Section 3.1 Dual Linear Programs

Symmetric form

Primal program	Dual program	
$\min \vec{c}^T\vec{x}$	$\max \vec{y}^T\vec{b}$	A $m\times n$
s.t. $A\vec{x} \geq \vec{b}$	s.t. $\vec{y}^T A \leq \vec{c}^T$	b $m\times 1$
$\vec{x} \geq \vec{0}$	$\vec{y} \geq \vec{0}$	x $n\times 1$
		c $n\times 1$
		y $m\times 1$

*the dual of the dual is the primal!

Transpose:
$\vec{c}^T\vec{x} = \vec{c}\cdot\vec{x} = \vec{x}\cdot\vec{c} = \vec{x}^T\vec{c}$
$(A^T)^T = A$
$(A\vec{x})^T = \vec{x}^T A^T$

rewrite dual as

Check: $\min(-\vec{b}^T\vec{y})$ s.t. $-A^T\vec{y} \geq -\vec{c}$, $\vec{y} \geq 0$

Dual of dual: $\max \vec{z}^T(-\vec{c})$ s.t. $\vec{z}^T(-A^T) \leq -\vec{b}$, $\vec{z} \geq \vec{0}$

Equiv: $\min \vec{c}^T\vec{z}$ s.t. $A\vec{z} \geq \vec{b}$, $\vec{z} \geq \vec{0}$

Asymmetric form

Primal	Dual
$\min \vec{c}^T\vec{x}$	$\max \vec{y}^T\vec{b}$
s.t. $A\vec{x} = \vec{b}$, $\vec{x} \geq \vec{0}$	s.t. $\vec{y}^T A \leq \vec{c}^T$

Example:

	Wood	Metal	Profit
x_1 Tables	8	5	80
x_2 Desks	6	4	60
x_3 Chairs	4	4	50

Have 100 wood, 60 metal
Max. profit

$$\min(-80x_1 - 60x_2 - 50x_3) \text{ s.t. } 8x_1 + 6x_2 + 4x_3 \leq 100$$
$$5x_1 + 4x_2 + 4x_3 \leq 60$$
$$x_1, x_2, x_3 \geq 0$$

R script

negative to flip to $\geq$

$$A = -\begin{bmatrix} 8 & 6 & 4 \\ 5 & 4 & 4 \end{bmatrix} \quad \vec{b} = -\begin{bmatrix} 100 \\ 60 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 80 \\ 60 \\ 50 \end{bmatrix}$$

Dual: $\max -100y_1 - 60y_2$ s.t. $8y_1 + 5y_2 \geq 80$
$\Leftrightarrow \min 100y_1 + 60y_2$ $\quad 6y_1 + 4y_2 \geq 60$ $\quad y_1, y_2 \geq 0$
(cost pt of view) $\quad 4y_1 + 4y_2 \geq 50$

Worksheet

Sol'n to primal problem: $\vec{x} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$, max profit \$960
Sol'n to dual problem: $\vec{y} = \begin{bmatrix} 0 \\ 16 \end{bmatrix}$, min cost \$960

Section 3.2 Duality Theorem

Asymmetric form

Primal

$\min \vec{c}^T \vec{x}$ s.t. $A\vec{x} = \vec{b}, \vec{x} \geq \vec{0}$

Dual

$\max \vec{y}^T \vec{b}$ s.t. $\vec{y}^T A \leq \vec{c}^T$

~~Lemma~~

Duality theorem of linear programming

If either the primal or dual problems has a finite optimal sol'n, then so does the other, with the optimal values of the objective fns equal. If either problem has an unbounded objective, the other problem has no feasible sol'n.

Example of 2nd case:

$$\max\ x_1 + 4x_2 + x_3 \text{ s.t. } \begin{array}{l} 2x_1 - 2x_2 + x_3 = 4 \\ x_1 - x_3 = 1 \\ x_2 \geq 0, x_3 \geq 0 \quad (x_1 \text{ free}) \end{array}$$

Eliminate x_1 to put in standard form: $x_1 = x_3 + 1$

$$\max\ 4x_2 + 2x_3 \text{ s.t. } -2x_2 + 3x_3 = 2$$
$$(\min -4x_2 - 2x_3) \qquad x_2 \geq 0, x_3 \geq 0$$

