Lecture 12:

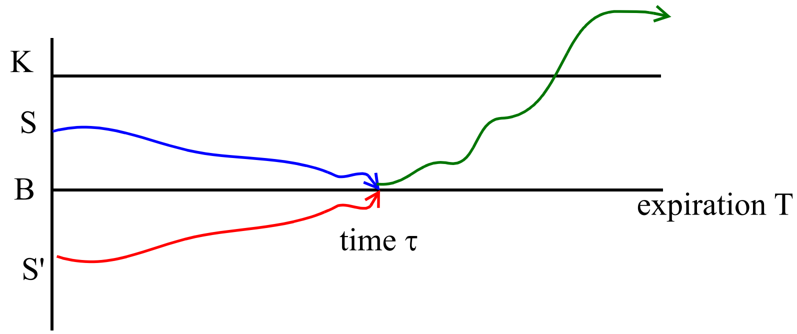
1. Weak Static Replication;
2. Extending the Binomial Mode

12.1 Valuing Barrier Options with K != B, assuming GBM

This is important because **the valuation under GBM suggests a method of replication**.

Valuing a Barrier Option for GBM with Zero Risk-Neutral Stock Drift

A **down-and-out** option with strike K and barrier B.



- **The Method of Images: cf. Electric Potentials or Mirror Images**
- Choose a "reflected" imaginary stock S' that evolves like S: The blue trajectory from S and the red trajectory from S' have equal probability to get to any point on B at time τ,.
- From there, they have equal probability of taking the future green trajectory that finishes in the money.
- For any green trajectory finishing in the money, the paths beginning at S and S' have the same probability of producing the green trajectory.
- Subtract the two probability densities to get a new density, and then above the barrier B, the contribution from every path emanating from S that touched the barrier at any time τ will be cancelled by a similar path emanating from S'. This is appropriate distribution for evaluating the payoff of a down-and-out knockout option.

Where is S'?

In GBM, the log of the stock price undergoes arithmetic Brownian motion. We saw that the returns $\ln S_T/S_t$ are normally distributed with a risk-neutral mean $r\tau - \frac{1}{2}\sigma^2\tau$ and a standard deviation $\sigma\sqrt{\tau}$, where $\tau = T - t$. The variable $x = \dfrac{\ln S_T/S_t - (r\tau - \frac{1}{2}\sigma^2\tau)}{\sigma\sqrt{\tau}}$ is normally distributed with mean 0, variance 1 with a probability density $N'(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$

The probability to get from S to B in a GBM world depends only on $\ln S/B$; The probability to get from S' to B depends on $\ln S'/B$. So we need the log distances from the higher to the lower to be the same:

$$\ln\frac{S}{B} = \ln\frac{B}{S'} \text{ or } S' = \frac{B^2}{S}$$

Now, if B = 100 and S = 120, then S′ = 83.33.

Thus assuming GBM, the correct density for getting from S to S_τ a time τ later, for $r = 0$, is something like

$$N'_{DO}(S_\tau) = N'\left(\frac{\ln\left(\frac{S_\tau}{S}\right) + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right) - \alpha N'\left(\frac{\ln\left(\frac{S_\tau S}{B^2}\right) + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right)$$

for some coefficient α, where $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$.

We want this density to vanish when $S_\tau = B$, which requires $\alpha = \left(\frac{S}{B}\right)$ independent of τ.

Thus the down-and-out option price under GBM is

$$C_{DO}(S, K, \sigma, \tau) = C_{BS}(S, K, \sigma, \tau) - \frac{S}{B}C_{BS}\left(\frac{B^2}{S}, K, \sigma, \tau\right)$$

Eq.12.1

Is this a reasonable answer?

Yes. Because solutions to backward PDEs depend on the PDE and the boundary conditions.

1. One can show that C_{DO} satisfies the Black-Scholes PDE.(This is a HW problem).

2. C_{DO} has the correct boundary conditions for a down-and-out option.

It vanishes on boundary S = B at any time τ.

It is worth an ordinary call at expiration. If S > K > B at expiration, $B^2/S < K^2/S < K$ and second option finishes out of the money.

Thus C_{DO} has the correct boundary condition for an ordinary call if it hasn't knocked out along the way.

Valuation for non-zero risk-neutral drift $\mu = r - 0.5\sigma^2$

When the drift is non-zero then probabilities for reaching B from both S and S' differ, since the drift distorts the symmetry. Try a superposition of densities and S and the same reflection $S' = B^2/S$.

