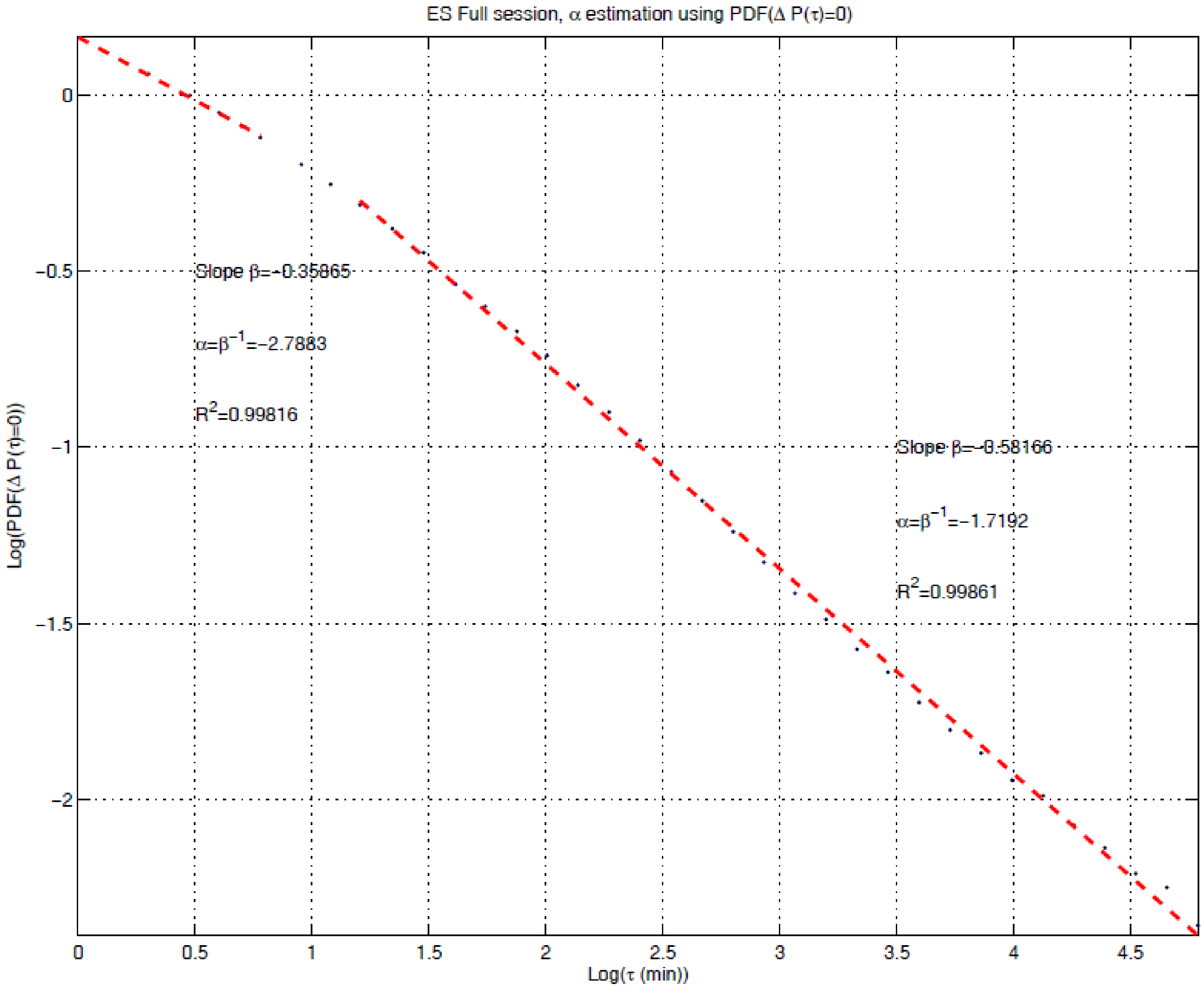
Math Methods – Financial Price Analysis

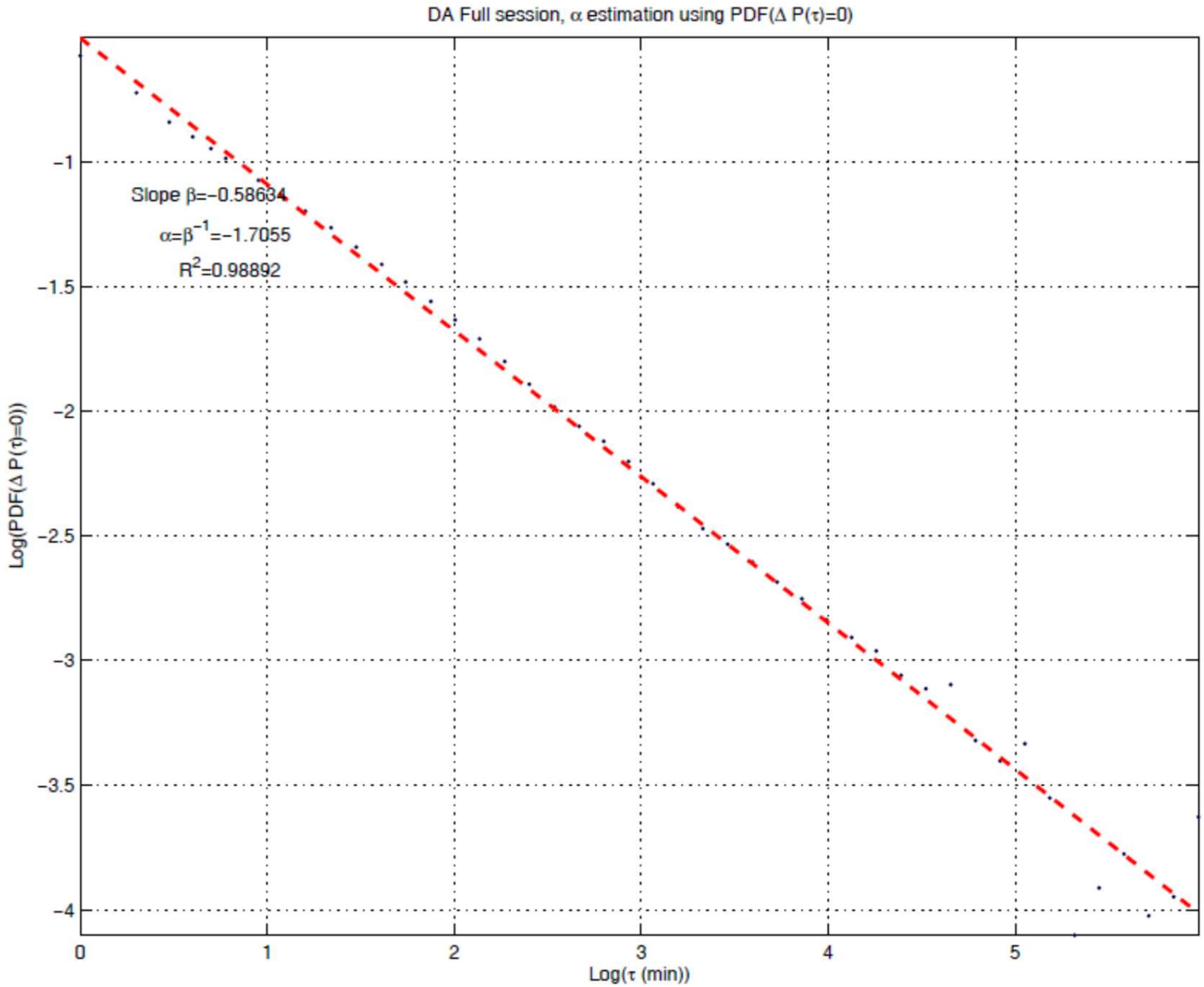
Mathematics GR5360

Instructor: Alexei Chekhlov

