Lecture 7:

P&L of Options Trading When:

Hedging continuously

Hedging discretely

Transactions costs:

Conclusion:

Accurate Replication is Very Difficult

7.1 The P&L of Any Hedged/Replicated Trading Strategy

$$(C_0 - \Delta_0 S_0)e^{r\tau} = (C_T - \Delta_T S_T) + \int_0^{\tau} e^{r(\tau - x)} S_x [d\Delta_x]_b \qquad \text{Eq 7.1}$$

You can integrate by parts using the relation

$$d\left[e^{r(\tau - x)} S_x \Delta_x\right] = -re^{r(\tau - x)} \Delta_x S_x dx + e^{r(\tau - x)} \Delta_x dS_x + e^{r(\tau - x)} S_x [d\Delta_x]_b$$ (see next section for backward Ito)

to obtain

Eq 7.2

$$(C_0 - \Delta_0 S_0)e^{r\tau} = (C_T - \Delta_T S_T) + \int_0^{\tau} d\left[e^{r(\tau - x)} S_x \Delta_x\right] + \int_0^{\tau} re^{r(\tau - x)} \Delta_x S_x dx - \int_0^{\tau} e^{r(\tau - x)} \Delta_x dS_x$$

$$(C_0 - \cancel{\Delta_0 S_0})e^{r\tau} = (C_T - \cancel{\Delta_T S_T}) + \left[\cancel{\Delta_T S_T} - \cancel{\Delta_0 S_0} e^{r\tau}\right] + e^{r\tau} \int_0^{\tau} e^{-rx} \Delta(S_x, x)[dS_x - rS_x dx]$$

$$C_0 = C_T e^{-r\tau} - \int_0^{\tau} \Delta(S_x, x)[dS_x - rS_x dx]e^{-rx} \qquad \text{Eq 7.3}$$

PV of payoff

PV of change in value of the hedged shares funded at the riskless rate

Equation 7.1 and Equation 7.3 provide a way to calculate the value of the option C_0 in terms of its final payoff and the hedging strategy. If you hedge perfectly and continuously to get a riskless position, Black-Scholes tells you that this value will be independent of the path the stock takes to expiration. Else the fair value of C_0 has a spread-out distribution.

Suppose that the stock really satisfies GBM with a drift equal to the riskless interest rate so that

$$dS - Srdt = \sigma SdZ.$$

Then

$$C_0 = C_T e^{-r\tau} - \int_0^{\tau'} \Delta(S_x, x)\sigma S_x e^{-rx} dZ_x$$

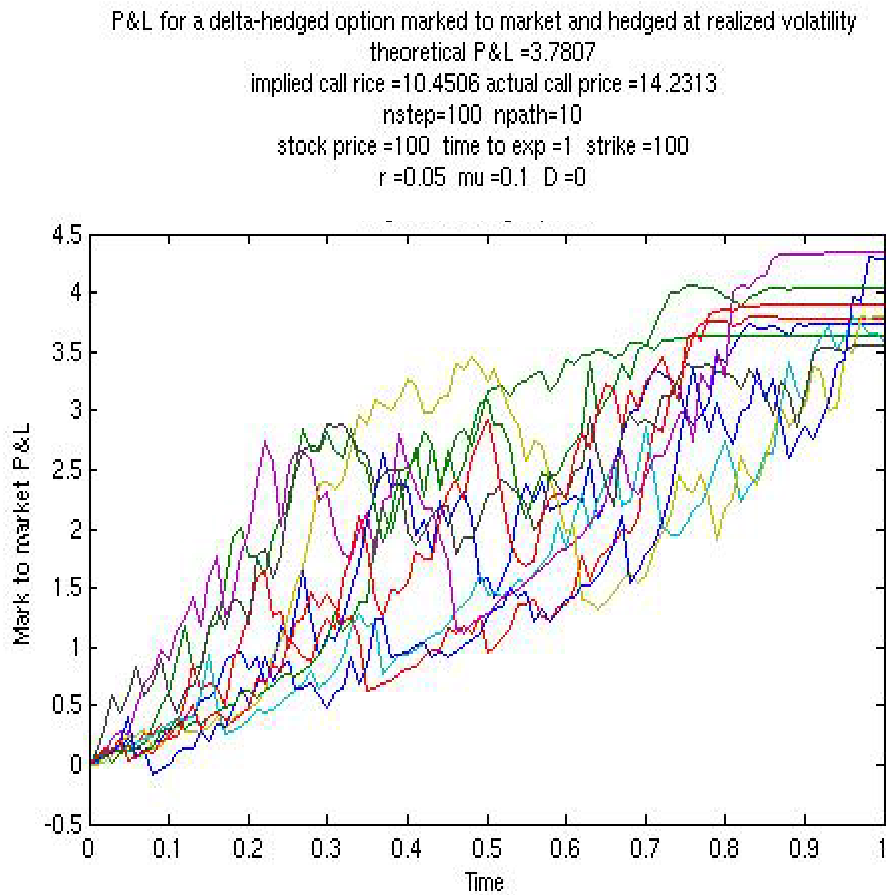
$$E[C_0] = E[C_T]e^{-r\tau}$$

independent of the volatility use to calculate Δ! But that’s a very special case that doesn’t happen.

7.2 Hedging at Realized

To illustrate this, plot cumulative **discounted P&L(I, R,R)** along ten random stock paths with100 steps

P&L starts at zero because initial position is totally financed

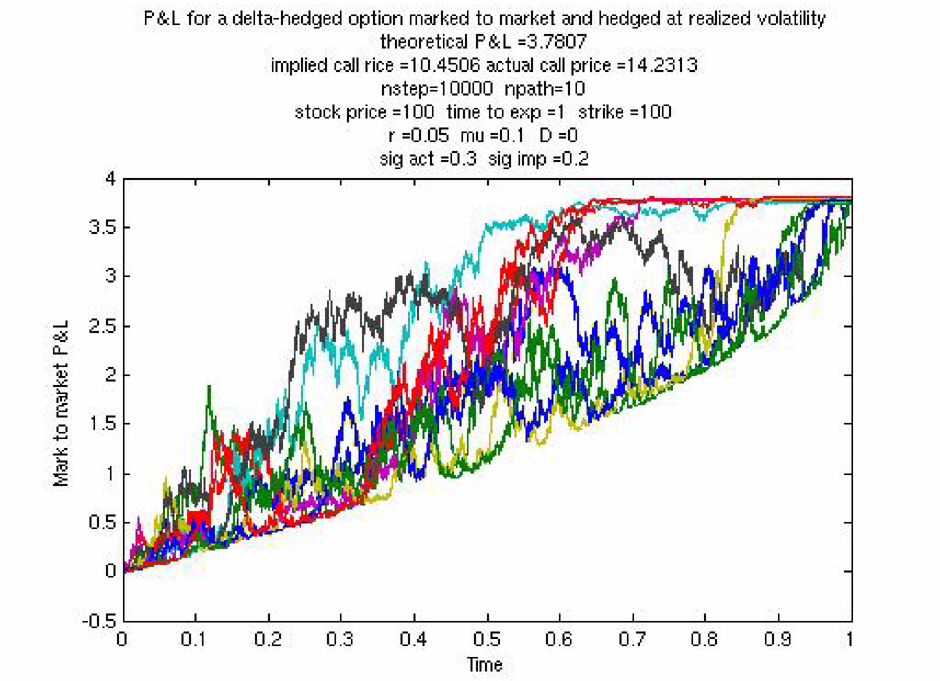


$\sigma_r = 0.3$

$\sigma_i = 0.2$

The **final P&L** is almost path-independent – almost, because 100 rehedgings per year is not quite the same as continuous hedging.

Rehedge 10,000 times, almost path-independent **final P&L**: Upper bound $(V_{R,0} - V_{I,0})$ occurs when



the second term is zero, when the option value is independent of volatility, which occurs at $S = 0$ or $S = \infty$, and the gamma of the option is zero.

