Section 3.6 Ergodic Markov Chains

Def'n: A Markov chain is called <u>ergodic</u> if it is

1) irreducible
2) aperiodic
3) all states have finite expected return times

<u>Theorem 3.8</u> Fundamental Limit Theorem for Ergodic Markov Chains

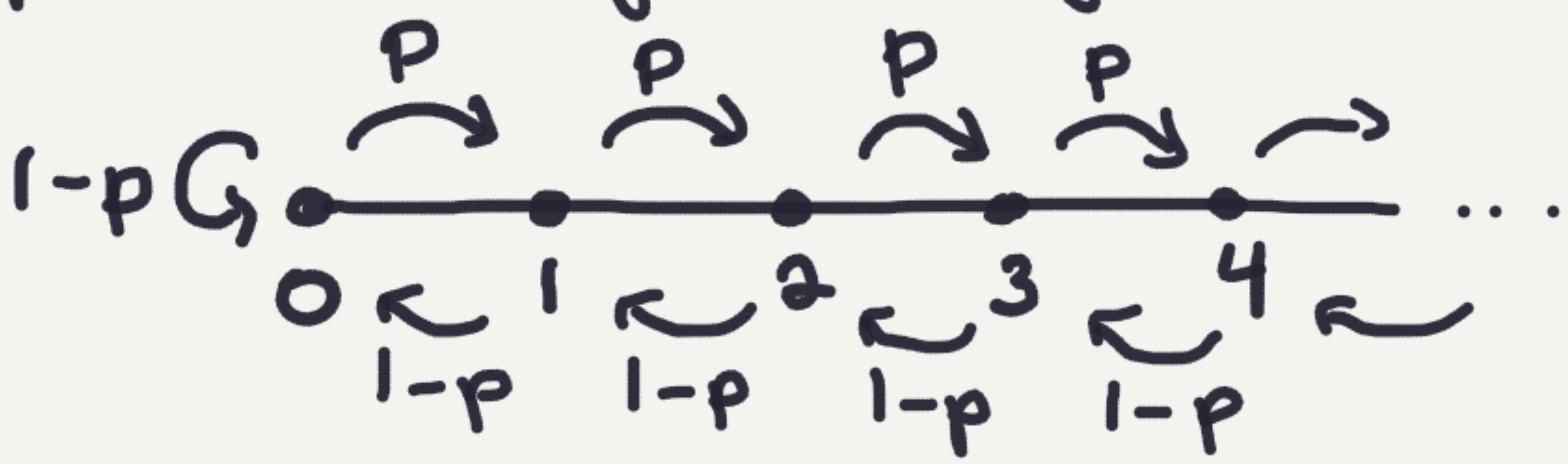
Let $X_0, X_1, X_2, \cdots$ be an ergodic MC. Then there exists a unique stationary distribution $\bar{\pi}$ with all positive components that is the

↳ ($\bar{\pi} P = \bar{\pi}$)

limiting distribution of the chain: $\pi_j = \lim_{n\to\infty} P_{ij}^n$ for all states i, j

(See proof in section 3.10. Essentially need to prove ergodic MC $\Leftrightarrow$ P is regular.)

Example: Partially reflecting walk on infinite state space



Ergodic?.

Find the stationary distribution: $\sum_{j=0}^{\infty} \pi_j = 1$ and $\underbrace{\sum_{i=0}^{\infty} \pi_i P_{ij} = \pi_j \text{ for all } j}$

$\pi_1(1-p) + \pi_0(1-p) = \pi_0$

$\pi_{j+1}(1-p) + \pi_{j-1}\, p = \pi_j, \ j>0$

Case $p = 1/2$: $\pi_1 = \pi_0, \ \pi_2 = \pi_0, \cdots$

Can't get $\sum \pi_j = 1$, so no sol'n

Not ergodic. (In fact, expected return times are infinite and states are null recurrent.)

