Lecture 4:

Variance and Volatility Swaps

4.1 The Second-Most Important Formula in Options Theory

How much profit should you expect to make when you hedge an option at the implied volatility? We can calculate it. To keep things simple, assume zero dividends and interest rates.

At time *t*: Buy the call for price C_i corresponding to a Black-Scholes implied volatility Σ, so

$$\frac{\partial C_i}{\partial t} + \frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C_i}{\partial S^2} = 0 \text{ or } \frac{\partial C_i}{\partial t} = -\frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C_i}{\partial S^2}$$

Now hedge it by shorting $\Delta_\Sigma = \frac{\partial C_i}{\partial S}$ shares of stock, so that $\pi(S, t) = C_i - \Delta_\Sigma S$ is riskless if the future volatility is Σ.

What happens when the stock moves to $S + dS$ at time $t + dt$ with a volatility $\sigma \neq \Sigma$, so that $(dS)^2 = \sigma^2 S^2 dt$ actually?

Then

$$d\pi = \frac{\partial C_i}{\partial t}dt + \cancel{\frac{\partial C_i}{\partial S}dS} + \frac{1}{2}\frac{\partial^2 C_i}{\partial S^2}(dS)^2 - \cancel{\Delta_\Sigma dS} = \frac{\partial C_i}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C_i}{\partial S^2}(dS)^2$$

$$= \frac{\partial C_i}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C_i}{\partial S^2}\sigma^2 S^2 dt = -\frac{1}{2}\Sigma^2 S^2 \frac{\partial^2 C_i}{\partial S^2} + \frac{1}{2}\frac{\partial^2 C_i}{\partial S^2}\sigma^2 S^2 dt = \frac{1}{2}\frac{\partial^2 C_i}{\partial S^2}S^2\left(\sigma^2 - \Sigma^2\right)dt$$

4.2 Hedging an Option Means Betting On Volatility

$$d\pi = \frac{1}{2}\frac{\partial^2 C_i}{\partial S^2}S^2(\sigma^2 - \Sigma^2)dt = \frac{1}{2}\Gamma_i S^2(\sigma^2 - \Sigma^2)dt$$

To profit, you need the realized volatility to be greater than the implied volatility.

A short position profits when the opposite is true.

Note: Black-Scholes uses a single unique volatility for all strikes K and expirations T, because the volatility, real or implied, is the volatility of the stock, not the option. If Black-Scholes is correct, then Σ is independent of K, t, T and S. But we know there is a smile …

Here is an illustration of the contributions to the P&L:

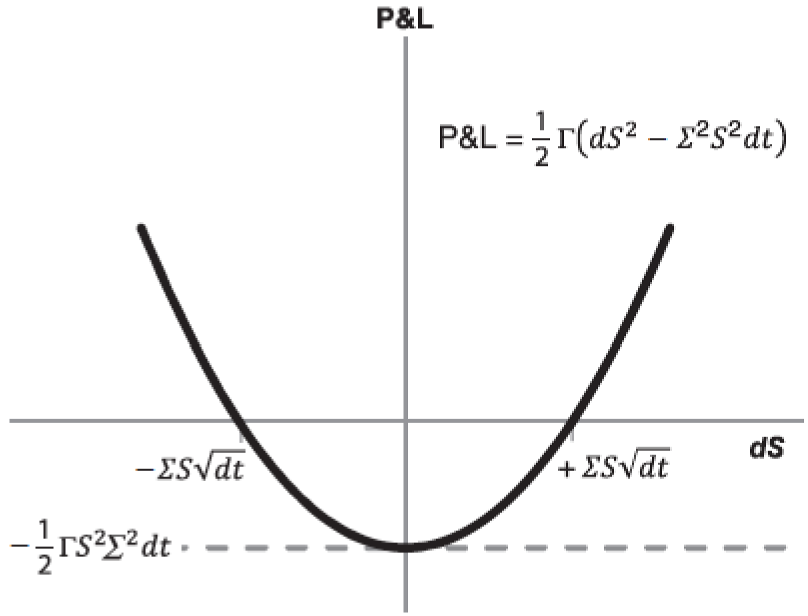


FIGURE 3.10 P&L from Implied versus Realized Volatility

To profit, you need the realized volatility to be greater than the implied volatility. A short position profits when the opposite is true.

Recall: Black-Scholes uses a single unique volatility for all strikes K and expirations T, because the volatility, real or implied, is the volatility of the stock, not the option. If Black-Scholes is correct, then Σ is independent of K, t, T and S.

One Vanilla Option is Not a Pure Bet on Volatility

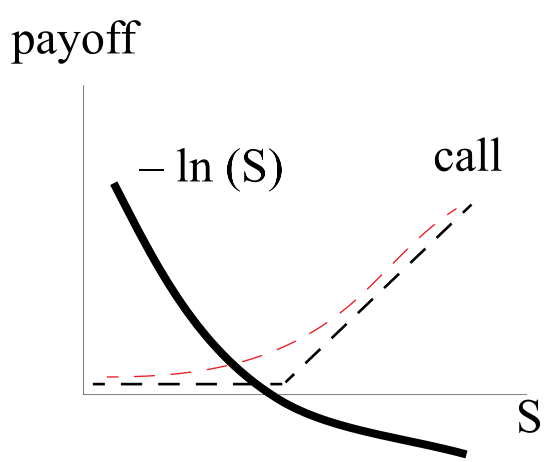
Net P&L from hedging an option $= \frac{1}{2}\int \Gamma S^2\left(\sigma^2 - \Sigma^2\right)dt$: sensitive to S and σ

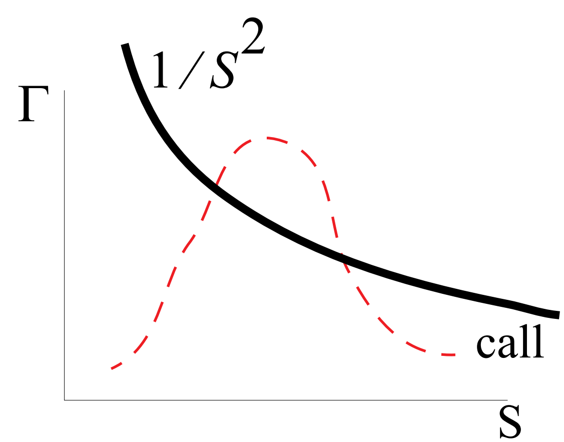
In a BS world, you can capture pure volatility if you own a derivative O whose curvature satisfies

$$\Gamma_O = 1/S^2 \quad \text{P\&L(O)} = \int \frac{1}{2}\left(\sigma^2 - \Sigma^2\right)\Delta t \text{ sensitive only to } \sigma$$

A security with this gamma is the “log contract” with BS value $O = -\ln S$ and $\Delta = -1/S$, **independent** of volatility, unlike an ordinary call! You delta-hedge it by owning $1 worth of stock always.

