

## OPTIMAL INVESTMENT STRATEGIES FOR CONTROLLING DRAWDOWNS

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We analyze the optimal risky investment policy for an investor who, at each point in time, wants to lose no more than a fixed percentage of the maximum value his wealth has achieved up to that time. In particular, if  $M_t$  is the maximum level of wealth  $W$  attained on or before time  $t$ , then the constraint imposed on his portfolio choice is that  $W_t \geq \alpha M_t$ , where  $\alpha$  is an exogenous number between 0 and 1. We show that, for constant relative risk aversion utility functions, the optimal policy involves an investment in risky assets at time  $t$  in proportion to the “surplus”  $W_t - \alpha M_t$ . The optimal policy may appear similar to the constant-proportion portfolio insurance policy analyzed in Black and Perold (1987) and Grossman and Vila (1989). However, in those papers, the investor keeps his wealth above a *nons stochastic* floor  $F$  instead of a stochastic floor  $\alpha M_t$ . The *stochastic* character of the floor studied here has interesting effects on the investment policy in states of nature when wealth is at an all-time high; i.e., when  $W_t = M_t$ . It can be shown that at  $W_t = M_t$ ,  $\alpha M_t$  is expected to grow at a faster rate than  $W_t$ , and therefore the investment in the risky asset can be expected to fall. We also show that the investment in the risky asset can be expected to rise when  $W_t$  is close to  $\alpha M_t$ . We conjecture that in an equilibrium model the stochastic character of the floor creates “resistance” levels as the market approaches an all-time high (because of the reluctance of investors to take more risk when  $W_t = M_t$ ).

KEY WORDS: drawdown, portfolio insurance, semimartingales, stochastic control

### 1. STATEMENT OF THE PROBLEM

We assume that there are assets in a financial market: one risky asset called “stock,” and one risk-free asset called “cash.” The financial market is frictionless. The rate of return on cash is a constant  $r$ , while the stock price process follows a geometric Brownian motion, with drift  $\mu + r$  and variance  $\sigma^2$ :

$$(1.1) \quad dP_t = P_t[(\mu + r) dt + \sigma dZ_t], \quad \text{and} \quad P_0 > 0,$$

where  $Z_t$  is a standard Brownian motion.<sup>1</sup>

Let  $\bar{W}_t$  be the wealth of a portfolio. Wealth dynamics is given by

$$(1.2) \quad d\bar{W}_t = r\bar{W}_t dt + \bar{x}_t[\mu dt + \sigma dZ_t],$$

with  $\bar{x}_t$  denoting the dollar holdings of stock. It is assumed that  $\bar{x}_t$  is a nonanticipating stochastic process such that  $\bar{W}_t$  is well defined by the equation  $d\bar{W}_t = r\bar{W}_t dt + \bar{x}_t(\mu dt + \sigma dZ_t)$  and such that the wealth is always nonnegative. Let  $M_0$  be a positive number which denotes the all-time-high value of discounted wealth that has occurred by time zero.

<sup>1</sup>The information revelation process is described by the filtration generated by the Brownian motion. The return from holding the stock is a pure capital gain; there are no dividends paid.

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Let  $\bar{\lambda}$  be a growth (or discounting) factor (to be further discussed). Define

$$(1.3) \quad \bar{M}_t = \max\{M_0 e^{\bar{\lambda}t}, \bar{W}_s e^{\bar{\lambda}(t-s)}; s \leq t\}.$$

For example, if  $\bar{\lambda} = 0$ , then  $\bar{M}_t$  is the higher of the previous all-time-high  $M_0$ , and the highest level of wealth  $\bar{W}_s$  achieved between  $s = 0$  and  $s = t$ . We also want to consider cases where the all-time high can grow because of an interest rate or decay. In particular, if  $\bar{\lambda} = -\infty$ , then the constraint  $\bar{W}_t \geq \alpha \bar{M}_t$  becomes  $\bar{W}_t \geq 0$ , and as we consider algebraically larger and larger values of  $\bar{\lambda}$ , the importance of the recent past grows. Note that at time  $t$  when  $\bar{W}_t \neq \max\{M_0 e^{\bar{\lambda}t}, \bar{W}_s e^{\bar{\lambda}(t-s)}; s \leq t\}$ ,  $\bar{M}_t$  locally grows at a rate of  $\bar{\lambda}$ ; and when  $\bar{W}_t = \max\{M_0 e^{\bar{\lambda}t}, \bar{W}_s e^{\bar{\lambda}(t-s)}; s \leq t\}$ , we set  $\bar{M}_t = \bar{W}_t$ .

Let  $\alpha \in (0, 1)$ . The wealth of a portfolio is never allowed to go below  $\alpha \bar{M}_t$ . That is, it is required that

$$(1.4) \quad \bar{W}_t \geq \alpha \bar{M}_t$$

almost surely for all  $t$ . We will only consider values of  $\bar{\lambda}$  such that  $\bar{\lambda} \leq r$ , because if  $\bar{\lambda} > r$  then there is no feasible strategy.

The term  $1 - \bar{W}_t/\bar{M}_t$  is called the *drawdown* from the previous all-time high (adjusted for a discount factor). The authors' knowledge and experience of the area of investment management where leverage is used extensively (such as the trading of futures, options, and foreign exchange) has convinced us that an essential aspect of the evaluation of investment managers and their strategies is the extent to which large drawdowns occur. It is not unusual for such managers to be fired subsequent to achieving a large drawdown, nor is it unusual for the managers to be told to avoid drawdowns larger than 25%. We make the extreme assumption that a drawdown larger than a prespecified amount,  $1 - \alpha$ , is unacceptable.

Another interpretation of the constraint in (1.4) faced by an investment manager can be understood as follows. His supervisor (the "owner") psychologically (and often physically) "books" (i.e., commits to other uses) a proportion of the profits made by the manager when he reaches an all-time high. So, if the manager loses  $1 - \alpha$  of the all-time high, it is as if the manager lost *all* of the wealth that can be used for investment purposes. The parameter  $\bar{\lambda}$  governs the rate at which the owner discounts booked profits from prior all-time highs (when  $\bar{\lambda} < 0$ ) or the rate at which he requires the manager to outperform an interest rate (when  $\bar{\lambda} > 0$ ) in order to keep money under management.

We assume that the portfolio manager has a constant relative risk aversion utility function  $U(W) = W^{1-A}/(1-A)$ , where  $A > 0$  and  $A \neq 1$ . The manager's objective is to choose a trading strategy to maximize the long-term growth rate of the expected utility of final wealth—that is, to maximize

$$(1.5) \quad \liminf_{T \rightarrow \infty} \frac{1}{(1-A)T} \ln E[(1-A)U(\bar{W}_T)]$$

subject to the portfolio dynamics in (1.2) and the constraint  $\bar{W}_t \geq \alpha \bar{M}_t$  for all  $t$ . Note that, if  $T$  is finite, maximizing  $[(1-A)T]^{-1} \ln E[(1-A)U(\bar{W}_T)]$  is equivalent to maximizing  $E[U(\bar{W}_T)]$ .

Let  $\bar{\mathcal{A}}$  denote the set of admissible trading strategies. That is,  $\bar{\mathcal{A}}$  is the set of all non-anticipating stochastic processes such that the wealth process is well defined by the stochastic differential equation  $d\bar{W}_t = r\bar{W}_t dt + \bar{x}(\mu dt + \sigma dZ_t)$  and such that  $\bar{W}_t \geq \alpha \bar{M}_t$ ,

for all  $t$ . We let  $\bar{\xi}$  denote the maximum long-term growth rate of the expected utility of final wealth:

$$(1.6) \quad \bar{\xi} \equiv \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)U(\bar{W}_T)].$$

The above optimization problem is strongly related to the problem of searching for an optimal stationary investment policy. For a justification of the objective function in (1.5), see Grossman and Vila (1989), Fleming et al. (1990), and Taksar et al. (1988).

### 1.1. A Benchmark Example: The Merton Case

For  $\alpha = 0$ , (1.4) becomes  $\bar{W}_t \geq 0$  for all  $t$ , which is the no-bankruptcy constraint. Merton (1971) has shown that for any finite  $T$  the optimal investment strategy for maximizing  $E[U(\bar{W}_T)]$ , subject to the no-bankruptcy constraint, is given by  $\bar{x}_t^* = (\mu/A\sigma^2)\bar{W}_t$ . Since  $\bar{x}_t^*$  does not depend on  $T$ , we can show that  $\bar{x}^* \in \bar{\mathcal{A}}$  also solves our problem defined in (1.5) when  $\alpha = 0$ .<sup>2</sup> The corresponding growth rate of utility can be computed to be  $\bar{\xi} = \mu^2/2A\sigma^2 + r$ .

It is shown in Appendix A that, if  $\alpha > 0$ , Merton's strategy will violate (1.4) with probability 1.

### 1.2. A Transformation of the Problem

Define  $W_t = \bar{W}_t e^{-\bar{\lambda}t}$  and  $M_t = \bar{M}_t e^{-\bar{\lambda}t} = \max\{M_0, W_s; s \leq t\}$ . Then we have the dynamics for  $W_t$ :  $dW_t = \lambda W_t dt + x_t(\mu dt + \sigma dZ_t)$ , where  $\lambda \equiv r - \bar{\lambda}$  and  $x_t = \bar{x}_t e^{-\bar{\lambda}t}$ . The portfolio constraint  $\bar{W}_t \geq \alpha \bar{M}_t$  becomes  $W_t \geq \alpha M_t$ . Let  $\mathcal{A}$  be the set of feasible strategies  $x_t$  such that  $W_t \geq \alpha M_t$  and  $W_t$  and  $M_t$  are as given above. We let  $\xi$  be the long-run growth rate in (1.6) when  $W_T$  replaces  $\bar{W}_T$ .

We concentrate on the problem under the no-bar notation. Note that replacing  $\bar{W}_T$  with  $e^{\bar{\lambda}T} W_t$  in (1.6) implies that  $\bar{\xi} = \xi + \bar{\lambda}$ . Once the problem under the no-bar notation is solved, the solution to the original bar problem is immediate:  $\bar{x}_t = x_t e^{\bar{\lambda}t}$  is the dollar investment in the stock and  $\bar{W}_t = W_t e^{\bar{\lambda}t}$  is the wealth process.

### 1.3. Our Solution Method

Our solution method is based on the following theorem, whose proof is given in Appendix F.

**THEOREM 1.1.** *If there exist a constant  $\xi$  and a function  $V(W, M)$  such that (i) for all  $t > 0$ ,  $V(W, M)$  satisfies the Bellman equation*

$$(1.7) \quad V(W, M) = \sup_{\mathcal{A}} E[V(W_t, M_t) e^{-(1-A)\xi t}],$$

<sup>2</sup>To see this, consider any strategy  $\bar{x} \in \bar{\mathcal{A}}$ . By the definition of  $\bar{x}^*$ , we have  $E[U(\bar{W}_T)] \leq E[U(\bar{W}_T^*)]$ , where  $\bar{W}_T$  and  $\bar{W}_T^*$  are associated with strategies  $\bar{x}$  and  $\bar{x}^*$  respectively. Note that  $E[U(\bar{W}_T)] \leq E[U(\bar{W}_T^*)]$  implies that

$$\liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)U(\bar{W}_T)] \leq \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)U(\bar{W}_T^*)],$$

which shows that  $\bar{x}^*$  solves the problem in (1.5).

where  $W_0 = W$  and  $M_0 = M$ , (ii) there exists a trading strategy  $x^* \in \mathcal{A}$  such that the above supremum over  $\mathcal{A}$  can be attained by  $x^*$  for all  $t > 0$ , and (iii) there exist positive constants  $C_1$  and  $C_2$  such that

$$(1.8) \quad C_1(1 - A)U(W) \leq (1 - A)V(W, M) \leq C_2(1 - A)U(W),$$

then  $\xi$  is the maximum long-term growth rate of the expected utility of final wealth, and the growth rate  $\xi$  is achieved by following the star strategy  $x^*$ . Moreover,  $\xi$  is also the growth rate of the finite horizon problem. In other words,

$$(1.9) \quad \xi = \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln [(1 - A)L(W, M, T)],$$

where  $L(W, M, T)$  is the value function of the finite horizon problem. That is,

$$(1.10) \quad L(W, M, T) = \sup_{\mathcal{A}} EU(W_T).$$

To construct a function  $V(W, M)$  and a constant  $\xi$  such that the assumptions of the theorem hold, we find it very helpful to heuristically consider an optimization problem twin to the growth optimization problem, which is to maximize the long-term discounted expected utility:

$$(1.12) \quad \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} E[U(W_T)e^{-(1-A)\xi T}],$$

where  $\xi$  is a constant. Note that there exists at most one  $\xi$  such that the supremum in (1.12) is neither zero nor infinity.<sup>3</sup> If such a  $\xi$  does exist, then it is the maximum growth rate.<sup>4</sup>

Define

$$(1.13) \quad \bar{V}(W, M) \equiv \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} E[U(W_T)e^{-(1-A)\xi T}].$$

We shall use properties of  $\bar{V}$  as a basis for guessing properties of  $V$  and  $\xi$  which satisfy the conditions of Theorem 1.1. Note that according to Theorem 1.1, we are providing a

<sup>3</sup>To see this, let  $\xi_1$  and  $\xi_2$  be two constants such that  $\xi_1 > \xi_2$ . Note that

$$E[U(W_T)e^{-(1-A)\xi_1 T}] = e^{-(1-A)(\xi_1 - \xi_2)T} E[U(W_T)e^{-(1-A)\xi_2 T}]$$

implies that if the supremum in (1.12) is neither zero nor infinity for  $\xi = \xi_2$ , then the supremum in (1.12) is either zero or infinity for  $\xi = \xi_1$ .

