payments. It has no such ability in another currency, and although it could create new pesos and exchange them for the currency owed, this might have a severe enough impact on the exchange rate between the currencies to call into question the country's willingness, and even its ability, to do so.

As the procedures for minimizing credit exposure to a counterparty become more complex, involving collateralization, netting, and margin calls, among other techniques, it becomes more difficult to represent the credit exposure in discounting procedures. Any oversimplification should definitely be avoided, such as discounting the flows owed by A to B on a swap at a discount rate appropriate for A's obligations and the flows owed by B to A on the same swap at a discount rate appropriate to B's obligations. This treats the gross amounts owed on the swap as if they were independent of one another, completely ignoring a primary motivation for structuring the transaction as a swap—the netting of obligations.

As credit exposure mitigation techniques grow in sophistication, they demand a parallel sophistication in valuation technology. This consists of initially treating all flows on a transaction to which credit exposure mitigation has been applied as if they were flows certain to be received. The actual credit exposure must then be calculated separately, taking into account the correlation between the net amount owed and the creditworthiness of the obligor. This methodology is more complex than we can tackle at this point in the book. We will return to this topic in Section 14.3.

10.2.2 Pricing Long-Dated Illiquid Flows by Stack and Roll

An issue that arises frequently for market-making firms is the need to provide value to customers by extending liquidity beyond the existing market. This need arises not only for bonds and single-currency swaps and forwards, but also for FX forwards and commodity forwards. A concrete example would be a firm trying to meet customer demand for 40-year swaps in a market that has liquidity only for swaps out to 30 years.

To see the actual profit and loss (P&L) consequences of a methodology for pricing these longer-term flows, we need to consider a well-known trading strategy: the *stack-and-roll hedge*. In our example, a stack-and-roll hedge would call for putting on a 30-year swap in the liquid market as a hedge against a 40-year swap contracted with a customer. Then, at the end of 10 years, the 20-year swap to which the 30-year swap has evolved will be offset and a new 30-year swap in the liquid market will be put on, which will completely offset the original 40-year swap, which is, at this point, a 30-year swap.

This stack-and-roll strategy can be characterized as a quasistatic hedge in that it requires one future rehedge at the end of 10 years. The results borrowing and an increase in its need for seven-year borrowings, both of which can be valued off the standard U.S. Treasury discount curve. However, a market maker has a lower credit rating and hence higher borrowing costs than the U.S. Treasury has. If the market maker tries to create the forward by buying a 10-year instrument, the price it would need to charge for the forward would be burdened by seven years' worth of the credit spread between the Treasury and the market maker.

To avoid this, the market maker needs to find a way to borrow for seven years at essentially a U.S. Treasury rate. Since it has the 10-year Treasury purchased to put up as collateral against its seven-year borrowing, this should be feasible. However, it is an institutional fact that a liquid market does not exist for borrowing against Treasury collateral at a fixed rate for seven years. It is certainly possible to borrow against Treasury collateral for short periods with great liquidity, and the market maker should feel no fear about the ability to continuously roll over this borrowing. However, this introduces a large variance in the possible funding costs due to uncertainty about the direction short-term repurchase rates will take over a seven-year period.

The way around this impasse is for the market maker to buy a 10-year Treasury, borrow a seven-year Treasury, and sell the seven-year Treasury. The 10-year Treasury is financed for seven years by a series of overnight repurchase agreements (RPs). The borrowing of the seven-year Treasury is financed by a series of overnight RPs. The market maker has succeeded in achieving the same cost of creating the forward that the Treasury would have, except for any net cost between the overnight RP rates at which the longer Treasury is financed and the overnight RP rate at which the borrowing of the shorter Treasury is financed.

In general, these two RP rates should not differ; on any given day, each represents a borrowing rate for the same tenor (overnight) and with the same quality collateral (a U.S. government obligation). However, the RP market is influenced by supply-and-demand factors involving the collateral preferences of the investors. Some of these investors are just looking for an overnight investment without credit risk, so they don't care which U.S. Treasury security they purchase as part of the RP. Other investors, however, are looking to receive a particular U.S. Treasury that they will then sell short—either as part of a strategy to create a particular forward Treasury or because they think this particular Treasury issue is overpriced and they want to take advantage of an anticipated downward price correction. The higher the demand by cash investors to borrow a particular security, the lower the interest rate they will be forced to accept on their cash. When RP rates on a particular Treasury issue decline due to the demand to borrow the issue, the RP for the issue is said to have *gone on special*.

So the market maker in our example will not know in advance what the relative RP rates will be on the shorter security on which it is receiving the RP rate and the longer security on which it is paying the RP rate. To properly value the Treasury forward created by a market maker, it is necessary to make a projection based on past experience with RP rates for similar securities. This source of uncertainty calls for risk controls, which could be a combination of limits to the amount of exposure to the spread between the RP rates and reserves on forward Treasuries, with reserve levels tied to the uncertainty of RP spreads.

Constant-maturity Treasury (CMT) swaps (see Hull 2012, Section 32.4) are popular products with rate resets based on U.S. Treasury yields. They are therefore valued in the same way as U.S. Treasury forwards. Control of the risk for this product focuses on creating long and short cash Treasury positions and managing the risk of the resulting RP spreads.

10.2.4 Indexed Flows

We will now examine how to extend our methods for handling fixed flows to handling nonfixed flows tied to certain types of indexes. Let's start with a simple example. To keep this clear, let's label all the times in our example with specific dates.

Let's say the current date is July 1, 2013. Bank XYZ is due to pay a single flow on July 1, 2015, with the amount of the flow to be determined on July 1, 2014, by the following formula: $100 million multiplied by the interest rate that Bank XYZ is offering on July 1, 2014, for $100 million deposits maturing on July 1, 2015. Since this interest rate will not be known for one year, we do not currently know the size of this flow. However, we can determine a completely equivalent set of fixed flows by the following argument and then value the fixed flows by the methodology already discussed.

We write our single flow as the sum of two sets of flows as follows:

	July 1, 2014	**July 1, 2015**
Set 1	–$100 million	+$100 million × (1 + Index rate)
Set 2	+$100 million	–$100 million
Contracted flow	0	+$100 million × Index rate

We will argue that the flows in set 1 should be valued at zero. If this is true, then the present value of our contracted flow must be equal to the present value of the second set of flows, which is a set of completely fixed flows.

of the excess above or below a strike, are also possible. This style of option, sometimes collectively known as *power options*, has largely fallen out of favor following the Bankers Trust (BT)/Procter & Gamble (P&G)/Gibson Greetings blowup of 1994, which is discussed in Section 4.3.1. The lawsuits and allegations prompted by large losses on contracts with complex payoff formulas with no discernible tie to any of the end user's economic motives led to a distrust of such derivatives. Currently, most market makers' client appropriateness rules permit such contracts only in very limited circumstances.

Nonetheless, some power options remain in active use. The most prominent are *log contracts*, which are of particular interest because of their link to valuing and hedging variance swaps, and a type of quanto option that is utilized in the foreign exchange (FX) and bullion markets. In addition, the convexity adjustments needed for valuing and hedging certain types of forward risk, which we discuss in Section 10.2.4, can usefully be viewed as a type of power option and managed by this technique. After examining each of these three cases, we will follow with an examination of the important case of binary options, which illustrates the issue of how to handle liquidity risk arising from static replication. Finally, we will show how binary options can be combined with vanilla options to create other exotics—a contingent premium option and an accrual swap.