Trial down-and-out density for reaching a stock price S_τ a time τ later is

$$N'_{DO} = N'\left(\frac{\ln\left(\frac{S_\tau}{S}\right) - \mu\tau}{\sigma\sqrt{\tau}}\right) - \alpha N'\left(\frac{\ln\left(\frac{S_\tau S}{B^2}\right) - \mu\tau}{\sigma\sqrt{\tau}}\right)$$

We want density to vanish on the barrier at any time:

$$N'\left(\frac{\ln\left(\frac{B}{S}\right) - \mu\tau}{\sigma\sqrt{\tau}}\right) - \alpha N'\left(\frac{\ln\left(\frac{S}{B}\right) - \mu\tau}{\sigma\sqrt{\tau}}\right) = 0$$

leads to $\alpha = \left(\frac{B}{S}\right)^{\frac{2\mu}{\sigma^2}} = \left(\frac{B}{S}\right)^{\frac{2r}{\sigma^2}-1}$ independent of τ, as we would like, where $\mu = r - 0.5\sigma^2$.

Integrating over the terminal stock price, we obtain

$$C_{DO}(S, K) = C_{BS}(S, K) - \left(\frac{B}{S}\right)^{\frac{2\mu}{\sigma^2}} C_{BS}\left(\frac{B^2}{S}, K\right)$$

12.2 Insight into Static Replication from the GBM Valuation Formula for Barrier Options in the Previous Section

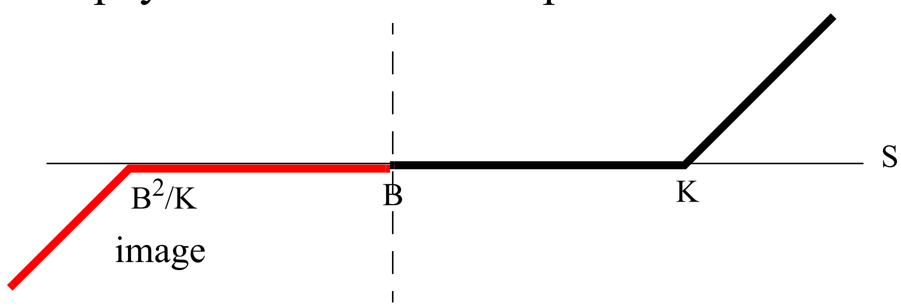
We showed above that, in a Black-Scholes world with zero rates, the fair value for a down-and-out call with strike K and barrier B is given by

$$C_{DO}(S,K) = C_{BS}(S,K) - \frac{S}{B}C_{BS}\left(\frac{B^2}{S}, K\right) \qquad \text{Eq.12.2}$$

Payoff of first term at expiration is: $\theta(S-K)(S-K)$

Payoff of $\frac{S}{B}C_{BS}\left(\frac{B^2}{S}, K\right)$ $\quad \frac{S}{B}\left(\frac{B^2}{S}-K\right)\theta\left(\frac{B^2}{S}-K\right) = \left(B-\frac{KS}{B}\right)\theta\left(\frac{B^2}{K}-S\right) = \frac{K}{B}\theta\left(\frac{B^2}{K}-S\right)\left(\frac{B^2}{K}-S\right)$

This second term represents the payoff of *K/B* standard puts with strike B^2/K.



Roughly speaking in GBM the payoff of a down-and-out-call is that of

1. long an ordinary call with strike K, and
2. short a put with its strike the image of K image as reflected (in log space rather than linear space) in the barrier.

$$\frac{K}{B} = \frac{B}{\text{image}} \qquad \text{image} = \frac{B^2}{K}$$

Think of replicating a down-and-out call by going long a call with the same strike and positive expected value, and short the right amount of puts (negative expected value) with reflected strike. The weighting must be such that the expected value on the barrier must be zero. In GBM the weighting is $\frac{K}{B}$.

This is weak replication because

you need a model to find the value of a standard option on the barrier;

you need to close out the position if the barrier is touched

Think: what strike-volatilities are you sensitive to in this portfolio? Although this insight was derived from the formula for valuation in a BSM world, this is a sensible way to think about replicating a down-and-out barrier option in general.

If you can go long a call with strike above the barrier and short the right amount of puts with strike below the barrier, you will have the correct payoff both at expiration and on the barrier:

- At expiration if the stock has never touched the barrier, the call with strike K will have the correct payoff of a down-and-out call, and the put will expire out-of-the-money.
- If the stock S does touch the barrier at B before expiration, then the net value of the long call and short put positions will be close to zero if you can short the correct amount of puts. At that point, you *must* close out the position to replicate the extinguishing of the down-and-out call option.
- The number of puts required is K /B only if the stock price undergoes geometric Brownian motion with constant volatility. More generally, the number will depend on how you model the smile, but the general picture still has validity even if we depart from a BSM world.

Extending the Binomial Model: Towards Local Volatility

- The binomial model as framework for modeling stock price evolution.
- The binomial model for option evaluation.
- Equivalence to the Black-Scholes-Merton model.
- Extending the binomial model to accommodate more general stock price evolution.

