Chapter 3: Fixed Income

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Introduction

- When you own a share of a stock, you have partial ownership in that company, which means you share in both the profits and losses (but not the liabilities). Nothing is guaranteed.
- When you buy a bond, you make a loan to the company. The corporation is obligated (unless it defaults...) to pay back the principal amount, called **par**, after a number of years (called **maturity**) specified by the loan, plus periodic payments called **coupon**
- (Coupons are similar to dividends paid by stocks, except that coupons are specified contractually whereas stock dividends are at the discretion of the company's management)
- You will receive a fixed stream of income
- Bonds are called fixed-income securities

Zero Coupon Bonds (3.2)

- Zero-coupon bonds pay no principal or interest until maturity
- The par value (face value) is the payment made to the bond holder at maturity
- A zero-coupon bond sells for less than par (if its yield is positive...)
- Any bond selling for less than par is called a **discount bond**
- A zero-coupon bond is also called a pure discount bond

Borrowers, Maturity

- Borrowers (a.k.a. issuers) can be corporations or governments (e.g., US Treasury, and the bonds it issues are called “Treasuries”
- Bonds might appear risk-free but actually this is not true
- Obviously, the issuer (borrower) may default on their payment. This is called **credit risk**
- The interest that lenders demand on top of prevailing interest rates to be compensated for that additional risk is called **credit spread**

Present Value

- The pricing of a bond (and any financial instrument, for that matter) is based on a simple principle called present valuation
- For a promised payment in the future, what amount should I be willing to pay today?
- One way to answer that question is to look at an alternative: Given prevailing interest rates, what amount would I need to invest today so that, after compounding, I get the same payment at the same future date?

Present Value

- We saw that the gross return equals value at the end of a period divided by value at the beginning, and that the gross return equals 1 + the net return
- Since x dollars today become $x(1+r)$ after one year if invested with annualized interest rate r, it follows that the value today (the **present value**) of 1,000 dollars to be received in one year is

$$\frac{1000}{1+r} = 1000 \times (1+r)^{-1}$$

- The rate used in present valuation is called **discount rate**
- The discount factor $(1+r)^{-1}$ is in general a **function** of time (implicitly one year in example above)
- The discount function depends today on the remaining time to maturity T, and we will denote it $D(T)$
- Note that the discount rate is the risk-free rate, or the risk-free rate plus the credit spread discussed earlier

Zero Coupon Bonds

- We just said that the value today (the **present value**) of 1,000 dollars to be received in one year is $\frac{1000}{1+r}$
- **General formula:** The price of a zero coupon bond is given by

$$\text{PRICE} = \frac{\text{PAR}}{(1+r)^T} = \text{PAR}(1+r)^{-T}$$

if we assume that T is the time to maturity in years

- The annual rate r such that the discounted PAR of a zero equals its current market price is called the **yield to maturity** (YTM) of that bond.

PV of a Zero Coupon Bond (3.2)

- For a 10 year zero coupon bond with a face value of \$1000 and interest rate at r=4%, the (fair) market price (present value, PV) is

$$\frac{1000}{(1+0.04)^{10}}$$

  if the interest is compounded annually

- The price of a 20-year zero with par value of \$1000 when the interest rate is r =6% and compounded annually is

$$\frac{1000}{(1+r)^{20}} = \frac{1000}{1.06^{20}} = 311.80$$

Credit Spread

- All the above (and what follows) assumes no risk of default
- But assume the same 20-year zero sells for \$290. What is its credit spread c?

$$\frac{1000}{(1+(r+c))^{20}} = 290$$

- There is a lot of research (incl. math and statistics research) to link c to the **probability of default** of the issuer and the **recovery rate**

Semi-Annual Compounding (3.2)

- If we assume the interest rate r is per year with semi-annual compounding, the price is

$$\text{PRICE} = \text{PAR}(1 + r/2)^{-2T}$$

- For a T-year zero coupon bond with interest compounded n times a year, the is price is

$$\text{Price} = \text{PAR}(1 + r/n)^{-nT}$$

- If the interest rate is r =6% and is compounded every six months, the present value of the 20-year bond is:

$$\frac{1000}{(1 + r/2)^{40}} = \frac{1000}{1.03^{40}} = 306.56$$

Continuous Compounding (3.2)