Case $p \neq 1/2$: Assume sol'n of form $\pi_j = x^j$:

$x^{j+1}(1-p) + x^{j-1}p = x^j$

$\Rightarrow \ x^2(1-p) - x + p = 0$

$\Rightarrow \ x = \dfrac{1 \pm \sqrt{(1-2p)^2}}{2(1-p)}$

$= 1, \ p/_{1-p}$

find constants that work

Possible sol'ns are $\pi_j = c_1 + c_2\left(\frac{p}{1-p}\right)^j$

If $p > 1/2$, then $\frac{p}{1-p} > 1$ and no sol'n exists with $\sum \pi_j = 1$. Not ergodic. (can show all states are transient)

If $p < 1/2$, let $c_1 = 0$, $c_2 = 1/\sum_{j=0}^{\infty}\left(\frac{p}{1-p}\right)^j = \frac{1-2p}{1-p}$

to get $\pi_j = \frac{1-2p}{1-p}\left(\frac{p}{1-p}\right)^j$ (can show is ergodic and all states are positive recurrent)

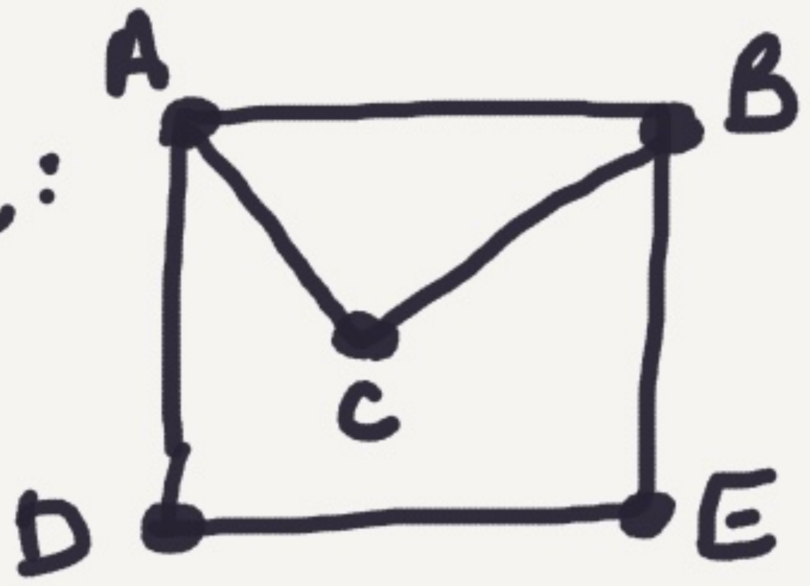
Section 3.7 Reversibility

Can you tell whether the chain is running forward or backward?

Defn: An irreducible MC with transition matrix P and stationary distr. π
($\pi P = \pi$)

is <u>time-reversible</u> if $\pi_i P_{ij} = \pi_j P_{ji}$ for all states i & j

Example:



Equally likely to follow any edge

$$P = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 0 & 1/3 & 0 & 1/3 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \end{bmatrix} \end{array}$$

Reversible? $\pi = [\, 3/12 \;\; 3/12 \;\; 2/12 \;\; 2/12 \;\; 2/12 \,]$ (degree of vertex divided by sum of all degrees)

If i & j are neighbors, $\pi_i P_{ij} = \dfrac{\#\text{edges}(i)}{\sum_k \#\text{edges}(k)} \cdot \dfrac{1}{\#\text{edges}(i)} = \dfrac{1}{2(\#\text{edges})}$
↖ total in graph

(state i has #edges(i) neighbors)

$\pi_j P_{ji}$ also equals $\dfrac{1}{2(\#\text{edges})}$

Yes, reversible.

Prop 3.9 Let P be the transition matrix of an irreducible MC.
If $\bar{x}$ is a prob distribution vector with $x_i P_{ij} = x_j P_{ji}$ for all i, j,
then $\bar{x}$ is the stationary distr $\bar{\pi}$ and the MC is reversible.

Proof: Need to show $\bar{x} = \bar{x}P$. Consider j^{th} component:

