Math 280, Spring 2022

Solutions to Homework #6

1. (24 points) Textbook Section 1.3.4, Problem 5 (expanded a bit):

(a) Use Prüfer's method to draw and label the trees with Prüfer sequences 1,2,3 and 3,4,1,2.

(b) Inspired by your answers in part (a), make a conjecture about which trees have Prüfer sequences consisting of all distinct terms.

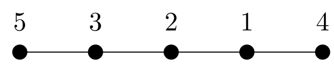
(c) Prove your conjecture from part (b).

Solution/Proof. (a): Call the first sequence $\sigma_0 = 1, 2, 3$. Since it has 3 entries, the corresponding tree has $n = 5$ vertices, so let $S_0 = \{1, 2, 3, 4, 5\}$.

$\boxed{i = 0}$ The smallest $j \in S_0$ not in σ_0 is 4, and the first entry in σ_0 is 1. so we add the edge $\boxed{1\text{–}4}$ to make T_1. Let $\sigma_1 = 2, 3$ and $S_1 = \{1, 2, 3, 5\}$.

$\boxed{i = 1}$ The smallest $j \in S_1$ not in σ_1 is 1, and the first entry in σ_1 is 2. so we add the edge $\boxed{1\text{–}2}$ to make T_2. Let $\sigma_2 = 3$ and $S_2 = \{2, 3, 5\}$.

$\boxed{i = 2}$ The smallest $j \in S_2$ not in σ_2 is 2, and the first entry in σ_2 is 3. so we add the edge $\boxed{2\text{–}3}$ to make T_3. Let $\sigma_3 = \varnothing$ and $S_3 = \{3, 5\}$. Since σ_3 is empty, we add the one more edge from S_3, namely $\boxed{3\text{–}5}$ yielding the following final graph:



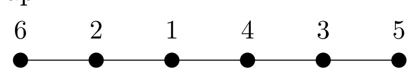
Call the second sequence $\sigma_0 = 3, 4, 1, 2$. Since it has 4 entries, the corresponding tree has $n = 6$ vertices, so let $S_0 = \{1, 2, 3, 4, 5, 6\}$.

$\boxed{i = 0}$ The smallest $j \in S_0$ not in σ_0 is 5, and the first entry in σ_0 is 3. so we add the edge $\boxed{3\text{–}5}$ to make T_1. Let $\sigma_1 = 4, 1, 2$ and $S_1 = \{1, 2, 3, 4, 6\}$.

$\boxed{i = 1}$ The smallest $j \in S_1$ not in σ_1 is 3, and the first entry in σ_1 is 4. so we add the edge $\boxed{3\text{–}4}$ to make T_2. Let $\sigma_2 = 1, 2$ and $S_2 = \{1, 2, 4, 6\}$.

$\boxed{i = 2}$ The smallest $j \in S_2$ not in σ_2 is 4, and the first entry in σ_2 is 1. so we add the edge $\boxed{1\text{–}4}$ to make T_3. Let $\sigma_3 = 2$ and $S_2 = \{1, 2, 6\}$.

$\boxed{i = 3}$ The smallest $j \in S_3$ not in σ_3 is 1, and the first entry in σ_3 is 2. so we add the edge $\boxed{1\text{–}2}$ to make T_4. Let $\sigma_4 = \varnothing$ and $S_4 = \{2, 6\}$. Since σ_4 is empty, we add the one more edge from S_4, namely $\boxed{2\text{–}6}$ yielding the following final graph:



(b): Let's conjecture that the trees with Prüfer sequence consisting of all distinct terms are precisely path graphs. That is, for any tree T with $n \geq 2$ vertices,

$$T \text{ has Prüfer sequence consisting of all distinct terms} \iff T \text{ is a path graph with } n \text{ vertices}$$

And in fact, let's add that the two vertices of degree 1 are the ones that do not appear in the Prüfer sequence.

(c): **Proof**. ($\Leftarrow$) We proceed by induction on $n \geq 2$. For $n = 2$, T is a path graph P_2 with two vertices, and Prüfer's algorithm gives the (empty) sequence σ, so neither vertex appears in $\sigma = \varnothing$.