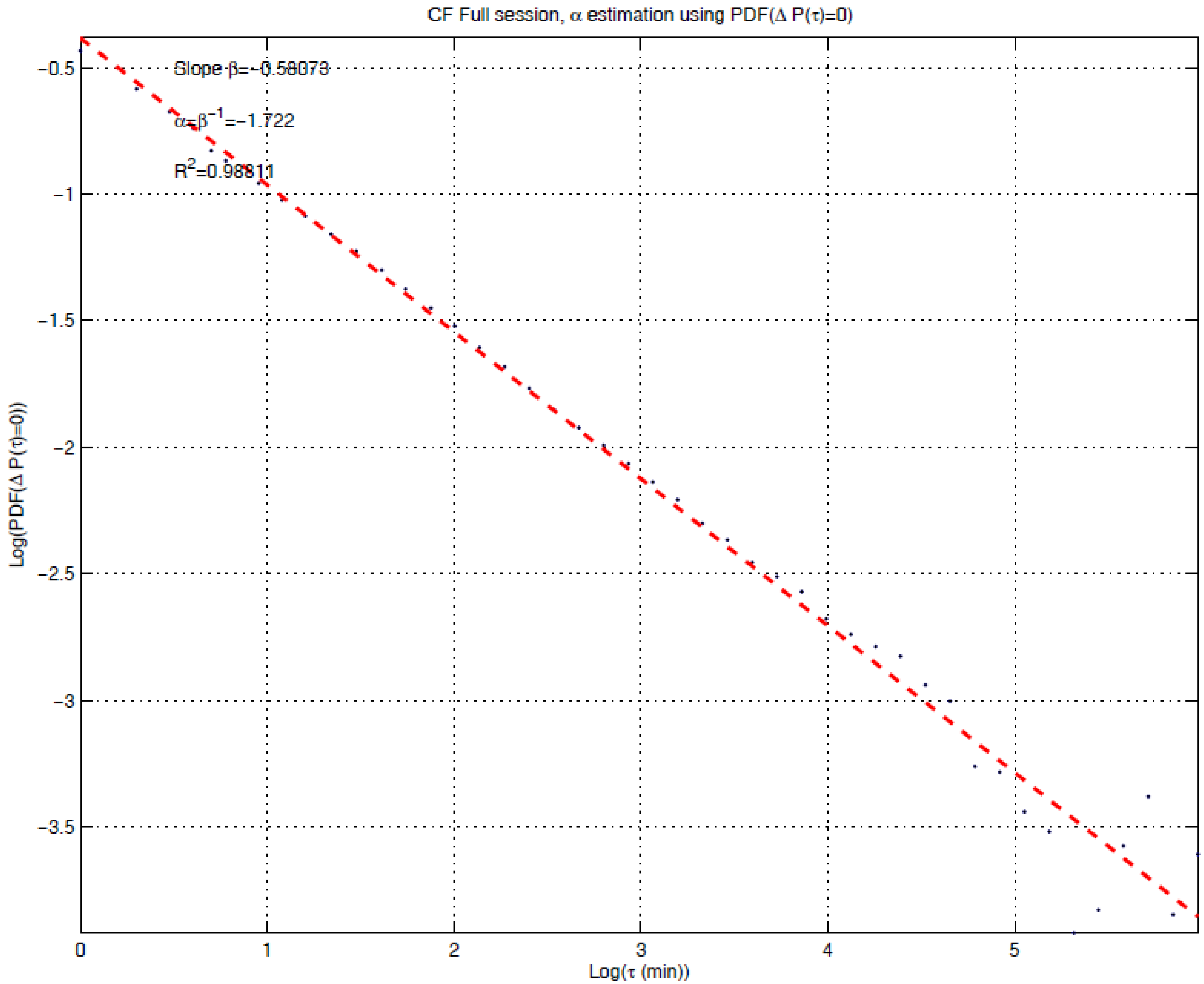
Universality of Algebraic Tails of PDFs: S&P 500



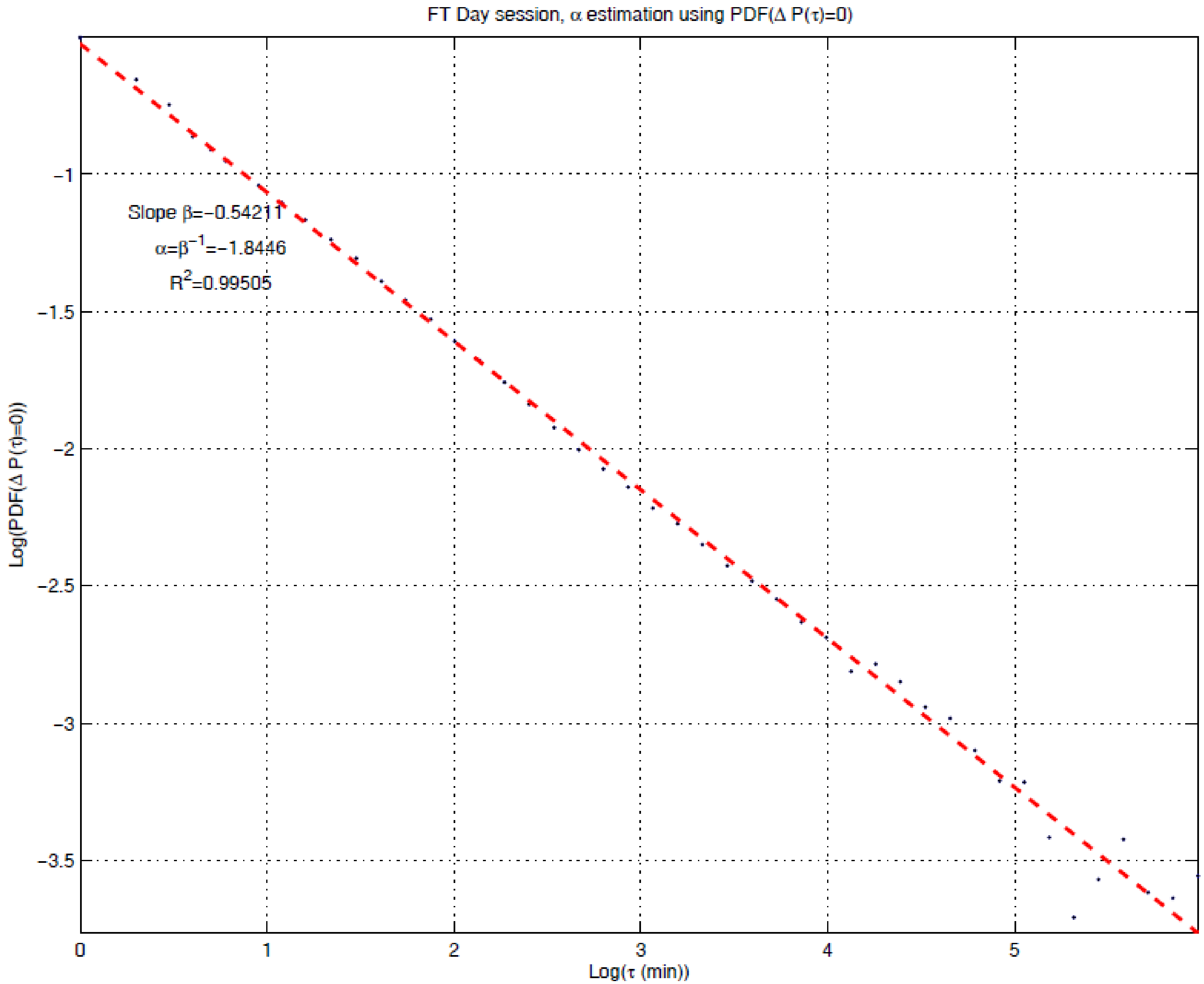
Universality of Algebraic Tails of PDFs: DAX