12.1.1 Log Contracts and Variance Swaps

A *variance swap* is a forward contract on annualized variance whose payout at expiry is:

$$(\sigma_\rho{}^2 - K_{\mathrm{VAR}}) \times N \tag{12.1}$$

where $\sigma_\rho{}^2$ is the realized stock variance (quoted in annualized terms) over the life of the contract, K_{VAR} is the delivery price for variance, and N is the notional amount of the swap in dollars per annualized volatility point squared. The holder of a variance swap at expiry receives N dollars for every point by which the stock's realized variance, $\sigma_\rho{}^2$, has exceeded the variance delivery price, K_{VAR}, and pays N dollars for every point by which the stock's realized variance, $\sigma_\rho{}^2$, falls short of the variance delivery price, K_{VAR}. This contract can be generalized to assets other than stocks and to amounts other than dollars.

Variance swaps give their holders a vega exposure similar to what they would have by purchasing a vanilla option. However, variance swaps differ from vanilla options in that their vega exposure remains constant over time, whereas vanilla options may go into or out of the money, reducing their vega exposure. This can be a significant advantage to a position taker whose main concern is to find an investment that expresses her economic view of future volatility. It also has the advantage of enabling her to avoid maintaining delta and gamma hedges, which will be seen as a distraction to the real intention, which is just to express a volatility view. The downside is the relative illiquidity of variance swaps versus vanilla options, leading to their being priced with wider bid-ask spreads. The log contract offers a means to link the hedging and valuation of the illiquid variance swap to that of liquid vanilla options, using the basket hedge methodology.

The link between the variance swap and the log contract comes from the following analytic formula for the value of a log contract:

$$\ln F - \tfrac{1}{2}\int_0^T \sigma^2 \, dT \tag{12.2}$$

where ln is the natural logarithm function, F is the current price of the underlying forward to contract expiry T, and σ^2 is actual realized variance over that time period. This formula is a direct consequence of Equations 10 and 11 in Demeterfli et al. (1999). A derivation can also be found in Neuberger (1996). Under the Black-Scholes assumptions of known constant volatility, this implies that the log contract should be valued at $\ln F - \tfrac{1}{2}\sigma^2 T$, an analytic formula used in the **BasketHedge** spreadsheet to check the value derived for the log contract when the volatility surface is flat.

Since we can use the spreadsheet to find a set of vanilla options to replicate the log contract, we now have a hedging strategy for a variance swap. Buy a replicating set of vanilla options for twice the volume of log contracts as the volume of variance swaps sold (twice the volume in order to counteract the ½ in front of the integral in the formula). Delta hedge these vanilla options. Since the log contract is losing value at exactly the rate of $\tfrac{1}{2}\int_0^T \sigma^2 \, dT$, the delta hedging should be producing profits at exactly the rate needed to cover payments on the variance swap.

In practice, this will not work exactly, due to jumps in underlying prices, as explained in Demeterfli et al. (1999, "Hedging Risks"). Monte Carlo simulation would be necessary to quantify the risk of this tracking error. However, the replication of the log contract still offers a good first-order hedge and valuation for the variance swap.

The section "The Difficulty with Volatility Contracts" in the same article discusses why this approach will not work for *volatility swaps*, which differ from variance swaps by having a payout of $(\sigma_\rho - K_{\mathrm{VOL}}) \times N$ rather than $(\sigma_\rho^2 - K_{\mathrm{VAR}}) \times N$. No static hedge for the volatility contract exists. In the categorization we are using in this chapter, it is path dependent and needs to be risk managed using the techniques of Section 12.3, utilizing local volatility or stochastic volatility models to determine dynamic hedges. However, its close relationship to the variance swap, and thus to the log contract, suggests the use of a liquid proxy approach: use dynamic hedging just for the difference between the volatility swap and log contract while static hedging the log contract.

For further reading on the modeling and risk management of variance and volatility swaps, I highly recommend Demeterfli et al. (1999) and Gatheral (2006, Chapter 11).

Exercise 12.1 asks you to utilize the **BasketHedge** spreadsheet to look at the impact of changes in the volatility surface on the valuation of log contracts and hence on variance swaps. Demeterfli et al. (1999) also has an instructive section on the "Effects of the Volatility Skew" on variance swaps. Log contracts and variance swaps require hedges over a very wide range of strikes and should therefore show valuation sensitivity across the whole volatility surface. This seems reasonable from an intuitive standpoint since changes in volatility impact variance swaps even when the underlying forward price has moved very far away from the current price, leaving a currently at-the-money option very insensitive to vega. So high- and low-strike vanilla options are needed to retain the vega sensitivity of the package.

12.1.2 Single-Asset Quanto Options

In Section 12.4.5, we discuss dual-currency quanto derivatives in which the percentage change of an asset denominated in one currency is paid out in another currency. For example, a 10 percent increase in the yen price of a Japanese stock will be reflected by a 10 percent increase in a dollar payment at a fixed-in-advance dollar/yen exchange rate. We will see that the forward price of a quanto is the standard forward multiplied by $\exp(\rho\sigma_S\sigma_F)$, where exp is the exponential to the base e, σ_S is the standard deviation of the asset price, σ_F is the standard deviation of the FX rate, and ρ is the correlation between them.

A related product is a single-currency quanto derivative in which the asset whose percentage change is to be calculated is also the asset whose exchange rate is fixed. Here are two examples:

1. A dollar/yen FX option, which, if the yen rises in value by 10 percent relative to the dollar, will be reflected by a 10 percent payout in yen. Since the yen has gone up in value by 10 percent versus the dollar, the payout in dollar terms is $110\% \times 10\% = 11\%$. In general, for a p percent increase, the payout is $(1 + p\%) \times p\% = p\% + p^2\%$.

in observed vanilla option prices may require changes to input parameters to fit current prices, and once parameters change, the hedge may need adjustment.

How stable is the resulting representation? To what degree does it require frequent and sizable adjustments in the options hedges that can result in hedge slippage as a result of both transaction costs (generally considerably higher for options than for the underlying) and the instability of the hedge against parameter changes? The more the price of a product is dependent on assumptions about volatility evolution, the greater the instability of hedges. Although trading desks may gain experience with the stability of particular models in particular markets through time, it is difficult to obtain a risk measure in advance. The projection of hedge changes through Monte Carlo simulation (as recommended by Derman [2001] as discussed in Section 8.2.6.2), which has proved very useful in establishing results for the hedging of vanilla options with other vanilla options, is orders of magnitude more difficult to achieve for exotics. This is because each step on each path of the Monte Carlo simulation requires recomputation of the hedge. When the only hedge change is in the underlying, this is a very simple calculation of the $N(d_1)$ in the Black-Scholes formula. When the hedge change is in an option, a complete recalculation of the model being used to link the vanilla options and the exotic option together is required.

12.3.3 Static Hedging Models for Barriers

The uncertainty surrounding the hedging costs of using dynamic hedging for barriers provides the motivation to search for static or near-static hedging alternatives. Static hedging models price barrier options based on the cost of a replication strategy that calls for an almost unvarying hedge portfolio (at least of the vanilla options; it would be possible to use a dynamic hedge of the underlying, although the particular static hedging models we discuss only utilize vanilla options in the hedge portfolio). These models utilize nearly static hedge portfolios both as a way to reduce transaction costs and as a way to reduce dependence on assumptions about the evolution of volatility. Chapter 9 of Gatheral (2006) analyzes these nearly static hedges of barrier options from a different vantage point than mine, but with broadly similar conclusions.

Three approaches to the static hedging of barriers can be distinguished:

1. The approach of Derman, Ergener, and Kani, which is broadly applicable to all exotic options whose payoff is a function of a single underlying asset, but has considerable exposure to being wrong about future volatility levels.

