LECTURE 16

THE LOCAL VOLATILITY MODEL CONTINUED …

DUPIRE EQUATION FOR LOCAL VOLATILITY
RULES OF THUMB FOR IMPLIED VOLATILITIES
VALUES OF HEDGE RATIOS
VALUES OF EXOTIC OPTIONS

Local vol and its consequences;
homework on local vol
(Corridor variance swap) -- more sophisticated variance swap replication
Stochastic vol …

16.1 Recall: Building the Tree

There are many ways to choose the central spine of a binomial tree:

- For every level with an odd number of nodes (1,3,5, etc.) choose the central node to be S. (CRR)
- For every period with even nodes (2,4,6 etc.) choose the two central nodes in those periods to lie above and below the initial stock price S exactly as in the CRR tree, given by

$$S_u = S e^{\sigma(S,t)\sqrt{dt}}$$

$$S_d = S e^{-\sigma(S,t)\sqrt{dt}}$$

Once you have the central nodes, you can generate the up and down nodes relative to the central node at each level of the tree by

$$S_u = F + \frac{S^2\sigma^2(S,t)dt}{F - S_d}$$

$$S_d = F - \frac{S^2\sigma^2(S,t)dt}{S_u - F}$$

You could equally well choose a tree whose spine corresponds to the forward price F of the stock, growing from level to level. Or anything else.

16.2 Dupire Equation

$\sigma(S, t)$ in the model can be found from **market prices of options** and their derivatives. Assuming zero interest rates and dividends, the local volatility at the stock price K is given by:

$$\frac{\sigma^2(K,T)}{2} = \frac{\dfrac{\partial C(S,t,K,T)}{\partial T}}{K^2 \dfrac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

We'll see that this is the continuous version of the procedure we used to construct a local volatility binomial tree.

If interest rates are non-zero, but for zero dividend yield,

,

$$\frac{\sigma^2(K,T)}{2} = \frac{\dfrac{\partial C(S,t,K,T)}{\partial T} + rK\dfrac{\partial C(S,t,K,T)}{\partial K}}{K^2 \dfrac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

Recall $T_1\sigma_1^2 + (T_2 - T_1)\sigma_2^2 = T_2\sigma_2^2$. This is the mathematically correct generalization of the notion of forward stock volatilities $\sigma(T)$ to local volatilities.

16.3 Understanding the Equation

$$\frac{\sigma^2(K,T)}{2} = \frac{\dfrac{\partial C(S,t,K,T)}{\partial T}}{K^2\dfrac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

We can interpret the numerator of the equation in economic terms.

$$\frac{\partial C(S,t,K,T)}{\partial T} = \lim_{dT \to 0} \frac{C(S,t,K,T+dT) - C(S,t,K,T)}{dT}$$

It is proportional to $1/dT$ infinitesimal calendar spreads for standard calls with strike K.

At expiration time T, the calendar spread has significant value only for $S_T \sim K$.

Far below $S_T \sim K$, both calls are worthless. Far above $S_T \sim K$, both calls are forward contracts with equal value.

At $S_T \sim K$ the relevant variance that determines the non-zero value of the calendar spread is $\sigma^2(K, T)$. This volatility determines whether the longer option is worth more than the shorter one.

Similarly the denominator:

$$\frac{\partial^2 C}{\partial K^2} = \frac{C(S, t, K + dK, T) - 2C(S, t, K, T) + C(S, t, K - dK, T)}{(dK)^2}$$

is proportional to an infinitesimal butterfly spread for standard calls with strike K.

Therefore the local variance $\sigma^2(K, T)$ at stock price K and time T is proportional to the ratio of the price of a calendar spread to a butterfly spread.

A calendar spread and a butterfly spread are **combinations of tradeable options**, and so the local volatility can be extracted from traded options prices (if they are available)!

16.3.1 More Intuition

The price at time t of a calendar spread

$$C(S, t, K, T + dT) - C(S, t, K, T)$$

measures the risk-neutral probability $p(S,t,K,T)$ of the stock moving from S at time t to K at time T, times the variance $\sigma^2(K, T)$ at K and T that is responsible for the adding option value. Roughly,

$$\text{calendar spread} \sim p(S, t, K, T)\sigma^2(K, T).$$

But, according to the Breeden-Litzenberger, the probability

$$p(S, t, K, T) \sim \frac{\partial^2}{\partial K^2} C(S, t, K, T) \sim \text{butterfly spread}$$

So, roughly speaking, combing the two equations above, we have

$$\text{calendar spread} \approx \text{butterfly spread} \times \sigma^2(K, T)$$

or

$$\sigma^2(K, T) \approx \frac{\text{calendar spread}}{\text{butterfly spread}}$$

16.3.2 About The Deceptive Appearance of the Equation Which We Will Derive

$$\frac{\sigma^2(K,T)}{2} = \frac{\frac{\partial}{\partial T}C(S,t,K,T) + rK\frac{\partial}{\partial K}C(S,t,K,T)}{K^2\frac{\partial^2 C}{\partial K^2}}$$

can be rewritten as

$$\frac{\partial C(S,t,K,T)}{\partial T} + rK\frac{\partial C(S,t,K,T)}{\partial K} - \frac{\sigma^2(K,T)}{2}K^2\frac{\partial C^2(S,t,K,T)}{\partial K^2} = 0$$

This resembles Black-Scholes equation with t replaced by T and S replaced by K. But …

- **Black-Scholes equation holds for any contingent claim on S**, relating the value of *any option* at S, t to the value of that option locally at $S + dS$, $t + dt$, **keeping strike and expiration fixed**. It was derived from no-arbitrage, by hedging the claim and then setting the value of the hedged portfolio equal to the riskless rate.
- **Dupire equation holds only for standard calls (or puts)** and relates the value of a *standard* option with strike and expiration at K, T **to the same option with a different strike and expiration** $K + dK$, $T + dT$ when S, t is kept fixed.