Assuming the conjecture is true for some $n \geq 2$, consider running Prüfer's first algorithm on an ($(n+1)$-vertex) path graph T. There are two leaves j and k, we we remove the lowest numbered leaf j. We then record j's neighbor m, which becomes a leaf of the new tree $T' = T - j$. (And the other leaf of T' is still k.

By our inductive hypothesis, the rest of Prüfer's algorithm gives us a sequence σ' of $(n - 2)$ distinct vertices, none of which are the leaves of T', and hence none of which are either m or k. (Note that σ' also does not include j, since j is not a vertex of T'.)

Thus, we finish the algorithm with the final Prüfer sequence σ being m followed by the list σ'. Since σ' has no repeats and none of the vertices j, k, m, it follows that σ also has no repeats, and it does not have j or m. That is, the final Prüfer sequence σ consists of $n-1=(n+1)-2$ distinct vertices of T, none of which are the two leaves j,k, as desired. QED ($\Leftarrow$)

($\Rightarrow$) By the previous implication, every sequence of length $n-2$ distinct entries comes from a path graph. Since Prüfer's algorithms give a one-to-one correspondence between trees and sequences, this means that no other tree can give a constant sequence. Thus, the path graphs are *precisely* the trees with constant Prüfer sequences. QED

Note: I was a little glib in the ($\Rightarrow$) implication, since I didn't explicitly verify that *every* sequence of length $n-2$ distinct entries comes from a path graph. So while I won't expect that verification as far as points are concerned on this problem, here's that missing argument:

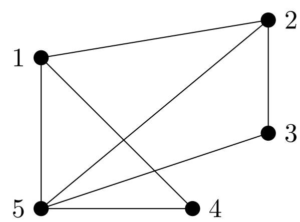
To draw a path graph with vertices labelled $\{1,\dots,n\}$, there are $n!/2$ ways to do it, because starting from the left we have n choices for the left-most vertex's label, then $n-1$ remaining choices for the one adjacent to it, and so on down the line, giving $n!$ choices. But of course, flipping the path end-to-end gives the same path graph, just labelled in the reverse order, so really, there are indeed $n!/2$ different labelled path graphs with labels $\{1,\dots,n\}$.

So it suffices to show that there are the same number of sequences of $n-2$ distinct elements of $\{1,\dots,n\}$. Well, there are n choices for the first element in the sequence, then $n-1$ choices for the second, and so on down the line, until we get to 3 choices for the last (i.e., $(n-2)$nd) element in the list. So there are $n(n-1)(n-2)\cdots(4)(3)=n!/2$ such sequences.

Since there are the same number of such path graphs as there are such sequences, and since Prüfer's algorithm maps the set of such path graphs one-to-one into the set of such sequences, then by the pigeonhole principle, every such sequence must indeed come from a path graph.

---

2. (8 points) Use the Matrix Tree Theorem to find the number of spanning trees of this graph:



Solution. The Laplacian matrix of this graph G is $A=\begin{bmatrix}3&-1&0&-1&-1\\-1&3&-1&0&-1\\0&-1&2&0&-1\\-1&0&0&2&-1\\-1&-1&-1&-1&4\end{bmatrix}$ and hence, computing the $(5,5)$-cofactor (which I chose to get the most zeros in the resulting submatrix), the Matrix Tree Theorem says there are

$$\begin{vmatrix}3&-1&0&-1\\-1&3&-1&0\\0&-1&2&0\\-1&0&0&2\end{vmatrix}=-(-1)\begin{vmatrix}-1&3&-1\\0&-1&2\\-1&0&0\end{vmatrix}+2\begin{vmatrix}3&-1&0\\-1&3&-1\\0&-1&2\end{vmatrix}$$

$$=(-1)\begin{vmatrix}3&-1\\-1&2\end{vmatrix}+2\Big[3\begin{vmatrix}3&-1\\-1&2\end{vmatrix}-(-1)\begin{vmatrix}-1&-1\\0&2\end{vmatrix}\Big]=-(6-1)+2\big[3(6-1)+(-2)\big]=-5+2(13)=$$

$\boxed{\text{21 spanning trees}}$

---

3. (10 points) Let G be a graph with Laplacian matrix Δ. Prove that $\det(\Delta)=0$.

Proof. Let $n=|V(G)|$, and let $\vec{x}$ be the n-entry vector of all 1's. We claim that $\Delta\vec{x}=\vec{0}$.

