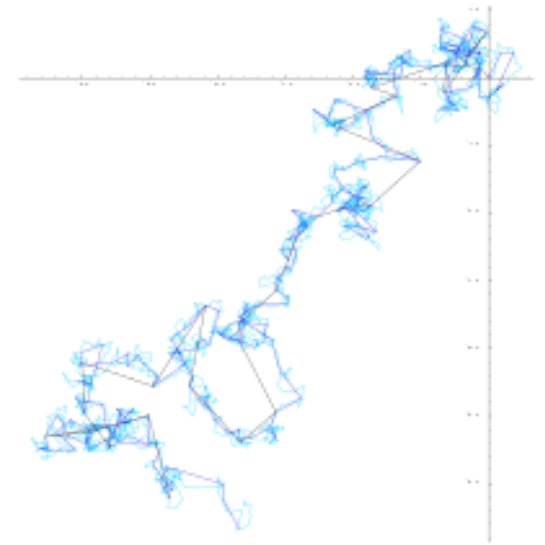


Brownian motion

Brownian motion or **pedesis** (from Ancient Greek: πήδησις /pɛːdɛːsɪs/ "leaping") is the random motion of particles suspended in a fluid (a liquid or a gas) resulting from their collision with the fast-moving molecules in the fluid.^[1]

This **pattern of motion** typically **alternates** random fluctuations in a particle's position inside a fluid sub-domain **with** a relocation to another sub-domain. Each relocation is followed by more fluctuations within the new closed volume. This pattern describes a fluid at thermal equilibrium, defined by a given temperature. Within such fluid there exists no preferential direction of flow as in transport phenomena. More specifically the fluid's overall linear and angular momenta remain null over time. It is important also to note that the kinetic energies of the molecular Brownian motions, together with those of molecular rotations and vibrations sum up to the caloric component of a fluid's internal energy.

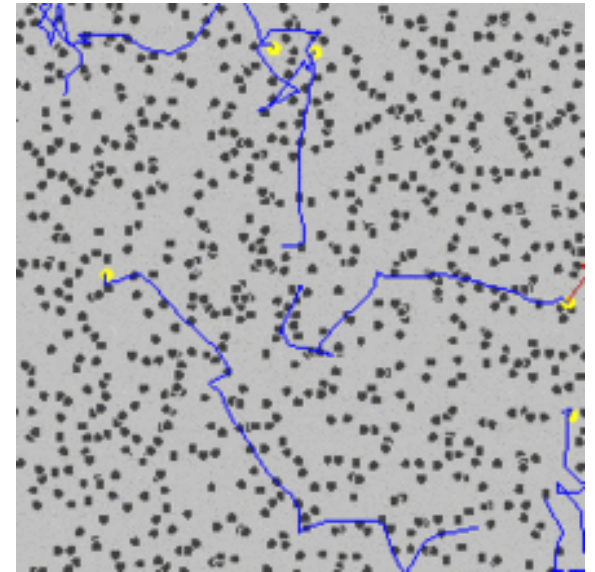
This motion is named after Robert Brown (botanist, born 1773). In 1827, while looking through a microscope at particles trapped in cavities inside pollen grains in water, he noted that the particles moved through the water; but he was not able to determine the mechanisms that caused this motion. Atoms and molecules had long been theorized as the constituents of matter, and Albert Einstein published a paper in 1905 that explained in precise detail how the motion that Brown had observed was a result of the pollen being moved by individual water molecules, making one of his first big contributions to science. This explanation of Brownian motion served as convincing evidence that atoms and molecules exist, and was further



Three different views of Brownian motion, with 32 steps, 256 steps, and 2048 steps denoted by progressively lighter colors

verified experimentally by Jean Perrin in 1908. Perrin was awarded the Nobel Prize in Physics in 1926 "for his work on the discontinuous structure of matter". The direction of the force of atomic bombardment is constantly changing, and at different times the particle is hit more on one side than another, leading to the seemingly random nature of the motion.