16.3.3 Note: The Local Variance is Always Positive, As It Should Be

Dupire Denominator: $\frac{\partial^2 C}{\partial K^2}$ is the positive price of a positive payoff. (No arbitrage)

strike grows at the forward rate

Dupire Numerator for option with strike $y = Ke^{rT}$ and expiration *T*, price $C(y, T)$:

$$\frac{d}{dT}C(y, T) = \frac{\partial C}{\partial y}\frac{\partial y}{\partial T} + \frac{\partial C}{\partial T} = \frac{\partial C}{\partial T} + ry\frac{\partial C}{\partial y}$$

We now show that this is also positive, when the strike increases at the riskless rate:

Go to discrete case: $C(S, t, Ke^{r(T+dT)}, T+dT) - C(S, t, Ke^{rT}, T)$

Evaluate this calendar spread at $t = T$ and $S_T > Ke^{rT}$:

Then first leg is in the money, worth $S_T - Ke^{rT}$ -- simply spot minus strike.

Second leg is $C\left(S_T, T, Ke^{r(T+dT)}, T+dT\right)$ always worth more than a forward by no arbitrage:

at time T

$$C\left(S_T, T, Ke^{r(T+dT)}, T+dT\right) \geq S_T - (Ke^{r(T+dT)})e^{-rdT} = S_T - Ke^{rT}$$

So the second leg is worth more than first, so the numerator at any earlier time *t* is always positive.

Using the Equation

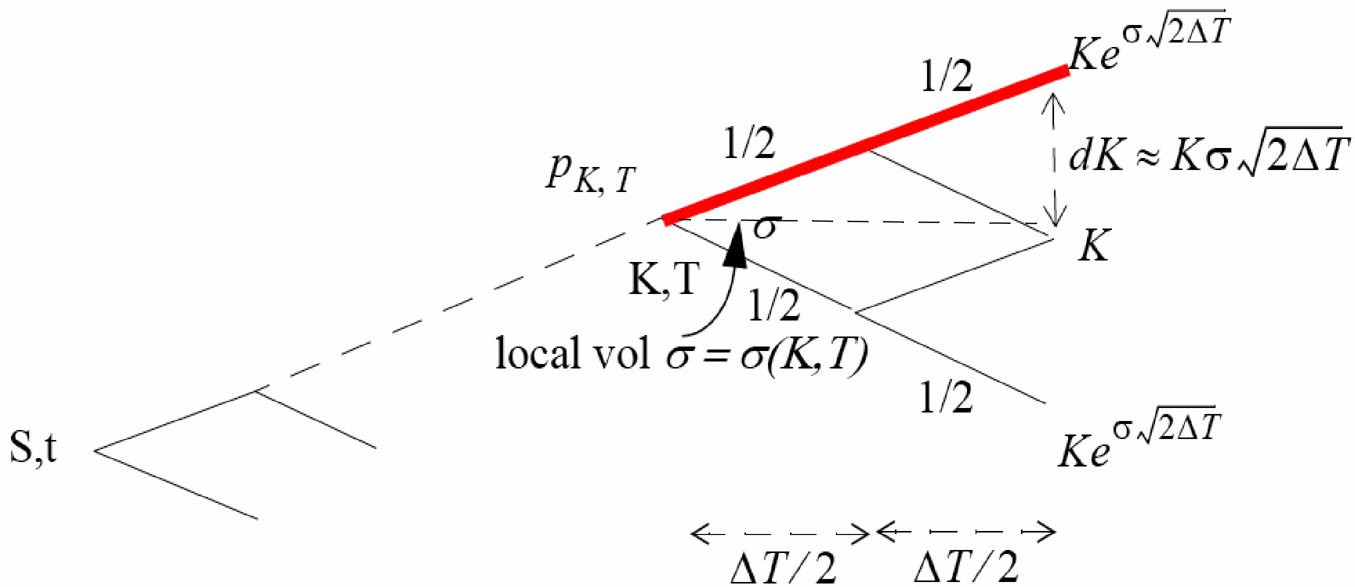
$$\frac{\sigma^2(K,T)}{2} = \frac{\frac{\partial}{\partial T}C(S,t,K,T) + rK\frac{\partial}{\partial K}C(S,t,K,T)}{K^2\frac{\partial^2 C}{\partial K^2}}$$

- Find $\sigma(K, T)$ from options prices and hence build local volatility tree from any options prices and their derivatives.
- Or you can do Monte Carlo over the stochastic process with the local volatility evolution.
- **This provides one consistent model** that values all standard options correctly rather than having to use several different inconsistent Black-Scholes models with different underlying volatilities.
- Volatility arbitrage trading. You can calculate the future local volatilities implied by options prices and then see if they seem reasonable. If some of them look too low or too high in the future, you can think about buying or selling future butterfly and calendar spreads to make a bet on future volatility.

16.4 A Poor Man's Derivation of the Dupire Equation in a Binomial Framework. (Zero Rates and Dividends)

$$\frac{\sigma^2(K,T)}{2} = \frac{\dfrac{\partial C(S,t,K,T)}{\partial T}}{K^2\dfrac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

Let's use a Jarrow-Rudd tree that goes from (S,t) to (K,T) through **two half-periods of time** $\Delta T/2$, keeping interest rates zero for pedagogical simplicity.



We now show that the calendar spread at time t obtains all its optionality from future nodes at $S_T = K$ that move up the heavy red line to a nonzero positive payoff, so that:

$$C(S, t, K, T + dT) - C(S, t, K, T) \equiv \frac{\partial C}{\partial T}\Delta T = \overset{\text{probability}}{p_{K,\,T}\frac{1}{4}} \times \overset{\text{payoff}}{dK}$$

Proof: Nodes S_T below (K,T) at expiration time T contribute zero value to $C(S, t, K, T)$ and produce transitions to nodes at time $T + dT$ that have zero payoff for $C(S, t, K, T + dT)$. Thus all nodes below (K,T) contribute zero to the calendar spread.

