Economics 361
Problem Set #2

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Question 1: Training Wheels

Let X be a random variable distributed Standard Normal, N(0,1). Let Y be a second random variable, also distributed Standard Normal. X and Y are distributed *independently* of each other.

Consider a third random variable $Z \equiv X + Y$. In this question, we will derive the distribution for Z from the distributions of X and Y

As X and Y are distributed Standard Normal, the marginal distributions of X and Y are

$$f(x) = \frac{1}{\sqrt{2\Pi}}\, e^{-\frac{1}{2}x^2} \qquad\qquad f(y) = \frac{1}{\sqrt{2\Pi}}\, e^{-\frac{1}{2}y^2}$$

(a) Using the above, briefly explain why the *joint* distribution of X and Y is

$$f(x,y) = \frac{1}{2\Pi}\, e^{-\frac{1}{2}(x^2+y^2)}$$

The *cumulative distribution function* for Z can be derived from the joint distribution of X and Y. Note that $\{Z \leq z\} = \{(x \in \mathcal{X}) \cap (y \in \mathcal{Y}) : x + y \leq z\}$. Therefore,

$$\begin{aligned}\underbrace{P_Z(Z \leq z)}_{F_Z(z)} &= P_{XY}(\{(x \in \mathcal{X}) \cap (y \in \mathcal{Y}) : x + y \leq z\}) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{???} f(x,y)\; dy\; dx\end{aligned}$$

(b) Fill in the ??? in the integral above.

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Using the Fundamental Theorem of Calculus, the *probability density function* of Z, $f(z)$, can be derived from its cumulative distribution function $F_Z(z)$

$$f(z) \quad = \quad \frac{d}{dz} F_Z(z)$$

(c) Use the above to show that

$$f(z) \quad = \quad \frac{1}{2\Pi}\, e^{-\frac{1}{4}z^2} \int_{-\infty}^{\infty} e^{-(x-\frac{1}{2}z)^2}\, dx$$

You may want to use Leibnitz's Rule:

$$\frac{d}{d\theta} \int_{a(\theta)}^{b(\theta)} f(x,\theta)\, dx \quad = \quad f(b(\theta),\theta)\frac{d}{d\theta}b(\theta) - f(a(\theta),\theta)\frac{d}{d\theta}a(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial\theta} f(x,\theta)\, dx$$

if $f(x,\theta), a(\theta), b(\theta)$ are differentiable with respect θ and $-\infty < \{a(\theta), b(\theta)\} < +\infty$ for all θ

Although not strictly correct, you may treat $+\infty$ and $-\infty$ as proper constants (i.e. finite real numbers) for this problem (avoids a technicality that requires some "dancing" when applying Leibnitz).

Also, remember to "complete the square" for $(x - \frac{1}{2}z)^2$ ...

(d) Now, show that

$$\int_{-\infty}^{\infty} e^{-(x-\frac{1}{2}z)^2} dx \quad = \quad \sqrt{\Pi}$$

HINT: What is the pdf of a random variable distributed $\mathrm{N}(\frac{1}{2}z, \frac{1}{2})$? And integrating a pdf over $(-\infty, \infty)$ gives you 1. You should know why.

(e) Use **(c)** and **(d)** to show that the pdf of Z is the pdf of a Normally distributed random variable. What are the mean and variance of Z?

Again, no peeking at other textbooks and websites. That ruins the "fun" and the instructional value of these problem sets

Question 2: Solo Time

A random variable Z distributed chi-squared with r degrees of freedom (χ^2_r) has the following probability density function (pdf):

$$f(z) \;=\; \frac{1}{\Gamma(r/2)2^{r/2}}\, z^{r/2-1}\, e^{-z/2} \quad \text{for} \quad z \in (0, +\infty)$$

where $\Gamma(\cdot)$ is the famed "Gamma" function

$$\Gamma(\alpha) \;=\; \int_0^{+\infty} y^{\alpha-1} e^{-y} dy$$

More specifically, the pdf of a χ^2_1 random variable Z is

$$f(z) \;=\; \frac{1}{\sqrt{2\Pi}}\, e^{-\frac{1}{2}z}\, \frac{1}{\sqrt{z}} \quad \text{for} \quad z \in (0, +\infty)$$

as $\Gamma(1/2) = \sqrt{\Pi}$ (Do not need to show)

Let X be a random variable distributed Standard Normal, N(0,1). Show that the pdf of $Z \equiv X^2$ is the same as that of a random variable distributed chi-squared with one degree of freedom (χ^2_1).