$$(\bar{x}P)_j = \sum_i x_i P_{ij}$$

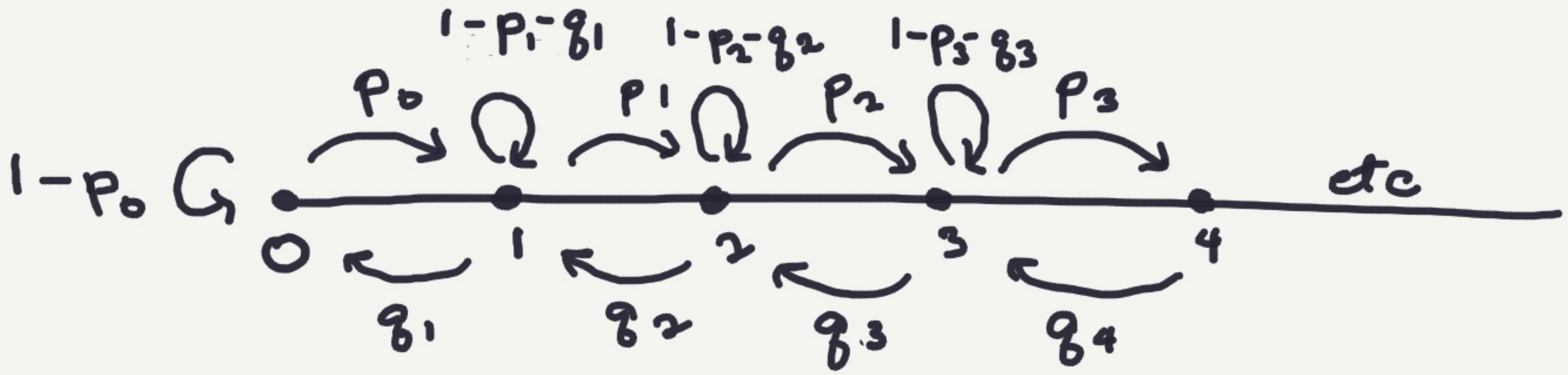
$$= \sum_i x_j P_{ji} \quad \text{using property given for } \bar{x}$$

$$= x_j \underbrace{\sum_i P_{ji}}_{=1} \quad (\text{summing across } j^{th} \text{ row of } P)$$

$$= x_j$$

So $\bar{x} = \bar{x}P$.

Example: birth and death chain



Solve for π_j satisfying $\pi_i P_{ij} = \pi_j P_{ji}$ to find when MC is reversible.

Only need to check moving over one spot:

$$\underbrace{\pi_i P_{i,i-1}}_{\text{move left}} = \underbrace{\pi_{i-1} P_{i-1,i}}_{\text{move right}} \longleftrightarrow \pi_i q_i = \pi_{i-1} p_{i-1} \text{ for } i \geq 1$$

So want $\pi_1 q_1 = \pi_0 p_0 \to \pi_1 = \frac{p_0}{q_1}\pi_0$

$$\pi_2 q_2 = \pi_1 p_1 \to \pi_2 = \frac{p_1}{q_2}\pi_1 = \frac{p_1}{q_2}\cdot\frac{p_0}{q_1}\pi_0$$

$$\vdots$$

$$\pi_k q_k = \pi_{k-1} p_{k-1} \to \pi_k = \frac{p_{k-1}}{q_k}\pi_{k-1} = \frac{p_{k-1}p_{k-2}\cdots p_1 p_0}{q_k q_{k-1}\cdots q_1}\pi_0$$

To get $\sum_{k=0}^{\infty}\pi_k = 1$, let $\pi_0 = 1\Big/1+\sum_{k=1}^{\infty}\prod_{i=1}^{k}\frac{p_{i-1}}{q_i}$, assuming series converges.

Birth&death chain being reversible requires that convergence.

Note: if all $p_i = p$ & $q_i = q$, then $\pi_k = \pi_0\left(\frac{p}{q}\right)^k$ and $\pi_0 = 1\Big/\sum_{k=0}^{\infty}\left(\frac{p}{q}\right)^k = \frac{1}{1/1-\frac{p}{q}} = 1-\frac{p}{q}$

Section 3.8 Absorbing chains

Idea: Decompose state space into recurrent classes $R_1, \cdots, R_m$ and a set of transient states. Treat each recurrent class as an absorbing state to calculate how long on average it takes for the MC to enter R_j, the prob of entering that particular R_j, how many visits to transient states the MC made before being absorbed into R_j, etc.

Reorganize transition matrix
$$P = \begin{array}{c|c} & \begin{array}{cccc} T & R_1 & \cdots & R_m \end{array} \\ \begin{array}{c} T \\ R_1 \\ \vdots \\ R_m \end{array} & \left[\begin{array}{c|c} Q & R \\ \hline O & \underline{1} \end{array}\right] \end{array}$$

treat each R_j as a single combined state

OR

$$P = \begin{array}{c|c} & \begin{array}{cccc} T & R_1 & \cdots & R_m \end{array} \\ \begin{array}{c} T \\ R_1 \\ \vdots \\ R_m \end{array} & \left[\begin{array}{c|c} Q & R \\ \hline O & \tilde{P} \end{array}\right] \end{array}$$

if keeping all states

$\tilde{P}$ is a stochastic matrix (rows sum to 1)

