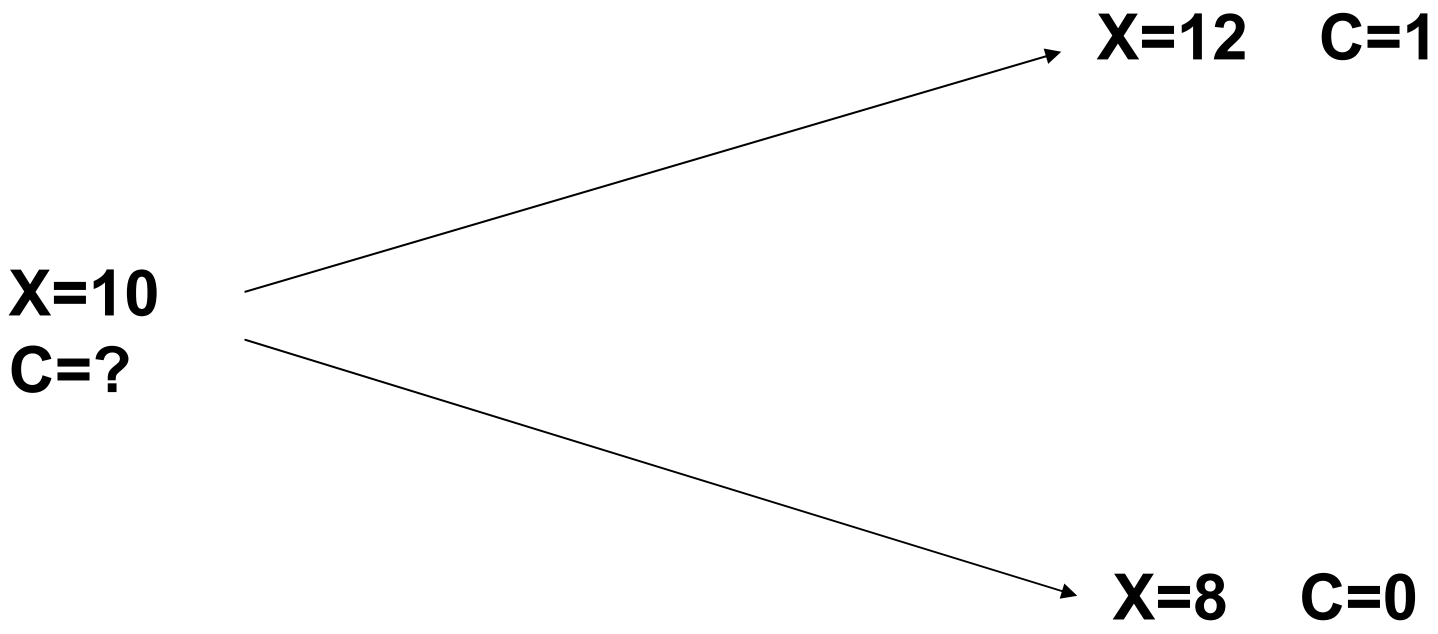
Risk-Free Portfolio Valuation
and
Risk-Neutral Probabilities Valuation

1. One-Step Binomial Model

Risk-Free Portfolio Valuation Method

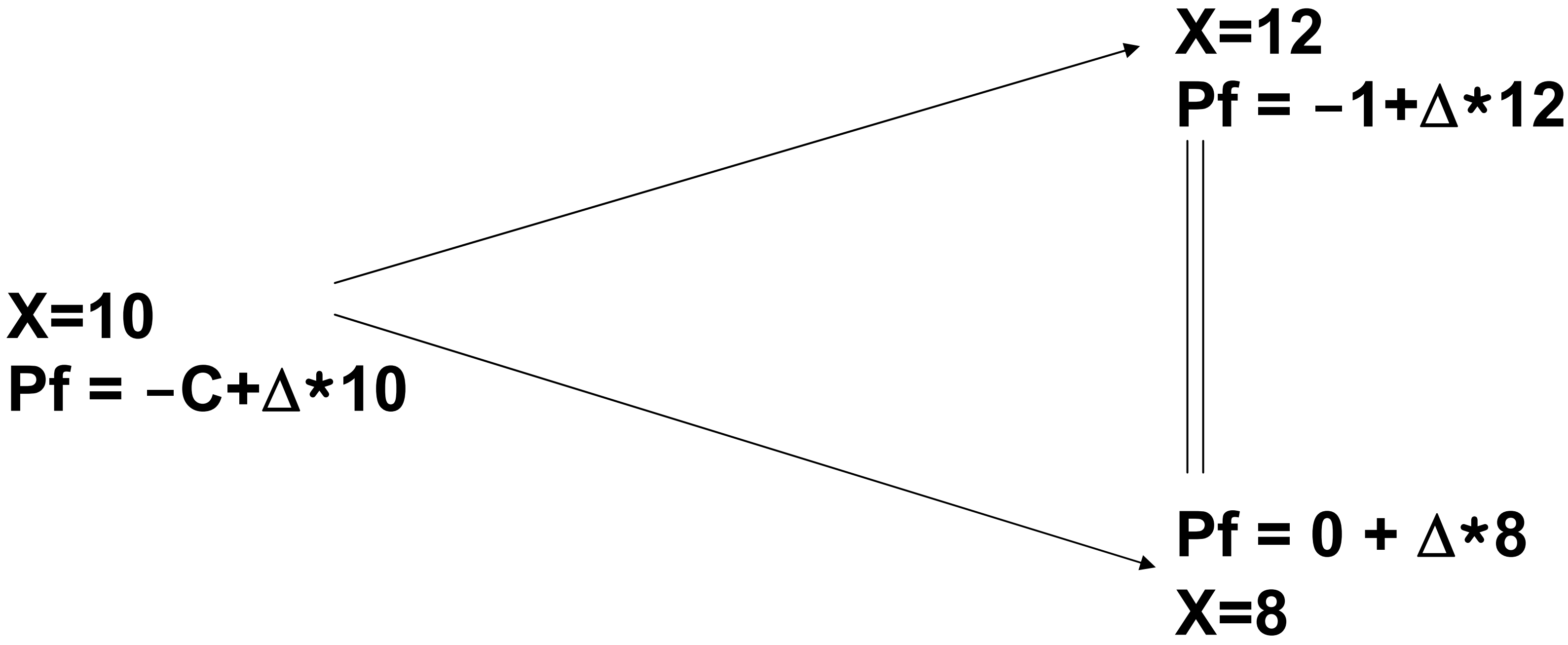


Stock **X** has a current price of $10.
In **T=1** year, the price of the stock will either rise to $12, or fall to $8.

Interest rate **r**=10% with annual compounding.