A log contract, hedged, will capture realized variance.





VARIANCE SWAPS
HOW TO TRADE PURE VOLATILITY
(A LESSON IN REPLICATION)

4.3 Volatility and Variance Swap Contracts

A Volatility swap is a forward contract on realized volatility. At expiration it pays the difference in dollars between the actual volatility realized by the index over the lifetime of the contract σ_R and some previously agreed upon "delivery" volatility K_{vol}:

$$(\sigma_R - K_{vol}) \times N \text{ where } N \text{ is the notional amount.}$$

Similarly, a **variance swap** is a forward contract on realized variance. It pays

$$\left(\sigma_R^2 - K_{var}\right) \times N$$

The fair delivery price is the value of $K_{...}$ K which makes the relevant contract worth zero.

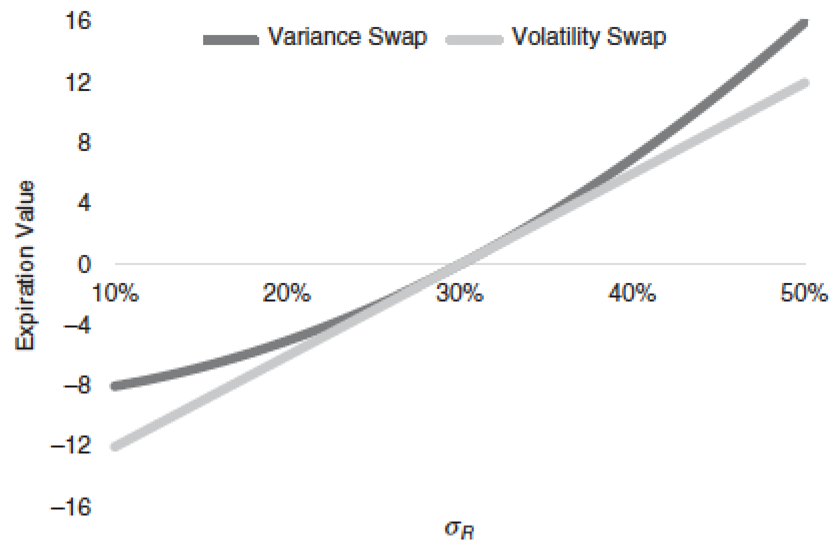


FIGURE 4.3 Comparison of a Volatility Swap with a Variance Swap

The variance swap is a derivative of the volatility swap. In theory, we could replicate the variance swap out of the volatility swap -- if we knew the evolution of volatility.

Common Sense: The fair delivery price of the volatility swap should be lower than the square root of the delivery price of variance, else the variance swap always does better than the volatility swap.

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4.4 Engineering Approach to Variance Replication in a BS World Using Vanilla Options

Zero interest rates and dividend yields for simplicity, so $C = C(S, K, v)$ where $v = \sigma\sqrt{\tau}$.

$$C_{BS} = SN(d_1) - KN(d_2) \qquad d_{1,2} = \frac{\ln S/K \pm v^2/2}{v}$$

Then the exposure to variance is given by

$$\kappa \equiv \frac{\partial C_{BS}}{\partial \sigma^2} = \frac{S\sqrt{\tau}}{2\sigma\sqrt{2\pi}} e^{-d_1^2/2}$$

$$S^* = Ke^{\frac{1}{2}v^2}$$

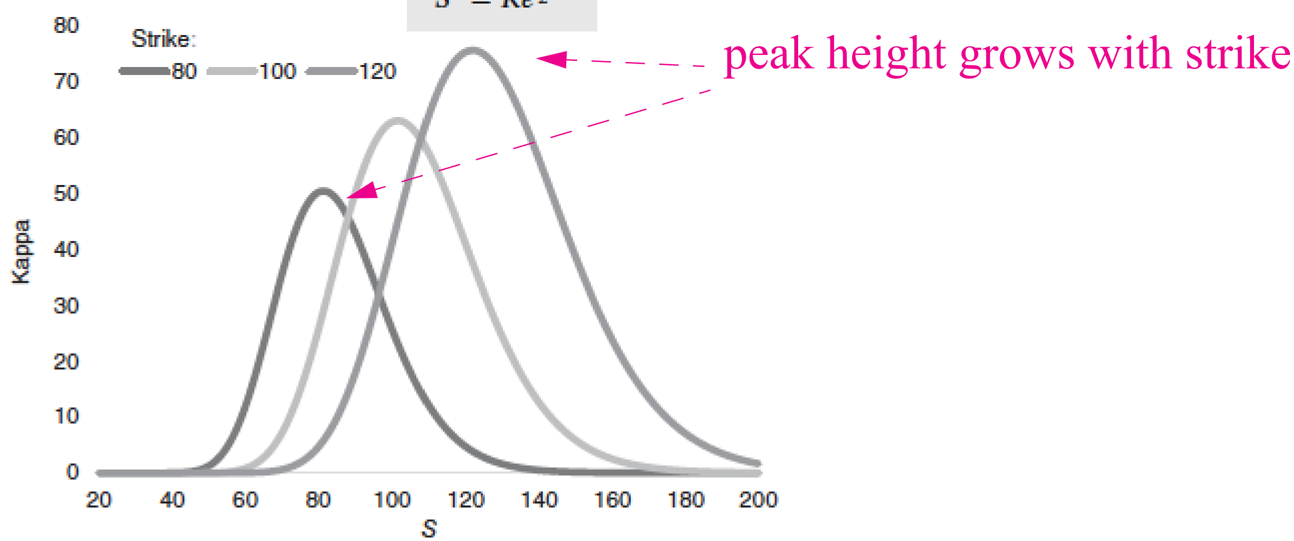

FIGURE 4.1 The Variance Vega for Three Strike Prices

You can see that the option has sensitivity to S and σ, and is therefore not a good way to make a clean bet on volatility. It peaks roughly at $S = K$, and so increases as the strike increases. What we want is a portfolio whose exposure κ to variance is independent of the stock price S, so that we can be exposed to volatility alone no matter what the stock price does.

