Homework 6: Monday March 6 2023
Due Friday March 24 2023

Total 100 points

Problem 1: **[20 points]**

A stock is worth \$100 today. Its volatility in the future varies with the stock price S according to the formula

$$\sigma(S) = 0.2 - \left(\frac{S}{100} - 1\right)$$

so that its volatility increases by one volatility point as S decreases by \$1, and vice versa, except that it can, of course, never go below zero. In the questions below this won't happen.

(i) Build a tree of stock prices with $\Delta t = 0.01$ for four periods out to 0.04 years. Assume interest rates are 50% per year, compounded continuously. Show all the stock prices in the tree and the associated risk-neutral binomial probabilities. [10 points]

(ii) Calculate the value of a European put on this stock that expires at the end of 0.04 years and struck at 98.

[5 points]

(iii) Show that this has approximately the same value as a similar put on CRR tree of constant volatility equal to 0.21. [5 points]

Problem 1: Solution follows on next two pages

Binomial stock tree with stock-dependent vol

DeltaT	0.010				
S	100.000				
sig function a + b*((S-100)/100)		a	0.20 b	-1.00	
r	0.500		one vol point per strike point		
fwd	1.005				

s	95		
vol	0.25		
slope in vol points per strike point =			

Part (i)

stock tree sigma(s,t)

				107.14
			105.57	
		103.86		103.86
	102.02		102.02	
100.00		100.00		100.00
	98.02		98.02	
		95.39		95.39
			93.31	
				89.94

			0.684
		0.665	
	0.656		0.656
p-tree	0.620		0.620	
		0.677		0.677
			0.544	
				0.703

Part (ii)

put struck at 98

0.04 years to expiration

strike 98

				0.000
			0.000	
		0.000		0.000
	0.108		0.000	
0.434		0.317		0.000
	0.972		0.838	
		2.361		2.607
			4.204	
				8.057

Four period risk-neutral binomial stock tree with equal time spacing and variable sig(t), one period per year

Force tree to stick to CRR pattern

DeltaT	0.01	
S	100	
sig		0.21
r	0.5	50% continuous
fwd	1.00501252	

Part (iii)

								strike		
stock tree						put struck at 98, 0.04 to expiration			98	
					108.76					0.000
				106.50					0.000	
			104.29		**104.29**			0.000		0.000
		102.12		**102.12**			0.120		0.000	
	100.00		**100.00**		**100.00**	0.424		0.312		0.000
		97.92		**97.92**			0.914		0.811	
			95.89		**95.89**			1.885		2.113
				93.89					3.617	
					91.94					6.057

				0.61
			0.61	
		0.61		**0.61**
p-tree	**0.61**		**0.61**	
		0.61		**0.61**
			0.61	
				0.61

				0.204
			0.204	
		0.204		**0.204**
vol computec	**0.204**		**0.204**	
		0.204		**0.204**
			0.204	
				0.204

Problem 2: **[20 points]**

Write a computer program to calculate the value of call with S = 100, K = 100, 0.05 years to expiration and zero rates and dividends, using a binomial model with 6 levels from inception to Run the program and show the trees it produces for the binomial tree's stock prices, local volatilities, probabilities and options prices. For local volatility use the formula from Problem 1 above.

Solution 2:

localbinomial(100, 0, 0.05, 100, 6)

local_vol = 0.2 - (spot/100-1);

time: 0 0.0100 0.0200 0.0300 0.0400 0.0500

stockprice =

100.0000	0	0	0	0	0
98.0199	102.0201	0	0	0	0
95.6757	100.0000	103.6857	0	0	0
93.3653	98.0199	102.0201	105.4037	0	0
90.6887	95.6757	100.0000	103.6857	106.7815	0
88.0487	93.3653	98.0199	102.0201	105.4037	108.2275

vols =

0.2000	0	0	0	0	0
0.2198	0.1798	0	0	0	0
0.2432	0.2000	0.1631	0	0	0
0.2663	0.2198	0.1798	0.1460	0	0
0.2931	0.2432	0.2000	0.1631	0.1322	0
0	0	0	0	0	0

opt_price =

1.8693	0	0	0	0	0
0.9349	2.8226	0	0	0	0
0.2691	1.4974	3.9152	0	0	0
0	0.5421	2.4720	5.4037	0	0
0	0	1.0000	3.6857	6.7815	0
0	0	0	2.0201	5.4037	8.2275