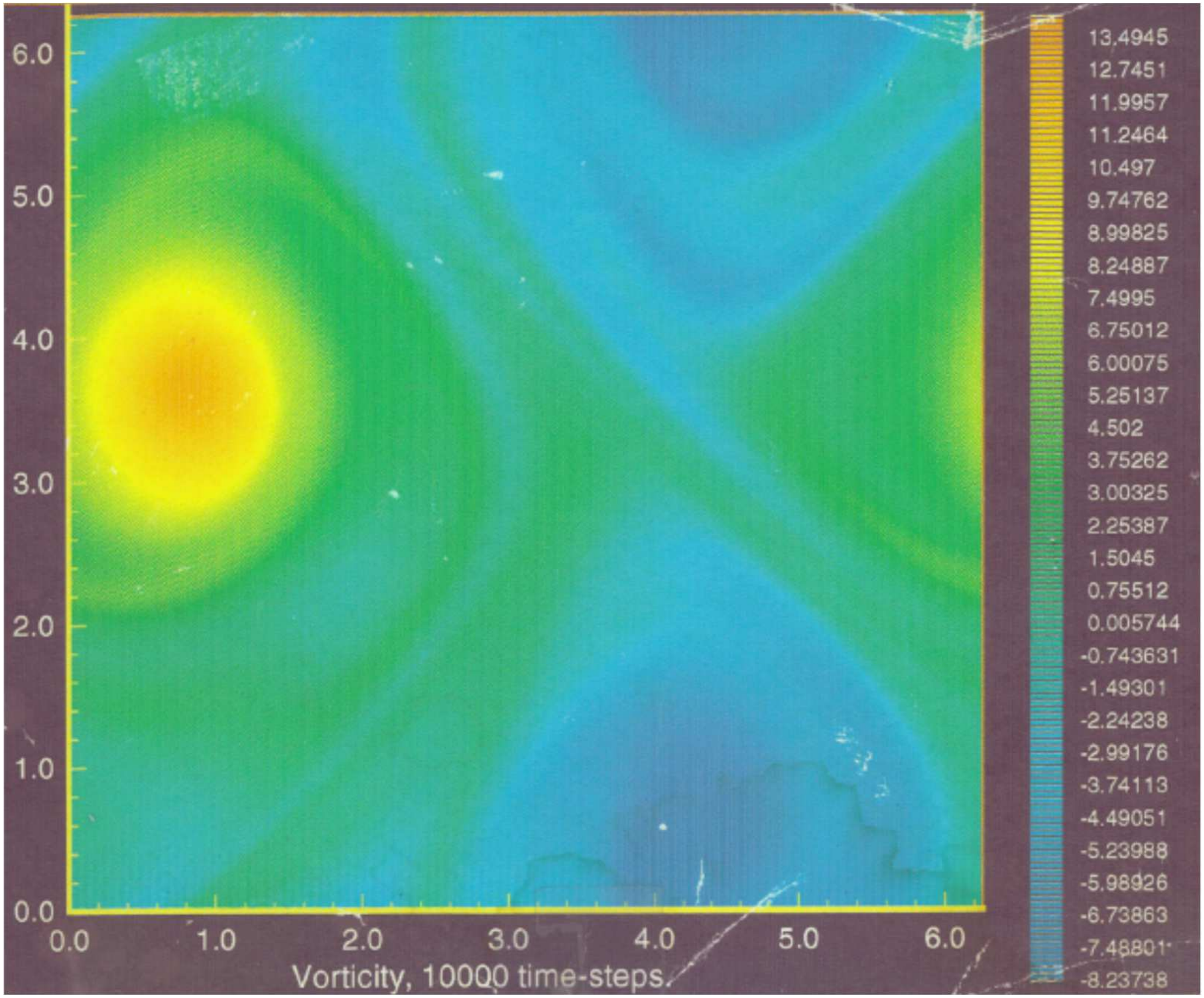
Universality of Algebraic Tails of PDFs: CAC 40



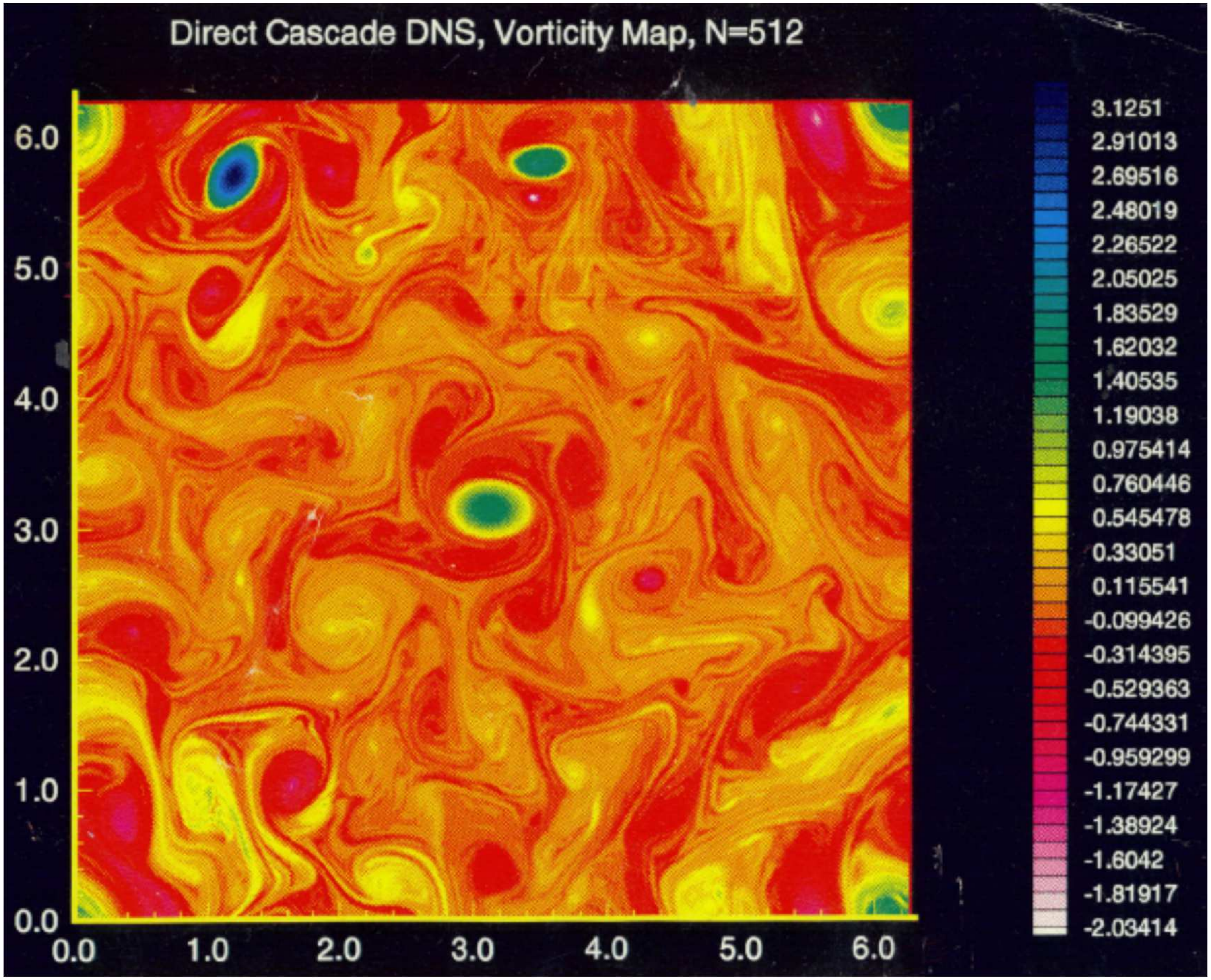
Universality of Algebraic Tails of PDFs: FTSE 100



Decaying 2-D Homogeneous Turbulence



Randomly-Forced 2-D Homogeneous Turbulence



Fluid Turbulence

- Turbulence is widely recognized as one of the outstanding problems of the physical sciences.
- Despite having attracted the sustained efforts of many leading scientists for well over a century, it still remains only partially understood.
- A U.S. theoretical physicist Robert Kraichnan, who made very significant contributions to the theory of turbulence, was the last postdoc with Albert Einstein as his scientific advisor. For his contributions he was awarded an Onsager Prize (1993) and a Dirac Medal (2003).
- Significant contributions into the theory of turbulence were made by the following deceased scientists (not a complete list by any means):
  - Osborne Reynolds (1842-1912) scientist and engineer;
  - Ludwig Prandtl (1875-1953) aerodynamist and engineer;
  - Theodore von Karman (1881-1963) aerodynamist and engineer;
  - Geoffrey Ingram Taylor (1886-1975) physicist, applied mathematician and engineer;
  - Lewis Fry Richardson (1881-1953) meteorologist and mathematician;
  - Andrey Nikolaevich Kolmogorov (1903-1987) mathematician and statistician;
  - Stanley Corrsin (1920-1986) fluid dynamicist;
  - George Keith Batchelor (1920-2000) fluid dynamicist;
  - Alan Townsend (1917-2010) physicist and fluid dynamicist;
  - Robert Kraichnan (1928-2008) mathematical physicist;
  - Philip Saffman (1931-2008) mathematician and fluid dynamicist;
  - Steven Alan Orszag (1943-2011) mathematician and fluid dynamicist.