The many-body interactions that yield the Brownian pattern cannot be solved by a model accounting for every involved molecule. In consequence only probabilistic models applied to molecular populations can be employed to describe it. Two such models of the statistical mechanics, due to Einstein and Smoluchowski are presented below. Another, pure probabilistic class of models is the class of the stochastic process models. There exist both simpler and more complicated stochastic processes which in extreme ("taken to the limit") may describe the Brownian Motion (see random walk and Donsker's theorem).



This is a simulation of the Brownian motion of 5 particles (yellow) that collide with a large set of 800 particles. The yellow particles leave 5 blue trails of random motion and one of them has a red velocity vector.

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Further reading

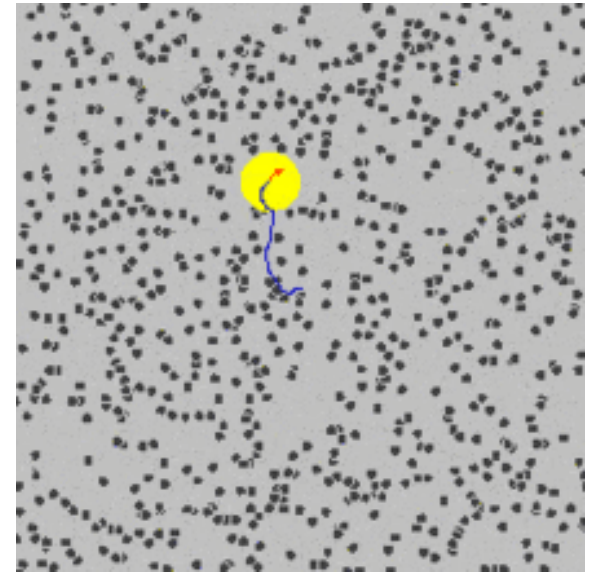
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History

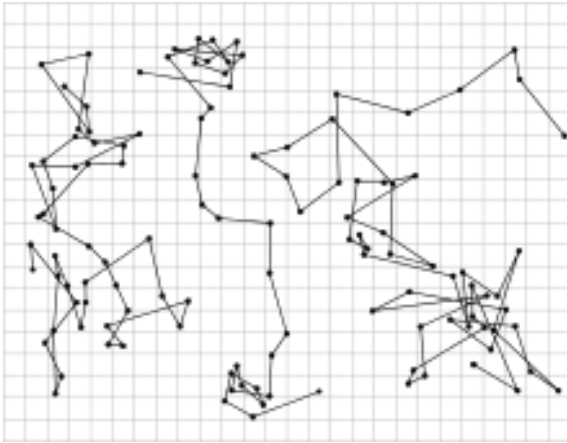
The Roman Lucretius's scientific poem "On the Nature of Things" (c. 60 BC) has a remarkable description of Brownian motion of dust particles in verses 113–140 from Book II. He uses this as a proof of the existence of atoms:

"Observe what happens when sunbeams are admitted into a building and shed light on its shadowy places. You will see a multitude of tiny particles mingling in a multitude of ways... their dancing is an actual indication of underlying movements of matter that are hidden from our sight... It originates with the atoms which move of themselves [i.e., spontaneously]. Then those small compound bodies that are least removed from the impetus of the atoms are set in motion by the impact of their invisible blows and in turn cannon against slightly larger bodies. So the movement mounts up from the atoms and gradually emerges to the level of our senses, so that those bodies are in motion that we see in sunbeams, moved by blows that remain invisible."

Although the mingling motion of dust particles is caused largely by air currents, the glittering, tumbling motion of small dust particles is, indeed, caused chiefly by true Brownian dynamics.



This is a simulation of the Brownian motion of a big particle (dust particle) that collides with a large set of smaller particles (molecules of a gas) which move with different velocities in different random directions.