- If interest rate $r = 6\%$ is compounded continuously, the present value of the 20-year bond is:

$$\frac{1000}{e^{20r}} = \frac{1000}{e^{20(0.06)}} = 301.19$$

Sensitivity of the Price of a Zero to Rate Increases (3.2.1)

- Assume you just bought for \$306.56 a 20-year zero with a face value of \$1000
- You assumed semi-annual compounding, which gave you an interest rate of 6%
- Six months later the interest rate increased to 7% . The present value is now:
$$\frac{1000}{1.035^{39}} = 261.41$$
- The value of your investment dropped by $306.56 - 261.41 = 45.15$
- You will get \$1000 if you keep the bond for another 19.5 years. However, if you sell it now, you lose \$45.15, a return of
$$\frac{-45.15}{306.56} = -14.73\%$$
for a half-year or -29.46% per year
- **The value of a bond decreases when interest rates increase**

Sensitivity of the Price of a Zero to Interest Rate Drops (3.2.1)

- assume semi-annual compounding
- you just bought the zero for \$306.56
- six months later the interest rate decreased to 5%. The present value now is:
$$\frac{1000}{1.025^{39}} = 381.74$$
- The value of your investment went up by $381.74 - 306.56 = 75.18$
- You will get \$1000 if you keep the bond for another 19.5 years. if you sell it now, you make a \$75.18 profit, a return of
$$\frac{75.18}{306.56} = 24.5\%$$
for a half-year or 49% per year (using the bond convention)
- **The value of a bond increases when interest rates decrease**

Coupon Bonds (3.3)

Most bonds make regular and fixed interest payments (**coupons**)

- Consider a 20-year bond with a par value of \$1000 and 6% annual **coupon rate** with semi-annual coupon payments
- This means that each coupon payment will be $6\%/2 \times \$1000 = \$30$
- The bond holder receives 40 payments of \$30..
- ..plus the par value, \$1000, after 20 years
- The ratio of dividend divided by par is called **coupon rate** or coupon yield
- The ratio of dividend divided by the current price of the bond is called **current yield**

The equivalence, also called replication, of coupon Bonds and a portfolio of zeroes is a general principal in finance that allows to price financial instruments based on others

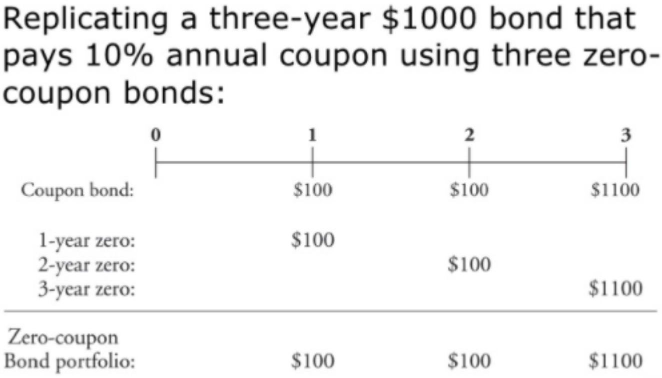


Figure: Source: Pearson Prentice Hall

Present Value of Coupon Bonds

- Each cash flow gets discounted, depending on the number of periods in the future that it will be paid out
- We assume a constant interest rate r, per half year
- The PV of the first coupon of C dollars is $\frac{C}{1+r}$, that of the second coupon is $\frac{C}{(1+r)^2}$, etc
- The general formula is given by

$$\text{bond price} = \sum_{t=1}^{2T} \frac{C}{(1+r)^t} + \frac{PAR}{(1+r)^{2T}}$$

where PAR = par value and T = maturity (in years)

The textbook also provides a very unusual expression for the price of a bond:

$$\text{bond price} \quad = \quad \frac{C}{r} + \left\{ \text{PAR} - \frac{C}{r} \right\} (1+r)^{-2T}$$

This formula is derived using the fact that, for $a \neq 1$:

$$1 + a + a^2 + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a}.$$

where $a = 1/(1+r)$
This formula is not used much in practice – practitioners use specialized software anyway. But most importantly... **Here's a piece of advice to survive in the financial industry: An inelegant formula that everyone knows is always preferable to a cute one that most people have never heard of.** That's in part because you'll have to spend time to prove the correctness of your equation to all parties.