Fluid Turbulence

- Nobel Laureate U.S. physicist Richard Feynman, following Albert Einstein, was quoted as saying "*Turbulence is the most important unsolved problem of classical physics*".
- Horace Lamb, the author of the great classic treatise "Hydrodynamics" is alleged to have said: "*When I meet the Creator, one of the first things I shall ask of Him is to reveal to me the solution to the problem of turbulence*".
- Why does the problem of turbulence exert such enduring fascination among scientists?
  - ✓ Perhaps because it is recognized as a prototype of problems in the physical sciences exhibiting both strong nonlinearity and irreversibility, which leads to great irregularity of the solution in both space and time.
  - ✓ May be because its complete solution has eluded the best scientific minds of the 20th century and it remains not completely solved to this date.
  - ✓ Also, a great span of applications of fluid dynamics has generated an ever growing need to achieve a better fundamental understanding of turbulence in practical circumstances: flight, meteorology, oceanography, planetary, inter-stellar magnetic fields evolution, plasma physics, chemical reactions, combustion, etc.

Fluid Turbulence and Symmetries

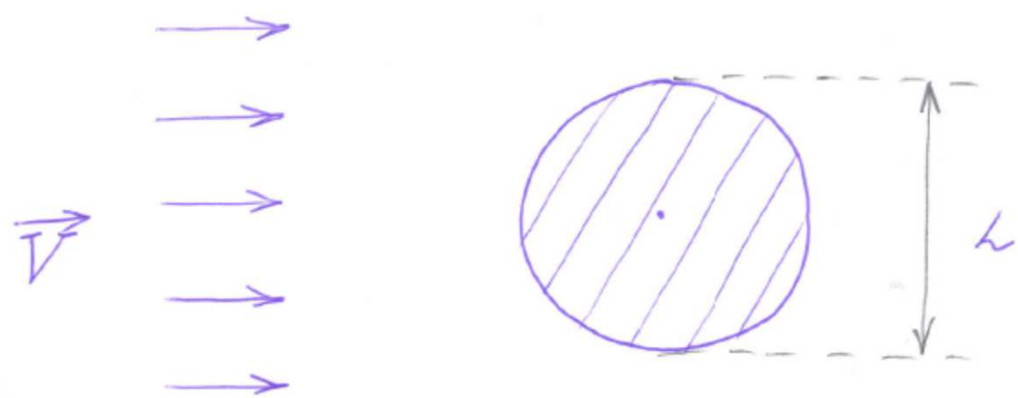
It is believed that "all of turbulence" is described by the equation known since at least 1823, generally referred to as the Navier - Stokes equation :

$$\begin{cases} \dfrac{\partial \vec{v}}{\partial t} + (\vec{v}, \nabla \vec{v}) = -\nabla p + \nu \cdot \Delta \vec{v}, \\ (\nabla, \vec{v}) = 0. \end{cases}$$

It must be supplemented by the initial and boundary conditions, such as $\vec{v} = 0$ at the rigid walls, etc. In addition to the knowledge of equations of motion, an enormous quantity of real experimental data exists and can be additionally produced.

One of the "simplest" examples of such experiments one can think of a flow around a cylinder.

Here is the basic set - up of such a problem : a uniform flow of incompressible liquid with velocity $\vec{V}$, incident on a cylinder of diameter L.



Fluid Turbulence and Symmetries

It can be shown by dimensional analysis that such problem has one, as opposed to several, control parameter, called Reynolds number :

$$\mathrm{Re} = \frac{L \cdot V}{\nu},$$

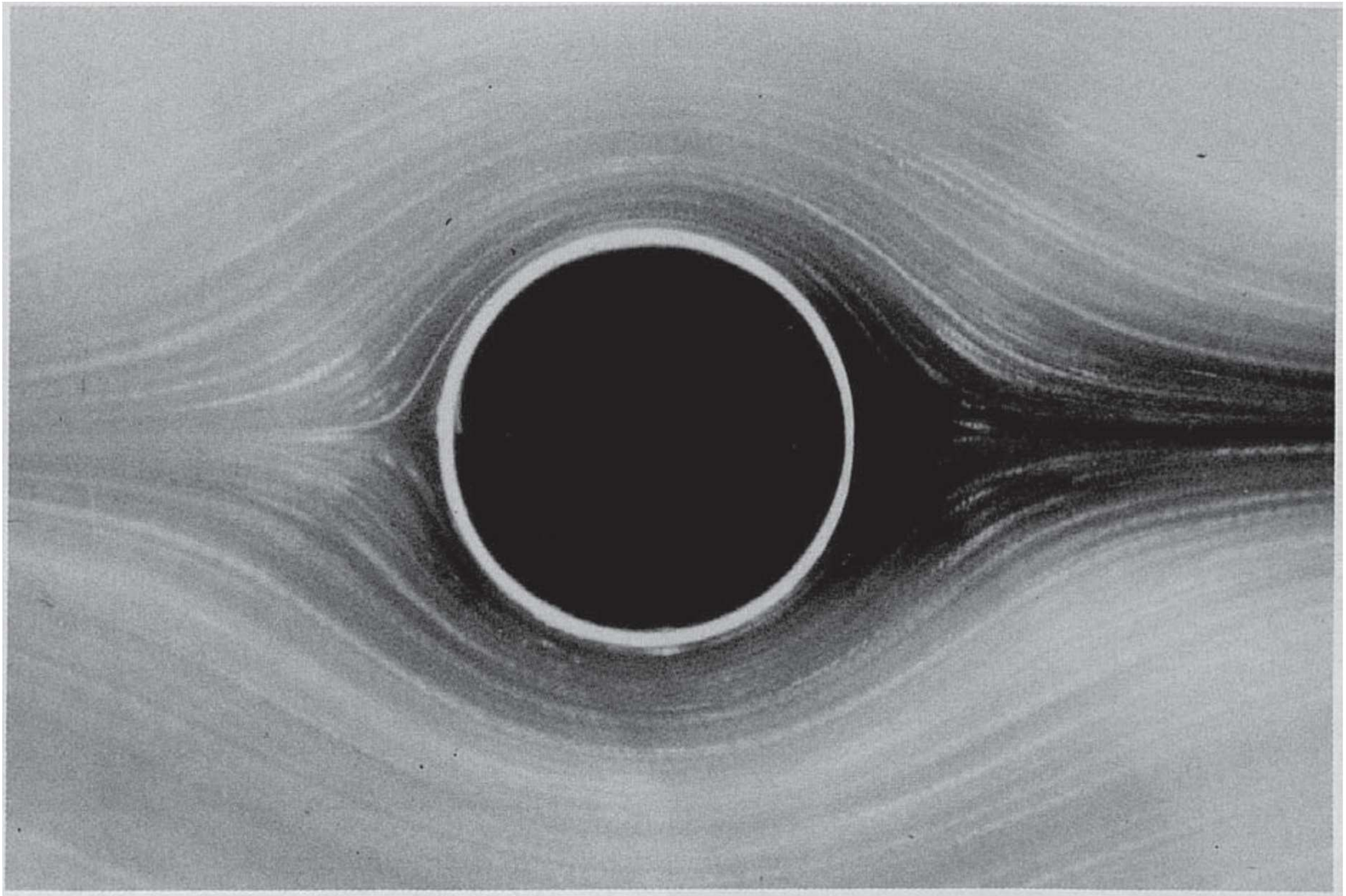
where L and V are characteristic scale and velocity of the flow, and ν is the kinematic viscosity of the fluid. The similarity principle for incompressible flow holds : for a given geometrical shape of the boundaries, the Reynolds number is the only control parameter of the flow problem. Chosing a cylinder as a geometrical shape leads to various symmetries.

