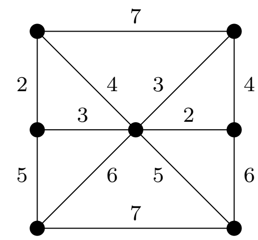
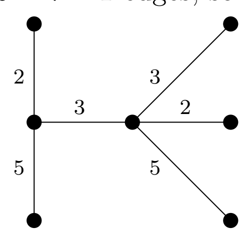
Math 280, Spring 2022

## Solutions to Homework #5

1. (8 points) Textbook Section 1.3.3, part of Problem 5:
Use Kruskal's algorithm to find a minimum weight spanning tree of the following graph. Be sure to (briefly!) show your steps.



**Solution**.
Steps 1 and 2: Add the smallest edges (both weight 2)
Steps 3 and 4: Add the next smallest edges (both weight 3)
Steps 5 and 6: We cannot add either edge of weight 4, since either one would form a cycle. So we add the two edges of weight 5.
There are 7 vertices, and we have added $6 = 7 - 1$ edges, so we are done. Here's the result:



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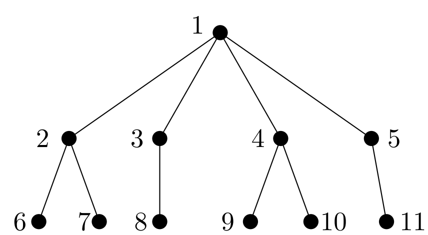
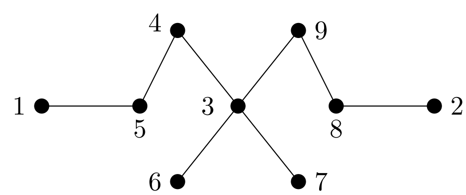
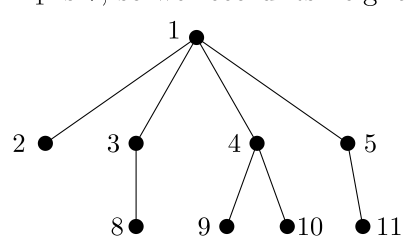
2. (16 points) Textbook Section 1.3.4, Problem 2: Use Prüfer's method to find the Prüfer sequences of the following two trees. As always, (briefly) show your steps.

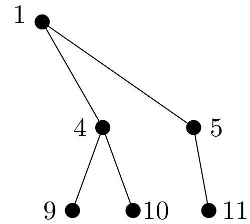




**Solution**. **First Tree**:
$\boxed{i=0}$ The smallest-labelled leaf of $T = T_0$ is 6, so we record its neighbor $\boxed{2}$ and let $T_1 = T_0 - 6$.
$\boxed{i=1}$ The smallest-labelled leaf of $T_1$ is 7, so we record its neighbor $\boxed{2}$ and let $T_2 = T_1 - 7$, which is:



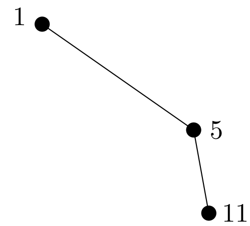
$\boxed{i=2}$ The smallest-labelled leaf of $T_2$ is 2, so we record its neighbor $\boxed{1}$ and let $T_3 = T_2 - 2$.
$\boxed{i=3}$ The smallest-labelled leaf of $T_3$ is 8, so we record its neighbor $\boxed{3}$ and let $T_4 = T_3 - 8$.
$\boxed{i=4}$ The smallest-labelled leaf of $T_4$ is 3, so we record its neighbor $\boxed{1}$ and let $T_5 = T_4 - 3$, which is:




$\boxed{i=5}$ The smallest-labelled leaf of $T_5$ is 9, so we record its neighbor $\boxed{4}$ and let $T_6 = T_5 - 9$.

$\boxed{i=6}$ The smallest-labelled leaf of $T_6$ is 10, so we record its neighbor $\boxed{4}$ and let $T_7 = T_6 - 10$.

$\boxed{i=7}$ The smallest-labelled leaf of $T_7$ is 4, so we record its neighbor $\boxed{1}$ and let $T_8 = T_7 - 4$, which is:




$\boxed{i=8}$ The smallest-labelled leaf of $T_8$ is 1, so we record its neighbor $\boxed{5}$ and let $T_9 = T_7 - 1$.

$\boxed{i=9}$ We are down to a two-vertex tree, so we stop.