Construct a portfolio $\pi(S) = \int_0^{\infty} \rho(K) C(S, K, v) dK$ such that $\kappa = \frac{\partial \pi}{\partial \sigma^2}$ is independent of S.

$$\frac{\partial \pi}{\partial \sigma^2} = \int_0^{\infty} \rho(K) \frac{S\sqrt{\tau}}{2\sigma} \frac{e^{-d_1^2/2}}{\sqrt{2\pi}} dK \sim \int_0^{\infty} \rho(K) S f\left(\frac{K}{S}, v\right) dK$$

We can make the S-dependence of $\rho(\ )$ explicit by changing variable to $x = K/S$ so that

$$\frac{\partial \pi}{\partial \sigma^2} = \int_0^{\infty} \rho(xS) S^2 f(x, v) dx$$

In order for this to be independent of S, we require that $\rho(K) \sim 1/K^2$

A density of options whose weights decrease as K^{-2} will give the correct volatility dependence.

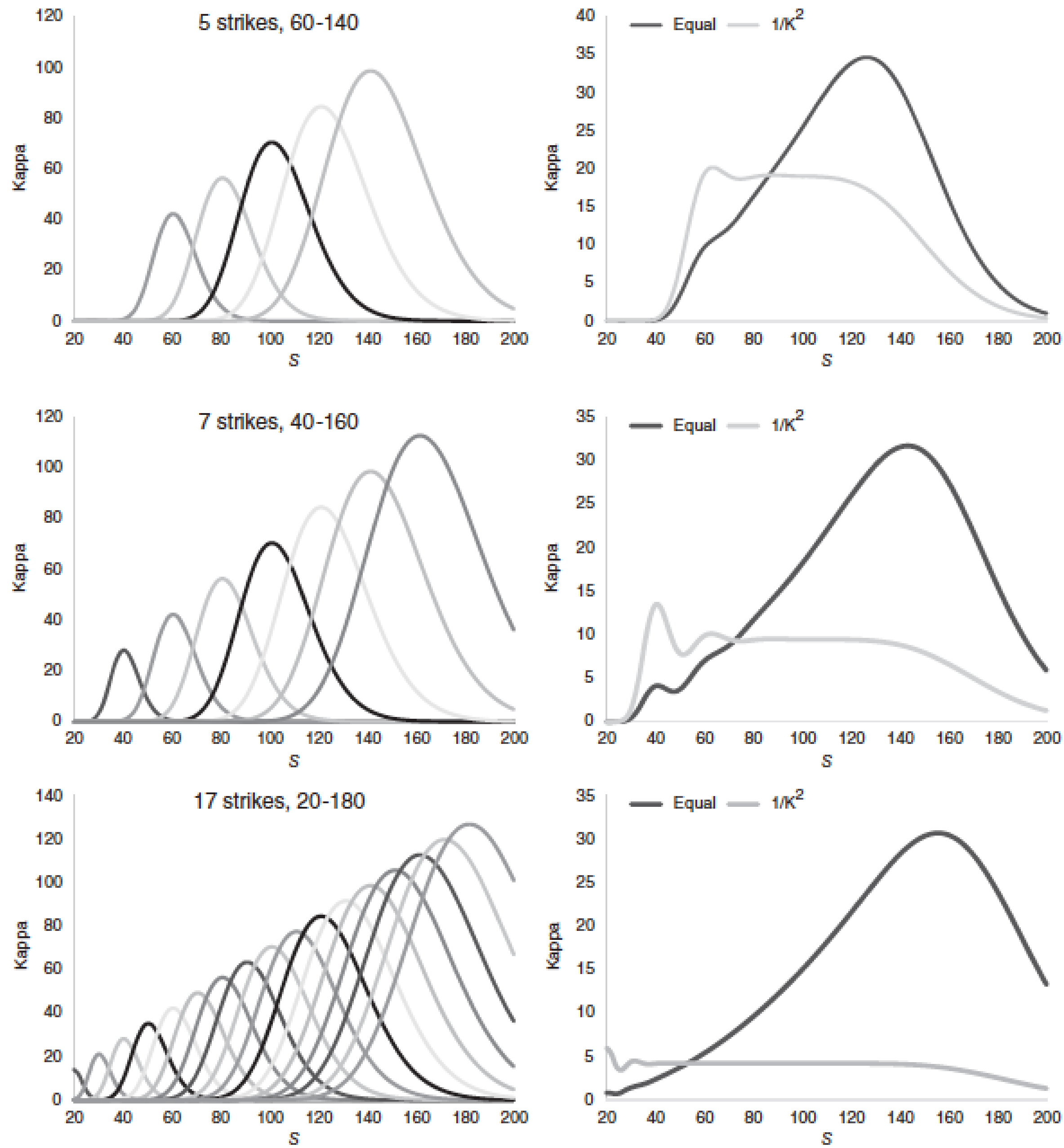

FIGURE 4.4 The Variance Vega of a Portfolio of Vanilla Options: Replicating a Variance Swap with Options Using Two Weighting Schemes

$1/K^2$ Calls and-or Puts Replicate a Log Payoff and a Forward

Use liquid puts below some strike S^* and use calls with strikes above S^*: $\pi\left(S, S^*, v\right)$.