With this in mind, let us observe what happens when one increases the Reynolds number of the flow around a cylinder.

For $\mathrm{Re} = 0.16$ we can observe the following symmetries of the flow :

1. Left - right, or x - reversal;
2. Up - down, or y - reversal;
3. Time - translation, or t - invariance;
4. Space - translation parallel to the axys of the cylinder, or z - invariance.

Fluid Turbulence and Symmetries



Flow around a smooth circular cylinder, Re=0.16, from Van Dyke (1982).

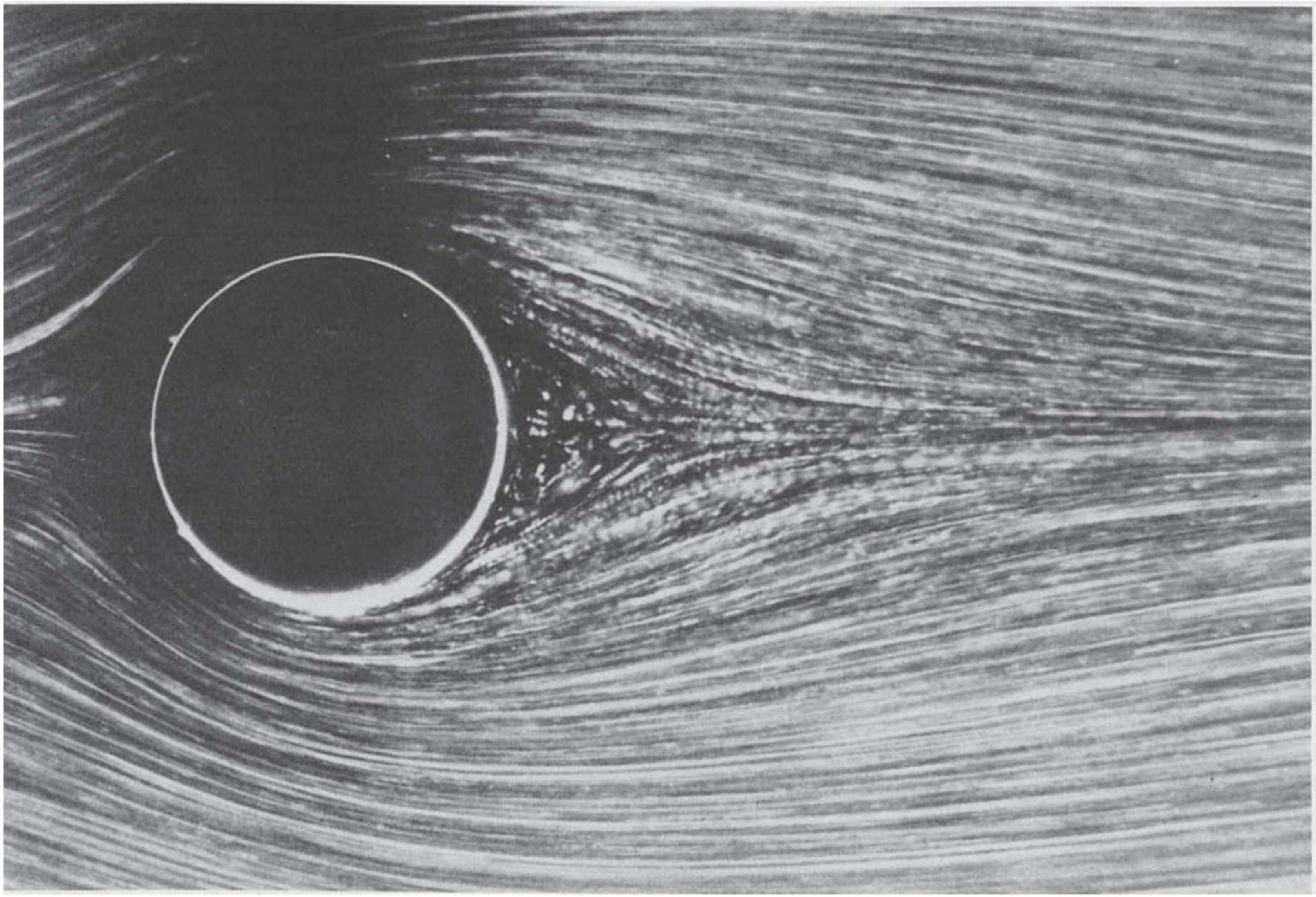
Fluid Turbulence and Symmetries

It is easy to see that the left - right symmetry is actually not consistent with the Navier - Stokes equation, specifically with the nonlinear term. Actually, a closer examination of the experimental picture reveals that even there one can see that this symmetry is imprecise.

Let us now analyze the experimental picture for $\mathrm{Re} = 9.6$ and beyond.

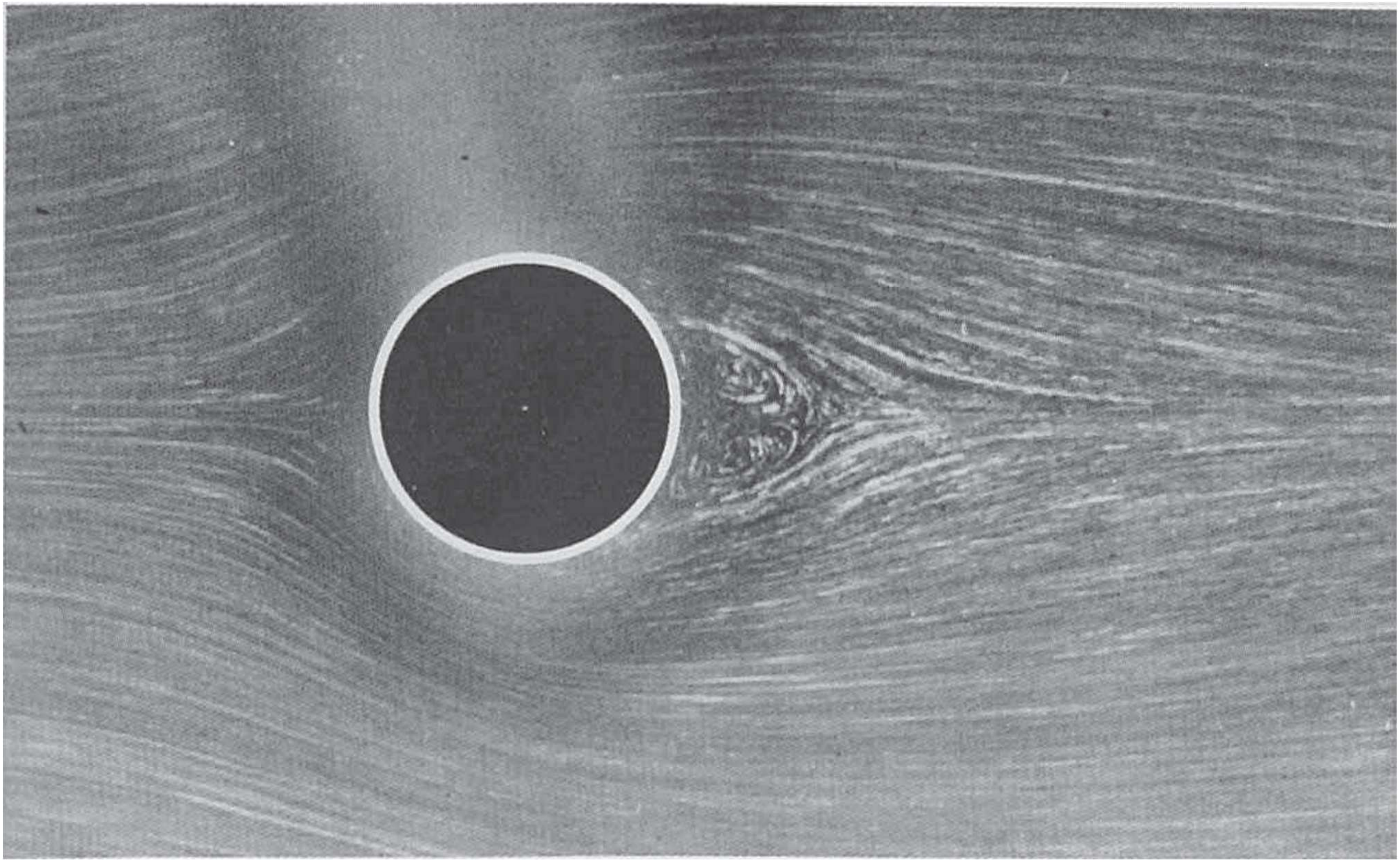
We can clearly see the left - right asymmetry. Actually, at around $\mathrm{Re} = 5.0$, the flow behind the cylinder is starting to separate. The topology of the flow is changing due to the creation of the recirculating standing eddies. Such topology is preserved approximately up to $\mathrm{Re} = 26$.

