Homework 2 -- Monday January 30 2023

Due Monday February 6 2023 **Total 100 points**

Problem 1. Risk Diminishment by Hedging **[20 points]**

Suppose the market consists of N stocks each of current market value S_i, and that there is an independent traded index **M** that represent the entire market and is correlated with every individual stock, and a riskless bond **B**. Assume the following stochastic processes:

$$\frac{dM}{M} = \mu_M dt + \sigma_M dZ_M \qquad \text{Eq. [1]}$$

$$\frac{dS_i}{S_i} = \mu_i dt + \sigma_i dZ_i$$
$$\frac{dB}{B} = r dt \qquad \text{Eq. [2]}$$

where the correlation between each stock's innovations and the market innovations is ρ_{iM} so that

$$dZ_i = \rho_{iM} dZ_M + \sqrt{1-\rho_{iM}^2}\varepsilon_i \qquad \text{Eq. [3]}$$

Here ε_i are random normal variables that represents the residual risk of each stock i, and are assumed to be uncorrelated with dZ_M and with each other. We assume that both $\varepsilon_i^2 = dt$ and $dZ_M^2 = dt$, so that $dZ_i^2 = dt$ and $dZ_i dZ_M = \rho_{iM} dt$.

For each stock S_i define a market-neutral version of the stock as $\tilde{S}_i = S_i - \Delta_i M$ Eq. [4]

where market-neutral means that its market risk is totally hedged away, so that its stochastic evolution has no dependence on dZ_m.

(i) Show that $\Delta_i = \beta_{im}\frac{S_i}{M}$ where $\beta_{im} = \frac{\rho_{iM}\sigma_i\sigma_M}{\sigma_M^2} = \frac{\sigma_{iM}}{\sigma_M^2}$ [10 points]

(ii) Show that $$\frac{d\tilde{S}_i}{\tilde{S}_i} = \tilde{\mu}_i dt + \tilde{\sigma}_i \varepsilon_i$$

where [10 points]

$$\tilde{\mu}_i = \frac{\mu_i - \beta_{iM}\mu_M}{1 - \beta_{iM}}$$

$$\tilde{\sigma}_i = \frac{\sigma_i\sqrt{1 - \rho_{iM}^2}}{1 - \beta_{iM}}$$

Solution 1.

Suppose the market consists of N stocks each of current market value S_i. Suppose further that there is a traded index M that represents the entire market. Assume that the price of M evolves lognormally according to the standard Wiener process

$$\boxed{\frac{dM}{M} = \mu_M dt + \sigma_M dZ_M} \qquad \text{Eq. [5]}$$

where μ_M is the expected return of M and σ_M is its volatility. We still assume that the prices of any stock S_i and the price of the riskless bond B evolve according to the equations

$$\boxed{\begin{aligned} \frac{dS_i}{S_i} &= \mu_i dt + \sigma_i dZ_i \\ \frac{dB}{B} &= r dt \end{aligned}} \qquad \text{Eq. [6]}$$

where

$$\boxed{dZ_i = \rho_{iM} dZ_M + \sqrt{1-\rho_{iM}^2}\varepsilon_i} \qquad \text{Eq. [7]}$$

Here ε_i is a random normal variable that represents the residual risk of stock *i*, uncorrelated with $\boxed{dZ_M}$. We assume that both $\boxed{\varepsilon_i^2 = dt}$ and $\boxed{dZ_M^2 = dt}$, so that $\boxed{dZ_i^2 = dt}$ and $\boxed{dZ_i dZ_M = \rho_{iM} dt}$.

Because all stocks are correlated with the market index M, you can create a market-neutral version of each stock S_i by shorting just enough of M to remove all market risk. Let $\boxed{\tilde{S}_i}$ denote the value of the market-neutral portfolio corresponding to the stock S_i, namely

$$\boxed{\tilde{S}_i = S_i - \Delta_i M} \qquad \text{Eq. [8]}$$

From Equations 5 – 8, the evolution of $\boxed{\tilde{S}_i}$ is given by

$$\boxed{\begin{aligned} d\tilde{S}_i &= dS_i - \Delta_i dM \\ &= S_i(\mu_i dt + \sigma_i dZ_i) - \Delta_i M(\mu_M dt + \sigma_M dZ_M) \\ &= \mu_i S_i dt + \sigma_i S_i(\rho_{iM} dZ_M + \sqrt{1-\rho_{iM}^2}\varepsilon_i) - \Delta_i M(\mu_M dt + \sigma_M dZ_M) \\ &= (\mu_i S_i - \Delta_i \mu_M M)dt + (\rho_{iM}\sigma_i S_i - \Delta_i \sigma_M M)dZ_M + \sigma_i S_i \sqrt{1-\rho_{iM}^2}\varepsilon_i \end{aligned}} \qquad \text{Eq. [9]}$$