The payoff at expiration at stock value S_T is $\pi\left(S_T, S^*, v\right)$ where at expiration $\nu = \sigma\sqrt{\tau} = 0$:

$$\begin{aligned}
\pi\left(S_T, S^*, 0\right) &= \int_{S^*}^{\infty} C(S_T, K, 0)\frac{dK}{K^2} + \int_{0}^{S^*} P(S_T, K, 0)\frac{dK}{K^2} \\
&= \int_{S^*}^{S_T} (S_T - K)\frac{dK}{K^2} + \int_{S_T}^{S^*} (K - S_T)\frac{dK}{K^2}
\end{aligned}$$

$$\int_{S^*}^{S_T} (S_T - K)\frac{dK}{K^2} + \int_{S_T}^{S^*} (K - S_T)\frac{dK}{K^2}$$

For $S_T > S^*$ only the first integral contributes, else the second integral. Each gives the same result:

$$\pi(S_T, S^*, T) = \left(\frac{S_T - S^*}{S^*}\right) - \ln\left(\frac{S_T}{S^*}\right) = \int_0^{S^*} \frac{1}{K^2} P(S_T, K, T)\, dK + \int_{S^*}^{\infty} \frac{1}{K^2} C(S_T, K, T)\, dK$$

In order to be exposed purely to volatility we need to own a forward contract with delivery price S^* (which has no volatility dependence), and be short a log contract L. The forward contract can be replicated statically, the log contract L must be hedged dynamically.

4.5 Value of Log Contract in a Black-Scholes World

Solve the Black-Scholes equation $\frac{\sigma^2 S^2}{2}\frac{\partial L}{\partial S^2} + \frac{\partial L}{\partial t} = 0$ for $r = 0$, with the boundary condition for the terminal log payoff $L(S, S^*, 0) = \ln\frac{S}{S^*}$.

Solution: At earlier times the log contract is worth

$$L(S, S^*, t, T) = \ln\left(\frac{S}{S^*}\right) - \frac{1}{2}\sigma^2(T-t)$$

check that it satisfies the BS PDE

$$L(S, S^*, t, T) = \ln\left(\frac{S}{S^*}\right) - \frac{1}{2}\sigma^2(T - t)$$

$\Delta = 1/S$. Hedge the log contract by owning by owning exactly \$1 worth of shares at any instant

$\Gamma = 1/S^2$ and $\kappa = \partial\pi/\partial\sigma^2 = -0.5(T-t)$ gives the correct volatility exposure.

When $t = 0$ and $\tau = T$, you need to be short 2/T contracts to have $\kappa = 1$, a variance exposure of \$1

Assuming zero rates and dividend yields, the value of the variance replicating portfolio at earlier times is (2/T) (short a log contract L and long a forward contract with delivery price S^*)

$$\pi(S, S^*, t, T) = \frac{2}{T}\left[\left(\frac{S - S^*}{S^*}\right) - \ln\left(\frac{S}{S^*}\right)\right] + \frac{T-t}{T}\sigma^2$$

If $S = S_0 = S^*$ at $t = 0$ $\qquad \pi(S_0, S_0, 0, T) = \sigma^2$

4.6 Explicit proof that the fair initial value of a hedged log contract with $S^*=S_0$ is actually the variance. (Assume r = 0)

Consider a log contract that pays out $\log(S_T/S_0)$ at expiration time T. Let its value today be denoted by the unknown amount L_0. The hedge ratio is $1/S$ so that one must always have a delta for which the value of the shares in the hedge is $\frac{1}{S} \times S = \$1$.

We now use simple accounting of profit and loss when we carry out a trading strategy to find L_0.

Look at the trading strategy below that starts with a short position in one log contract and long \$1 worth of shares to hedge it, and then maintains this dollar value of shares by rehedging as below.

TABLE 4.1 Before Rebalancing, Part I

Time	Stock Price	No. of Shares of Stock	Value of Stock	Value of One Log Contract	Bank Balance	Total Value of Position
t_0	S_0	$\frac{1}{S_0}$	1	L_0	0	$1 - L_0$
$t_{1\,(\text{pre})}$	S_1	$\frac{1}{S_0}$	$\frac{S_1}{S_0}$	L_1	0	$\frac{S_1}{S_0} - L_1$

Now at time 1 rebalance to own \$1 worth of shares: buy $(1/S_1 - 1/S_0)$ shares by borrowing $(1/S_1 - 1/S_0)S_1 = (S_0 - S_1)/S_0$ dollars. You then own $1/S_1$ shares worth \$1, and you have borrowed (that is, you are short) $(S_0 - S_1)/S_o$ dollars.

TABLE 4.2 Rebalancing, Part II

Time	Stock Price	No. of Shares of Stock	Value of Stock	Value of One Log Contract	Bank Balance	Total Value of Position
$t_{1\,(\text{post})}$	S_1	$\frac{1}{S_1}$	1	L_1	$-\frac{S_0 - S_1}{S_0}$	$1 - L_1 - \frac{S_0 - S_1}{S_0}$

Now move to time t_2 and rebalance again, to get

TABLE 4.3 Rebalancing, Part III

Time	Stock Price	No. of Shares of Stock	Value of Stock	Value of One Log Contract	Bank Balance	Total Value of Position
$t_{2\,(\text{post})}$	S_2	$\frac{1}{S_2}$	1	L_2	$-\frac{S_0 - S_1}{S_0} - \frac{S_1 - S_2}{S_1}$	$1 - L_2 - \frac{S_0 - S_1}{S_0} - \frac{S_1 - S_2}{S_1}$

Repeat rehedging N times to expiration where we know the terminal value of the log contract:

$$
\begin{aligned}
V_N &= 1 - L_N - \sum_{i=0}^{N-1} \frac{S_i - S_{i+1}}{S_i} \\
&= 1 - \ln\left(\frac{S_N}{S_0}\right) + \sum_{i=0}^{N-1} \frac{\Delta S_i}{S_i} \\
&= 1 - \sum_{i=0}^{N-1} \ln\left(\frac{S_{i+1}}{S_i}\right) + \sum_{i=0}^{N-1} \frac{\Delta S_i}{S_i}
\end{aligned}
$$

because $\ln\frac{a}{b} + \ln\frac{b}{c} = \ln\frac{a}{c}$. Now recall that $\ln\left(\frac{S+\Delta S}{S}\right) = \ln\left(1+\frac{\Delta S}{S}\right) \approx \frac{\Delta S}{S} - \frac{1}{2}\left(\frac{\Delta S}{S}\right)^2$

Assume we can ignore shifts in the stock return with power greater than 2:

Taking a second-order Taylor expansion of the terms in the first summation, we have

$$
\begin{aligned}
V_N &= 1 - \sum_{i=0}^{N-1} \left[\frac{\Delta S_i}{S_i} - \frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2\right] + \sum_{i=0}^{N-1} \frac{\Delta S_i}{S_i} \\
&= 1 + \sum_{i=0}^{N-1} \frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2 \qquad (4.26) \\
&= 1 + \sum_{i=0}^{N-1} \frac{\sigma_i^2 \Delta t_i}{2}
\end{aligned}
$$

Thus, if you assume zero interest rates, we've shown that an initial investment at time t = 0 of value $-L_0 + 1$, by dynamic rehedging, leads to a final value at time t = T of $1 + \sum \frac{\sigma_i^2 \Delta t_i}{2}$.