Nodes S_T above (K,T) at expiration time T produce a payoff $S_T - K$. Each of these nodes transition to three nodes S_{T+dT} above K at time $T + dT$ that have the same risk-neutral expected value $S_T - K$ at time T, (assuming interest rates are zero). So these nodes above (K,T) also contribute zero to the calendar spread.

So, only the node at (K,T) matters.

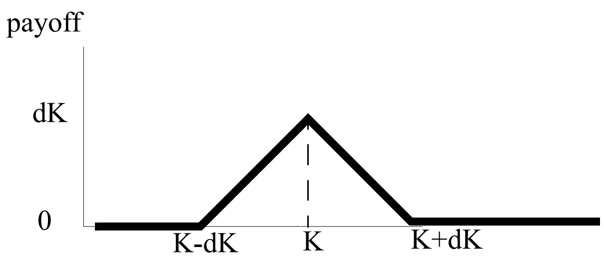
The value of the calendar spread per unit time is $$\frac{\partial C}{\partial T} = \frac{1}{4}p_{K,\,T} \times \frac{dK}{dT}$$

where $p_{K,\,T}$ is the risk-neutral probability of getting to (K, T) and $dK \approx \Delta K = K\sigma\sqrt{2\Delta T}$.

We can get $p_{K,\,T}$ from a **butterfly spread portfolio**

$p_{K,\,T}$ is the value of a portfolio that pays \$1 if the stock price is at node K, and zero for all other nodes at time T.

The butterfly spread $C(S, t, K + dK, T) - 2C(S, t, K, T) + C(S, t, K - dK, T)$ pays dK rather than \$1:



Dividing by dK produces a payoff which is \$1 at the node K and zero at adjacent nodes.

$$
\begin{aligned}
p_{K,T} &= \frac{C(S,t,K-dK,T) - 2C(S,t,K,T) + C(S,t,K+dK,T)}{dK} \\
&= \frac{C(S,t,K+dK,T) - C(S,t,K,T)}{dK} - \frac{C(S,t,K,T) - C(S,t,K-dK,T)}{dK} \\
&\approx \frac{\partial C(S,t,K,T)}{\partial K} - \frac{\partial C(S,t,K-dK,T)}{\partial K} \\
&\approx \frac{\partial^2 C(S,t,K,T)}{\partial K^2} dK
\end{aligned}
$$

Combining the expression for $p_{K,\,T}$ with $\frac{\partial C}{\partial T} = \frac{1}{4}p_{K,\,T} \times \frac{dK}{dT}$

$$\begin{aligned} \frac{\partial C}{\partial T} &= \frac{1}{4} p_{K,T} \frac{dK}{dT} \\ &= \frac{1}{4} \frac{\partial^2 C(S,t,K,T)}{\partial K^2} \frac{dK^2}{dT} \end{aligned} \qquad dK = K\sigma\sqrt{2dT},$$

So

$$\frac{\partial C(S,t,K,T)}{\partial T} = \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C(S,t,K,T)}{\partial K^2}$$

so that the local volatility $\sigma(K, T)$ is given by

$$\frac{\sigma^2(K,T)}{2} = \frac{\dfrac{\partial C(S,t,K,T)}{\partial T}}{K^2 \dfrac{\partial^2 C(S,t,K,T)}{\partial K^2}}$$

You can regard this as a *definition* of the effective local volatility from options prices and has meaning beyond the model, even with stochastic volatility, as we will see later.

16.5 An Ito-Formal Proof of Dupire's Equation

The stochastic PDE for the risk-neutral stock price with stochastic σ: $\frac{dS_t}{S_t} = rdt + \sigma(S_t, t, \ldots)dZ_t$

Call value at time t: $C_t(K, T) = e^{-r(T-t)}E\{[S_T - K]\theta(S_T - K)\}$

where E denotes the q-measure risk-neutral expectation over S_T and **all other hedged stochastic variables**.

Examine the derivatives of the call value that enters the Dupire equation.

$$\left.\frac{\partial C}{\partial K}\right|_T = -e^{-r(T-t)}E\{\theta(S_T - K)\}$$

$$\left.\frac{\partial^2 C}{\partial K^2}\right|_T = e^{-r(T-t)}E\{\delta(S_T - K)\}$$

To find $\left.\frac{\partial C}{\partial T}\right|_K$ we need to take account of both the change in T and the corresponding change in S_T through Ito's Lemma.

$$C_t(K, T) = e^{-r(T-t)} E\{[S_T - K]\theta(S_T - K)\}$$ Do Ito on terminal time and terminal stock price

$$d_T C\Big|_K = E\left\{\frac{\partial C}{\partial T} dT + \frac{\partial C}{\partial S_T} dS_T + \frac{1}{2}\frac{\partial^2 C}{\partial S_T^2}(dS_T)^2\right\}$$ expectation in risk-neutral measure over S_T

$$= E\left\{-rCdT + e^{-r\tau}\theta(S_T - K)dS_T + \frac{1}{2}e^{-r\tau}\delta(S_T - K)\sigma^2(S_T, T, .)S_T^2 dT\right\}$$

1 $$= E\left\{-rCdT + e^{-r\tau}\theta(S_T - K)(rS_T dT) + \frac{1}{2}e^{-r\tau}\delta(S_T - K)\sigma^2(S_T, T, .)K^2 dT\right\}$$

$$= E\left\{-re^{-r\tau}\theta(S_T - K)(S_T - K)dT + e^{-r\tau}\theta(S_T - K)rS_T dT + \frac{1}{2}e^{-r\tau}\delta(S_T - K)\sigma^2(K, T, .)K^2 dT\right\}$$

$$= e^{-r\tau} E\left\{rK\theta(S_T - K)dT + \frac{1}{2}\delta(S_T - K)\sigma^2(K, T, .)K^2 dT\right\}$$

$$= -rK\frac{\partial C}{\partial K}dT + \frac{1}{2}E\left\{\sigma^2(K, T, .)\right\}\frac{\partial^2 C}{\partial K^2}K^2 dT$$