The lower bound to the P&L occurs when second term $[V(\sigma, S, m) - V(\Sigma, S, m)] \sim \frac{\partial V}{\partial \sigma}(\Sigma - \sigma)$ is a maximum, i.e. when vega is largest, close to at-the money, which turns out to be at $S = Ke^{-(r - 0.5\sigma\Sigma)(T - t)}$

7.3 P&L When Hedging with Implied Volatility

Cumulative P&L along 10 random stock paths, 100 hedging steps to expiration

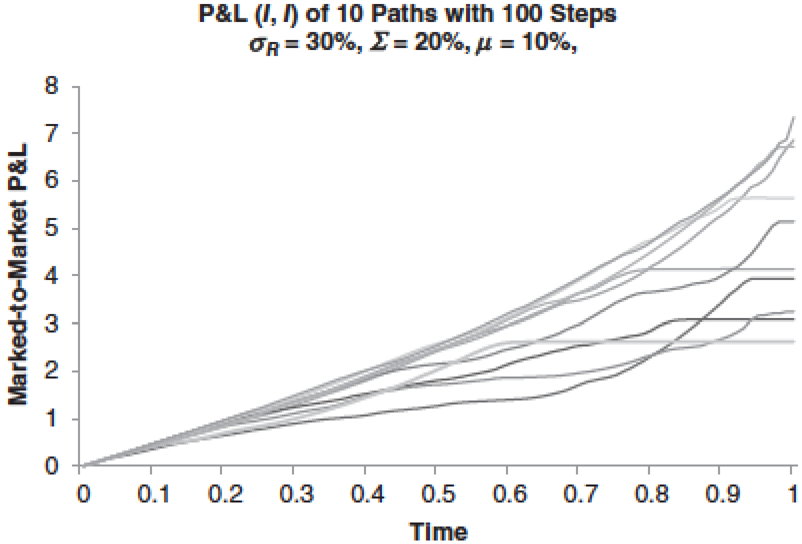


FIGURE 5.4 Hedging with Implied Volatility: Cumulative Discounted P&L with 100 Steps, Drift = 10%, Time to Expiration = 1 Year

most P&L when gamma stays large, i.e. when the stock stays near the strike

with high drift the gamma quickly becomes close to zero

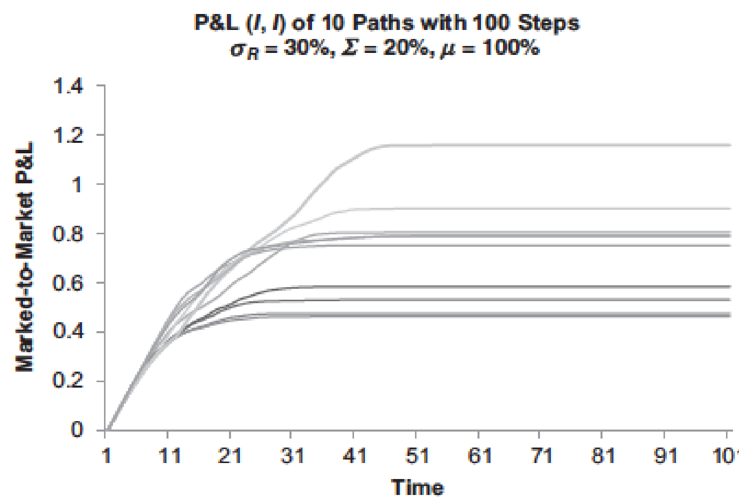


FIGURE 5.5 Hedging with Implied Volatility: Cumulative Discounted P&L with 100 Steps, Drift = 100%, Time to Expiration = 1 Year

7.4 What Happens to Cash From Dividends etc in P&L??

time		PV at time t	PV[P&L]
t	B_t	B	
$t+dt$	B_{t+dt}	B	0

The PV of the P&L is zero so we omitted it in previous calculations.

7.5 Hedging Errors from Discrete Hedging at $\sigma_h = \sigma_r$

- Hedging perfectly and continuously at no cost is a Platonic ideal with no error term dZ.
- In real life, you can rebalance the hedge only a finite number of times.
- You are mishedged in the intervals, and the P&L picks up a random component.
- The more often you hedge, the smaller the deviation from perfection.
- Transaction costs affect things, too, but that's later.

A Simulation Approach

You cannot hedge continuously, and therefore it is important to understand the errors that creep into your P&L when you hedge at discrete intervals. Some traders hedge at regularly spaced time intervals; others hedge whenever the delta changes by more than a certain amount. In what follows here we will discuss replication at regular time intervals as the stock evolves at σ_r.

$$(C_0 - \Delta_0 S_0)e^{r\tau} = (C_T - \Delta_T S_T) + \int_0^{\tau} e^{r(\tau - x)} S_x [d\Delta_x]_b$$ $\Delta(\sigma_h)$ is the replication/hedging volatility.

Sample code: Evolving the stock through time on each path that corresponds to given drift and vol:

```
Stockprice(i,:) = Stockprice(i-1,:).*exp((mu-div_rate-0.5*sig_act^2)*dt + sig_r*sqrt(dt)*Z(i-1,:))
```

Summing up all the gamma contributions along each path:

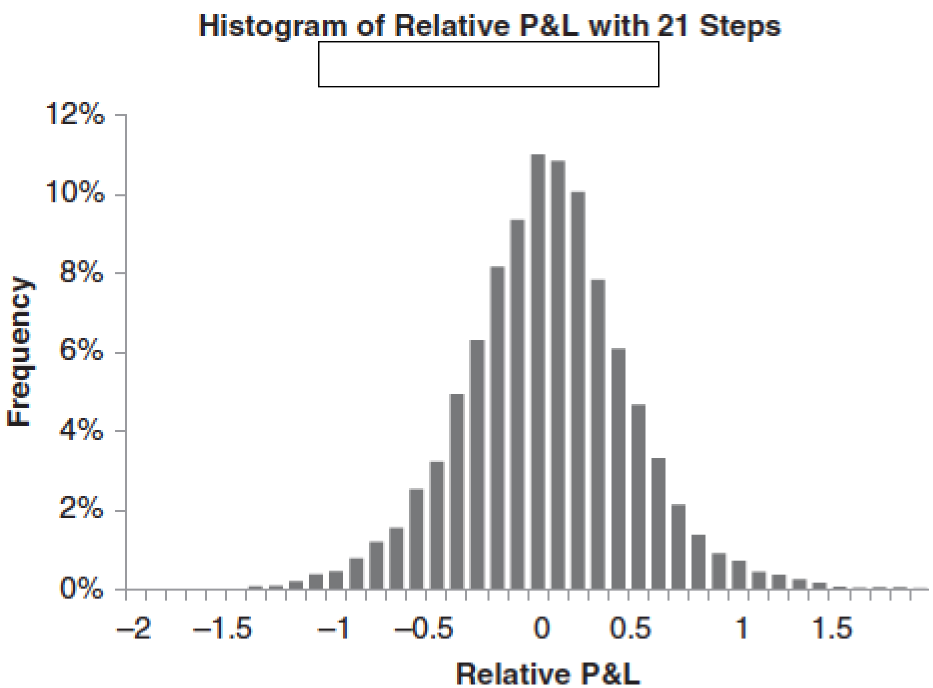
```
Time_integral = Time_integral + Stockprice(i,:).*(Delta(i,:) - Delta(i-1,:))*exp(rate *(time_to_exp))
```

Calculating initial call value from the path integral in Equations 7.1:

Because we hedge or replicate discretely, different paths will give different values and so we'll get a histogram of call values, and an average with a standard deviation, the hedging error.

Monte Carlo: ATM option, expiration 1 month, the realized volatility is 20%, $\mu = r = 0$, hedged or replicated at an implied volatility of 20% equal to the realized volatility. $\sigma_r = \sigma_h = 0.2$

Plot: Relative P&L = Present Value of Payoff – BSM Fair Value.



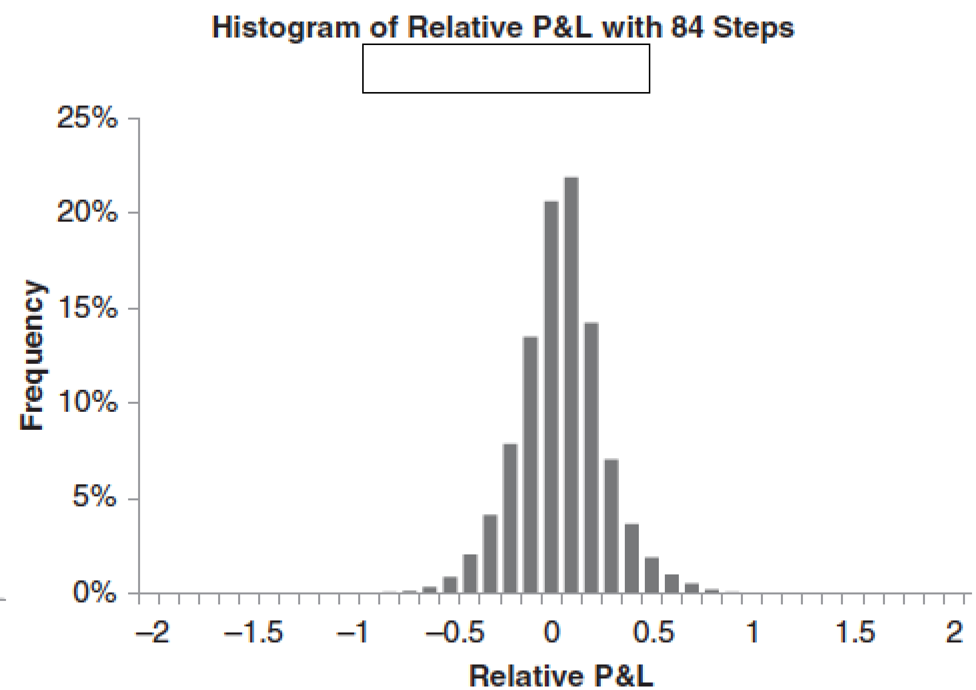


FIGURE 6.1 Distribution of Relative P&L for a One-Month At-the-Money Call Option When Hedging Volatility = Realized Volatility, $\mu = r$ (Relative P&L = Present Value of Payoff – BSM Fair Value)

The mean deviation from BS is zero; When we quadruple the number of hedgings, the standard deviation of the P&L halves. We do better by hedging/replicating more frequently.