Discounting at the Coupon Rate

- Assume discounting at the coupon rate
- The present value of all payments, when discounting at the 6% annual rate (3% semi-annual), equals \$1000:

$$\sum_{t=1}^{40} \frac{30}{1.03^t} + \frac{1000}{1.03^{40}} = 1000$$

- In other words, if the prevailing interest rate (which is used to calculate present values) equals the coupon yield, and you discount semi-annually, the fair value of the bond **has to** equal the par amount

Discounting at the Coupon Rate

- After 6 months, if the discount rate is unchanged, then the bond (including the first coupon payment, which is now due) is worth

$$\sum_{t=0}^{39} \frac{30}{1.03^t} + \frac{1000}{1.03^{39}} = 1030$$

- Above, the expression still has 40 coupons, but the first time is zero because the first coupon is due now
- We can easily verify that the expression above equals:

$$\left(\sum_{t=1}^{40} \frac{30}{1.03^t} + \frac{1000}{1.03^{40}} \right) \times 1.03 = 1030$$

- Think for a minute about the consistency of these results: At bond issuance, a $30 coupon was due in 6 months, and was discounted like all the other payments. But it's due now, so it's not discounted
- One instant after the coupon is paid, 39 coupons remain due. With the same discount, the present value comes back to $1,000

Sensitivity of Present Value to Interest Rate Increases

- Assume the interest (discount) rate increases to 7% six months after the bond is issued
- Then, after six months, the bond is worth

$$\sum_{t=0}^{39} \frac{30}{1.035^t} + \frac{1000}{1.035^{39}} = 924.48$$

and the annual return is

$$2\left(\frac{924.48 - 1000}{1000}\right) = -15.1\%$$

- **The value of a bond decreases when interest rates increase**

Sensitivity of Present Value to a Drop in Interest Rate

- If the interest rate drops to 5% after six months, then the investment is worth

$$\sum_{t=0}^{39} \frac{30}{1.025^t} + \frac{1000}{1.025^{39}} = 1153.70$$

which results in an annual return of:

$$2\left(\frac{1153.70 - 1000}{1000}\right) = 30.7\%$$

- **The value of a bond increases when interest rates decrease**

Yield to Maturity (3.4)

- Suppose a bond with remaining time to maturity $T = 30$ years, and we assume annual compounding.
- The annual coupon is $C = 40\$$ and PAR$= 1000$
- Let's assume the current price is \$1200, \$200 above the par value. (At maturity, you will get only \$1000 of \$1200 invested.)
- We saw that $40 \ / \ 1000 = 4\%$ is the **coupon rate**
- But you're getting \$40 for \$1200 invested, or a **current yield** of 3.33% per year
- The **yield to maturity** (YTM) is the one discount rate that makes the present value equal to the actual price:

$$\sum_{t=1}^{30} \frac{40}{(1+YTM)^t} + \frac{1000}{(1+YTM)^{30}} = \text{current price}$$

- In this case the YTM is 0.0324 per half year, or 6.48% annualized

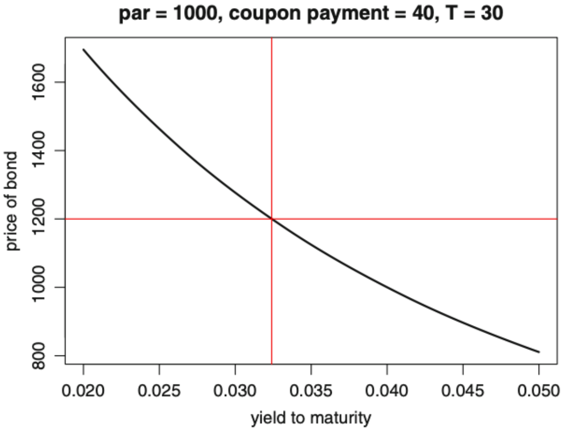
General Formula (3.4.2)