<sup>4</sup>If there exists a constant  $\xi$  such that the supremum in (1.12) is neither zero nor infinity, then the identity

$$\frac{1}{(1 - A)T} \ln E[(1 - A)U(W_T)] = \xi + \frac{1}{(1 - A)T} \ln E[(1 - A)U(W_T)e^{-(1-A)\xi T}]$$

implies that the maximum growth rate is equal to  $\xi$ , because

$$\sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)U(W_T)e^{-(1-A)\xi T}] = 0.$$

solution to the optimization problem in (1.5) as long as we can construct  $\xi$  and  $V(W, M)$  satisfying all the assumptions of that theorem. The only reason for us to study the problem in (1.13) is that it will help us construct such  $\xi$  and  $V(W, M)$ .

To motivate the Bellman equation in (1.7) for  $V(W, M)$ , we observe that for  $h > 0$ , (1.13) implies

$$\begin{aligned}\tilde{V}(W, M) &= \sup_{(x)_0^\infty} \liminf_{T \rightarrow \infty} E[U(W_T)e^{-(1-A)\xi T}] \\ &= \sup_{(x)_0^\infty} \liminf_{T \rightarrow \infty} E[U(W_{h+T})e^{-(1-A)\xi(h+T)}] \\ &= \sup_{(x)_0^\infty} e^{-(1-A)\xi h} \liminf_{T \rightarrow \infty} E_0\{E_h[U(W_{h+T})e^{-(1-A)\xi T}]\} \\ &= \sup_{(x)_0^h} e^{-(1-A)\xi h} \sup_{(x)_h^\infty} \liminf_{T \rightarrow \infty} E_0\{E_h[U(W_{h+T})e^{-(1-A)\xi T}]\}.\end{aligned}$$

If  $\liminf_{T \rightarrow \infty}$  and  $E_0$  can change positions in the above equation, then we have<sup>5</sup>

$$\begin{aligned}\tilde{V}(W, M) &= \sup_{(x)_0^h} e^{-(1-A)\xi h} E_0\{\sup_{(x)_h^\infty} \liminf_{T \rightarrow \infty} [E_h U(W_{h+T})e^{-(1-A)\xi T}]\} \\ &= \sup_{(x)_0^h} E_0[e^{-(1-A)\xi h} \tilde{V}(W_h, M_h)],\end{aligned}$$

which is the same functional equation that  $V$  in (1.7) satisfies.

In order to help us guess a  $V$  and  $\xi$  which satisfy conditions of Theorem 1.1, we use the following two properties of  $\tilde{V}$ .

**PROPOSITION 1.1.** *If there exists a constant  $\xi$  such that  $\tilde{V}(W, M)$  is finite for  $W \geq \alpha M$ , then, for  $W \geq \alpha M$ ,  $\tilde{V}(W, M)$  is homogeneous of degree  $1 - A$  in  $W$  and  $M$ . That is, for all positive  $k$ ,  $\tilde{V}(kW, kM) = k^{1-A}\tilde{V}(W, M)$ .*

*Proof.* Let  $x$  be a feasible trading strategy associated with an initial state  $(W_0, M_0)$ . Then for the initial state  $(kW_0, kM_0)$ , the trading strategy  $kx$  is feasible because the wealth process associated with this strategy,  $dW_{kt} = \lambda W_{kt} dt + kx(\mu dt + \sigma dZ_t)$ , satisfies  $W_{kt} = kW_{1t}$  so that  $W_{kt} = kW_{1t} \geq k\alpha M_{1t} = \alpha M_{kt}$ . Since  $W_{kT}^{1-A} = k^{1-A}W_{1T}^{1-A}$ , we have just established that  $\tilde{V}(kW, kM) \geq k^{1-A}\tilde{V}(W, M)$ . Observe that

$$\tilde{V}(W, M) = \tilde{V}(k^{-1}kW, k^{-1}kM) \geq (k^{-1})^{1-A}\tilde{V}(kW, kM),$$

so we have  $\tilde{V}(kW, kM) \leq k^{1-A}\tilde{V}(W, M)$ . The proof is complete.  $\square$

**PROPOSITION 1.2.** *Assume that there exists a constant  $\xi$  such that  $\tilde{V}(W, M)$  is finite for  $W \geq \alpha M$ . Holding  $M$  constant,  $\tilde{V}(W, M)$  is an increasing and concave function of  $W$ . Holding  $W$  constant,  $\tilde{V}(W, M)$  is a decreasing function of  $M$ .*

<sup>5</sup>Note that Fatou's lemma can guarantee a one-sided inequality. Since our verification theorems in Section 4 do not depend upon the derivation of the Bellman equation, we do not state conditions under which  $\liminf$  and the expectation operator can change their positions.

*Proof.* It is trivial to see that  $\bar{V}(W, M)$  is increasing in  $W$  if  $M$  is held constant. It is also trivial to show that  $\bar{V}(W, M)$  is decreasing in  $M$  if  $W$  is held constant. To show concavity, let  $(W_{10}, M_0)$  and  $(W_{20}, M_0)$  be two initial states, and  $z \in (0, 1)$ . Let  $x_1$  and  $x_2$  be feasible strategies for the initial states  $(W_{10}, M_0)$  and  $(W_{20}, M_0)$ , respectively. Define  $x = zx_1 + (1 - z)x_2$ . Then  $x$  is a feasible strategy for the initial state  $(zW_{10} + (1 - z)W_{20}, M_0)$  because

$$\begin{aligned} zW_{1t} + (1 - z)W_{2t} &\geq z\alpha M_{1t} + (1 - z)\alpha M_{2t} \\ &\geq \alpha \max\{M_0, zW_{1s} + (1 - z)W_{2s}; s \leq t\}. \end{aligned}$$

Since the concavity of  $U(\cdot)$  implies that  $U[\lambda W_{1T} + (1 - \lambda)W_{2T}] \geq \lambda U(W_{1T}) + (1 - \lambda)U(W_{2T})$ , the concavity of  $\bar{V}(W, M)$  in  $W$  follows because, for any two series  $x_n$  and  $y_n$ , the  $\liminf$  of  $x_n + y_n$  is no less than the sum of  $\liminf$  of  $x_n$  and the  $\liminf$  of  $y_n$ .  $\square$

In the next section, we provide a proposed closed-form solution for  $V$ ,  $\xi$ , and the optimal trading strategy when  $\lambda = 0$ . In Section 3 we deal with the case  $\lambda > 0$ . We must caution the reader that the next two sections should be taken heuristically. Section 4 proves the verification theorems, showing that the trading strategies in Sections 2 and 3 indeed maximize the objective in (1.5) subject to (1.4).

## 2. THE CASE $\lambda = 0$

In this section we assume that  $\lambda = 0$ , i.e., that  $\bar{\lambda} = r$ . Under this assumption,

$$(2.1) \quad dW_t = x_t(\mu dt + \sigma dZ_t).$$

We use (1.7) and the characteristics of  $\bar{V}$  in Propositions 1.1 and 1.2 to derive a differential equation for  $V$  and an equation for  $\xi$ . We will explicitly solve the equations for  $V$  and  $\xi$  and find the candidate trading strategy. Theorem 4.1 will show that  $\xi$  is the maximum growth rate and that the candidate trading strategy is indeed the optimal trading strategy for achieving  $\xi$ .

Note that by definition,  $W \in [\alpha M, M]$ . Whenever  $W_t = \alpha M_t$ , we claim that the portfolio manager must put all the money into bonds in order to be certain that  $W_t \geq \alpha M_t$ . An argument runs as follows: suppose  $W_t = \alpha M_t$  and the dollar investment in the stock is  $x$  for a short time  $[t, t + h]$ . Then the constraint

$$W_{t+h} = W_t + x[\mu h + \sigma(Z_{t+h} - Z_t)] \geq \alpha M_{t+h} \geq \alpha M_t = W_t$$

implies that with probability 1,  $x[\mu h + \sigma(Z_{t+h} - Z_t)] \geq 0$ , which is impossible unless  $x = 0$ .

In the interior region,  $W \in (\alpha M, M)$ , we have  $dM = 0$ . Thus, for  $W \in (\alpha M, M)$  and small  $h > 0$ , (1.7) implies

$$(2.2) \quad 0 = \max_x E[V(W_h, M)e^{-(1-A)\xi h} - V(W, M)].$$

Applying Ito's formula to  $E[V(W_h, M)e^{-(1-A)\xi h} - V(W, M)]$ , dividing by  $h$ , and letting  $h \rightarrow 0$ , we get the usual Bellman equation:

$$(2.3) \quad 0 = \max_x [-(1-A)\xi V + \frac{\partial V}{\partial W}\mu x + \frac{1}{2}\frac{\partial^2 V}{\partial W^2}\sigma^2 x^2].$$

The first-order condition implies that the optimal investment  $x$  equals

$$(2.4) \quad x^* = -\frac{\partial V}{\partial W}\mu \bigg/ \frac{\partial^2 V}{\partial W^2}\sigma^2.$$

Substituting  $x^*$  back into (2.3), we obtain the partial differential equation

$$(2.5) \quad -(1-A)\xi V - \frac{1}{2}\left(\left(\frac{\partial V}{\partial W}\right)^2\mu^2 \bigg/ \frac{\partial^2 V}{\partial W^2}\sigma^2\right) = 0.$$

## 2.1. Analysis of $V(W, M)$ for $W \in [\alpha M, M)$

Because of Proposition 1.1, we search for solutions to (2.5) which are homogeneous; that is,

$$V(W, M) = V\left(M \cdot \frac{W}{M}, M \cdot 1\right) = M^{1-A}f(u),$$

where  $u \equiv W/M$  and  $f(u) \equiv V(u, 1)$ . Note that

$$(2.6) \quad \frac{\partial V}{\partial W} = M^{-A}f'(u),$$

$$(2.7) \quad \frac{\partial^2 V}{\partial W^2} = M^{-A-1}f''(u),$$

$$(2.8) \quad \frac{\partial V}{\partial M} = (1-A)M^{-A}f(u) - M^{-A}uf'(u).$$

Substituting (2.6)–(2.8) into (2.5) yields the differential equation

$$(2.9) \quad 0 = -(1-A)\xi f(u) - \frac{1}{2}\frac{\mu^2[f'(u)]^2}{\sigma^2 f''(u)} \quad \text{for } u \in (\alpha, 1).$$

LEMMA 2.1. *For nonnegative  $a$ , the differential equation*

$$(2.10) \quad 0 = f(u) + a\frac{[f'(u)]^2}{f''(u)}$$

has a general solution of the form  $f(u) = \gamma(u + c)^{1/(1+a)}$ , where  $c$  and  $\gamma (\neq 0)$  are constants.

*Proof.* We can rewrite (2.9) as

$$(2.11) \quad \frac{f''(u)}{f'(u)} = -a \frac{f'(u)}{f(u)}.$$

Taking indefinite integrals on both sides of the equation, we obtain  $\ln f'(u) = -a \ln f(u) + c_1$ , or, equivalently,  $f'(u)[f(u)]^a = e^{c_1}$ . Integrating this expression gives  $[f(u)]^{1+a} = (1+a)(e^{c_1}u + c_2)$ , which implies  $f(u) = \gamma(u + c)^{1/(1+a)}$  for  $\gamma \equiv [(1+a)e^{c_1}]^{1/(1+a)}$ , and  $c \equiv c_2e^{-c_1}$ .  $\square$

Using Lemma 2.1, we can write the solution to (2.9) in the form  $f(u) = \gamma(u + c)^\beta$  for nonzero  $\gamma$ , where  $\beta$  satisfies

$$(2.12) \quad (1 - A)\xi = \frac{1}{2} \frac{\mu^2}{\sigma^2} \frac{\beta}{1 - \beta}.$$

From (2.4) and the form of  $f(u)$ , we obtain that, for  $W < M$ , the optimal investment in stocks is given by  $x = [\mu/\sigma^2(1 - \beta)](u + c)M$ . Since  $x = 0$  when  $u = \alpha$ , we have  $c = -\alpha$ . Thus,

$$(2.13) \quad x = \frac{\mu}{\sigma^2(1 - \beta)}(W - \alpha M),$$

which is a constant-proportion investment plan. The only unknown constant,  $\beta$ , will be determined in the next subsection.

## 2.2. Analysis of $V(W, M)$ for $W = M$

We have analyzed  $V(W, M)$  for  $u \in [\alpha, 1)$ . It remains for us to analyze  $V(W, M)$  at  $u = 1$ , i.e., at  $W = M$ . This requires an analysis of the process  $M_t$ . Note that  $M_t$  is a continuous increasing process and thus is a finite-variation process. Therefore both  $W_t$  and  $M_t$  are continuous semimartingales.<sup>6</sup> To see what the Bellman equation looks like at  $W = M$ , we need the following lemma.

**ITO'S LEMMA FOR SEMIMARTINGALES.** *Let  $X = (X^1, \dots, X^n)$  be an  $n$ -tuple of continuous semimartingales, and let  $f: R^n \rightarrow R$  have continuous second-order partial derivatives. Then  $f(X)$  is a semimartingale and the following formula holds:*

$$(2.14) \quad f(X_t) - f(X_0) = \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(X_s) dX_s^i + \frac{1}{2} \sum_{1 \leq i, j \leq n} \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s) d[X^i, X^j]_s,$$

where  $[X^i, X^j]$  is the quadratic covariation of  $X^i$  and  $X^j$ .

<sup>6</sup>A semimartingale is a local martingale plus a finite-variation process (see Protter 1990). In particular, a finite-variation process itself is a semimartingale. It is well known that diffusion processes are semimartingales.



This version of Ito's lemma can be found in Protter (1990, p. 74). In addition, Protter (1990, p. 68) contains a theorem which implies that if  $X$  is a continuous finite-variation process, then, for any semimartingale  $Y$ , the quadratic covariation of  $X$  and  $Y$ ,  $[X, Y]_t$ , is equal to  $X_0 Y_0$  for all  $t$ . Therefore,  $d[W, M]_t = 0$  and  $d[M, M]_t = 0$ .