Now let's see what happens $\sigma_h \neq \sigma_r$ and we hedge at realized. Choose a replication volatility of 40%.

P&L is relative to BS valued at realized. The mean is still close to the BS value.

There is no longer the same reduction in standard deviation when the number of rebalancings quadruple. Both distributions are more or less symmetric though.

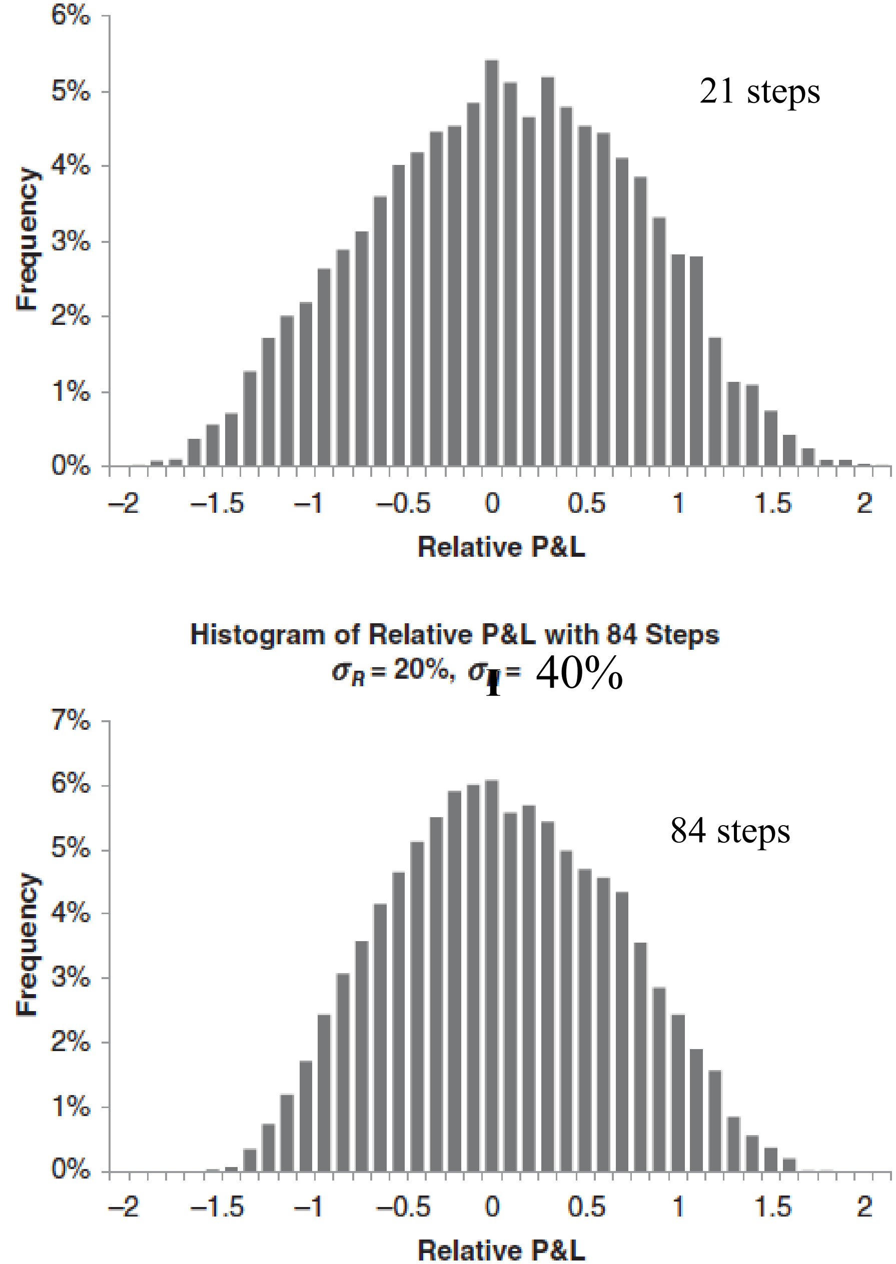


FIGURE 6.2 Distribution of Relative P&L for a One-Month At-the-Money Call Option When Hedging Volatility ≠ Implied Volatility, $\mu = r$ (Relative P&L = Present Value of Payoff – BSM Fair Value)

7.6 Understanding ATM Discrete Hedging Error Analytically when $\sigma_i = \sigma_r \equiv \sigma$. (Assuming we know the future volatility)

Discrete time means dt is larger than infinitesimal, -- the stock goes through a finite move.

$$\frac{dS}{S} = \mu dt + \sigma Z\sqrt{dt}$$

$$Z \sim N(0,1)$$ i

Hedged portfolio $\pi = C - \left(\frac{\partial C}{\partial S}\right)S$; Initial long π bought with borrowed money. If we hedged continuously the P&L would be zero.

Hedging error owing to mismatch between a **continuous** hedge ratio and a **discrete** time step:

$$
\begin{aligned}
HE_{dt} &= \pi + d\pi - \pi e^{rdt} \\
&\approx d\pi - r\pi dt \\
&\approx \left[\frac{\partial C}{dt}dt + \cancel{\frac{\partial C}{\partial S}dS} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 Z^2 dt - \cancel{\frac{\partial C}{\partial S}dS}\right] - rdt\left[C - \frac{\partial C}{\partial S}S\right] \\
&\approx \left[\frac{\partial C}{dt} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 Z^2 - r\left(C - \frac{\partial C}{\partial S}S\right)\right]dt
\end{aligned}
$$

discrete continuous

Now from Black-Scholes

$$r\left(C - \frac{\partial C}{\partial S}S\right) = \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2$$

$$HE_{dt} \approx \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \left(Z^2 - 1\right) dt$$

Gamma distribution Eq.7.1

Over n steps to expiration, the total HE is

$$HE = \sum_{i=1}^{n-1} \frac{1}{2}\Gamma_{i-1} S_{i-1}^2 \sigma^2 \left(Z_i^2 - 1\right) dt \qquad \text{Eq.7.2}$$

Z is normal with $E(Z^2) = 1$ and $E(Z^4) = 3$ so $E[HE] = 0$ with a χ^2 distribution.

The variance of the hedging error is

$$E\left[\sum_{i=1}^{n-1} \frac{1}{2}\Gamma_{i-1} S_{i-1}^2 \sigma^2 \left(Z_i^2 - 1\right) dt\right]^2 = E\left[\sum_{i=1}^{n-1} \left(\frac{1}{2}\Gamma_{i-1} S_{i-1}^2 \sigma^2 dt\right)^2 \left(Z_i^2 - 1\right)^2\right]$$

$$\sigma_{HE}^2 = E\left[\sum_{i=1}^{n} \frac{1}{2}\left[\Gamma_{i-1} S_{i-1}^2\right]^2 (\sigma^2 dt)^2\right] \text{ over all paths} \qquad \text{Eq.7.3}$$

Integrating over all returns starting from S_0 **for a single atm option**

$$E\left[\Gamma_i S_i^2\right]^2 = S_0^4 \Gamma_0^2 \sqrt{\frac{T^2}{T^2 - t_i^2}}$$

Thus for constant volatility

$$\begin{aligned}
\sigma_{HE}^2 &\approx \sum_{i=1}^{n} \frac{1}{2} S_0^4 \Gamma_0^2 \sqrt{\frac{T^2}{T^2 - t_i^2}} \left(\sigma^2 dt\right)^2 \\
&\approx \frac{1}{2} S_0^4 \Gamma_0^2 \left(\sigma^2 dt\right)^2 \sum_{i=1}^{n} \sqrt{\frac{T^2}{T^2 - t_i^2}} \\
&\approx \frac{1}{2} S_0^4 \Gamma_0^2 \left(\sigma^2 dt\right)^2 \frac{1}{dt} \int_t^T \sqrt{\frac{T^2}{T^2 - \tau^2}} d\tau \\
&\approx S_0^4 \Gamma_0^2 \left(\sigma^2 dt\right)^2 \frac{\pi (T - t)}{4dt} \\
&\approx \frac{\pi}{4} n \left(S_0^2 \Gamma_0 \sigma^2 dt\right)^2
\end{aligned}$$

$$n = \frac{T-t}{dt}. \qquad \text{\# of steps}$$

From BS we can interpret $S_0^2\Gamma_0 = \dfrac{1}{\sigma(T-t)}\dfrac{\partial C}{\partial\sigma}$

$$\sigma_{HE}^2 \approx \frac{\pi}{4}n\left(\frac{1}{\sigma(T-t)}\frac{\partial C}{\partial\sigma}\sigma^2 dt\right)^2$$

$$\approx \frac{\pi}{4}n\left(\sigma\frac{1}{n}\frac{\partial C}{\partial\sigma}\right)^2 \qquad \frac{dt}{T-t} = n$$

$$\approx \frac{\pi}{4n}\left(\sigma\frac{\partial C}{\partial\sigma}\right)^2$$

$$\sigma_{HE} \approx \sqrt{\frac{\pi}{4}}\frac{\partial C}{\partial\sigma}\frac{\sigma}{\sqrt{n}} \qquad \text{Eq.7.4}$$

Thus, the hedging error is approximately $\dfrac{\partial C}{\partial\sigma}\times\dfrac{\sigma}{\sqrt{n}}$. In order to halve the error we must quadruple the number of rehedgings. What does this mean?