To prove the claim, for each $i=1,\dots,n$, let $m=\deg(v_i)$. Then there are m appearances of -1 in row i of Δ, one for each vertex adjacent to v_i; and the i-th entry of the row is m; and all the other entries in the row are 0. So taking the dot product with $\vec{x}$, i.e., summing the entries of the row, gives $m+(-1)+\cdots+(-1)=0$. This is true for every row, so every entry of $\Delta\vec{x}$ is 0, proving our claim that $\Delta\vec{x}=\vec{0}$.

Thus, the matrix Δ has a nontrivial nullspace (a.k.a. kernel), since $\vec{x} \neq \vec{0}$ is an element of this nullspace. Therefore, by properties of determinants, we have $\det(\Delta) = 0$. QED

---

4. (10 points) Textbook Section 1.4.2, Problem 7(b): Determine the values of $m, n \geq 1$ such that the complete bipartite graph $K_{m,n}$ is Eulerian. Prove your answer.

Solution/Proof. We claim $K_{m,n}$ is Eulerian if and only if $\boxed{m \text{ and } n \text{ are both even}}$

Here's the proof. Write $V(K_{m,n}) = X \cup Y$, where $|X| = m$ and $|Y| = n$, and there is an edge from every $x \in X$ to every $y \in Y$, and no other edges. Thus, we have $\deg(x) = n$ for every $x \in X$, and $\deg(y) = m$ for every $y \in Y$.

By Theorem 1.20, the graph (which we already know is connected) is Eulerian if and only every vertex has even degree. By the above paragraph, this happens for $K_{m,n}$ if and only if m and n are both even, since they are the degrees of the vertices of $K_{m,n}$. QED

---

5. (14 points) Textbook Section 1.4.2, Problem 7(a): Determine the precise set of values of $m, n \geq 1$ such that the complete bipartite graph $K_{m,n}$ has an Eulerian trail. Prove your answer.

Solution/Proof. We claim $K_{m,n}$ is has an Eulerian trail if and only if

$$\boxed{\text{either } m \text{ and } n \text{ are both even; or one is odd and the other is 2; or } m = n = 1} \qquad (\star)$$

Here's the proof. Write $V(K_{m,n}) = X \cup Y$, where $|X| = m$ and $|Y| = n$, and there is an edge from every $x \in X$ to every $y \in Y$, and no other edges. Thus, we have $\deg(x) = n$ for every $x \in X$, and $\deg(y) = m$ for every $y \in Y$. That is, there are m vertices of degree m and n vertices of degree m

By Corollary 1.21, there is an Eulerian trail if and only if there are at most two vertices of odd degree. It suffices to show this happens if and only if condition $(\star)$ holds.

To prove this claim, we consider several cases.

Case 1: $\boxed{m, n \text{ are both even}}$ Then every vertex has even degree, i.e., there are $0 < 2$ vertices of odd degree. In addition, condition $(\star)$ holds, proving the claim in this case.

Case 2: $\boxed{m \text{ is odd and } n = 2}$ Then all of the m vertices in X have degree 2, which is even, and both of the 2 vertices in Y have odd degree m. Thus, there are 2 vertices of odd degree. In addition, condition $(\star)$ holds, proving the claim in this case.

Case 3: $\boxed{n \text{ is odd and } m = 2}$ Same as Case 2, with the roles of m and n reversed, so the claim holds in this case.

Case 4: $\boxed{m = n = 1}$ Then there are only 2 total vertices, both of which have degree 1. Thus, there are 2 vertices of odd degree. In addition, condition $(\star)$ holds, proving the claim in this case.

Case 5: $\boxed{m \text{ is odd and } n \geq 3}$ Then condition $(\star)$ is false, and in addition, there are at least 3 vertices of degree m, which is odd, so by Corollary 1.21, there is no Eulerian trail, proving the claim in this case.

Case 6: $\boxed{n \text{ is odd and } m \geq 3}$ Same as Case 5, with the roles of m and n reversed, so the claim holds in this case.

To finish the proof, we only need to verify that the above cases cover all the possibilities for integers $m, n \geq 1$. So consider arbitrary integers $m, n \geq 1$:

If we are not in Case 1, then at least one of m and n must be odd; without loss m is odd.