Applying Ito's lemma for semimartingales to the Bellman equation for  $V(W, M)$ , we obtain

$$(2.15) \quad V(W, M) = \sup_{(x)_0^h} E[e^{-(1-A)\xi h} V(W_h, M_h)].$$

We conclude that when  $W = M$ , the Bellman equation becomes

$$(2.16) \quad 0 = \sup_{(x)_0^h} \left\{ E \int_0^h \left[ -(1-A)\xi V + \frac{\partial V}{\partial W} \mu x + \frac{1}{2} \frac{\partial^2 V}{\partial W^2} \sigma^2 x^2 \right] ds + E \int_0^h \frac{\partial V}{\partial M} dM \right\}.$$

LEMMA 2.2.

$$(2.17) \quad E_0[M_h - M_0 \mid W_0 = M_0] = \sqrt{2/\pi} \sigma |x| \sqrt{h} + O(h).$$

*Proof.* Note that  $E[M_h - M_0] = E[\max_{0 \leq s \leq h} \int_0^s (\mu dl + \sigma dZ_l)]$ , which is  $O(h) + \sigma |x| \cdot E(\max_{0 \leq s \leq h} Z_s)$ . Note that  $\max_{0 \leq s \leq h} Z_s$  has the same distribution as the reflected Brownian motion  $|Z_h|$  (see Harrison 1985, p. xii for details). Therefore,  $Z_h \sim N(0, h)$  implies

$$E(\max_{0 \leq s \leq h} Z_s) = E|Z_h| = \sqrt{2/\pi} \sqrt{h} + O(h).$$

Thus,  $E[M_h - M_0]$  equals  $\sqrt{2/\pi} \sigma |x| \sqrt{h}$ . □

Note that  $x = 0$  is not optimal at  $W_0 = M_0$  because this would keep  $W_t = M_t = W_0$  forever and there would be no growth. Since the optimal  $x$  is not zero when  $W = M$ , we claim that at  $W = M$ ,  $\partial V / \partial M = 0$ . This is true because if  $\partial V / \partial M < 0$  there exist a positive  $\varepsilon$  and a positive  $h$  such that,  $\forall s \in [0, h]$ ,  $(\partial V / \partial M)_s \leq -\varepsilon$ . So

$$E \left[ \int_0^h \frac{\partial V}{\partial M} dM \right] \leq -\varepsilon E[M_h - M_0] = -\varepsilon \sqrt{\frac{2}{\pi}} \sigma |x| \sqrt{h} + O(h).$$

The Bellman equation (2.16) simply could not hold since the  $\sqrt{h}$  term in  $E[\int_0^h \partial V / \partial M dM]$  dominates all other terms for small  $h$ .

Note that Protter (1985, p. 54) asserts that if a semimartingale  $X$  has paths of finite variation on compact sets, then a stochastic integral with respect to  $X$  is indistinguishable from the Lebesgue-Stieltjes integral, computed path by path. Therefore, we can show

LEMMA 2.3. *Since  $\partial V/\partial M = 0$  at  $W = M$ , then almost surely*

$$(2.18) \quad \int_0^h \frac{\partial V}{\partial M} dM = 0.$$

*Proof.* Note that  $M_s$  can increase only when  $W_s = M_s$  and that, when  $W_s = M_s$  the integrand is zero. So the integral is zero. To be precise, let  $g(s) \equiv (\partial V/\partial M)_s$  and let  $m$  denote the measure induced by the finite-variation function  $M_s$ . Define  $H \equiv \{s \in [0, h] : W_s < M_s\}$  and  $H^c \equiv \{s \in [0, h] : W_s = M_s\}$ . Then

$$\int_0^h \frac{\partial V}{\partial M} dM = \int_H g(s) dm + \int_{H^c} g(s) dm.$$

The second integral on the RHS is zero because, on  $H^c$ ,  $g(s)$  is identically zero. To show that the first integral on the RHS is zero, we need only verify that  $m(H) = 0$ . It is easy to see that  $H$  is an open set, so  $H$  can be expressed as a union of countably many disjoint open intervals. Denote a typical such open interval by  $(a, b)$ . By the definition of measure  $m$ ,  $m[(a, b)] = M_b - M_a = 0$ . Therefore  $m(H) = 0$ , and the proof is complete.  $\square$

With Lemma 2.3, we can divide (2.16) by  $h$  and let  $h$  tend to zero to obtain

$$0 = \max_x \left[ -(1 - A)\xi V + \frac{\partial V}{\partial W} \mu x + \frac{1}{2} \frac{\partial^2 V}{\partial W^2} \sigma^2 x^2 \right].$$

Hence, the optimal investment at  $W = M$  is given by the same formula as in the case  $W < M$ , because the Bellman equation can be expressed in the same form.

The fact that  $\partial V/\partial M = 0$  is used in the following proposition to determine the long-run growth rate.

PROPOSITION 2.1. *Since  $\partial V/\partial M = 0$  at  $W = M$ , we have*

$$(2.19) \quad \beta = (1 - A)(1 - \alpha)$$

and

$$(2.20) \quad \xi = \frac{\mu^2}{2\sigma^2} \frac{1 - \alpha}{\alpha + (1 - \alpha)A}.$$

*Proof.* Note that, at  $u = 1$ ,

$$\frac{\partial V}{\partial M} = (1 - A)M^{-A}f(u) - M^{-A}uf'(u) = 0$$

implies that  $(1 - A)f(1) - f'(1) = 0$ . With  $f(u) = \gamma(u - \alpha)^\beta$ ,  $(1 - A)f(1) - f'(1) = 0$  implies that  $\beta = (1 - A)(1 - \alpha)$ . The expression for  $\xi$  is obtained from (2.12).  $\square$

### 2.3. Optimal Policy When $\lambda = 0$

Summarizing the results of Sections 2.1 and 2.2, we see that the optimal investment strategy is

$$(2.21) \quad X_t = k(W_t - \alpha M_t),$$

where

$$(2.22) \quad k \equiv \frac{\mu}{\sigma^2} \frac{1}{(1 - \alpha)A + \alpha}.$$

Recall that the fraction of wealth invested in stocks in the Merton case (i.e., where  $\alpha = 0$ ) is a constant and equal to  $\mu/\sigma^2 A$ . The ratio of the fraction invested in stocks in our case to the fraction when  $\alpha = 0$  is

$$(2.23) \quad q \equiv \frac{A(1 - \alpha/u_t)}{(1 - \alpha)A + \alpha},$$

which is always between 0 and 1. At  $u_t = 1$ ,  $q$  equals  $1 - \alpha/[(1 - \alpha)A + \alpha]$ . The number  $\alpha/[(1 - \alpha)A + \alpha]$  reflects the extent to which a more conservative investment policy is followed (even at an all-time high) due to an attempt to control drawdowns.

The analysis of the previous section can be repeated for the standard portfolio insurance problem where  $\alpha M_t$  is replaced by a fixed floor  $K$ , and it can be shown (see Appendix B) that the optimal dollar investment in the risky asset is

$$(2.24) \quad y_t = \frac{\mu}{\sigma^2 A} (W_t - K).$$

With our path-dependent floor, the optimal trading strategy (2.21) replaces  $K$  with  $\alpha M_t$ , where  $\alpha$  represents the degree of protection for the gains earned by trading the portfolio, and  $A$  is replaced by  $(1 - \alpha)A + \alpha 1$ , which is a convex combination of  $A$  and 1 with weights given by the degree of protection.

In both cases, the optimal investment strategies are of CPPI (that is, constant-proportion portfolio insurance) type. For a fixed floor, the position in risky assets is inversely related to the risk aversion coefficient. For a path-dependent floor, the position in risky assets can be understood in the following way: before wealth hits an all-time high, the portfolio manager behaves as if he faces a fixed floor but with a modified risk aversion equal to  $(1 - \alpha)A + \alpha 1$ ; when wealth hits all-time highs, the floor changes continuously but the risk aversion coefficient is still the same. The modified risk aversion is always between  $A$  and 1: when the degree of protection is close to 100%, the modified risk aversion gets close to 1 (which is the risk aversion of the logarithmic utility function); when the degree of protection is low, the modified risk aversion is close to  $A$ .

### 2.4. The Cost of Drawdown Control

Let  $\xi_\alpha$  be the growth rate when the degree of protection is  $\alpha$ . Thus,  $\xi_0$  corresponds to the growth rate in the Merton case. Obviously,  $\xi_\alpha$  is a strictly decreasing function of  $\alpha$ .

Observe that

$$\frac{\xi_\alpha}{\xi_0} = 1 - \left[ 1 + \left( \frac{1 - \alpha}{\alpha} \right) A \right]^{-1},$$

which is close to 1 for large  $A$  or for small  $\alpha$ . In other words, when the insurance requirement is imposed, the loss in growth rate is minimal if the portfolio manager is very risk averse or if the degree of protection is slight. On the other extreme, the loss in growth rate is maximal if the portfolio manager is close to risk neutrality or if the degree of protection is set close to 100%.

## 2.5. Characterization of the Optimal Wealth Process

We have shown that the optimal stock holdings should be  $x = k(W - \alpha M)$  for  $k = \mu/\sigma^2(1 - \beta)$ . Our main results in this subsection include an *explicit* representation of the optimal wealth process in terms of the Brownian motion and an *explicit* representation of the floor process. We establish that  $\ln(W_t/M_t - \alpha)$  follows a *regulated Brownian motion* process.<sup>7</sup>

Integrating (2.1), using  $x = k(W - \alpha M)$ , yields

$$(2.25) \quad W_t = W_0 + \int_0^t k(W_s - \alpha M_s)(\mu ds + \sigma dZ_s).$$

Using Ito's lemma for semimartingales, we obtain

$$(2.26) \quad \ln\left(\frac{W_t}{M_t} - \alpha\right) = \ln\left(\frac{W_0}{M_0} - \alpha\right) + \left(k\mu - \frac{1}{2}k^2\sigma^2\right)t + k\sigma Z_t - \int_0^t \frac{dM_s}{(1 - \alpha)M_s}.$$

Since  $M_t$  has finite variation,

$$(2.27) \quad \int_0^t \frac{dM_s}{(1 - \alpha)M_s} = \frac{1}{1 - \alpha} \ln \frac{M_t}{M_0}.$$

Thus,

$$(2.28) \quad \ln\left(\frac{W_t}{M_t} - \alpha\right) = \ln\left(\frac{W_0}{M_0} - \alpha\right) + \left(k\mu - \frac{1}{2}k^2\sigma^2\right)t + k\sigma Z_t - \frac{1}{1 - \alpha} \ln \frac{M_t}{M_0}.$$

We observe that  $\ln(W_t/M_t - \alpha)$  is bounded from above by  $\ln(1 - \alpha)$ . So we conjecture that  $(1 - \alpha)^{-1} \ln(M_t/M_0)$  serves as a regulator to keep the Brownian motion,  $\ln(W_0/M_0 - \alpha) + (k\mu - \frac{1}{2}k^2\sigma^2)t + k\sigma Z_t$ , from going above  $\ln(1 - \alpha)$ . Under the conjecture, we get

<sup>7</sup>For a definition of regulated Brownian motion, see Harrison (1985, p. 14).

$$(2.29) \quad \frac{1}{1-\alpha} \ln \frac{M_t}{M_0} = L_t,$$

where we define

$$(2.30) \quad L_t = \max_{s \leq t} \left[ \ln \left( \frac{W_0}{M_0} - \alpha \right) + \left( k\mu - \frac{1}{2}k^2\sigma^2 \right) s + k\sigma Z_s - \ln(1-\alpha) \right]^+,$$

or, equivalently,

$$(2.31) \quad M_t = M_0 e^{(1-\alpha)L_t}.$$

From (2.28), we obtain

$$(2.32) \quad W_t = \alpha M_0 e^{(1-\alpha)L_t} + (W_0 - \alpha M_0) \exp[-\alpha L_t + (k\mu - \frac{1}{2}k^2\sigma^2)t + k\sigma Z_t].$$

The conjecture is stated as Proposition 2.2 and proved in Appendix C.

**PROPOSITION 2.2.** *The solution to the stochastic differential equation*

$$(2.33) \quad W_t = W_0 + \int_0^t k(W_s - \alpha M_s)(\mu ds + \sigma dZ_s),$$

where  $M_t$  is defined to be  $\max_{s \leq t} [M_0, W_s]$ , is

$$(2.34) \quad W_t = \alpha M_0 e^{(1-\alpha)L_t} + (W_0 - \alpha M_0) \exp[-\alpha L_t + (k\mu - \frac{1}{2}k^2\sigma^2)t + k\sigma Z_t],$$

where we define

$$(2.35) \quad L_t = \max_{s \leq t} \left[ \ln \left( \frac{W_0}{M_0} - \alpha \right) + \left( k\mu - \frac{1}{2}k^2\sigma^2 \right) s + k\sigma Z_s - \ln(1-\alpha) \right]^+.$$

We observe that the growth rate of utility is  $(1-A)^{-1}$  multiplied by the growth rate of  $E[M_t^{1-A}]$ , because at every  $t$  the ratio of  $W_t$  over  $M_t$  is between  $\alpha$  and 1. Note that (2.31) and (2.19) imply

$$(2.36) \quad E[M_t^{1-A}] = M_0^{1-A} E\{\exp[\beta \max_{s \leq t} ((k\mu - \frac{1}{2}k^2\sigma^2)s + k\sigma Z_s)]\},$$

where, for the purpose of calculating the growth rate, we can assume  $W_0 = M_0$  without loss of generality. Replacing  $\max_{s \leq t} ((k\mu - \frac{1}{2}k^2\sigma^2)s + k\sigma Z_s)$  by  $(k\mu - \frac{1}{2}k^2\sigma^2)t + k\sigma Z_t$  in (2.36) gives that the growth rate of utility is at least as large as

$$(2.37) \quad \frac{1}{1-A} [\beta(k\mu - \frac{1}{2}k^2\sigma^2) + \frac{1}{2}\beta^2 k^2\sigma^2],$$

which, after simple substitution for  $k = \mu/\sigma^2(1 - \beta)$  and for  $\beta = (1 - A)(1 - \alpha)$ , coincides with the growth rate  $\xi$  given in (2.20). It is not surprising that if we treat  $k$  as a parameter to be chosen to maximize (2.37), the optimal  $k$  is given by  $k = \mu/\sigma^2(1 - \beta)$ .