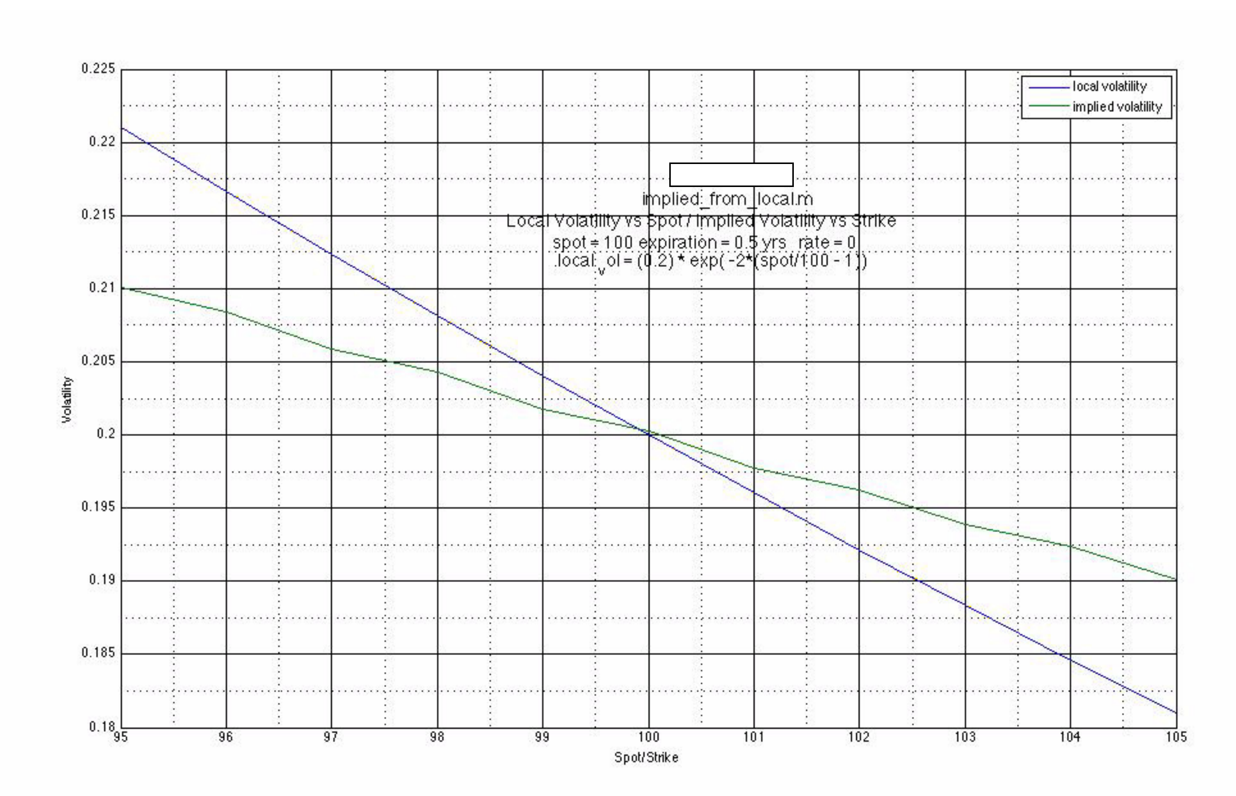
delta = 0.4719________________

Problem 3: **[20 points]**

Write a program to calculate the Black-Scholes implied volatilities from strikes of 95 to 105 when S = 100 for options with 0.5 years to expiration, zero rates and dividends, and a local volatility function given by $\sigma(s,t) = (0.2) * \exp(-2*(s/100 - 1))$. Use 100 periods (101 levels) in the tree. Then plot the implied volatilities as a function of strike and the local volatilities as a function of stock price.