Let $x_2 = \frac{3}{2}x_3 - 1$ (solve constraint), so objective fn becomes $8x_3 + \text{const.}$
$\rightarrow$ unbounded, no max

Dual: $\vec{c} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$ $\vec{b} = [2]$ $A = [-2\ 3]$

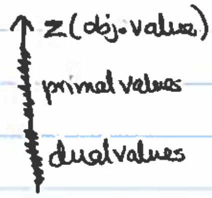
$$\max\ 2y \text{ s.t. } \begin{array}{l} -2y \leq -4 \\ 3y \leq -2 \end{array} \Leftrightarrow \begin{array}{l} y \geq 2 \\ y \leq -2/3 \end{array} \quad \text{infeasible}$$

Weak Duality Lemma: If $\vec{x}$ and $\vec{y}$ are feasible for the primal & dual problems, resp., then $\vec{c}^T\vec{x} \geq \vec{y}^T\vec{b}$.

Proof: We have $A\vec{x}=\vec{b}$, $\vec{x}\geq\vec{0}$, and $\vec{y}^TA \leq \vec{c}^T$.
Then $\vec{y}^T\vec{b} = \vec{y}^TA\vec{x} \leq \vec{c}^T\vec{x}$.
↑ using both $\vec{x}\geq\vec{0}$ and $\vec{y}^TA\leq\vec{c}^T$

This lemma shows that a feasible vector to either problem yields a bound on the optimal value of the other problem.

Values associated with the primal are all larger than those for the dual.
Since the primal seeks a minimum and the dual seeks a maximum,

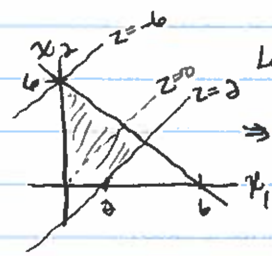


if values meet each other, that must be the optimal value for both.

Corollary: If $\vec{x}_0$ and $\vec{y}_0$ are feasible for primal & dual problems, resp., and if $\vec{c}^T\vec{x}_0 = \vec{y}_0^T\vec{b}$, then $\vec{x}_0$ and $\vec{y}_0$ are optimal.

Example: primal problem

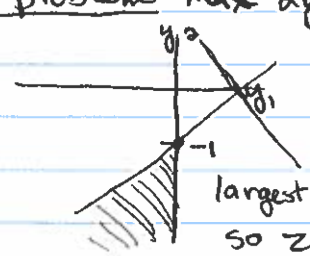
$$\min\ x_1 - x_2 \text{ s.t. } \begin{aligned} x_1 - x_2 &\leq 2 \\ x_1 + x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Level sets $z = x_1 - x_2$
$\Rightarrow$ objective values $z \geq -6$

Standard form:
$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ x_1 + x_2 + x_4 &= 6 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

dual problem $\max\ 2y_1 + 6y_2$ s.t.
$$\begin{aligned} y_1 + y_2 &\leq 1 \\ -y_1 + y_2 &\leq -1 \\ y_1 &\leq 0 \\ y_2 &\leq 0 \end{aligned}$$



largest value at $(0,-1)$, so $z \leq -6$

$\Rightarrow$ optimal value is -6 for both problems

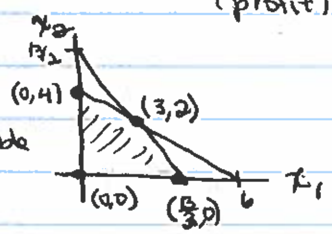
Section 3.3 Geometric and economic interpretations

Example: Manufacturer produces nails & bolts, using two machines "A" and "B". Each set of nails uses 3 hrs on A and 2 on B, while each set of bolts uses 2 hrs on A and 3 on B. Assuming profit on set of nails is \$3 and bolts \$4, how many sets of each to make each day if A can run at most 13 hrs & B at most 12 hrs?

x_1 = daily # sets of nails, x_2 = daily # sets of bolts produced

LP: max $3x_1+4x_2$ (profit) subject to $3x_1+2x_2 \le 13$, $2x_1+3x_2 \le 12$ (available time on machines), $x_1 \ge 0$, $x_2 \ge 0$

feasible set



Check profit at each extreme pt.:
$(0,0) \to 0$
$(\frac{13}{3},0) \to 13$
$(0,4) \to 16$
$\boxed{(3,2) \to \$17}$

Entrepreneur wants to rent machines, so needs to negotiate fair price per hour: R dollars per hour for A, S for B. (manufacturer won't have use of them)
They offer $3R+2S$ dollars for each set of nails produced, which would have given manufacturer profit of \$3: $3R+2S \ge 3$
Similarly, offer $2R+3S$ for each set of bolts produced: $2R+3S \ge 4$

Manufacturer says wants to rent full time on machines, not just part of day; costing $13R+12S$ for the entrepreneur.