## 2.6. Some Properties of the Optimal Trading Strategy

Note that the fraction of wealth  $f_t$  invested in stocks is

$$(2.38) \quad f_t \equiv x_t/W_t = k(1 - \alpha/u_t).$$

Clearly  $f_t$  is an increasing function of  $u_t$ , with  $f_t = k(1 - \alpha)$  when  $u_t = 1$  (i.e., when wealth is at an all-time high), and with  $f_t = 0$  when  $u_t = \alpha$  (i.e., when wealth hits the floor  $\alpha M_t$ ). The stochastic character of the floor  $\alpha M_t$  creates very sharp resistance to increases in  $f_t$  when  $u_t$  is close to 1. Further,  $u_t$  can be expected to fall when  $u_t = 1$  because  $u_t = 1$  is a reflecting barrier for the  $u_t$  process.

The above remarks suggest that  $E_t[df_t|u_t]$  is a declining function around  $u_t = 1$ . To see this, note that (2.28) and (2.29) imply

$$(2.39) \quad u_t = \alpha + (W_0/M_0 - \alpha) \exp[-L_t + (k\mu - \frac{1}{2}k^2\sigma^2)t + k\sigma Z_t],$$

from which we have  $du_t = (u_t - \alpha)(k\mu dt + k\sigma dZ_t - dL_t)$ . Applying Ito's lemma for semimartingales yields

$$(2.40) \quad E_t f_{t+h} - f_t = -k\alpha E_t \int_t^{t+h} \frac{u_s - \alpha}{u_s^2} \left[ -k\mu ds + dL_s + \left( \frac{u_s - \alpha}{u_s} \right) k^2 \sigma^2 ds \right].$$

If  $u_t$  is below 1 and  $h$  is small, then  $dL_s$  is close to zero for  $s \in (t, t+h)$ , and (2.40) implies that<sup>8</sup>

$$(2.41) \quad \lim_{h \rightarrow 0} \frac{1}{h} E_t[f_{t+h} - f_t] = -k\alpha \left( \frac{u_t - \alpha}{u_t^2} \right) \left[ -k\mu + \left( \frac{u_t - \alpha}{u_t} \right) k^2 \sigma^2 \right].$$

<sup>8</sup>The key in verifying (2.41) is to show that

$$\lim_{h \rightarrow 0} \frac{1}{h} E_t \int_t^{t+h} \frac{u_s - \alpha}{u_s^2} dL_s = 0.$$

Note that  $(u_s - \alpha)/u_s^2 \leq (1 - \alpha)/\alpha^2$ , so it suffices to show that  $\lim_{h \rightarrow 0} (1/h) E_t(L_{t+h} - L_t) = 0$ . Observe that the memoryless property of a Brownian motion regulator (see Harrison 1985, p. 21) implies

$$\begin{aligned} L_{t+h} - L_t &= \sup_{s \leq h} [\ln(u_t - \alpha) + \mu's + \sigma'(Z_{t+s} - Z_t) - \ln(1 - \alpha)]^+ \\ &\leq \sup_{s \leq h} [\mu's + \sigma'(Z_{t+s} - Z_t)] 1_\Lambda, \end{aligned}$$

where  $\mu' = k\mu - \frac{1}{2}k^2\sigma^2$ ,  $\sigma' = k\sigma$ , and  $1_\Lambda$  denotes the indicator function of set  $\Lambda$  for

$$\Lambda \equiv \{ \sup_{s \leq h} [\mu's + \sigma'(Z_{t+s} - Z_t)] \geq \ln(1 - \alpha) - \ln(u_t - \alpha) \}.$$

Thus, when  $u_t < 1$ , the sign of  $E_t d(x_t/W_t)$  is the same as the sign of  $\mu - (u_t - \alpha)k\sigma^2/u_t$ , or the sign of  $\mu - (1 - \alpha/u_t)\mu/(1 - \beta)$ , since  $k\sigma^2 = \mu/(1 - \beta)$ . Assume  $\mu > 0$ . Then  $E_t d(x_t/W_t)$  is positive iff  $\beta < \alpha/u_t$ , i.e., iff  $(1 - \alpha)(1 - A) < \alpha/u_t$ . This inequality will clearly hold for  $u_t$  close to  $\alpha$  as long as  $A > 0$ . Thus there exists a  $u^*$  such that  $E_t d(x_t/W_t)$  is positive iff  $u_t \in (\alpha, u^*)$ . Note that  $u^* = 1$  if  $\alpha > (1 - \alpha)(1 - A)$ , which is always true if  $\alpha \geq 0.5$  or  $A \geq 1$ .

We next show that even if  $u^* = 1$ ,  $E_t(f_{t+h} - f_t)$  will be negative for small  $h$  when  $u_t = 1$ . We will do this by showing that the negative term  $-k\alpha \int_t^{t+h} ((u_s - \alpha)/u_s^2) dL_s$  in (2.40) dominates all the other terms when  $u_t = 1$ . To see this, note that when  $u_t = 1$ , (2.28) and (2.29) imply

$$\ln(1 - \alpha) = \ln(u_0 - \alpha) - L_t + (k\mu - \frac{1}{2}k^2\sigma^2)t + k\sigma Z_t,$$

which can be rewritten as

$$L_t = \ln(u_0 - \alpha) + (k\mu - \frac{1}{2}k^2\sigma^2)t + k\sigma Z_t - \ln(1 - \alpha),$$

which implies that the right-hand side of the above equation is nonnegative.

Recall the definition of  $L_t$ :

$$L_t = \max_{s \leq t} [\ln(u_0 - \alpha) + (k\mu - \frac{1}{2}k^2\sigma^2)s + k\sigma Z_s - \ln(1 - \alpha)]^+.$$

Note that

$$\sup_{s \leq h} [\mu' s + \sigma'(Z_{t+s} - Z_t)] \leq |\mu'|h + |\sigma'| \sup_{s \leq h} (Z_{t+s} - Z_t),$$

so

$$L_{t+h} - L_t \leq [|\mu'|h + |\sigma'| \sup_{s \leq h} (Z_{t+s} - Z_t)] 1_\Gamma,$$

where

$$\Gamma \equiv \{|\mu'|h + |\sigma'| \sup_{s \leq h} (Z_{t+s} - Z_t) \geq \ln(1 - \alpha) - \ln(u_t - \alpha)\}.$$

Using Cauchy-Schwarz inequality gives

$$\frac{1}{h} E_t(L_{t+h} - L_t) \leq \left\{ E_t[|\mu'|h + |\sigma'| \sup_{s \leq h} (Z_{t+s} - Z_t)]^2 \right\}^{1/2} \left\{ \frac{1}{h} [P_t(\Gamma)]^{1/2} \right\}.$$

Thus it suffices to show that  $\lim_{h \rightarrow 0} P_t(\Gamma)/h^2 = 0$ . Note that  $P_t(\Gamma) = P_t[\sup_{s \leq h} (Z_{t+s} - Z_t) \geq a]$ , where

$$a \equiv [\ln(1 - \alpha) - \ln(u_t - \alpha) - |\mu'|h] / |\sigma'| \geq b \equiv [\ln(1 - \alpha) - \ln(u_t - \alpha)] / 2|\sigma'| > 0$$

for small  $h$ . Thus,  $P_t(\Gamma) \leq P_t[\sup_{s \leq h} (Z_{t+s} - Z_t) \geq b]$ . Since  $\sup_{s \leq h} (Z_{t+s} - Z_t)$  has the same distribution as  $|Z_{t+h} - Z_t|$ , we have

$$P_t[\sup_{s \leq h} (Z_{t+s} - Z_t) \geq b] = P_t[|Z_{t+h} - Z_t| \geq b] = \sqrt{\frac{2}{\pi}} \int_{b/\sqrt{h}}^{\infty} e^{-t^2/2} dt,$$

which is  $o(h^2)$ . This implies that  $\lim_{h \rightarrow 0} P_t(\Gamma) / h^2 = 0$ .

Thus, the maximum in the definition of  $L_t$  occurs at time  $t$  when  $u_t = 1$ . For  $h > 0$ , we have

$$\begin{aligned} L_{t+h} - L_t &= \max_{t \leq s \leq t+h} [\ln(u_0 - \alpha) + (k\mu - \tfrac{1}{2}k^2\sigma^2)s + k\sigma Z_s - \ln(1 - \alpha)] \\ &\quad - [\ln(u_0 - \alpha) + (k\mu - \tfrac{1}{2}k^2\sigma^2)t + k\sigma Z_t - \ln(1 - \alpha)] \\ &= \max_{t \leq s \leq t+h} [(k\mu - \tfrac{1}{2}k^2\sigma^2)(s - t) + k\sigma(Z_s - Z_t)]. \end{aligned}$$

As a consequence of Lemma 2, we have that for small  $h$ ,

$$\begin{aligned} E_t(L_{t+h} - L_t) &= E_t\left\{ \max_{t \leq s \leq t+h} [(k\mu - \tfrac{1}{2}k^2\sigma^2)(s - t) + k\sigma(Z_s - Z_t)] \right\} \\ &= \sqrt{2/\pi} |k|\sigma\sqrt{h} + O(h). \end{aligned}$$

Equation (2.40) thus gives us that for  $u_t = 1$ ,

$$(2.42) \quad E_t(f_{t+h} - f_t) = -k\alpha(1 - \alpha)[\sqrt{2/\pi} |k|\sigma\sqrt{h} + O(h)],$$

which is negative and is in the order of  $\sqrt{h}$  for positive  $\mu$ .

Indeed, by the same argument we can show that the *level* of the investment  $x_t$  is also expected to fall at  $u_t = 1$ . To see this, note that

$$(2.43) \quad E_t(x_{t+h} - x_t) = E_t[(f_{t+h} - f_t)W_{t+h}] + f_t E_t(W_{t+h} - W_t).$$

Note that from (2.33),  $E_t(W_{t+h} - W_t)$  is  $O(h)$ . Therefore by (2.42), if  $u_t = 1$ ,  $E_t(f_{t+h} - f_t)$  is negative and will dominate the rest of the terms in (2.43).

It may seem paradoxical that  $E_t(f_{t+h} - f_t) < 0$  if  $u_t = 1$ , while  $E_t df_t > 0$  for  $u_t < 1$  (assuming for example  $\alpha > 0.5$ ). This effect arises because

$$\lim_{h \rightarrow 0} \lim_{u \rightarrow 1} \frac{1}{h} E_t(f_{t+h} - f_t) \neq \lim_{u \rightarrow 1} \lim_{h \rightarrow 0} \frac{1}{h} E_t(f_{t+h} - f_t).$$

For fixed  $h$ , as  $u_t$  approaches 1, the term involving  $dL_s$  in (2.40) grows in importance and it is of order  $\sqrt{h}$ . For a fixed  $u_t < 1$ ,  $E_t(f_{t+h} - f_t)$  is of order  $h$ , because as  $h$  gets small the terms not involving  $dL_s$  in (2.40) dominate.

In Appendix D, we provide an explicit calculation of  $E_t[f_{t+h} - f_t | u_t = u]$  for arbitrary  $u$  and  $h$ . The results are graphed in Figures 2.1 and 2.2. The figures make clear that there is a sharp expected decrease in holdings of the risky asset as  $u_t$  approaches 1.