Understanding The Results Intuitively

Hedging discretely introduces uncertainty in the hedging outcome but no bias: $E[HE] = 0$

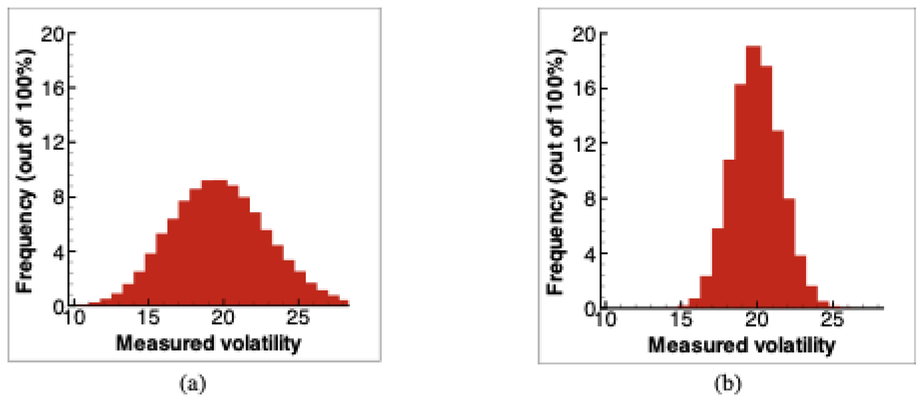
Simple analytic rule $$\sigma_{HE} \sim \sqrt{\frac{\pi}{4}}\frac{\partial C}{\partial \sigma} \times \frac{\sigma}{\sqrt{n}}$$

For $S \sim K$, more simply $$.C \approx \frac{S\sigma\sqrt{\tau}}{\sqrt{2\pi}} \qquad \frac{\partial C}{\partial \sigma} \approx \frac{S\sqrt{\tau}}{\sqrt{2\pi}}$$

Therefore $$\frac{\sigma_{HE}}{C} \approx \sqrt{\frac{\pi}{4n}} = \frac{0.89}{\sqrt{n}}:$$ 9% error for 100 rehedges

Think of this as statistical sampling error: discrete hedging samples/estimates the volatility discretely and is therefore subject to error owing to only n observations. The standard deviation or standard error in a Monte Carlo simulation using constant volatility σ but sampled with discrete steps is $\frac{\sigma}{\sqrt{2n}}$. Do a simulation and measure the volatility on each path with n steps per path:

Exhibit 2: Histograms showing the volatilities estimated from (a) 21 and (b) 84 simulated returns.



The error in the observed call price owing to the error in the volatility estimate is

$$\sigma_{HE} \approx dC \approx \frac{\partial C}{\partial \sigma} d\sigma \approx \frac{\sigma}{\sqrt{n}} \frac{\partial C}{\partial \sigma}$$

This is quite a large error even assuming we know the future volatility with certainty. In real life your hedge ratio is incorrect not just because hedging is discrete, but because you don't actually know the appropriate volatility to use.

What happens when the hedging volatility and the realized volatility are different for discrete hedging? More frequent hedging doesn't reduce the error.

Example for $r = D = 0$ and $\sigma_r = 30\%$, one-month atm option.

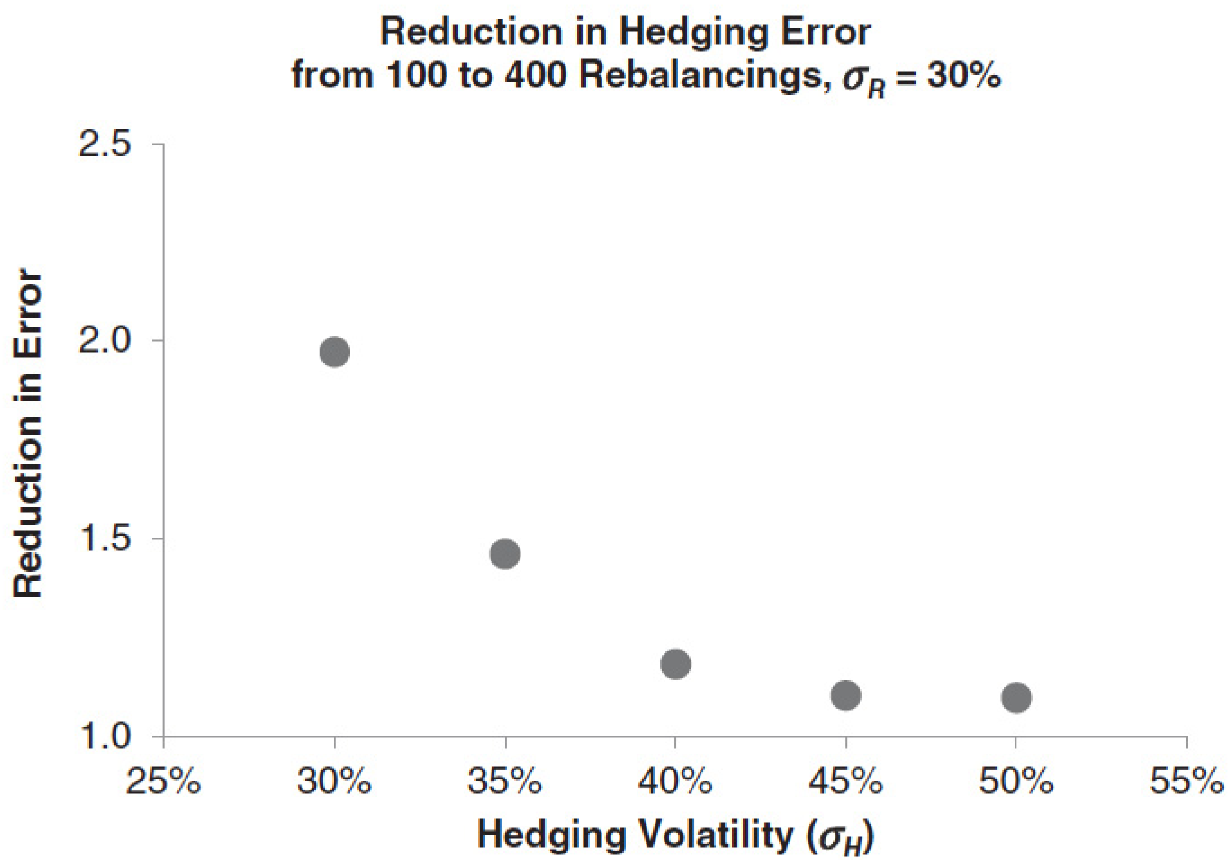


FIGURE 6.5 Reduction in Replication Error with Fourfold Increase in the Number of Rebalancings

Conclusion: Accurate Replication and Hedging are Very Difficult

- In our BS laboratory, we assumed that we could know future realized volatility with certainty.
- In fact, you know the implied volatility from the market price of the option, but you can only try to predict future volatility. Therefore, when you hedge an option, you usually have to choose between hedging at implied volatility and hedging using a guess for the future realized volatility.
- If you estimate future realized volatility correctly and hedge (or replicate) **continuously** at that volatility, your P&L will capture the exact value of the option.
- If you hedge discretely at the realized volatility, your P&L will have a random component. You will get closer and closer to the exact BSM value the more often you hedge, with the discrepancy decreasing proportional to $1/\sqrt{n}$ where n is the number of rehedgings.

- If implied volatility is not equal to realized volatility and you hedge continuously at implied volatility, your P&L will be path-dependent and unpredictable. The P&L will be a maximum when the gamma of the option is a maximum, which occurs when the stock price stays close to the strike price on its path to expiration.