What are fair prices R & S? Solve min $13R+12S$ (what entrepreneur pays) s.t. $3R+2S \ge 3$, $2R+3S \ge 4$ (manufacturer wants equal or better profits), $R,S \ge 0$

dual feasible set

$(2,0) \to 26$
$(0,\frac{3}{2}) \to 18$
$\boxed{(\frac{1}{5},\frac{6}{5}) \to \frac{13+72}{5} = \$17}$

(note obj fn $\to \infty$ as $R,S \to \infty$)

This 2nd system is in fact the dual of the 1st system:

$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ $b = \begin{bmatrix} 13 \\ 12 \end{bmatrix}$ $c = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\to$ dual is max $[R\ S]\begin{bmatrix} -13 \\ -12 \end{bmatrix}$ s.t. $[R\ S]\begin{bmatrix} -3 & -2 \\ -2 & -3 \end{bmatrix} \le [-3\ -4]$

$\Leftrightarrow$ min $13R+12S$, $3R+2S \ge 3$, $2R+3S \ge 4$, $R,S \ge 0$

R & S are called "shadow prices" or "marginal costs"

Why are optimal values of primal & dual problems the same?

<u>Standard primal form</u>	<u>Dual problem</u>
$\min \vec{c}^T\vec{x}$ s.t. $A\vec{x}=\vec{b}$, $\vec{x}\geq\vec{0}$	$\max \vec{y}^T\vec{b}$ s.t. $\vec{y}^TA\leq\vec{c}^T$

Let B be the square matrix of lin ind columns of A used in forming the optimal basic feasible sol'n $\vec{x}=\begin{bmatrix}\vec{x}_B\\ \vec{0}\end{bmatrix}$ to the primal problem.
Partion $A=[B\ D]$ (reorder columns if necessary), so $\vec{x}_B=B^{-1}\vec{b}$
Define $\vec{y}^T=\vec{c}_B^{\,T}B^{-1}$ ($\vec{c}_B$ has components of $\vec{c}$ associated with columns of B in A)
If $\vec{y}$ is dual feasible ($\vec{y}^TA\leq\vec{c}^T$), then it's a basic feasible sol'n for the dual, with $\vec{y}^T\vec{b}=\vec{c}_B^{\,T}B^{-1}\vec{b}=\vec{c}_B^{\,T}\vec{x}_B=\vec{c}^T\vec{x}$, so has same optimal value as primal ~~problem~~ sol'n.

<u>Thm</u> Suppose the standard primal problem have an optimal basic feasible sol'n corresponding to matrix B (as described above).
Then the vector $\vec{y}$ satisfying $\vec{y}^T=\vec{c}_B^{\,T}B^{-1}$ is an optimal sol'n to the dual problem if it is dual feasible. In this case, the optimal values of both problems are equal.

Example: (Worksheet) $\max x_1+2x_2$ s.t. $x_1+3x_2+x_3=4$, $2x_2+x_3=2$, $x_1,x_2,x_3\geq 0$

1) Find all basic sol'ns: $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}2\\0\\2\end{bmatrix}$
2) Determine optimal basic sol'n: $\vec{x}=\begin{bmatrix}1\\1\\0\end{bmatrix}$, optimal value 3
3) $\vec{x}_B=\begin{bmatrix}1\\1\end{bmatrix}$ $B=\begin{bmatrix}1&3\\0&2\end{bmatrix}$ check $\vec{x}_B=B^{-1}\vec{b}$: $\frac{1}{2}\begin{bmatrix}2&-3\\0&1\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$

4) $\vec{y}^T=\vec{c}_B^{\,T}B^{-1}=[1\ 2]\begin{bmatrix}1&-3/2\\0&1/2\end{bmatrix}=[1\ -1/2]$
5) $\vec{y}^T\vec{b}=[1\ -1/2]\begin{bmatrix}4\\2\end{bmatrix}=3$ ✓