Then the change in the value of C when S_T and T change is given by

$$\left.\frac{\partial C}{\partial T}\right|_K = -rK\left.\frac{\partial C}{\partial K}\right|_T + \frac{1}{2}E\left\{\sigma^2(K, T, .)\right\}\left.\frac{\partial^2 C}{\partial K^2}\right|_T K^2$$

$$E\left\{\sigma^2(K, T, ...)\right\} = \frac{\left(\left.\frac{\partial C}{\partial T}\right|_K + rK\left.\frac{\partial C}{\partial K}\right|_T\right)}{\left.\frac{1}{2}\frac{\partial^2 C}{\partial K^2}\right|_T K^2}$$

Define the local volatility as: $\sigma^2(K, T) \equiv E\left\{\sigma^2(K, T, ...)\right\}$ and this is the Dupire equation.

Denominator is positive, and we showed earlier via no-arbitrage arguments that the numerator is positive too.

For another more classic proof using the Fokker-Planck equation, see the Appendix of Derman, E. & I. Kani. “Riding on a Smile.” RISK, 7(2) Feb.1994, pp. 139-145, or on my website at http://emanuelderman.com/the-volatility-smile-and-its-implied-tree/

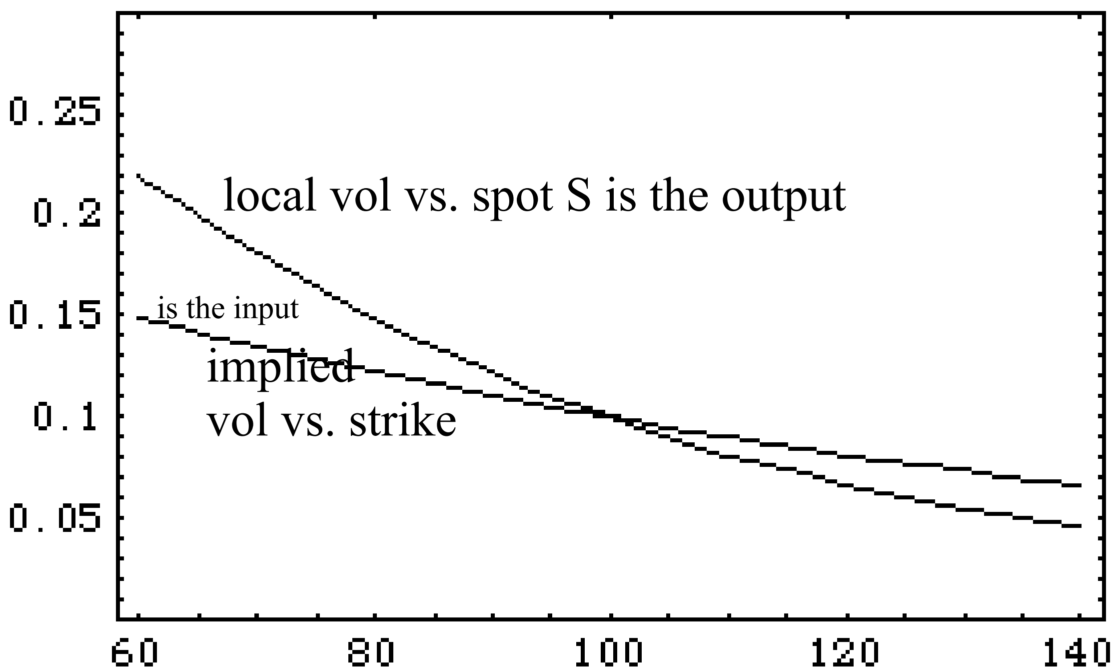
16.6 An Example Of Using The Dupire Equation To Calibrate the Local Volatilities

These graphs are produced inversely. **Given a skew**, we compute the local volatilities, i.e. we solve the inverse problem. (In the previous lectures, we proceeded from local volatilities to implieds.)

Assume $\Sigma(K, T) = 0.1\exp\left[-\left(\frac{K}{100} - 1\right)\right]$ with no term structure.

I used Mathematica to evaluate the derivatives of the Black-Scholes options prices using this skew to calculate the local volatility.

Implied and Local Vols

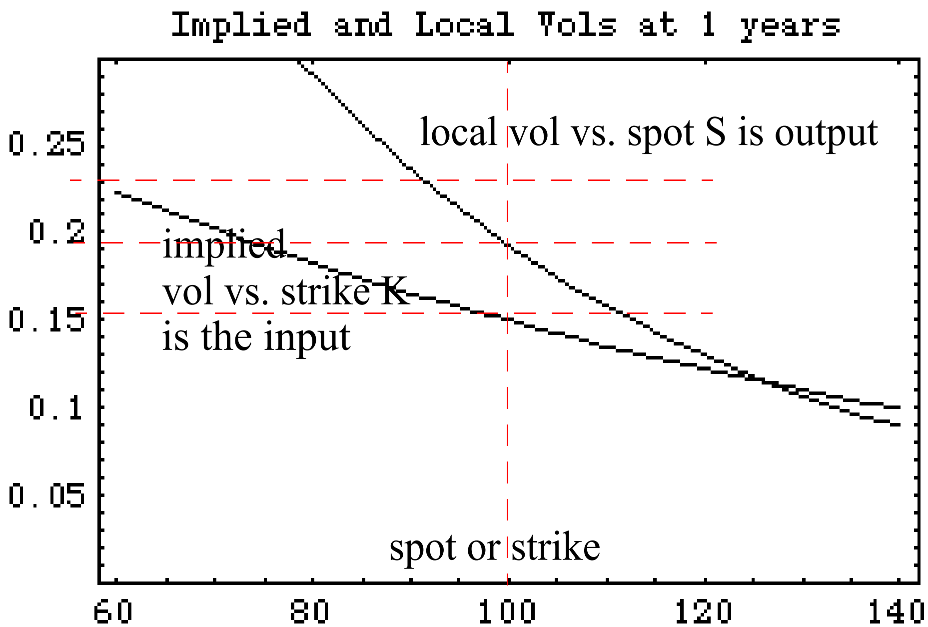


Local volatility does increase roughly twice as fast with spot as implied volatility varies with strike.