2. The approach of Carr, which is more specifically tailored to barrier options, utilizing an analysis of the Black-Scholes formula to form a hedge portfolio that is immune to changes in overall volatility level and volatility smile. However, the Carr approach is still vulnerable to changes in the volatility skew. It is easier to implement than the Derman-Ergener-Kani approach for barriers in the absence of drift (that is, forward equal to spot) and produces a very simple hedging portfolio that helps develop intuitive understanding of the risk profile of the barrier.
3. Approaches that utilize optimal fitting give solutions close to those provided by the Carr approach for single barriers in the absence of drift, but are more flexible in handling drift and are less vulnerable to changes in volatility skew. Optimal fitting can be generalized to broader classes of exotics, but with less ease than the Derman-Ergener-Kani approach.

All three approaches are based on the idea of finding a basket of vanilla options that statically replicate the differences between the barrier option and a closely related vanilla option. To facilitate the discussion, we will confine ourselves to the case of a knock-out call, since a knock-in call can be handled as a vanilla call less a knock-out call, and all options can be treated as call options to exchange one asset for another (refer back to the introductory section of Chapter 11). The idea is to purchase a vanilla call with the same strike and expiration date as the knock-out being sold and then reduce the cost of creating the knock-out by selling a basket of vanilla options (this basket may have purchases as well as sales, but the net initial cash flow on the basket is positive to the barrier option seller). The basket of vanilla options must be constructed so that:

- It has no payoff if the barrier is never hit. In this case, the payout on the barrier option, which has not been knocked out, is exactly offset by the pay-in from the vanilla call that was purchased, so nothing is left over to make payments on the basket.
- Its value when the barrier is hit is an exact offset to the value of the vanilla call. When the barrier is hit, you know you will not need to make any payments on the barrier option, so you can afford now to sell the vanilla call you purchased. You do not want to later be vulnerable to payouts on the basket of vanilla options you sold, so you must purchase this basket. In order for cash flows to be zero, the basket purchase price must equal the vanilla call sale price.

You can guarantee the first condition by only using calls struck at or above the barrier in the case of a barrier higher than the current price and by only using puts struck at or below the barrier in the case of a barrier lower than the current price. If the barrier is never hit, then you certainly won't be above the up barrier at expiration, so you won't owe anything on a call, and you certainly won't be below the down barrier at expiration, so you won't owe anything on a put.

All three static hedging techniques take advantage of knowing that at the time you are reversing your position in these vanilla options, the underlying must be at the barrier. A useful analogy can be made between these approaches to static hedging and the one we examined for forward-start options in Section 12.2. For forward-start options, we purchased an initial set of vanilla options and then had a fixed date on which we would make a single switch of selling our initial package of vanilla options and buying a new vanilla option. For barrier options, we cannot know in advance what the time of the switch will be, but we can know what the forward price of the underlying will be at the time of the switch. As with forward starts, we confine ourselves to one single switch out of the initial vanilla option hedge package. All of these approaches therefore share many of the advantages we saw for the static hedge technique for forward starts:

- A clear distinction between the portion of expected cost that can be locked in at current market prices of vanilla options (including current volatility surface shape) versus the portion that requires projections of what the volatility surface shape will be at the time of the switch.
- An estimate of uncertainty for establishing limits and reserves can be based on readily observable historical market data for possible volatility surface shapes. The impact of uncertainty is easy to calculate since it only needs to be computed at one particular point.
- Future liquidity costs, such as the potential payment of bid-ask spread, are confined to a single switch.
- Although it is to be expected that trading desks will, in practice, adjust the static hedge as market circumstances evolve, it remains useful as a risk management technique to evaluate the consequences of an unadjusted hedge.

The three approaches differ in how they attempt to ensure that the option package will be equal in value to the vanilla call at the time the barrier is hit. The Derman-Ergener-Kani approach (see Derman, Ergener, and Kani 1995) uses a package of vanilla options that expire at different times. The algorithm works backward, starting at a time close to the expiration of the barrier option. If the barrier is hit at this time, the only vanilla options still outstanding will be the vanilla call and the very last option to expire in the package. Since both the underlying price is known (namely, the barrier) and the time to expiry is known, the only remaining factor in determining the values of the vanilla options is the implied volatility, which can be derived from a local or stochastic volatility model (if it is derived from a stochastic volatility model, it will be based on expected values over the probability distribution). Thus, the Derman-Ergener-Kani approach can be viewed as the static hedging analog of the dynamic hedging approaches we have been considering.

Once the prices of the vanilla options at the time the barrier is hit are calculated, you can easily determine the amount of the option that is part of the basket that needs to be sold in order to exactly offset the sale of the vanilla call with the purchase of the option in the basket. You then work backward time period by time period, calculating the values of all vanilla options if the barrier is hit at this time period and calculating the volume of the new option in the basket that is needed to set the price of the entire basket equal to the price of the vanilla call. At each stage, you only need to consider unexpired options, so you only need to consider options for which you have already computed the volumes held.

The following points about the Derman-Ergener-Kani approach should be noted:

- If the barrier is hit in between two time periods for which vanilla options have been included in the package, the results are approximated by the nearest prior time period. The inaccuracy of this approximation can be reduced as much as you want by increasing the number of time periods used.
- The approach can easily accommodate the existence of drift (dividend rate unequal to risk-free rate), since a separate computation is made for each time the barrier could potentially be hit.
- Since the approach relies on the results of a local or stochastic volatility model to forecast future volatility surface levels and shapes, it is vulnerable to the same issue as when these models are used for dynamic hedging—the hedge works only to the extent that the assumptions underlying the model prove to be true. As Derman, Ergener, and Kani state, "The hedge is only truly static if the yield curve, the dividend, and the volatility structures remain unchanged over time. Otherwise, the hedge must be readjusted." This is illustrated in Table 12.6, which shows the potential mismatch in unwind cost at a period close to expiry based on differences between model-assumed volatilities and actual volatilities at the time the barrier is hit.

Note that the Derman-Ergener-Kani approach is vulnerable to model errors relating to both the level of volatility surface and the shape of volatility surface.

TABLE 12.6 Unwind Costs of Derman-Ergener-Kani Hedge of Barrier Option

Strike at-the-money, barrier at 95 percent of forward, and three months to expiry.
Down-and-out call value at initial 20 percent volatility is 3.1955.
Unwind with one month to expiry.

Volatility at Unwind	Unwind Gain or Loss
10.00%	0.4479
15.00%	0.2928
20.00%	0.0000
25.00%	–0.3595
30.00%	–0.7549

The Carr approach (see Carr, Ellis, and Gupta 1998) avoids this dependence on projecting future volatility surfaces and is much simpler to implement, but at a price—it cannot handle volatility skews (though it can handle volatility smiles) and its simplicity depends on the absence of drift (dividend rate equals risk-free rate).

The Carr approach achieves a degree of model independence by using a framework that corresponds directly with the Black-Scholes equation and determining a hedge package that will work, providing no drift or volatility skew is present. In these circumstances, one can calculate exactly a single vanilla put that will be selling at the same price as the vanilla call in the case that a down barrier is hit. It is based on the principle of put-call symmetry. In the boxes, we first explain how the principle of put-call symmetry can be derived from the Black-Scholes equation and then show how the exact Carr hedges can be derived from put-call symmetry.