We will consider stock evolution in the binomial model and then progress to modifying it for local volatility.

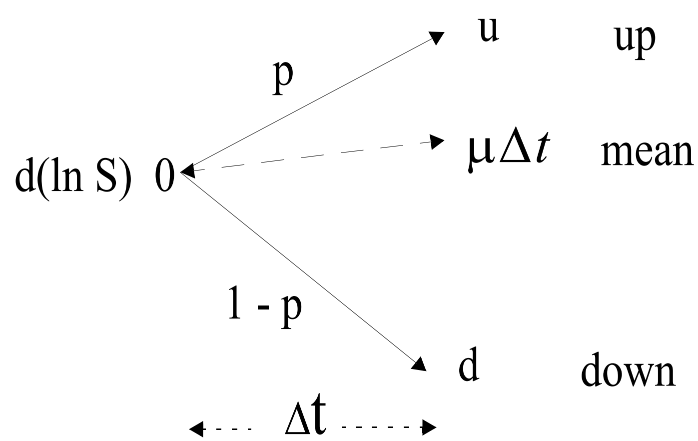
12.3 The Binomial Model for Stock Evolution

Search for models of stock price evolution that can account for the smile. It's easiest to begin in the binomial framework where intuition is clearer.

In Black-Scholes framework $d(\ln S) = \mu dt + \sigma dZ$ with no dividends paid.

Expected return on the stock price is $\mu + \sigma^2/2$. The total variance after time t is $\sigma^2 t$.

We model the actual evolution of the stock price over an instantaneous time Δt by means of a one-period binomial tree.



How do we choose p, u and d to match the continuous-time $d(\ln S) = \mu dt + \sigma dZ$?

Match the mean and variance of the return:

$$pu + (1-p)d = \mu\Delta t$$

$$p[u - \mu\Delta t]^2 + (1-p)[d - \mu\Delta t]^2 = \sigma^2\Delta t$$

$$pu + (1-p)d = \mu\Delta t$$

$$p(1-p)(u-d)^2 = \sigma^2\Delta t$$

Two constraints on the three variables *p, u,* and *d.* Pick convenient ones.

First Solution: The Cox-Ross-Rubinstein Convention

Choose $u + d = 0$: stock price always returns to the same level; center of the tree fixed.

$$\begin{aligned}(2p-1)u &= \mu\Delta t \\ 4p(1-p)u^2 &= \sigma^2\Delta t\end{aligned} \qquad \text{Eq.12.3}$$

Squaring the first equation and subtracting the second leads to $u^2 = (\mu\Delta t)^2 + \sigma^2\Delta t$.

In the limit $\Delta t \to 0$, $u = \sigma\sqrt{\Delta t}$ and $d = -\sigma\sqrt{\Delta t}$

Then from Equation 12.3

$$p \approx \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta t} \qquad \begin{aligned}u &= \sigma\sqrt{\Delta t} \\ d &= -\sigma\sqrt{\Delta t}\end{aligned} \qquad \text{Eq.12.4}$$

Check: mean return of the process is $\left(\frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\Delta t}\right)(\sigma\sqrt{\Delta t}) - \left(\frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{\Delta t}\right)(\sigma\sqrt{\Delta t}) = \mu\Delta t$ perfect!

The variance is $p(1-p)(u-d)^2 \approx \frac{1}{4}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\left(1 - \frac{\mu}{\sigma}\sqrt{\Delta t}\right)4\sigma^2\Delta t \approx \sigma^2\Delta t - \mu^2(\Delta t)^2$ a little too small.

The convergence to the continuum limit is a little slower than if it matched the variance exactly.
For small enough Δt there is no riskless arbitrage with this convention.

Another Solution: The Jarrow-Rudd (JR) Convention

We must satisfy the constraints

$$\begin{aligned} pu + (1-p)d &= \mu\Delta t \\ p(1-p)(u-d)^2 &= \sigma^2\Delta t \end{aligned}$$

Choose $p = 1/2$, so that the up and down moves have equal probability:

$$\begin{aligned} u + d &= 2\mu\Delta t \\ u - d &= 2\sigma\sqrt{\Delta t} \end{aligned}$$

$$\begin{aligned} u &= \mu\Delta t + \sigma\sqrt{\Delta t} \\ d &= \mu\Delta t - \sigma\sqrt{\Delta t} \end{aligned}$$

The mean return is exactly μ; the volatility of returns is exactly σ, convergence is faster than CRR.

$$E[S] = \frac{(e^u + e^d)}{2}S = e^{\mu\Delta t}\frac{\left(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}\right)}{2} \approx e^{\mu\Delta t}\left(1 + \frac{\sigma^2\Delta t}{2}\right) \approx e^{\left(\mu + \frac{\sigma^2}{2}\right)\Delta t} \qquad \text{Ito}$$