Q has rows that sum to ≤ 1

Block matrix multiplication

$$\begin{array}{cc} & \begin{array}{cc} T & R_1 \cdots R_m \end{array} \\ \begin{array}{c} T \\ R_1 \\ \vdots \\ R_m \end{array} & \begin{bmatrix} Q & R \\ 0 & \tilde{P} \end{bmatrix} \end{array} \begin{bmatrix} Q & R \\ 0 & \tilde{P} \end{bmatrix} = \begin{bmatrix} Q^2 & QR + R\tilde{P} \\ 0 & \tilde{P}^2 \end{bmatrix}$$

$$\begin{bmatrix} Q & R \\ 0 & \tilde{P} \end{bmatrix}^n = \begin{bmatrix} Q^n & * \\ 0 & \tilde{P}^n \end{bmatrix}$$

States in T are transient so $Q^n \to 0$ as $n \to \infty$

This implies no eigenvalue of Q can equal 1 $(\lambda^n \to 0$ as $n \to \infty)$

and $|\lambda| < 1$ for all eigenvalues of Q.

This in turn implies $I - Q$ is invertible (no $\lambda = 1 \Leftrightarrow \det(I-Q) \neq 0$)

Let $M = (I-Q)^{-1}$. This matrix will provide info about the # of steps expected to take to go from a transient state to a recurrent class, as well as the # of visits to each transient state before getting absorbed.

First we need a linear algebra result relating sums of powers of a matrix to matrices of form $(I-A)^{-1}$.

Lemma 3.10 $\sum_{n=0}^{\infty} A^n = (I-A)^{-1}$ if $A^n \to 0$ as $n \to \infty$

Proof: Check $(I-A)(I+A+A^2+\cdots) = I$

Now let's see what $M = (I-Q)^{-1}$ can tell us about visits to transient states.

Let i be a transient state.

Let Y_i = total # visits to i = $\sum_{n=0}^{\infty} \underbrace{I\{X_n = i\}}_{\text{indicator fn} = \begin{cases} 1 \text{ if } X_n = i \\ 0 \text{ if } X_n \neq i \end{cases}}$

$\mathbb{E}[I\{X_n=i\}] = \mathbb{P}\{X_n = i\}$

Start in some transient state j, $X_0 = j$

$$\begin{aligned}
\mathbb{E}[Y_i \mid X_0 = j] &= \mathbb{E}\left[\sum_{n=0}^{\infty} I\{X_n = i\} \mid X_0 = j\right] \\
&= \sum_{n=0}^{\infty} \mathbb{P}\{X_n = i \mid X_0 = j\} \\
&= \sum_{n=0}^{\infty} P^n_{\underbrace{ji}_{\text{both transient states in } T}} = \sum_{n=0}^{\infty} Q^n_{ji} = (I-Q)^{-1}_{ji} = M_{ji}
\end{aligned}$$

$\Rightarrow M_{ji}$ = expected # visits to transient state i if start at transient state j

M_{ji} = expected # visits to i if start at j (i & j transient states) before MC is absorbed into a recurrent class

$\sum_{i \in T} M_{ji}$ = expected # steps before MC is absorbed into a recurrent class if start at transient state j

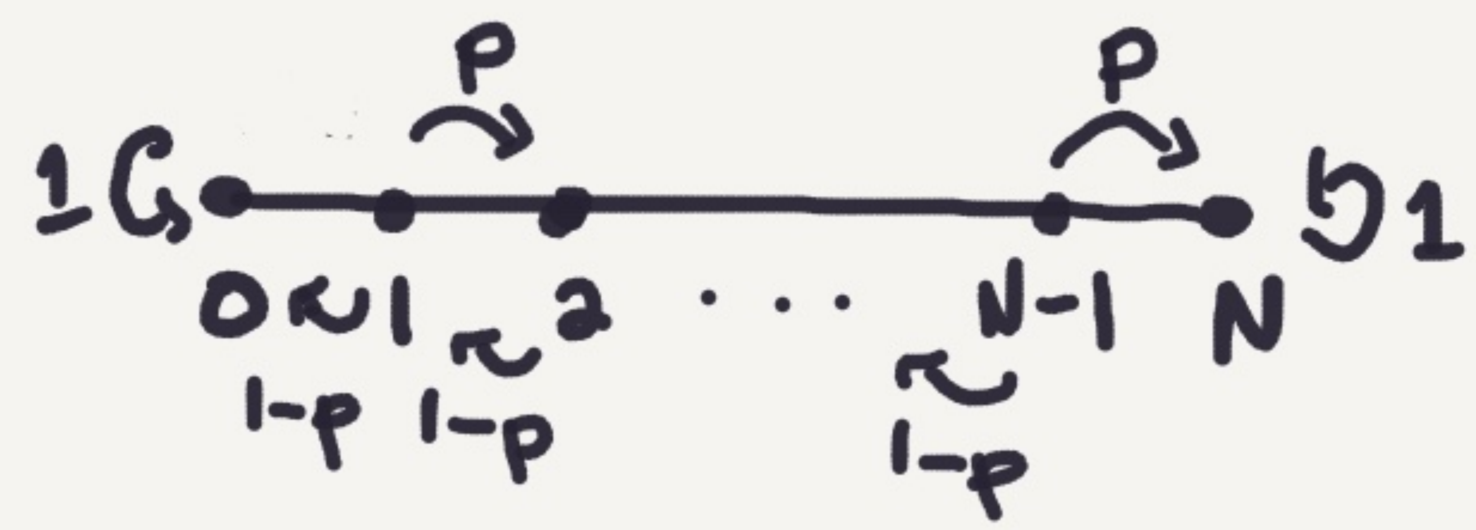
(summed over transient states)

