Math Methods – Financial Price Analysis

Mathematics GR5360

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The Random Walk*

- In the previous lectures we have summarized some basic results of probability theory that are used in the statistical description of time-dependent random processes.
- In order to apply some of that "abstract mathematics" to a real physical problem we here consider and solve the problem of random walk.
- This random walk model is aimed to imitate, in a very idealized and perhaps unrealistic way, the motion of a large particle in a fluid of light molecules.
- The large particle will be called Brownian, in honor of botanist R. Brown who first observed the motion of such particles under a microscope.
- We will limit ourselves to one-dimensional problem.
- We will further assume that the motion of Brownian particle can be described by the following idealization:
  - I. At any moment, the Brownian particle sits at the sites of a one-dimensional lattice with coordinates: $0, \pm a, \pm 2a, \pm 3a, \ldots$ etc.
  - II. At the times $t=\tau, 2\tau, 3\tau, \ldots$ etc, the Brownian particle jumps from one site to a neighboring site, with an equal probability of going to the left or to the right.

* - with some changes from "Classical Kinetic Theory of Fluids" by Resibois and De Leener, ref. B30.

The Random Walk*

Let $p_1(m,n)$ denote the probability that the Brownian particle sits at the site $x = ma$ at time $t = n\tau$. The Brownian particle can be at site $x = ma$ at moment $t = n\tau$ only if it was either at previous site $x = (m-1)a$ or at next $x = (m+1)a$ site at previous moment of time $t = (n-1)\tau$:

$$p_1(m,n) = \frac{1}{2} p_1(m-1, n-1) + \frac{1}{2} p_1(m+1, n-1), \qquad (*)$$

which is a discrete analog of the Chapman - Kolmogorov equation.

We will be looking for a solution of the above equation through the Fourier transform method. The Fourier transform for the probability is:

$$p_{1,l}(n) = \sum_{m=-\infty}^{\infty} p_1(m,n) \cdot e^{-ilm}, \text{ for } -\pi \le l < \pi, \text{ so that}$$

$$p_1(m,n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} dl \cdot p_{1,l}(n) \cdot e^{ilm}.$$

This can be verified by inserting it in the previous equation and performing the $l-$integral:

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} dl \cdot e^{il(m-m')} = \delta_{m,m'}.$$

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Multiplying (*) by e^{-ilm} and summing over m, we get

$$p_{1,l}(n) = \frac{e^{-il} + e^{il}}{2} \cdot p_{1,l}(n-1) = \cos(l) \cdot p_{1,l}(n-1).$$

Iterating this relationship we get the solution :

$$p_{1,l}(n) = (\cos(l))^n \cdot p_{1,l}(0).$$

Let us use the initial condition that at $t = 0$ the Brownian particle was at position $x = 0$:

$$p_{l,0}(0) = \sum_{m=-\infty}^{+\infty} \delta_{m,0} \cdot e^{-ilm} = 1.$$

Using this, we can arrive to the following representation of solution :

$$p_1(m,n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} dl \cdot (\cos l)^n \cdot \cos(lm) \qquad (**).$$

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For small finite values of n this integral can be calculated exactly.

For example, for $n=1$ we have:

$$p_1(m,1) = \frac{1}{8\pi}\int_{-\pi}^{+\pi} dl \cdot \left[e^{il} + e^{-il}\right]\cdot\left[e^{ilm} + e^{-ilm}\right] = \frac{\delta_{m,-1} + \delta_{m,1}}{2}.$$

For general values of n this method is very tedious.

Let us recall that the interval τ represents some average time between the collisions of the Brownian particle with the fluid molecules; which is very short compared to times over which macroscopic observations are made. This corresponds to the limit $n \to +\infty$.