### 3. THE CASE $\lambda > 0$

In Section 2, we dealt with the case  $\lambda = 0$ . Note that  $\lambda = 0$  implies that the floor process increases at the rate of risk-free rate. With a no-arbitrage argument, one can easily verify that, starting with a finite wealth, if  $M_t$  increases at a rate in excess of the risk-free rate, there is no feasible strategy satisfying  $W_t \geq \alpha M_t$  for all  $t$ . The case that  $\lambda > 0$  corresponds



$$100 \cdot E_t \left[ \frac{f_{t+h}}{f_t} - 1 \mid u_t = u \right]$$

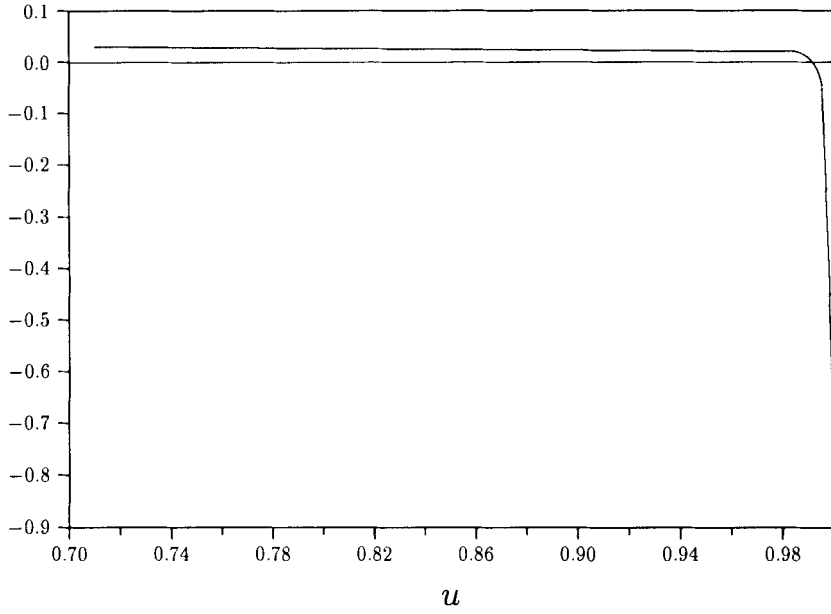


FIGURE 2.1. Expected daily percentage change in the fraction of wealth invested in stock ( $\lambda = 0$ ,  $\alpha = 0.7$ ,  $A = 2$ ,  $\mu = 0.06$ ,  $\sigma = 0.2$ ,  $h = 0.004$ ).

$$100 \cdot E_t \left[ \frac{f_{t+h}}{f_t} - 1 \mid u_t = u \right]$$

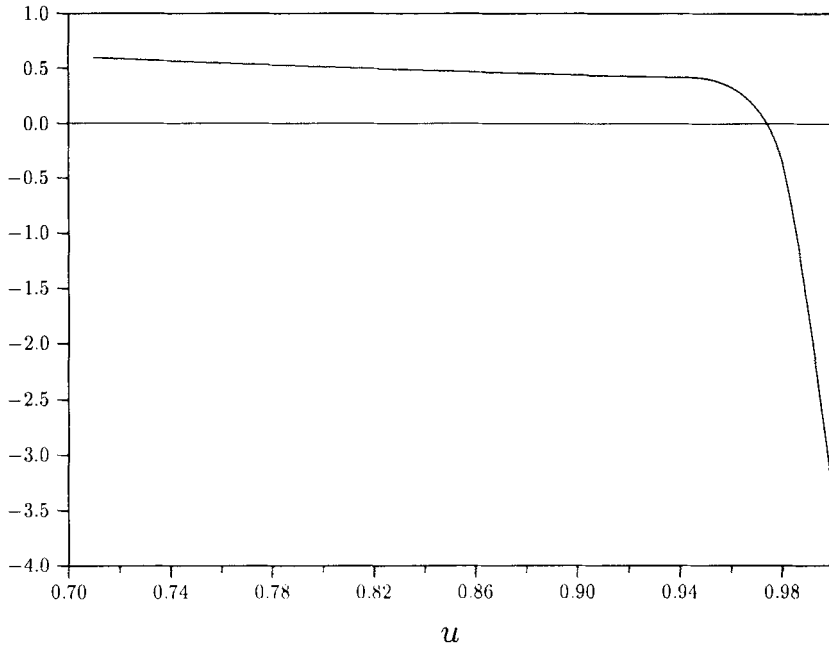


FIGURE 2.2. Expected monthly percentage change in the fraction of wealth invested in stock ( $\lambda = 0$ ,  $\alpha = 0.7$ ,  $A = 2$ ,  $\mu = 0.06$ ,  $\sigma = 0.2$ ,  $h = 0.088$ ).

to the assumption that the floor process increases at a rate less than  $r$ . As we shall see, even a small positive  $\lambda$  can significantly improve the expected value of the growth rate.

The solution method in this section is the same as that in the last section. We will use the Bellman equation (1.7) and the properties of  $\bar{V}$  in Propositions 1.1 and 1.2 to derive equations for  $V$  and  $\xi$ . We will then explain how to actually carry out numerical calculations for  $\xi$  and the candidate trading strategy. Theorem 4.1 will show that the  $\xi$  as constructed in this section is indeed the maximum growth rate, and that  $\xi$  can be achieved by following the candidate trading strategy as constructed in this section.

Recall that when  $\lambda > 0$ , the wealth process is given by

$$(3.1) \quad dW_t = \lambda W_t dt + x(\mu dt + \sigma dZ_t).$$

From the Bellman equation (1.7) and Ito's lemma, we have

$$(3.2) \quad 0 = \max_x \left[ -(1 - A)\xi V + \frac{\partial V}{\partial W} \lambda W + \frac{\partial V}{\partial W} \mu x + \frac{1}{2} \frac{\partial^2 V}{\partial W^2} \sigma^2 x^2 \right].$$

The first-order condition yields

$$(3.3) \quad x = - \mu \frac{\partial V}{\partial W} / \sigma^2 \frac{\partial^2 V}{\partial W^2}.$$

Substituting (3.3) into (3.2) gives

$$(3.4) \quad 0 = -(1 - A)\xi V + \frac{\partial V}{\partial W} \lambda W - \frac{\mu^2}{2\sigma^2} (\partial V / \partial W)^2 / \partial^2 V / \partial W^2.$$

Because of Proposition 1.1, we are motivated to search for  $V(W, M)$  that is homogeneous of degree  $1 - A$  in  $W$  and  $M$ . Thus, we write  $V(W, M) = M^{1-A} f(u)$  for some function  $f(u)$ , where  $u \equiv W/M$  is between  $\alpha$  and 1. Equation (3.3) then becomes

$$(3.5) \quad x = - \frac{\mu f'(u)}{\sigma^2 f''(u)} M,$$

and (3.4) can be rewritten as

$$(3.6) \quad 0 = -(1 - A)\xi f(u) + \lambda u f'(u) - \frac{\mu^2}{2\sigma^2} \frac{[f'(u)]^2}{f''(u)}.$$

Let  $y(u) \equiv f'(u)/f''(u)$ . Then, (3.6) becomes

$$(3.7) \quad 0 = -(1 - A)\xi f(u) + \lambda u f'(u) - \frac{\mu^2}{2\sigma^2} f'(u) y(u),$$

which, after differentiation, turns out to be

$$(3.8) \quad 0 = \frac{\mu^2}{2\sigma^2} y(u) y'(u) + \left[ (1 - A)\xi + \frac{\mu^2}{2\sigma^2} - \lambda \right] y(u) - \lambda u.$$

The general solution to (3.8) is

$$(3.9) \quad |y(u) - \theta_1 u|^{1-\delta} |y(u) - \theta_2 u|^{1+\delta} = c$$

for some constant  $c$ , where  $\theta_1 < 0$  and  $\theta_2 > 0$  satisfy

$$(3.10) \quad 0 = (\mu^2/2\sigma^2)\theta^2 + [(1-A)\xi + \mu^2/2\sigma^2 - \lambda]\theta - \lambda,$$

and where  $\delta$  is defined to be

$$(3.11) \quad \delta = -\frac{(1-A)\xi + \mu^2/2\sigma^2 - \lambda}{\sqrt{((1-A)\xi + \mu^2/2\sigma^2 - \lambda)^2 + 2\lambda\mu^2/2\sigma^2}}.$$

Now let us consider boundary conditions. At  $u = \alpha$ , the stake in risky assets should be zero. This implies that  $y(\alpha) = 0$ . Substituting  $y(\alpha) = 0$  into (3.9) gives an expression for  $c$ :

$$(3.12) \quad c = (-\theta_1 \alpha)^{1-\delta} (\theta_2 \alpha)^{1+\delta}.$$

At  $u = 1$ , we can use the heuristic in Section 2 to argue that  $\partial V/\partial M = 0$ . That is,  $f'(1) = (1-A)f(1)$ . This condition, together with (3.7), implies that

$$(3.13) \quad \xi - \lambda = -(\mu^2/2\sigma^2)y(1).$$

Equations (3.9)–(3.13) determine the value of the growth rate in the following way: from (3.13), we can write  $y(1)$  as a function of  $\xi$ ; from (3.10), we can represent  $\theta_1$  and  $\theta_2$  in terms of  $\xi$ ; (3.11) and (3.12) make  $\delta$  and  $c$  functions of  $\xi$ . Substituting  $\theta_1$ ,  $\theta_2$ ,  $\delta$ ,  $c$ , and  $y(1)$  (all as functions of one variable  $\xi$ ) into (3.9) evaluated at  $u = 1$ ,

$$|y(1) - \theta_1|^{1-\delta} |y(1) - \theta_2|^{1+\delta} = c,$$

gives an algebraic equation for  $\xi$ . We can solve  $\xi$  numerically from the algebraic equation. Some numerical results will be reported in Section 3.2.

With the growth rate computed, we can numerically calculate the optimal trading strategy  $y(u)$  through (3.9). Section 3.3 will discuss this issue.

### 3.1. Logarithmic Utility Function

In the case of logarithmic utility function,  $U(W) = \ln(W)$ , the drawdown control problem becomes one of choosing a feasible dynamic trading strategy to maximize the expected long-term growth of wealth. We will see that with only slight modifications, the procedure above can be carried through to solve this problem of maximizing the expected growth rate of wealth. We define the maximum growth rate to be

$$(3.14) \quad \theta \equiv \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{T} E[\ln(W_T)].$$

Simply put, our objective is to find the maximum growth rate  $\theta$  and to search for a

feasible trading strategy that will give rise to  $\theta$ . If we can construct a constant  $\theta$  and a function  $I(W, M)$  such that (i) for all  $t > 0$ ,  $I(W, M)$  satisfies the Bellman equation

$$(3.15) \quad I(W, M) = \sup_{\mathcal{A}} E[I(W_t, M_t) - \theta t],$$

where  $W_0 = W$  and  $M_0 = M$ , (ii) there exists a trading strategy  $x^* \in \mathcal{A}$  such that the above supremum over  $\mathcal{A}$  can be attained by  $x^*$  for all  $t > 0$ , and (iii) there exist constants  $C_1$  and  $C_2$  such that

$$(3.16) \quad C_1 + U(W) \leq I(W, M) \leq C_2 + U(W),$$

then we can construct a proof similar to the proof of Theorem 1.1 to show that  $\theta$  is indeed the maximum long-term growth rate of wealth, and the growth rate  $\theta$  can be achieved by following the star strategy  $x^*$ .

To help construct such  $\theta$  and  $I(W, M)$ , we define

$$(3.17) \quad \tilde{I}(W, M) = \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} E[\ln(W_T) - \theta T].$$

We use the properties of  $\tilde{I}(W, M)$  as a basis for guessing the properties of  $I(W, M)$ . For example,  $\tilde{I}(W, M)$  can be used to motivate (3.15) for  $I(W, M)$ . Since for any  $k > 0$ ,  $\tilde{I}(kW, kM) = \ln k + \tilde{I}(W, M)$ , we can take  $k = 1/M$  and write  $\tilde{I}(W, M) = \ln M + \tilde{I}(W/M, 1)$ . We are thus motivated to search for  $I(W, M)$  that can be written as  $I(W, M) = \ln M + h(u)$ , where  $h(u) \equiv I(u, 1)$  and  $u \equiv W/M$ .

Note that in the interior region,  $W \in (\alpha M, M)$ , we have  $dM = 0$ . Applying Ito's lemma to the Bellman equation (3.15) gives

$$(3.18) \quad 0 = \max_z \left[ -\theta + \frac{\partial I}{\partial W} \lambda W + \frac{\partial I}{\partial W} \mu z + \frac{1}{2} \frac{\partial^2 I}{\partial W^2} \sigma^2 z^2 \right],$$

where  $z$  denotes the dollar amount of investment in the risky asset. The first-order condition gives

$$(3.19) \quad z = -\frac{\partial I}{\partial W} \mu \bigg/ \frac{\partial^2 I}{\partial W^2} \sigma^2.$$

Substituting (3.19) into (3.18) gives

$$(3.20) \quad 0 = -\theta + \frac{\partial I}{\partial W} \lambda W - \frac{\mu^2}{2\sigma^2} (\partial I / \partial W)^2 / \partial^2 I / \partial W^2.$$

Substituting  $I(W, M) = \ln M + h(u)$  into (3.19) gives

$$z = -\frac{\mu}{\sigma^2} \frac{h'(u)}{h''(u)} M,$$

so the fraction of wealth invested in the risky asset equals  $-(\mu/\sigma^2) [h'(u)/uh''(u)]$ . Substituting  $I(W, M) = \ln M + h(u)$  into (3.20) yields the ODE

$$(3.21) \quad 0 = -\theta + \lambda u h'(u) - \frac{\mu^2 [h'(u)]^2}{2\sigma^2 h''(u)}.$$

For  $y(u) = h'(u)/h''(u)$ , (3.21) becomes

$$(3.22) \quad 0 = -\theta + \lambda u h'(u) - (\mu^2/2\sigma^2) h'(u) y(u),$$

which, after differentiation, turns out to be

$$(3.23) \quad 0 = (\mu^2/2\sigma^2) y(u) y'(u) + [\mu^2/2\sigma^2 - \lambda] y(u) - \lambda u.$$

As in (3.9), the solution to (3.23) is given by

$$(3.24) \quad |y(u) - \theta_1 u|^{1-\delta} |y(u) - \theta_2 u|^{1+\delta} = c$$

for some constant  $c$ , where  $\theta_1 < 0$  and  $\theta_2 > 0$  satisfy

$$(3.25) \quad 0 = (\mu^2/2\sigma^2) \theta^2 + (\mu^2/2\sigma^2 - \lambda) \theta - \lambda,$$

and where  $\delta$  is defined to be

$$(3.26) \quad \delta = -\frac{\mu^2/2\sigma^2 - \lambda}{\sqrt{(\mu^2/2\sigma^2 - \lambda)^2 + 2\lambda\mu^2/2\sigma^2}}.$$

Now let us consider boundary conditions. At  $u = \alpha$ , the stake in the risky asset should be zero. This implies that  $y(\alpha) = 0$ . Substituting  $y(\alpha) = 0$  into (3.24) gives

$$(3.27) \quad c = (-\theta_1 \alpha)^{1-\delta} (\theta_2 \alpha)^{1+\delta}.$$

Note that (3.24) determines  $y(u)$ , which in turn gives the investment strategy: the fraction of wealth invested in the risky asset equals  $-(\mu/\sigma^2) (y(u)/u)$ .