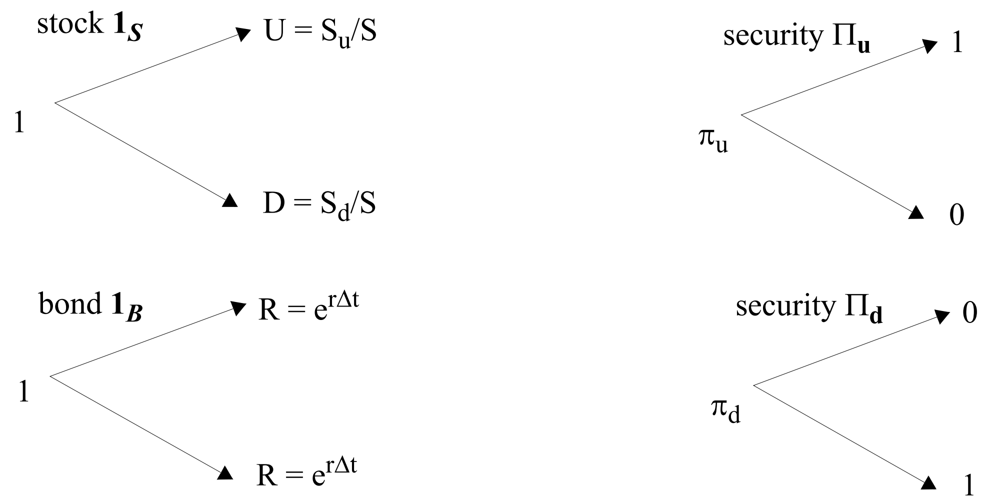
In the limit $\Delta t \to 0$, both the CRR and the JR convention describe the same process, and there are many other choices of u, d, and q that do so too.

We will use these binomial processes as a basis for modeling more general local volatility processes that can match the smile.

12.4 The Binomial Model for Options

Options Valuation in the q-measure

One can decompose the stock S and the bond B into two securities Π_u and Π_d that pay out only in the up or down state.We are denominating securities in dollars rather than anything else!



Define $\Pi_u = \alpha \times \vec{1}_S + \beta \times \vec{1}_B$. Note that because it is riskless, the sum $\Pi_u + \Pi_d = 1/R$

Then $\begin{aligned} \alpha U + \beta R &= 1 \\ \alpha D + \beta R &= 0 \end{aligned}$ so that $\begin{aligned} \alpha &= \frac{1}{(U-D)} \\ \beta &= \frac{-D}{R(U-D)} \end{aligned}$

$$\vec{\Pi}_u = \frac{R \times \vec{1}_S - D \times \vec{1}_B}{R(U-D)} \qquad \vec{\Pi}_d = \frac{U \times \vec{1}_B - R \times \vec{1}_S}{R(U-D)}$$

The current values are $\pi_u = \frac{R-D}{R(U-D)} \equiv \frac{q}{R}$ $\qquad \pi_d = \frac{U-R}{R(U-D)} \equiv \frac{1-q}{R}$

$$q = \frac{R-D}{U-D} \qquad 1-q = \frac{U-R}{U-D}$$

are the no-arbitrage pseudo-probabilities that don't depend on p.

The first equation can be rewritten as $qU + (1-q)D = R$, or

$$S = \frac{qS_u + (1-q)S_d}{R}$$

so that in this q-measure the expected future stock price is the forward price.

Any option C which pays C_u (C_d) in the up (down)-state is replicated by $C = C_u\Pi_u + C_d\Pi_d$ with

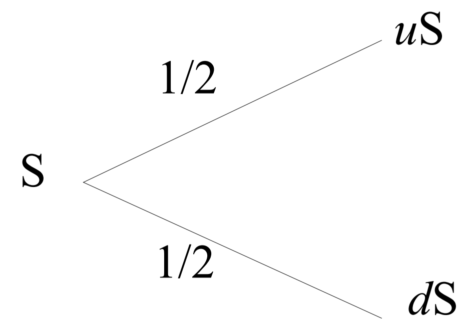
$$C = \frac{qC_u + (1-q)C_d}{R}$$

Regard the stock equation as *defining* the measure q, given the values of S, S_u and S_d; the second equation specifies the value C in terms of the option payoffs and the value of p. This is why probability theory seems to be important in options pricing, because of complete markets.

The Black-Scholes Partial Differential Equation from the Binomial Model

The BS PDE can be obtained by taking the limit of the binomial pricing equation in the *q*-measure as $\Delta t \to 0$.