- If you hedge discretely at implied volatility, not only will your P&L be path-dependent and unpredictable, but in addition your P&L will pick up a random component that occurs because the hedge is accurate only instantaneously, but not during the intervals between rebalancing.
- In practice, traders are most likely to hedge at implied volatility. The more implied volatility differs from the realized volatility, the more they will lose the benefit of increasing the number of rehedgings.

7.7 The Effect of Transactions Costs

It costs money to hedge:

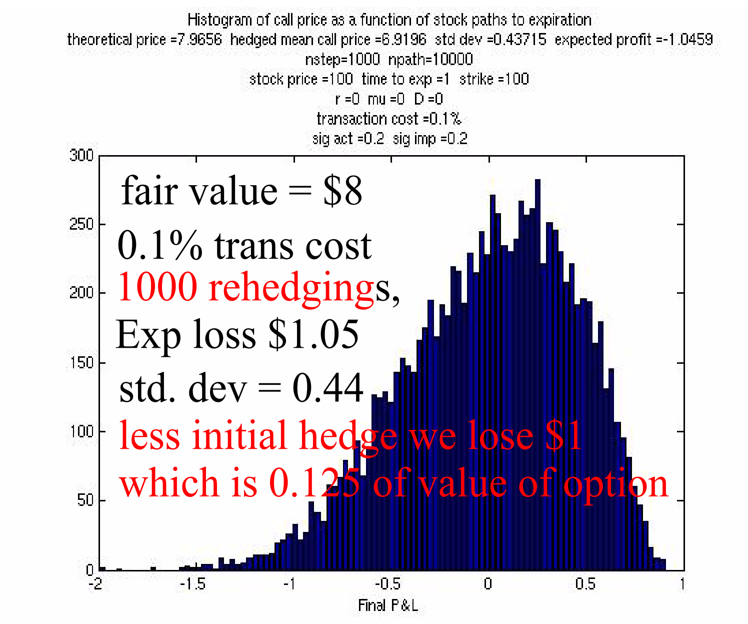
- That means that if you buy the option, you will be required to spend some extra cash to hedge, and therefore the option is worth less to you than the BSM value. If we use the BSM formula to calculate the implied volatility of the option, this lower price corresponds to a lower implied volatility than we would get without transaction costs.
- Similarly, if you are short the option, you will also have to spend extra cash to hedge it, and therefore you should have sold it for a greater price than the pure BSM value. This corresponds to a greater implied volatility in the BSM formula. Transaction costs, in short, introduce a natural bid-ask spread into option valuation.
- When there are no transaction costs, the value of a portfolio of two BSM options is equal to the sum of their individual values. This is not true when you have to pay a fee to buy or sell stocks.
- If you combine two options into a portfolio, their hedge ratios may partially cancel, and hence the transaction costs required to hedge two options together are not necessarily the sum of the transaction costs required to hedge each option separately. The transaction costs for a portfolio are **nonlinear** in the number of options, and you cannot unambiguously isolate the transaction costs for a single option if that option is part of a portfolio.
- It's important to understand that there is a natural tension in hedging with transaction costs: The more often you hedge, the smaller the hedging error, but the more often you hedge, the greater the cost and the smaller the expected net payoff.

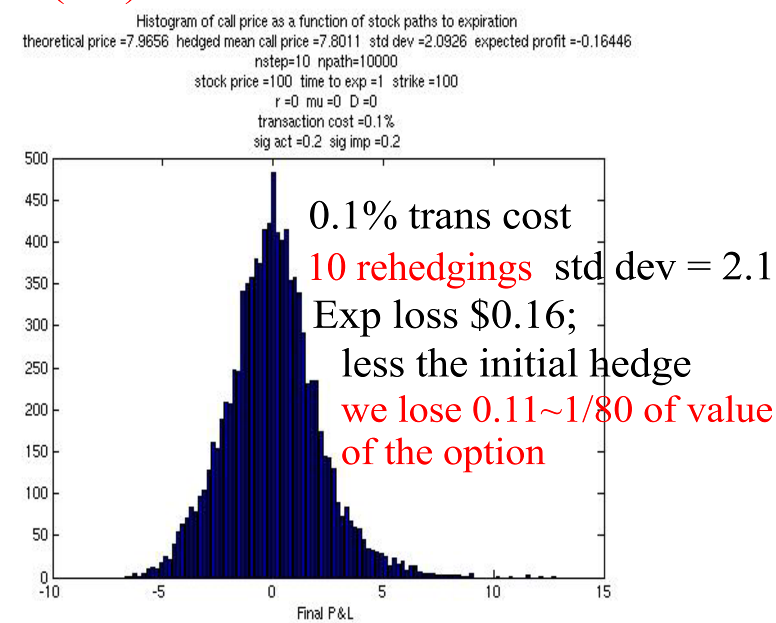
Simulation of Transaction Costs: Rehedging at regular intervals with all volatilities the same

In the examples that follow, we assume that the realized volatility σ is known, and that we replicate the call using a number of shares equal to the BSM hedge ratio $\Delta_{BS}(\sigma)$. We will also assume that transaction costs are proportional to the price of the shares traded, whether those shares are bought or sold.

Let k = 0.1%, the cost per dollar of share price. Also, assume $\sigma = 0.2$

cost of initial hedge is k(Δ)S=0.001(0.5)100=\$0.05





Tension between diminishing hedging error and reducing cost! What is optimal rebalancing?

Remember: we assumed that the realized volatility was known, but that too is uncertain.

On a trading desk you could do simulations with more possibilities

Relative P&L, same situation:

