Chapter 2: Returns

Professor: Hammou El Barmi
Columbia University

Introduction

- Let P_t be the price of an asset at time t. Assuming no dividends are paid, the net return is defined as

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1.$$

- The simple gross return is defined as

$$\frac{P_t}{P_{t-1}} = 1 + R_t$$

- Example: If $P_{t-1} = 10$ and $P_t = 10.3$ then

$$1 + R_t = \frac{P_t}{P_{t-1}} = \frac{10.3}{10} = 1.03$$

and $R_t = 0.03$ or $R_t = 3\%$.

Introduction

- The gross return over k periods $t-k, t-k+1, \ldots, t$ is defined as

$$\begin{aligned} 1+R_t(k) &= \frac{P_t}{P_{t-k}} \\ &= \frac{P_t}{P_{t-1}}\frac{P_{t-1}}{P_{t-2}}\ldots\frac{P_{t-k+1}}{P_{t-k}} \\ &= (1+R_t)(1+R_{t-1})\ldots(1+R_{t-k+1}) \end{aligned}$$

Introduction

Note that the returns are:

- scale-free, meaning that they do not depend on monetary units (dollars, cents, etc.)
- not unit-less in time; they depend on the units of t (hour, day, etc.)

Example

time	t-3	t-2	t-1	t
P	200	210	206	212
1+R		1.05	0.981	1.03
1+ R(2)			1.03	1.01
1+ R(3)				1.06

1+R

$$1.05 = 210/200 \quad 0.981 = 206/210 \quad 1.03 = 212/206$$

1+R(2)

$$1.03 = 206/210 \quad 1.01 = 212/210$$

1+R(3)

$$1.06 = 212/200$$

Log returns

- log prices are defined by

$$p_t = \log(P_t)$$

where $\log(x)$ is the natural logarithm of x.

- The continuously compounded or log returns are the logarithms of the gross returns:

$$r_t = \log(1 + R_t) = \log\left(\frac{P_t}{P_{t-1}}\right) = p_t - p_{t-1}.$$

- Notice that

$$P_t = P_{t-1}e^{r_t}.$$

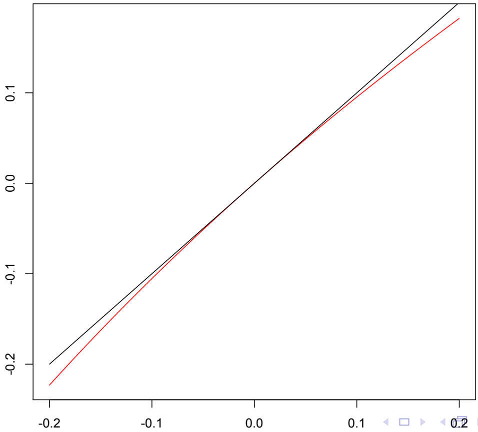
- In addition

$$\begin{aligned} r_t(k) &= \log(1 + R_t(k)) \\ &= \log((1 + R_t)(1 + R_{t-1}) \ldots (1 + R_{t-k+1})) \\ &= \log(1 + R_t) + \log(1 + R_{t-1}) + \ldots + \log(1 + R_{t-k+1}) \\ &= r_t + r_{t-1} + \ldots + r_{t-k+1} \end{aligned}$$

Log returns

Log returns are approximately equal to net returns since for small $x, \log(1+x) \approx x$. In this case $r_t = \log(1+R_t) \approx R_t$

Figure: graphs of $f(x) = x$ (black) and $g(x) = \log(1+x)$ (red)



Normal Model

At time $t-1, P_t$ and R_t are not only unknown but we also do not know their probability distributions.
A common model assumes that the returns are independent, identically distributed and normal. That is, if $R_1, R_2, \ldots, R_t =$ returns from a single asset then

- $R_1, R_2, \ldots, R_t$ mutually independent
- identically distributed
- normally distributed

Normal Model

There are two problems with this model

1. This model implies the possibility of unlimited losses but the liability is usually limited ($R_t \geq -1$) since you cannot lose more than your investment
2. Also

$$1 + R_t(k) = \prod_{i=0}^{k-1}(1 + R_{t-i})$$

is not normal. Under some conditions, sums of independent normals are normal but not their products.