Therefore, the fair value of L_0 at the beginning must be $L_0 = -\sum_i \frac{\sigma_i^2 \Delta t_i}{2}$.

Being short a log contract with strike S_0 and being long \$1 worth of stock, dynamically rehedging as the stock moves, will guarantee you a final payoff equal to the realized volatility (assuming GBM even with a variable volatility).

Problems with Replication

If you could buy a log contract you'd have exactly what you want. Instead you have to buy a continuum of calls and puts, which isn't perfectly possible. You can only buy a discrete number in a discrete range, so you have no sensitivity to volatility outside the strike range.

4.7 Replication of Variance when Volatility is Stochastic

As long as there is continuous diffusion (**no jumps**), the log contract still captures realized volatility even if the volatility is time dependent.

$$\frac{dS_t}{S_t} = \mu dt + \sigma(t) dZ_t$$

$$d\ln S_t = \left(\mu - \frac{\sigma(t)^2}{2}\right)dt + \sigma dZ_t$$

$$\frac{dS_t}{S_t} - d\ln S_t = \frac{1}{2}\sigma(t)^2 dt \qquad \text{Eq 4.1}$$

$$\text{average total variance} = \frac{1}{T}\int_0^T \sigma(t)^2 dt = \frac{2}{T}\left[\underbrace{\int_0^T \frac{dS_t}{S_t}}_{\text{cost of rebalancing hedge } 1/S \text{ shares dynamically}} - \underbrace{\ln\frac{S_T}{S_0}}_{\text{static short log contract}}\right]$$

Now $\ln(S_T/S_0) = \ln(S_*/S_0) + \ln(S_T/S_*)$

And $\ln(S_T/S_*)$ is related to the payoff of a sum of calls and puts.

and Eq 4.2

$$\left(\frac{S_T - S^*}{S^*}\right) - \ln\left(\frac{S_T}{S^*}\right) = \int_0^{S^*} \frac{1}{K^2} P(S_T, K, T)\, dK + \int_{S^*}^{\infty} \frac{1}{K^2} C(S_T, K, T)\, dK$$

Therefore from the previous page,

money spent in rebalancing exposure to \$1

$\ln S_T / S_0$

options value at expiration T

$$\int_0^T \sigma^2(t)dt = \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_*}{S_0} - \frac{(S_T - S_*)}{S_*} + \int_0^{S^*} P(S_T, K, T)\frac{dK}{K^2} + \int_{S^*}^{\infty} C(S_T, K, T)\frac{dK}{K^2}\right]$$

Eq 4.3

Everything on the RHS is a derivative of S_T and so can be valued by risk-neutral hedging.

Now take the expected risk-neutral discounted value e^{-rT} of Equations 4.3. The PV value of the LHS = the expected discounted value of the RHS in a risk-neutral world where $S_0 = e^{-rT}E[S_T]$ and the expected discounted value of the call payoff is today's call value.

$$
{}^{-rT}\left[\frac{1}{T}\int_0^T \sigma^2(t)dt\right] = \frac{2}{T}\left[e^{-rT}(rT) - e^{-rT}\ln\frac{S_*}{S_0} - \left(\frac{S_0}{S_*} - 1e^{-rT}\right) + \overbrace{\int_{S^*}^{\infty} C(S_0, K, 0)\frac{dK}{K^2}}^{\text{call value at time } 0} + \overbrace{\int_0^{S^*} P(S_0, K, 0)\frac{dK}{K^2}}^{\text{put value at time } 0}\right.
$$

Thus the fair value of the total variance is obtained by multiplying by e^{rT}:

$$
\int_0^T \sigma^2 dt = \frac{2}{T}\left[rT - \ln\frac{S_*}{S_0} - \left(\frac{S_0 e^{rT}}{S_*} - 1\right) + e^{rT}\int_{S^*}^{\infty} C(S_0, K, 0)\frac{dK}{K^2} + e^{rT}\int_0^{S^*} P(S_0, K, 0)\frac{dK}{K^2}\right.
$$

Every option's price can be taken from the marketplace, even with a skew, and we can value the variance (almost) independent of theory.

If $S_* = S_0$ then we get the simpler formula

$$
\int_0^T \sigma^2 dt = \frac{2}{T}\left[rT - \left(e^{rT} - 1\right) + e^{rT}\int_{S_0}^{\infty} C(S_0, K, 0)\frac{dK}{K^2} + e^{rT}\int_0^{S_0} P(S_0, K, 0)\frac{dK}{K^2}\right.
$$

$$
\approx \frac{2}{T}\left[e^{rT}\int_{S_0}^{\infty} C(S_0, K, 0)\frac{dK}{K^2} + e^{rT}\int_0^{S_0} P(S_0, K, 0)\frac{dK}{K^2}\right]
$$

The initial value of the portfolio of puts and calls is the value of the time-averaged total variance.

4.8 Approximating the Fair Variance in a Skew

From page 15 we need to the payoff $\frac{2}{T}\left[\frac{(S_T - S_0)}{S_0} - \ln\frac{S_T}{S_0}\right]$ to capture future volatility.