HINT: The steps are similar to those in Question 1. Start with

$$F_Z(z) \;=\; P_Z(\{Z \le z\}) \;=\; P_X(\;\;???\;\;) \quad \text{for } z \ge 0$$

ASIDE: More generally, you can show that for some continuous random variable X with a well defined distribution, the random variable $Y \equiv X^2$ has the pdf

$$f_Y(y) \;=\; \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right) \quad \text{for } y > 0$$

where $f_X(\cdot)$ is the pdf of X. This should be evident from your answer above.

Question 3: Simple Change of Variables

A function $g(X)$ is considered a **monotonic** transformation if

(a) $u > v$ necessarily implies $g(u) > g(v)$
– or –
(b) $u > v$ necessarily implies $g(u) < g(v)$

(either (a) or (b), not both)

Monotonic transformations that satisfy (a) are considered **increasing** and those that satisfy (b) are considered **decreasing**.

The calculus technique of **change of variables** can be used to show that for a transformation $Y = g(X)$ that is differentiable and monotonic

$$f_Y(y) \quad = \quad f(y) \quad = \quad f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

where $g^{-1}(\cdot)$ is the inverse function of $g(\cdot)$, the function that, for a given y, returns the value of x such that $g(x) = y$. Example: if $g(x) = 3x + 5$ then $g^{-1}(y) = \frac{y-5}{3}$.

The term $\left| \frac{d}{dy} g^{-1}(y) \right|$ is referred to as the *Jacobian.* It is the *absolute value* of the derivative of the inverse function (in the single variable case).

A random variable X that is distributed Normal with mean μ and variance σ^2, $\mathrm{N}(\mu, \sigma^2)$, has the following probability density function (pdf)

$$f_X(x) \quad = \quad f(x) \quad = \quad \frac{1}{\sqrt{2\Pi\sigma^2}} \, e^{-\frac{1}{2} \, \frac{(x-\mu)^2}{\sigma^2}}$$

Let $Y \equiv a + b \; X$ where a, b are known real valued constants.

Use the change of variables technique to show that the pdf of Y is the same as that of a random variable distributed Normal with mean $a + b\mu$ and variance $b^2\sigma^2$, $\mathrm{N}(a + b\mu, b^2\sigma^2)$

BONUS: Use change of variables to show the result in Question 2: the square of a random variable distributed N(0,1) is itself distributed χ^2_1. This problem is optional (but recommended)

Question 4: Not Quite Normal

The Normal distribution is, by far, the most popular/utilized distribution for continuous random variables – for reasons we will discuss later. But not all random experiments are natural candidates for representation by a Normally distributed random variable.

Consider stock prices. Generally speaking, stock prices are non-negative.[1] While it is natural to represent the random experiment of the exchange value of a corporation (at some point in time) with a random variable that simply returns the corporation's stock price

$$\text{e.g.}\quad X = 52.48 \quad \text{when the corporation's stock price is \$52.48}$$

it is unreasonable to assume that such random variable would be distributed Normal. Recall that random variables distributed Normal can take any real value, including negative values!

Many stock analysts assume that stock price (rather, the random variable that returns the stock price) is distributed *log Normal.* A random variable distributed log Normal is assumed only to realize *non-negative* values.

The pdf of the log Normal (for given parameter values α and β) is

$$f_X(x) \quad = \quad \frac{1}{\sqrt{2\pi\beta}} \ \frac{e^{\frac{-(ln(x)-\alpha)^2}{2\beta}}}{x} \qquad \text{for } x \geq 0$$

α may take any real value but β must be positive.