We'll use the JR choice of *u* & *d* and set $q = 1/2$ and set the log(S) growth to be $\mu = r - 0.5\sigma^2$ so that the stock price grows at the riskless rate *r,* as required for the q-measure.



$$u = e^{(r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} \qquad d = e^{(r - 0.5\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}$$

Since q = 1/2, option value is given by this **backward equation:**

$$e^{r\Delta t}C(S, t) = \frac{1}{2}C(S_u, t + \Delta t) + \frac{1}{2}C(S_d, t + \Delta t)$$

Performing a Taylor series expansion to leading order in Δt, one can show that

$$Cr\Delta t = \frac{\partial C}{\partial S}\{rS\Delta t\} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\left\{S^2\sigma^2\Delta t\right\} + \frac{\partial C}{\partial t}\Delta t \qquad \text{Future Homework Problem}$$

The PDE is equivalent to a backward equation.

: It did not assume that σ is constant through time, it just used its value at that instant.

12.5 Step 1. Extending Black-Scholes to Time-dependent Deterministic Volatility to Account for the Term Structure of Skew.

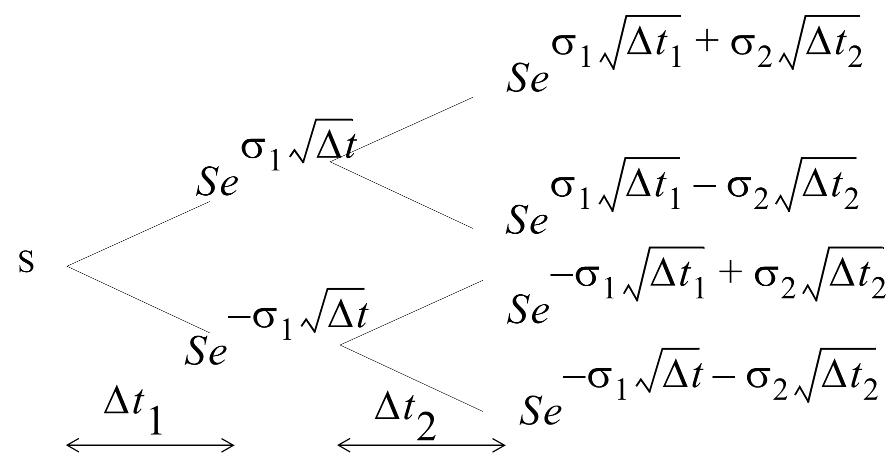
Black-Scholes and the binomial model assume that σ is constant.
Suppose now that the stock volatility σ is a function of (future) time t.

$$\frac{dS}{S} = \mu dt + \sigma(t)dZ$$

How do we modify Black-Scholes or the binomial tree method for term structure $\sigma(t)$?

Suppose we try to build a CRR tree with σ_1 in period 1 and σ_2 in period 2.

The tree doesn't "close" in the second period unless σ_i is constant. It doesn't have to close but it's computationally convenient if it does.



To make the tree close

$$\sigma_1\sqrt{\Delta t_1} = \sigma_2\sqrt{\Delta t_2} = \dots = \sigma_N\sqrt{\Delta t_N}$$

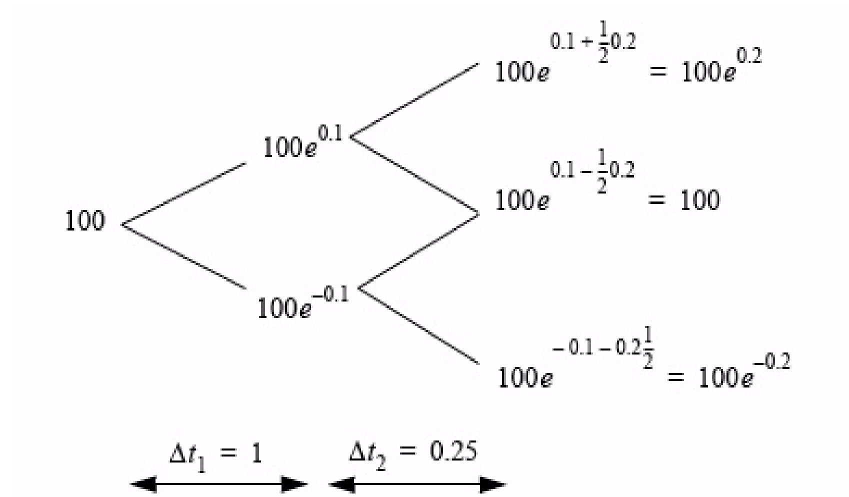
Thus, though the tree looks the same from a topological point of view, each step between levels involves a step in time that is smaller when volatility in the period is larger, and vice versa.

How many time steps needed? Given the term structure of volatilities, solve for the number of time steps needed.