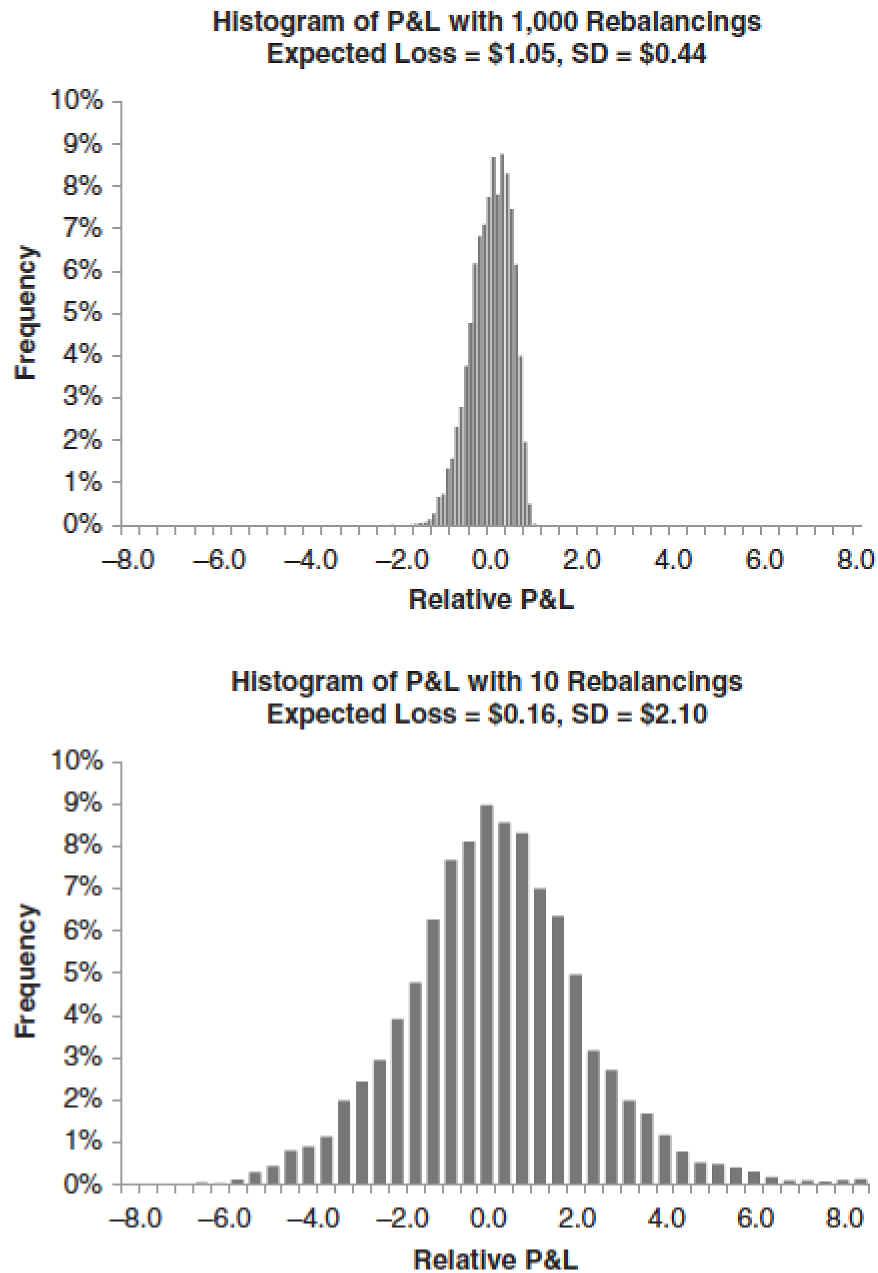
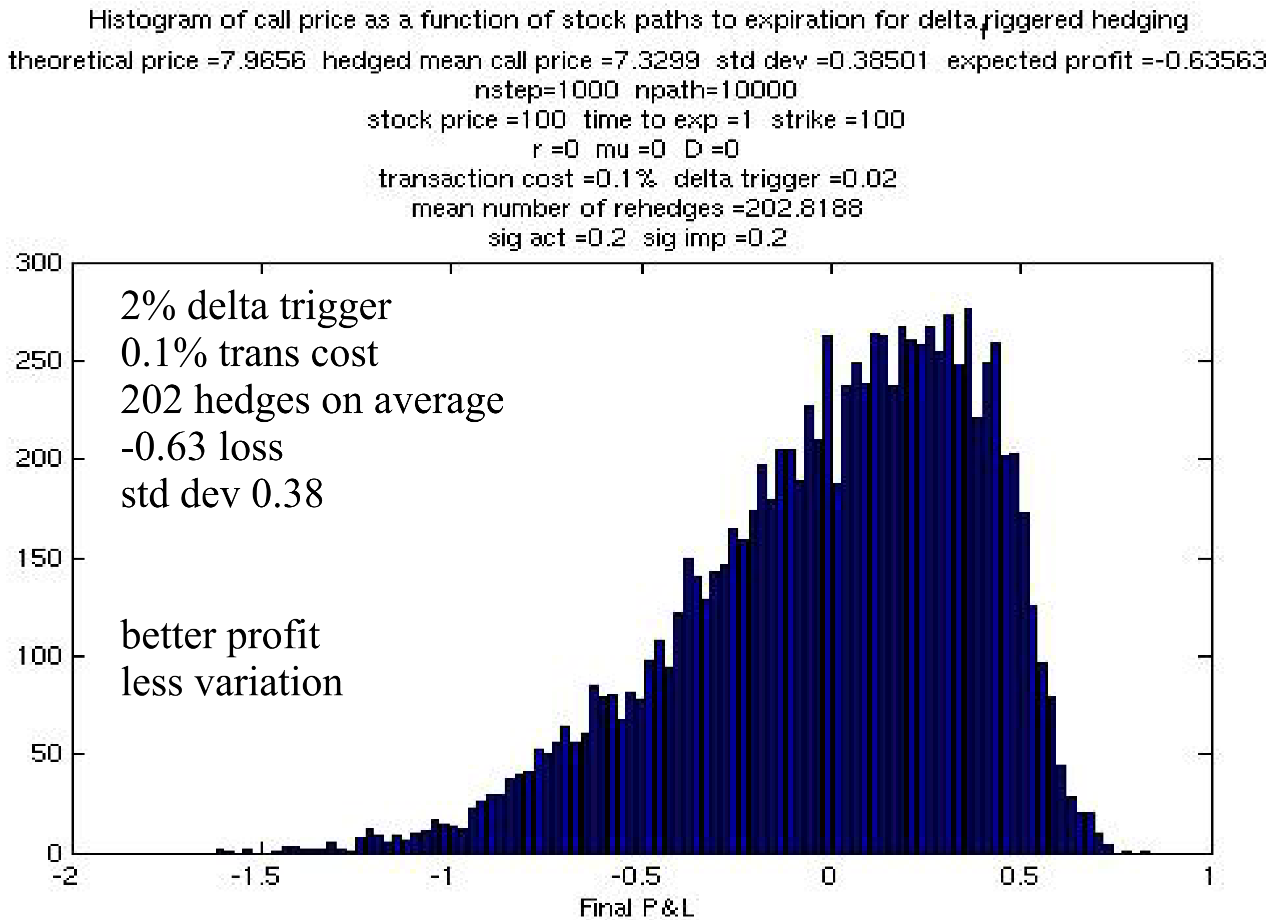


FIGURE 7.2 The Effect of Transaction Costs on the Value of Hedged Option Payoffs

7.7.1 Rehedging Triggered By Changes In The Hedge Ratio

We can hedge more efficiently by triggering on a substantial change in delta.

Hedging 1 yr at-the-money call with a delta trigger of 0.02 or 2% and a transactions cost of 0.1%.



Relative P&L, same situation.

The loss owing to the transactions cost is smaller; the standard deviation of the P&L is smaller too.

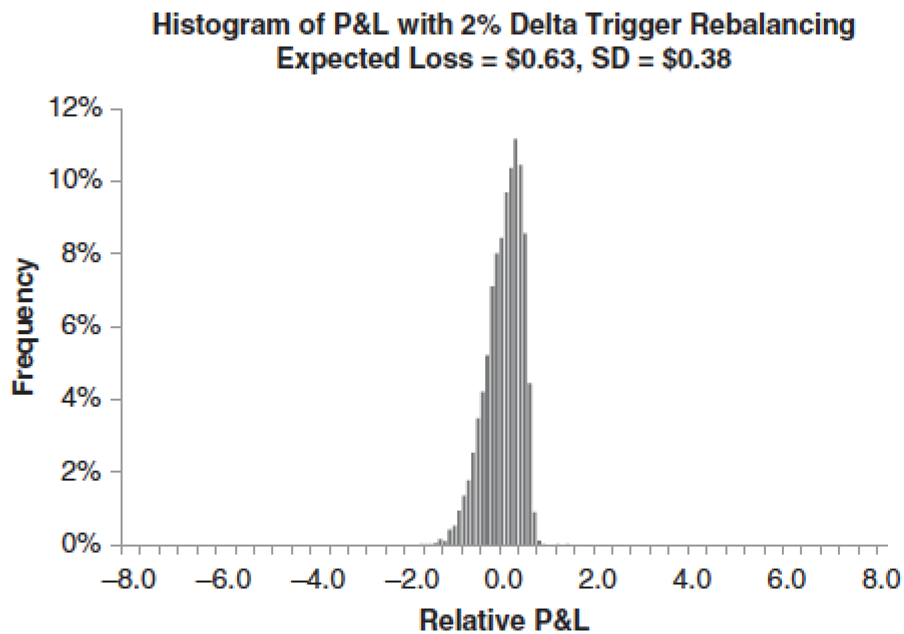
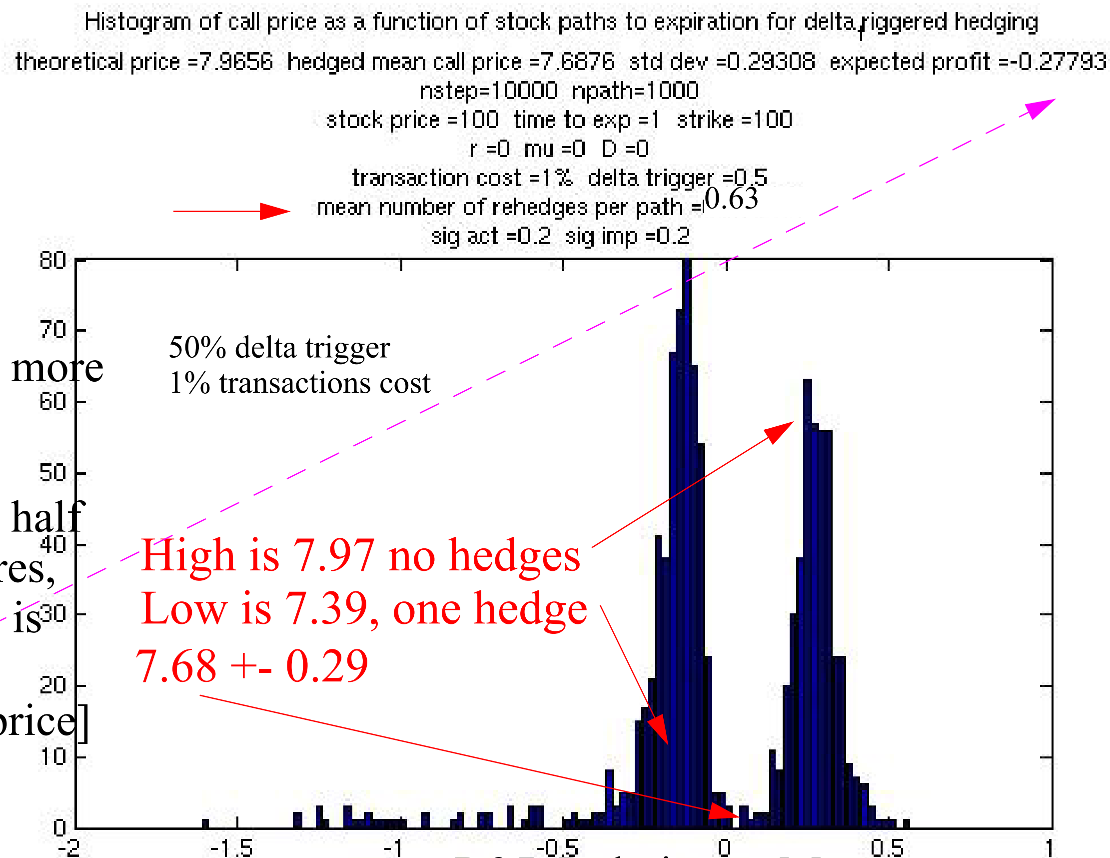


FIGURE 7.3 Rebalancing with 0.02 Delta Trigger, 0.1% Transaction Cost

Testing extreme case for illustration: rehedge only when the delta changes by 50 percentage points and with a transactions cost of 1%.