Consider a European call option **C** with strike price $11, expiring in one year.

How much should the option **C** cost?



Let’s try to construct a risk-less portfolio so that its value will be the same should stock go up to 12 or down to 8.

Let us sell short one call option **C** and buy Δ shares of stock**.**

Pf = −C+Δ*10 is the value of portfolio at time 0.

Here Δ is a parameter to be determined.

Pf(at X=12) = −1+Δ*12 = Pf(at X=8) = 0+Δ*8

−1+Δ*12 = 0+Δ*8 from where Δ=**1/4**

X=12

$$Pf = -1 + \frac{1}{4} * 12 = 2$$

X=10

$$Pf = -C + \frac{1}{4} * 10$$

$$Pf = 0 + \frac{1}{4} * 8 = 2$$

X=8

$$r = 10\%, \quad T = 1$$

Our portfolio value **Pf(at t=1) = 2** and it is independent whether stock price is 8 or 12.

If we have an asset whose value is 2 with certainty at time t=1, we can find its value at time t=0 as a present value.

$$\mathbf{Pf(at\ t = 0)} = \frac{2}{(1+r)^T} = \frac{2}{(1+0.1)^1} = 1.818181...$$

$$Pf = 2$$

$$Pf = -C + \frac{1}{4} * 10$$

$$Pf = 1.8181$$

$$Pf = 2$$

$$r = 10\%, \quad T = 1$$

On one hand our portfolio value at time T=0 as a present value is $\mathbf{Pf(at\ t=0) = \frac{2}{(1+r)^1} = \frac{2}{(1+0.1)^1} = 1.818181...}$

On another hand our portfolio $\mathbf{Pf\ (at\ t=0) = -C + \frac{1}{4} * 10}$

So $\mathbf{-C + \frac{1}{4} * 10 = 1.8181}$ so $\mathbf{C = 2.5 - 1.8181 = 0.681818}$

T=1, Time step in years in binomial model.

X_o=10, Starting stock price at time 0.

U=12, Up stock price at time T.

D=8, Down stock price at time T.

K=11, Strike price.

r=0.1, Annually compounded interest rate.

C, Call price with strike K.

$$C = 2.5 - \frac{2}{(1+0.1)^1} = \frac{10}{4} - \frac{\frac{1}{4}\cdot 8}{(1+0.1)^1} = 10\cdot\frac{1}{12-8} - \frac{\frac{1}{12-8}\cdot 8}{(1+0.1)^1} =$$

$$= X_0 \cdot \frac{U-K}{U-D} - \frac{\frac{U-K}{U-D}\cdot D}{(1+r)^T} = \frac{\frac{U-K}{U-D}\cdot X_0 \cdot (1+r)^T - \frac{U-K}{U-D}\cdot D}{(1+r)^T} =$$

$$= \left(\frac{U-K}{U-D}\right)\cdot\frac{X_0\cdot(1+r)^T - D}{(1+r)^T}, \quad \text{here} \quad \Delta = \frac{U-K}{U-D}.$$

T=1, Time step in years in binomial model.

X_0=10, Starting stock price at time 0.

U=12, Up stock price at time T.

D=8, Down stock price at time T.

K=11, Strike price.

r=0.1, Annually compounded interest rate.

C, Call price with strike K.

$$\mathbf{C} = \left(\frac{\mathbf{U}-\mathbf{K}}{\mathbf{U}-\mathbf{D}}\right) \cdot \frac{\mathbf{X_0} \cdot (\mathbf{1}+\mathbf{r})^{\mathbf{T}} - \mathbf{D}}{(\mathbf{1}+\mathbf{r})^{\mathbf{T}}} = \left(\frac{\mathbf{12}-\mathbf{11}}{\mathbf{12}-\mathbf{8}}\right) \cdot \frac{\mathbf{10} \cdot (\mathbf{1}+\mathbf{0.1})^{\mathbf{1}} - \mathbf{8}}{(\mathbf{1}+\mathbf{0.1})^{\mathbf{1}}} = \mathbf{0.681818...}$$

So here is a theoretical Call price formula in our binomial model.

Important Remark:

The probabilities of upward and downward movements of the stock prices are not involved in the valuation of the option.

Why is C = 0.681818 is the correct price for the option? Because if the price is not 0.6818 , there is an opportunity to make an **arbitrage** - a risk-less profit.

1.Suppose the price of the call option is not $0.6818 but instead is higher: $0.80

What is the arbitrage strategy we should take?

a) Sell a call option for $0.80

b) Create a **synthetic long call** option position by trading stock and cash only. The creation of the synthetic long call option position will cost us exactly $0.6818

Position of a short real call and synthetic long call present no risk. If short real call loses 1 dollar when stock goes to 12 synthetic long call makes 1 dollar.

Arbitrage profit is $0.80-$0.6818 = $0.11818

1.Synthetic LONG call option position. Cost $0.6818

Take $0.6818

Borrow 2.50 - 0.6818 = **1.8181** from the bank at 10%.
Buy 1/4 shares of stock at $10.00 paying
2.50= 1.8181(that you borrowed)+ 0.6818 you received.

Wait one year.

If the stock price increases to $12.00 1/4 share is worth $3. Sell it for $3 repay the bank principal and interest which is $2=$1.8181*(1+0.1). **You have $1** (same as real call would have when stock price is $12)

If the stock price decreases to $8.00, 1/4 share is worth $2 Sell it for $2 repay the bank principal and interest which is $2=$1.8181*(1+0.1). **You have $0** (same as real call would have when stock price is $8)

2. Suppose that the price of the call option is lower than the theoretical price 0.6818 . Say price is $0.40.
What is the arbitrage strategy we should take?

a) Buy a call option for $0.40.

b) Create a synthetic short call option position by trading stock and cash only. The creation of the synthetic short call option position will bring us exactly $0.6818 of cash.

Position of long real call and synthetic short call present no risk. If short synthetic call loses 1 dollar when stock goes to 12 real long call makes 1 dollar.

Our arbitrage profit is $0.6818-$0.40=$0.2818

2. Synthetic SHORT call option position. Profit $0.6818

Sell short 1/4 shares of stock at $10.00 receiving 2.50. Out of these 2.50 take away $0.6818 which would be your profit.

Put remainder 1.8181..=2.50 - 0.6818 **in the bank** at 10%. In one year these $1.8181 will become $2

Wait one year.

If the stock price increases to $12.00 1/4 share is worth $3. You need to close short position and pay $3 to buy 1/4 share. You have $2 in a bank and is short $1
You have value -1$ (same as short real call would have when the stock price is $12).