Reading off what we recorded, the Prüfer sequence is $\boxed{2,2,1,3,1,4,4,1,5}$

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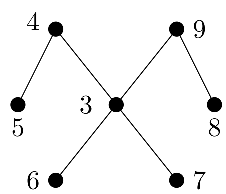
**Second Tree**:

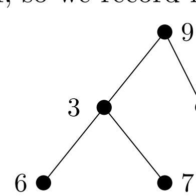
$\boxed{i=0}$ The smallest-labelled leaf of $T = T_0$ is 1, so we record its neighbor $\boxed{5}$ and let $T_1 = T_0 - 1$.

$\boxed{i=1}$ The smallest-labelled leaf of $T_1$ is 2, so we record its neighbor $\boxed{8}$ and let $T_2 = T_1 - 2$, which is:



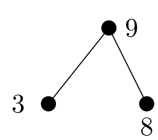

$\boxed{i=2}$ The smallest-labelled leaf of $T_2$ is 5, so we record its neighbor $\boxed{4}$ and let $T_3 = T_2 - 5$.

$\boxed{i=3}$ The smallest-labelled leaf of $T_3$ is 4, so we record its neighbor $\boxed{3}$ and let $T_4 = T_3 - 4$, which is:




$\boxed{i=4}$ The smallest-labelled leaf of $T_4$ is 6, so we record its neighbor $\boxed{3}$ and let $T_5 = T_4 - 6$.

$\boxed{i=5}$ The smallest-labelled leaf of $T_5$ is 7, so we record its neighbor $\boxed{3}$ and let $T_6 = T_5 - 7$, which is:




$\boxed{i=6}$ The smallest-labelled leaf of $T_6$ is 3, so we record its neighbor $\boxed{9}$ and let $T_7 = T_6 - 3$.

$\boxed{i=7}$ We are down to a two-vertex tree, so we stop.

Reading off what we recorded, the Prüfer sequence is $\boxed{5,8,4,3,3,3,9}$

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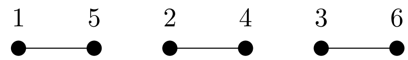
3. (8 points) Textbook Section 1.3.4, Problem 3: Use Prüfer's method to draw and label a tree with Prüfer sequence 5,4,3,5,4,3,5,4,3. As always, (briefly) show your steps.

**Solution**. Call the sequence $\sigma_0$. Since $\sigma_0$ has 9 entries, we must have $n = 9 + 2 = 11$ vertices, so let $S_0 = \{1, 2, \ldots, 11\}$.

$\boxed{i=0}$ The smallest $j \in S_0$ not in $\sigma_0$ is 1, and the first entry in $\sigma_0$ is 5. so we add the edge $\boxed{1\text{–}5}$ to make $T_1$. Let $\sigma_1 = 4,3,5,4,3,5,4,3$ and $S_1 = \{2,3,\dots,11\}$.

$\boxed{i=1}$ The smallest $j \in S_1$ not in $\sigma_1$ is 2, and the first entry in $\sigma_1$ is 4. so we add the edge $\boxed{2\text{–}4}$ to make $T_2$. Let $\sigma_2 = 3,5,4,3,5,4,3$ and $S_2 = \{3,4,\dots,11\}$.

$\boxed{i=2}$ The smallest $j \in S_2$ not in $\sigma_2$ is 6, and the first entry in $\sigma_2$ is 3. so we add the edge $\boxed{3\text{–}6}$ to make $T_3$, which looks like this:

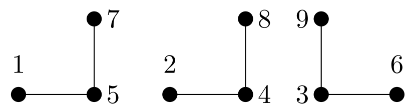



Let $\sigma_3 = 5,4,3,5,4,3$ and $S_3 = \{3,4,5,7,8,9,10,11\}$.

$\boxed{i=3}$ The smallest $j \in S_3$ not in $\sigma_3$ is 7, and the first entry in $\sigma_3$ is 5. so we add the edge $\boxed{5\text{–}7}$ to make $T_4$. Let $\sigma_4 = 4,3,5,4,3$ and $S_4 = \{3,4,5,8,9,10,11\}$.

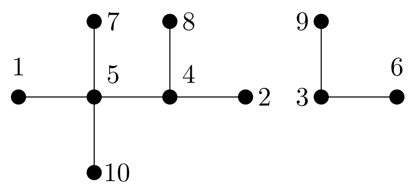
$\boxed{i=4}$ The smallest $j \in S_4$ not in $\sigma_4$ is 8, and the first entry in $\sigma_4$ is 4. so we add the edge $\boxed{4\text{–}8}$ to make $T_5$. Let $\sigma_5 = 3,5,4,3$ and $S_5 = \{3,4,5,9,10,11\}$.