We can eliminate all of the risk of $\tilde{S}_i$ with respect to market moves dZ_m by choosing $\rho_{iM}\sigma_i S_i - \Delta_i \sigma_M M = 0$, so that the short position in M at any instant is given by

$$\Delta_i = \frac{\rho_{iM}\sigma_i S_i}{\sigma_M M} = \frac{\rho_{iM}\sigma_i \sigma_M S_i}{\sigma_M^2 M} = \beta_{im}\frac{S_i}{M} \qquad \text{Eq. [10]}$$

where

$$\beta_{im} = \frac{\rho_{iM}\sigma_i \sigma_M}{\sigma_M^2} = \frac{\sigma_{iM}}{\sigma_M^2} \qquad \text{Eq. [11]}$$

is the traditional beta, the ratio of the covariance σ_{iM} of stock *i* with the market to the variance of the market σ_M^2.

By substituting the value of Δ_i in Equation 10 into Equation 8 one finds that the value of the market-neutral version of S_i is

$$\tilde{S}_i = (1 - \beta_{iM})S_i \qquad \text{Eq. [12]}$$

By using the same value of Δ_i in the last line of Equation 9 one can write the evolution of $\tilde{S}_i$ as

$$\frac{d\tilde{S}_i}{\tilde{S}_i} = \tilde{\mu}_i dt + \tilde{\sigma}_i \varepsilon_i \qquad \text{Eq. [13]}$$

where

$$\begin{aligned} \tilde{\mu}_i &= \frac{\mu_i - \beta_{iM}\mu_M}{1 - \beta_{iM}} \\ \tilde{\sigma}_i &= \frac{\sigma_i\sqrt{1 - \rho_{iM}^2}}{1 - \beta_{iM}} \end{aligned} \qquad \text{Eq. [14]}$$

Problem 2: Exotic Log Contract **[20 points]**

Assume geometric Brownian motion for the stock **S** and the Black-Scholes formalism for options on **S.** Assume interest rates and dividend yields are zero.

A client of your bank thinks volatility is going to get higher as the market rises, and so wants you to sell him a European-style exotic derivative EL(S) constructed so its volatility sensitivity increases linearly with stock price *S*, so that it is relatively insensitive to volatility at low stock prices and very sensitive at high ones.

(i) What density $\rho(K)$ of calls and/or puts with strike *K* must be chosen to replicate EL(S)? [10]

(ii) Show that they payoff of the contract at expiration *T* is

$$EL(S_T, S^*) = S_T \ln\frac{S_T}{S^*} + S^* - S_T$$

where S^* is a strike level above which we use only calls and below which we use only puts in the replication. [5]

(iii) Show that at an earlier time *t* when the stock price is *S*, the value of *EL* given by

$$EL(S, S^*, t, T) = S\left(\ln\frac{S}{S^*} + \sigma^2\frac{(T-t)}{2}\right) + S^* - S$$

satisfies the Black-Scholes PDE for zero interest rates and zero dividend yield. [5]

Solution 2. Exotic Log Contract

(i) Following the method we used in class for the log contract, the volatility exposure of a call or put is related to $\boxed{\kappa}$, which is a function $\boxed{Sf\left(\frac{K}{S}, \sigma\right)}$. We can then write the volatility exposure of the derivative *L* that has been replicated with a density $\boxed{\rho(K)}$ of calls or puts as

$$\boxed{\frac{\partial}{\partial\sigma}EL = \int_0^\infty Sf\left(\frac{K}{S}, \sigma\right)\rho(K)dK}$$

You can rewrite this by changing variables from K to x = K/S

$$\boxed{\frac{\partial}{\partial\sigma}EL = \int_0^\infty S^2 f(x, \sigma)\rho(xS)dx}$$

Then this will grow linearly with S if $\boxed{\rho(K) = K^{-1}}$.