Example where volatility varies with strike and expiration, increasing with expiration T but decreasing with strike K according to the formula

$$\Sigma(K, T) = (0.1 + 0.5T)\exp\left[-\left(\frac{K}{100} - 1\right)\right]$$

This smile has both term structure and skew.



At one year in the future, the local volatility still has the characteristic variation with spot, but its value is higher. This illustrates our previous argument that if the term structure of implied volatilities is increasing, then local volatilities must grow about twice as fast with time as well as decrease twice as fast with spot

16.7 An Exact Relationship Between Local and Implied Volatilities and Its Consequences

For zero interest rates and dividend yields, we derived $\sigma^2(K, T) = \left(2\frac{\partial C}{\partial T}\bigg|_K\right) \Big/ \left(K^2 \frac{\partial^2 C}{\partial K^2}\bigg|_T\right)$

Quoting in terms of BS implied vols: $C(S, t, K, T) = C_{BS}(S, t, K, T, \Sigma(S, t, K, T))$

By carefully using the chain rule for differentiation and the formulas for the Black-Scholes Greeks:

$$\sigma^2(K,\tau) = \frac{2\frac{\partial \Sigma}{\partial \tau} + \frac{\Sigma}{\tau}}{K^2\left[\frac{\partial^2 \Sigma}{\partial K^2} - d_1\sqrt{\tau}\left(\frac{\partial \Sigma}{\partial K}\right)^2 + \frac{1}{\Sigma}\left(\frac{1}{K\sqrt{\tau}} + d_1\frac{\partial \Sigma}{\partial K}\right)^2\right]}$$

where $d_1 = \frac{\ln(S/K)}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2}$, and $\Sigma = \Sigma(S, t, K, T)$ is a function of S, t, K, T.

This formula is the generalization of the notion of forward volatilities in a no-skew world to local volatilities in a skewed world.

We can now prove rigorously the previous relations we intuited between implied volatility and local volatility.

16.7.1 Implied variance is average of local variance over life of the option if there is no skew.

$\Sigma(S, t, K, T)$ is independent of strike K, $\frac{\partial\Sigma}{\partial K} = 0$ with no skew at all. Then, writing $\tau = T - t$

$$\frac{1}{2}\sigma^2(K, T) = \frac{\frac{\partial\Sigma}{\partial\tau} + \frac{\Sigma}{2\tau}}{K^2\frac{1}{\Sigma}\left\{\frac{1}{K\sqrt{\tau}}\right\}^2} = \tau\Sigma\frac{\partial\Sigma}{\partial\tau} + \frac{\Sigma^2}{2}$$

$$\sigma(\tau)^2 = \frac{\partial}{\partial\tau}(\Sigma^2\tau) \qquad \text{integrate from } \tau = 0 \quad \text{to} \quad \infty$$

$$\tau\Sigma^2(\tau) = \int_0^{\tau}\sigma^2(u)du$$

the standard result that expresses the total variance as an average of forward variances.

16.7.2 The Rule of Two: Near the money, for flat term structure, the slope of the skew w.r.t strike is 1/2 the slope of the local volatility w.r.t. spot, *for weak skew close to atm*

$\Sigma = \Sigma(K)$ alone, independent of expiration, and $\dfrac{\partial\Sigma}{\partial\tau} = 0$.

Assume a *weak linear* dependence of the skew on K, so that we keep only terms of order $\dfrac{\partial\Sigma}{\partial K}$,

assuming $\left(\dfrac{\partial\Sigma}{\partial K}\right)^2$ and $\dfrac{\partial^2\Sigma}{\partial K^2}$ are negligible.

Then

$$\sigma^2(K, T) = \frac{\cancel{2\frac{\partial\Sigma}{\partial T}} + \frac{\Sigma}{T-t}}{K^2\left(\cancel{\frac{\partial^2\Sigma}{\partial K^2}} - d_1\sqrt{T-t}\,\cancel{\left(\frac{\partial\Sigma}{\partial K}\right)}^2 + \frac{1}{\Sigma}\left\{\frac{1}{K\sqrt{T-t}} + d_1\frac{\partial\Sigma}{\partial K}\right\}^2\right)} \approx \frac{\frac{\Sigma}{\tau}}{\frac{K^2}{\Sigma}\left(\left\{\frac{1}{K\sqrt{\tau}} + d_1\frac{\partial\Sigma}{\partial K}\right\}^2\right)} = \frac{\Sigma^2}{\left\{1 + d_1 K\sqrt{\tau}\frac{\partial\Sigma}{\partial K}\right\}^2}$$

and so

$$\sigma(K) = \frac{\Sigma(K)}{1 + d_1 K\sqrt{\tau}\frac{\partial\Sigma}{\partial K}}$$

Close to at-the-money, $K = S + \Delta K$. Then for small Σ

$$d_1 \approx \frac{\ln S/K}{\Sigma\sqrt{\tau}} \approx -\frac{(\Delta K)}{S(\Sigma\sqrt{\tau})} \approx -\frac{(\Delta K)}{K(\Sigma\sqrt{\tau})}$$

so that

$$\sigma(K) \approx \frac{\Sigma(K)}{1 - \frac{(\Delta K)}{\Sigma}\frac{\partial\Sigma}{\partial K}} \approx \Sigma(K)\left(1 + \frac{(\Delta K)}{\Sigma}\frac{\partial\Sigma}{\partial K}\right) \approx \Sigma(K) + (\Delta K)\frac{\partial\Sigma}{\partial K}$$