Fluid Turbulence and Symmetries



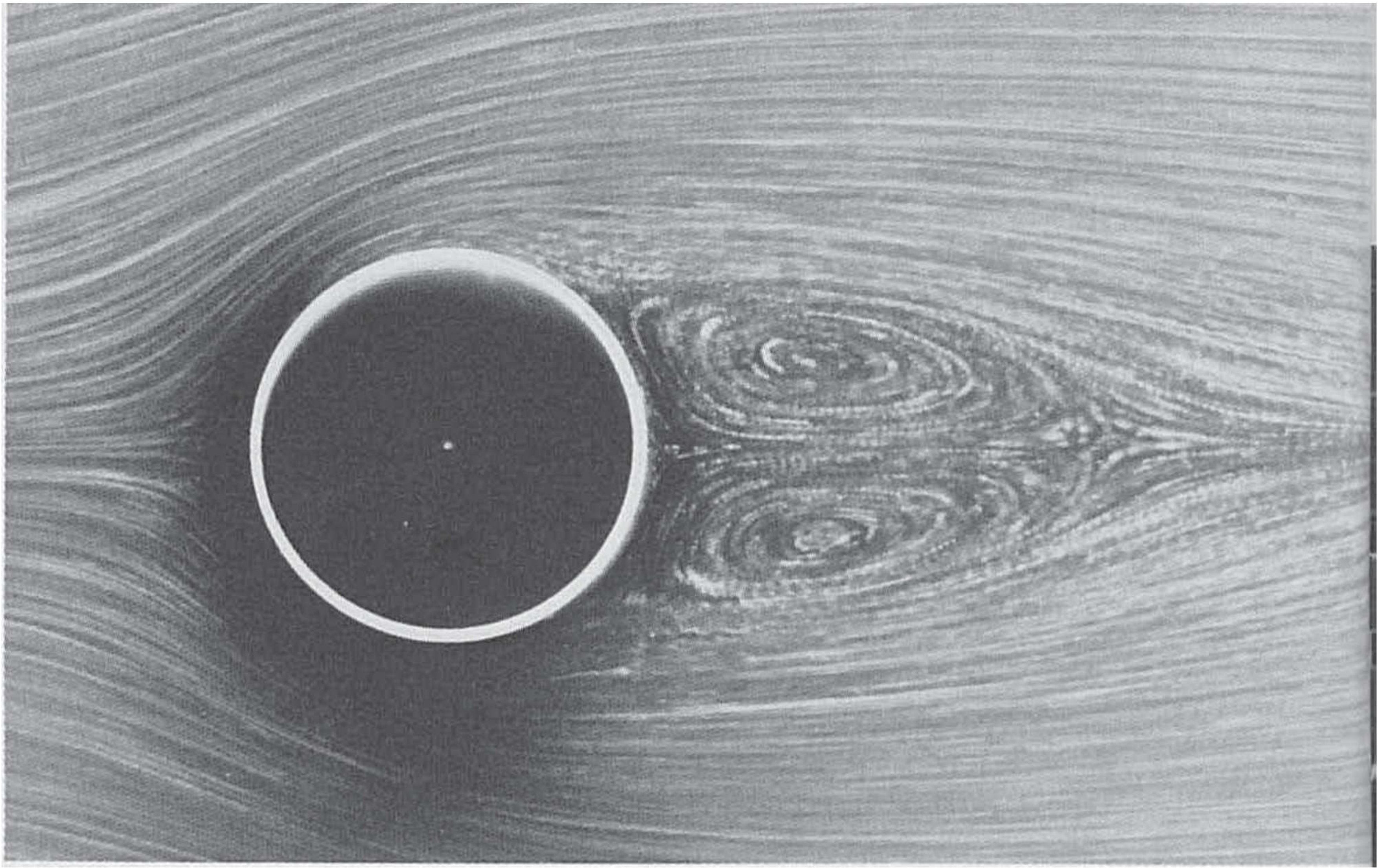
Flow around a smooth circular cylinder, Re=9.6, from Van Dyke (1982).

Fluid Turbulence and Symmetries



Flow around a smooth circular cylinder, Re=13.1, from Van Dyke (1982).

Fluid Turbulence and Symmetries



Flow around a smooth circular cylinder, Re=26, from Van Dyke (1982).