Reproduced from the book of Jean Baptiste Perrin, *Les Atomes*, three tracings of the motion of colloidal particles of radius $0.53\text{ }\mu\text{m}$, as seen under the microscope, are displayed. Successive positions every 30 seconds are joined by straight line segments (the mesh size is $3.2\text{ }\mu\text{m}$).^[2]

While Jan Ingenhousz described the irregular motion of coal dust particles on the surface of alcohol in 1785, the discovery of this phenomenon is often credited to the botanist Robert Brown in 1827. Brown was studying pollen grains of the plant *Clarkia pulchella* suspended in water under a microscope when he observed minute particles, ejected by the pollen grains, executing a jittery motion. By repeating the experiment with particles of inorganic matter he was able to rule out that the motion was life-related, although its origin was yet to be explained.

The first person to describe the mathematics behind Brownian motion was Thorvald N. Thiele in a paper on the method of least squares published in 1880. This was followed independently by Louis Bachelier in 1900 in his PhD thesis "The theory of speculation", in which he presented a stochastic analysis of the stock and option markets. The Brownian motion model of the stock market is often cited, but Benoit Mandelbrot rejected its applicability to stock price movements in part because these are discontinuous.^[3]

Albert Einstein (in one of his 1905 papers) and Marian Smoluchowski (1906) brought the solution of the problem to the attention of physicists, and presented it as a way to indirectly confirm the existence of atoms and molecules. Their equations describing Brownian motion were subsequently verified by the experimental work of Jean Baptiste Perrin in 1908.

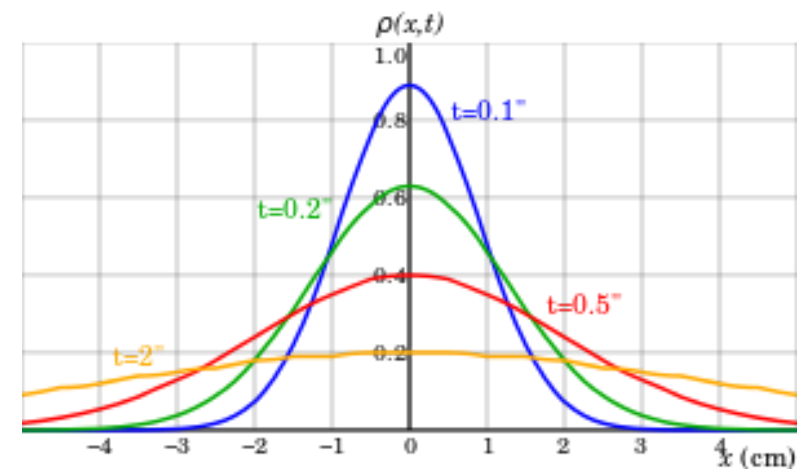
Statistical Mechanics Theories

Einstein's theory

There are two parts to Einstein's theory: the first part consists in the formulation of a diffusion equation for Brownian particles, in which the diffusion coefficient is related to the mean squared displacement of a Brownian particle, while the second part consists in relating the diffusion coefficient to measurable physical quantities.^[4] In this way Einstein was able to determine the size of atoms, and how many atoms there are in a mole, or the molecular weight in grams, of a gas.^[5] In accordance to Avogadro's law this volume is the same for all ideal gases, which is 22.414 liters at standard temperature and pressure. The number of atoms contained in this volume is referred to as Avogadro's constant, and the determination of this number is tantamount to the knowledge of the mass of an atom since the latter is obtained by dividing the mass of a mole of the gas by Avogadro's constant.

The first part of Einstein's argument was to determine how far a Brownian particle travels in a given time interval.^[6] Classical mechanics is unable to determine this distance because of the enormous number of bombardments a Brownian particle will undergo, roughly of the order of 10^{14} collisions per second.^[1] Thus Einstein was led to consider the collective motion of Brownian particles.