Let us also use the fact that:

$$(\cos l)^n \cdot \cos(lm) = (-1)^{n+m}\left[\cos(\pi - l)\right]^n \cos\left[(\pi - l)m\right].$$

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Let us consider two cases:

(i) if n and m have the same parity (either both odd or even),

i.e. $n+m$ is even. Then the domain of integration in (**) can

be reduced to $\left(-\pi/2,+\pi/2\right)$:

$$p_1(m,n)=\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} dl\cdot\left(\cos l\right)^n\cdot\cos(lm),\ \text{if}\ n+m\ \text{is even;}$$

(ii) if n and m have different parities, then we get:

$p_1(m,n)=0$, if $n+m$ is odd.

Moreover, we can calculate the mean and mean square deviation

exactly. Indeed:

$$\left\langle m\right\rangle=\sum_{m=-\infty}^{+\infty} m\cdot p_1(m,n)=0,$$

since $p_1(m,n)$ is an even function of m.

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For the mean square deviation we have:

$$\langle m^2 \rangle = \sum_{m=-\infty}^{+\infty} m^2 p_1(m,n) = \sum_{m\,\text{even}} -\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} dl \cdot (\cos l)^n \cdot \frac{\partial^2 \cos(lm)}{\partial l^2}, \quad (***)$$

where for simplicity we have assumed that n is even.

The same final result holds for n odd. Integrating this

by parts, we get:

$$-\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} dl \cdot (\cos l)^n \cdot \frac{\partial^2 \cos(lm)}{\partial l^2} = -\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} dl \cdot \left[\frac{\partial^2}{dl^2}(\cos l)^n\right]\cos(lm) =$$

$$= -\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} dl \cdot \left[n(n-1)(\cos l)^{n-2}\sin^2 l - n(\cos l)^n\right] \cdot \cos(lm) =$$

$$= \frac{n(n-1)}{\pi}\int_{-\pi/2}^{\pi/2} dl \cdot \left[(\cos l)^n - (\cos l)^{n-2}\right] \cdot \cos(lm) + \frac{n}{\pi}\int_{-\pi/2}^{\pi/2} dl \cdot (\cos l)^n \cos(lm) =$$

$$= n(n-1) \cdot \left[p_1(m,n) - p_1(m,n-2)\right] + np_1(m,n).$$

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Inserting this result into (***) and using *(i,ii)* above and normalization condition we obtain a very simple result:

$$\langle m^2 \rangle = n.$$

This result means that the probability distribution $p_1(m,n)$, as a function of m, has a width of order $\sqrt{n}$.

In that case, instead of evaluating *(i)* for arbitrary values of m as $n \to +\infty$, we can do that for those $m \le \sqrt{n}$. We can thus introduce a small variable

$$\alpha_h = \frac{m}{\sqrt{n}}, \text{ which stays finite in the limit } n \to +\infty.$$

To do this, we use a variable $y = l\sqrt{n}$ and write

$$p_1(m,n) = \frac{1}{\pi\sqrt{n}} \cdot \int_{-\pi\sqrt{n}/2}^{+\pi\sqrt{n}/2} dy \cdot \left[\cos\left(\frac{y}{\sqrt{n}}\right)\right]^n \cdot \cos(\alpha_h y) \text{ for } m+n \text{ even.}$$

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For finite α_h it is intuitively clear that the dominant contribtion to the integrand will come from finite values of y and $n \to +\infty$:

$$\lim_{n\to+\infty}\left[\cos\left(\frac{y}{\sqrt{n}}\right)\right]^n = \lim_{n\to+\infty}\left(1-\frac{y}{\sqrt{n}}\right)^n = e^{-\frac{y^2}{2}}.$$

Now, letting the limits of the integral go to $\pm\infty$ with negligible error for $n \to +\infty$, we get:

$$p_1(m,n) = \frac{1}{\pi\sqrt{n}}\int_{-\infty}^{+\infty} dy \cdot e^{-\frac{y^2}{2}}\cos(\alpha_h y), \text{ for } n \to +\infty \text{ such that}$$

$\dfrac{m}{\sqrt{n}} = \alpha_h$ is finite and is $m+n$ is even.