PUT-CALL SYMMETRY

The principle of put-call symmetry says that if you have two strikes, K_1 and K_2, whose geometric average is the forward price, that is, $\sqrt{K_1 K_2} = F$, then the current price of a call strike at K_1 for expiry T, $C(K_1,T)$, and the current price of a put struck at K_2 for the same expiry T, $P(K_2,T)$, are related by the equation:

$$C(K_1, T) / \sqrt{K_1} = P(K_2, T) / \sqrt{K_2}$$

This formula is a direct and easy consequence of the Black-Scholes formula. From Hull (2012, Section 17.8), the Black-Scholes formula for the price of a call and put based on the forward price is:

Atuhor: The parentheses are unbalanced in both equations

$$C(K_1,T)=e^{-rt}(FN((\ln(F/K_1)+\sigma^2T/2)/\sigma\sqrt{T}$$
$$-K_1N((\ln(F/K_1)-\sigma^2T/2/\sigma\sqrt{T}))$$

$$P(K_2,T)=e^{-rt}(K_2N((\ln(K_2/F)+\sigma^2T/2)/\sigma\sqrt{T}$$
$$-FN((\ln(K_2/F)-\sigma^2T/2)/\sigma\sqrt{T}))$$

But since $F=\sqrt{K_1K_2}$,

$$K_2/F=K_2/\sqrt{K_1K_2}=\sqrt{K_2}/\sqrt{K_1}=\sqrt{K_1K_2}/K_1=F/K_1$$

So,

$$C(K_1,T)/\sqrt{K_1}=e^{-rt}(\sqrt{K_2}N((\ln(F/K_1)+\sigma^2T/2)/\sigma\sqrt{T})$$
$$-\sqrt{K_1}N((\ln(F/K_1)-\sigma^2T/2/\sigma\sqrt{T}))$$

And substituting F/K_1 for K_2/F,

$$P(K_2,T)/\sqrt{K_2}=e^{-rt}(\sqrt{K_2}N((\ln(F/K_1)+\sigma^2T/2)/\sigma\sqrt{T})$$
$$-\sqrt{K_1}N((\ln(F/K_1)-\sigma^2T/2)/\sigma\sqrt{T}))$$

$$=C(K_1,T)/\sqrt{K_1}$$

Since we have utilized the Black-Scholes formula in our derivation, this result holds only under the Black-Scholes assumption of a flat volatility surface for the expiry time T or if the deviation from flat volatility surface is exactly the same at strike K_1 and K_2. However, since the forward is the geometric average of these two strikes, this is equivalent to saying that one strike is the same percentage above the forward as the percentage the other strike is below the forward. For their volatilities to be equal, the volatility surface must have a smile shape, not a skew shape, using the terminology of Section 11.6.2.

DERIVING THE CARR HEDGE

Since no drift is present, the forward price is equal to the spot price, which is the barrier level, H. Since the call is struck at K, we can find a reflection strike, R, such that $\sqrt{KR} = H$ and, by put-call symmetry, $\sqrt{R}Call\ (k) = \sqrt{K}Put\ (R)$. Since $\sqrt{KR} = H, R = H^2 / K, \sqrt{R} = H / \sqrt{K}$, you need to purchase $\frac{\sqrt{K}}{\sqrt{R}} = \frac{K}{H}$ puts struck at H^2 / K.

For an up barrier, one must separately hedge the intrinsic value and the time value of the vanilla call at the time the barrier is hit. The intrinsic value can almost be perfectly offset by selling binary options that pay $2 \times I$, where I is the intrinsic value. Any time the barrier is hit, there will be nearly a 50–50 chance that the binary will finish in-the-money, so its value is close to $50\% \times 2 \times I = I$. In fact, the standard lognormal pricing of a binary results in assuming slightly less than a 50 percent chance of finishing above the barrier, so we need to supplement the binary with I / H of a plain-vanilla call struck at the barrier. The exact value of the binary is $2 \times I \times N\left(-\frac{\sigma\sqrt{T}}{2}\right)$, and the value of the vanilla call struck at the barrier, and hence exactly at-the-money when the barrier is hit, is

$$(I / H) \times H \times \left(N\left(\frac{\sigma\sqrt{T}}{2}\right) - N\left(-\frac{\sigma\sqrt{T}}{2}\right)\right)$$

$$= I \times \left(1 - 2 \times N\left(-\frac{\sigma\sqrt{T}}{2}\right)\right)$$

The sum of these two terms is then I.

The Carr approach has several advantages:

- It shows that it is at least plausible to price the barrier based on options with tenor equal to the final tenor of the barrier, indicating that this is probably where most of the barrier's risk exposure is coming from.
- Having a large binary component of the hedge is an excellent means of highlighting and isolating the pin risk contained in this barrier that dies in-the-money. Techniques we have already developed for managing pin risk on binaries can now easily be brought into play. For example, we could establish a reserve against the pin risk of the binary

(see Section 12.1.4). This approach is quite independent of whether the trading desk actually sells a binary as a part of the hedge—the risk of the binary is present in any case.

- Because the Carr approach uses a small number of options in the hedge package, it is very well suited for developing intuition about how changes in the shape of the volatility surface impact barrier prices.
- Even if you choose to hedge and price using a dynamic hedging approach, the Carr methodology can still be useful as a liquid proxy. Dynamic hedging can be employed for the difference between the barrier and the static hedge determined by the Carr approach. By choosing an initial hedge that, on theoretical grounds, we expect to be close to a good static hedge, we expect to minimize the degree to which changes in option hedges are required. However, by using dynamic hedging, we allow for as much protection as the accuracy of the model provides against uncertainty in skew and drift.
- Neither the presence of volatility smiles nor the uncertainty of future volatility smiles impacts the Carr approach. Since it deals with options that are symmetrically placed relative to the at-the-money strike, all smile effects cancel out.

The simplicity of the Carr approach is lost in the presence of drift or volatility skew. See the appendix to Carr and Chou (1996) for a method of using a large number of vanilla options to create a volatility-independent static hedge of barrier options in the presence of drift. See Carr (2001) for a method of handling volatility skew.

To appreciate how the Carr model performs and to gain the benefit of its insight into the risk structure of barriers, you should study the **CarrBarrier** spreadsheet provided on the website for this book. The spreadsheet shows the hedge structure for all eight possible simple barrier structures and the result of the barrier unwind for a specified scenario. Exercise 12.3 guides you through some sample runs. Here are some of the points you should be looking for:

- The one common element in all eight variants is the use of the reflection option—the one that utilizes the principle of put-call symmetry. It captures the time value of the barrier option at the point the barrier is hit.
- The sample run displayed in Table 12.7 shows that on unwind, for the down call and up put cases, the reflection option exactly offsets the value of the option that needs to be purchased for the in cases and needs to be sold for the out cases. For the up call and down put cases, a binary piece also needs to be offset, but the reflection option offsets the entire time value. In Table 12.8, in which the only change from Table 12.7 is

that the volatility at unwind has been raised, the binary piece (the sum of the binary and binary correction) is unchanged from Table 12.7, but the time value has increased exactly equally for the vanilla option and the reflection option.

- The time value when the barrier is hit depends on how far the barrier is from the strike. In the Table 12.7 example, the up barrier of 110 is further from the 100 strike than the 95 down barrier is, so the up reflection options have far less value than the down reflection options. You can think of the reflection option as taking value away from the out option and transferring it to the in option.
- The up call and down put cases are ones with binary components, since these in options will begin life already in-the-money and these out options cause an in-the-money component to be extinguished. The size of the binary component at the time the barrier is hit is the exact difference between the strike and barrier. It is divided into two pieces: the principal piece is the binary option and the secondary piece is the vanilla option used to supplement the binary. The total value of these two components at initiation will be less than the potential value on hitting the barrier, precisely reflecting the (risk-neutral) probability that the barrier will be hit.
- By trying different values for barrier-hitting scenarios, you will see that as long as volatility skew and drift are both equal to zero, the total impact of buys and sells in all eight cases is always zero. That is, the hedge works perfectly regardless of the assumptions made as to the time remaining when the barrier is hit, the at-the-money volatility, the volatility smile, or the risk-free rate. However, if either drift or volatility skew differs from zero, gains and losses will occur when the barrier is hit, varying by case. Examples are shown in Tables 12.9 and 12.10. It would clearly be a relatively easy task to calculate the size of potential losses based on assumptions about how adverse drift and skew could be at different possible times the barrier is hit. This could serve as input for the determination of reserves and limits.
- When the initial volatility skew, volatility smile, and drift are set equal to zero, pricing given by the standard analytic formula for barriers (shown on the top line in each column) exactly equals the total creation cost of the Carr hedges, as can be seen from the zero on the line labeled "difference." When any of these values is different from zero, the Carr hedge gives a different value than the analytic formula. For example, Tabe 12.11 shows a case that corresponds to the one analyzed in Table 12.5, showing a 3.104 value for the up-and-out call in the presence of a volatility skew compared with a 2.7421 value using the analytic formula. Note that the presence of volatility skew (or drift) in the initial conditions does not imply that the Carr hedge will

not work. Only conditions at the time the barrier is hit determine the efficiency of the hedge.