We can replicate it by a linear-payoff of calls and puts that dominate it, starting from the minimum value at $K_0 = S_0$,

with strikes $K_i^{c,\,p}$ for the calls and puts

.

$$V(t) = \cdots + (\lambda_1^P - \lambda_0^P)\,P(K_P^1) + \lambda_0^P P(K_0) + \lambda_0^C C(K_0) + (\lambda_1^C - \lambda_0^C)\,C(K_C^1) + \cdots$$

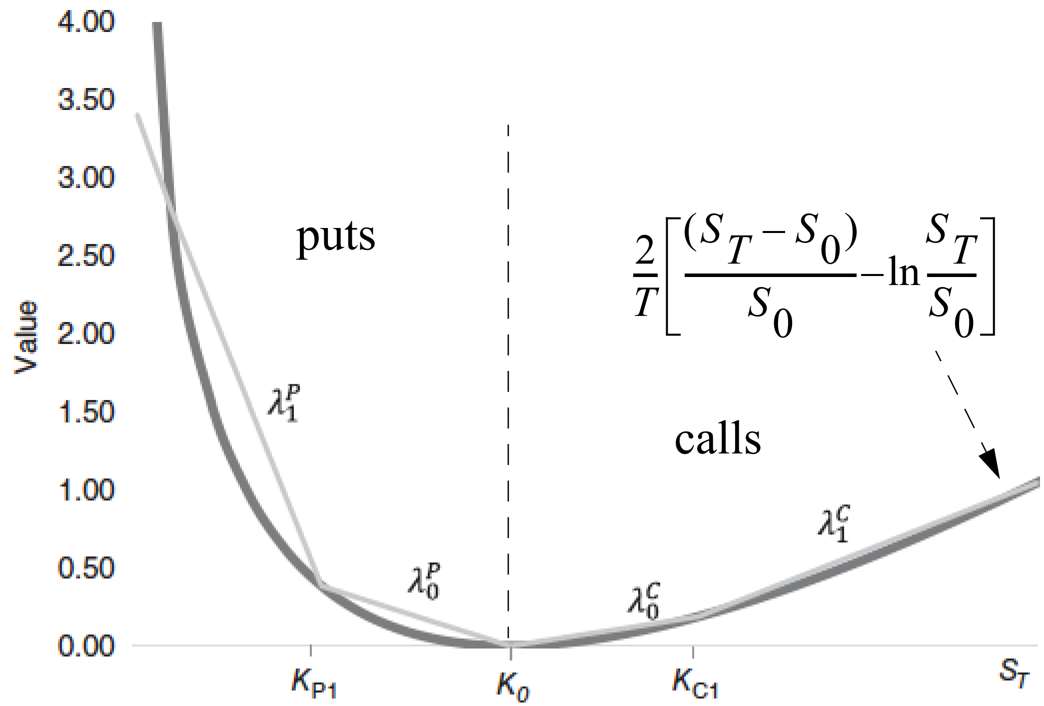

FIGURE 4.6 Piecewise-Linear Replication of a Variance Swap

See Demeterfi et al paper: *More Than You Ever Wanted To Know About Volatility Swaps,* posted on Courseworks, or in my textbook.

Example

Find the value of a one-year variance swap on the S&P 500. Assume that the riskless rate is zero, and the current level of the S&P is 2,000. The market prices of one-year options on the S&P 500 are listed in the following table.

K_i	C_i	P_i
1,200	802.91	2.91
1,400	614.38	14.38
1,600	445.31	45.31
1,800	305.44	105.44
2,000	198.95	198.95
2,200	123.81	323.81
2,400	74.12	474.12
2,600	42.97	642.97
2,800	24.28	824.28

We can use the sum of calls and puts to approximate the value of the variance swap. We begin by calculating the value $\pi(K_i)$, of the replicating portfolio at each of the available strike prices $S = K_i$,

using $-\ln\dfrac{S}{S^0} + \left(\dfrac{S - S^0}{S^0}\right)$.

Next we calculate the slopes, λ_i, for our piecewise-linear function, $\lambda_i = [\pi(K_i) - \pi(K_{i-1})]/[K_i - K_{i-1}]$. We then use these slopes to calculate the weights w_i for the options, $w_i = \lambda_{i+1} - \lambda_i$.

In the rightmost column, we multiply the weights by the option prices. We use puts below the current market level and calls at and above the current market level. By adding the values in the rightmost column we obtain our price for the variance swap. Prices for variance swaps are typically quoted in terms of volatility. Our final answer is then 25.15%.

K_i	$\pi(K_i)$	λ_i	w_i	C_i	P_i	$w_i \times O_i$
1,000	0.386					
1,200	0.222	0.000823	0.000282		2.91	0.0008
1,400	0.113	0.000542	0.000206		14.38	0.0030
1,600	0.046	0.000335	0.000157		45.31	0.0071
1,800	0.011	0.000178	0.000124		105.44	0.0131
2,000	0.000	0.000054	0.000054		198.95	0.0107
2,000	0.000	0.000047	0.000047	198.95		0.0093
2,200	0.009	0.000130	0.000083	123.81		0.0103
2,400	0.035	0.000200	0.000070	74.12		0.0052
2,600	0.075	0.000259	0.000059	42.97		0.0026
2,800	0.127	0.000310	0.000051	24.28		0.0012
3,000	0.189					
					Variance	0.0632
					Vol	0.2515

The option prices were actually generated from a vol of 25% and we are not far off. A little high because we interpolate above the curve.

4.9 Imperfections in Valuation by Replication

- **Discrete strikes** with a limited range capture less variance than the true variance. You gamble by omitting some strikes because when/if the stock price gets to those strikes, you have no options strikes and therefore no option gamma to capture the variance.
- **Effect of jumps**
  The log contract doesn't capture the true variance if jumps occur, for two reasons.

  1. Jumps can move the stock price out of the range of strikes you use for replication.
  2. A jump contributes to the rigorous definition of the realized variance which depends on the second moment of the return, i.e. J^2, but jumps contribute to the Taylor series expansion of the log contract with a J^3 term too, plus higher orders.
  "
  The log contract hedging strategy

$$\text{otal variance} = \frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_T}{S_0}\right] \text{captures}$$

$$\sum \frac{\Delta S_i}{Si} - \log\frac{S_T}{S_0} = \sum\left[\frac{\Delta S_i}{Si} - \log\frac{S_{i+1}}{S_i}\right] = \sum\left[\frac{\Delta S_i}{Si} - \log\left(1 + \frac{\Delta S_i}{S_i}\right)\right] = \quad \approx \sum \frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2 - \frac{1}{3}\left(\frac{\Delta S_i}{S_i}\right)^3 + \dots$$

- The first term is the true realized variance contribution; the second is normally negligible, but for a large jump $(\Delta S_i)/S_i = J$ will add an asymmetric term to the P&L that is absent from the true variance

4.10 Valuing Volatility Swaps: Negative Convexity

Volatility is a derivative, the square root of variance which we know how to replicate with a combination of calls and puts. You can replicate volatility using the continuous dynamic trading of portfolios of variance swaps, just as you can replicate $\sqrt{S}$ by trading S.