He regarded the increment of particle positions in unrestricted one dimensional (x) domain as a random variable (Δ or x , under coordinate transformation so that the origin lies at the initial position of the particle) with some probability density function $\varphi(\Delta)$. Further, assuming conservation of particle number, he expanded the density (number of particles per unit volume) change in a Taylor series,



The characteristic bell-shaped curves of the diffusion of Brownian particles. The distribution begins as a Dirac delta function, indicating that all the particles are located at the origin at time $t=0$, and for increasing times they become flatter and flatter until the distribution becomes uniform in the asymptotic time limit.

$$\begin{aligned}
\rho(x, t) + \tau \frac{\partial \rho(x)}{\partial t} + \dots &= \rho(x, t + \tau) = \int_{-\infty}^{+\infty} \rho(x + \Delta, t) \cdot \varphi(\Delta) d\Delta \\
&= \rho(x, t) \cdot \int_{-\infty}^{+\infty} \varphi(\Delta) d\Delta + \frac{\partial \rho}{\partial x} \cdot \int_{-\infty}^{+\infty} \Delta \cdot \varphi(\Delta) d\Delta \\
&\quad + \frac{\partial^2 \rho}{\partial x^2} \cdot \int_{-\infty}^{+\infty} \frac{\Delta^2}{2} \cdot \varphi(\Delta) d\Delta + \dots \\
&= \rho(x, t) \cdot 1 + 0 + \frac{\partial^2 \rho}{\partial x^2} \cdot \int_{-\infty}^{+\infty} \frac{\Delta^2}{2} \cdot \varphi(\Delta) d\Delta + \dots
\end{aligned}$$

where the second equality in the first line is by definition of φ . The integral in the first term is equal to one by the definition of probability, and the second and other even terms (i.e. first and other odd moments) vanish because of space symmetry. What is left gives rise to the following relation:

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2} \cdot \int_{-\infty}^{+\infty} \frac{\Delta^2}{2\tau} \cdot \varphi(\Delta) d\Delta + \text{higher-order even moments}$$

Where the coefficient before the Laplacian, the second moment of probability of displacement Δ , is interpreted as mass diffusivity D :

$$D = \int_{-\infty}^{+\infty} \frac{\Delta^2}{2\tau} \cdot \varphi(\Delta) d\Delta$$

Then the density of Brownian particles ρ at point x at time t satisfies the diffusion equation:

$$\frac{\partial \rho}{\partial t} = D \cdot \frac{\partial^2 \rho}{\partial x^2},$$

Assuming that N particles start from the origin at the initial time $t=0$, the diffusion equation has the solution

$$\rho(x, t) = \frac{N}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}.$$

This expression (which is a normal distribution with the mean $\mu = 0$ and variance $\sigma^2 = 2Dt$ usually called Brownian motion B_t) allowed Einstein to calculate the moments directly. The first moment is seen to vanish, meaning that the Brownian particle is equally likely to move to the left as it is to move to the right. The second moment is, however, non-vanishing, being given by

$$\overline{x^2} = 2Dt.$$

This expresses the mean squared displacement in terms of the time elapsed and the diffusivity. From this expression Einstein argued that the displacement of a Brownian particle is not proportional to the elapsed time, but rather to its square root.^[4] His argument is based on a conceptual switch from the "ensemble" of Brownian particles to the "single" Brownian particle: we can speak of the relative number of particles at a single instant just as well as of the time it takes a Brownian particle to reach a given point.^[7]

The second part of Einstein's theory relates the diffusion constant to physically measurable quantities, such as the mean squared displacement of a particle in a given time interval. This result enables the experimental determination of Avogadro's number and therefore the size of molecules. Einstein analyzed a dynamic equilibrium being established between opposing forces. The beauty of his argument is that the final result does not depend upon which forces are involved in setting up the dynamic equilibrium.

In his original treatment, Einstein considered an osmotic pressure experiment, but the same conclusion can be reached in other ways.