- Suppose that a coupon bond pays **semi-annual** payments of C, has par value of PAR, has T years until maturity
- Suppose $r_1, r_2, \ldots, r_{2T}$, are the half-year rate for 1/2, 1, 1.5, ..., T years
- The yield to maturity is the value of y that satisfies

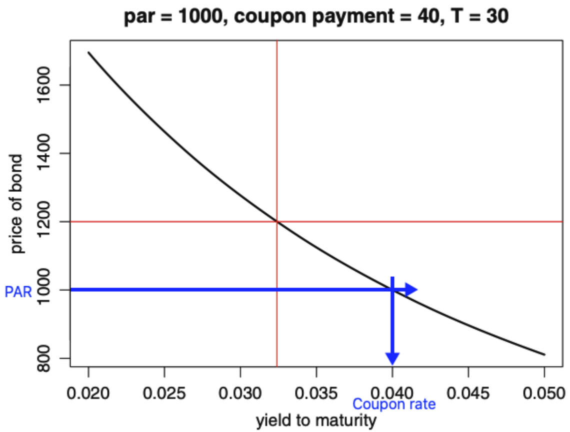
$$\frac{C}{1+r_1} + \frac{C}{(1+r_2)^2} + \ldots + \frac{C}{(1+r_{2T-1})^{2T-1}} + \frac{C}{(1+r_{2T})^{2T}} + \frac{PAR}{(1+r_{2T})^{2T}}$$
$$= \frac{C}{1+y} + \frac{C}{(1+y)^2} + \ldots + \frac{C}{(1+y)^{2T-1}} + \frac{C}{(1+y)^{2T}} + \frac{PAR}{(1+y)^{2T}}$$

Yield to Maturity - Figure 3.1

Figure: Bond price versus yield to maturity. The horizontal red line is at the bond price of $1,200. The price/yield curve intersects this line at 0.0324 as indicated by the vertical red line. Therefore, 0.0324 is the bond's yield to maturity (per half-year)



If price > par, then $\frac{1}{price} < \frac{1}{par}$, so current yield < coupon rate



Comparison of the three rates or yields (3.4)

A key property, illustrated by the previous graph, is that:

$$\textbf{price} > \textbf{par} \Longrightarrow \textbf{coupon rate} > \textbf{current yield}$$

Reciprocally:

$$\textbf{price} < \textbf{par} \Longrightarrow \textbf{coupon rate} < \textbf{current yield}$$

Comparison of the three rates or yields (3.4)

A broader property, illustrated by the previous graphs, is that:

$$\textbf{price} > \textbf{par} \Longrightarrow \textbf{coupon rate} > \textbf{current yield} > \textbf{yield to maturity}$$

Reciprocally:

$$\textbf{price} < \textbf{par} \Longrightarrow \textbf{coupon rate} < \textbf{current yield} < \textbf{yield to maturity}$$

Yield to Maturity – Introducing Spot Rates (3.4.1)

We had see the general expression for the price of a coupon bond paying semi-annually (which is the case of most bonds) is linked to its YTM y:

$$\text{bond price} \quad = \quad \sum_{t=1}^{2T} \frac{C}{(1+y)^t} + \frac{PAR}{(1+y)^{2T}}$$

For a zero-coupon bond, $C = 0$ and the equation above becomes

$$\text{PRICE} \quad = \quad \text{PAR}(1+y)^{-2T}$$

This implies that, for a zero-coupon bond free of credit risk, the yield to maturity is the interest rate, for a borrowing of T years, prevailing today – "on the **spot**"

Spot rates (3.4.2)

- Zero-coupon bonds allow to derive a "pure" or "clean" discount rates / YTMs unencumbered of intermediate coupon payments
- Their YTMs are called **spot rates**
- But interest rates vary depending on the time horizon (maturities), and through time
- More precisely, the YTM of a zero coupon bond of maturity n years is called the ***n*-year spot rate**
- In the textbook, it is denoted as y_n
- The present value (or net present value) of a payment of \$1 to be made n years from now is $\frac{\$1}{(1+y_n)^n}$, assuming an annual rate