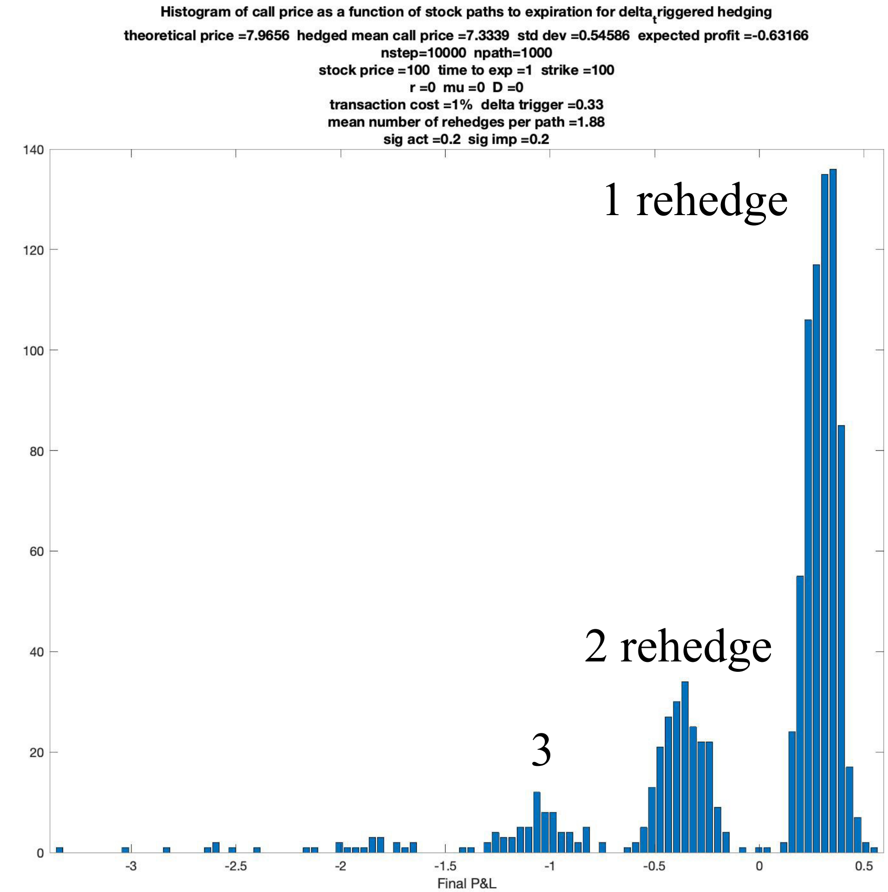
Some moves lead to no rehedging and high value; some moves lead to one or more rehedges and loss in value below the mean:
If you rehedge once, about half the time, and trade 0.5 shares, *then expected* loss in value is probability x cost =
(0.63)[k x shares traded x price]
=(0.63)(0.01)(0.5)100
=0.31, close to simulation



The distribution is bimodal. The reason is that if you rehedge only when the delta of the option changes by 50 points, then rehedges only occur when the stock makes a substantial move up or down in order to achieve such a large change in the delta. Hence one set of final call prices involve no transactions costs for rehedging (over the paths where delta changed by less than 50 points) and hence lie above the mean; the other set of call final call prices involve one rehedging and its cost (over the paths where delta did change by 50bp or more) and hence lie below the mean.

Always try to make sense of your numerical results.

Delta Trigger 0.33 leads to three modes.



2 rehedges at average stock price 100 times 1% cost, but you trade 1/3 of a share, the average cost is

2 [100 x 1% x 1/3] = 0.67 dollars is the reduction in value.

7.7.2 Analytical Approximations to Transactions Cost

First consider the case with no transactions cost, where we showed:

$$HE \approx \sum_{i=1}^{n} \frac{1}{2}\Gamma_i \sigma_i^2 S_i^2 (Z_i^2 - 1)dt$$

$$E[HE] = 0 \qquad \text{Eq.7.5}$$

$$\sigma_{HE}^2 \sim O([dt]^2)$$

The total number of rehedgings is $T/(dt)$

$$\sigma_{HE}^2 \sim O\left(\frac{T}{dt}[dt]^2\right) \sim O(Tdt) \to 0 \qquad \text{as } dt \to 0$$

Hedging continuously captures exactly the value of the option. (Story).

Now include transactions costs. Assume that you rehedge an option with value C every time dt passes. Every time you trade the stock (buying *or* selling), you **pay** a fraction k of the cost of the shares traded.

Then, every time you rehedge, you have to trade a number of shares equal to

$$N = \Delta(S + dS, t + dt) - \Delta(S, t) \approx \frac{\partial^2 C}{\partial S^2} dS + \text{terms of order } dt$$

But to leading order

$$dS \sim \sigma S\sqrt{dt}Z.$$

$$N \approx \frac{\partial^2 C}{\partial S^2}\sigma S\sqrt{dt}Z$$

The trading cost is value of number of shares traded times the fraction k, that is

$$\begin{aligned}\text{Cost} &= |NS|\,k \\ &= \left|\frac{\partial^2 C}{\partial S^2}\sigma S^2 Z\sqrt{dt}\right| k \\ &= \left|\frac{\partial^2 C}{\partial S^2}Z\right| \sigma S^2 k\sqrt{dt}\end{aligned}$$

where the absolute value reflects the fact that you pay a positive transaction cost irrespective of whether you buy or sell shares. (Therefore nonlinear!)

There are $T/(dt)$ rehedgings to expiration.

Total cost of order $\frac{T}{dt}\sqrt{dt} \sim \frac{1}{\sqrt{dt}} \to \infty$ as the time between rehedgings goes to zero! You don't want to hedge continuously even if you could.

A PDE Model of Transactions Costs

One can approach transactions costs even more precisely in the framework of Hoggard, Whaley & Wilmott. (There are many other treatments, the first originally tackled by Leland.) In this way you can estimate the effect of transactions costs by simply adjusting the BSM volatility.

Assume zero dividend yield, and

$$dS = \mu S dt + \sigma S Z\sqrt{dt}$$

where Z is drawn from a standard normal distribution. Calculate the change in value of hedged position for non-infinitesimal dt:

$$
\begin{aligned}
d\text{P\&L} &= dV - \Delta dS - \text{cash spent on transactions costs} \\
&\approx \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 dt - \Delta dS - \kappa S|N| \\
&= \frac{\partial V}{\partial t}dt + \left(\frac{\partial V}{\partial S} - \Delta\right)(\mu S dt + \sigma S Z\sqrt{dt}) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 dt - \kappa S|N| \\
&= \cancel{\left(\frac{\partial V}{\partial S} - \Delta\right)}\sigma S Z\sqrt{dt} + \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \mu S\cancel{\left(\frac{\partial V}{\partial S} - \Delta\right)} + \frac{\partial V}{\partial t}\right)dt - \kappa S|N|
\end{aligned}
$$

Choose the continuous hedge ratio $\Delta = \frac{\partial}{\partial S}V(S, t)$ to eliminate the first term.

$$\text{dP\&L} = \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \frac{\partial V}{\partial t}\right) dt - \kappa S |N|$$

Then

$$= \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} Z^2 + \frac{\partial V}{\partial t}\right) dt - \kappa \sigma S^2 \left|\frac{\partial^2 V}{\partial S^2} Z\right| \sqrt{dt}$$

N itself is stochastic and related to Γ of course.

The P&L is not riskless, unfortunately, but we can calculate its expected value.

