PROBABILITY GU4155: Spring 2023

ASSIGNMENT # 1

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Read Chapters 1 and 2 in Walsh (2012), as well as Chapter 1 in Stirzaker (2003).

Exercise # 1: For arbitrary events $A_1, \cdots, A_n$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, prove the so-called *Inclusion-Exclusion Formula*

$$\mathbb{P}\left(\cup_{i=1}^n A_i\right) = \sum_{i=1}^n \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1}\, \mathbb{P}\left(\cap_{i=1}^n A_i\right) .$$

This formula can be written a bit more compactly as

$$\mathbb{P}\left(\bigcup_{i=1}^n A_i\right) = \sum_{\mathcal{J} \subseteq \{1, \cdots, n\},\ \mathcal{J} \neq \emptyset} \big(-1\big)^{|\mathcal{J}|+1}\, \mathbb{P}\Big(\bigcap_{j \in \mathcal{J}} A_j\Big) .$$

Exercise # 2: Let X, Y be real-valued, measurable functions on $(\Omega, \mathcal{F})$. If c is a real number, show that the functions below are also measurable:

$$c\,X\,, \qquad X^2\,, \qquad X+Y\,, \qquad X\,Y\,, \qquad |X|\,, \qquad X^{\pm}\,.$$

Exercise # 3: Let Ω be an arbitrary nonempty set, and denote by $\mathcal{C}$ the collection of all its "singletons", that is, all subsets that consist of exactly one element of Ω. Show that

$$\sigma(\mathcal{C}) = \mathcal{A} := \{\, A \subset \Omega \,:\, A \ \text{ or } \ A^c \ \text{ is countable} \,\}\,.$$

Exercise # 4: Stirling's Formula. Establish the celebrated Stirling formula

$$n! \sim \sqrt{2\pi n}\, n^n\, e^{-n}\,,$$

actually in its stronger version

$$n\,! = \sqrt{2\pi}\; n^{n+1/2} e^{-n+\varepsilon_n} \qquad \text{with} \qquad \frac{1}{12n+1} < \varepsilon_n < \frac{1}{12n}\,. \tag{0.1}$$

Hint: Establish the double inequality

$$n \log n - n \;<\; \log n\,! \;<\; (n+1)\,\log(n+1) - n\,,$$

which suggests considering the quantity

$$C_n \;:=\; \log n\,! - \Big(n + \frac{1}{2}\Big)\,\log n + n$$

(the difference of the middle term from the average of the upper and lower bounds).
Show that $C = \lim_{n\to\infty} C_n$ exists in $(0,\infty)$, in fact that we have

$$C + \frac{1}{12n+1} \;<\; C_n \;<\; C + \frac{1}{12n}$$

for all $n \in \mathbb{N}$. This constant can be identified as $C = \log \sqrt{2\pi}$, but do not worry too much about that – see the next Assignment.

Exercise # 5: A sub-collection $\mathcal{C} \subseteq \mathcal{B}$ of a $\sigma-$algebra $\mathcal{B}$ is called a *generating system* (for $\mathcal{B}$), if $\mathcal{B} = \sigma(\mathcal{C})$.

Show that each of the collections

$$\mathcal{C}_1 \;=\; \{(a,b)\,|\,a \in \mathbb{R}\,,\, b \in \mathbb{R}\,,\, a < b\,\}\,, \quad \mathcal{C}_2 \;=\; \{[a,b]\,|\,a \in \mathbb{R}\,,\, b \in \mathbb{R}\,,\, a < b\,\}\,,$$

$$\mathcal{C}_3 \;=\; \{(a,b]\,|\,a \in \mathbb{R}\,,\, b \in \mathbb{R}\,,\, a < b\,\}\,, \quad \mathcal{C}_4 \;=\; \{[a,b)\,|\,a \in \mathbb{R}\,,\, b \in \mathbb{R}\,,\, a < b\,\}\,,$$

$$\mathcal{C}_5 \;=\; \{(a,\infty)\,|\,a \in \mathbb{R}\,\}\,, \quad \mathcal{C}_6 \;=\; \{\,O \subset \mathbb{R}\,|\;O \text{ open in } \mathbb{R}\,\}$$

is a generating system for the $\sigma-$algebra of Borel subsets $\mathcal{B}(\mathbb{R})$ of the real line.

Exercise # 6: In a sequence of independent coin tosses, what is the probability that n successes (heads) materialize before m failures (tails) do?

Exercise # 7: Do Problem 1.31 in Walsh (2012).

Exercise # 8: Do Problem 1.32 in Walsh (2012).

Exercise # 9: Do Problem 1.68 in Walsh (2012).

Exercise # 10: Do Problem 1.69 in Walsh (2012).

Exercise # 11: Do Problem 1.70 in Walsh (2012).

Exercise # 12: Do Problem 1.71 in Walsh (2012).

Exercise # 13: Do Problem 1.72 in Walsh (2012).