$\boxed{\text{R demo}}$ 7-state example to practice

Common strategy: make a particular state of interest absorbing so all other states are transient (to obtain Q)

- # Analysis of Random Walk on a Circle

First we need a Gambler's Ruin result



Let a_j = prob of getting absorbed into state N if start at state j

$a_0 = 0$ and $a_N = 1$.

For $0 < j < N$, $a_j = (1-p)\,a_{j-1} + p\,a_{j+1}$

(take one step forward)

If $p = 1/2$, then $\boxed{a_j = j/N}$

If $p \neq 1/2$, then seek sol'n of form x^j: $x^j = (1-p)\,x^{j-1} + p\,x^{j+1}$

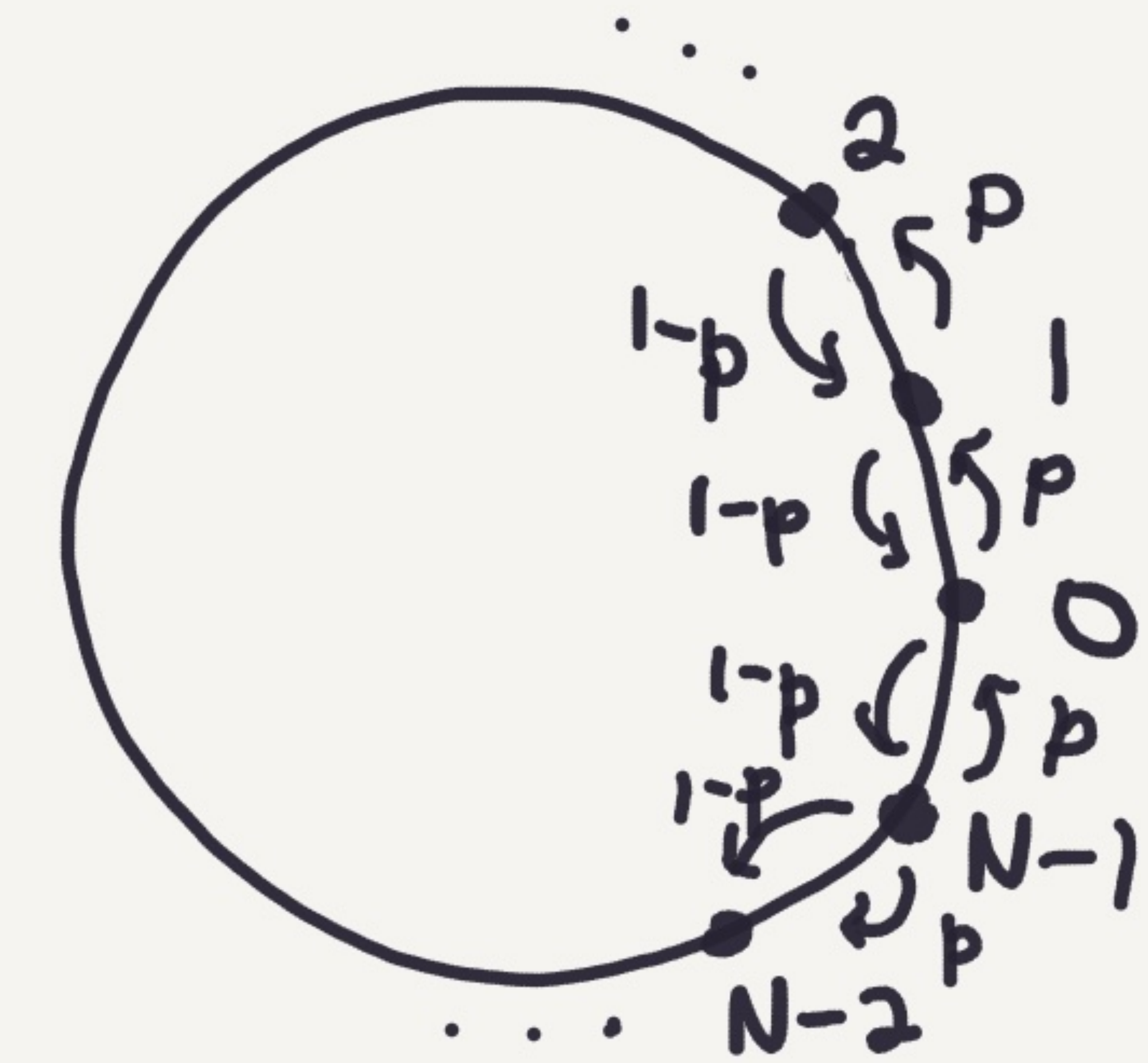
$$\Rightarrow p x^2 - x + (1-p) = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} = \frac{1 \pm (1-2p)}{2p} = 1,\ \frac{1-p}{p}$$