It can be shown that this is an exact result in the limit $n \to +\infty$. The last integral above can be taken exactly and leads to the final solution:

$$p_1(m,n) = \left(\frac{2}{\pi n}\right)^{1/2} \cdot e^{-\frac{m^2}{2n}}, \text{ for } m+n \text{ even, as } n \to +\infty, \text{ that } \frac{m}{\sqrt{n}} = \alpha_h \text{ is finite, and}$$

$p_1(m,n) = 0$, for $m+n$ odd.

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The Random Walk*

We now can consider passage to the continuous limit. Indeed, as interval τ is small compared to macroscopic times of our interest, the distance a between two sites is small compared to macroscopic distances. In the continuous limit, we need to know the probability $P_1(x,t)\cdot\Delta x$ of a particle is at time t located within an interval Δx around point x that includes a large enough number Δm of sites, such that :

$$\Delta m = \left[\frac{\Delta x}{a}\right] >> 1, \Delta m << \sqrt{n},$$

If $[\cdots]$ denotes a truncated integer part of any number, then

$$P_1(x,t)\cdot\Delta x = \sum_{m\le m_1\le m+\Delta m} p_1\left(m_1,\left[\frac{t}{\tau}\right]\right).$$

If we notice that $p_1(m,n)$ varies slowly within interval Δm,

$$P_1(x,t)\cdot\Delta x = \frac{1}{2}\left(\frac{2\tau}{a^2\pi t}\right)^{1/2}\cdot e^{-\frac{x^2\tau}{2a^2t}}\cdot a\cdot\Delta m,\ \text{for } t\to+\infty,$$

so that $\dfrac{\mathrm{x}}{\mathrm{a}\sqrt{\mathrm{t}/\tau}} = \alpha_{\mathrm{h}}$ remains finite.

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If we now introduce a diffusion coefficent D by $D = \frac{a^2}{2\tau}$, we get :

$$P_1(x,t) = \frac{1}{2\sqrt{\pi Dt}} \cdot e^{-\frac{x^2}{4Dt}},$$

a result which is valid in the limit $t \to +\infty$, such that $\frac{x}{\sqrt{Dt}}$ remains finite.

It is easy to verify that this distribution density is the solution of the following PDE (diffusion or heat equation) :

$$\frac{\partial \mathrm{P}_1(\mathrm{x},\mathrm{t})}{\partial \mathrm{t}} = D\frac{\partial^2 P_1(x,t)}{\partial x^2}$$

with the initial condition given by a delta function : $P_1(x,0) = \delta(x)$.

This equation governs the time evolution of the probability density $P_1(x,t)$.

It is easy to show that the mean - squared displacement of a Brownian particle can be calculated as :

$$\left\langle x^2 \right\rangle_t = \int_{-\infty}^{+\infty} dx \cdot x^2 \cdot P_1(x,t) = 2Dt$$

in the limit $t \to +\infty$, which is a well - known Einstein's formula.

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The Random Walk*

Let us generalize somewhat our Random Walk results.

The following features of the Random Walk model we developed can and are universally applied in statistical studies of various physical applications (including price fluctuations) :

(i) the probabilistic description of a "microscopic" phenomenon;
(ii) the derivation of macroscopic equations from a "microscopic" model;
(iii) the important role of asymptotic limits in the justification of these macroscopic equations.

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The Random Walk*

At last, we would like to refine our Random Walk model to include the ability to study the statistical properties of the velocity of the Brownian particle. Indeed, in our discrete Random Walk model, only the discrete position of the Brownian particle was relevant, its velocity was not even defined.

Classical Brownian Motion theory is based on a stochastic approximation to those discrete equations. Suppose that the Brownian particle, with mass M, coordinates $\vec{r}$ and velocity $\vec{v}$ is so large that its motion can be described by a hydrodynamic approximation, where the fluid is described as a continuum:

$$\frac{d\vec{r}}{dt} = \vec{v},$$

$$\frac{d\vec{v}}{dt} = -\frac{\zeta_B}{M} \cdot \vec{v},$$

where ζ_B is the friction coefficient of the Brownian particle.