(ii) Thus the payoff of EL at expiration replicated out of puts and calls, where $\boxed{S_T}$ is the stock price at expiration, is

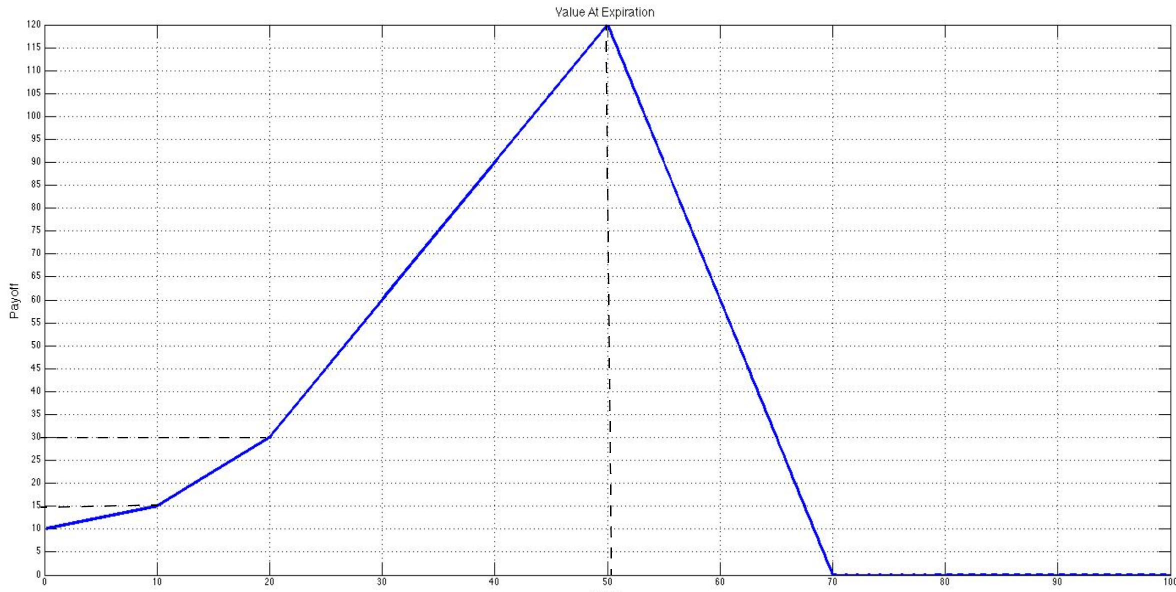
$$\boxed{EL(S) = \underset{\text{if } S_T < S^*}{\left(\int_{S_T}^{S^*}(K - S_T)\frac{dK}{K}\right)} \text{ or } \underset{\text{if } S_T > S^*}{\left(\int_{S^*}^{S_T}(S_T - K)\frac{dK}{K}\right)} = S_T \ln\frac{S_T}{S^*} + S^* - S_T}$$

(iii)

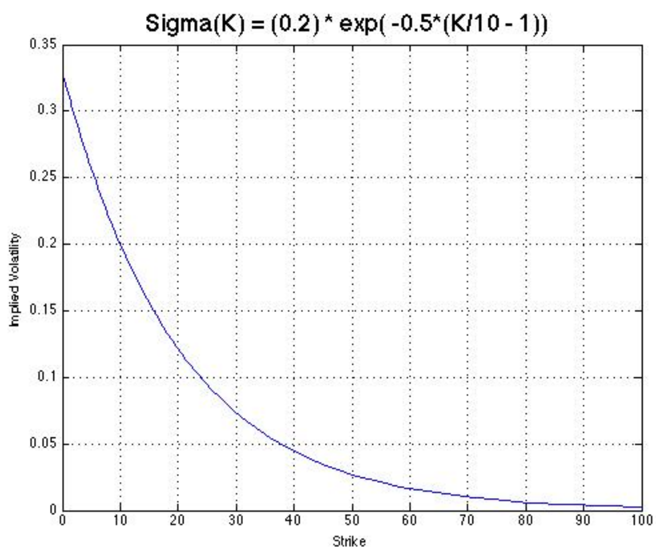
$$\begin{aligned}
EL(S, S^*, t, T) &= S\left(\ln\frac{S}{S^*} + \sigma^2\frac{(T-t)}{2}\right) + S^* - S \\
\frac{\partial}{\partial t}EL &= -S\frac{\sigma^2}{2} \\
\frac{\partial}{\partial S}EL &= \ln\frac{S}{S^*} + \sigma^2\frac{(T-t)}{2} + 1 - 1 = \ln\frac{S}{S^*} + \sigma^2\frac{(T-t)}{2} \\
\frac{\partial^2}{\partial S^2}EL &= \frac{1}{S} \\
\frac{(\sigma S)^2}{2}\frac{\partial^2}{\partial S^2}EL + \frac{\partial}{\partial t}EL &= \frac{(\sigma S)^2}{2}\frac{1}{S} - -S\frac{\sigma^2}{2} = 0
\end{aligned}$$

Problem 3: Valuation With a Skew **[20 points]**

Shown below is the European payoff of an exotic option expiring 5 years from today. The x-axis represents the final stock price. The payoff is zero everywhere above $70. The stock pays zero dividends, and the five-year riskless interest rate is 5% per year compounded annually.