YTM vs Spot rates

- But that works only for zeroes
- Otherwise, the YTM of a coupon bond applies to a specific bond: It is the single rate such that, discounting all cash flows over time (coupons and final par payment) at that one single rate, the present value of the bond equals the current market price
- In contrast, spot rates are "universal" but apply today for different periods. There is a market place for these rates
- A coupon bond can be seen as a bundle of zeros, each with a different maturity and therefore discounted at a different spot rate

Spot rates to Price Bonds

Suppose that

- the one-half year spot rate is 5% per year
- the one year spot rate is 6% per year

We can think of a \$40 semi-annual coupon bond as being composed of two zero coupon bonds

- one with $T = 1/2$ and par value of 40
- one with $T = 1$ and par value of 1040
- the price of the bond is the sum of the prices of these two zero coupons and is given by

$$\frac{40}{1+0.025} + \frac{1040}{(1+0.03)^2} = 1019.$$

- the yield to maturity on this coupon is the value of y that satisfies

$$\frac{40}{1+y} + \frac{1040}{(1+y)^2} = 1019.$$

In this case $y = 0.0299$.

YTM vs Spot rates

Example of a 4-yr bond with par $100 and a $5 annual coupon:

Years	0	1	2	3	4
Coupon		5	5	5	5
Par		0	0	0	100
Spot rates		4%	4.30%	4.20%	3.59%
PV of coupon		4.81	4.60	4.42	91.18
Bond price	105.00				

Years	0	1	2	3	4
Coupon		5	5	5	5
Par		0	0	0	100
YTM		3.63%	3.63%	3.63%	3.63%
PV of coupon		4.82	4.66	4.49	91.04
Bond price	105.02				

Spot rates - Example

Other Example: Given the spot rates r_1 equals 8% and r_2 equals 10%, what should a 5%-coupon two-year bond cost? **Assume yearly compounding.**

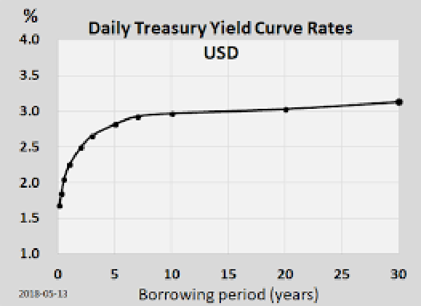
Solution

Solution to the example: Given the spot rates r_1 equals 8% and r_2 equals 10%, what should a 5%-coupon two-year bond cost? Assuming yearly compounding:

$$\text{PRICE} = \frac{50}{1+0.08} + \frac{1050}{(1+0.10)^2} = 914.06$$

Term Structure and Yield Curve (3.5)

- Even if you're guaranteed to be paid back, the longer you lend your money, the higher the interest rate you should expect
- If you plot interest rates offered by one issuer as a function of maturities, the graph is called a **term structure**
- In the special case of interest rates offered by Treasuries, that term structure is called the **yield curve**



Example

The interest rate of a bond depends on the **maturity** of the bond – that is, on how long you have to wait to get your investment back

On March 28, 2001, the interest rate of Treasuries were:

- 4.23% for 3-month bills
- 4.81% on 10-year notes
- 5.46% on 30-year bonds

On March 22, 2006:

- 4.69% on 3-month
- 4.86% on 10-year
- 4.73% on 30-year

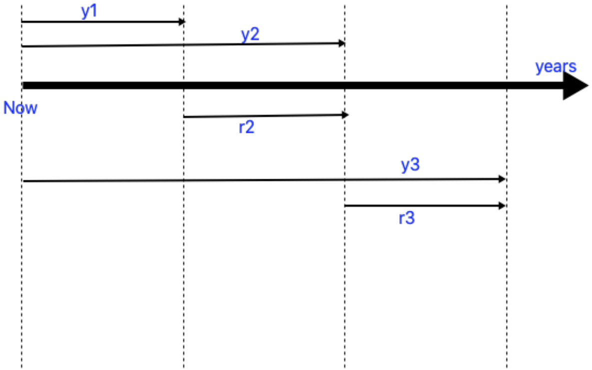
Question: what do you observe? What do you think was happening?