$\boxed{i=5}$ The smallest $j \in S_5$ not in $\sigma_5$ is 9, and the first entry in $\sigma_5$ is 3. so we add the edge $\boxed{3\text{–}9}$ to make $T_6$. Let $\sigma_6 = 5,4,3$ and $S_6 = \{3,4,5,10,11\}$, which looks like this:

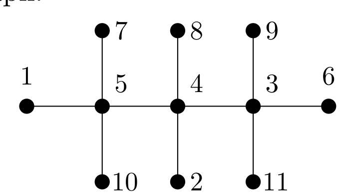



$\boxed{i=6}$ The smallest $j \in S_6$ not in $\sigma_6$ is 10, and the first entry in $\sigma_6$ is 5. so we add the edge $\boxed{5\text{–}10}$ to make $T_7$. Let $\sigma_7 = 4,3$ and $S_7 = \{3,4,5,11\}$.

$\boxed{i=7}$ The smallest $j \in S_7$ not in $\sigma_7$ is 5, and the first entry in $\sigma_7$ is 4. so we add the edge $\boxed{4\text{–}5}$ to make $T_8$. Let $\sigma_8 = 3$ and $S_8 = \{3,4,11\}$, which looks like this, after some rearranging of our previous diagram:




$\boxed{i=8}$ The smallest $j \in S_8$ not in $\sigma_8$ is 4, and the first entry in $\sigma_8$ is 3. so we add the edge $\boxed{3\text{–}4}$ to make $T_9$. Let $\sigma_9 = \varnothing$ and $S_9 = \{3,11\}$. Since $\sigma_9$ is empty, we add the one more edge from $S_9$, namely $\boxed{3\text{–}11}$ yielding the following final graph:

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4. (16 points) Textbook Section 1.3.4, Problem 1: Let $T$ be a labeled tree, and let $\sigma$ be its Prüfer sequence. For each vertex $v \in V(T)$, prove that $v$ appears in $\sigma$ exactly $\deg(v) - 1$ times.

**Proof**. We proceed by induction on $n = |V(T)| \geq 2$.
For $n = 2$, both vertices are leaves, and hence both satisfy $\deg(v) - 1 = 0$, and also both appear 0 times in the corresponding (empty) Prüfer sequence.

For the inductive step, for $n \geq 3$ assume the desired statement is true for trees with $n-1$ vertices. Let $T$ be a tree with $|V(T)| = n$, and let $v \in V(T)$. Let $d = \deg(v)$. We consider three cases.

**Case 1**: If $v$ is the lowest-numbered leaf of $T$, then it will be removed at the first stage of Prüfer's algorithm (and $v$'s neighbor, but not $v$ itself, will be recorded), and then will not be part of the remaining tree $T'$. Thus, $v$ will appear 0 times in the Prüfer sequence, and sure enough, since $v$ is a leaf, $d - 1 = 0$ as well.

**Case 2**: $v$ is adjacent to the lowest-numbered leaf $j$ of $T$. Then $j$ is removed at the first stage of Prüfer's algorithm, and we record $v$ and make a new tree $T'$ with $j$ removed (along with the edge $j$–$v$). The new tree $T'$ has order $n-1$, and because of the edge removed, we have

$$d = \deg_T(v) = 1 + \deg_{T'}(v).$$

By the inductive hypothesis, $v$ shows up $\deg_{T'}(v) - 1 = d - 2$ times in the Prüfer sequence $\sigma'$ for $T'$. Note that the Prüfer sequence for $T$ is $v$ followed by $\sigma'$, and hence $v$ shows up $1 + (d-2) = d-1$ times in the Prüfer sequence for $T$, as desired.

**Case 3**: Otherwise, $v$ is neither the lowest-numbered leaf $j$ of $T$, nor the unique neighbor of $j$. Thus, in the tree $T' = T - j$, which removes $j$ and its unique edge, no edges have been removed from $v$, and hence

$$d = \deg_T(v) = \deg_{T'}(v).$$

In addition, the vertex $w$ recorded in the first stage of Prüfer's algorithm is *not* $v$. By the inductive hypothesis, $v$ shows up $\deg_{T'}(v) - 1 = d - 1$ times in the Prüfer sequence $\sigma'$ for $T'$. Note that the Prüfer sequence for $T$ is $w$ followed by $\sigma'$, and hence $v$ shows up $0 + (d-1) = d-1$ times in the Prüfer sequence for $T$, as desired. QED

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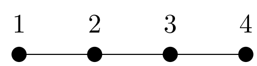
5. (18 points) For each of the following four graphs, write down its Laplacian matrix, and then use the Matrix Tree Theorem to find its number of spanning trees.