Example: suppose we believe volatility will be 10% in year 1 and 20% in year 2. We choose the first period to be one year long and then solve for the second period.

	period 1	period 2
σ	0.1	0.2
σ^2	0.01	0.04
Δt	1	**1/4**

CRR convention up & down moves given by $\sigma_i\sqrt{\Delta t_i}$:



In essence, we build a standard binomial tree with price moves generated by $e^{\pm\sigma\sqrt{\Delta t}}$, where $\sigma\sqrt{\Delta t}$ is constant, and then choose σ to match the term structure of volatility, and then adjust our notion of Δt. The tree and node prices are topologically identical to a constant volatility tree. However, we reinterpret the times at which the levels occur, and the volatilities that took them there. One tree with same prices at each node can represent different stochastic processes with different volatilities moving through different amounts of time.

The tree in the illustration above extended to 1.25 years. We would need a total of 4 periods to span the entire second year at a volatility of 0.2, but only one period for the first year, so that 5 steps are necessary to span two years.

More generally, if you have a definite time T to expiration, then $T = \sum_{i=1}^{N} \Delta t_i = \Delta t_1 \sum_{i=1}^{N} \frac{\sigma_1^2}{\sigma_i^2}$

Solve for N.

Note 1: Even though the nodes in the tree above have *prices* corresponding to a CRR tree with $\sigma_i \sqrt{\Delta t_i} = 0.1$, irrespective of volatility term structure, *the binomial no-arbitrage probabilities vary with* Δt_i, because for each fork in the tree, and so q-measure option prices change:

$$q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

Even though $e^{\sigma\sqrt{\Delta t}}$ is the same over all time steps Δt, the factor $e^{r\Delta t}$ varies from step to step with the value of Δt, so that q varies from level to level.

Note 2: The total variance at the terminal level of the tree is the same as before:

$$\Sigma^2(t, T) \times (T-t) \equiv \sum_{i=1}^{N} \sigma_i^2 \Delta t_i \rightarrow \int_t^T \sigma^2(s) ds \approx N\sigma_1^2 \Delta t_1$$

Valuing an option on this tree leads to the Black-Scholes formula with the relevant time to expiration, the relevant interest rates and dividends at each period, and a total variance

$$\Sigma^2(t, T) = \frac{1}{T-t}\int_t^T \sigma^2(s) ds$$

Note: If we know the term structure of volatility, we can extract the forward (local) volatilities:

$$\sigma^2(T) = \frac{\partial}{\partial T}[\tau\Sigma^2(\tau)] \text{ where } \tau = T-t$$

Forward vol (*local in time*) is related to the time derivative of the implied vol.

We will see the generalization to local vol $\sigma(S, t)$ in time and stock price later.

Constant volatility of 20%

sigma		0.2	0.2	0.2	0.2	0.2
delta t		0.25	0.25	0.25	0.25	0.25
Time	0	0.25	0.5	0.75	1	1.25
sig*sqrt(delta	0.1	0.1	0.1	0.1	0.1	0.1
u		1.105	1.105	1.105	1.105	1.105
r_annual	0.05					

risk neutral stock tree CRR-style with constant sigma(t)

					164.87
				149.18	
			134.99		134.99
		122.14		122.14	
	110.52		110.52		110.52
100.00		100.00		100.00	
	90.48		90.48		90.48
		81.87		81.87	
			74.08		74.08
				67.03	
					60.65

same tree in terms of prices, but different total variance and time

Time	0	0.25	0.5	0.75	1	1.25
					0.536	
				0.536		
			0.536		0.536	
		0.536		0.536		
p-tree	0.536		0.536		0.536	
		0.536		0.536		
			0.536		0.536	
				0.536		
					0.536	
				0.200		
			0.200			
		0.200		0.200		
vol computed	0.200		0.200			
		0.200		0.200		
			0.200			
				0.200		

from p, u, d

First 3-months volatility is 10%

sigma		0.1	0.2	0.2	0.2	0.2
delta t		1	0.25	0.25	0.25	0.25
Time	0	1	1.25	1.5	1.75	2
sig*sqrt(delta	0.1	0.1	0.1	0.1	0.1	0.1
u		1.105	1.105	1.105	1.105	1.105
r_annual	0.05					

risk neutral stock tree CRR-style with variable sigma(t)

					164.87
				149.18	
			134.99		134.99
		122.14		122.14	
	110.52		110.52		110.52
100.00		100.00		100.00	
	90.48		90.48		90.48
		81.87		81.87	
			74.08		74.08
				67.03	
					60.65

Time	0	1	1.25	1.5	1.75	2
					0.536	
				0.536		
			0.536		0.536	
		0.536		0.536		
p-tree	0.725		0.536		0.536	
		0.536		0.536		
			0.536		0.536	
				0.536		
					0.536	
				0.200		
			0.200			
		0.200		0.200		
vol computed	0.089		0.200			
		0.200		0.200		
			0.200			
				0.200		

from p, u, d

a little inaccurate for large steps, should be 0.1

$$\sigma^2 = \frac{0.275(0.725)(1.1052-0.9048)^2}{1} = (0.089)^2 \equiv \frac{p(1-p)(u-d)^2}{\Delta t}$$

Calibrating a binomial tree to term structures

How do we build a binomial tree to price options that's consistent with yield and vol term structures? This is important for American-style options and early exercise. We have to make sure to use the right forward rate and the right forward volatility at each node, with $\sigma\sqrt{\Delta t}$ constant.