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The effect of the randomly hitting Brownian particle molecules can be modeled by the random fluctuating force, which leads to the Langevin equation:

$$\frac{d\vec{v}}{dt} = -\frac{\zeta_B}{M}\cdot\vec{v} + \frac{\vec{f}(t)}{M},$$

where $\vec{f}(t)$ is a stochastic force that depends on some probability law. Such problems can be solved either by simulation (Monte Carlo), or one can solve the PDEs for the probability density function, or Fokker-Plank equation. The fluctuating force is assumed ot have the following properties:

$\left\langle \vec{f}(t) \right\rangle = 0$, and

$\left\langle \vec{f}(t)\vec{f}(t') \right\rangle = \xi_B \cdot U \cdot \delta(t-t')$ for a constant ξ_B and a unit matrix U.

Such force is designed to simulate that molecules hitting the Brownian particle and thus producing the fluctuating force act completely independently of one another, except for when they act at the same exact moment.

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log-Brownian and log-Ornstein-Uhlenbeck* Processes

We can add a lot of "reality" to the continuous Random Walk model process if we account for two features which seem to be a part of real-life financial time series:

- the fact that prices are non-negative and price returns may resemble the stationary process more than the price changes;
- the fact that due to behavioral bias of over-reaction many financial time series additionally posess the property of mean-reversion.

The Ornstein-Uhlenbeck process is an example of a Gaussian random process with a bounded variance, in contrast to the Random Walk. The difference in the equations governing these processes lies in the "drift" term: for Random Walk the drift term is constant, whereas for the Ornstein-Uhlenbeck process it is dependent on the current value of the process: if the current value of the process is less than the long-term mean, the drift will be positive; if the current value of the process is greater than the long-term mean, the drift will be negative. The mean acts as an equilibrium level for this process, which leads to its name, "mean-reverting".

* Uhlenbeck, G.E., Ornstein, L.S. (1930) “On the Theory of Brownian Motion”. Phys. Rev. 36: 823-841.

log-Brownian and log-Ornstein-Uhlenbeck* Processes

The continuous version of log - Brownian process is :

$$\frac{dS}{S} = \mu \cdot dt + \sigma \cdot dW,$$

which is a "Gaussian noise with a trend" and constant parameters

μ and σ are average return and volatility.

We can discretize it in the following way :

$$S_{i+1} = S_i \cdot \left\{ 1 + \frac{\mu}{253} + \frac{\sigma}{\sqrt{253}} \cdot \varepsilon_i \right\}.$$

Here, we are thinking of S_i as of the stock price at the end of the

$i-$th day, we have used parameters μ and σ as annual average return

and volatility, and 253 being the number of trading days per year.

ε_i is an independent Gaussian - distributed random variable such that :

$\overline{\varepsilon_i} = 0$ and $\overline{\varepsilon_i^2} = 1.$

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log-Brownian and log-Ornstein-Uhlenbeck* Processes

Similarly, the continuous version of the Ornstein - Uhlenbeck process is:

$$\frac{dS}{S} = -a \cdot \left(S - \bar{S}\right) \cdot dt + \sigma \cdot dW,$$

where the new parameter $\bar{S}$ has the meaning of the mean price.

We can discretize it in a similar to Random walk way:

$$S_{i+1} = S_i \cdot \left\{1 - a \cdot \left(S_i - \bar{S}\right) + \frac{\sigma}{\sqrt{253}} \cdot \varepsilon_i\right\}.$$

Here, again, ε_i are Gaussian - distributed independent random variable such that: $\overline{\varepsilon_i} = 0$, and $\overline{\varepsilon_i^2} = 1$.

Below we have simulated these variables on a sample of 10,000 "days" to illusrate the following point: as the volatility coefficient increases to some close to real market values (many markets have larger than 20% annualized volatility), it becomes very non - trivial to correctly "guess" which sample path is trend - following and which one is mean - reverting.