If the stock price decreases to $8.00, You need to close short position and pay $2 to buy 1/4 share. You have $2 in a bank and has no further liabilities.
You have value 0$ (same as short real call would have when the stock price is $8).

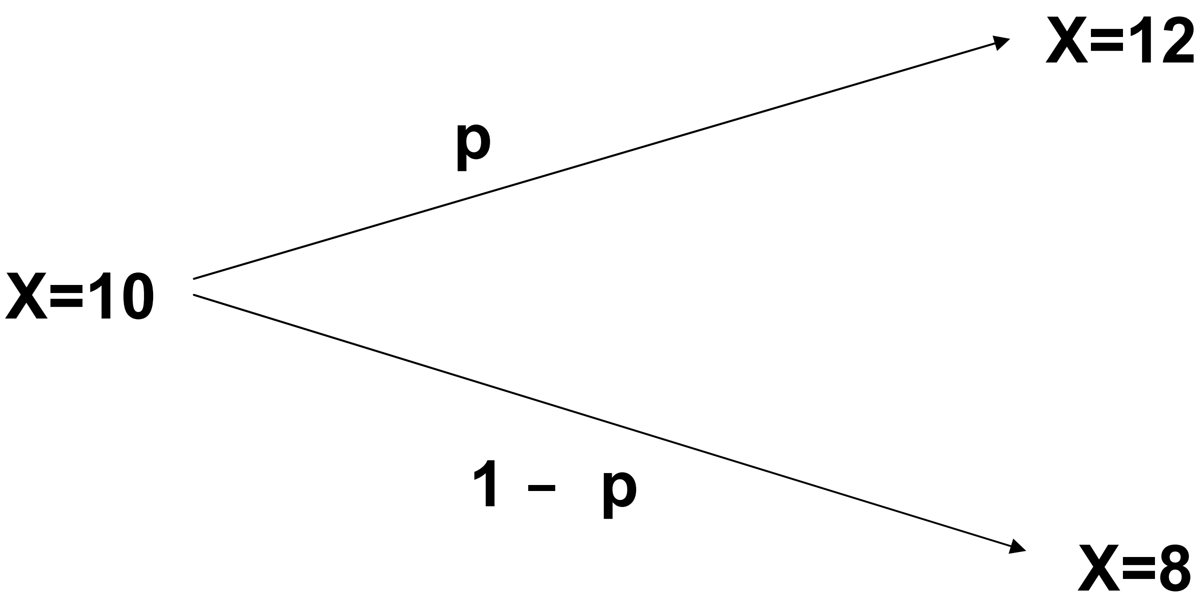
2. One-Step Binomial Model

Risk-Neutral Probabilities Valuation Method

Risk Neutral Probabilities Valuation.

is a second “artificial” method to calculate option price.It gives the answer that is the same as a “true” risk-free portfolio method.

It is artificial because it assigns “artificial” probabilities for upward and downward movements of stock, so that the average (or “expected”) annual growth rate of the stock equals to the interest rate **r**.

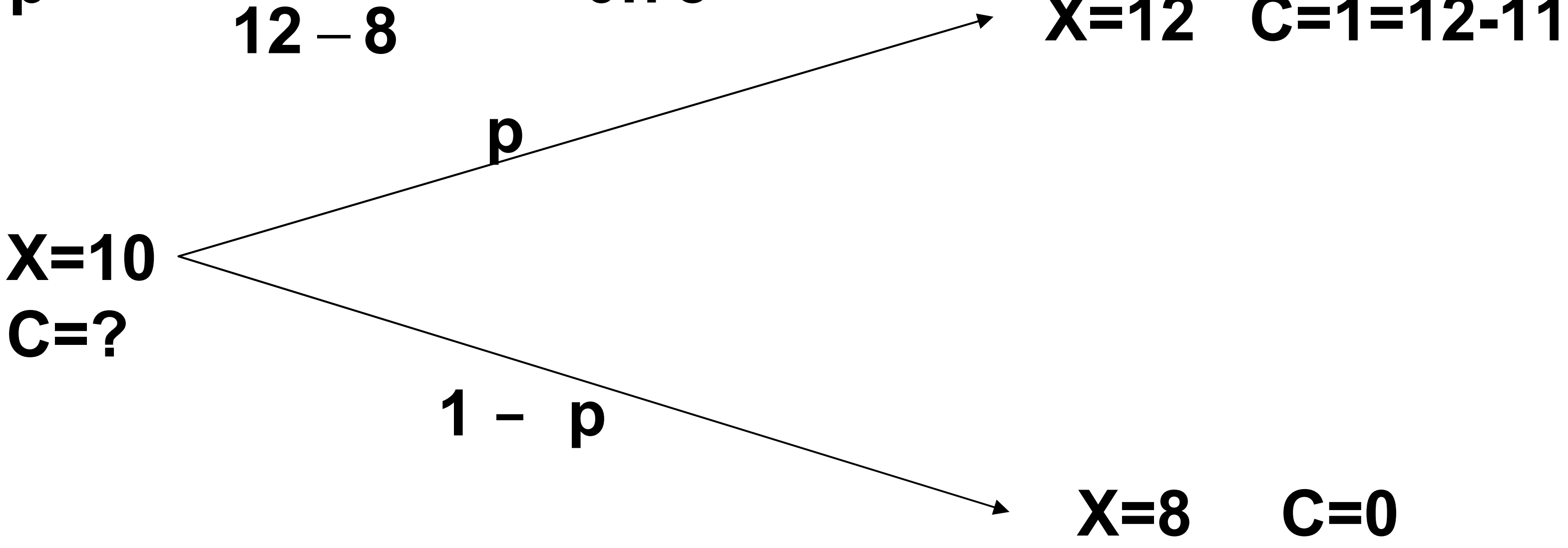


$$10 \cdot (1+r)^{T} = p \cdot 12 + (1-p) \cdot 8$$

Risk Neutral Probabilities Valuation Continued.

$$10\cdot(1+r)^{T} = p\cdot 12+(1-p)\cdot 8, \qquad \text{here } r = 0.1$$

$$p = \frac{10\cdot(1+r)^{T}-8}{12-8} = 0.75$$



The value of the option C is the present value of its Expected Value calculated using this “artificial” probability p.

$$\text{Exp.Val. } (C \text{ at } t=1) = 1\cdot p + 0\cdot(1-p) = 1\cdot\frac{10\cdot(1+r)^{T}-8}{12-8} = \frac{11-8}{12-8} = 0.75$$

$$C = P.V.(0.75) = \frac{0.75}{(1+r)^{T}} = \frac{0.75}{(1+0.1)^{1}} = 0.681818\ldots$$

$$p = \frac{10 \cdot (1+r)^T - 8}{12-8}$$

The value of the option C is the present value of its Expected Value calculated using this “artificial” probability p.