Log-normal Model

In this case we assume instead that

$$r_t = \log(1 + R_t)$$

are IID normals. That is,

$$\log(1 + R_t) \sim N(\mu, \sigma^2)$$

As a consequence

1. $1 + R_t = \exp(\text{normal random variable}) \geq 0$
2. $R_t \geq -1$

- Notice that

$$\begin{aligned} 1 + R_t(k) &= \prod_{i=0}^{k-1} (1 + R_{t-i}) \\ &= \exp(r_t) \exp(r_{t-1}) \ldots \exp(r_{t-k+1}) \\ &= \exp(r_t + r_{t-1} + \ldots + r_{t-k+1}) \end{aligned}$$

therefore

$$\log(1 + R_t(k)) = \sum_{i=0}^{k-1} r_{t-i}$$

and since sums of independent normals are normal, normality of single period log returns implies normality of log multiple period returns.

Log-normal Model

Example: Suppose a simple gross return $(1+R)$ is lognormal(0, 0.01), that is $\log(1+R) \sim N(\mu = 0, \sigma^2 = 0.01)$. Find $P(R \leq 0.05)$.
Answer:

$$
\begin{aligned}
P(R \leq 0.05) &= P(1 + R \leq 1.05) \\
&= P(\log(1+R) \leq \log(1.05)) \\
&= P(N(0, 0.01) \leq \log(1.05) \\
&= \Phi((\log(1.05) - 0)/0.1) = 0.68719
\end{aligned}
$$

Log-normal Model

Formula for the *kth* period return. Assume that

- $1 + R_t(k) = (1 + R_t)(1 + R_{t-1}) \ldots (1 + R_1)$
- $\log(1 + R_i) \sim N(\mu, \sigma^2)$
- the $\{R_i\}$ are independent then

$$\log(1 + R_t(k)) \sim N(k\mu, k\sigma^2)$$

and

$$P(R_t(k) < x) = \Phi\left(\frac{\log(1 + x) - k\mu}{\sqrt{k}\sigma}\right).$$

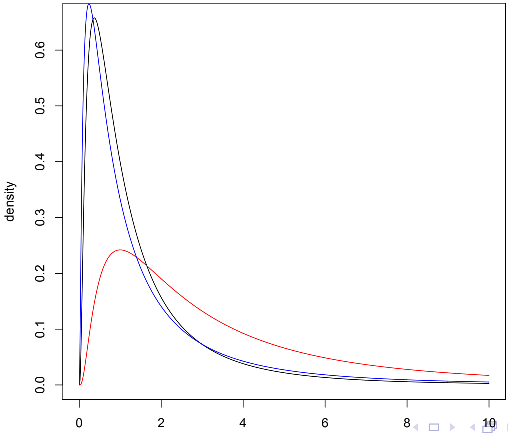
Log-normal Model

Example: Assume again that $(1+R)$ is lognormal(0, 0.01). Find the probability that a simple gross two period return is less that 1.05.
Answer: The two period gross return is lognormal(0, 0.02). Therefore

$$\begin{aligned} P(R \leq 0.05) &= P(1+R \leq 1.05) \\ &= P(\log(1+R) \leq \log(1.05)) \\ &= P(N(0, 0.01) \leq \log(1.05) \\ &= \Phi((\log(1.05) - 0)/\sqrt{0.02}) = 0.6349 \end{aligned}$$

Lognormal Density

Figure: graphs of lognormal(0,1) (black), lognormal(1, 1) (red) and lognormal(0, 1.2) (blue)



Adjustment for Dividends

Many stocks pay dividends that must be accounted for when computing returns. If a dividend D_t is paid prior to time t, then the gross return at time t is defined as

$$1 + R_t = \frac{P_t + D_t}{P_{t-1}}$$

and the net return return is

$$R_t = \frac{P_t + D_t - P_{t-1}}{P_{t-1}}$$

and

$$r_t = \log(1 + R_t) = \log(P_t + D_t) - \log(P_{t-1})$$

and multiple-period gross returns are products of single period returns

$$\begin{aligned} 1 + R_t(k) &= \left(\frac{P_t + D_t)}{P_{t-1}}\right)\left(\frac{P_{t-1} + D_{t-1}}{P_{t-2}}\right)\dots\left(\frac{P_{t-k+1} + D_{t-k+1}}{P_{t-k}}\right) \\ &= (1 + R_t)(1 + R_{t-1})\dots(1 + R_{t-k+1}) \end{aligned}$$