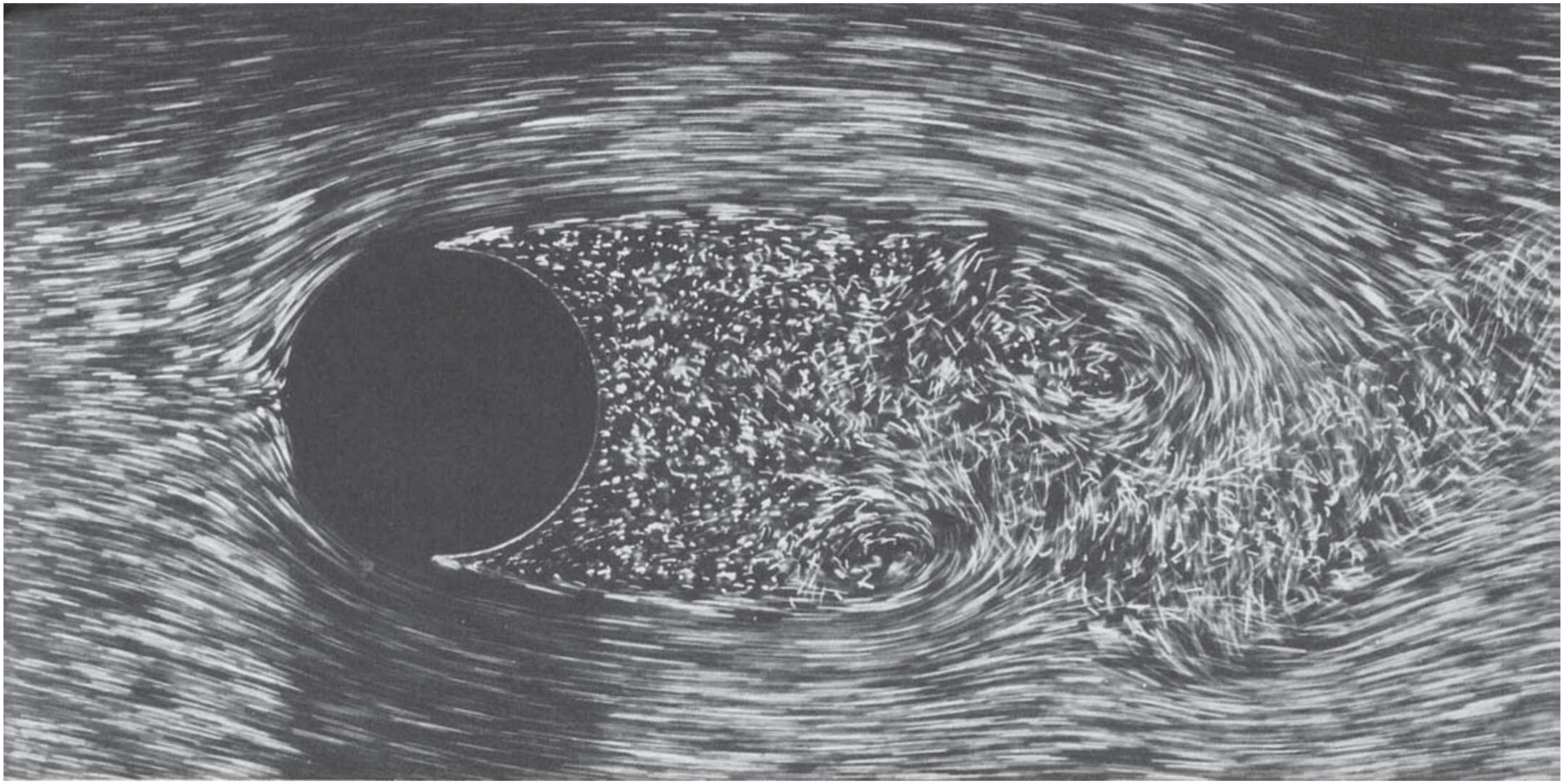
Fluid Turbulence and Symmetries

If we keep increasing the Reynolds number, around Re = 40, the first true loss of symmetry occurs: Andronov-Hopf bifurcation. It makes the flow time-periodic. The continuous t-invariance is broken and a discrete t-invariance is established. At even higher Reynolds numbers the shedding of the recirculation eddies becomes very pronounced and leads to the formation of the "Karman street" of alternating vortices.

There is numerical evidence that the z-invariance is spontaneousely broken somewhere between Reylonds numbers of 40 and 75. A symmetry is said to be spentaneousely broken if it is consistent with the equations of motion and the boundary conditions, but is not present in the solution.

With further increase in Reynolds number the flow becomes chaotic and time-dependent.

Fluid Turbulence and Symmetries



Flow around a smooth circular cylinder, Re=2,000, from Van Dyke (1982).

Fluid Turbulence and Symmetries

Thus we observe that with the increase of the control parameter, Reynolds number, the flow is continuously losing all of its original symmetries. The far downsteam flow behind an object at very high Reynolds numbers will look like a randomly fluctuating " turbulent" wake.

Suprisingly enough, at high Reynolds numbers, re - stating the problem from continuous functions formulation such as above into a statistical functions formulation, will lead to restoration of all or some of the lost symmetries.

For example, if statistical properties of turbulence do not change under space transaltions and rotations, such turbulent flow is called homogeneous isotropic turbulence. Invariance and symmetries need to be undertstood now in statistical sense. Turbulence at very high Reynolds numbers, when all or some of the possible symmetries are restored in a statistical sense, is known as fully developed turbulence.

Experimental Results for Fully Developed Turbulence

Empirically two laws were measured for fully developed turbulence:

1. 2/3s - law (or, 5/3 - law).

In a turbulent flow at very high Reynolds number, the mean square velocity increment $\left\langle (\partial v(l))^2 \right\rangle$ between two points separated by a distance l behaves approximately as the 2/3 power of the distance:

$$S_2(l) = \left\langle (\partial v(l))^2 \right\rangle \propto l^{2/3}.$$

Using the properties of Fourier transform, one can show that when the energy spectrum is a lower - law:

$E(k) \propto k^{-n}$, for $1 < n < 3$,

then the velocity field is homogeneous and the second order structure function is also a power - law:

$$S_2(l) = \left\langle \left| \vec{\mathrm{v}}(\vec{\mathrm{r}} + \vec{\mathrm{l}}) - \vec{\mathrm{v}}(\vec{\mathrm{r}}) \right|^2 \right\rangle \propto \left| \vec{l} \right|^{n-1}.$$

From this follows that, if the $S_2(l) \propto l^{2/3}$, then it follows that $E(k) \propto k^{-5/3}$.

2. Finite energy dissipation.

If all the control parameters are kept the same, except for the viscosity, which is lowered as much as possible, the energy dissipation per unit mass dE/dt tends to a finite positive limit.

Experimental Results for Fully Developed Turbulence

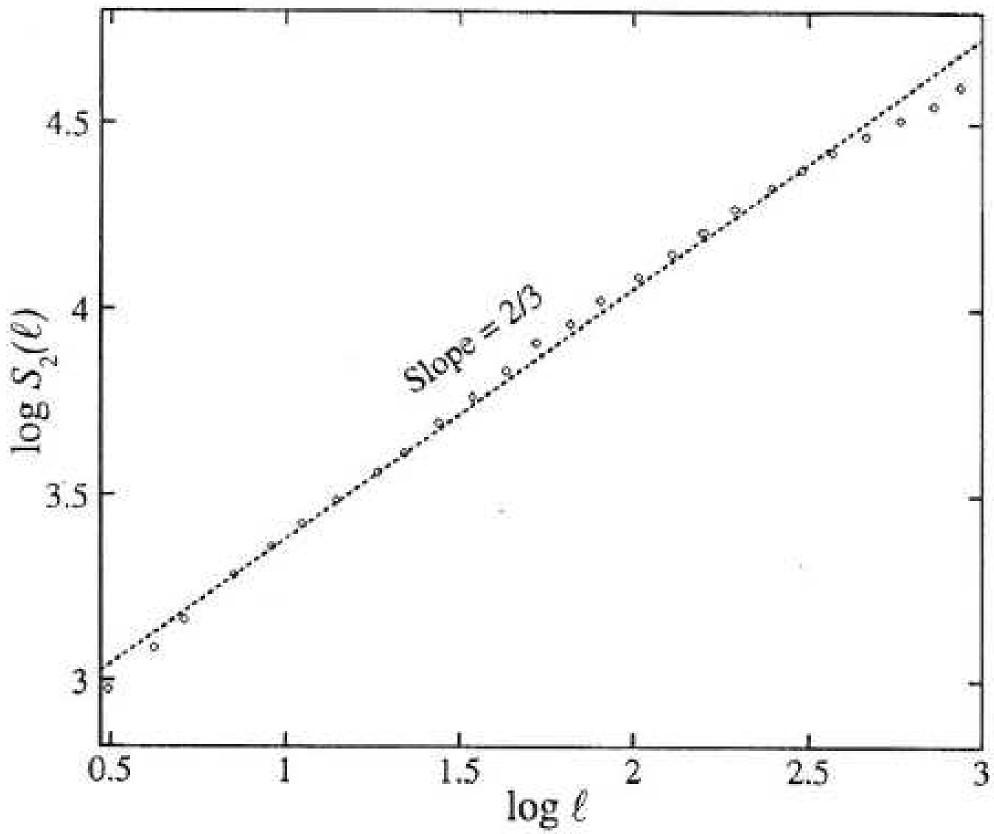


Fig. 5.1. log–log plot of the second order structure function in the time domain for data from the S1 wind tunnel of ONERA. Courtesy Y. Gagne and E. Hopfinger.

