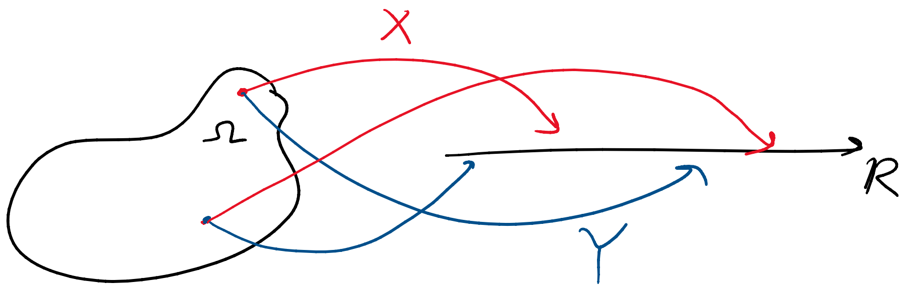
Suppose X and Y are two discrete random variables.



Def: The joint PMF of (X, Y) is the function

$$p_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y), \quad \text{for } x, y \in \mathbb{R}.$$

Comma means "and": $\mathbb{P}(X=x, Y=y) = \mathbb{P}(\{X=x\} \cap \{Y=y\})$

That is, $p_{X,Y} : \mathbb{R}^2 \to [0,1]$.

Similarly, for 3 r.v.'s (X, Y, Z) the joint PMF is the function $p_{X,Y,Z} : \mathbb{R}^3 \to [0,1]$ defined by

$$p_{X,Y,Z}(x, y, z) = \mathbb{P}(X = x, Y = y, Z = z).$$

Ex1: Let $\Omega = \{1, 2, \ldots, 6\}^2$ = 2 dice rolls.

Let X = sum of 2 rolls,

Y = maximum of 2 rolls.

Ex:

$X(2,3) = 2 + 3 = 5$ $\qquad$ $X(5,1) = 6$

$Y(2,3) = \max(2,3) = 3$ $\qquad$ $Y(5,1) = 5$

Let's find Joint PMF:

$Y \backslash X$	2	3	4	5	6	7	8	9	10	11	12	p_Y
1	$1/36$	0	0	0	0	0	0	0	0	0	0	$p_Y(1) = \frac{1}{36} = \mathbb{P}(Y=1)$
2	0	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	0	0	0	0	$p_Y(2) = \frac{3}{36}$
3	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	0	0	$p_Y(3) = \frac{5}{36}$
4	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	$\frac{7}{36}$
5	0	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	$9/36$

Y \ X	2	3	4	5	6	7	8	9	10	11	12	p_Y
4	0	0	0	$\frac{[illegible]}{36}$	36	36	36	0	0	0	0	36
5	0	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	$\frac{9}{36}$
6	0	0	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{11}{36}$
p_X	$p_X(2)=\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	

In box $(x,y)=(2,1)$, put the value $p_{X,Y}(2,1)=\mathbb{P}(X=2, Y=1)$.

$$\mathbb{P}(X=2, Y=1) = \mathbb{P}((1,1)) = \frac{1}{36}.$$

$$\mathbb{P}(X=3, Y=1) = 0, \qquad \mathbb{P}(X=x, Y=1)=0 \quad \text{for } x \geq 3$$

Note: $X \leq 2Y$

$$\mathbb{P}(X=2, Y=2) = 0$$
$$\mathbb{P}(X=3, Y=2) = \mathbb{P}(\{(1,2),(2,1)\}) = \frac{2}{36}$$
$$\mathbb{P}(X=4, Y=2) = \mathbb{P}((2,2)) = \frac{1}{36}$$

Def: The <u>marginal</u> PMF of X is just the (individual) PMF of X, given by

$$p_X(x) = \sum_y p_{X,Y}(x,y).$$

Marginal PMF of Y is $\quad p_Y(y) = \sum_x p_{X,Y}(x,y)$.

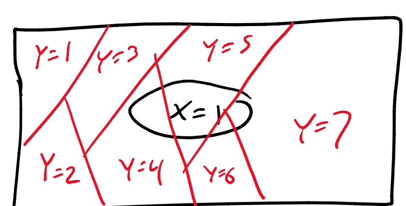
Notation Convention:
Capital letter = random var.
Lowercase = nonrandom

Hidden here is the statement that

$$\mathbb{P}(X=x) = \sum_y \mathbb{P}(X=x, Y=y) \quad \text{for all } x.$$

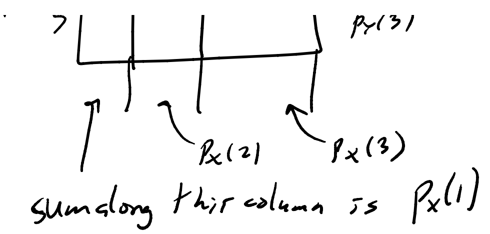
equivalent

↳ why? Law of total probability.



Note: If we depict $p_{X,Y}$ via a table, we get $p_X(x)$ by summing along x column, and $p_Y(y)$ by summing along y rows.

