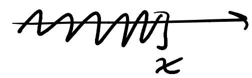
Class 15: Continuous RVs

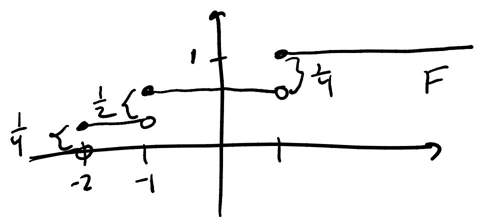
Tuesday, October 29, 2019 11:42 AM

- HW 6 due Nov 7, Thursday, 1pm
- Midterm 2 Nov 12

Def: The cumulative distribution function (CDF) of a r.v. X is the function $F(x) = \mathbb{P}(X \le x)$.



- Discrete case:



Recover PMF of X:

$\mathbb{P}(X=k) =$ size of jump at k

$\mathbb{P}(X=-2) = \frac{1}{4}$

$\mathbb{P}(X=-1) = \frac{1}{2}$

$\mathbb{P}(X=1) = \frac{1}{4}$

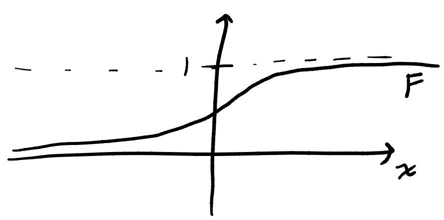
$\mathbb{P}(X=0) = 0$

$\mathbb{P}(X=k) = 0$ unless $k = -2, -1, 1$.

CDF:

$$F(x) = \begin{cases} 0 & \text{if } x < -2 \\ 1/4 & \text{if } -2 \le x < -1 \\ \frac{1}{4} + \frac{1}{2} = \frac{3}{4} & \text{if } -1 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

- Continuous case: F is continuous & differentiable

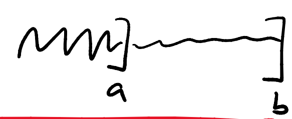


By def, a continuous r.v. is one for which $f = F'$ exists.

$$f(x) = \frac{d}{dx}F(x).$$

$\hookrightarrow$ $\mathbb{P}(a \le X \le b) = \int_a^b f(x)\,dx = F(b) - F(a)$ for $a < b$

Note $\mathbb{P}(X=a) = 0$.



Note

$$\boxed{f(x) \neq \mathbb{P}(X=x)}$$

General properties of CDFs: (discrete or continuous)

1. F is non-decreasing
2. $0 \le F \le 1$
3. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

③ $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} \ldots$

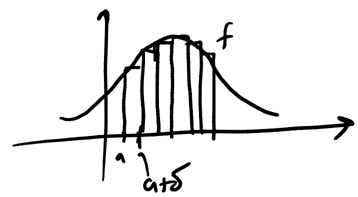
Continuous RVs:

The PDF (probability density function) $f(x) = F'(x)$ has many uses:

① $\mathbb{P}(a \le X \le b) = \int_a^b f(x)\,dx$. (Note $f \ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.)

② The mean or expectation is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx.$$



why? $\mathbb{P}(a \le X \le a+\delta) = \int_a^{a+\delta} f(x)\,dx \simeq \delta f(a)$ and $\delta a f(a) \approx \int_a^{a+\delta} x f(x)\,dx$

③ $Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

↳ same as discrete case, but calculation will be different

④ $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

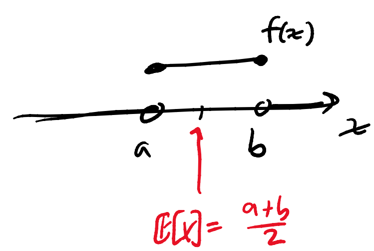
↳ Parallels the discrete case, but with integrals instead of sums

---

Examples

① Uniform distribution: $X \sim Unif[a, b]$

PDF: $f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$



CDF: $F(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f(y)\,dy$

If $a < x < b$ then

$$F(x) = \int_a^x f(y)\,dy$$

since $f(y) = 0$ for $y < a$

$$= \int_a^x \frac{1}{b-a}\,dy$$

$$= \frac{x-a}{b-a}.$$

In general: $F(x) = \begin{cases} 0 & \text{if } x \le a \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } x \ge b \end{cases}$

· Note $F(a)=0$ implies $F(x)=0$ for all $x \le a$
and $F(b)=1$ implies $F(x)=1$ for all $x \ge b$.