In Exercise 12.4 you will run a Monte Carlo simulation of the cost of a hedging strategy that hedges a barrier option with the Carr hedge, utilizing the spreadsheet **CarrBarrierMC**.

A more general approach to static hedging that can handle all drift and volatility shape conditions is optimization, in which a set of vanilla options is chosen that fits as closely as possible the unwind of the barrier option at different possible times, drifts, volatility levels, and volatility surface shapes that may prevail when the barrier is hit. The optimization approach is discussed in Dembo (1994). Often no perfect static hedge can be found, but in these cases the optimization produces information on the distribution of possible hedge errors that can be useful input for determining a reasonable reserve. A similar approach can be taken to many different types of exotic structures.

The **OptBarrier** spreadsheet illustrates how optimization can be used to find a static hedge for a barrier option. If the possible conditions when the barrier is hit are restricted to zero drift and volatility smile but no skew, then the Excel Solver will find a set of vanilla options that almost exactly matches the barrier unwind for all volatility levels and times to expiry (although the particular set of hedges chosen may lack the clarity of insight that the Carr hedges offer). Of course, this is not a surprise since we know from the Carr approach that a perfect static hedge is possible under these circumstances. When different nonzero drift and volatility skew conditions are allowed, the match of the barrier unwind is no longer as exact.

The spreadsheet determines how much this slippage can be across all the specified cases of hitting time, skew, and drift. As with the Carr approach, this information can then be used to set reserves and limits. The difference from the Carr approach is the objective to find a hedge that minimizes the amount of this slippage. Exercise 12.3 guides you through some sample runs.

As a concluding note, observe that there is a lower limit on the uncertainty of unwind costs for any static hedging approach. Any dynamic hedging model can be used to compute the unwind cost of a selected static hedging strategy. So any difference in the pricing of barrier options between different dynamic hedging models translates into uncertainty of unwind costs. Practical experience with dynamic hedging models shows that differences in assumptions (for example, stochastic volatility versus local volatility and the frequency of jumps) give rise to substantial differences in barrier options prices utilizing the same input for current vanilla options prices. So you can search for static hedges that minimize the uncertainty of unwind costs, but an irreducible uncertainty will always remain that can be controlled only through limits and reserves. Static hedging greatly simplifies the calculations needed for limits and reserves.

Atuhor: can you verify with the author that these are acronyms and so call caps?

			CDO	CDI	CUO	CUI	PDO	PDI	PUO	PUI
Price	100.00									
Strike	100.00		3.1955	0.7923	0.6343	3.3535	0.0778	3.9100	3.8791	0.1087
Up barrier	110.00	Vanilla	–3.9878	0	–3.9878	0	–3.9878	0	–3.9878	0
Down barrier	95.00	Digital	0	0	3.1581	–3.1581	3.2171	–3.2171	0	0
Time to expiry	0.25	Correct dig	0	0	0.0867	–0.0867	–0.0994	0.0994	0	0
Rate	0.00%	Reflect	0.7923	–0.7923	0.1087	–0.1087	0.7923	–0.7923	0.1087	–0.1087
Drift	0.00%	Total	–3.1955	–0.7923	–0.6343	–3.3535	–0.0778	–3.9100	–3.8791	–0.1087
ATM volatility	20.00%	Difference	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Vol smile	0.00%	Reflect point	90.25	90.25	121	121	90.25	90.25	121	121
Vol skew	0.00%									
		At barrier								
Time to expiry	0.25	Vanilla	–1.8881	1.8881	–10.9539	10.9539	–6.8881	6.8881	–0.9539	0.9539
Rate	0.00%	Digital	0	0	9.6012	–9.6012	5.1994	–5.1994	0	0
Drift	0.00%	Correct dig	0	0	0.3988	–0.3988	–0.1994	0.1994	0	0
ATM volatility	20.00%	Reflect	1.8881	–1.8881	0.9539	–0.9539	1.8881	–1.8881	0.9539	–0.9539
Vol smile	0.00%	Total	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Vol skew	0.00%									

Forward	100
Up forward	110
Down forward	95

TABLE 12.7 Carr Static Hedge

			CDO	CDI	CUO	CUI	PDO	PDI	PUO	PUI
Price	100.00		CDO	CDI	CUO	CUI	PDO	PDI	PUO	PUI
Strike	100.00		3.1955	0.7923	0.6343	3.3535	0.0778	3.9100	3.8791	0.1087
Up barrier	110.00	Vanilla	–3.9878	0	–3.9878	0	–3.9878	0	–3.9878	0
Down barrier	95.00	Digital	0	0	3.1581	–3.1581	3.2171	–3.2171	0	0
Time to expiry	0.25	Correct dig	0	0	0.0867	–0.0867	–0.0994	0.0994	0	0
Rate	0.00%	Reflect	0.7923	–0.7923	0.1087	–0.1087	0.7923	–0.7923	0.1087	–0.1087
Drift	0.00%	Total	–3.1955	–0.7923	–0.6343	–3.3535	–0.0778	–3.9100	–3.8791	–0.1087
ATM volatility	20.00%	Difference	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Vol smile	0.00%	Reflect point	90.25	90.25	121	121	90.25	90.25	121	121
Vol skew	0.00%									
		At barrier								
Time to expiry	0.25	Vanilla	–5.5195	5.5195	–14.2920	14.2920	–10.5195	10.5195	–4.2920	4.2920
Rate	0.00%	Digital	0	0	9.2034	–9.2034	5.3983	–5.3983	0	0
Drift	0.00%	Correct dig	0	0	0.7966	–0.7966	–0.3983	0.3983	0	0
ATM volatility	40.00%	Reflect	5.5195	–5.5195	4.2920	–4.2920	5.5195	–5.5195	4.2920	–4.2920
Vol smile	0.00%	Total	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Vol skew	0.00%									

Forward	100
Up forward	110
Down forward	95

TABLE 12.8 Carr Static Hedge with Higher Volatility at Unwind

Price	100.00		CDO	CDI	CUO	CUI	PDO	PDI	PUO	PUI
Strike	100.00		3.1955	0.7923	0.6343	3.3535	0.0778	3.9100	3.8791	0.1087
Up barrier	110.00	Vanilla	–3.9878	0	–3.9878	0	–3.9878	0	–3.9878	0
Down barrier	95.00	Digital	0	0	3.1581	–3.1581	3.2171	–3.2171	0	0
Time to expiry	0.25	Correct dig	0	0	0.0867	–0.0867	–0.0994	0.0994	0	0
Rate	0.00%	Reflect	0.7923	–0.7923	0.1087	–0.1087	0.7923	–0.7923	0.1087	–0.1087
Drift	0.00%	Total	–3.1955	–0.7923	–0.6343	–3.3535	–0.0778	–3.9100	–3.8791	–0.1087
ATM volatility	20.00%	Difference	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Vol smile	0.00%	Reflect point	90.25	90.25	121	121	90.25	90.25	121	121
Vol skew	0.00%									
		At barrier								
Time to expiry	0.25	Vanilla	–1.9757	1.9757	–10.8303	10.8303	–6.9757	6.9757	–0.8303	0.8303
Rate	0.00%	Digital	0	0	9.2028	–9.2028	5.0002	–5.0002	0	0
Drift	0.00%	Correct dig	0	0	0.3988	–0.3988	–0.1994	0.1994	0	0
ATM volatility	20.00%	Reflect	1.8010	–1.8010	1.0830	–1.0830	1.8010	–1.8010	1.0830	–1.0830
Vol smile	0.00%	Total	–0.1746	0.1746	–0.1457	0.1457	–0.3739	0.3739	0.2527	–0.2527
Vol skew	10.00%									