To compute the expected growth rate of wealth, we can argue that at  $u = 1$ ,  $\partial I/\partial M = 0$ , which gives  $h'(1) = 1$ . This condition, together with equation (3.22), implies

$$(3.28) \quad \theta - \lambda = -(\mu^2/2\sigma^2) y(1).$$

From (3.28), we can compute the expected growth rate of wealth after we obtain  $y(1)$  from (3.24).

**REMARKS.** When  $\lambda = 0$ , we can explicitly solve for  $y(u)$  in (3.23), which can then be used to determine the optimal investment strategy and the expected growth rate of wealth. The results are listed as follows:  $y(u) = -(u - \alpha)$ ; the fraction of wealth invested in the risky asset equals  $(\mu/\sigma^2) (1 - \alpha/u)$ ; the expected growth rate of wealth equals  $\theta = (\mu^2/2\sigma^2) (1 - \alpha)$ .

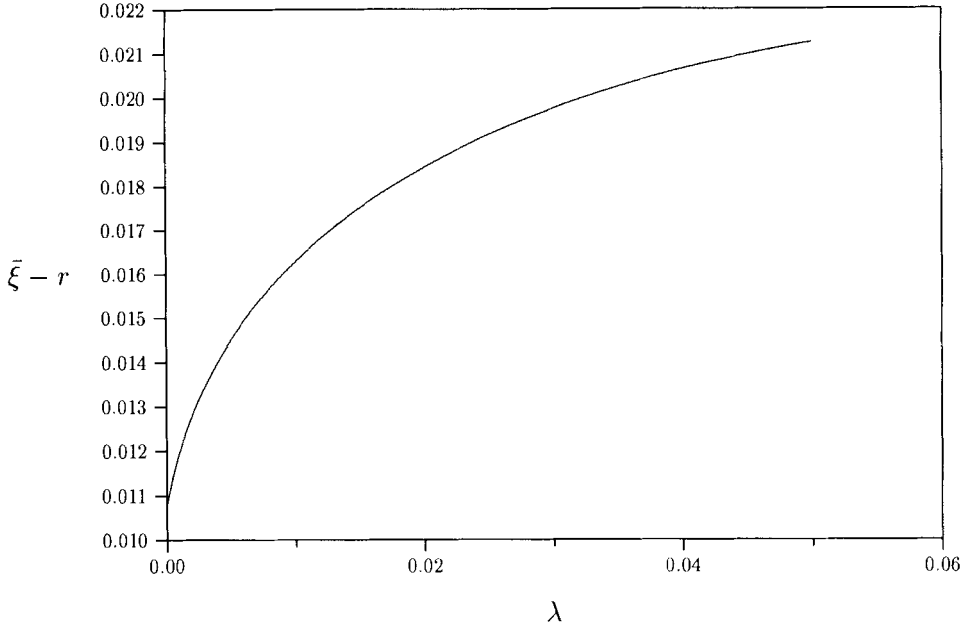


FIGURE 3.1. Growth rate as a function of  $\lambda$  ( $A = 2$ ,  $\mu/\sigma = 0.3$ ,  $\alpha = 0.7$ ).

### 3.2. Computation of Growth Rates

In principle, (3.9)–(3.13) should determine the value of the growth rate. For obvious reasons, we are unable to obtain an explicit expression for  $\xi$ .

In the next section, we are able to show that given  $\xi$  and a negative  $y(u)$  which solve (3.9)–(3.13), then  $\xi$  is necessarily *the* growth rate, so there is a unique solution to (3.9)–(3.13). Thus, when we try to solve (3.9)–(3.13) numerically, we get a unique solution.

Figure 3.1 presents numerical results which show the impact of  $\lambda$  on the growth rate. The horizontal axis in Figure 3.1 is  $\lambda$ , and the vertical axis is  $\xi - \lambda$ , which is  $\bar{\xi} - r$ . Note that  $\xi - \lambda$  should be between

$$\frac{\mu^2}{2\sigma^2} \frac{1 - \alpha}{\alpha + (1 - \alpha)A} \quad \text{and} \quad \frac{\mu^2}{2A\sigma^2}.$$

These bounds for  $\xi - \lambda$  can help us in searching for numerical solutions. In Figure 3.1, we assign  $A = 2$ ,  $\mu/\sigma = 0.3$ , and  $\alpha = 0.7$ . Note that a positive  $\lambda$  can improve growth rate significantly. The marginal improvement is bigger for smaller  $\lambda$ .

### 3.3. Computation of the Optimal Trading Strategies

Having computed the growth rate in the last subsection, we are now in a position to calculate the optimal trading strategies. Given a  $\lambda$  and the corresponding growth rate  $\xi$ , Appendix E shows that we can uniquely determine the optimal fraction of wealth that is invested in stocks from (3.9). At the end of Appendix E, we show that the fraction of

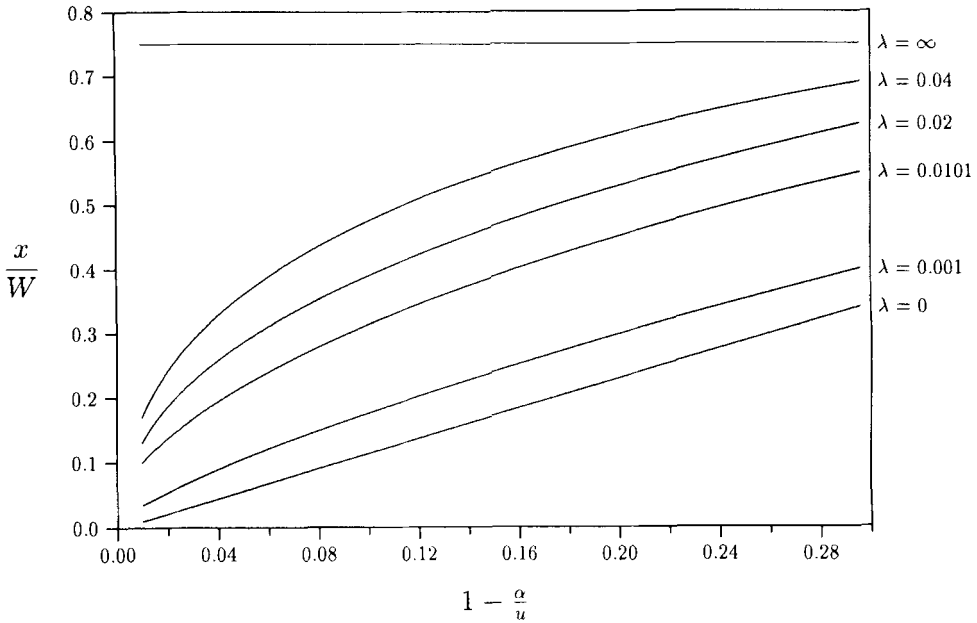


FIGURE 3.2. Fraction of wealth invested in stock ( $\alpha = 0.7$ ,  $A = 2$ ,  $\mu = 0.06$ ,  $\sigma = 0.2$ ).

wealth invested in the risky asset is an increasing function of  $u$ . Figure 3.2 presents the optimal fraction of wealth invested in the risky asset for different values of  $\lambda$ .

#### 4. VERIFICATION THEOREMS

The goal of this section is to verify that the heuristic solution to the drawdown control problem is indeed valid. We will first prove the validity of the solution for the case that  $\lambda > 0$ ; then we will take limit as  $\lambda$  tends to zero to show the validity of the solution for the case that  $\lambda = 0$ . The verification for the case of logarithmic utility function can be done parallel.

If  $\lambda > 0$ , we need to show that if there exist a constant  $\xi$  and negative  $y(u)$  that satisfy (3.9)–(3.13), then  $\xi$  is the maximum growth rate and  $\xi$  can be achieved by using the following trading strategy  $x^*$  (henceforth called star strategy): the fraction of wealth invested in the risky asset is set equal to  $(-\mu/\sigma^2)(y(u)/u)$ , where  $u$  is defined to be  $W/M$ . To verify this result by Theorem 1.1, we need to construct  $V(W, M)$  that satisfies the conditions of Theorem 1.1.

Given a constant  $\xi$  and negative  $y(u)$  that satisfy (3.9)–(3.13), we define

$$(4.1) \quad V(W, M) \equiv M^{1-A} f(u),$$

where  $f(u)$  is defined by

$$(4.2) \quad f(u) \equiv \frac{1}{1-A} \exp \left[ \int_{\alpha}^u \frac{(1-A)\xi}{\lambda v - (\mu^2/2\sigma^2)y(v)} dv \right].$$

To motivate the definition of  $f(u)$ , we note that dividing (3.7) by  $f'(u)$  gives

$$(4.3) \quad (1 - A)\xi \frac{f(u)}{f'(u)} = \lambda u - \frac{\mu^2}{2\sigma^2} y(u),$$

or, equivalently,

$$(4.4) \quad \frac{f'(u)}{f(u)} = \frac{(1 - A)\xi}{\lambda u - (\mu^2/2\sigma^2)y(u)}.$$

Integrating (4.4) yields

$$(4.5) \quad f(u) = c \exp \left[ \int_{\alpha}^u \frac{(1 - A)\xi}{\lambda v - (\mu^2/2\sigma^2)y(v)} dv \right],$$

where  $c$  is a constant. We chose  $c = (1 - A)^{-1}$  to make sure that  $V(W, M)$  has the same sign as  $U(W)$ .

Note that by definition,  $V(W, M)$  is a smooth function of  $W$  and  $M$  for  $W \in [\alpha M, M]$ . Moreover, we can verify (1) when  $W = M$ ,  $\partial V/\partial M = 0$ ; (2)  $V(W, M)$  is concave in  $W$ ; (3) (3.2) holds; and (4) the maximum in (3.2) is attained by using the star strategy. Based on these properties of function  $V(W, M)$ , the following lemma proves that  $V(W, M)$  satisfies the first condition of Theorem 1.1.

LEMMA 4.1. *For all  $t > 0$ , function  $V(W, M)$  as defined in (4.1) and (4.2) satisfies the Bellman equation*

$$(4.6) \quad V(W, M) = \sup_{\mathcal{A}} E[V(W_t, M_t)] e^{-(1-A)\xi t}.$$

*Proof.* Consider any  $x \in \mathcal{A}$ . By Ito's lemma for semimartingales, we obtain

$$\begin{aligned} E[V(W_t, M_t)] e^{-(1-A)\xi t} &= V(W, M) \\ &= E \int_0^t \left[ -(1 - A)\xi V + \frac{\partial V}{\partial W} \lambda W + \frac{\partial V}{\partial W} \mu x + \frac{1}{2} \frac{\partial^2 V}{\partial W^2} \sigma^2 x^2 \right] e^{-(1-A)\xi s} ds \\ &\quad + E \int_0^t \frac{\partial V}{\partial M} dM \\ &= E(I) + E(II), \end{aligned}$$

where we define

$$I = \int_0^t \left[ -(1 - A)\xi V + \frac{\partial V}{\partial W} \lambda W + \frac{\partial V}{\partial W} \mu x + \frac{1}{2} \frac{\partial^2 V}{\partial W^2} \sigma^2 x^2 \right] e^{-(1-A)\xi s} ds$$

and

$$II = \int_0^t \frac{\partial V}{\partial M} dM.$$



Note that the integrand in  $I$  is less than or equal to zero by (3.2), so  $E(I) \leq 0$ . Recall that when  $W = M$ ,  $\partial V/\partial M = 0$ , so  $II = 0$  and thus  $E(II) = 0$  according to Lemma 2.3. Therefore,

$$(4.7) \quad V(W, M) \geq \sup_{\mathcal{A}} E[V(W_t, M_t)e^{-(1-A)\xi t}].$$

Since for the trading strategy  $x^*$  described in Section 3.3, we have that

$$\begin{aligned} & E[V(W_t^*, M_t^*)e^{-(1-A)\xi t}] - V(W, M) \\ &= E \int_0^t \left[ -(1-A)\xi V + \frac{\partial V}{\partial W} \lambda W_s^* + \frac{\partial V}{\partial W} \mu x_s^* + \frac{1}{2} \frac{\partial^2 V}{\partial W^2} \sigma^2 (x_s^*)^2 \right] e^{-(1-A)\xi s} ds \\ &\quad + E \int_0^t \frac{\partial V}{\partial M} dM \\ &= 0, \end{aligned}$$

where  $W_t^*$  denotes for the wealth process associated with  $x^*$  and  $M_t^*$  is defined through  $W_t^*$ . Hence,

$$(4.8) \quad V(W, M) \leq \sup_{\mathcal{A}} E[V(W_t, M_t)e^{-(1-A)\xi t}].$$

The proof is now complete. □

The proof of the lemma also reveals that the second condition of Theorem 1.1 holds. That is, the supremum in the Bellman equation (1.7) can be attained by the star strategy for all  $t > 0$ . The next lemma confirms the third condition of Theorem 1.1.

LEMMA 4.2. *For  $\lambda > 0$ , there exist positive constants  $C_1$  and  $C_2$  such that*

$$(4.9) \quad C_1(1-A)U(W) \leq (1-A)V(W, M) \leq C_2(1-A)U(W).$$

*Proof.* Note that  $V(W, M) = M^{1-A}f(u)$ , where  $f(u)$  is given by (4.2). It is easy to see that  $V(W, M)$  and  $U(W)$  have the same sign. Since  $y(v)$  is negative by construction, we have that  $\lambda v - (\mu^2/2\sigma^2)y(v)$  is no less than  $\lambda\alpha$ . Thus  $(1-A)f(u)$  is bounded by two positive numbers. The rest of the proof is obvious. □

The main theorem is given below.