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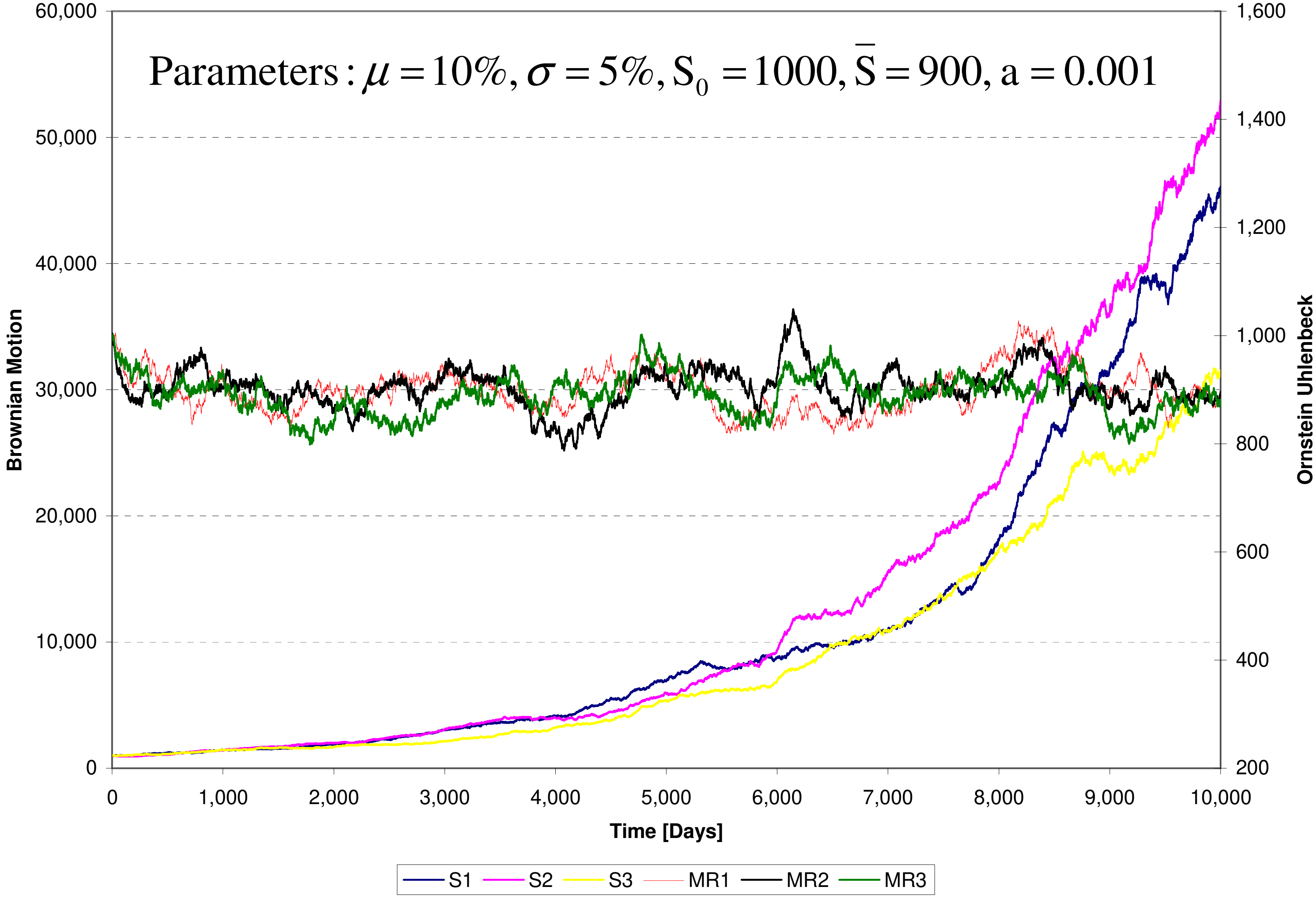
log-Brownian and log-Ornstein-Uhlenbeck* Processes



