Homework #5

Due **Wednesday, March 23** in Gradescope by **11:59 pm ET**

READ Textbook Sections 1.3.4 and 1.4.1

WRITE AND SUBMIT solutions to the following problems.

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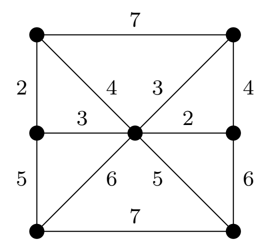
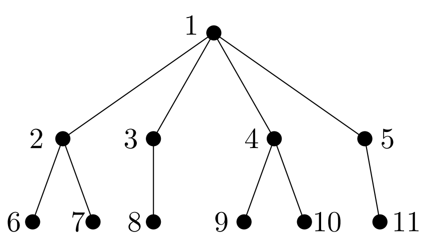
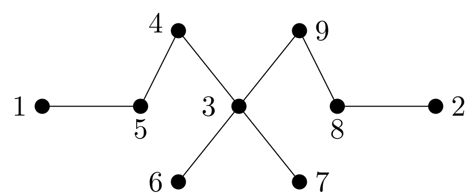
1. (8 points) Textbook Section 1.3.3, part of Problem 5:
Use Kruskal's algorithm to find a minimum weight spanning tree of the following graph. Be sure to (briefly!) show your steps.



2. (16 points) Textbook Section 1.3.4, Problem 2:
Use Prüfer's method to find the Prüfer sequences of the following two trees. As always, (briefly) show your steps.





3. (8 points) Textbook Section 1.3.4, Problem 3:
Use Prüfer's method to draw and label a tree with Prüfer sequence 5,4,3,5,4,3,5,4,3. As always, (briefly) show your steps.

4. (16 points) Textbook Section 1.3.4, Problem 1:
Let T be a labeled tree, and let σ be its Prüfer sequence. For each vertex $v \in V(T)$, prove that v appears in σ exactly $\deg(v) - 1$ times.

(*Suggestion*: Do an induction on $n \geq 2$, where n is the order of the tree.)

(*Note*: As a special case, this means that none of the leaves of T appear in the sequence σ at all. The textbook states that as a separate fact, but since it's just a special case of the above statement, you only need to prove the above statement.)

(continued next page)

5. (18 points) For each of the following four graphs, write down its Laplacian matrix, and then use the Matrix Tree Theorem to find its number of spanning trees.

P_4 $\qquad$ C_4 $\qquad$ K_4 $\qquad$ $K_{2,3}$

6. (20 points) Textbook Section 1.3.4, Problem 4 (expanded a bit):

(a) Use Prüfer's method to draw and label the trees with Prüfer sequences 1,1,1,1,1 and 3,3,3,3.

(b) Inspired by your answers in part (a), make a conjecture about which trees have constant Prüfer sequences.

(c) Prove your conjecture from part (b).

Optional Challenges (do NOT hand in):
Textbook Section 1.3.4, Problems 6, 7.

Prove the linear algebra fact stated in the last paragraph of page 49:

If M is an $n \times n$ matrix with the property that all of its rows and all of its columns sum to 0, then all cofactors of M have the same value.

(*Suggestion*: Start by proving that the $(1, 1)$ and $(1, 2)$ cofactors are the same.)

Questions? You can ask in:

Class: MWF 11:00–11:50am, SMUD 205
Tu 9:00–9:50am, SMUD 205

My office hours: Mon 2:30–3:30pm, Tue 2–3:30pm, and Thu, 1–2:30pm, SMUD 406

Anna's Math Fellow office hours:
Sundays, 7:30–9:00pm, and Tuesdays, 6:00–7:30pm,
SMUD 007

Also, you may email me any time at `rlbenedetto@amherst.edu`