Y \ X	1	2	3	
1				$p_Y(1)$
2				$p_Y(2)$
3				$p_Y(3)$



Expectations:

- Recall: $\mathbb{E}[f(X)] = \sum_x f(x) p_X(x)$ for $f: \mathbb{R} \to \mathbb{R}$

- Similarly: $\mathbb{E}[f(X,Y)] = \sum_{x,y} f(x,y)\, p_{X,Y}(x,y)$ for $f: \mathbb{R}^2 \to \mathbb{R}$

$$\mathbb{E}[f(X,Y,Z)] = \sum_{x,y,z} f(x,y,z)\, p_{X,Y,Z}(x,y,z), \quad f: \mathbb{R}^3 \to \mathbb{R}$$

Ex 1 revisited: 2 dice rolls, X = sum Y = max.

- $\mathbb{P}(X = Y+1)$ = sum $p_{X,Y}(x,y)$ for all possible (x,y) satisfying $x = y+1$

(x,y): $(2,1)$, $(3,2)$, $(4,3)$, $(5,4)$, $(6,5)$, $(7,6)$

$$= p_{X,Y}(2,1) + p_{X,Y}(3,2) + p_{X,Y}(4,3) + p_{X,Y}(5,4) + p_{X,Y}(6,5) + p_{X,Y}(7,6)$$

$$= \frac{1}{36} + \frac{2}{36} + \frac{2}{36} + \frac{2}{36} + \frac{2}{36} + \frac{2}{36}$$

$$= \frac{11}{36}.$$

- $\mathbb{E}[XY] = \sum_{x=2}^{12} \sum_{y=1}^{6} xy\, p_{X,Y}(x,y) = \ldots$

Linearity of Expectation:

We saw before: $\mathbb{E}[aX+b] = a\mathbb{E}[X] + b$ for constants a, b.

We saw before: $\mathbb{E}[aX+b] = a\mathbb{E}[X]+b$ for constants a, b.

More generally:

$$\text{✪} \quad \mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

for constants a, b, c

or

$$\mathbb{E}[aX+bY+c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$$

Proof of ✪:

Use $f(x,y) = x+y$, then

$$\mathbb{E}[X+Y] = \mathbb{E}[f(X,Y)] = \sum_x \sum_y f(x,y)\, p_{X,Y}(x,y)$$

$$= \sum_x \sum_y (x+y)\, p_{X,Y}(x,y)$$

$$= \sum_x \sum_y x\, p_{X,Y}(x,y) + \sum_x \sum_y y\, p_{X,Y}(x,y)$$

$$= \sum_x x \underbrace{\left(\sum_y p_{X,Y}(x,y)\right)}_{\text{marginal of } X} + \sum_y y \underbrace{\left(\sum_x p_{X,Y}(x,y)\right)}_{\text{marginal of } Y}$$

$$= \sum_x x\, p_X(x) + \sum_y y\, p_Y(y)$$

$$= \mathbb{E}[X] + \mathbb{E}[Y].$$

Note: $\mathbb{E}[XY] \neq \mathbb{E}[X]\mathbb{E}[Y]$

The Hat Problem:

n people

Each person puts their hat in the box.

Randomly line up and pick a random hat from box.

What is the expected # people who get their own hat?

Let Y be the number of people who get own hat.

<u>Goal:</u> Find $\mathbb{E}[Y]$.

But we don't know the PMF of Y.

$$\text{Let } X_i = \begin{cases} 1 & \text{if person } i \text{ gets own hat} \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, n$.

(we don't know joint PMF.)

$$\text{Then } Y = \sum_{i=1}^{n} X_i .$$

$$\text{So } \mathbb{E}[Y] = \sum_{i=1}^{n} \mathbb{E}[X_i] .$$

$$\begin{aligned} \text{Also } \mathbb{E}[X_i] &= 1 \cdot \mathbb{P}(\text{person } i \text{ gets own hat}) \\ &\quad + 0 \cdot \mathbb{P}(\text{person } i \text{ does not get own hat}) \\ &= \mathbb{P}(\text{person } i \text{ gets own hat}) \\ &= \frac{1}{n} . \end{aligned}$$

$$\Rightarrow \quad \mathbb{E}[Y] = \sum_{i=1}^{n} \frac{1}{n} = 1 .$$

$\mathbb{E}[Y]$ = Expected # people who get own hat $= 1$.

Conditioning:

- For a r.v. X and either an event A or a r.v. Y, "conditioning" on A or Y will change the probabilities involving X, i.e., the PMF.

  ↳ conditional PMF

- For an event A, the conditional PMF of X given A is

$$p_{X|A}(x) = \mathbb{P}(X=x \mid A) \qquad \text{for } x \in \mathbb{R}$$
$$= \frac{\mathbb{P}(\{X=x\} \cap A)}{\mathbb{P}(A)}.$$

- Ex: Let $X = \#$ of heads in 3 coin flips $\sim Bin(3, \frac{1}{2})$

x	$p_X(x)$
0	1/8
1	3/8
2	3/8
3	1/8

Let A be the event that we get at least 2 heads.

↳ $A = \{X \geq 2\}$

$\mathbb{P}(A) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$.

Conditional PMF:

$$p_{X|A}(x) = 0 \quad \text{if } x=0 \text{ or } x=1,$$

$$p_{X|A}(2) = \mathbb{P}(X=2 \mid A) = \frac{\mathbb{P}(X=2, X \geq 2)}{\mathbb{P}(X \geq 2)} = \frac{\mathbb{P}(X=2)}{\mathbb{P}(X \geq 2)}$$

$$= \frac{3/8}{1/2} = \frac{3}{4}$$

$$p_{X|A}(3) = \frac{1}{4}$$

x	$p_{X|A}(x)$
0	0
1	0
2	3/4
3	1/4