Mean:

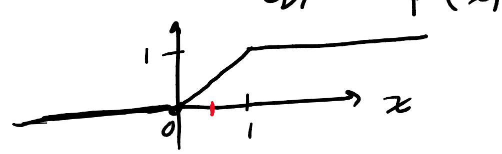
$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx = \frac{1}{b-a} \left. \frac{1}{2}x^2 \right|_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}.$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a} \left. \frac{1}{3}x^3 \right|_a^b = \frac{b^3-a^3}{3(b-a)}.$$

$$\hookrightarrow \quad Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{b^3-a^3}{3(b-a)} - \left(\frac{a+b}{2}\right)^2 = \text{simplify...}$$

Ex: $X \sim Unif[0,1]$ ($a=0$ and $b=1$)



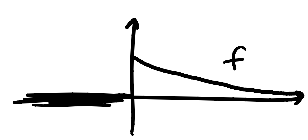
$\hookrightarrow$ $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{else.} \end{cases}$

$\hookrightarrow$ $\mathbb{E}[X] = \frac{1}{2}$, $Var(X) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$

---

② Exponential: $X \sim Exp(\lambda)$ $\lambda > 0$ "rate"

PDF: $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{else.} \end{cases}$



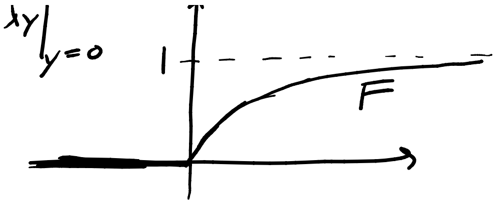
CDF: $F(x) = 0$ if $x \le 0$

If $x > 0$, $F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_0^x f(y)\,dy$

$$= \int_0^x \lambda e^{-\lambda y}\,dy = \left. -e^{-\lambda y} \right|_{y=0}^{x}$$

$$= \int_0^{x} \lambda e^{-\lambda y}\,dy = -e^{-\lambda y}\Big|_{y=0}$$

$$= 1 - e^{-\lambda x}$$



Mean:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\,dx$$

$$= -xe^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx$$

$$= 0 + -\frac{1}{\lambda}e^{-\lambda x}\Big|_0^{\infty}$$

$$= \frac{1}{\lambda}.$$

Integrate by parts:

$u = x \to du = dx$

$dv = \lambda e^{-\lambda x}\,dx \to v = -e^{-\lambda x}$

$\int u\,dv = uv\Big|_0^{\infty} - \int v\,du$

Check:

$\lim_{x\to\infty} x e^{-\lambda x} = 0$

Think of λ as "rate" of arrivals.

Memoryless: (saw for Geometric)

For all $t, s > 0$, $\underbrace{\mathbb{P}(X > t+s \mid X > s)}_{\text{prob. of waiting an additional } t \text{ units of time given that we already waited } s} = \underbrace{\mathbb{P}(X > t)}_{\text{prob of waiting } > t \text{ units of time}}.$

---

(3) Normal / Gaussian distribution:

$X \sim N(\mu, \sigma^2)$ $\qquad$ $\mu \in \mathbb{R}$ mean parameter

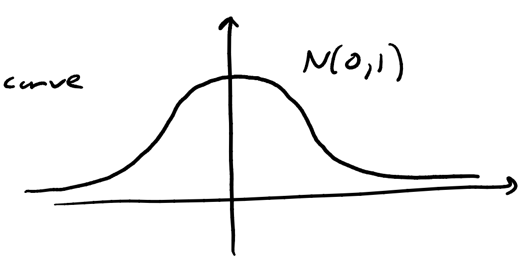
$\qquad\qquad\qquad\qquad$ $\sigma^2 > 0$ variance parameter

In a few weeks: central limit theorem

PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \text{for } x \in \mathbb{R}.$$

bell curve



Standard Normal:

$N(0,1)$, $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$

check: $f \geq 0$ ✓

$\int^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = 1$?