To estimate the effect, expand about V_E, the expected value of the distribution of variance:

$$
\begin{aligned}
\sigma &= \sqrt{\sigma^2} = \sqrt{V} \equiv \sqrt{V_E + \{V - V_E\}} \\
&= \sqrt{V_E}\left(1 + \frac{V - V_E}{V_E}\right)^{1/2} \\
&\approx \sqrt{V_E}\left[1 + \frac{V - V_E}{2V_E} - \frac{1}{8}\left(\frac{V - V_E}{V_E}\right)^2 + \dots\right] \\
&\approx \sqrt{V_E} + \frac{V - V_E}{2\sqrt{V_E}} - \frac{1}{8}\frac{(V - V_E)^2}{V_E^{3/2}}
\end{aligned}
$$

The square root has negative convexity therefore worth less.

Taking risk-neutral expectation to value the volatility: $E(\sigma) \approx \sqrt{V_E} - \frac{1}{8}\frac{E\left[(V - V_E)^2\right]}{V_E^{3/2}}$

Thus the fair volatility is smaller than the square root of the variance, and depends on the volatility of variance, like an option on variance.

Fair Volatility Must Be Smaller Than Square Root Of Variance

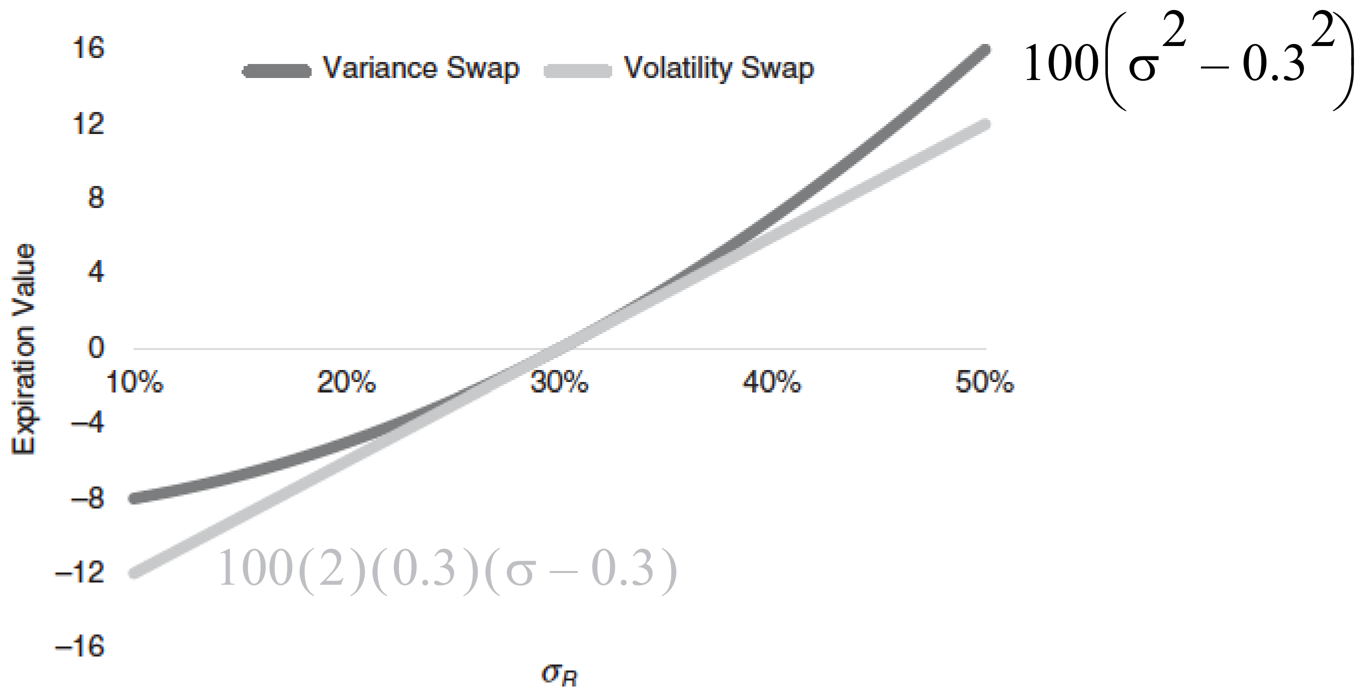

FIGURE 4.3 Comparison of a Volatility Swap with a Variance Swap

We are comparing a variance swap with notional 100 and a vol swap with equivalent exposure at the strike. $\sigma^2 - 0.3^2 = (\sigma - 0.3)(\sigma + 0.3) \approx (2)(0.3)(\sigma - 0.3)$

The variance swap dominates the vol swap and must be worth more and would generate an arbitrage-free profit if we bought the variance swap and sold the vol swap at the same delivery price. This cannot be.

Thus the fair strike of the volatility swap, the value of the volatility for which the swap is worth zero, must be lower. How much depends on the vol of vol.

4.11 The VIX Volatility Index

The VIX, from 1993 - 2003, was defined as the weighted average of various atm and otm implied volatilities of the S&P. This was a rather arbitrary average of parameters. In 2003 the CBOE changed the definition of the VIX to be the square root of the fair delivery price of variance as captured by a variance swap, using the formula from our paper, extended to account for stock dividends.

$$\int_0^T \sigma^2 dt = \frac{2}{T}\left[(r-d)T - \ln\frac{S_*}{S_0} - \left(\frac{S_0 e^{(r-d)T}}{S_*} - 1\right) + e^{rT}\int_{(K > S^*)} C(S, K, 0)\frac{dK}{K^2} + e^{rT}\int_{(K < S^*)} P(S, K, 0)\frac{dK}{K^2}\right]$$

Writing F as the forward price of the S&P, $F = S_0 e^{(r-d)T}$, the RHS is

$$\frac{2}{T}\left\{\ln\frac{F}{S_0} - \ln\frac{S_*}{S_0} - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls above S* plus puts below S*}]\right\}$$

$$= \frac{2}{T}\left\{\ln\frac{F}{S^*} - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls and puts}]\right\}$$

$$= \frac{2}{T}\left\{\ln\left(1 + \frac{F}{S^*} - 1\right) - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls and puts}]\right\}$$

$$\approx \frac{2}{T}\left\{e^{rT}[\text{sum of calls and puts}] - \frac{1}{2}\left(\frac{F}{S_*} - 1\right)^2\right\}$$

The CBOE uses a finite sum over traded options at two expirations near 30 days, and then interpolates/extrapolates to thirty day volatility.