$$\textbf{Exp.Val. (C at T} = 1) = 1 \cdot p + 0 \cdot (1-p) = 1 \cdot \frac{10 \cdot (1+r)^T - 8}{12-8} = \frac{11-8}{12-8} = 0.75$$

The present value of that

$$C = \frac{1}{(1+r)^T}(1 \cdot p + 0 \cdot (1-p)) = \frac{1}{(1+r)^T}(12-11) \cdot \frac{10 \cdot (1+r)^T - 8}{12-8} =$$

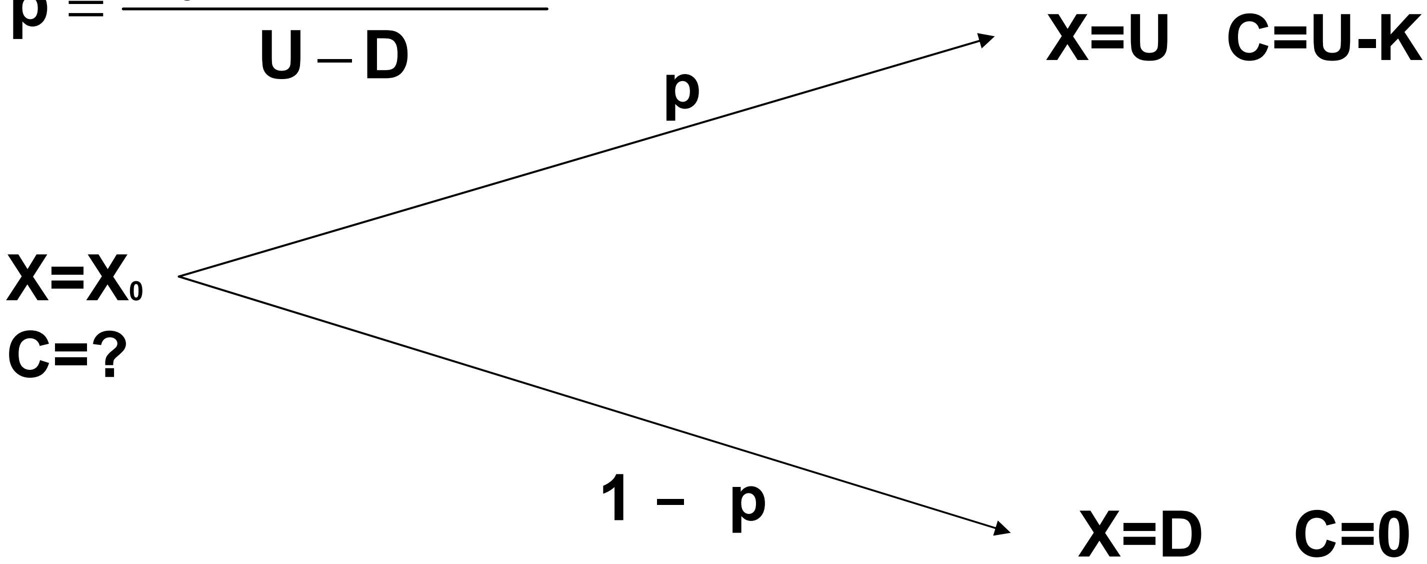
$$= \left(\frac{12-11}{12-8}\right) \cdot \frac{10 \cdot (1+0.1)^1 - 8}{(1+0.1)^1} = 0.681818...$$

The value of the option C calculated this way is the same 0.681818 as calculated using risk-free portfolio with delta hedging. Moreover the formula is the same.

In the general case:

$$X_0 \cdot (1+r)^T = p \cdot U + (1-p) \cdot D,$$

$$p = \frac{X_0 \cdot (1+r)^T - D}{U - D}$$



The value of the option C is the present value of its Expected Value calculated using this “artificial” probability p.

$$\text{Exp.Val. (C at T)} = (U-K) \cdot p + 0 \cdot (1-p) = (U-K) \cdot \frac{X_0 \cdot (1+r)^T - D}{U-D} = \frac{U-K}{U-D} \cdot (X_0 \cdot (1+r)^T - D)$$

$$C = \text{P.V.}\left(\text{Exp.Val. (C at T)}\right) = \left(\frac{U-K}{U-D}\right) \frac{(X_0 \cdot (1+r)^T - D)}{(1+r)^T}$$

Now we see that in the one step binomial model artificial **risk-neutral probabilities** give the **same results** and the **same formulas** as a **no arbitrage risk-free portfolio** method.

In the general case it is also true that the two methods give the same result.

Calculating the present value of expected value of option under risk-neutral probabilities usually is an easier task.

We can now put some philosophy.

In a risk-neutral world, the expected return on a non-dividend paying stock is equal to the risk-free interest rate, regardless of the risk involved in the investment.

In the real world investors do require a higher return for risky investment. So **real world probabilities** of stock going up are likely **higher** than in the **risk-neutral world**.

However, we showed that the artificial **risk-neutral probabilities model** will enable us to correctly determine the real-world values of options that do not allow for arbitrage.

The “true” risk free portfolio model does not involve probabilities at all !!

So risk free method no arbitrage option price is the same in any “world” with any probabilities of up and down move.

So risk free method no arbitrage option price is the same in any “world” with any probabilities of up and down move.

While in such “worlds” expected value of the option may be higher than theoretical no-arbitrage value, that expected value has uncertainty.

The no-arbitrage value is certain with probability 1.

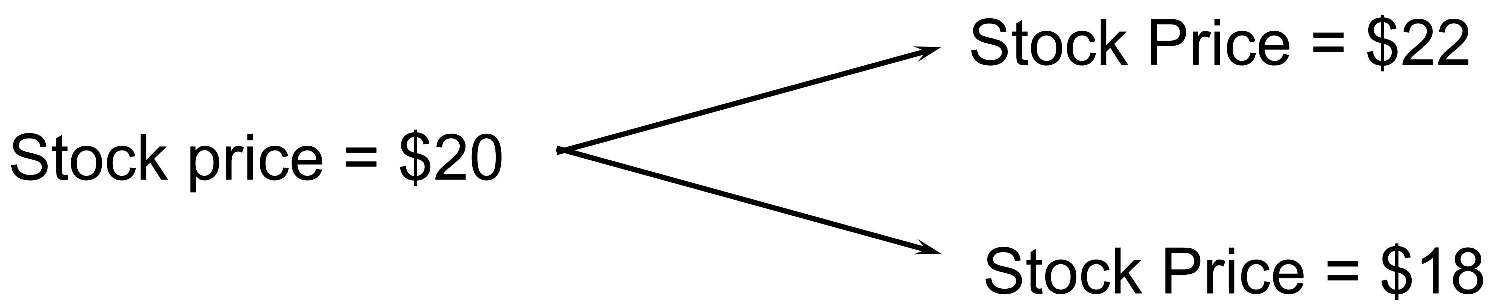
So there is no contradiction. You can get higher expected value by buying and holding call option in a real world but that is a premium for the risk you are taking.