Consider, for instance, particles suspended in a viscous fluid in a gravitational field. Gravity tends to make the particles settle, whereas diffusion acts to homogenize them, driving them into regions of smaller concentration. Under the action of gravity, a particle acquires a downward speed of $v = \mu mg$, where m is the mass of the particle, g is the acceleration due to gravity, and μ is the particle's mobility in the fluid. George Stokes had shown that the mobility for a spherical particle with radius r is $\mu = \frac{1}{6\pi\eta r}$, where η is the dynamic viscosity of the fluid. In a state of dynamic equilibrium, and under the hypothesis of isothermal fluid, the particles are distributed according to the barometric distribution

$$\rho = \rho_0 e^{-\frac{mgh}{k_B T}},$$

where $\rho - \rho_0$ is the difference in density of particles separated by a height difference of h , k_B is Boltzmann's constant (namely, the ratio of the universal gas constant, R , to Avogadro's number, N), and T is the absolute temperature. It is Avogadro's number that is to be determined.

Dynamic equilibrium is established because the more that particles are pulled down by gravity, the greater is the tendency for the particles to migrate to regions of lower concentration. The flux is given by Fick's law,

$$J = -D \frac{d\rho}{dh},$$

where $J = \rho v$. Introducing the formula for ρ , we find that

$$v = \frac{Dmg}{k_B T}.$$

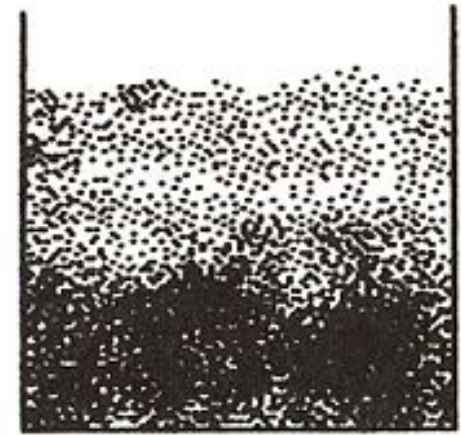
In a state of dynamical equilibrium, this speed must also be equal to $v = \mu mg$. Notice that both expressions for v are proportional to mg , reflecting how the derivation is independent of the type of forces considered. Equating these two expressions yields a formula for the diffusivity:

$$\frac{\overline{x^2}}{2t} = D = \mu k_B T = \frac{\mu RT}{N} = \frac{RT}{6\pi\eta r N}.$$

Here the first equality follows from the first part of Einstein's theory, the third equality follows from the definition of Boltzmann's constant as $k_B = R / N$, and the fourth equality follows from Stokes's formula for the mobility. By measuring the mean squared displacement over a time interval along with the universal gas constant R , the temperature T , the viscosity η , and the particle radius r , Avogadro's number N can be determined.

The type of dynamical equilibrium proposed by Einstein was not new. It had been pointed out previously by J. J. Thomson^[8] in his series of lectures at Yale University in May 1903 that the dynamic equilibrium between the velocity generated by a concentration gradient given by Fick's law and the velocity due to the variation of the partial pressure caused when ions are set in motion "gives us a method of determining Avogadro's Constant which is independent of any hypothesis as to the shape or size of molecules, or of the way in which they act upon each other".^[8]

An identical expression to Einstein's formula for the diffusion coefficient was also found by Walther Nernst in 1888^[9] in which he expressed the diffusion coefficient as the ratio of the osmotic pressure to the ratio of the frictional force and the velocity to which it gives rise. The former was equated to the law of van 't Hoff while the latter was given by Stokes's law. He writes $k' = p_0 / k$ for the diffusion coefficient k' , where p_0 is the osmotic pressure and k is the ratio of the frictional force to the molecular viscosity which he assumes is given by Stokes's formula for the viscosity. Introducing the ideal



The equilibrium distribution for particles of gamboge shows the tendency for granules to move to regions of lower concentration when affected by gravity.