Experimental Results for Fully Developed Turbulence

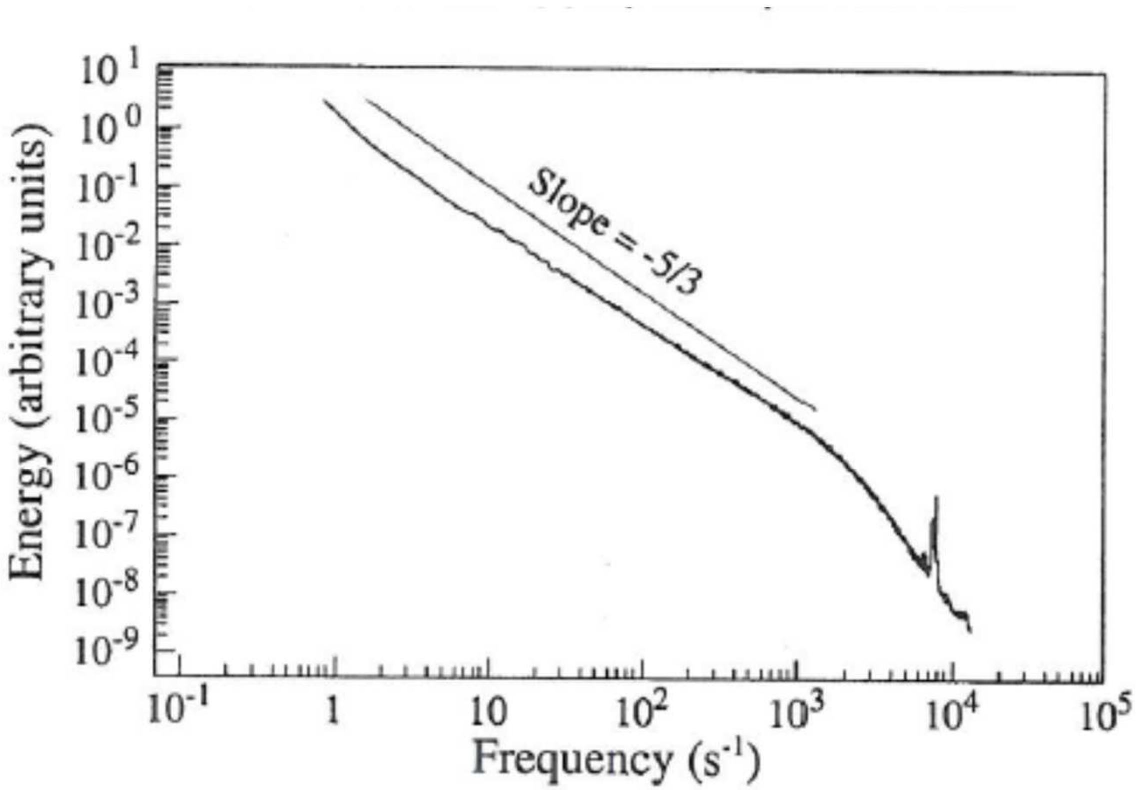


Fig. 5.4. Energy spectrum in the time domain for data from S1. Reynolds number $R_\lambda = 2720$. Courtesy Y. Gagne and M. Marchand.

Kolmogorov 1941 Theory

There is currently no fully deductive theory starting with the Navier - Stokes equations of motion and leading to these two experimentally observed laws.

In 1941, however, Kolmogorov using simple assumptions about statistical flow symmetries and dimensional arguments, was able to "derive" these experimental laws.

H1. In the limit of infinite Reynolds number, all the possible symmetries of the Navier - Stokes equation, broken in the turbulent flow, are restored in a statistical sense at small scales and away from the boundaries.

For example, small - scale homogeneity is understood as follows. If we define velocity increments as :

$\partial\vec{v}(\vec{r},\vec{l}\,) = \vec{v}(\vec{r}+\vec{l}\,) - \vec{v}(\vec{r}),$

then the increments $\partial\vec{v}(\vec{r}+\vec{\rho},\vec{l}\,)$ are statistically equivalent to $\partial\vec{v}(\vec{r},\vec{l}\,)$, i.e. have the same statistics that describes them.

Similarly, isotropy is understood as velocity increments are statistcally invariant under simultaneous rotations of both $\vec{l}$ and $\partial\vec{v}$.

Kolmogorov 1941 Theory

H2. Under the same assumptions as in H1, the turbulent flow is self - similar at small scales, that is it posesses a unique scaling exponent h such that :

$$\partial\vec{v}(\vec{r},\lambda\vec{l}\,) = \lambda^{h}\cdot\partial\vec{v}(\vec{r},\vec{l}\,),$$

for any real - valued λ.

As shown by Kolmogorov, if one assumes that energy dissipation is constant, the scaling exponent h is equal to $1/3$. Indeed, consider the second order structure function $\left\langle\left(\partial\vec{\mathrm{v}}(l)\right)^{2}\right\rangle$. A straightforward dimensional analysis shows that it has dimensions $[L]^{2}/[T]^{2}$, where $[L]$ and $[T]$ are dimensions of length and time. Since the mean energy dissipation per unit mass, ε, has the dimensions $[L]^{2}/[T]^{3}$, it then follows that :

$$\left\langle\left(\partial\vec{\mathrm{v}}(l)\right)^{2}\right\rangle = C\cdot\varepsilon^{2/3}\cdot l^{2/3},$$

where C is a universal dimensionless constant.

On the other hand, according to H2, $\left\langle\left(\partial\vec{\mathrm{v}}(l)\right)^{2}\right\rangle$ should scale as l^{2h}, from which follows that $h = 1/3$.

Fluid Turbulence and Financial Markets

	Turbulence	Financial Markets
Elementary fluctuating quantity that needs to be described or studied statistically	Velocity difference at the same moment of time between two locations in space, $\delta\vec{v} = \vec{v}(\vec{x}+\delta\vec{x},t) - \vec{v}(\vec{x},t)$.	Price differences between two moments of time, $\delta p = p(t+\delta t) - p(t)$.
Independent variable(s)	Location is space $\vec{x}$ and moment of time t.	Time t.
Energy transfer or distribution across various scales	In 3-dimensional (and 1-dimensional) case the energy is injected at the Large or Box Scale and through non-linear interactions gets distributed across all scales all the way through the smallest or Dissipation or Viscous Scale. 2-dimensional case is having an inverse energy transfer.	"Information" is injected into the system at a Large time Scale and through individual traders-agents interactions with one another via trading, gets transmitted across all time scales, all the way to the smallest Limit Order Book Event time Scale.
Relationship to some exact "equations of motion"	The exact equations of motion are known, however, that does not help in solving the problem, which is too complex to be solved precisely. Despite that knowledge, strictly speaking, this physics problem remains to be analytically unsolved.	The exact equations of motion are not known and the exact analytical description is not yet exactly known (may never be).

Fluid Turbulence and Financial Markets

	Turbulence	Financial Markets
Non-Gaussian behavior of fat tails of two-point probability distribution functions	Strongly pronounced for strong turbulence regimes. Algebraic decay of the probability density function tails.	Strongly pronounced for highly liquid, mature markets. Algebraic decay of the probability density function tails.
Auto-correlation and energy spectrum	$\delta\vec{v}$ is anti-correlated with the power law energy spectrum: $E(k) \propto k^{-5/3}$.	For δp - nearly absent correlations, and a very close to a Random Walk power law energy spectrum: $E(\omega) \propto \omega^{-2}$.
Meaning of the second order moment of the fluctuating quantity	(Kinetic) energy of the turbulent liquid.	Volatility.
Large scale structures and intermittency	Deviations from Gaussian manifest themselves through coherent structures, shock-waves for 1-dimensional case and coherent vortices for 2- and 3-dimensional cases.	Deviations from Gaussian manifest themselves in higher than normal frequency of gaps or shocks (up- or down-).
Stationary process?	Non-stationary at small time scales but asymptotically stationary at long time scales.	Non-stationary at small time scales but asymptotically stationary at long time scales.
Convergence to a Gaussian	As the time separation δt becomes increasingly larger.	As the time separation δt becomes increasingly larger.
Power law behavior	Yes.	Yes.