Check: $f \geq 0$ ✓

How to compute $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$?

$$\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right)^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} - \frac{y^2}{2}} dx\, dy$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} r\, dr\, d\theta$$

Polar coordinates:
$r^2 = x^2 + y^2$
θ = angle of (x,y)
$dx\,dy \to r\,dr\,d\theta$

$\frac{d}{dr}\left(-e^{-\frac{r^2}{2}}\right) = re^{-\frac{r^2}{2}}$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[-e^{-\frac{r^2}{2}} \right]_0^{\infty} d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} 1\, d\theta = \frac{1}{2\pi} \cdot 2\pi = 1.$$

Mean: $X \sim N(0,1)$

Use $\frac{d}{dx}\left(-e^{-\frac{x^2}{2}}\right) = xe^{-\frac{x^2}{2}}$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[-e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty}$$

$$= 0.$$

Variance: $X \sim N(0,1)$. Since $\mathbb{E}[X] = 0$,

$$Var(X) = \mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

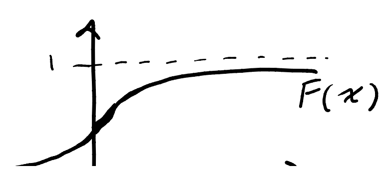
Integrate by parts:
$u = x \to du = dx$
$dv = xe^{-\frac{x^2}{2}} dx \to v = -e^{-\frac{x^2}{2}}$

$$= \frac{1}{\sqrt{2\pi}} \left[-xe^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$$

$$= 0 + \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}$$

Just saw this is 1

$$= 1.$$

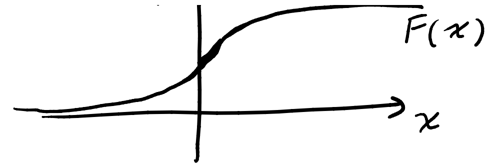


CDF: $X \sim N(0,1)$

$F(x) = P(X \leq x) = \int^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$

CDF: $X \sim N(0,1)$

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy.$$



Special function: "error function"

General Case: $X \sim N(0,1)$.

Let $Y = \mu + \sigma X$ where $\mu \in \mathbb{R}$ and $\sigma > 0$.

Then $Y \sim N(\mu, \sigma^2)$.

Proof: show Y has PDF $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$.

Can't treat PDF like probabilities!

Use the CDF.

$$\mathbb{P}(Y \leq y) = \mathbb{P}(\mu + \sigma X \leq y) = \mathbb{P}\left(X \leq \frac{y-\mu}{\sigma}\right)$$

$$= F_X\left(\frac{y-\mu}{\sigma}\right)$$ where F_X is CDF of $X \sim N(0,1)$.

$\Rightarrow$ PDF of Y is

$$f_Y(y) = \frac{d}{dy} \mathbb{P}(Y \leq y) = \frac{d}{dy}\left[F_X\left(\frac{y-\mu}{\sigma}\right)\right]$$

$$= F_X'\left(\frac{y-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \cdot \frac{1}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}.$$

$\leftarrow$ PDF of X is derivative of CDF, $F_X'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$.

Ex: $Y = \mu + \sigma X \rightarrow \mathbb{E}[Y] = \mu + \sigma \mathbb{E}[X] = \mu$

$Y \sim N(\mu, \sigma^2)$ $\quad Var(Y) = Var(\mu + \sigma X) = \sigma^2 Var(X) = \sigma^2$