Here is today's implied volatility smile for 5-year options on the underlying stock. The volatilities are quoted annually.

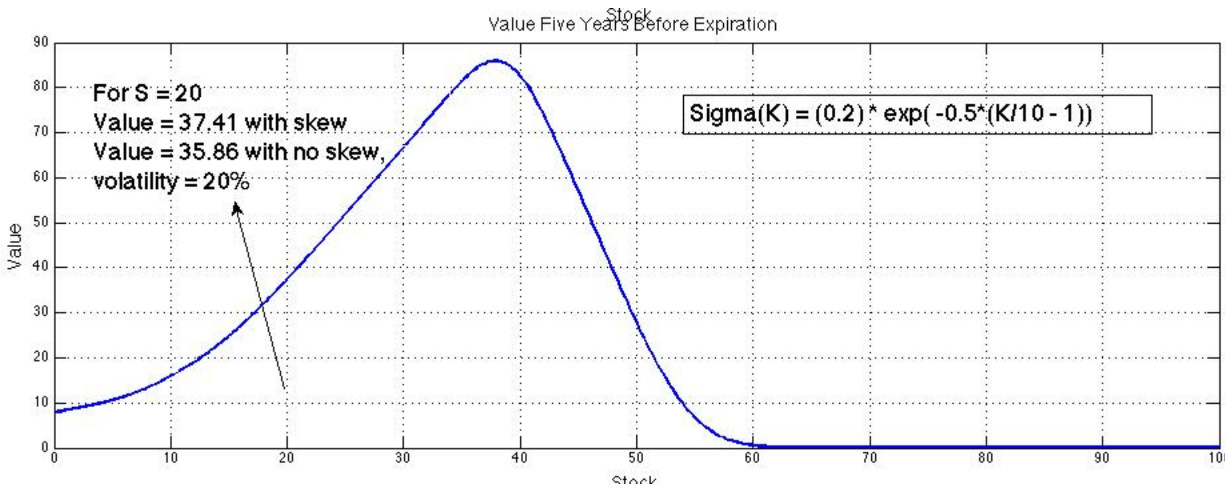


(i) Find the value of the exotic option today when the stock price is $20. [10 points]

(ii) Find the value that the exotic option would have today if there is no skew and all current implied volatilities were 20%. [10 points]

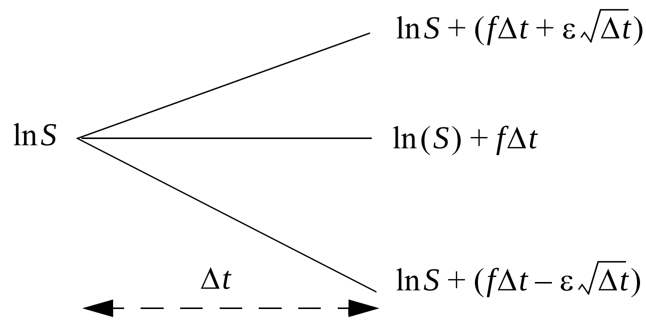
Solution 3: You can replicate the payoff of the option with the following portfolio: Long 1 Zero Coupon bond with face 10 and maturity 5 years.
0.5 shares stock
1 call on the stock struck at K = 10 and expiration in 5 years
1.5 calls on the stock struck at K = 20 and expiration in 5 years
-9 calls on the stock struck at K = 50 and expiration in 5 years
6 calls struck at K = 70 and expiration in 5 years

	With skew:	Without skew, all vols = 20%
S = 20		
Zero coupon bond =	7.84	7.84
1/2 share of stock =	10	10
1 call struck at 10 =	12.2	12.2
1.5 calls struck at 20 =	1.5 * 4.88	1.5 * 5.83
-9 calls struck at 50 =	0	-9 * 0.37
6 calls struck at 70 =	0	6 * 0.06
	sum =37.4	**35.9**



Problem 4. **[20 points]**

Consider a stock whose log price evolves trinomially over time Δt:



where each branch of the tree has equal probability 1/3.

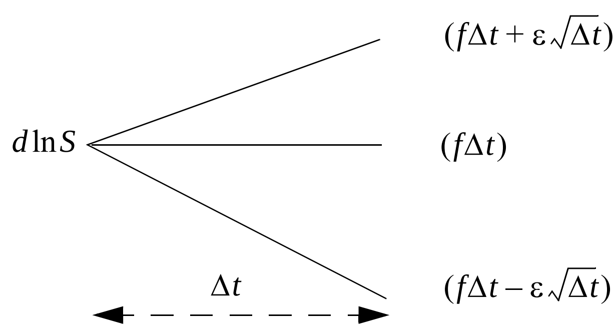
In the continuum limit as Δt vanishes, one can write the stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dZ$$

Find μ and σ in terms of f and ε.