Adjustment for Dividends

Similarly, a k-period log return is a sum of of the log of single period returns

$$\begin{aligned}
\log(1+R_t(k)) &= \log\left(\frac{P_t + D_t)}{P_{t-1}}\right) + \log\left(\frac{P_{t-1} + D_{t-1}}{P_{t-2}}\right) + \dots \\
&+ \log\left(\frac{P_{t-k+1} + D_{t-k+1}}{P_{t-k}}\right) \\
&= \log(1+R_t) + \log(1+R_{t-1}) + \dots + \log(1+R_{t-k+1})
\end{aligned}$$

Random Walk

Let $Z_1, Z_2, \ldots, Z_n$ be IID with mean μ and standard deviation σ. Let Z_0 be an arbitrary starting point and define

$$S_0 = Z_0, \quad \text{and} \quad S_t = Z_0 + Z_1 + \ldots + Z_t, t \geq 1.$$

The process $S_0, S_1, \ldots$ is called a random walk and $Z_1, Z_2, \ldots$ are called it steps. The expectation and the variance of S_t conditional on S_0, are

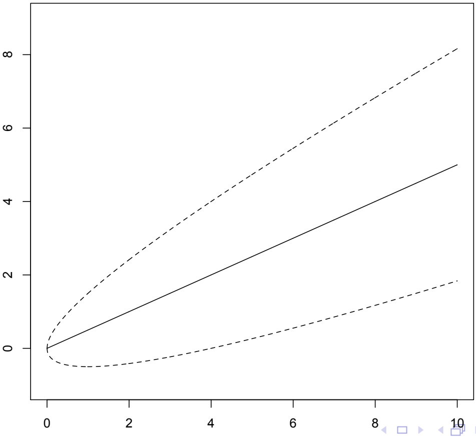
- $E(S_t|Z_0) = Z_0 + t\mu$
- $Var(S_t|Z_0) = \sigma^2 t$
- $SD(S_t|Z_0) = \sigma\sqrt{t}$

Random Walk

- The parameter μ is called the drift and it determines the general direction of the random walk
- . The parameter σ is the volatility and determines how much the random walk fluctuates about the conditional mean $S_0 + \mu t$
- Since the standard deviation of S_t given S_0 is $\sigma\sqrt{t}, (S_0 + \mu t) \pm \sigma\sqrt{t}$ gives the mean plus and minus on standard deviation, which, for a normal random walk (i.e. the Z_is are normal) , gives a 68% confidence interval for the conditional mean.
- The width of this interval grows proportionally to $\sqrt{t}$ as can be see in the next graph.

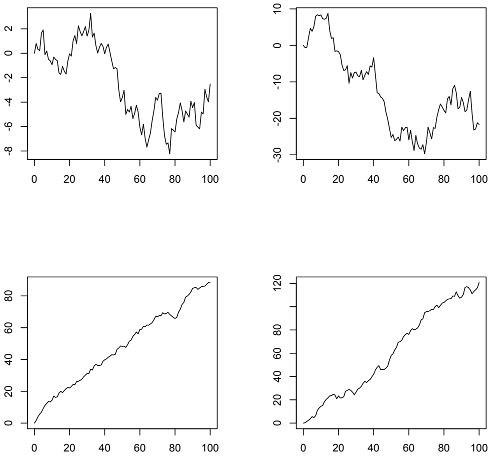
Random Walk

Figure: Mean and probability bounds on a random walk with $S_0 = 0, \mu = .5$ and $\sigma = 1$. At any given time, the probability of being between the probability bounds (dashed curves) is 68%. The Z_is here normal



Random Walk

Figure: random walks with $(\mu, \sigma) = (0, 1), (0, 2), (1, 1)$ and $(1,\ 2)$



Random Walk

r-commands used to generate these plots are

```
T=100
x = 0:T
y=c(0, cumsum(rnorm(T)))
plot(x,y, lty=1, type="l", xlab="", ylab="")
```