Consider a random variable Y that is distributed Normal with mean μ and variance σ^2, $N(\mu, \sigma^2)$.

(a) Show that the pdf of random variable $X \equiv e^Y$ is distributed log Normal. State what (α, β) are in terms of (μ, σ^2)

(b) Briefly explain why the log Normal distribution is called, well, the log Normal distribution!

(c) Based on what you know about the shape of the Normal pdf (when graphing $f(y)$ against y), provide an intuitive reason why the log Normal pdf is uni-modal but not symmetric.

(d) Based on our discussion so far, what might a stock analyst be predicting about stock price when s/he claims that s/he is "betting on a far tail outcome."

(e) Consider a second random variable, Z, that is also distributed log Normal. Further, let X and Z be distributed independently of each other. Recall the result we derived in Question 1. Intuitively, explain why $W \equiv X \times Z$ is also distributed log Normal. (You do not need to show formally, although you could using the tools we have developed in this problem set)

[1] Non-negativity stems, in part, from the limited liability status and bankruptcy protection afforded to corporations

Question 5: Patent Race, Part I

Consider two research and development (R&D) firms who are both working on an improved pain reliever. Let us dub the two firms "incumbent" (**I**) and "entrant" (**E**).

Let X_t^I be the random variable indicating whether the incumbent had successfully innovated in the t^{th} year. Let X_t^E be the random variable indicating whether the entrant had successfully innovated in the t^{th} year. The random variables take the value of 1 with success and 0 with failure.

$$X_t^I = \begin{cases} 1 & \text{if R\&D succeeds in } t^{th} \text{ for } \mathbf{I} \\ 0 & \text{if R\&D fails in } t^{th} \text{ year for } \mathbf{I} \end{cases} \qquad X_t^E = \begin{cases} 1 & \text{if R\&D succeeds in } t^{th} \text{ year for } \mathbf{E} \\ 0 & \text{if R\&D fails in } t^{th} \text{ year for } \mathbf{E} \end{cases}$$

For simplicity, assume that previous R&D failures have **no impact** on current R&D. The firm neither learns from nor gets discouraged by earlier research failures. So X_t^I is statistically independent of X_s^I for all $s < t$. Similarly for X_t^E.

Consider the following conditional probability for the incumbent

$$P(X_t^I = x |\text{ no earlier R\&D success}) = \begin{cases} \theta_I & \text{if } x = 1 \\ 1 - \theta_I & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

and similarly for the entrant

$$P(X_t^E = x |\text{ no earlier R\&D success}) = \begin{cases} \theta_E & \text{if } x = 1 \\ 1 - \theta_E & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

$\{\theta_I, \theta_E\}$ are known as "transition probability" (for the incumbent and entrant, respectively). They denote the probability that the firm will "transition" from one state (here, R&D failure) to another (here, R&D success).

(a) Briefly explain why the following probability mass function statements are correct:

$$\begin{aligned} P(\{X_1^E = 0, X_2^E = 1\}) &= P(X_1^E = 0) \cdot P(X_2^E = 1 \mid X_1^E = 0) \\ P(\{X_1^E = 0, X_2^E = 0, X_3^E = 1\}) &= P(X_1^E = 0) \cdot P(X_2^E = 0 \mid X_1^E = 0) \cdot P(X_3^E = 1 \mid \{X_1^E = 0, X_2^E = 0\}) \end{aligned}$$

(b) Let Y^E denote the random variable that indicates the number of years of research until the entrant achieves R&D success. So the event $Y^E = 5$ corresponds to $\{X_1^E = 0, X_2^E = 0, \ldots X_5^E = 1\}$ Explain why the probability mass function for Y^E is as follows:

$$P(Y^E = y) = \begin{cases} (1 - \theta_E)^{y-1}\, \theta_E & \text{if } y \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

(c) Write down the cumulative distribution function for Y^E

We will continue this problem in the next problem set.