THEOREM 4.1. *Let  $\lambda$  be positive. If there exist a constant  $\xi$  and negative  $y(u)$  that satisfy (3.9)–(3.13), then (i)  $\xi$  is the maximum long-run growth rate of expected utility. (ii) The optimal trading strategy is as constructed in Section 3.3. Moreover, (iii)  $\xi$  is also the growth rate of the finite horizon problem. In other words,*

$$(4.10) \quad \xi = \liminf_{T \rightarrow \infty} \frac{1}{(1-A)T} \ln[(1-A)L(W, M, T)],$$

where  $L(W, M, T)$  is the value function of the finite horizon problem. That is,

$$(4.11) \quad L(W, M, T) = \sup_{\mathcal{A}} EU(W_T).$$

*Proof.* The proof follows trivially from Theorem 1.1 and Lemmas 4.1 and 4.2.  $\square$

To emphasize the dependence of  $\xi$  on  $\lambda$ , let us use  $\xi(\lambda)$  to denote the long-run growth rate for  $\lambda$ . Theorem 4.1 validates that  $\xi(\lambda)$  is as given in Section 3.2. The next theorem states that  $\xi(0)$  is given by (2.20) and that the optimal trading strategy is given by (2.13) and (2.19).

**THEOREM 4.2.** *Let  $\lambda = 0$ . Then (i)*

$$(4.12) \quad \xi(0) = \frac{\mu^2}{2\sigma^2} \frac{1 - \alpha}{\alpha + (1 - \alpha)A},$$

*and (ii) the optimal trading strategy is given by*

$$(4.13) \quad x_t^* = \frac{\mu}{\sigma^2(1 - \beta)}(W_t - \alpha M_t),$$

*with  $\beta = (1 - A)(1 - \alpha)$ .*

*Proof.* To prove (i), note that at the end of Section 2.5, we have calculated that using strategy  $x^*$  given above, the long-run growth rate is at least as large as  $(\mu^2/2\sigma^2)[(1 - \alpha)/(\alpha + (1 - \alpha)A)]$ . So,

$$(4.14) \quad \xi(0) \geq \frac{\mu^2}{2\sigma^2} \frac{1 - \alpha}{\alpha + (1 - \alpha)A}.$$

Note that  $\xi(0) \leq \xi(\lambda)$  for all positive  $\lambda$ . Note also that  $\xi(\lambda)$  is an increasing function of  $\lambda$ . Thus,  $\xi(0) \leq \xi^+$ , where  $\xi^+$  is defined to be the right limit of  $\xi(\lambda)$  as  $\lambda$  tends to zero. To complete the proof of part (i), we need only show that

$$(4.15) \quad \xi^+ \leq \frac{\mu^2}{2\sigma^2} \frac{1 - \alpha}{\alpha + (1 - \alpha)A},$$

which implies the continuity of  $\xi(\lambda)$  at  $\lambda = 0$ .

To this end, we have to revisit (3.9)–(3.13). When  $\lambda$  gets close to zero, we observe that  $\delta$ , as defined by (3.11), tends to  $-1$  because the numerator  $(1 - A)\xi(\lambda) + \mu^2/2\sigma^2 - \lambda$  is positive for small  $\lambda$  for the following reasons: when  $A < 1$ , it is obvious that the numerator is positive for small  $\lambda$ ; when  $A > 1$ , we recall that the growth rate for the original bar problem  $\bar{\xi}(\lambda) = \xi(\lambda) + \bar{\lambda} \approx \xi(\lambda) + r - \lambda$  should be less than or equal to Merton's growth rate  $\mu^2/2A\sigma^2 + r$ . Thus we have  $\xi(\lambda) \leq \lambda + \mu^2/2A\sigma^2$ , and the numerator is no less than

$$\frac{\mu^2}{2\sigma^2} - \frac{\mu^2}{2\sigma^2} \left( \frac{A - 1}{A} \right) - (1 - A)\lambda - \lambda,$$

which is positive for small  $\lambda$ . It is easy to verify that as  $\lambda$  approaches zero,  $\theta_1$  tends to  $-(2\sigma^2/\mu^2)[(1-A)\xi^+ + \mu^2/2\sigma^2]$ ,  $\theta_2$  tends to zero, and  $y(1)$  tends to  $-2\sigma^2\xi^+/\mu^2$ . Taking the limit as  $\lambda \rightarrow 0$  in the expression (which is (3.9) evaluated at  $u = 1$ )

$$(4.16) \quad |y(1) - \theta_1|^{1-\delta}|y(1) - \theta_2|^{1+\delta} = (-\theta_1\alpha)^{1-\delta}(\theta_2\alpha)^{1+\delta},$$

we obtain

$$(4.17) \quad \left| -\frac{2\sigma^2}{\mu^2}\xi^+ + \frac{2\sigma^2}{\mu^2}\left[(1-A)\xi^+ + \frac{\mu^2}{2\sigma^2}\right] \right| = \left| \frac{2\sigma^2}{\mu^2}\left[(1-A)\xi^+ + \frac{\mu^2}{2\sigma^2}\right]\alpha \right|,$$

from which it follows easily that

$$(4.18) \quad \xi^+ = \frac{\mu^2}{2\sigma^2} \frac{1-\alpha}{\alpha + (1-\alpha)A}.$$

We have thus shown that

$$(4.19) \quad \xi(0) = \frac{\mu^2}{2\sigma^2} \frac{1-\alpha}{\alpha + (1-\alpha)A}.$$

To prove (ii), we note that in Section 2.5 it is shown that using the star strategy yields a growth rate no less than  $(\mu^2/2\sigma^2)[(1-\alpha)/\alpha + (1-\alpha)A]$ , which is equal to the maximal growth rate  $\xi(0)$  by part (i). Therefore, the star strategy is optimal. This completes the proof.  $\square$

## APPENDIX A

*Proof.* Note that

$$(A.1) \quad \overline{M}_t = W_0 e^{rt} \exp[\max_{s \leq t} \{(p-r)s + qZ_s\}],$$

so  $P[\tau < \infty]$  is also equal to the probability of  $pt + qZ_t$  ever hitting  $\ln \alpha + rt + \max_{s \leq t} \{(p-r)s + qZ_s\}$  or the probability of  $(p-r)t + qZ_t$  ever hitting  $\ln \alpha + \max_{s \leq t} \{(p-r)s + qZ_s\}$ .

Observe that

$$(A.2) \quad P[\tau = \infty] = P[\forall t : (p-r)t + qZ_t > \ln \alpha + \max_{s \leq t} \{(p-r)s + qZ_s\}],$$

which is no larger than (note:  $n$  denotes for positive integers)

$$(A.3) \quad P[\forall n \forall t \in (0, 1] : (p-r)(n+t) + qZ_{n+t} > \ln \alpha + (p-r)n + qZ_n],$$

which is equal to

$$(A.4) \quad P[\forall n \forall t \in (0, 1] : (p - r)t + q(Z_{n+t} - Z_n) > \ln \alpha].$$

Due to the fact that Brownian motion exhibits independent increments, the above probability is equal to zero because

$$(A.5) \quad 1 > P[\forall t \in (0, 1] : (p - r)t + qZ_t > \ln \alpha].$$

This completes the proof that with probability 1, Merton's strategy will violate (1.4).  $\square$

## APPENDIX B

The analysis in Section 2 can be repeated for the standard portfolio insurance problem where the stochastic floor  $\alpha M_t$  is replaced by a fixed floor  $K$ . An admissible wealth process satisfies  $dW_t = y_t(\mu dt + \sigma dZ_t)$  for some nonanticipating process  $y$  and satisfies  $W_t \geq K$  for all  $t$ . We define  $\eta$  to be the supremum of  $\liminf_{T \rightarrow \infty} 1/[(1 - A)T] \ln E[(1 - A)U(W_T)]$  over all admissible wealth processes, and define  $\tilde{J}(W)$  to be the supremum of  $\liminf_{T \rightarrow \infty} EU(W_T)e^{-(1-A)\eta T}$  over all admissible wealth processes. Motivated by the properties of  $\tilde{J}$ , we want to construct  $J(W)$  that satisfies the Bellman equation

$$(B.1) \quad 0 = \max_y \left[ -(1 - A)\eta J(W) + \frac{dJ}{dW}\mu y + \frac{1}{2} \frac{d^2 J}{dW^2} \sigma^2 y^2 \right],$$

with the boundary condition that at  $W = K$  the investment in the risky asset is zero.

The solution to (B.1) is

$$(B.2) \quad J(W) = c(W - K)^\delta, \quad y = \frac{\mu}{\sigma^2(1 - \delta)}(W - K),$$

where  $c$  is a constant and  $\delta$  satisfies

$$(B.3) \quad (1 - A)\eta = \frac{1}{2} \frac{\mu^2}{\sigma^2} \frac{\delta}{1 - \delta}.$$

The wealth process is

$$dW_t = \frac{\mu}{\sigma^2(1 - \delta)}(W_t - K)(\mu dt + \sigma dZ_t),$$

the solution of which can be obtained easily:

$$(B.4) \quad W_t - K = (W_0 - K) \exp \left[ \frac{\mu^2(1 - 2\delta)}{2\sigma^2(1 - \delta)^2} t + \frac{\mu}{\sigma(1 - \delta)} Z_t \right].$$

The growth rate of utility can be computed directly from (B.4) to be

$$(B.5) \quad \eta = \frac{\mu^2}{\sigma^2(1 - \delta)} - \frac{A}{2} \frac{\mu^2}{\sigma^2(1 - \delta)^2},$$

which, together with the equation

$$(1 - A)\eta = \frac{1}{2} \frac{\mu^2}{\sigma^2} \frac{\delta}{1 - \delta}$$

implies that  $\eta$  and  $\delta$  do not depend on the magnitude of  $K$ . So  $\eta$  is the same as the growth rate for the case  $K = 0$  (Merton case):

$$(B.6) \quad \eta = \mu^2/2A\sigma^2.$$

Similarly,  $\delta = 1 - A$ , and the optimal investment in the risky asset is given by  $y = (\mu/A\sigma^2)(W - K)$ .

## APPENDIX C

*Proof.* Define

$$(C.1) \quad \hat{W}_t = \alpha M_0 e^{(1-\alpha)L_t} + (W_0 - \alpha M_0) \exp[-\alpha L_t + (k\mu - \frac{1}{2}k^2\sigma^2)t + k\sigma Z_t],$$

$$(C.2) \quad \hat{M}_t = M_0 e^{(1-\alpha)L_t}.$$

Notice that the definitions of  $\hat{W}_t$  and  $\hat{M}_t$  imply

$$(C.3) \quad \ln\left(\frac{\hat{W}_t}{\hat{M}_t} - \alpha\right) = \ln\left(\frac{W_0}{M_0} - \alpha\right) + \left(k\mu - \frac{1}{2}k^2\sigma^2\right)t + k\sigma Z_t - L_t,$$

and

$$(C.4) \quad \frac{1}{1 - \alpha} \ln \frac{\hat{M}_t}{M_0} = L_t.$$

Thus,

$$(C.5) \quad \ln\left(\frac{\hat{W}_t}{\hat{M}_t} - \alpha\right) = \ln\left(\frac{W_0}{M_0} - \alpha\right) + \left(k\mu - \frac{1}{2}k^2\sigma^2\right)t + k\sigma Z_t - \frac{1}{1 - \alpha} \ln \frac{\hat{M}_t}{M_0}.$$

and note that  $\ln(\hat{W}_t/\hat{M}_t - \alpha) \leq \ln(1 - \alpha)$ , implying  $\hat{W}_t \leq \hat{M}_t$ . Since  $\hat{M}_t$  is nondecreasing, we have  $\hat{M}_t \geq \max_{s \leq t} [M_0, \hat{W}_s]$ .

Now let us show that  $\hat{M}_t \leq \max_{s \leq t} [M_0, \hat{W}_s]$ . If  $\hat{W}_s \leq M_0$  for all  $s \leq t$ , the proof is

trivial. For sample paths of wealth passing through  $M_0$ , let  $\tau$  be the first time that  $\hat{W}_\tau = M_0$ . For  $s \in [\tau, t]$ , (C.1) and (C.2) imply

$$(C.6) \quad \begin{aligned} \hat{M}_s &= \hat{M}_\tau e^{(1-\alpha)L_s^\tau}, \\ \hat{W}_s &= \alpha \hat{M}_\tau e^{(1-\alpha)L_s^\tau} \\ &\quad + (\hat{W}_\tau - \alpha \hat{M}_\tau) \exp[-\alpha L_s^\tau] \\ &\quad + \left( k\mu - \frac{1}{2}k^2\sigma^2 \right) (s - \tau) + k\sigma(Z_s - Z_\tau), \end{aligned}$$

where we define

$$(C.7) \quad L_t^\tau \equiv \max_{\tau \leq s \leq t} [(k\mu - \frac{1}{2}k^2\sigma^2)(s - \tau) + k\sigma(Z_s - Z_\tau)].$$

Pick a point of time  $s$  such that  $(k\mu - \frac{1}{2}k^2\sigma^2)(s - \tau) + k\sigma(Z_s - Z_\tau) = L_t^\tau$ . Then at  $s$ ,  $L_s^\tau = L_t^\tau$ . It is straightforward to see that  $\hat{W}_s = \hat{M}_s$ , thus establishing that  $\hat{M}_t \leq \max_{s \leq t} [M_0, \hat{W}_s]$ . So we have verified that  $\hat{M}_t = \max_{s \leq t} [M_0, \hat{W}_s]$ .