Therefore since, $K = S + \Delta K$ $\quad \sigma(S + \Delta K) \approx \Sigma(S + \Delta K) + (\Delta K)\frac{\partial\Sigma}{\partial K}$

and so, since $\sigma(S + \Delta K) \approx \sigma(S) + (\Delta K)\frac{\partial\sigma}{\partial S} = \sigma(S) + (\Delta K)\frac{\partial\sigma}{\partial S}$ and $\Sigma(S + \Delta K) \approx \Sigma(S) + (\Delta K)\frac{\partial\Sigma}{\partial K}$

we obtain $\sigma(S) + \frac{\partial\sigma(S)}{\partial S}\Delta K \approx \Sigma(S) + 2\frac{\partial\Sigma(S)}{\partial S}\Delta K$ and so $\frac{\partial}{\partial S}\sigma(S) \approx 2\left(\frac{\partial\Sigma}{\partial K}\right)\Big|_{K=S}$

The local volatility $\sigma(S)$ grows twice as fast with stock price *S* as the implied volatility $\Sigma(K)$ grows with strike!

16.7.3 Implied volatility is an harmonic average over local volatility *at short expirations*.

For zero rates and dividends: $$\sigma^2(K,T) = \frac{2\frac{\partial\Sigma}{\partial\tau} + \frac{\Sigma}{\tau}}{K^2\left(\frac{\partial^2\Sigma}{\partial K^2} - d_1\sqrt{\tau}\left(\frac{\partial\Sigma}{\partial K}\right)^2 + \frac{1}{\Sigma}\left\{\frac{1}{K\sqrt{\tau}} + d_1\frac{\partial\Sigma}{\partial K}\right\}^2\right)}$$

Multiplying top and bottom by τ: $$\sigma^2(K,T) = \frac{\cancel{2\tau\frac{\partial\Sigma}{\partial\tau}} + \Sigma}{K^2\left(\cancel{\tau\frac{\partial^2}{\partial K^2}(\Sigma)} - \cancel{d_1\tau\sqrt{\tau}\left(\frac{\partial\Sigma}{\partial K}\right)^2} + \frac{1}{\Sigma}\left\{\frac{1}{K} + \sqrt{\tau}d_1\frac{\partial\Sigma}{\partial K}\right\}^2\right)}$$

As $\tau \to 0$, this becomes the o.d.e. $$\sigma^2(K,T) = \frac{\Sigma}{K^2\left(\frac{1}{\Sigma}\left\{\frac{1}{K} + \sqrt{\tau}d_1\frac{d\Sigma}{dK}\right\}^2\right)} = \frac{\Sigma^2}{\left\{1 + \sqrt{\tau}Kd_1\frac{d\Sigma}{dK}\right\}^2}$$

Now $\sqrt{\tau}Kd_1 \to \frac{K\ln(S/K)}{\Sigma}$ as $\tau \to 0$, and we obtain $$\sigma(K) = \frac{\Sigma}{1 + \frac{K}{\Sigma}\frac{d\Sigma}{dK}\ln(S/K)}$$ as a function of K for fixed S.

Transforming from K into the new variable $x = \ln(K/S)$ we can rewrite this as the o.d.e.

$$\frac{\Sigma}{1-\frac{x}{\Sigma}\frac{d\Sigma}{dx}} = \sigma(x) \qquad \text{Regard } \sigma \text{ as function of } x$$

We can solve this as follows. Define $L = \frac{1}{\Sigma}$. Then $\frac{d\Sigma}{dx} = -\frac{1}{L^2}\frac{dL}{dx}$ and so you can write the equation

$$\frac{1}{L\left[1+\frac{x}{L}\frac{dL}{dx}\right]} = \sigma(x)$$

or

$$L + x\frac{dL}{dx} = \frac{1}{\sigma(x)} \qquad \text{Eq.16.1}$$

which is relatively simple. In fact you can rewrite this as

$$\frac{d}{dx}[Lx] = \frac{1}{\sigma(x)}$$

Integrating from $x = 0$, i.e. $K = S$, to $x = \ln\frac{K}{S}$ we obtain

$$\left(\ln\frac{K}{S}\right)L(S, K) = \int_{o}^{\ln\frac{K}{S}} \frac{1}{\sigma(x)}dx \qquad \text{Eq.16.2}$$

So

$$\frac{\ln\left(\frac{K}{S}\right)}{\Sigma\left(\frac{K}{S}\right)} = \int_0^{\ln\left(\frac{K}{S}\right)} \frac{1}{\sigma(x)}dx$$

$$\frac{1}{\Sigma\left(\frac{K}{S}\right)} = \frac{1}{\left(\ln\frac{K}{S}\right)} \int_{o}^{\ln\frac{K}{S}} \frac{1}{\sigma(x)}dx$$

In other words, at very short times to expiration, the implied volatility is the harmonic mean of the local volatility as a function of $\ln K/S$ between spot and strike.

This is intuitively reasonable, more sensible than an arithmetic mean.

Suppose that $\sigma(y)$ falls to zero above a certain level *K*, so that the stock price can never diffuse higher. Then the implied volatility of any option with a strike above that level should be zero.