If in addition we are not in Case 2, we have $n \neq 2$.

If in addition we are not in Case 5, we have $n \not\geq 3$, so $n = 1$.

If in addition we are not in Case 4, we have $m \neq 1$; since m is odd, we have $m \geq 3$. But then we are in Case 6. Thus, the above cases do indeed cover all possibilities. QED

---

6. (16 points) Textbook Section 1.4.2, Problem 1: For each of the following, draw an Eulerian graph the satisfies the conditions, or prove that no such graph exists.

(a) An even number of vertices, and an even number of edges.

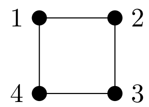
(b) An even number of vertices, and an odd number of edges.

(c) An odd number of vertices, and an even number of edges.

(d) An odd number of vertices, and an odd number of edges.

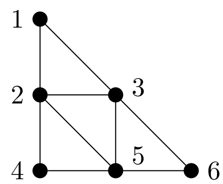
Solution. In fact, all four cases are possible. Here are some examples; there are others.

(a):



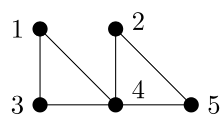
is connected with 4 vertices all of even degree, and 4 edges.

(b):



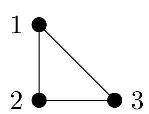
is connected with 6 vertices all of even degree, and 9 edges.

(c):



is connected with 5 vertices all of even degree, and 6 edges.

(d):



is connected with 3 vertices all of even degree, and 3 edges.

---

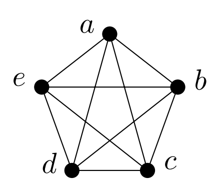
7. (8 points) For the graph $G = K_5$, determine:

(a) is it Eulerian? (b) is it Hamiltonian?

(c) is it traceable? (d) what is its independence number $\alpha(G)$?

As always, be sure to (briefly) justify your answers.

Solution/Proof. Here is the graph:



(a): $\boxed{\text{Yes, Eulerian}}$ by Theorem 1.20, because G is connected and every vertex has degree 4, which is even.

(b): $\boxed{\text{Yes, Hamiltonian}}$ because the cycle a, b, c, d, e, a hits every vertex.

(c): $\boxed{\text{Yes, traceable}}$ because the path a, b, c, d, e hits every vertex once.

(d): $\boxed{\alpha(G) = 1}$ because $\{a\}$ is a (one-element) independent set of vertices, but for any set $S \subseteq V(G)$ with $|S| \geq 2$, there are $x \neq y \in S$, and we must have x and y adjacent because G is complete. So the largest independent set has 1 element, as claimed.

---

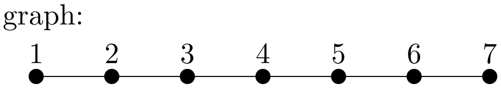
8. (10 points) For the graph $G = P_7$, determine:

(a) is it Eulerian? (b) is it Hamiltonian?

(c) is it traceable? (d) what is its independence number $\alpha(G)$?

As always, be sure to (briefly) justify your answers.

Solution/Proof. Here is the graph:



(a): $\boxed{\text{No, not Eulerian}}$ by Theorem 1.20, because $\deg(1) = 1$ is odd.

(b): $\boxed{\text{No, not Hamiltonian}}$ Any cycle passing through a vertex v must use at least two different edges incident with v. Since $\deg(1) = 1$, there cannot be any cycle passing through the vertex 1, so there cannot be a Hamiltonian cycle.

(c): $\boxed{\text{Yes, traceable}}$ because the path $1, 2, 3, 4, 5, 6, 7$ hits every vertex once.

(c): $\boxed{\alpha(G) = 4}$ because $\{1, 3, 5, 7\}$ is a (four-element) independent set of vertices. For any set $\{a_1, \ldots, a_k\}$ of k vertices to be independent, assuming without loss that $a_1 < a_2 < \cdots < a_k$, we must have $a_2 \geq a_1 + 2$ (since a_2 is not adjacent to a_1), and $a_3 \geq a_2 + 2$ (since a_3 is not adjacent to a_2), and so on. Since $a_1 \geq 1$, if $k \geq 5$, we would have $a_5 \geq 9$, which is impossible because the highest-numbered vertex is 7. So the largest independent set has 4 elements, as claimed.