So $a_j = c_1 + c_2\left(\frac{1-p}{p}\right)^j$, $a_0 = 0$, $a_N = 1$,

leading to $\boxed{a_j = \dfrac{1-\left(\frac{1-p}{p}\right)^j}{1-\left(\frac{1-p}{p}\right)^N}}$

Apply to Random Walk on a Circle



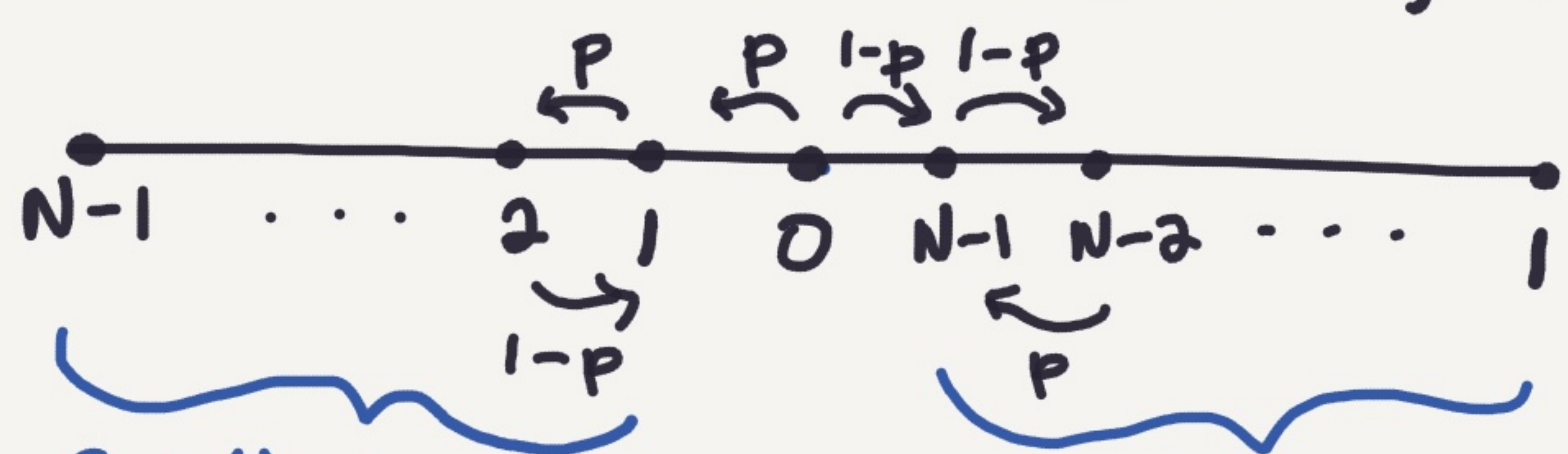
Assume $N \geq 2$ and $X_0 = 0$

$p = \mathbb{P}\{\text{take a step CCW}\}$

$1-p = \mathbb{P}\{\text{take a step CW}\}$

We want to find the prob of visiting all other states before returning to state 0

Treat as double Gambler's Ruin, conditioning on first step:



Gambler's ruin, want to hit $N-1$ before 0, starting at 1

Gambler's ruin, want to hit 1 before 0, starting at $N-1$

(finish in Monday's lab)