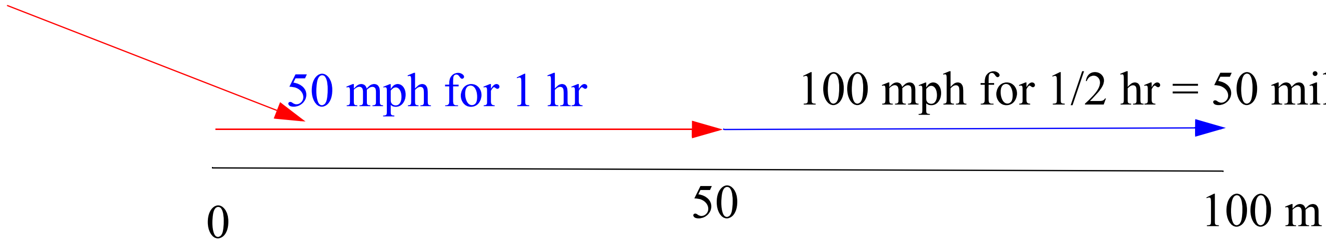
If $\Sigma(x) = \frac{1}{x}\int_{o}^{x} \sigma(y)dy$, an ordinary arithmetic mean, then its value would be non-zero, which is impossible if the stock can never reach the strike.

In contrast, for the harmonic mean, if $\sigma(y)$ becomes zero anywhere in the range between spot and strike, then the implied volatility for that strike, $\Sigma(x)$, becomes zero too, which is as to be expected.

There is an intuitive way to understand this by thinking of volatility as the speed of log diffusion.

Think of a car with a speed $v(x)$ that varies locally with position x. A local velocity.

Specifically, say a car travels at 50 mph for the first 50 miles, and 100 mph for the second 50 miles to cover 100 miles in total.



Total distance = 100m. Total time = 1.5 hrs. Average velocity is not 75 mph because the car spends less time traveling at 100 mph to cover the distance. The average velocity is 100m/1.5 hrs = 66.7.

Average velocity is *not* average of local velocities because high velocity means less time spent.

The **average velocity *V* is the total distance *D* divided by the total time *T***.

Times are additive. The **total time** T for the trip is the sum of the local times = dist/speed

$$T = \int_0^D \frac{dx}{v(x)} \equiv \frac{D}{V} \quad \text{by definition}$$

$$\frac{1}{V} = \frac{1}{D}\int_0^D \frac{dx}{v(x)}$$

Average velocity is the harmonic average of the local velocity!

That was for linear motion, not diffusion.

Now think about diffusion. Think of the stock as diffusing through as a medium with a volatility that is analogous to speed. The greater the volatility, the faster and wider the diffusion. (Brownian motion). If the stock's volatility were infinite, the medium would immediately be transparent to diffusion.

σ has the dimension $time^{-1/2}$. $1/\sigma^2$ has the dimensions of time. Think of $1/(\sigma^2(S))$ as the time taken for *dlnS* to diffuse at each stock price *S*.

Times are additive.

In velocity, it's the times that are additive. In diffusion it's the square roots of time that effectively are additive.

Then $\int_{\ln S}^{\ln K} 1/\sigma^2$ is the total diffusion time from ln(S) to ln(K).

Or better, think of $\int_{\ln S}^{\ln K} 1/\sigma$ as the *square root* of the total diffusion time from ln(S) to ln(K), because it's square roots that matter in Brownian motion.

Let $x = \ln K/S$. Then $\int_o^x \frac{1}{\sigma(y)} dy$ is *roughly* the total $\sqrt{\text{diffusion time}}$ computed from the sum of local $\sqrt{\text{diffusion times}}$ under lognormal diffusion.

Define **Average Volatility** $\Sigma(S, K)$ by the relation between total $\sqrt{\text{diffusion time}}$ and total log distance $\ln K/S$:

Total $\sqrt{\text{diffusion time}} = \frac{\text{total log distance}}{\Sigma} = \frac{\ln K/S}{\Sigma} \equiv \frac{x}{\Sigma(x)}$ - this is definition of the average volatility $\Sigma(S, K)$

In velocity, it's the times that are additive. In diffusion it's the square roots of time that effectively are additive. So total $\sqrt{\text{diffusion time}}$ is the integral of the local $\sqrt{\text{diffusion time}}$: $\int_o^x \frac{1}{\sigma(y)} dy$

$\frac{x}{\Sigma(x)} = \int_o^x \frac{1}{\sigma(y)} dy$ The average volatility is found from the total time and total distance.

16.8 Why Does the Implied Vol as the Average of Local Vols from Spot to Strike Work Quite Well? More Rigorously: Spot-to-Strike is the Path With The Highest Gamma, Most P&L From Hedging. ...

Assume zero rates for didactic simplicity. Value an option with strike K, expiration T.

Local vol PDE for price $C_\sigma(S, t, K, T)$: assume local vol $dS = S\sigma(S, t)dZ$ is the actual evolution of the stock:

$$\frac{\partial C_\sigma}{\partial t} + \frac{1}{2}\frac{\partial^2 C_\sigma}{\partial S^2}S^2\sigma^2(S, t) = 0$$

Let the local volatility solution be quoted in terms of the Black-Scholes implied volatility Σ_{KT}, different for different strikes K, T, so that

$$C_\sigma(S, t, K, T) \equiv C_{\Sigma_{KT}}(S, t, K, T) \quad \text{using Black-Scholes solution}$$

$C_\Sigma(S, t, K, T)$ satisfies the BS equation for each strike, by definition:

$$\frac{\partial C_{\Sigma_{KT}}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{\Sigma_{KT}}}{\partial S^2}S^2\Sigma_{KT}^2 = 0 \qquad \text{BS Equation} \qquad \text{Eq 16.1}$$

Now use Ito to figure out how $C_{\Sigma_{K,T}}$ varies as the stock price does the actual local vol evolution:

$$dC_{\Sigma} = \frac{\partial C_{\Sigma}}{\partial t}dt + \frac{\partial C_{\Sigma}}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C_{\Sigma}}{\partial S^2}S^2\sigma^2(S,t)dt = \frac{\partial C_{\Sigma_{KT}}}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C_{\Sigma_{KT}}}{\partial S^2}S^2\left[\sigma^2(S,t) - \Sigma_{KT}^2\right]dt \quad \text{Eq 16.2}$$

because from Equations 16.1 $\frac{\partial C_{\Sigma_{KT}}}{\partial t} = -\frac{1}{2}\frac{\partial^2 \Sigma_{K,T}}{\partial S^2}S^2\Sigma^2_{KT}$