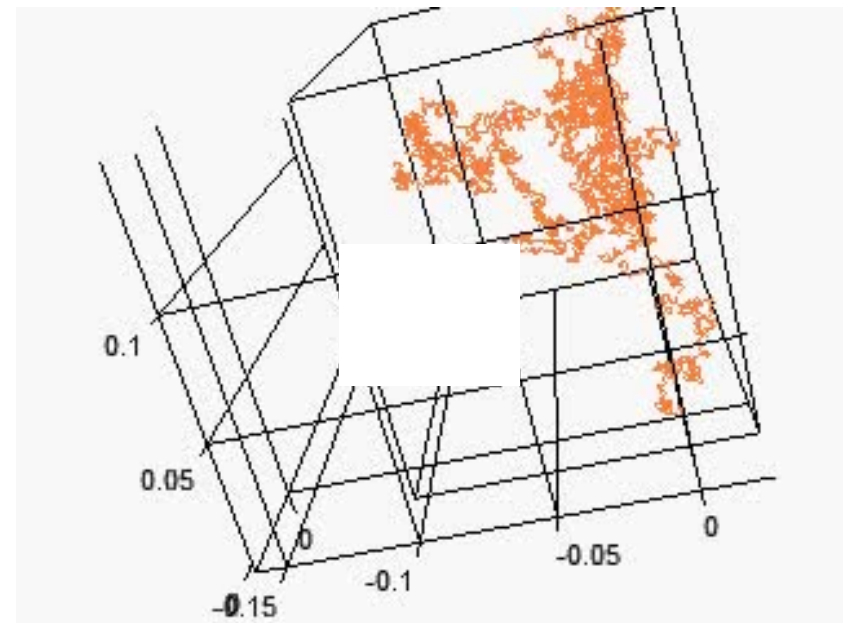
gas law per unit volume for the osmotic pressure, the formula becomes identical to that of Einstein's.^[10] The use of Stokes's law in Nernst's case, as well as in Einstein and Smoluchowski, is not strictly applicable since it does not apply to the case where the radius of the sphere is small in comparison with the mean free path.^[11]

At first, the predictions of Einstein's formula were seemingly refuted by a series of experiments by Svedberg in 1906 and 1907, which gave displacements of the particles as 4 to 6 times the predicted value, and by Henri in 1908 who found displacements 3 times greater than Einstein's formula predicted.^[12] But Einstein's predictions were finally confirmed in a series of experiments carried out by Chaudesaigues in 1908 and Perrin in 1909. The confirmation of Einstein's theory constituted empirical progress for the kinetic theory of heat. In essence, Einstein showed that the motion can be predicted directly from the kinetic model of thermal equilibrium. The importance of the theory lay in the fact that it confirmed the kinetic theory's account of the second law of thermodynamics as being an essentially statistical law.^[13]

Smoluchowski model

Smoluchowski's theory of Brownian motion^[14] starts from the same premise as that of Einstein and derives the same probability distribution $\rho(x, t)$ for the displacement of a Brownian particle along the x in time t . He therefore gets the same expression for the mean squared displacement: $\overline{(\Delta x)^2}$. However, when he relates it to a particle of mass m moving at a velocity u which is the result of a frictional force governed by Stokes's law, he finds

$$\overline{(\Delta x)^2} = 2Dt = t \frac{32}{81} \frac{\mu^2}{\pi \mu a} = t \frac{64}{27} \frac{\frac{1}{2} \mu^2}{3 \pi \mu a},$$



Brownian motion model of the trajectory of a particle of dye in water.

where μ is the viscosity coefficient, and a is the radius of the particle. Associating the kinetic energy $\mu^2/2$ with the thermal energy RT/N , the expression for the mean squared displacement is $64/27$ times that found by Einstein. The fraction $27/64$ was commented on by Arnold Sommerfeld in his necrology on Smoluchowski: "The numerical coefficient of Einstein, which differs from Smoluchowski by $27/64$ can only be put in doubt."^[15]

Smoluchowski^[16] attempts to answer the question of why a Brownian particle should be displaced by bombardments of smaller particles when the probabilities for striking it in the forward and rear directions are equal. In order to do so, he uses, unknowingly, the ballot theorem, first proved by W.A. Whitworth in 1878.^[17] The ballot theorem states that if a candidate A scores m votes and candidate B scores $n - m$ that the probability throughout the counting that A will have more votes than B is

$$\frac{[m - (n - m)]}{[m + n - m]} = \frac{2m - n}{n},$$

no matter how large the total number of votes n may be. In other words, if one candidate has an edge on the other candidate he will tend to keep that edge even though there is nothing favoring either candidate on a ballot extraction.