Forward	100
Up forward	110
Down forward	95

TABLE 12.9 Carr Static Hedge with Nonzero Skew at Unwind

			CDO	CDI	CUO	CUI	PDO	PDI	PUO	PUI
Price	100.00									
Strike	100.00		3.1955	0.7923	0.6343	3.3535	0.0778	3.9100	3.8791	0.1087
Up barrier	110.00	Vanilla	–3.9878	0	–3.9878	0	–3.9878	0	–3.9878	0
Down barrier	95.00	Digital	0	0	3.1581	–3.1581	3.2171	–3.2171	0	0
Time to expiry	0.25	Correct dig	0	0	0.0867	–0.0867	–0.0994	0.0994	0	0
Rate	0.00%	Reflect	0.7923	–0.7923	0.1087	–0.1087	0.7923	–0.7923	0.1087	–0.1087
Drift	0.00%	Total	–3.1955	–0.7923	–0.6343	–3.3535	–0.0778	–3.9100	–3.8791	–0.1087
ATM volatility	20.00%	Difference	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Vol smile	0.00%	Reflect point	90.25	90.25	121	121	90.25	90.25	121	121
Vol skew	0.00%									
		At barrier								
Time to expiry	0.25	Vanilla	–1.6691	1.6691	–10.2694	10.2694	–7.3790	7.3790	–1.0913	1.0913
Rate	0.00%	Digital	0	0	9.0052	–9.0052	5.4974	–5.4974	0	0
Drift	–3.00%	Correct dig	0	0	0.3610	–0.3610	–0.2179	0.2179	0	0
ATM volatility	20.00%	Reflect	2.1120	–2.1120	0.8243	–0.8243	2.1120	–2.1120	0.8243	–0.8243
Vol smile	0.00%	Total	0.4429	–0.4429	–0.0789	0.0789	0.0125	–0.0125	–0.2671	0.2671
Vol skew	0.00%									

Forward	100
Up forward	109.178086
Down forward	94.29016521

TABLE 12.10 Carr Static Hedge with Nonzero Drift at Unwind

			CDO	CDI	CUO	CUI	PDO	PDI	PUO	PUI
Price	100.00									
Strike	100.00		3.9244	0.0633	2.7421	1.2457	0.8479	3.1399	3.9874	0.0003
Up barrier	120.00	Vanilla	–3.9878	0	–3.9878	0	–3.9878	0	–3.9878	0
Down barrier	90.00	Digital	0	0	0.8708	–0.8708	3.7355	–3.7355	0	0
Time to expiry	0.25	Correct dig	0	0	0.0130	–0.0130	–0.0935	0.0935	0	0
Rate	0.00%	Reflect	0.1266	–0.1266	0.0000	0.0000	0.1266	–0.1266	0.0000	0.0000
Drift	0.00%	Total	–3.8611	–0.1266	–3.1040	–0.8838	–0.2192	–3.7686	–3.9878	0.0000
ATM volatility	20.00%	Difference	0.0633	–0.0633	–0.3619	0.3619	0.6287	–0.6287	–0.0003	0.0003
Vol smile	0.00%	Reflect point	81	81	144	144	81	81	144	144
Vol skew	–10.95%									
		At barrier								
Time to expiry	0.25	Vanilla	–0.7124	0.7124	–20.1473	20.1473	–10.7124	10.7124	–0.1473	0.1473
Rate	0.00%	Digital	0	0	19.2024	–19.2024	10.3988	–10.3988	0	0
Drift	0.00%	Correct dig	0	0	0.7976	–0.7976	–0.3988	0.3988	0	0
ATM volatility	20.00%	Reflect	0.7124	–0.7124	0.1473	–0.1473	0.7124	–0.7124	0.1473	–0.1473
Vol smile	0.00%	Total	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
Vol skew	0.00%									

Forward	100
Up forward	120
Down forward	90

TABLE 12.11 Carr Static Hedge with Nonzero Skew at Initiation

12.3.4 Barrier Options with Rebates, Lookback, and Ladder Options

We will show how to use barrier options to create a static hedge for barrier options with rebates, lookback, and ladder options. Thus, we can transfer the techniques we have studied for using vanilla options to represent and hedge barrier option positions to create vanilla option representations and hedges of barrier options with rebates, lookback, and ladder options.

The use of a rebate feature in a barrier option can be regarded as a binary option triggered by a barrier. For example, suppose you have a down-and-out call that pays a rebate of $2 million if the down barrier is hit and the call is canceled. This can be viewed as the sum of a down-and-out call with no rebate and a down-and-in binary option that pays $2 million if the barrier is hit. However, since a binary option can be represented by being long one vanilla call and short another vanilla call, as discussed in Section 12.1.4, a down-and-in binary can also be treated as being long one down-and-in call and short another down-and-in call. So the rebate can be hedged and valued through the methodology we have already developed for barriers without rebates.

Lookback options come in two varieties: those that pay the difference between the maximum price that an asset achieves during a selected period and the closing price and those that pay the difference between the maximum price that an asset achieves during a selected period and a fixed strike. Symbolically, the lookback pays either $S_{max} - S_T$ or max $(0, S_{max} - K)$. We can reproduce the payoffs of a lookback of the first type exactly by buying a lookback of the second type with a strike equal to the current price of the asset (S_0), selling the asset forward to time T, and buying a forward delivery of S_0 dollars at time T. Since S_{max} is certainly $\geq S_0$, $\max(0, S_{max} - S_0) = S_{max} - S_0$, the total payoff of this combination at time T is:

$$\max(0, S_{max} - S_0) - S_T + S_0 = (S_{max} - S_0) - S_T + S_0 = S_{max} - S_T \quad (12.4)$$

So if we can hedge the second type of lookback option by static hedging with barriers, we can create the first type of lookback option by static hedging with barriers as well.

Lookback options have a closely related product called *ladder options* that pay max $(0, S_{max} - K)$ rounded down by a specified increment. For example, if $K = 100$ and $S_{max} = 117.3$, the lookback call of the second type would pay 17.3, a ladder with increments of 1 would pay 17, a ladder with increments of 5 would pay 15, and a ladder with increments of 10 would pay 10. Since a lookback call can be approximated as closely as we want by a ladder with a small enough increment, it is sufficient to show how to statically hedge a ladder with barriers.

It is easy to create a static hedge for a ladder option using up-and-in binary options. For each ladder rung, you buy an up-and-in binary option of the same tenor that pays the increment conditional on the rung being breached at some point during the life of the option. For example, if $K = 100$ and we have a ladder with increments of 5, we buy an up-and-in binary option having a payoff of 5 and a barrier of 105, another with a payoff of 5 and a barrier of 110, and so on. If the highest level the underlying reaches during the life of the ladder option is 12, then 10 will be owed on the ladder option, but the binary up-and-ins with barriers of 105 and 110 will both have been triggered for a payment of 5 + 5 = 10.

12.3.5 Broader Classes of Path-Dependent Exotics

Now that we have looked at several dynamic hedging and static hedging alternatives for managing risk on standard barrier options, we want to examine how these approaches can be generalized to the full universe of single-asset exotics. We will focus most of our attention on double barriers and partial-time barriers, since these are reasonably popular products and since any techniques that are flexible enough to handle these variants would be flexible enough to handle any product.