Similarly if you just put your money in stock in the real world you get higher expected value than from cash. But you take additional risk.

A One Step Binomial Model Revisited. Example 2 of Risk Free Portfolio.

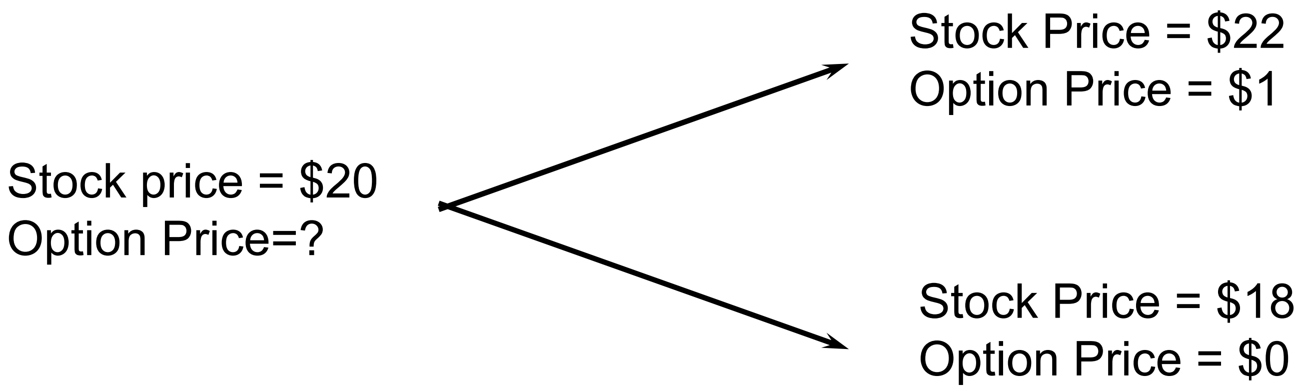
A stock price is currently \$20

In 3 months it will be either \$22 or \$18



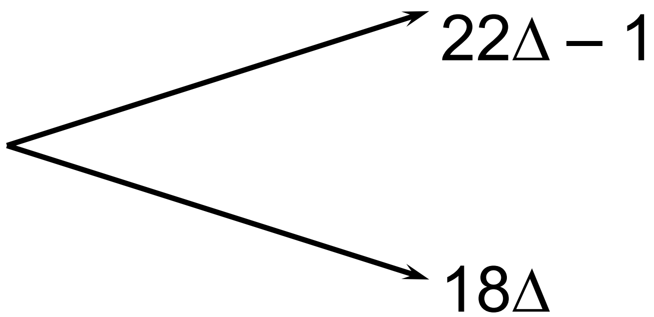
A Call Option

A 3-month call option on the stock has a strike price of 21.



Setting Up a Riskless Portfolio

For a portfolio that is long Δ shares and a short 1 call option values are



Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 1/4$

Valuing the Portfolio

Risk-Free Rate is 12% with continuous compounding. Time to maturity 0.25 year

The risk-less portfolio is:

long 1/4 shares

short 1 call option

The value of the portfolio in 3 months is

When stock price is 22

$22 \times 1/4 - 1 = 4.50$

When stock price is 18

$18 \times 1/4 - 0 = 4.50$

The value of the portfolio today is

$4.5e^{-0.12\times 0.25} = 4.3670$

Valuing the Option

The portfolio that is

long 1/4 shares

short 1 option

is worth 4.367

The value of the shares is

5.00 = 1/4 × 20

The value of the short option position is therefore

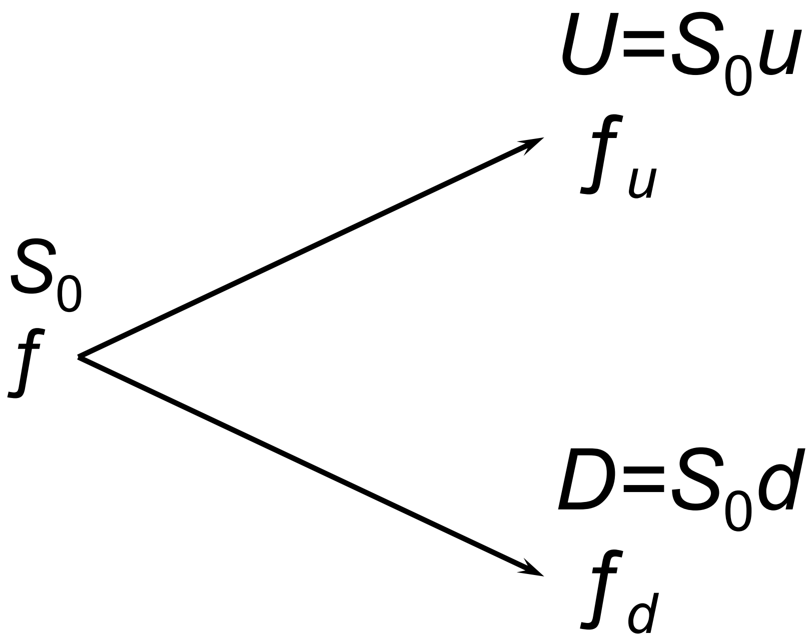
-0.633 (4.367 -5.000 = 0.633)

The value of the option is therefore +0.633

Generalization

A derivative **f** lasts for time **T** and is dependent on the stock price.

T, Time step in years in binomial model.
S_0 also denoted in the past by **X_0**, starting stock price at time 0.
U= u*S_0, Up stock price at time **T**, here **u** is a multiplier for up move.
D= d*S_0, Down stock price at time **T**, here **d** is a multiplier for down move.
r, continuously compounded interest rate.
f, derivative price at time 0 (can be call option, put option, forward, other type of derivative.
f_u, derivative price at time T when stock price moved up to **U= u*S_0**.
f_d, derivative price at time T when stock price moved down to **D= d*S_0**.



Generalization continued.

Value of a portfolio that is long Δ shares and short 1 derivative:

$$S_0 u\Delta - f_u$$

$$S_0 d\Delta - f_d$$

The portfolio is riskless when $\boldsymbol{S_0 u\Delta - f_u = S_0 d\Delta - f_d}$ or

$$\Delta = \frac{\mathbf{f_u - f_d}}{\mathbf{S_0 u - S_0 d}}$$

change in derivative price divided by change in stock price.