Forward Rates (3.5.2)

The term structure for all maturities up to n years can be described by any one of the following sets:

- **forward rates** $r_1, r_2, \ldots, r_n$ where r_k is the interest that you can lock in now, to borrow (here, assumed for one year) in the k-th year in the future
- With that notation, r_1 is the spot rate for the coming year
- Note that, if you lend money for 2 years, you should be indifferent between receiving an interest rate of y_2 during two years, or r_1 the first year then r_2 the second year
- This is a key principle in finance, called the arbitrage argument
- It follows that the present value of \$1 is $\frac{\$1}{(1+y_2)^2}$ but is also $\frac{\$1}{(1+r_1)(1+r_2)}$
- Thus $(1+y_2)^2 = (1+r_1)(1+r_2)$

Forward Rates (3.5.2)



Term Structure of Interest Rates (3.5.2)

The term structure for all maturities up to n years can be described by any one of the following sets:

- prices of zero coupon bonds of maturities 1-year, 2-years, $\ldots$, n-years denoted here by $P(1), P(2), \ldots, P(n)$
- **spot rates** (yields of maturity of zero coupon bonds) of maturities 1-year, 2-years, $\ldots$, n-years denoted by $y_1, y_2, \ldots, , y_n$
- **forward rates** $r_1, r_2, \ldots, r_n$ where r_k is the interest that you can lock in now for borrowing in the k-th year in the future

Finding Prices from Forward Rates (3.5.2)

- Consider a \$1000-par zero with a maturity of 2 years
- In one year, its PV (present value, hence its price) will be the par amount discounted at the rate that will prevail **then** for the following year:

$$\frac{1000}{1+r_2}$$

- That was the bond's value one year from now. But its value today should be discounted at the current rate, so the price today is

$$P(2) = \frac{1000}{(1+r_1)(1+r_2)}$$

- In general, the present value of \$1000 paid n periods from now is

$$P(n) = \frac{1000}{(1+r_1)(1+r_2)\dots(1+r_n)}$$

where $r_1, r_2, \dots, r_n$ are the **forward interest rates** during the periods $1, 2, \dots, n$, respectively

Term Structure of Interest Rates (3.5.2)

- Each of the above sets (zero prices, spot rates and forward rates) can be computed from either of the other sets.
- We're going to calculate YTMs from forward rates...
- Forward rates from spot rates...
- Prices from forward rates...
- YTMs from prices...

Finding YTMs from Forward Rates (3.5.2)

- The yield to maturity y_n is the one rate that satisfies

$$\frac{1}{(1+r_1)(1+r_2)\dots(1+r_n)} = \frac{1}{(1+y_n)^n}$$

So we have

$$y_n = \sqrt[n]{(1+r_1)(1+r_2)\dots(1+r_n)} - 1$$

$(1+y_n$ is the geometric mean of $(1+r_1),(1+r_2),\dots,(1+r_n))$

Finding Forward Rates from Spot Rates / YTMs

- $r_1 = y_1$
- and:

$$r_n = \frac{(1+y_n)^n}{(1+y_{n-1})^{n-1}} - 1$$

Finding prices from forward interest rates (Example 3.2)

Example:

Year (i)	Forward interest rate (r_i)
1	6%
2	7%
3	8%

Finding Prices from Forward Rates (Example 3.2)

- A one-year zero, with par value \$1000, would sell for

$$P(1) = \frac{1000}{1+r_1} = \frac{1000}{1+0.06} = 943.40$$

- A two-year zero coupon would sell for

$$P(2) = \frac{1000}{(1+r_1)(1+r_2)} = \frac{1000}{(1+0.06)(1+0.07)} = 881.68$$

- A three-year zero coupon would sell for

$$P(3) = \frac{1000}{(1+r_1)(1+r_2)(1+r_3)} = \frac{1000}{(1+0.06)(1+0.07)(1+0.08)} = 816.37$$