Double barriers knock out (or knock in) if either a higher or a lower barrier is crossed. An example would be a one-year call option struck at 100 that knocks out if the price during the year is ever either above 120 or below 80. Partial-time barriers have a restricted time period during which the barrier provision applies. An example would be a one-year call option struck at 100 that knocks out if the price is below 90 any time between the end of month 3 and the end of month 9. If the price goes below 90 prior to month 3 but then goes back above 90 by the end of month 3, no knock-out occurs. Similarly, if the first time the price goes below 90 is after month 9, no knock-out occurs.

The greatest flexibility is offered by dynamic hedging, using either local volatility or stochastic volatility models, and by the Derman-Ergener-Kani approach to static hedging. Both can be easily generalized to double barriers and partial-time barriers. Local volatility models that solve for the exotic option values on a tree constructed to fit vanilla option prices can be easily adapted to solve for virtually any set of payoffs. Stochastic volatility models, which may require Monte Carlo simulation solutions, can easily handle any deterministic payout. The Derman-Ergener-Kani static hedging algorithm can solve for hedge packages that give zero unwind costs for double barriers and partial-time barriers just as easily as for standard barriers. The **DermanErgenerKaniDoubleBarrier** and **DermanErgenerKaniPartialBarrier** spreadsheets illustrate this computation. An interested reader could use these spreadsheets as a guide to program a general calculator for applying the Derman-Ergener-Kani method to more complex barriers.

The drawbacks of dynamic hedging and Derman-Ergener-Kani static hedging that we analyzed for standard barriers apply in a more general setting as well. It will still be difficult to project the potential effects of hedge slippage for dynamic hedging. This is a heightened concern for double barriers since they have a reputation among exotics traders as particularly treacherous to dynamically hedge since they are almost always threatening to cross one barrier or the other. The dependence of Derman-Ergener-Kani on the model used to calculate the hedge ratios, and hence its vulnerability to being wrong about future volatility levels, remains true for the expanded product set.

Peter Carr and his collaborators have done a lot to expand the applicability of his static hedging approach beyond standard barriers. In particular, Carr, Ellis, and Gupta (1998, Section 3.1) have developed a static hedge for double barriers, and Carr and Chou (1997) have developed a static hedge for partial-time barriers. Similar results are presented in Andersen and Andreasen (2000). These hedges offer one of the major advantages of the Carr hedge for standard barriers—protection against shifts in volatility levels. However, they do not offer another major advantage of the Carr hedge for standard barriers: They are not simple to compute and do not provide much intuitive insight into the risk structure of the exotic being hedged. The specialized nature of each construction does not offer significant guidance as to how to build hedges for other exotics.

Optimal fitting would seem to offer the best hope for an easy-to-generalize static hedge that will minimize sensitivity to model assumptions. However, unlike the Derman-Ergener-Kani method, which automates the selection of the vanilla options to be used in hedging a particular exotic, the optimal fitting approach relies on practitioner insight to generate a good set of hedge candidates. A poor choice of possible hedges results in a poorly performing static hedge. A possible solution is to try to generalize the Derman-Ergener-Kani approach to fit to a range of volatility surfaces rather than to a single one. Some promising results along these lines have been obtained by Allen and Padovani (2002, Section 6). A copy of this paper is on the book website.

12.4 CORRELATION-DEPENDENT OPTIONS

Valuation and hedging strategies for derivatives whose payoff is a function of more than one underlying asset are critically dependent on assumptions about correlation between the underlying assets. With only a few exceptions (which are discussed in Section 12.4.3), there is an absence of sufficiently liquid market prices to enable implied correlations to be inferred in the way implied volatilities can be derived from reasonably liquid prices of vanilla options. So much of the focus of risk management for these derivatives revolves around controlling the degree of exposure to correlation assumptions and building reserves and limits against the differences between actual realized and estimated correlations.

An important distinction within derivatives with multiasset payoffs should be made between those whose payoff is based on a linear combination of asset prices (for example, the average of a set of prices or the difference between two prices) and those whose payoff is based on a nonlinear combination of asset prices (for example, the maximum of a set of prices or the product of two prices). When the payoff is based on a linear combination of asset prices, risk management is considerably simpler, even if the payoff itself is a nonlinear function of the linear combination of asset prices, such as an option on the average of a set of prices. We therefore discuss these two types of derivatives in separate sections. A final section discusses options that depend on a different type of correlation—the correlation between underlying asset value and the probability of option exercise.

12.4.1 Linear Combinations of Asset Prices

Derivatives whose payoff depends on a linear combination of asset prices share several important characteristics that simplify their risk management:

- If the payoff function is a linear function of the linear combination of asset prices, then the derivative does not have any option characteristics and can be perfectly hedged with a static portfolio of the underlying assets. In such cases, the valuation of the derivative is independent of correlation assumptions. This is not true of derivatives whose payoff function is a linear function of a nonlinear combination of asset prices, such as a forward based on the product of an asset price and an FX rate (a so-called quanto) that requires dynamic hedging.
- Even when the payoff function is a nonlinear function of the linear combination of asset prices, such as an option on the average of a set of prices, and therefore requires dynamic hedging, the rules for dynamic hedging are particularly simple to calculate.
- Even when dynamic hedging is required, it is often possible to make very good approximations of valuation and the risk of incorrect correlation assumptions using a standard Black-Scholes model.

accompanied by higher correlation than ordinary price moves. This can result in baskets being priced at higher volatility skews than individual components of the basket since it increases correlation and hence increases volatility at lower price levels. For further discussion of this point, see Derman and Zou (2001).

The Monte Carlo approach affords great flexibility, including the incorporation of stochastic volatility and price jump assumptions. Its drawback is difficulty in valuing American-style options that require the determination of optimal early exercise strategies. Further developments in Monte Carlo modeling do allow approximations of American option valuation; see, for example, Broadie, Glasserman, and Jain (1997).

The alternative approach for American-style options on baskets is the three-dimensional tree approach described in Hull and White (1994). This approach enables the combination of two trinomial trees that have been fitted to the full implied volatility surface, using the techniques discussed in Section 12.3.2, to be combined into a single tree based on assumed correlations, which can vary by node. Basket values can then be computed on the combined tree and option values determined by working backwards on the tree. This approach has the advantage of greater precision in determining early exercise strategies. The disadvantages are that it is only computationally feasible for baskets involving two assets and it is restricted to using local volatility models to replicate the implied volatility surface, which lacks the flexibility to incorporate stochastic volatility or price jumps. A possible combination of the two methods for more than two assets would be to determine the option price for the final exercise using the more precise Monte Carlo method and estimating the extra value due to possible early exercise using the three-dimensional tree technique using the first two principal components of the assets as the two variables to be modeled on the tree.

12.4.2 Risk Management of Options on Linear Combinations

We will now take advantage of the simple formula available to approximate the value of an option on a linear combination of assets to examine how risks arising from positions in these options should be managed.

One possible risk management technique is pure dynamic hedging of options positions in a particular linear combination. This is operationally straightforward, as discussed in Section 12.4.1.2. However, it encounters the same deficiencies of reliance on the delta-hedging strategy that we discussed in Section 11.1. The same arguments favoring the use of other options in hedging that were given in Section 11.1 apply, but it is unusual to find any liquidity in options on asset combinations. This suggests the use of options on individual assets comprising the basket as part of the hedge.