Generalization continued.

Value of the portfolio at time T is $S_0 u \Delta - f_u$

Value of the portfolio today is $(S_0 u \Delta - f_u)e^{-rT}$

Another expression for the portfolio value today is $S_0\Delta - f$

Hence

$$f = S_0\Delta - (S_0 u \Delta - f_u)e^{-rT}$$

$$\Delta = \frac{\mathbf{f_u - f_d}}{\mathbf{S_0 u - S_0 d}}$$

Generalization continued.

$$f = S_0\Delta - (S_0u\,\Delta - f_u\,)e^{-rT}$$

Substituting $\Delta = \dfrac{\mathbf{f_u - f_d}}{\mathbf{S_0u - S_0d}}$ we obtain

$$f = [\,pf_u + (1 - p)f_d\,]e^{-rT}$$

where

$$\mathbf{p} = \frac{\mathbf{e^{rT} - d}}{\mathbf{u - d}}$$

Risk Neutral Probabilities Valuation.

It is natural to interpret p and 1-p as probabilities of up and down movements

The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate

We can check that

$$\begin{tikzcd} & \begin{matrix} S_0u \\ f_u \end{matrix} \\ \begin{matrix} S_0 \\ f \end{matrix} \arrow[ur, "p"] \arrow[dr, "(1-p)"'] & \\ & \begin{matrix} S_0d \\ f_d \end{matrix} \end{tikzcd}$$

$$f = [\, p f_u + (1 - p) f_d \,] e^{-rT}$$

Risk-Neutral Valuation. Continued.

When the probability of an up and down movements are p and 1-p the expected stock price at time T is $\mathbf{S_0e^{rT}}$

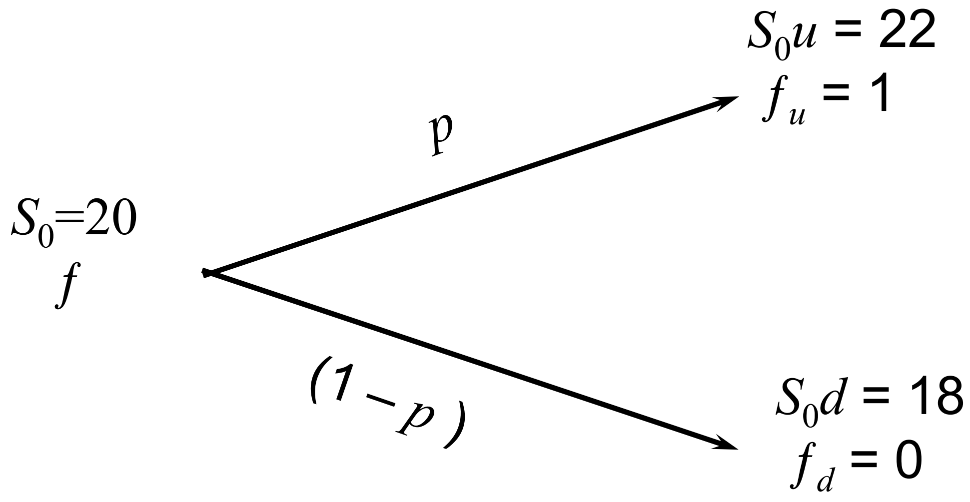
$$\mathbf{S_0e^{rT}} = \frac{\mathbf{e^{rT} - d}}{\mathbf{u - d}}\mathbf{S_0u} + \left(\mathbf{1} - \frac{\mathbf{e^{rT} - d}}{\mathbf{u - d}}\right)\mathbf{S_0d}$$

This shows that the stock price grows at the risk-free rate with these probabilities.

Binomial trees illustrate the general result that to value a derivative we can assume that the expected return on the underlying asset is the risk-free rate and discount at the risk-free rate

This is known as using risk-neutral valuation

Example 2 Revisited. Risk Neutral Valuation



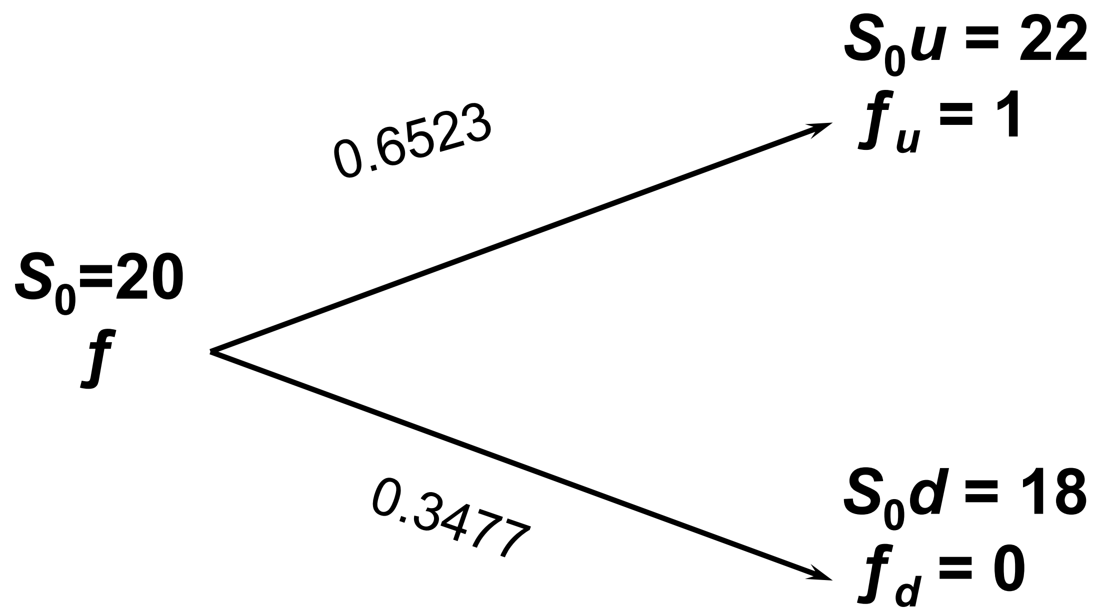
p is the probability that gives a return on the stock equal to the risk-free rate:

$$20e^{0.12\times 0.25} = 22p + 18(1-p) \text{ so that } p = 0.6523$$

Alternatively:

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12\times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Valuing the Option Using Risk-Neutral Valuation



The value of the option is

$$e^{-0.12\times0.25}\,(0.6523\times1 + 0.3477\times0) \;=\; 0.633$$

Irrelevance of Stock Expected Return

When we are valuing an option in terms of the price of the underlying asset, the probability of up and down movements in the real world are irrelevant

