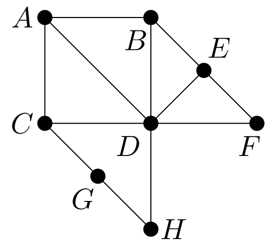
Math 280, Spring 2022

Solutions to Homework #2

1. (12 points) Let X be the following graph:



(a) Find the degree sequence of X.
(b) Find a trail in X of longest possible length. Justify why there are no longer trails.
(c) Find a path in X of longest possible length. Justify why there are no longer paths.

Solution. (a): Here are the degrees of the various vertices:

A	B	C	D	E	F	G	H
3	3	3	6	3	2	2	2

So, putting these in numerical order, the degree sequence is $\boxed{6,3,3,3,3,2,2,2}$

(b): There are 12 edges in the graph, but we claim no trail can use all 12. After all, with four vertices (namely, A, B, C, D) of degree 3, at least two of these four vertices are not endpoints of the trail. Call one such vertex v, so v is in the middle of the trail, and $\deg(v) = 3$. Then the trail must first arrive at v along one edge xv, and then leave it along another edge vy. Then the trail can never use the third edge vz, because it can only use that edge if it then ends at v; but we said v was *not* an endpoint of the trail. This proves our claim, that no trail can use all 12 edges.
Here is a trail of length 11, which therefore must be of maximum length: $E, F, D, E, B, D, A, C, D, H, G, C$

(c): With 8 vertices, none of which can be repeated, certainly no path can have length more than 7. And here is a path of length 7, which therefore must be of maximum length: D, F, E, B, A, C, G, H

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2. (12 points) Textbook, Section 1.1.2, Problem 14:
Let G be a 2-connected graph. Prove that G contains at least one cycle.

Proof. Since $2 \le \kappa(G) \le |V(G)| - 1$, the order of G must be at least 3, i.e., there must be at least 3 distinct vertices.
We may pick $a \in V(G)$, since this set is nonempty. By a theorem from class, we have $\delta(G) \ge \kappa(G)$, and hence $\deg(a) \ge \delta(G) \ge \kappa(G) = 2$. Thus, there are at least two distinct vertices v, w adjacent to a, with a, v, w all distinct.
[See below for an alternative proof of the above fact.]
Because $G - a$ is connected, there is a path from v to w in $G - a$. That is, there is a path $v = x_1, x_2, \ldots, x_k = w$ of (distinct) vertices, none of which are a. Since a is adjacent to both $v = x_1$ and $w = x_k$, it follows that

$$a, x_1, x_2, \ldots, x_k, a$$

is a cycle in G. QED

Note: Here is an alternative, direct proof of the fact $\deg(a) \ge 2$ used above, without quoting the theorem from class:

Since $|V(G)| \ge 3$, there are at least two other different vertices $b, c \in V(G) \setminus \{a\}$. Because G is connected, there must be a path from a to $b \ne a$. Let v be the first vertex in this path after a; so $v \ne a$ is adjacent to a. Now v might be b or c, but it cannot be both, since $b \ne c$; so without loss, assume that $v \ne c$. Because G is 2-connected, we know that $G - v$ is connected, and hence there is a path from a to b in $G - v$. This path cannot use the edge av and so must begin with an edge aw for some vertex $w \ne v, a$. That is, we have at least two vertices $v \ne w$ adjacent to a.

3. (20 points) Textbook, Section 1.1.2, Problem 16:

(a) Let G be a graph of order n such that $\delta(G) \geq (n-1)/2$. Prove that G is connected.

(b) For any positive even integer $n = 2m \geq 2$, find a graph G of order n such that $\delta(G) \geq (n-2)/2$ but G is *not* connected.

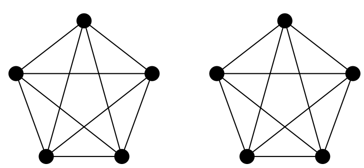
Proof/Solution. (a): For any connected component H of G, we claim that $|V(H)| \geq (n+1)/2$. To prove this claim, consider a vertex v in such a connected component H. Since $\deg(v) \geq \delta(G) \geq (n-1)/2$, there are at least $k = (n-1)/2$ distinct vertices $w_1, \ldots, w_k$ adjacent to v. Thus, the $k+1 = (n+1)/2$ distinct vertices $v, w_1, \ldots, w_k$ are all in the same connected component H that v is in, since they are all connected to v. That is, H has at least $(n+1)/2$ vertices, as claimed.

Suppose G were not connected. Then G would have at least two different connected components H_1 and H_2, with no vertices in common. Since $V(G) \supseteq V(H_1) \cup V(H_2)$, then by the claim and the fact that $V(H_1) \cap V(H_2) = \varnothing$, we have

$$n = |V(G)| \geq |V(H_1)| + |V(H_2)| \geq \frac{n+1}{2} + \frac{n+1}{2} = n+1,$$

which is a contradiction. By this contradiction, it follows that our supposition is wrong, i.e., G must be connected. QED

(b): For each such $n = 2m$, let G be the graph that consists of two separate copies of the complete graph K_m. So for example, for $n = 10$, I mean the graph is two disjoint copies of K_5, which looks like this:



Since K_m has m vertices, G does indeed have $2m = n$ vertices. And every vertex in K_m, and hence in G, has degree $m - 1 = (n-2)/2$. And G is not connected, because no vertex in the one copy of K_m can be connected to any vertex in the other copy of K_m. QED

Note: To dream up the answer for (b), I thought about not only the result of part (a), — which tells me that *some* vertex v of G must have $\deg(v) < (n-1)/2$ — but the proof of (a), and I realized that I probably would need to have *every* vertex x of G have $\deg(x) < (n-1)/2$. And since part (b) says $\delta(G) \geq (n-2)/2$, that would mean $\deg(x) = (n-2)/2 = m-1$ for every $x \in V(G)$. And I recognized that as the signature of K_m, which led me to think of two disjoint copies of K_m.