$P_4$ $\qquad$ $C_4$ $\qquad$ $K_4$ $\qquad$ $K_{2,3}$

**Solution**. $\boxed{P_4}$ The graph is:




So the Laplacian is $\Delta = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ and hence, computing the (2,1)-cofactor (which I chose because there are a lot of zeros in row 1), the Matrix Tree Theorem says there is

$$(-1)\begin{vmatrix} -1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = (-1)(-1)(2-1) = \boxed{1 \text{ spanning tree}}$$

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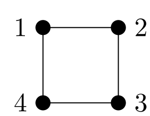
$\boxed{C_4}$ The graph is:




So the Laplacian is $\Delta = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$ and hence, computing the (1,1)-cofactor, the Matrix Tree Theorem says there are

$$\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} = 2(4-1) + (-2-0) = 6 - 2 = \boxed{4 \text{ spanning trees}}$$

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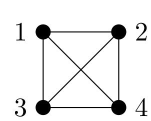
$\boxed{K_4}$ The graph is:



So the Laplacian is $\Delta = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$ and hence, computing the (1,1)-cofactor, the Matrix Tree Theorem says there are $\begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix} = 3\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 3 \\ -1 & -1 \end{vmatrix}$
$= 3(9-1) + (-3-1) - (1+3) = 24 - 4 - 4 = \boxed{16 \text{ spanning trees}}$

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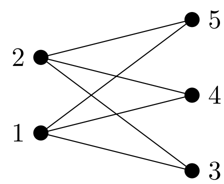
$\boxed{K_{2,3}}$ The graph is:




So the Laplacian is $\Delta = \begin{bmatrix} 3 & 0 & -1 & -1 & -1 \\ 0 & 3 & -1 & -1 & -1 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 0 & 2 \end{bmatrix}$ and hence, computing the (2,1)-cofactor (which I chose to preserve as many zeros as possile), the Matrix Tree Theorem says there are

$$(-1)\begin{vmatrix} 0 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{vmatrix} = (-1)\left( -(-1)\begin{vmatrix} -1 & -1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} + 2\begin{vmatrix} 0 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} \right)$$

$= (-1)\big((-4) + 2(-2-2)\big) = -(-12) = \boxed{12 \text{ spanning trees}}$

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6. (20 points) Textbook Section 1.3.4, Problem 4 (expanded a bit):

(a) Use Prüfer's method to draw and label the trees with Prüfer sequences 1,1,1,1,1 and 3,3,3,3.

(b) Inspired by your answers in part (a), make a conjecture about which trees have constant Prüfer sequences.

(c) Prove your conjecture from part (b).

**Solution/Proof**. (a): Call the first sequence $\sigma_0 = 1,1,1,1,1$. Since it has 5 entries, the corresponding tree has $n = 7$ vertices, so let $S_0 = \{1,2,3,4,5,6,7\}$.

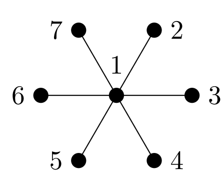
$\boxed{i=0}$ The smallest $j \in S_0$ not in $\sigma_0$ is 2, and the first entry in $\sigma_0$ is 1. so we add the edge $\boxed{1\text{–}2}$ to make $T_1$. Let $\sigma_1 = 1,1,1,1$ and $S_1 = \{1,3,4,5,6,7\}$.

$\boxed{i=1}$ The smallest $j \in S_1$ not in $\sigma_1$ is 3, and the first entry in $\sigma_1$ is 1. so we add the edge $\boxed{1\text{–}3}$ to make $T_2$. Let $\sigma_2 = 1,1,1$ and $S_2 = \{1,4,5,6,7\}$.

$\boxed{i=2}$ The smallest $j \in S_2$ not in $\sigma_2$ is 4, and the first entry in $\sigma_2$ is 1. so we add the edge $\boxed{1\text{–}4}$ to make $T_3$. Let $\sigma_3 = 1,1$ and $S_3 = \{1,5,6,7\}$.

$\boxed{i=3}$ The smallest $j \in S_3$ not in $\sigma_3$ is 5, and the first entry in $\sigma_3$ is 1. so we add the edge $\boxed{1\text{–}5}$ to make $T_4$. Let $\sigma_4 = 1$ and $S_4 = \{1,6,7\}$.