Finally, we want to use (C.5) to show that

$$(C.8) \quad \hat{W}_t = W_0 + \int_0^t k(\hat{W}_s - \alpha \hat{M}_s)(\mu ds + \sigma dZ_s).$$

To this end, we notice that Ito's lemma for semimartingales gives

$$(C.9) \quad \int_0^t \frac{d\hat{M}_s}{(1-\alpha)\hat{M}_s} = \frac{1}{1-\alpha} \ln \frac{\hat{M}_t}{M_0},$$

so

$$(C.10) \quad \ln \left( \frac{\hat{W}_t}{\hat{M}_t} - \alpha \right) = \ln \left( \frac{W_0}{M_0} - \alpha \right) + \left( k\mu - \frac{1}{2}k^2\sigma^2 \right) t + k\sigma Z_t - \int_0^t \frac{d\hat{M}_s}{(1-\alpha)\hat{M}_s}.$$

Using (C.10) and Ito's lemma for semimartingales, we get

$$(C.11) \quad \hat{W}_t = W_0 + \int_0^t k(\hat{W}_s - \alpha \hat{M}_s)(\mu ds + \sigma dZ_s).$$

This establishes that  $\hat{W}_t$  is the solution to the stochastic differential equation

$$(C.12) \quad W_t = W_0 + \int_0^t k(W_s - \alpha M_s)(\mu ds + \sigma dZ_s). \quad \square$$

## APPENDIX D

We are interested in the calculation of

$$(D.1) \quad E[f_{t+h} - f_t | u_t = u] = -k\alpha E[1/u_{t+h} - 1/u_t | u_t = u].$$

Let  $\mu' = k\mu - \frac{1}{2}k^2\sigma^2$  and  $\sigma' = k\sigma$ . Note that

$$\begin{aligned} v_t &\equiv \ln(1 - \alpha) - \ln(u_t - \alpha) \\ &= \ln(1 - \alpha) - \ln(u_0 - \alpha) - \mu't - \sigma'Z_t + L_t \\ &= \ln(1 - \alpha) - \ln(u_0 - \alpha) - \mu't - \sigma'Z_t \\ &\quad + \sup_{s \leq t} [\ln(u_0 - \alpha) + \mu's + \sigma'Z_s - \ln(1 - \alpha)]^+ \\ &= \ln(1 - \alpha) - \ln(u_0 - \alpha) - \mu't - \sigma'Z_t \\ &\quad + \sup_{s \leq t} [\ln(1 - \alpha) - \ln(u_0 - \alpha) - \mu's - \sigma'Z_s]^- \end{aligned}$$

is a regulated  $(-\mu', \sigma')$  Brownian motion with lower barrier at zero and with initial value  $v_0 = \ln(1 - \alpha) - \ln(u_0 - \alpha)$ .

The distribution function of  $v_t$  is given by Harrison (1985, p. 49) as

$$(D.2) \quad P[v_{t+h} > y | v_t = v] = \Phi\left(\frac{-y + v - \mu'h}{\sigma'\sqrt{h}}\right) + e^{-2\mu'y/(\sigma')^2} \Phi\left(\frac{-y - v + \mu'h}{\sigma'\sqrt{h}}\right),$$

where  $\Phi(\cdot)$  denotes the cumulative distribution function of a standard normal random variable. Therefore,

$$\begin{aligned} E[f_{t+h} - f_t | u_t = u] &= -k\alpha E\left[\frac{1}{\alpha + (1 - \alpha)e^{-v_{t+h}}} - \frac{1}{\alpha + (1 - \alpha)e^{-v_t}} \middle| v_t = v\right] \\ &= -k\alpha \int_0^\infty \frac{1}{\alpha + (1 - \alpha)e^{-y}} dP[v_{t+h} \leq y | v_t = v] + \frac{k\alpha}{\alpha + (1 - \alpha)e^{-v}}, \end{aligned}$$

where  $v = \ln(1 - \alpha) - \ln(u - \alpha)$ .

## APPENDIX E

This appendix explains how to compute the optimal investment strategy from (3.9). Recall (3.9):

$$(E.1) \quad |y(u) - \theta_1 u|^{1-\delta} |y(u) - \theta_2 u|^{1+\delta} = c,$$

where  $y(u) = f'(u)/f''(u)$ . Define  $k(u)$  to be the optimal fraction of wealth invested in the risky assets. According to (3.5), we have

$$(E.2) \quad k(u) \equiv \frac{x}{W} = -\frac{\mu}{\sigma^2} \frac{y(u)}{u}.$$

Thus, (3.9) can be written in terms of  $k(u)$ :

$$(E.3) \quad |k(u) + \frac{\mu}{\sigma^2}\theta_1|^{1-\delta}|k(u) + \frac{\mu}{\sigma^2}\theta_2|^{1+\delta} = \frac{c}{u^2}\left(\frac{\mu}{\sigma^2}\right)^2.$$

Define a function  $F(x)$  by

$$(E.4) \quad F(x) = \left|x + \frac{\mu}{\sigma^2}\theta_1\right|^{1-\delta}\left|x + \frac{\mu}{\sigma^2}\theta_2\right|^{1+\delta},$$

so (E.3) becomes  $F(k(u)) = (c/u^2)(\mu/\sigma^2)^2$ . Note that  $F(x)$  can be rewritten as

$$(E.5) \quad F(x) = \begin{cases} \left(-\frac{\mu}{\sigma^2}\theta_1 - x\right)^{1-\delta}\left(-x - \frac{\mu}{\sigma^2}\theta_2\right)^{1+\delta} & \text{if } x < -\frac{\mu}{\sigma^2}\theta_2, \\ \left(-\frac{\mu}{\sigma^2}\theta_1 - x\right)^{1-\delta}\left(x + \frac{\mu}{\sigma^2}\theta_2\right)^{1+\delta} & \text{if } x \in \left[-\frac{\mu}{\sigma^2}\theta_2, -\frac{\mu}{\sigma^2}\theta_1\right], \\ \left(\frac{\mu}{\sigma^2}\theta_1 + x\right)^{1-\delta}\left(x + \frac{\mu}{\sigma^2}\theta_2\right)^{1+\delta} & \text{if } x > -\frac{\mu}{\sigma^2}\theta_1. \end{cases}$$

**PROPOSITION E.1.** *In the region  $x \in [-(\mu/\sigma^2)\theta_2, -(\mu/\sigma^2)\theta_1]$ , we have  $\arg \max F(x) = 0$ .*

*Proof.* Since  $F(x)$  is nonnegative over  $[-(\mu/\sigma^2)\theta_2, -(\mu/\sigma^2)\theta_1]$  and vanishes at the endpoints, it suffices to show that the only critical point of  $F(x)$  is  $x = 0$ . Or, equivalently, it suffices to show that if  $[\ln F(x)]' = 0$ , then  $x = 0$ . Note that  $[\ln F(x)]' = 0$  implies

$$(E.6) \quad 0 = (1 - \delta)\left(x + \frac{\mu}{\sigma^2}\theta_2\right) + (1 + \delta)\left(x + \frac{\mu}{\sigma^2}\theta_1\right),$$

which implies

$$(E.7) \quad x = -\frac{1}{2}\left[(1 - \delta) + (1 + \delta)\frac{\theta_1}{\theta_2}\right]\frac{\mu}{\sigma^2}\theta_2 = 0,$$

where the last equality follows because, by definition,  $\theta_1/\theta_2 = (\delta - 1)/(\delta + 1)$ .  $\square$

We assume  $\mu > 0$ . The case  $\mu < 0$  is similar. The proposition shows that  $F(x)$  is strictly decreasing in the region  $[0, -(\mu/\sigma^2)\theta_1]$ . Note that the image under  $F(x)$  of the region  $[0, -(\mu/\sigma^2)\theta_1]$  is  $F([0, -(\mu/\sigma^2)\theta_1]) = [0, (c/\alpha^2)(\mu/\sigma^2)^2]$ . From now on, we will treat  $F(x)$  as a one-to-one mapping from  $[0, -(\mu/\sigma^2)\theta_1]$  onto  $[0, (c/\alpha^2)(\mu/\sigma^2)^2]$ .

Here is the procedure for finding the optimal investment strategy. For any given  $u$  between  $\alpha$  and 1, we compute a number called  $\omega(u)$  by  $\omega(u) = (c/u^2)(\mu/\sigma^2)^2$ . Since



$\omega(u) \in [0, (c/\alpha^2)(\mu/\sigma^2)^2]$ ,  $F^{-1}(\omega(u))$  is well defined, and we set  $k(u) = F^{-1}(\omega(u))$  to be the fractional investment in the risky asset.

Note that  $k(u) \in [0, -(\mu/\sigma^2)\theta_1]$ . Since  $\omega(u)$  is a strictly decreasing function of  $u$  and  $F(\cdot)$  is also a strictly decreasing function, we conclude that  $k(u)$  is a strictly increasing function of  $u$  with  $k(\alpha) = 0$ . That is, the fraction of wealth invested in the risky assets increases as  $u$  increases.

## APPENDIX F

*Proof.* To prove (i), let us consider an arbitrary feasible strategy  $x \in \mathcal{A}$ . It is easy to see that  $(1 - A)V(W, M) \geq C_1(1 - A)U(W)$  implies

$$(F.1) \quad E[(1 - A)V(W_T, M_T)e^{-(1-A)\xi T}] \geq C_1 E[(1 - A)U(W_T)e^{-(1-A)\xi T}],$$

from which we obtain that, for  $A < 1$ ,

$$\begin{aligned} & \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)V(W_T, M_T)e^{-(1-A)\xi T}] \\ & \geq \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln \{C_1 E[(1 - A)U(W_T)e^{-(1-A)\xi T}]\}. \end{aligned}$$

Note that (1.7) implies that, if  $A < 1$ ,

$$\frac{1}{(1 - A)T} \ln E[(1 - A)V(W_T, M_T)e^{-(1-A)\xi T}] \leq \frac{1}{(1 - A)T} \ln[(1 - A)V(W, M)],$$

from which one infers

$$(F.2) \quad \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)V(W_T, M_T)e^{-(1-A)\xi T}] \leq 0.$$

Thus,

$$(F.3) \quad \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln \{C_1 E[(1 - A)U(W_T)e^{-(1-A)\xi T}]\} \leq 0.$$

Note that

$$\begin{aligned} & \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln \{C_1 E[(1 - A)U(W_T)e^{-(1-A)\xi T}]\} \\ & = \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)U(W_T)] - \xi. \end{aligned}$$

So for  $A < 1$ ,

$$(F.4) \quad \xi \geq \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)U(W_T)].$$

Note that for the particular strategy  $x^*$ ,  $(1 - A)V(W, M) \leq C_2(1 - A)U(W)$  implies that

$$\begin{aligned} E[C_2(1 - A)U(W_T^*)e^{-(1-A)\xi T}] &\geq E[(1 - A)V(W_T^*, M_T^*)e^{-(1-A)\xi T}] \\ &= (1 - A)V(W, M), \end{aligned}$$

from which it follows that

$$(F.5) \quad \xi \leq \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)U(W_T^*)],$$

and thus

$$(F.6) \quad \xi \leq \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)U(W_T)].$$

Therefore, we have shown that, for  $A < 1$ ,

$$(F.7) \quad \xi = \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)U(W_T)].$$

For  $A > 1$ , we need only slightly change the above argument. It is easy to verify that for  $A > 1$ ,  $(1 - A)V(W, M) \leq C_2(1 - A)U(W)$  implies

$$\begin{aligned} \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)V(W_T, M_T)e^{-(1-A)\xi T}] \\ \geq \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln \{C_2 E[(1 - A)U(W_T)e^{-(1-A)\xi T}]\}, \end{aligned}$$

Observe that (1.7) implies

$$(F.8) \quad \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)V(W_T, M_T)e^{-(1-A)\xi T}] \leq 0,$$

and thus

$$(F.9) \quad \xi \geq \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)U(W_T)].$$

Based on the particular strategy  $x^*$  and the inequality  $(1 - A)V(W, M) \geq C_1(1 - A)U(W)$ , we can show

$$(F.10) \quad \xi \leq \liminf_{T \rightarrow \infty} \frac{1}{(1 - A)T} \ln E[(1 - A)U(W_T^*)],$$

and thus

$$(F.11) \quad \xi \leq \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1-A)T} \ln E[(1-A)U(W_T)].$$

Therefore, we have just established that, for  $A > 1$ ,

$$(F.12) \quad \xi = \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1-A)T} \ln E[(1-A)U(W_T)].$$

To prove (ii), we have noted (in (F.5) and (F.10)) that using the star strategy yields a growth rate no less than  $\xi$ :

$$(F.13) \quad \xi \leq \liminf_{T \rightarrow \infty} \frac{1}{(1-A)T} \ln E[(1-A)U(W_T^*)].$$

On the other hand, (F.4) and (F.9) have shown

$$(F.14) \quad \xi \geq \sup_{\mathcal{A}} \liminf_{T \rightarrow \infty} \frac{1}{(1-A)T} \ln E[(1-A)U(W_T)],$$

and, in particular,

$$(F.15) \quad \xi \geq \liminf_{T \rightarrow \infty} \frac{1}{(1-A)T} \ln E[(1-A)U(W_T^*)].$$

Therefore,

$$(F.16) \quad \xi = \liminf_{T \rightarrow \infty} \frac{1}{(1-A)T} \ln E[(1-A)U(W_T^*)].$$

That is, the maximal growth rate is achieved by following the star strategy.

To prove (iii), note that according to (1.8) there exist two positive constants,  $D_1$  and  $D_2$ , such that  $D_1 U(W) \leq V(W, M) \leq D_2 U(W)$ . Thus, for any feasible strategy  $x \in \mathcal{A}$ ,

$$(F.17) \quad D_1 U(W_T) e^{-(1-A)\xi T} \leq V(W_T, M_T) e^{-(1-A)\xi T} \leq D_2 U(W_T) e^{-(1-A)\xi T}.$$

Taking expectations and then taking the supremum over  $\mathcal{A}$ , we have

$$(F.18) \quad D_1 L(W, M, T) e^{-(1-A)\xi T} \leq V(W, M) \leq D_2 L(W, M, T) e^{-(1-A)\xi T},$$

from which it follows easily that

$$(F.19) \quad \xi = \liminf_{T \rightarrow \infty} \frac{1}{(1-A)T} \ln[(1-A)L(W, M, T)].$$

This completes the proof. □

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