The expected value of the change in the P&L is therefore given by

$$E\left[Z^2\right] = 1$$

$$E\left[|Z|\right] = \sqrt{\frac{2}{\pi}}$$

$$E[d\text{P\&L}] = \left[\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \sqrt{\frac{2}{\pi \delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right|\kappa\sigma S^2\right)dt\right]$$

This isn't riskless. **Nevertheless** let's assume we expect to earn the riskless rate on the hedge, on average.

$$E[d\text{P\&L}] = r\left(V - S\frac{\partial V}{\partial S}\right)dt.$$

Combining, we obtain

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \sqrt{\frac{2}{\pi \delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right|\kappa\sigma S^2 + rS\frac{\partial V}{\partial S} - rV = 0$$

Modified BS equation with nonlinear extra term proportional to the value of $\Gamma = \frac{\partial^2 V}{\partial S^2}$.

The sum of two solutions to the equation is not necessarily a solution too; you cannot assume that the transactions costs for a portfolio of options is the sum of the transactions costs for hedging each option in isolation.

For a single long position in a call or a put, $\frac{\partial^2 V}{\partial S^2} \geq 0$, so we can drop the absolute value sign.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\hat{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad \text{Eq.1.6}$$

where

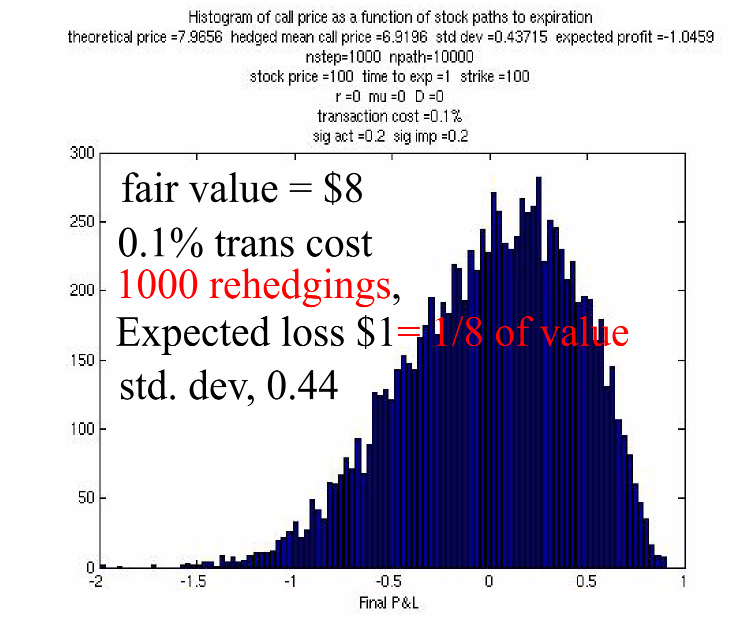
$$\hat{\sigma}^2 = \sigma^2 - 2\kappa\sigma\sqrt{\frac{2}{\pi\delta t}} \qquad \hat{\sigma} \approx \sigma - \kappa\sqrt{\frac{2}{\pi\delta t}}$$

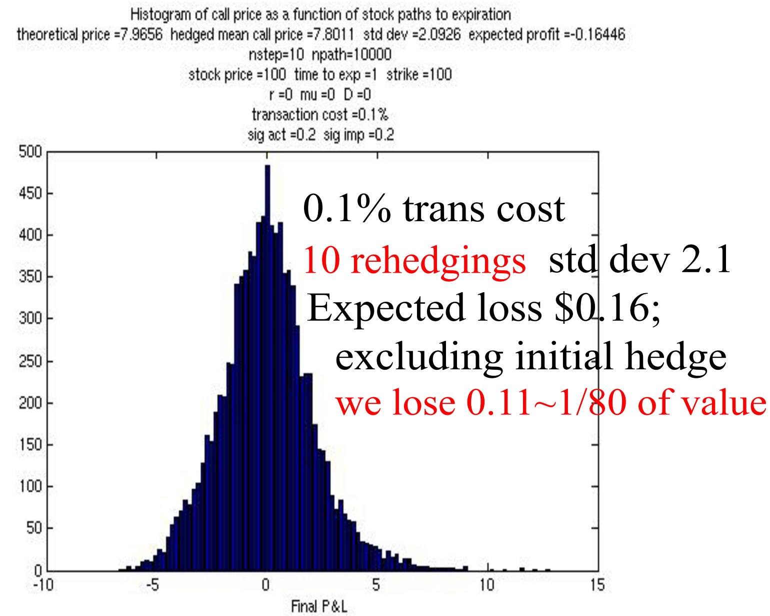
This is the Black-Scholes equation with a modified reduced volatility.For a short position, the effective volatility is enhanced, given by

$\hat{\sigma} \approx \sigma + \kappa\sqrt{\frac{2}{\pi\delta t}}$ and $\kappa \ll \sigma\sqrt{\delta t}$ so that the correction is actually small.

Compare Above Theory with our Earlier Simulations

cost of initial hedge is $k(\Delta)S = 0.001(0.5)100 = \$0.05$





ATM Call $C \approx \dfrac{S\sigma\sqrt{\tau}}{\sqrt{2\pi}}$. Percentage change in ATM option is $\dfrac{\hat{\sigma}-\sigma}{\sigma} = \dfrac{\kappa}{\sigma}\sqrt{\dfrac{2}{\pi\delta t}}$

Case 1: $\dfrac{0.001}{0.2}\sqrt{\dfrac{2}{\pi\frac{1}{1000}}} \approx 0.005(25) = 0.125 \approx \dfrac{1}{8}$

- Case 2: $\dfrac{0.001}{0.2}\sqrt{\dfrac{2}{\pi\frac{1}{10}}} \approx 0.005(2.5) = 0.0125 \approx \dfrac{1}{80}$ **excluding** initial trans. cost of setting up hedge, but hedging sparsely.

7.8 Back to: Hedging at an Arbitrary Constant Volatility

We don't know the future volatility. Suppose we just choose some hedging volatility. PV(I,R,H)

Buy an option at implied vol Σ, hedge it to expiration at volatility σ_h, while realized volatility σ_r

Table 1: Position Values when Hedging with an Arbitrary Volatility

Time	Option Position, Value	Stock Position, Value	Value of Cash Position	Net Position Value
t	$\vec{V}_i, V_i$	$-\Delta_h\vec{S}$	$\Delta_h S - V_i =$ $(\Delta_h S - V_h) + (V_h - V_i)$	0
t + dt	$\vec{V}_i, V_i + dV_i$	$-\Delta_h\vec{S}, -\Delta_h(S + dS)$	$(\Delta_h S - V_i)(1 + rdt)$ $-\Delta_h DSdt$	$(V_i + dV_i - \Delta_h(S + dS))$ $+ (\Delta_h S - V_i)(1 + rdt)$ $-\Delta_h DSdt$

P&L:

$$dP\&L(I,R,H) = dV_i - \Delta_h dS - \Delta_h SDdt + \{(\Delta_h S - V_h) + (V_h - V_i)\}rdt$$

$$= dV_h - \Delta_h dS - \Delta_h SDdt + (dV_i - dV_h) + \{(\Delta_h S - V_h) + (V_h - V_i)\}rdt$$

Now the BS

$$= \left\{\Theta_h + \frac{1}{2}\Gamma_h S^2 \sigma_r^2 + (r - D)S\Delta_h - rV_h\right\}dt + (dV_i - dV_h) + (V_h - V_i)rdt$$

solution with σ_h satisfies the p.d.e $dP\&L[H,H,H] = 0 \qquad \Theta_h + (r-D)S\Delta_h + \frac{1}{2}\Gamma_h S^2 \sigma_h^2 - rV_h = 0$

Substituting this last equation into the previous one

$$dP\&L(I,R,H) = \frac{1}{2}\Gamma_h S^2\left(\sigma_r^2 - \sigma_h^2\right)dt + (dV_i - dV_h) + (V_h - V_i)rdt$$

$$= \frac{1}{2}\Gamma_h S^2\left(\sigma_r^2 - \sigma_h^2\right)dt + e^{rt}d\left\{e^{-rt}(V_i - V_h)\right\}$$

Taking present values $e^{-r(t-t_0)}$ to time t_0 leads to

$$dPV(P\&L(I,R,H) = e^{-r(t-t_0)}\frac{1}{2}\Gamma_h S^2\left(\sigma_r^2 - \sigma_h^2\right)dt + e^{rt_0}d\left\{e^{-rt}(V_i - V_h)\right\}$$

Integrate:

$$PV[\mathrm{P\&L(I,H)}] = V_h - V_i + \frac{1}{2}\int_{t_0}^{T} e^{-r(t-t_0)}\Gamma_h S^2\left(\sigma_r^2 - \sigma_h^2\right)dt$$ Eq.7.7

- Note that $V_h = V_i$ have equal values at expiration. When σ_h is set equal to either σ_r or σ_i, Equation 7.2 reduces to our previous results.

Next: Back to the Smile in Various Markets, Graphing the Smile, Bounds on the Smile from No Arbitrage.