$\boxed{i=4}$ The smallest $j \in S_4$ not in $\sigma_4$ is 6, and the first entry in $\sigma_4$ is 1. so we add the edge $\boxed{1\text{–}6}$ to make $T_4$. Let $\sigma_5 = \varnothing$ and $S_5 = \{1,7\}$. Since $\sigma_5$ is empty, we add the one more edge from $S_5$, namely $\boxed{1\text{–}7}$ yielding the following final graph:



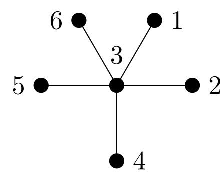
Call the second sequence $\sigma_0 = 3, 3, 3, 3$. Since it has 4 entries, the corresponding tree has $n = 6$ vertices, so let $S_0 = \{1, 2, 3, 4, 5, 6\}$.

$\boxed{i = 0}$ The smallest $j \in S_0$ not in $\sigma_0$ is 1, and the first entry in $\sigma_0$ is 3. so we add the edge $\boxed{1\text{–}3}$ to make $T_1$. Let $\sigma_1 = 3, 3, 3$ and $S_1 = \{2, 3, 4, 5, 6\}$.

$\boxed{i = 1}$ The smallest $j \in S_1$ not in $\sigma_1$ is 2, and the first entry in $\sigma_1$ is 3. so we add the edge $\boxed{2\text{–}3}$ to make $T_2$. Let $\sigma_2 = 3, 3$ and $S_2 = \{3, 4, 5, 6\}$.

$\boxed{i = 2}$ The smallest $j \in S_2$ not in $\sigma_2$ is 4, and the first entry in $\sigma_2$ is 3. so we add the edge $\boxed{3\text{–}4}$ to make $T_3$. Let $\sigma_3 = 3$ and $S_2 = \{3, 5, 6\}$.

$\boxed{i = 3}$ The smallest $j \in S_3$ not in $\sigma_3$ is 5, and the first entry in $\sigma_3$ is 3. so we add the edge $\boxed{3\text{–}5}$ to make $T_4$. Let $\sigma_4 = \varnothing$ and $S_4 = \{3, 6\}$. Since $\sigma_4$ is empty, we add the one more edge from $S_4$, namely $\boxed{3\text{–}6}$ yielding the following final graph:

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(b): Let's conjecture that the trees with constant Prüfer sequence are precisely star graphs (with the constant being the label on the center vertex). That is, for any tree $T$ with $n \geq 2$ vertices,

$$T \text{ has constant Prüfer sequence } m, \dots, m \quad \Longleftrightarrow \quad T \text{ is a star graph with center vertex } m$$

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(c): **Proof**. ($\Leftarrow$) We proceed by induction on $n \geq 2$. For $n = 2$, $T$ may be considered a star graph $S_2$ with either of the two vertices being considered the center. Prüfer's algorithm gives the (empty) sequence $\sigma$, and indeed whichever vertex we considered the center point of $T$ can be considered as being the only vertex that shows up in $\sigma = \varnothing$.

Assuming the conjecture is true for some $n \geq 2$, consider running Prüfer's first algorithm on the ($(n+1)$-vertex) star graph $T$ with $n$ vertices and central vertex $m$. We remove the lowest numbered leaf $j$, i.e., the lowest-numbered vertex that is *not* $m$. We then record $j$'s neighbor, which is $m$.

We are left with a star graph $S_n$ with central vertex $m$. By our inductive hypothesis, the rest of Prüfer's algorithm gives us a sequence $m, \dots, m$ of $(n-2)$ $m'$s. With the one we recorded at the start, that means we finish the algorithm with the final Prüfer sequence being a list of $n - 1 = (n + 1) - 2$ copies of $m$, as desired. QED ($\Leftarrow$)

($\Rightarrow$) By the previous implication, every constant sequence $m, \dots, m$ comes from a star graph. Since Prüfer's algorithms give a one-to-one correspondence between trees and sequences, this means that no other tree can give a constant sequence. Thus, the star graphs are *precisely* the trees with constant Prüfer sequences. QED

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**Alternative method for ($\Leftarrow$) proof**: Given a star graph $T$ with $n \geq 2$ and with center vertex $m$, observe that the center vertex has $\deg(m) = n - 1$, and every other vertex has degree 1. Thus, by problem 4, in the Prüfer sequence for $T$, the label $m$ shows up $(n-1)-1 = n-2$ times, and every other vertex $i$ shows up $1 - 1 = 0$ times. That is, the Prüfer sequence for $T$ is $m, m, \dots m$, i.e., a constant sequence of $n - 2$ copies of $m$, as desired. QED ($\Leftarrow$)