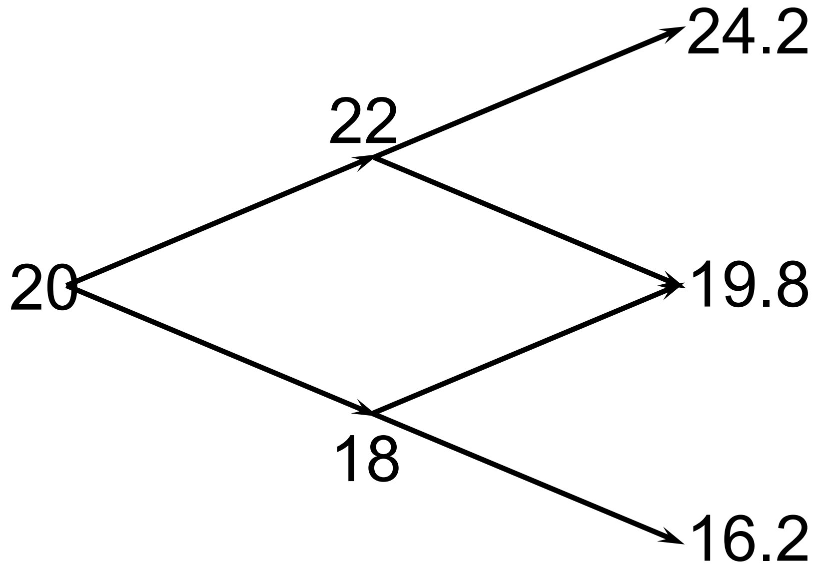
This is an example of a more general result stating that the expected return on the underlying asset in the real world is irrelevant.

For Risk Neutral valuation probabilities of the up and down move are just abstract numbers that allow us to value derivative consistently with the no-arbitrage principle.

A Two-Step Example

Call with strike K=21, r = 12% continuously compounded

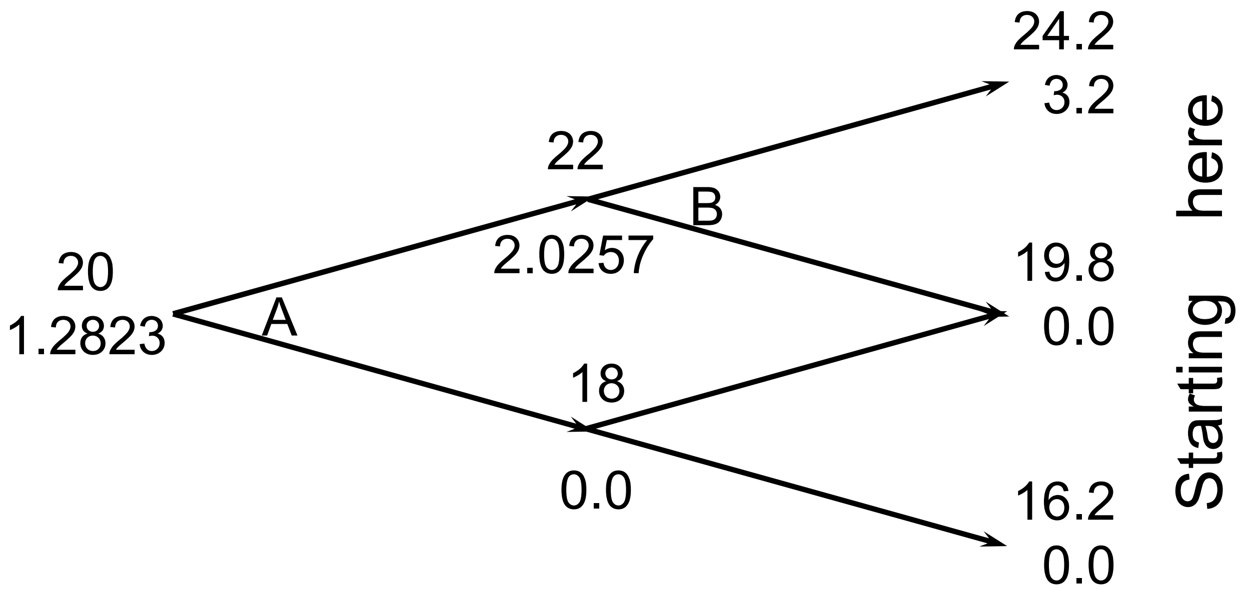
Each time step is 3 months=0.25 year. 2 time steps.



u=1.1, d=0.9

$$\mathbf{p} = \frac{\mathbf{e}^{\mathbf{rT}} - \mathbf{d}}{\mathbf{u} - \mathbf{d}} = \frac{\mathbf{e}^{\mathbf{0.12} \times \mathbf{0.25}} - \mathbf{0.9}}{\mathbf{1.1} - \mathbf{0.9}} = \mathbf{0.6523}$$

Valuing a call option using risk neutral probabilities.
Starting at the end time nodes



Value at node B

$$f_B = e^{-0.12\times0.25}(0.6523\times3.2 + 0.3477\times0) = 2.0257$$

Value at node A

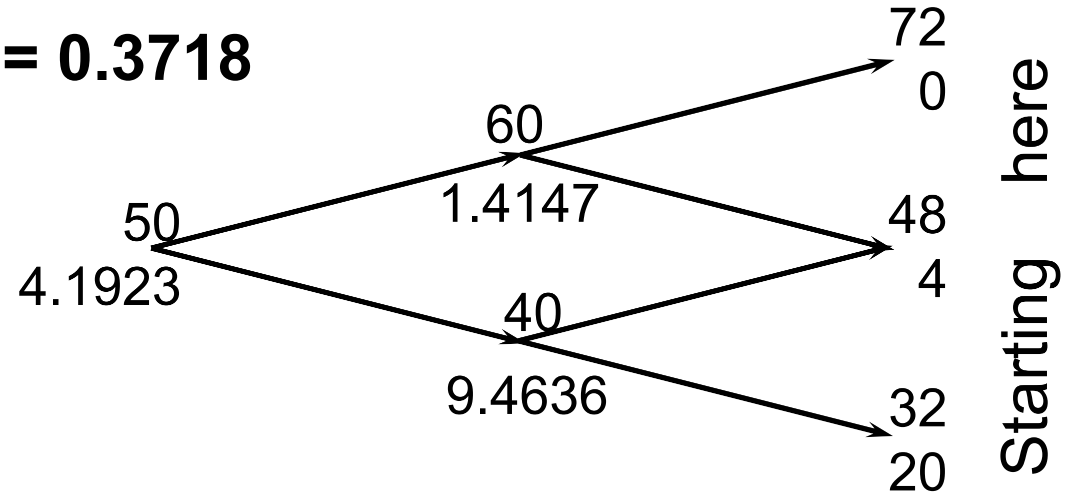
$$f = f_A = e^{-0.12\times0.25}(0.6523\times2.0257 + 0.3477\times0) = 1.2823$$

