)$ where at expiration $v = \sigma\sqrt{\tau} = 0$:

$$\pi\left(S_T, S^*, 0\right) = \int_{S^*}^{\infty} C(S_T, K, 0)\frac{dK}{K^2} + \int_{0}^{S^*} P(S_T, K, 0)\frac{dK}{K^2}$$

$$= \int_{S^*}^{S_T} (S_T - K)\frac{dK}{K^2} + \int_{S_T}^{S^*} (K - S_T)\frac{dK}{K^2}$$

$$\int_{S^*}^{S_T} (S_T - K)\frac{dK}{K^2} + \int_{S_T}^{S^*} (K - S_T)\frac{dK}{K^2}$$

For $S_T > S^*$ only the first integral contributes, else the second integral. Each gives the same result:

$$\pi(S_T, S^*, T) = \left(\frac{S_T - S^*}{S^*}\right) - \ln\left(\frac{S_T}{S^*}\right) = \int_0^{S^*} \frac{1}{K^2} P(S_T, K, T)\, dK + \int_{S^*}^{\infty} \frac{1}{K^2} C(S_T, K, T)\, dK$$

In order to be exposed purely to volatility we need to own a forward contract with delivery price S* (which has no volatility dependence), and be short a log contract L. The forward contract can be replicated statically, the log contract L must be hedged dynamically.

The Payoff at Expiration of the Replicating Portfolio made of Calls and Puts

$$\left(\frac{S_T - S^*}{S^*}\right) - \ln\left(\frac{S_T}{S^*}\right)$$

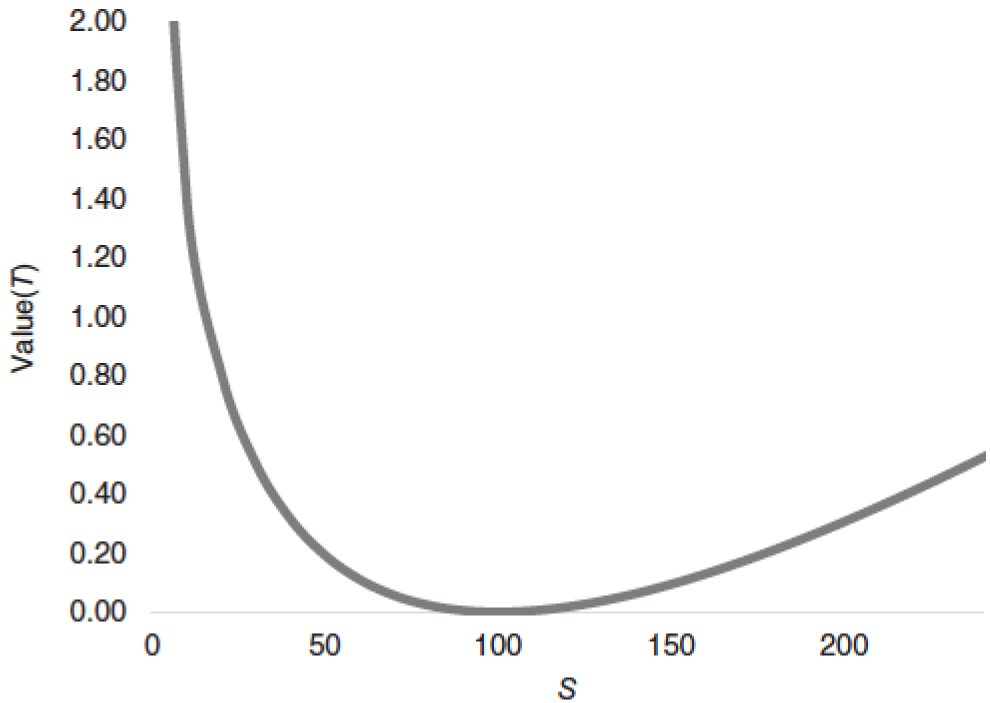

FIGURE 4.5 Value of Replicating Portfolio at Expiration, $S^* = 100$