Finding YTMs from Prices and Forward Rates (Example 3.3)

Example: Finding yield to maturity from prices (using the prices we derived in Example 3.2) and the same forward rates as earlier

Year (i)	Forward interest rate (r_i)
1	6%
2	7%
3	8%

Finding YTMs from Prices (Example 3.3)

- We had

$$P(1) = \frac{1000}{1 + r_1} = \frac{1000}{1 + 0.06} = 943.40$$

- Solving for YTM y_1 gives 0.06
- Getting y_2:

$$P(2) = \frac{1000}{(1 + y_2)^2} = 881.68$$

thus

$$y_2 = \sqrt{\frac{1000}{P(2)}} - 1 = 0.0650$$

Finding YTMs from Prices (Example 3.3)

- Getting y_3:

$$P(3) = \frac{1000}{(1+y_3)^3} = 816.37$$

thus

$$y_3 = \sqrt[3]{\frac{1000}{P(3)}} - 1 = 0.070$$

- In general, YTMs equal:

$$y_n = \sqrt[n]{\frac{1000}{P(n)}} - 1$$

Finding YTMs from Forward Rates (Example 3.3)

- We are applying what we saw earlier to Example 3.3 of the Textbook
- We saaw that the yield to maturity rate y_n of a zero satisfies

$$\frac{1}{(1+r_1)(1+r_2)\ldots(1+r_n)} = \frac{1}{(1+y)^n},$$

we have

$$y_n = \sqrt[n]{(1+r_1)(1+r_2)\ldots(1+r_n)} - 1$$

- $1 + y_n$ is the geometric mean of $(1+r_1), (1+r_2), \ldots, (1+r_n)$
- In our example, we have

$$y_1 = r_1 = 0.06, \quad y_2 = \sqrt{(1+r_1)(1+r_2)} - 1 = \sqrt{(1.06)(1.07)} - 1 = 0.0649$$

and

$$y_3 = \sqrt{(1+r_1)(1+r_2)(+r_3)} - 1 = \sqrt{(1.06)(1.07)(1.08)} - 1 = 0.0700$$

Term Structure of Interest Rates

In summary, we can derive the following equations relating YTMs, prices and forward rates:

- the yield to maturity from bond prices:

$$y_n = \sqrt[n]{\frac{1000}{P(n)}} - 1$$

- the yield to maturity from forward rates

$$y_n = \sqrt[n]{(1+r_1)(1+r_2)\ldots(1+r_n)} - 1$$

- bond prices from yields to maturity

$$P(n) = \frac{1000}{(1+y_n)^n}.$$

- r_n from the prices of zero coupon bonds

$$r_n = \frac{P(n-1)}{P(n)} - 1$$

Finding yields and forward rates from prices (Example 3.4)

Suppose that one-, two-, and three-year par-\$1,000 zeros are priced as given in this table

Maturity (years)	Price
1	\$920
2	\$830
3	\$760

Finding YTMs from prices (Example 3.4)

Yields to maturity:

$$y_1 = \frac{1000}{920} - 1 = 0.087,$$

$$y_2 = \left\{\frac{1000}{830}\right\}^{1/2} - 1 = 0.0976,$$

$$y_3 = \left\{\frac{1000}{760}\right\}^{1/3} - 1 = 0.096.$$

Forward rates from YTMs (Example 3.4)

Forward rates:

$$
\begin{aligned}
r_1 &= y_1 = 0.087, \\
r_2 &= \frac{(1+y_2)^2}{(1+y_1)} - 1 = \frac{(1.0976)^2}{1.0876} - 1 = 0.108, \text{ and} \\
r_3 &= \frac{(1+y_3)^3}{(1+y_2)^2} - 1 = \frac{(1.096)^3}{(1.0976)^2} - 1 = 0.092.
\end{aligned}
$$

Forward rates from prices (Example 3.5)

Forward rates:

$$r_1 = \frac{1000}{920} - 1 = 0.087,$$

$$r_2 = \frac{920}{830} - 1 = 0.108,$$

$$r_3 = \frac{830}{760} - 1 = 0.092.$$

Continuous Compounding (3.6)

Continuous compounding simplifies the relationships between forward rates, yields to maturity of zeros (spot rates) and prices of zeros.