A Put Option Example from Hull p.283

A 2 year European put with strike 52. *K* = 52, time step T =1yr, *r* = 5%, *u* =1.2, *d* = 0.8, f put price.

$$\mathbf{p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05 \times 1} - 0.8}{1.2 - 0.8} = 0.6282}$$

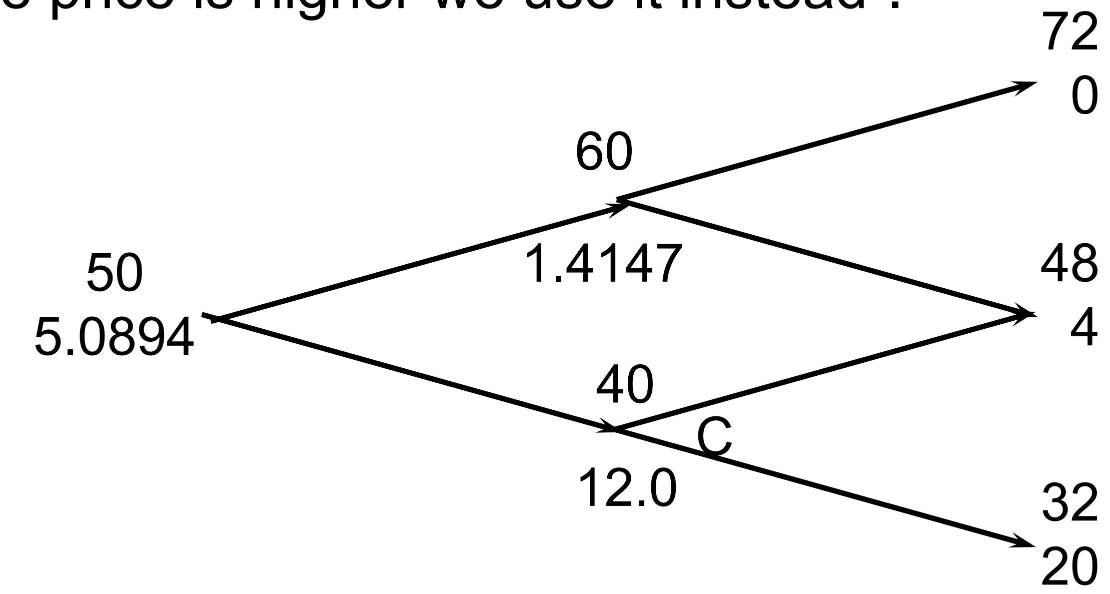
$$\mathbf{1 - p = 0.3718}$$



$$\mathbf{f} = e^{-0.05\times 2}(0.6282^2\times 0 + 2\times 0.6282\times 0.3718\times 4 + 0.3718^2\times 20) = 4.1923$$

What Happens When the Put Option is American

At each node (like node C) we must compare the binomial price calculated from previous 2 nodes to early exercise price. If the early exercise price is higher we use it instead .



At the node C binomial price calculated from the previous 2 nodes is 9.4636
Early exercise price is 12 = 52 - 40. (Stock price S=40 strike K=52)

The American feature increases the value at node C from 9.4636 to 12.0

This increases the value of the option from 4.1923 to 5.0894.

Delta

Delta Δ is the ratio of the change in the price of a stock option to the change in the price of the underlying stock

The value of Δ varies from node to node

Similarly we can calculate Δ at each node

Choosing u and d to match volatility

One way of matching the volatility in the tree is to set

$$\mathbf{u} = \mathbf{e}^{\sigma\sqrt{\Delta \mathbf{t}}}$$

$$\mathbf{d} = \mathbf{1}/\mathbf{u} = \mathbf{e}^{-\sigma\sqrt{\Delta \mathbf{t}}}$$

where σ is the volatility and Δt is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein

One of the important observable characteristics of stocks describing their variability and risk is Annualized Volatility also denoted by σ.

Calculated from history using past daily stock returns.

$$\sigma = \mathbf{Stdev(r_1, r_2, ..., r_n)} \cdot \sqrt{\mathbf{250}}$$

It is expressed in percent, for example 30% or 0.3.
Typical values of volatility are 10%-40% with major us equity index S&P500 having volatility 15-20%

It is proved that we can measure volatility in the real world and use it to build a tree for the an asset in the risk-neutral world

Assets Other than Non-Dividend Paying Stocks

For options on stock indices, currencies and futures the basic procedure for constructing the tree is the same except for the calculation of p

The Probability of an Up Move

$$p = \frac{a-d}{u-d}$$

$a = e^{r\Delta t}$ for a nondividend paying stock

$a = e^{(r-q)\Delta t}$ for a stock index where q is the dividend yield on the index

$a = e^{(r-r_f)\Delta t}$ for a currency where r_f is the foreign risk - free rate

$a = 1$ for a futures contract

Proving Black-Scholes-Merton from Binomial Trees (Hull Appendix to Chapter 13)

$$c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K,\ 0)$$

Option is in the money when $j > \alpha$ where

$$\alpha = \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$$

so that

$$c = e^{-rT}(S_0 U_1 - K U_2)$$

where

$$U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j} u^j d^{n-j}$$

$$U_2 = \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j}$$

The expression for U_1 can be written

$$\mathbf{U_1} = [\mathbf{pu} + (\mathbf{1} - \mathbf{p})\mathbf{d}]^{\mathbf{n}} \sum_{j>\alpha} \frac{\mathbf{n!}}{(\mathbf{n}-\mathbf{j})!\,\mathbf{j}!} \left(\mathbf{p}^{*}\right)^{\mathbf{j}} \left(\mathbf{1} - \mathbf{p}^{*}\right)^{\mathbf{n}-\mathbf{j}} = \mathbf{e}^{\mathbf{rT}} \sum_{j>\alpha} \frac{\mathbf{n!}}{(\mathbf{n}-\mathbf{j})!\,\mathbf{j}!} \left(\mathbf{p}^{*}\right)^{\mathbf{j}} \left(\mathbf{1} - \mathbf{p}^{*}\right)^{\mathbf{n}-\mathbf{j}}$$

where $$\mathbf{p}^{*} = \frac{\mathbf{pu}}{\mathbf{pu} + (\mathbf{1} - \mathbf{p})\mathbf{d}}$$

Both U_1 and U_2 can now be evaluated in terms of the cumulative binomial distribution

We now let the number of time steps tend to infinity and use the result that a binomial distribution tends to a normal distribution