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4. (12 points) Textbook, Section 1.1.3, Problem 3: Is K_4 a subgraph of $K_{4,4}$? If yes, exhibit one explicitly. If no, prove no such subgraph exists.

Answer/Proof: $\boxed{\text{No}}$. If there were such a subgraph K_4, pick any 3 distinct vertices $a, b, c \in V(K_4)$. Then the triangle a, b, c, a is a cycle of length 3 inside this subgraph, and hence a cycle of length 3 inside the larger graph $K_{4,4}$.

However, by a theorem, no bipartite graph can contain a cycle of odd length. Since $K_{4,4}$ is bipartite, this is a contradiction; so no subgraph isomorphic to K_4 exists.

Alternative proof: Suppose H were a subgraph of $K_{4,4}$ isomorphic to K_4. Write $V(K_{4,4}) = X \cup Y$, where X and Y are the two sets of four vertices each, with all edges running from a vertex of X to a vertex of Y.

Since the four vertices of $V(H)$ all lie in $X \cup Y$, at least one of X or Y contains at least two different vertices v, w of H. Without loss of generality, suppose $v, w \in X$. Then because H is isomorphic to K_4, the edge vw must belong to $E(H)$ and hence to $E(K_{4,4})$. But because $v, w \in X$, the edge vw is *not* in $E(K_{4,4})$. This is a contradiction, so no such subgraph H exists. QED

5. (8 points) Textbook, Section 1.1.3, Problem 8:
Let G and H be isomorphic graphs. Prove that their complements $\overline{G}$ and $\overline{H}$ are also isomorphic.

Proof. By hypothesis, there is a function $f : V(G) \to V(H)$ that is bijective, and such that for any distinct $a, b \in V(G)$, we have

$$ab \in E(G) \iff f(a)f(b) \in E(H).$$

By definition, we have $V(\overline{G}) = V(G)$ and $V(\overline{H}) = V(H)$. Thus, f is also a bijective function from $V(\overline{G})$ to $V(\overline{H})$.
In addition, for any distinct $a, b \in V(\overline{G}) = V(G)$, we have

$$ab \in E(\overline{G}) \iff ab \notin E(G) \iff f(a)f(b) \notin E(H) \iff f(a)f(b) \in E(H),$$

where the first iff is by definition of $\overline{G}$, the second is by the property of f above, and the third is by definition of $\overline{H}$.
Thus, f is an isomorphism from $\overline{G}$ to $\overline{H}$ QED

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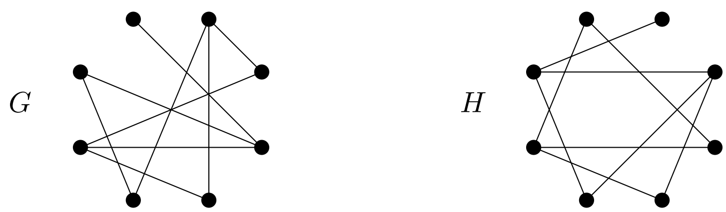
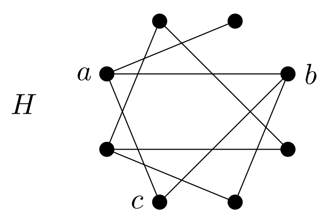
6. (12 points) Textbook, Section 1.1.3, Problem 9. Consider the following two graphs:



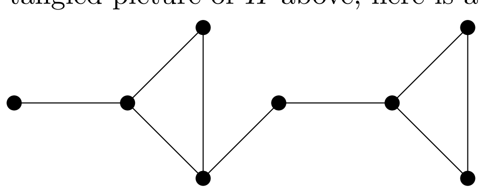
Verify that G and H have the same order, same size, and same degree sequence. Then prove that in spite of that, G and H are *not* isomorphic.

Answer/Proof: By inspection, the orders are $|V(G)| = 8 = |V(H)|$, the sizes are $|E(G)| = 9 = |E(H)|$, and the degree sequences are both 3, 3, 3, 2, 2, 2, 2, 1
However, if we label some of the vertices of H as follows:



we see that the vertices a, b, c form a cycle of length 3, involving two of the vertices of degree 3.
However, there is no cycle of length 3 in graph G, and there certainly isn't one involving two of the vertices of degree 3. Indeed, only two of the degree-3 vertices have an edge between them at all (the one horizontal-running edge in the picture of G), and these two vertices have no adjacent vertices in common.
An isomorphism would take a cycle a, b, c, a of length 3 to another cycle x, y, z, x of length 3, so since G doesn't have such a cycle, there cannot be an isomorphism. QED

Side note: There are other ways to see G and H are not isomorphic. For example, H has a second cycle of length three that could be used instead. Or, rearranging the picture by starting at the degree 1 vertex and kind of unrolling the tangled picture of H above, here is another drawing of H:



This new picture makes it clear that H has three bridges (single edges that can be removed to disconnect the graph), whereas G only has the one (from the vertex of degree 1).