If the options prices from the local volatility tree, which are always approximate in a binomial model, somehow don't have implied Black-Scholes volatilities because they lie outside reasonable values, then try different numbers of periods rather than 100 and hopefully you will get a reasonable answer for some number of levels. Do the best you can. Play around and don't worry too much.

Solution 3:

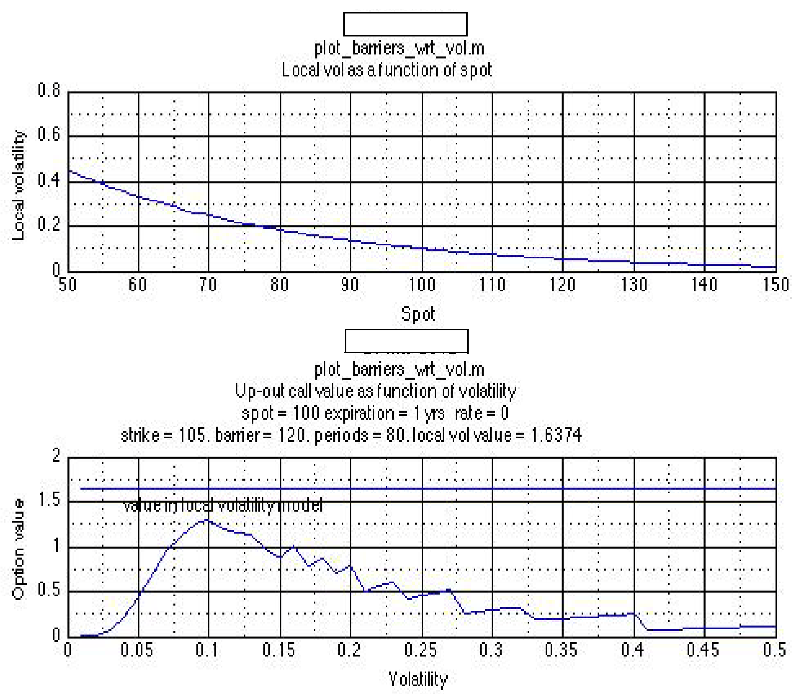


Problem 4. **[20]**

The local volatility of a stock is given $\sigma(s,t) = (0.1) * \exp(-3*(s/100 - 1))$. Write a program to build a binomial local volatility tree, and then calculate the value of a an up-and-out barrier call option when S = 100, K = 105, the barrier B = 120, and the time to expiration is one year, and all rates and dividend yields are zero. Also plot the value of the same option in the Black-Scholes world (i.e. a binomial world with constant volatility at each node) for a range of volatilities and see if you can find the constant implied volatility that matches the local volatility model value for the barrier option.

(Even though simple binomial models are not the most efficient way to compute the value of barrier options, whose value converges very slowly as the number of levels in the tree increases, don't worry about this. Just use the binomial model for the local volatility computation, and, if you like, use the same program for the constant volatility model too by setting the local volatility to be a constant. I used about 80 periods for a rough calculation.)

Solution 4.



Approximate value of barrier option in this local volatility framework is 1.64. The graph below shows that the value of the barrier option with any implied volatility never exceeds approximately 1.3

Problem 5. From implied to local volatility. [20 points]

We proved in class that you can find the local volatility from calendar and butterfly spreads for zero rates and dividend yields; i.e.

$$\sigma^2(K, T) = \frac{2\left.\dfrac{\partial C}{\partial T}\right|_K}{K^2\left.\dfrac{\partial^2 C}{\partial K^2}\right|_T} \qquad \text{Eq.H6.1}$$

where the numerator is a *partial derivative w.r.t T that keeps K constant*, and vice versa in the denominator.

If options prices are really quoted in terms of their Black-Scholes implied volatilities Σ, then we can write

$$C(S, t, K, T) = C_{BS}(S, t, K, T, \Sigma(S, t, K, T)) \qquad \text{Eq.H6.2}$$

where we are assuming zero rates and dividends in this case, for simplicity.