Geometric Random Walk

Recall that

$$\frac{P_t}{P_{t-k}} = 1 + R_t(k) = \exp(r_t + r_{t-1} + \ldots + r_{t-k+1})$$

if we take $k = t$, we get

$$P_t = P_0 \exp(r_t + r_{t-1} + \ldots + r_1)$$

If $r_1, r_2, \ldots, r_t$ are IID $N(\mu, \sigma^2)$ (i.e gross returns are lognormal and independent), then

- $\log(1 + R_t(k)) = r_1 + r_2 + \ldots + r_{t-k+1}$ is a random walk
- $\{P_t, t = 1, 2, \ldots, k\}$ is exponential of random walk
- such a process is known as a geometric random walk.
- $1 + R_t(k) = \exp(r_t + r_{t-1} + \ldots + r_1)$ is lognormal and also skewed.

Geometric Random Walk

The effect of the drift μ

- the geometric random walk does not imply that one cannot make money
- since μ is positive, there is upward drift
- the log returns on the US stock market as a whole have mean of about 10% and a standard deviation of about 20%

Distribution of the returns?

Are the log returns normally distributed? The ways to check this include

- normal plot (approximately a straight line)
- sample skewness and kurtosis: Check if their values are near those of the normal. Skewness and Kurtosis measure the shape of the distribution and are independent of the mean and the variance. Every normal distribution has skewness coefficient equal to 0 and kurtosis equal to 3.

Distribution of the returns

Let $r_1, r_2, \ldots, r_T$ be log returns on some asset, let $\hat{\mu}$ be the sample mean $\hat{\sigma}$ be the sample standard deviation. The sample skewness is

$$\hat{S} = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{r_t - \hat{\mu}}{\hat{\sigma}}\right)^3$$

and the sample kurtosis is

$$\hat{K} = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{r_t - \hat{\mu}}{\hat{\sigma}}\right)^4$$

The excess kurtosis is defined as $\hat{K} - 3$. It measures the deviation from 3, the kurtosis of the normal distribution. Both skewness and excess kurtosis should be near 0.

Testing for normality: Let $\hat{F}$ be the empirical CDF.

$$\hat{F}(x) = \frac{1}{T}\sum_{t=1}^{T} I(r_t \leq t)$$

$\hat{F}$ is an estimator of the true distribution. Normality is tested by comparing $\hat{F}(x)$ and $\Phi((x-\hat{\mu})/\hat{\sigma})$

Distribution of the returns

The three commonly used tests of normality that compare $\hat{F}(x)$ with $\Phi((x-\hat{\mu})/\hat{\sigma})$ are

- Anderson-Darling test
- Shapiro-Wilks test
- Kolmogorov-Smirnov test

We will discuss further these test in the near future.

The Kolmogorov-Smirnov test based on

$$\sup_x |\hat{F}(x) - \Phi((x - \hat{\mu})/\hat{\sigma})|$$

the maximum distance between $\hat{F}(x)$ with $\Phi((x - \hat{\mu})/\hat{\sigma})$

- If one dollar is invested for one year at 5% rate of simple interest, it is worth \$1.05 at end of year.
- If the 5% interest is compounded semi-annually, then the dollar worth after one year

$$\left(1 + \frac{0.05}{2}\right)^2 = 1.050625$$

- If the compounding is daily, then the dollar is worth after one year

$$\left(1 + \frac{0.05}{365}\right)^{365} = 1.0512675.$$

- If the 5% interest is compounded every hour, then the dollar's worth after one year would be

$$\left(1+\frac{0.05}{(24)(365)}\right)^{(24)(365)} = 1.0512709.$$

- As the compounding becomes more and more frequent, the dollar' s worth after one year is in the limit:

$$\lim_{N\to\infty}\left(1+\frac{0.05}{N}\right)^{N} = \exp(0.05) = 1.0512711$$

- This limit is called continuous compounding

- If P dollars are invested at a continuously compounded rate r for one year, then value after year is $P\exp(r)$
- The return is $\exp(r)$ so that the log return is r.
- For this is the reason, the log return on an asset is called the continuously compounded rate.

Here is another way of looking at continuous compounding. Let t be time in years and let P_t be the value of a deposit that is growing according to the following differential equation

$$\frac{dP_t}{dt} = rP_t$$

then the solution to this differential equation is

$$P_t = P_0 \exp(rt)$$

In particular, the value after one year is

$$P_1 = P_0 \exp(r)$$