Now take expected value E_{σ} of both sides of Equations 16.2 over the distribution of S given by local vols $dS = S\sigma(S,t)dZ$, i.e. over the realized paths, where remember that Σ means Σ_{KT}:

$$E_{\sigma}[dC_{\Sigma}] = E_{\sigma}\left\{\frac{\partial C_{\Sigma}}{\partial S}\cancel{dS} + \frac{1}{2}\frac{\partial^2 C_{\Sigma}}{\partial S^2}S^2[\sigma^2(S,t) - \Sigma^2]dt\right\} \text{ where } E_{\sigma}[dS] = 0$$

Now integrate from 0 to expiration T:

$$E_{\sigma}[C_{\Sigma}(S_T,T) - C_{\Sigma}(S_0,0)] = E_{\sigma}[C_{\Sigma}(S_T,T)] - \underbrace{C_{\Sigma}(S_0,0)}^{\text{known}} = E_{\sigma}\left\{\int_0^T \frac{1}{2}\frac{\partial^2 C_{\Sigma}}{\partial S^2}S^2[\sigma^2(S,t) - \Sigma^2]dt\right\}$$

The LHS is zero because the value at expiration of $C_\Sigma(S_T, T)$ is independent of volatility, and by definition of Σ the expected value of the payoffs equals the current Black-Scholes value $C_\Sigma(S_0, 0)$.

$$E_\sigma\left\{\int_0^T \frac{\partial^2 C_\Sigma}{\partial S^2} S^2[\sigma^2(S, t) - \Sigma^2]dt\right\} = 0$$

So

$$\Sigma^2 = \left(E_\sigma\left\{\int_0^T \frac{\partial^2 C_\Sigma}{\partial S^2} S^2[\sigma^2(S, t)]dt\right\}\right) \Big/ \left(E_\sigma\left\{\int_0^T \frac{\partial^2 C_\Sigma}{\partial S^2} S^2 dt\right\}\right)$$

So we see that the implied vol squared, $\Sigma^2{}_{KT}$, is average of local vol squared over all actual paths of the stock, weighted by **implied** vol gamma x S^2. This is an implicit equation because the LHS and the RHS both involve $\Sigma^2{}_{K,\,T}$.

This gives insight but doesn't make computation easy.

What is the distribution of $\frac{\partial^2 C_\Sigma}{\partial S^2} S^2$ under the evolution of S and σ?

Graphing The Gamma Weighting Function

Assume zero rates. Value an option with strike K, expiration T. We showed

$$\Sigma^2 = \left(E_\sigma\left\{\int_0^T \frac{\partial^2 C_\Sigma}{\partial S^2} S^2 [\sigma^2(S,t)]dt\right\}\right) \Big/ \left(E_\sigma\left\{\int_0^T \frac{\partial^2 C_\Sigma}{\partial S^2} S^2 dt\right\}\right)$$

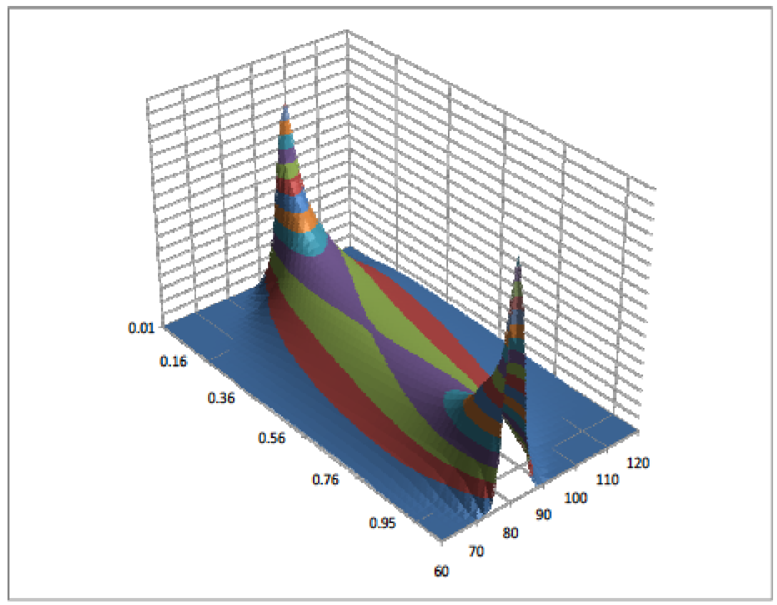
Implied vol squared is average of local vol squared weighted by implied vol gamma. Implicit equation. $dS = S\sigma(S,t)dZ$ describes the evolution of local vol in E_σ.

Where does this peak?

The evolution in E_σ starts out from S_0 with probability distribution function:

$$\frac{\exp\left(-\dfrac{\left(\dfrac{\ln S_t/S_0 - (rt - \frac{1}{2}\sigma^2 t)}{\sigma\sqrt{t}}\right)^2}{2}\right)}{\sqrt{2\pi t}\sigma S_t}$$

and is a Dirac delta function at $t = 0$ at S_0.



Gamma plotted over distribution of lognormal paths for the stock price

FIGURE 1. Graph of $q_\sigma(t, f)$ when the forward is lognormal; $\sigma = 30\%$, $f_0 = 100$, $K = 80$, $T = 1$.

The Gamma in the integral for the implied volatility is

$$\frac{\partial^2 C_\Sigma}{\partial S^2} = \frac{\exp\left(-\frac{d_1^2}{2}\right)}{S\Sigma\sqrt{2\pi\tau}}$$

$$d_1 = \frac{\ln S/K}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2}$$

which peaks at $S = K$ as the time to expiration $\tau \to 0$.

So path from S to K is the dominant one for determining the implied volatility.