Use the chain rule for differentiation to evaluate $\left.\dfrac{\partial C}{\partial T}\right|_K$ and $\left.\dfrac{\partial^2 C}{\partial K^2}\right|_T$ in terms of their Black-Scholes derivatives (which you can evaluate in terms of $N(d_1)$ etc., since you know the Black-Scholes formula) and in terms of the derivatives of Σ w.r.t T and K, in order to show that there is a direct relationship between the local volatility σ and implied volatility Σ, namely

$$\sigma^2(K, T) = \frac{2\dfrac{\partial \Sigma}{\partial T} + \dfrac{\Sigma}{T-t}}{K^2\left(\dfrac{\partial^2 \Sigma}{\partial K^2} - d_1\sqrt{T-t}\left(\dfrac{\partial \Sigma}{\partial K}\right)^2 + \dfrac{1}{\Sigma}\left\{\dfrac{1}{K\sqrt{T-t}} + d_1\dfrac{\partial \Sigma}{\partial K}\right\}^2\right)}$$

where $d_1 = \dfrac{\ln(S/K)}{\Sigma\sqrt{T-t}} + \dfrac{\Sigma\sqrt{T-t}}{2}$, and (don't forget) that $\Sigma = \Sigma(S, t, K, T)$ is a function of *S, t, K, T*. When you take the chain rule to derive this you have to be very careful.

Solution 5.

$$\boxed{\sigma^2(K,T) = \frac{2\left.\frac{\partial C}{\partial T}\right|_K}{K^2\left.\frac{\partial^2 C}{\partial K^2}\right|_T}} \qquad \text{Eq.H6.3}$$

If options prices are really quoted in terms of their Black-Scholes implied volatilities $\boxed{\Sigma}$, then we can write

$$\boxed{C(S,t,K,T) = C_{BS}(S,t,K,T,\Sigma(S,t,K,T))} \qquad \text{Eq.H6.4}$$

For r = 0 and d = 0 here are some useful formulas:

$$\boxed{\begin{aligned} C_{BS} &= SN(d_1) - KN(d_2) \\ d_{1,2} &= \frac{\ln\frac{S}{K}}{\Sigma\sqrt{\tau}} \pm \frac{\Sigma\sqrt{\tau}}{2} \\ SN'(d_1) &= KN'(d_2) \\ N'(d_1) &= \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-d_1^2}{2}\right) \\ N''(d_1) &= -N'(d_1)d_1 \end{aligned}}$$

Also it's easy to show that

$$\boxed{\left.\frac{\partial C_{BS}}{\partial T}\right|_{K,\Sigma,\ldots} = \frac{KN'(d_2)\Sigma}{2\sqrt{\tau}}} \qquad \text{Eq.H6.5}$$

$$\boxed{\left.\frac{\partial C_{BS}}{\partial \Sigma}\right|_{K,T,\ldots} = KN'(d_2)\sqrt{\tau}} \qquad \text{Eq.H6.6}$$

and so from the chain rule applied to Equation H6.4 we get the numerator for Dupire

$$\boxed{\left.\frac{\partial C}{\partial T}\right|_K = \left.\frac{\partial C_{BS}}{\partial T}\right|_{K,\Sigma,\ldots} + \left.\frac{\partial C_{BS}}{\partial \Sigma}\right|_{K,T,\ldots} \times \frac{\partial \Sigma}{\partial T} = KN'(d_2)\sqrt{\tau}\left[\frac{\partial \Sigma}{\partial \tau} + \frac{\Sigma}{2\tau}\right]} \qquad \text{Eq.H6.7}$$