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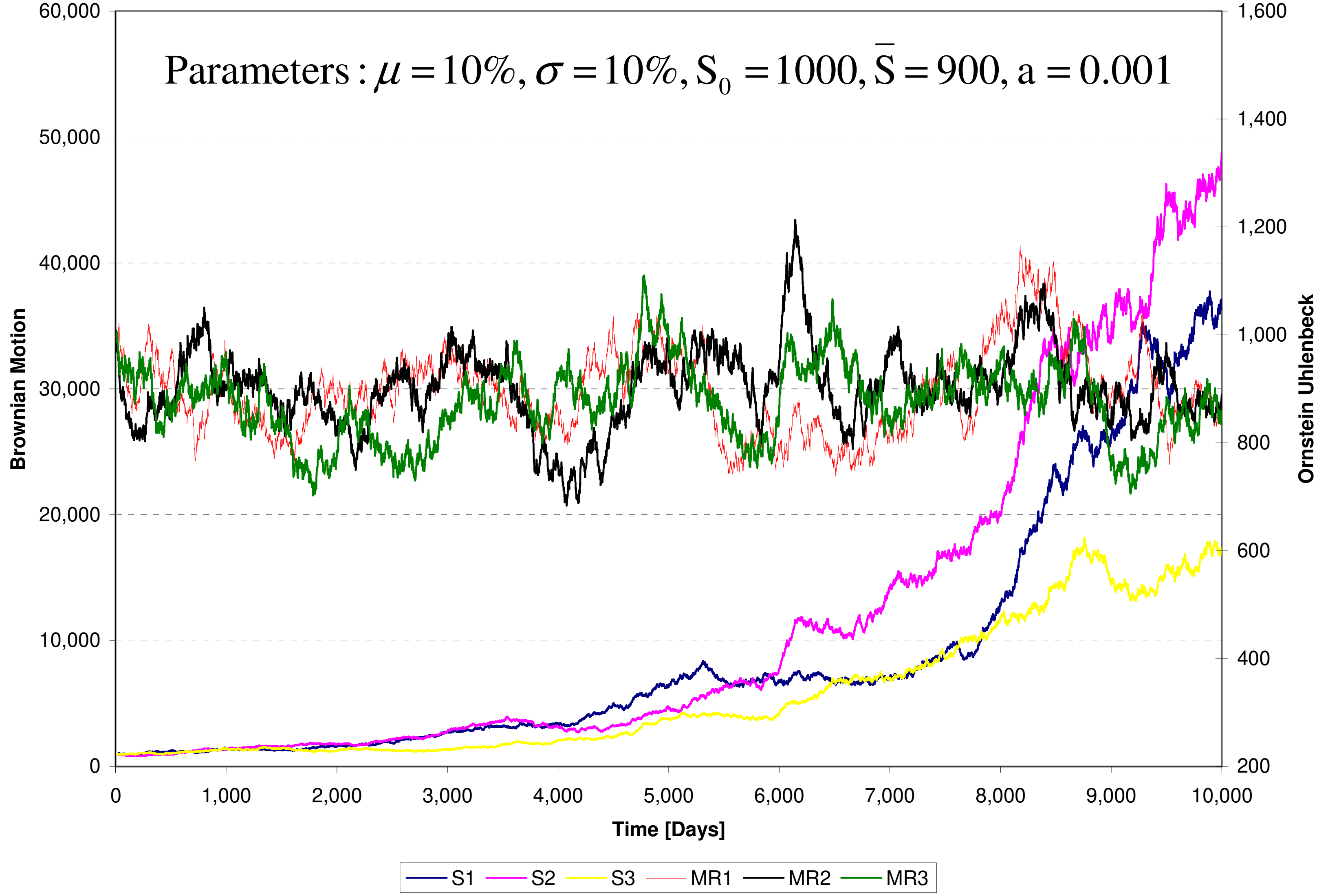

