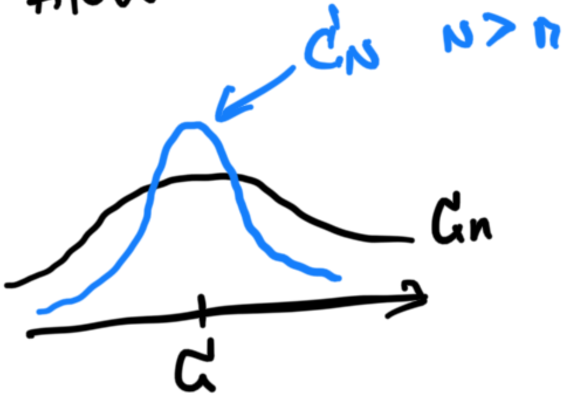
Call option example

theoretical BS price $= C$

MC estimator $= C_n = \frac{1}{n}\sum_{i=1}^{n} f(z_i)$

Trial	samples drawn		simulated price
#1	$z_1^{(1)}, \dots, z_n^{(1)}$	$\longrightarrow$	$C_n^{(1)}$
#2	$z_1^{(2)}, \dots, z_n^{(2)}$	$\longrightarrow$	$C_n^{(2)}$
⋮			
#1000			$C_n^{(1000)}$

$\left.\begin{matrix} C_n^{(1)} \\ C_n^{(2)} \\ \vdots \\ C_n^{(1000)} \end{matrix}\right\}$ empirical distrib. of C_n



variance reduction

Find an unbiased estimator of C, $\tilde{C}_n$ s.t.

$$\mathrm{var}(\tilde{C}_n) < \mathrm{var}(C_n)$$

Antithetic sampling

Ref. Glasserman Chap 4.2

call option example

sample	asset at T	disc. payoff
$z_1 = 0.1$	$S(z_1) = 181$	$f(z_1) = 0.99 \times 21$
$-z_1 = -0.1$	$S(-z_1) = 178$	$f(-z_1) = 0.99 \times 18$
z_2	$S(z_2)$	$f(z_2)$
$-z_2$	$S(-z_2)$	$f(-z_2)$
⋮	⋮	⋮

$K = 160$

$$\tilde{C}_n = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{f(z_i) + f(-z_i)}{2}\right]$$

Can prove that $\mathrm{var}(\tilde{C}_n) \le \mathrm{var}(C_n)$ (Glasserman P. 208)

Indeed in call option case.

$$\mathrm{var}(\tilde{C}_n) \le \mathrm{var}(C_{2n})$$

This is true if $f(z)$ is montonic increasing
or " decreasing.

Options on multiple assets

① Options on spread of 2 asset prices
$S_1(t)$, $S_2(t)$

call payoff $= \max(S_1(T) - S_2(T) - K, 0)$
at expiry T

- crack spread (NYMEX)
  heating oil vs. WTI crude oil.
- location spread (NYMEX)
  WTI crude oil vs Brent oil

② Options on a basket of assets
$S_1(t), S_2(t), \cdots, S_k(t)$

- currency market / equity mkt

eg. Basket : 50% AAPL, 30% GOOG, 20% MFST

- call option on % change in basket price

$$P(0) = \sum_{i=1}^{k} w_i S_i(0) \quad \text{initial weighted avg price}$$

$$P(T) = \sum_{i=1}^{k} w_i S_i(T)$$

payoff rate at expiry T

$$= \max\left(\frac{P(T) - P(0)}{P(0)} - K, 0\right)$$

Assume $S_1(t), S_2(t), \cdots, S_k(t)$ follows correlated GBM.

$$\frac{dS_i(t)}{S_i(t)} = \mu_i\,dt + \sigma_i\,dW_i(t)$$

$$\boxed{\begin{aligned} &d\ln S_i(t) \\ &= \left(\mu_i - \frac{\sigma_i^2}{2}\right)dt \\ &\quad + \sigma_i\,dW_i(t) \end{aligned}}$$

$W_1(t), \dots, W_k(t)$ correlated std. BM.

$$\boxed{\begin{gathered} S_i(T) = S_i(0)\exp\left(\left(\mu_i - \frac{\sigma_i^2}{2}\right)T + \underline{\sigma_i\sqrt{T}}\, z_i\right) \\ (z_1, \dots, z_k) \sim MVN(0, \Sigma) \end{gathered}}$$

Σ ← corr. matrix (cov)

vol, corr : historical asset prices
→ log returns → estimate for vol, corrs.

<u>One simulation</u>

<u>random sample</u> | asset prices

$(z_1, \dots, z_k) \longrightarrow S_1(z_1), S_2(z_2), \dots, S_k(z_k)$

$\longrightarrow$ discounted payoff $f(\vec{z})$

var reduction

$(-z_1, \dots, -z_k) \longrightarrow S_1(-z_1), \dots, S_k(-z_k)$

$\longrightarrow f(-\vec{z})$

<u>Path dependent options</u>

- Asian options
  - options on daily avg price over fixed period.
  - FX, commodity, energy.

eg. daily avg of closing prices over month of April.
- WTI crude, electricity ...

$S(t)$ = asset price at t

$0 \quad t_1 \quad t_2 \quad t_3 \quad \dots \quad t_m = T$ expiry.

$$A(t) = \frac{1}{m}\sum_{i=1}^{m} S(t_i)$$

call payoff at T $= \max(A(T) - K, 0)$

Suppose $S(t)$ follows GBM.

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dW(t)$$

Simulate a path $S(0), S(t_1), S(t_2), \cdots, S(t_m)$

1 $S(t_1) = S(0) \exp\left((\mu - \frac{\sigma^2}{2}) t_1 + \sigma \sqrt{t_1}\, z_1 \right)$, $z_1 \sim N(0,1)$

Now $S(t_1)$ known. How to sample $S(t_2)$?

2 $S(t_2) = S(t_1) \exp\left((\mu - \frac{\sigma^2}{2})(t_2 - t_1) + \sigma \sqrt{t_2 - t_1}\, z_2 \right)$

$z_2 \sim N(0,1)$

$d \ln S(t) = (\mu - \frac{\sigma^2}{2}) dt + \sigma\, dW(t)$

integrate over $[t_1, t_2]$

$\ln S(t_2) - \ln S(t_1) = (\mu - \frac{\sigma^2}{2})(t_2 - t_1) + \sigma [W(t_2) - W(t_1)]$

- sample $z_1 \sim N(0,1) \rightarrow S(t_1, z_1)$ 1
- sample $z_2 \sim N(0,1) \rightarrow S(t_2, z_2)$ 2

$\vdots$

- $z_m \sim N(0,1) \rightarrow S(t_m, z_m)$

$A(T) = A(t_1, \cdots, t_m)$

Recursively, we get a path of asset prices

$$\boxed{\begin{aligned} S(t_i) = S(t_{i-1}) \exp\Big((\mu - \tfrac{\sigma^2}{2})(t_i - t_{i-1}) \\ + \sigma \sqrt{t_i - t_{i-1}}\, z_i \Big) \\ z_1, z_2, \cdots, z_m \text{ indept. samples of } N(0,1) \end{aligned}}$$