Now do the K derivatives:

$$\left.\frac{\partial C_{BS}}{\partial K}\right|_{T,\Sigma,\ldots} = -N(d_2)$$

$$\left.\frac{\partial C}{\partial K}\right|_{T} = \left.\frac{\partial C_{BS}}{\partial K}\right|_{T,\Sigma,\ldots} + \left.\frac{\partial C_{BS}}{\partial \Sigma}\right|_{K,T,\ldots} \times \frac{\partial \Sigma}{\partial K} = -N(d_2) + KN'(d_2)\sqrt{\tau}\frac{\partial \Sigma}{\partial K}$$

$$\begin{aligned}\left.\frac{\partial^2 C}{\partial K^2}\right|_{T} &= -N'(d_2)\left.\frac{\partial d_2}{\partial K}\right|_{T} + N'(d_2)\sqrt{\tau}\frac{\partial \Sigma}{\partial K} + KN''(d_2)\sqrt{\tau}\left.\frac{\partial \Sigma}{\partial K}\frac{\partial d_2}{\partial K}\right|_{T} + KN'(d_2)\sqrt{\tau}\frac{\partial^2 \Sigma}{\partial K^2} \\ &= -N'(d_2)\left.\frac{\partial d_2}{\partial K}\right|_{T} + N'(d_2)\sqrt{\tau}\frac{\partial \Sigma}{\partial K} - KN'(d_2)d_2\sqrt{\tau}\left.\frac{\partial \Sigma}{\partial K}\frac{\partial d_2}{\partial K}\right|_{T} + KN'(d_2)\sqrt{\tau}\frac{\partial^2 \Sigma}{\partial K^2}\end{aligned} \qquad \text{Eq.H6.8}$$

where it's not hard to show that

$$\frac{\partial d_2}{\partial K} = -\frac{1}{K\Sigma\sqrt{\tau}} - \frac{\ln(S/K)}{\Sigma^2\tau}\frac{\partial \Sigma}{\partial K} - \frac{\sqrt{\tau}}{2}\frac{\partial \Sigma}{\partial K} = -\left[\frac{1}{\Sigma}\frac{\partial \Sigma}{\partial K}d_1 + \frac{1}{K\Sigma\sqrt{\tau}}\right] \qquad \text{Eq.H6.9}$$

Combining the last two equations we obtain

$$\begin{aligned}\left.\frac{\partial^2 C}{\partial K^2}\right|_{T} &= -N'(d_2)\left.\frac{\partial d_2}{\partial K}\right|_{T} + N'(d_2)\sqrt{\tau}\frac{\partial \Sigma}{\partial K} - KN'(d_2)d_2\sqrt{\tau}\left.\frac{\partial \Sigma}{\partial K}\frac{\partial d_2}{\partial K}\right|_{T} + KN'(d_2)\sqrt{\tau}\frac{\partial^2 \Sigma}{\partial K^2} \\ &= N'(d_2)\left\{\left(1 + Kd_2\sqrt{\tau}\frac{\partial \Sigma}{\partial K}\right)\left[\frac{1}{\Sigma}\frac{\partial \Sigma}{\partial K}d_1 + \frac{1}{K\Sigma\sqrt{\tau}}\right] + \sqrt{\tau}\frac{\partial \Sigma}{\partial K} + K\sqrt{\tau}\frac{\partial^2 \Sigma}{\partial K^2}\right\}\end{aligned}$$

and since $\boxed{d_2 = d_1 - \Sigma\sqrt{\tau}}$ we obtain

$$\left.\frac{\partial^2 C}{\partial K^2}\right|_{T} = N'(d_2)\left\{\left(1 + Kd_1\sqrt{\tau}\frac{\partial \Sigma}{\partial K} - K\Sigma\tau\frac{\partial \Sigma}{\partial K}\right)\left[\frac{1}{\Sigma}\frac{\partial \Sigma}{\partial K}d_1 + \frac{1}{K\Sigma\sqrt{\tau}}\right] + \sqrt{\tau}\frac{\partial \Sigma}{\partial K} + K\sqrt{\tau}\frac{\partial^2 \Sigma}{\partial K^2}\right\} \qquad \text{Eq.H6.10}$$

Now from Equation H6.3, Equation H6.7 and Equation H6.10 we can cancel out the $\boxed{N'(d_2)}$ that appears in both numerator and denominator and simplify a little to obtain Equation H7.12 that gives the local volatility in terms of the implied volatility and its derivatives.