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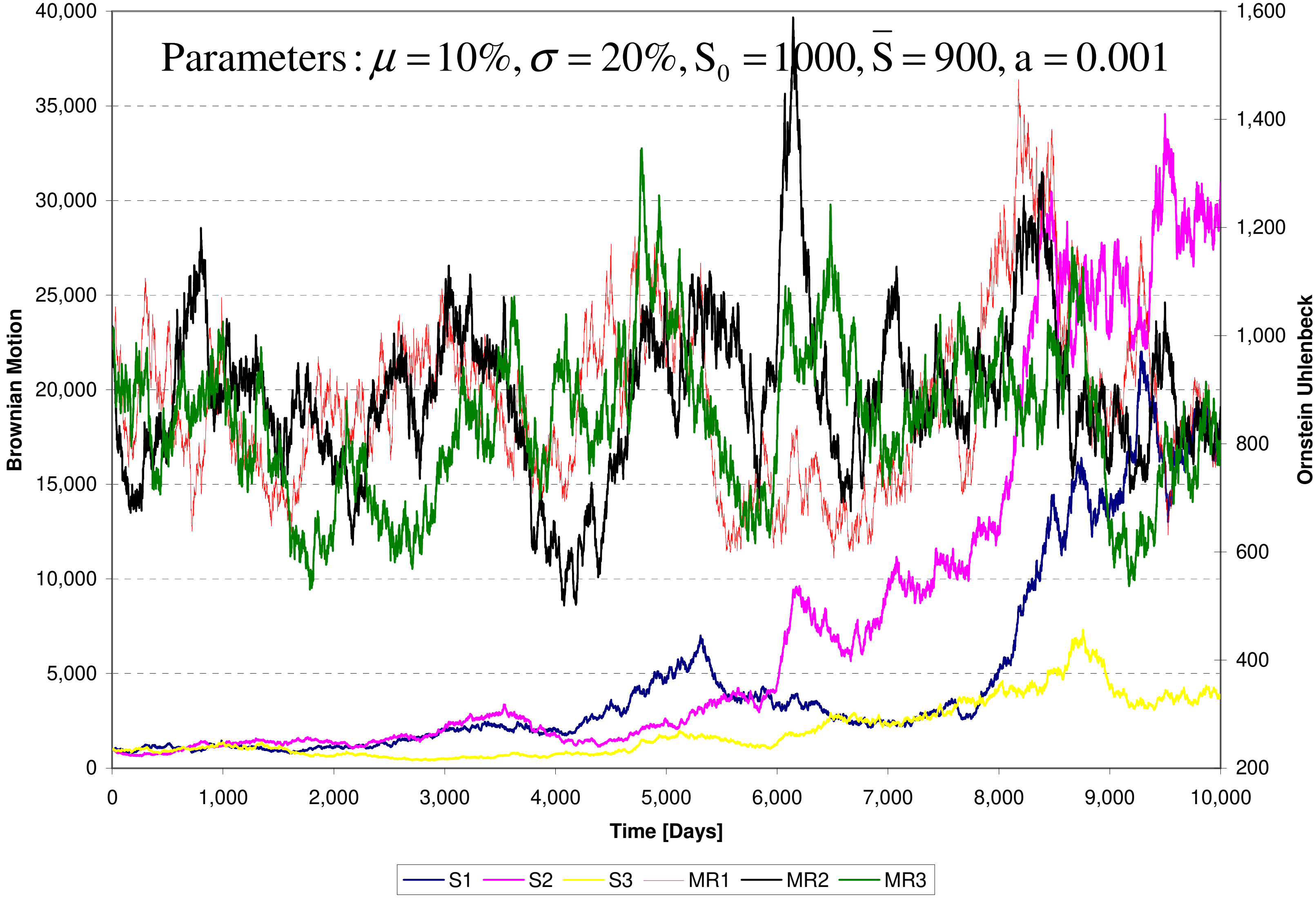

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