Example:

	Year 1	Year 2	Year 3
Term structure of zero coupons:	5%	7.47%	9.92%
Forward rates:	5%	$10\% = \frac{(1.0747^2)}{1.05} - 1$	15%
Term structure of Implied vols:	Σ_1 20%	Σ_2 25.5%	Σ_3 31.1%
Forward vols:	Σ_1 20%	$\Sigma_{12} = \sqrt{2\Sigma_2^2 - \Sigma_1^2}$ 30%	$\Sigma_{23} = \sqrt{3\Sigma_3^2 - 2\Sigma_2^2}$ 40%

Note that the forward volatility rises twice as fast with time as the implied volatility does with expiration.

Now build a (toy) tree with different forward rates/vols:

r:	5%	10%	15%
σ	20%	30%	40%
Δt	Δt_1	$\Delta t_2 = \left(\frac{\sigma_1}{\sigma_2}\right)^2 \Delta t_1$	$\Delta t_3 = \left(\frac{\sigma_1}{\sigma_3}\right)^2 \Delta t_1$
		$= 0.44\Delta t_1$	$0.25\Delta t_1$

A possible scheme:

For the first year use $\Delta t_1 = 0.1$ and take 10 periods of 0.1 years per step. $\sigma\sqrt{\Delta t} = 0.2(0.316)$

Then $\Delta t_2 = 0.044$ and we need about 23 periods for the second year.

Finally, $\Delta t_3 = 0.025$ and we need 40 periods for the third year.

In each period up and down moves in the tree are generated by

$$e^{\sigma\sqrt{\Delta t}} = e^{(0.2)0.316} = 1.065.$$

Using forward rates and forward volatilities over three years produces a very different tree from using just the three-year rates and volatilities over the whole period, but the total variance is all that matters for European options. That’s not the case for American-style exercise.

12.6 Time-Dependent Deterministic Volatility Another Way

Suppose volatility is time-dependent (slightly local) and deterministic, and, just to make it simple, interest rates are zero.

Then the Black-Scholes equation is $\frac{\partial V}{\partial t} + \frac{1}{2}\sigma(t)^2 S^2 \frac{\partial^2 V}{\partial S^2} = 0$. (cf local vol)

Let's combine $\sigma(t)^2 dt$. So change from t to a new dimensionless variable $\tau = \int_t^T \sigma(u)^2 du$:

total variance to expiration will be the new variable in our PDE.

Note that τ decreases as t increases. T is the expiration, t is the current time.

Now change variables from t to τ in the pde, and remember how to do partial derivatives.

Write $V(S, t) = W(S, \tau)$. Then find the pde for W. Note that $\frac{\partial \tau}{\partial t} = -\sigma(t)^2$

$$\frac{\partial V}{\partial t} = \frac{\partial W}{\partial S}\frac{\partial S}{\partial t} + \frac{\partial W}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\partial W}{\partial \tau}\sigma(t)^2$$

$$\frac{\partial V}{\partial S} = \frac{\partial W}{\partial S}$$

Now substitute these into the pde for V to get pde for W. The variance cancels.

$$-\frac{\partial W}{\partial \tau}\cancel{\sigma(t)^2} + \frac{1}{2}\cancel{\sigma(t)^2} S^2 \frac{\partial^2 W}{\partial S^2} = 0$$

$$-\frac{\partial W}{\partial \tau} + \frac{1}{2} S^2 \frac{\partial^2 W}{\partial S^2} = 0$$

This equation is the equivalent Black-Scholes equation for W. It's the BS equation for an option, and in τ the stock has volatility equal to 1. The minus sign is there because it's quoted in terms of the time to expiration.

We know the solution: it's the BS solution with time to expiration $\int_t^T \sigma(u)^2 du$ and volatility = 1.

The BS European solution depends on the total variance which is $vol^2\tau = 1^2 \times \int_t^T \sigma(u)^2 du$.

You can write this instead as $\left[\frac{1}{(T-t)}\int_t^T \sigma(u)^2 du\right] \times (T-t)$ and interpret the first term as the **time-average variance** and the second as the actual time to expiration. Then it's as though the volatility squared were $\bar{\sigma}^2 = \frac{1}{T-t}\int_t^T \sigma(u)^2 du$ in the BS solution with the usual $T-t$ time to expiration.