TABLE 12.12 The Impact of Hedging Basket Options with Single-Stock Options

	Standard Deviation	Transaction Costs
Dynamically hedge with underlying stocks only	28.7%	2.3%
Purchase at-the-money options on stocks A and B and then dynamically hedge	14.0%	1.9%

Consider the following simple example. An option has been written on the average of two assets, A and B. Compare the simulation results of a pure dynamic hedge with the underlying stocks with the simulation results of a hedge that involves first purchasing options on assets A and B and then dynamically hedging the resulting position with the underlying stocks.

Suppose a one-year at-the-money option has been written on the average of the prices of two stocks, A and B. Assume that both A and B have 20 percent volatility on average with a 33 percent standard deviation of volatility and that correlation between the two assets averages 0 percent with a 33 percent standard deviation. We will simulate two hedging strategies: Use a pure dynamic hedge with the underlying stocks, or first purchase an at-the-money option on A and an at-the-money option on B and then dynamically hedge the resulting position with the underlying stocks. The ratio of the notional of purchased options on individual stocks to the notional of the sold basket option we will use is 70 percent, split equally between the option on A and the option on B. This 70 percent ratio is suggested by the average volatility of the basket option being $\sqrt{(50\% \times 20\%)^2 + (50\% \times 20\%)^2} = 14.14\%$, which is just a little bit more than 70 percent of the 20 percent average volatility of the individual stocks. Simulation starting with different ratios of individual stock options to the basket options confirms that 70 percent is the ratio that results in the lowest standard deviation of the dynamic hedging results. Table 12.12 compares the results between the two hedging strategies.

Although a substantial reduction in uncertainty and transaction costs results from utilizing an option in the constituent stocks as a hedge, it is not as large a reduction as was shown for hedging vanilla options with vanilla options at other strikes in Table 11.2. Even if we were certain of the correlation, the static hedge utilizing the purchase of at-the-money options on stocks A and B can only reduce the standard deviation to 12.2 percent. The intuitive reason for this is that the relationship of one strike being located midway between two other strikes is obviously stable, whereas the underlying stock options can move into or out of the money without a similar move on the part of the basket option. For example, if stock A's price rises by 20 percent and stock B's price falls by 20 percent, the previously at-the-money call options on stock A and B will now be substantially in-the-money and out-of-the-money, respectively. In both cases, their sensitivity to volatility will be considerably reduced from the time of initiation. This is not true for the basket option, which will still have its same initial sensitivity to volatility since it is still at-the-money relative to the average price of A and B.

A possible remedy would be to dynamically change the amount of single stock options being used to hedge in response to changes in relative volatility sensitivity of the basket option and single stock options. This has many similar virtues and drawbacks with the proposal to dynamically hedge barrier options with vanilla options that was considered in Section 12.3.2. One advantage in this case is that it is considerably easier to calculate the required option hedges in the Monte Carlo simulation, provided you are willing to accept the degree of approximation of the simple formula.

Whether employing static hedging or dynamic hedging with single-asset options, the following rules should apply:

- Any residual exposure to the uncertainty of correlation should be reflected in reserve policies and limits, since this is an exposure that cannot be hedged with liquid instruments.
- Residual unhedgeable exposure to the uncertainty of single-asset volatility should be quantified, as shown in the Monte Carlo example in Table 12.12, and reflected in reserve policies and limits.
- Valuation procedures and risk measurement should be in agreement. If implied volatilities of individual assets are used as an input to the valuation of a basket option, then the exposure to changes in each constituent asset's implied volatility should be reflected, either statically or dynamically, in price-vol matrix reports and other volatility exposure measures computed for the individual asset. Similarly, delta exposure should be reflected in individual underlying asset position reports. If this principle is not followed, valuation exposure to changes in the price or volatility of an asset can grow without control by being included in more and more basket products.
- In some cases, individual asset volatility may be so slight a contribution to the risk of a basket option that it is not worth the effort of utilizing the implied volatility as an input to valuation or reflecting exposure to volatility changes in individual asset risk reports. The basket option will then effectively be managed as if it was an option on a separate underlying unrelated to the single-asset options. Note that this does not change the use of the individual underlying to perform delta hedging.

TABLE 12.13 Sensitivities of Option on Basket

Correlation Level	1% Shift in Volatilities	10% Shift in Correlation
90%	0.97%	0.51%
75%	0.94%	0.53%
50%	0.87%	0.57%
25%	0.79%	0.62%
0%	0.71%	0.69%
–25%	0.61%	0.79%
–50%	0.50%	0.95%
–75%	0.35%	1.30%
–90%	0.22%	1.85%
–95%	0.16%	2.31%
–98%	0.10%	2.90%

The **BasketOption** spreadsheet on the website for this book shows the calculation of basket option exposures to changes in correlation and individual asset volatility under the approximation of the simple formula. Table 12.13 shows some sample results for an equally weighted two-asset basket with both assets having a 20 percent volatility. The impacts shown are for a 1 percent shift in the volatilities of both assets (for example, 20% + 1% = 21%) and a 10 percent shift in correlation (for example, 75% + 10% = 85%).

Note how the relative contribution of individual stock volatility relative to correlation declines sharply as correlation levels become negative. This is very relevant for options on the spread between two asset prices, since the hedge basket then consists of a positive position in one asset and a negative position in the other. If the assets are strongly correlated, their positions in the basket will show high negative correlation. In these cases, hedging the individual option volatilities is questionable.

One reporting issue for all multiasset derivatives is whether to take correlation into account when reporting delta and vega exposure of the derivative. As a concrete example, consider a forward on the average of two stocks, A and B, whose prices are 90 percent correlated. If the overall basket position has an exposure of $1 million for a 10 percent rise in the average price, should you show the exposure to A as $500,000 or as something closer to $1 million to reflect the probability that a rise in the price of A will be accompanied by a rise in the price of B? Clearly, for purposes of the firm's consolidated risk-management reports, $500,000 is the right figure since the consolidated reports will also be showing a $500,000 exposure to B and these two positions will contribute to the consolidated reporting of total exposure to a 10 percent increase in stock prices. If you used a position closer to \$1 million for the A exposure, it would have the absurd result, when combined with exposure to B, of showing an exposure greater than \$1 million to a 10 percent increase in stock prices. However, including a correlation may be appropriate for specially tailored reports for traders who want a quick rule of thumb about how much the basket price will move when stock A's price moves (perhaps because A's price is more liquid than B's). A particular example that has attracted industry attention is the sensitivity of convertible bond prices to changes in the underlying stock price, which we discuss further in Section 12.4.4.

A particular example of a basket option is an Asian option on a single asset. An Asian option is an option on the average price of the asset over a specified set of observations. This is equivalent to an option on a basket of forwards where all the forwards are for the same underlying asset. Obviously, one would expect correlations on such forwards to be quite high. In fact, the conventional Asian option pricing formula assumes a correlation of 100 percent (see Hull 2012, Section 25.12), which is equivalent to assuming constant interest rates, which is slightly inaccurate. Note that the time period over which each forward will contribute volatility to the basket is different, which is a key element to be taken into account in the pricing of the option.

12.4.3 Index Options

As a generalization, we have stated that most multiasset derivatives are illiquid. But this rule has clear exceptions—most prominently, options on interest rate swaps and options on equity indexes. Options on interest rate swaps, also known as *swaptions*, are mathematically and financially equivalent to options on a basket of forwards so they reflect an implied correlation. This special case is treated at length in Section 12.5. Options on stock indexes, such as the S&P 500, NASDAQ, FTSE, and Nikkei, are among the most widely traded of all options. Comparing implied volatilities of stock index options with implied volatilities of options on single stocks that are constituents of the index will therefore yield implied correlation levels. We look at the risk management consequences, which can also be applied to other liquid index options such as options on commodity baskets and FX baskets.

The first principle is that the valuation of a reasonably liquid index option should always be directly based on market prices for the index option and not derived from prices for options on individual stocks in the index and a correlation assumption. Correlation assumptions, no matter how