Solution 4.

The trinomial step for $\boxed{d\ln S}$ is



From the branches, the expected value $\boxed{E[d\ln S] = f\Delta t}$ and $\boxed{var[d\ln S] = \frac{2}{3}\varepsilon^2\Delta t}$ so that, since $\boxed{dZ^2 = dt}$ we can write $\boxed{d\ln S = f\Delta t + \sqrt{\frac{2}{3}}\varepsilon dZ}$ where $\boxed{\Delta t = dt}$ in the continuum limit, and $\boxed{\sigma = \sqrt{\frac{2}{3}}\varepsilon}$

Then, by Ito's lemma, since $\boxed{d\ln S = \frac{1}{S}dS - \frac{1}{2}\frac{1}{S^2}\sigma^2 S^2 dt}$, we have that

$$\boxed{\frac{dS}{S} = d\ln S + \frac{\sigma^2}{2}dt = \left(f + \frac{1}{3}\varepsilon^2\right)dt + \sqrt{\frac{2}{3}}\varepsilon dZ}$$

Problem 5: Symmetry of the BS equation **[10 points]**

If $C(S,t)$ satisfies the BS equation with zero interest rates, show that so $S\frac{\partial C}{\partial S}$ satisfies it too. (Then, of course, $S\frac{\partial}{\partial S}$ applied repeatedly to BS solutions generates further solutions.)

Solution 6: Symmetry of the BS equation

The simplest way to do it is as follows.

The BS equation for zero rates is $\boxed{\frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} + \frac{\partial C}{\partial t} = 0}$.

Define $\boxed{V = SC_S}$ where the subscript S denotes $\boxed{\frac{\partial}{\partial S}}$

Then the B-S equation for V is $\boxed{\frac{\sigma^2 S^2}{2}V_{SS} + V_t = 0}$, which is what we want to prove holds.

Substituting $\boxed{V = SC_S}$ we obtain

$$\boxed{\begin{aligned} V_t &= SC_{St} \\ V_S &= C_S + SC_{SS} \\ V_{SS} &= 2C_{SS} + SC_{SSS} \end{aligned}}$$

Then the B-S equation for V rewritten in terms of C is

$$\boxed{\frac{\sigma^2 S^2}{2}[2C_{SS} + SC_{SSS}] + SC_{St} = 0}$$ and this is what we want to prove.

Divide this equation by S to obtain:

$$\boxed{\frac{\sigma^2}{2}[2SC_{SS} + S^2 C_{SSS}] + C_{St} = 0}$$

which you can write as

$\boxed{\frac{\partial}{\partial S}\left[\frac{\sigma^2 S^2}{2} C_{SS} + C_t\right] = 0}$ and this is true because the expression in the square bracket is equal to zero because it is the BS equation for C itself.

Problem 6: Dominance **[10 points]**

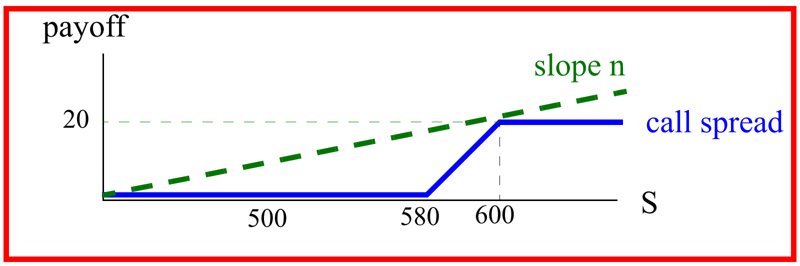
Suppose Google's current price is $500. You work on a trading desk and an investor wants to buy a European call spread on Google from you, i.e. they want to buy a one-year call struck at 580 and sell a one-year call struck at 600. He is betting that Google will end up above 580 but not much more than 600. You sell it to them. How many shares of Google stock must you statically own to guarantee that, if you sell them, you'll have enough money to pay off the investor one year from now, no matter how the stock price behaves? (The answer can be a fractional number of shares.)

Solution 6: Google

[10 points]

The call spread has the payoff of the solid line shown on the right.

If you own *n* shares of stock whose payoff is shown by the dashed line, then it will dominate the call spread for all stock prices provided



$\boxed{n \times 600 = 20}$ so the number of shares you need is 1/30.