Local Volatility Models

- In a local volatility model, the instantaneous stock volatility $\sigma(S, t)$ is a function of stock price and future time.
- How to build and use a binomial tree with variable local volatility.
- The BSM implied volatility of a standard option in a local volatility model is approximately the average of the local volatilities between the initial stock price and the strike.

Where we're going with local volatility:

- Building a local volatility tree
- The implied volatility surface that results
- Calibrating a local volatility tree to the implied volatility surface
- The Dupire equation for local vols
- The relation between local and implied vol
- Hedge ratios of vanilla options
- Exotics

  etc

Local Volatility Models

We just extended the constant-volatility geometric Brownian motion picture underlying the Black-Scholes model to account for a volatility $\sigma(t)$ that can vary with future time and found a relation between forward and implied volatility..

$$\Sigma^2(t,T) = \frac{1}{T-t}\int_t^T \sigma^2(s)ds$$

So, given a term structure of implied volatility, we know how to determine the forward (local) volatilities.

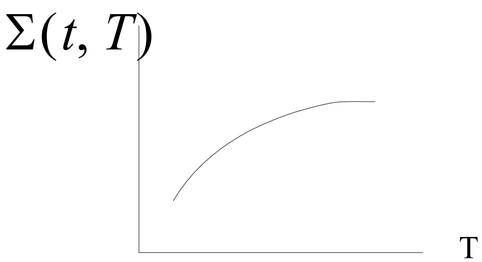
How to make $\sigma = \sigma(S, t)$ a function of future stock price S and future time t to account for the strike structure of the smile as well?

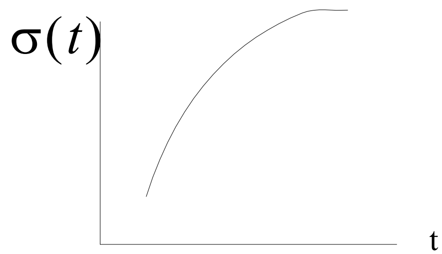
Modeling a stock with a variable volatility $\sigma(S, t)$: The Idea

Model a stock with a variable volatility $\sigma(S, t)$, value options, examine $\Sigma(S, t, K, T)$.

Pure term structure $\Sigma(t, T)$, calibrate the forward volatilities $\sigma(t)$, $\Sigma^2(t, T) = \frac{1}{T-t}\int_t^T \sigma^2(s)ds$.

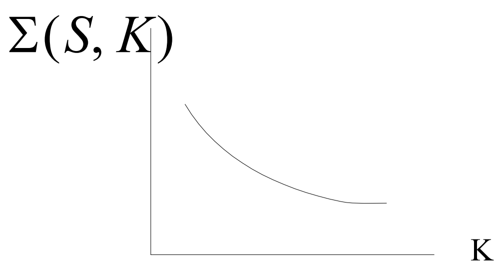
forward vol rises twice as fast with time as implied vol with expiration

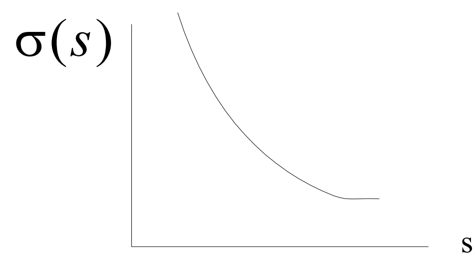




"Sideways" volatilities $\Sigma(S, t, K, T)$ to $\sigma(S, t)$?

sideways vol rises twice as fas by stock price as implied vol by strike





Questions

1. Can we find a unique local volatility function or surface $\sigma(S, t)$ to match the observed implied volatility surface $\Sigma(S, t, K, T)$? If we can, that means that we can explain the observed smile by means of a local volatility process for the stock.
2. But is the explanation meaningful? Does the stock actually evolve according to an observable local volatility function? There are, as we will see, many different models that can match the implied volatility surface, but achieving a match doesn't mean that model is "correct."
3. What does the local volatility model tell us about the hedge ratios of vanilla options and the values of exotic options? How do the results differ from those of the classic BSM model?

Some references on Local Volatility Models (there are many more).

- *The Volatility Smile*, Derman and Miller (2016)

- *The Volatility Smile and Its Implied Tree*, Derman and Kani, RISK, 7-2 Feb.1994, pp. 139-145, pp. 32-39 (see www.emanuelderman.com for a PDF copy of this.

- *The Local Volatility Surface by Derman, Kani and Zou, Financial Analysts Journal, (July-Aug 1996), pp. 25-36* (see www.emanuelderman.com for a PDF copy of this). Read this to get a general idea of where we're going.

Local volatility models are widely used because they are so easy to understand and yet still add features beyond Black-Scholes. And one can layer stochastic volatility on top of them.