- Prices from forward rates

$$P(1) = \frac{1000}{exp(r_1)} = 1000e^{-r_1}$$

and in general

$$P(n) = \frac{1000}{e^{r_1+r_2+\ldots+r_n}}$$

- Forward rates from prices

$$\frac{P(n-1)}{P(n)} = \frac{e^{r_1+r_2+\ldots+r_n}}{e^{r_1+r_2+\ldots+r_{n-1}}},$$

therefore

$$r_n = \log\left\{\frac{P(n-1)}{P(n)}\right\}$$

Continuous Compounding (3.6)

- The yield to maturity of an n-year zero-coupon bond solves the equation

$$P(n) = \frac{1000}{e^{ny_n}}$$

- That is, the discount function is

$$e^{-ny_n}$$

- Hence, we get the yield to maturity from forward rates:

$$y_n = (r_1 + r_2 + \ldots + r_n)/n$$

- $r_1, r_2, \ldots, r_n$ are found from $y_1, y_2, \ldots, y_n$ by

$$r_1 = y_1 \quad \text{and} \quad r_n = ny_n - (n-1)y_{n-1}$$

for $n > 1$.

Continuously compounded forward rates and yields from prices (Example 3.6)

Using the same prices as before

Maturity (years)	Price
1	\$920
2	\$830
3	\$760

We derive the continuously compounded rates as follows:

$$r_1 = \log\left\{\frac{1000}{920}\right\} = 0.083,$$

$$r_2 = \log\left\{\frac{920}{830}\right\} = 0.103,$$

$$r_3 = \log\left\{\frac{830}{760}\right\} = 0.088.$$

Continuous Forward Rates (3.7)

We have assumed that forward interest rates

- are constant within each year.
- have a fixed starting time with all maturities some integer number of years from this date.

Continuous Compounding (3.7)

- To be realistic, we assume that there is a forward rate function $r(t)$
- We just saw the discount function as a function of the YTM:

$$e^{-ny_n}$$

- It is then natural that the continuous version is:

$$D(T) = e^{-\int_0^T r(t)dt}$$

- Of course, the price of the bond is still $P(T) = \text{PAR} \times D(T)$

Continuous Compounding (3.7)

- The yield to maturity of a zero-coupon bond with maturity T is defined as

$$y_T = \frac{1}{T} \int_0^T r(t)dt$$

- As a consequence the price is

$$P(T) = \text{PAR} \quad e^{-Ty_T}$$

- You can imagine that practitioners like this one number y_T and prefer the above expression to

$$P(T) = \text{PAR} \quad e^{-\int_0^T r(t)dt}$$

The Case of Piecewise Constant

- The book discusses a special case, without much explanation, of piecewise constant forward rates
- The example works only if each rate applies to a full year
- Otherwise, the integral would be slightly more complicated

if $r(t) = r_k$ for $k - 1 \leq t \leq k$ then

$$\int_0^T r(t)dt = r_1 + r_2 + \ldots + r_T$$

so that the price is

$$P(T) = \text{PAR} \times e^{-(r_1 + r_2 + \ldots + r_T)}$$

Linear Forward Rates (3.7, Example 3.7)

Example: Suppose the forward rate is

$$r(t) = 0.03 + 0.0005t$$

find $r(10), y_{10}$ and the price $P(10)$ of a zero with PAR 1,000 dollars and maturity 10 years
Answer:

- Forward rate, 10 years from now: $r(10) = 0.03 + 0.0005(10) = 0.035$
- YTM:

$$y_{10} = \frac{1}{10}\int_0^{10}(0.03+0.0005t)dt = \frac{1}{10}\left(0.03(10) + 0.0005\frac{100}{2}\right) = 0.0325$$

- The price is par times the discount function for the yield to maturity we just computed:

$$P(10) = 1000e^{-10(0.0325)} = 772.5$$