Some advantages of the "new VIX":

- The VIX is an estimate of one-month future realized volatility based on listed options prices.
- The estimate is independent of volatility at one particular market level because it involves the sum of different options prices.
- It is *relatively* insensitive to model issues, because it assumes only continuous underlier movement, but doesn't assume Black-Scholes no-smile.
- It can be replicated and hedged because it involves a portfolio of listed options.
- The VIX is the most liquid measure of short-term implied volatility. People tend to regard it as an indicator of future realized volatility. "The fear index"

Future Extensions

Many variance swaps are capped and implicitly contain embedded volatility options.

Valuing options on volatility is the big challenge. More on volatility of volatility later.

With a model for volatility of the VIX, one can price futures, forwards and options on the VIX. The CBOE offers listed futures and options on the VIX.

4.12 Perspective: The Black-Scholes Equation is Also a Statement about Sharpe Ratios

Valuation by perfect replication. We assume

- Continuous stock price movements (one-factor geometric Brownian motion) with constant volatility; no other sources of randomness; no jumps.
- Ability to hedge continuously by taking on arbitrarily large long or short positions.
- No bid-ask spreads.
- No transactions costs.
- No forced unwinding of positions.

$$dS = \mu_S S dt + \sigma_S S dZ$$
$$dB = Brdt$$

The option price $C(S_t, t)$ whose evolution is given by

$$\begin{aligned} dC &= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\sigma_S S)^2 dt \\ &= \left\{\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu_S S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\sigma_S S)^2\right\}dt + \frac{\partial C}{\partial S}\sigma_S S dZ \\ &= \mu_C C dt + \sigma_C C dZ \end{aligned}$$

where by definition

$$\mu_C = \frac{1}{C}\left\{\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu_S S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\sigma_S S)^2\right\}$$

$$\sigma_C = \frac{S}{C}\frac{\partial C}{\partial S}\sigma_S = \frac{\partial \ln C}{\partial \ln S}\sigma_S \qquad \text{Eq 4.4}$$

Riskless portfolio $\pi = \alpha S + C$

Then

$$\begin{aligned} d\pi &= \alpha(\mu_S S dt + \sigma_S S dZ) + (\mu_C C dt + \sigma_C C dZ) \\ &= (\alpha\mu_S S + \mu_C C)dt + (\alpha\sigma_S S + \sigma_C C)dZ \end{aligned} \qquad \textit{Eq 4.5}$$

Riskless necessitates dZ coefficient must be zero:

$$\begin{aligned} \alpha\sigma_S S + \sigma_C C = 0 \\ \alpha = -\frac{\sigma_C C}{\sigma_S S} \end{aligned} \qquad \text{Eq 4.6}$$

Therefore

$$d\pi = (\alpha\mu_S S + \mu_C C)dt$$

No riskless arbitrage: $d\pi = \pi r dt$.

Requires

$$\alpha \mu_S S + \mu_C C = (\alpha S + C) r$$

Rearranging the terms we obtain

$$\alpha S(\mu_S - r) = -C(\mu_C - r)$$

Substituting for α from Equation 4.6 leads to the relation

$$\frac{(\mu_C - r)}{\sigma_C} = \frac{(\mu_S - r)}{\sigma_S} \qquad \text{Equal Sharpe Ratios of Stock and Option} \qquad \text{Eq 4.7}$$

This is the argument by which Black originally derived the Black-Scholes equation.

Substituting from Equation 4.4 into Equation 4.7 for μ_C and σ_C we obtain

$$\frac{\frac{1}{C}\left\{\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu_S S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\sigma_S S)^2\right\} - r}{\frac{1}{C}\frac{\partial C}{\partial S}\sigma_S S} = \frac{(\mu_S - r)}{\sigma_S}$$

which leads to

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$ Black-Scholes equation, no drift Eq 4.8

The solution, the Black-Scholes formula and its implied volatility, is the quoting currency for trading prices of vanilla options.

You can get a great deal of insight into more complex models by regarding them as perturbations or mixtures of different Black-Scholes solutions.

$$C(S,K,t,T,\sigma,r) = e^{-r(T-t)}[S_F N(d_1) - KN(d_2)]$$
$$S_F = e^{r(T-t)}S$$
$$d_1 = \frac{\ln\left(\frac{S_F}{K}\right) + \left(\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \quad d_2 = \frac{\ln\left(\frac{S_F}{K}\right) - \left(\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}y^2}dy$$ Eq 4.9

Notice that except for the $r(T-t)$ term, time to expiration and volatility always appear together in the combination $\sigma^2(T-t)$. If you rewrite the formula in terms of the prices of traded securities –

the present value of the bond K_{PV} and the stock price S – then indeed time and volatility always appear together:

$$C(S, K, t, T, \sigma) = [SN(d_1) - K_{PV}N(d_2)$$

$$K_{PV} = e^{-r(T-t)}K$$

$$d_{1,2} = \frac{\ln(S/K_{PV}) \pm 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} \qquad \text{Eq 4.10}$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-y^2/2}\,dy$$

Note that $\sigma^2(T-t)$ is the total future variance. Smart users of the formula can enter their estimates of the total variance, which on average may be smaller on weekends than on weekdays, for example.

The Effective Instantaneous Volatility of a Call -- Example

According to analysts at your firm, the expected return on Microsoft (MSFT) is 11%. MSFT is currently trading at \$50. 3-month at-the-money calls on MSFT have a delta of 0.52 and trade at \$2.00. The volatility of MSFT is 15%. What is current volatility of the call options?

$$\begin{aligned}\sigma_C &= \frac{S}{C}\frac{\partial C}{\partial S}\sigma_S \\ &= \frac{50}{2}\times 0.52 \times 0.15 \\ &= 1.95\end{aligned}$$

The instantaneous current volatility of the option is 195%, much riskier than the stock itself.