If the probability of m gains and $n - m$ losses follows a binomial distribution,

$$P_{m,n} = \binom{n}{m} 2^{-n},$$

with equal *a priori* probabilities of $1/2$, the mean total gain is

$$\overline{2m - n} = \sum_{m=\frac{n}{2}}^n (2m - n) P_{m,n} = \frac{nn!}{2^n \left[\left(\frac{n}{2} \right)! \right]^2}.$$

If n is large enough so that Stirling's approximation can be used in the form

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n},$$

then the expected total gain will be

$$\overline{2m - n} \approx \sqrt{\frac{2n}{\pi}},$$

showing that it increases as the square root of the total population.

Suppose that a Brownian particle of mass M is surrounded by lighter particles of mass m which are traveling at a speed u . Then, reasons Smoluchowski, in any collision between a surrounding and Brownian particles, the velocity transmitted to the latter will be mu/M . This ratio is of the order of 10^{-7} cm/s. But we also have to take into consideration that in a gas there will be more than 10^{16} collisions in a second, and even greater in a liquid where we expect that there will be 10^{20} collision in one second. Some of these collisions will tend to accelerate the Brownian particle; others will tend to decelerate it. If there is a mean excess of one kind of collision or the other to be of the order of 10^8 to 10^{10} collisions in one second, then velocity of the Brownian particle may be anywhere between 10 and 1000 cm/s. Thus, even though there are equal probabilities for forward and backward collisions there will be a net tendency to keep the Brownian particle in motion, just as the ballot theorem predicts.

These orders of magnitude are not exact because they don't take into consideration the velocity of the Brownian particle, U , which depends on the collisions that tend to accelerate and decelerate it. The larger U is, the greater will be the collisions that will retard it so that the velocity of a Brownian particle can never increase without limit. Could such a process occur, it would be tantamount to a perpetual motion of the second type. And since equipartition of energy applies, the kinetic energy of the Brownian particle, $MU^2/2$, will be equal, on the average, to the kinetic energy of the surrounding fluid particle, $mu^2/2$.

In 1906 Smoluchowski published a one-dimensional model to describe a particle undergoing Brownian motion.^[18] The model assumes collisions with $M \gg m$ where M is the test particle's mass and m the mass of one of the individual particles composing the fluid. It is assumed that the particle collisions are confined to one dimension and that it is equally probable for the test particle to be hit from the left as from the right. It is also assumed that every collision always imparts the same magnitude of ΔV . If N_R is the number of collisions from the right and N_L the number of collisions from the left then after N collisions the particle's velocity will have changed by $\Delta V(2N_R - N)$. The multiplicity is then simply given by:

$$\binom{N}{N_R} = \frac{N!}{N_R!(N - N_R)!}$$

and the total number of possible states is given by 2^N . Therefore, the probability of the particle being hit from the right N_R times is:

$$P_N(N_R) = \frac{N!}{2^N N_R!(N - N_R)!}$$

As a result of its simplicity, Smoluchowski's 1D model can only qualitatively describe Brownian motion. For a realistic particle undergoing Brownian motion in a fluid many of the assumptions cannot be made. For example, the assumption that on average there occurs an equal number of collisions from the right as from the left falls apart once the particle is in motion. Also, there would be a distribution of different possible ΔV s instead of always just one in a realistic situation.

Other Physics Models Using Partial Differential Equations

The diffusion equation yields an approximation of the time evolution of the probability density function associated to the position of the particle going under a Brownian movement under the physical definition. The approximation is valid on short timescales.

The time evolution of the position of the Brownian particle itself is best described using Langevin equation, an equation which involves a random force field representing the effect of the thermal fluctuations of the solvent on the particle.

The displacement of a particle undergoing Brownian motion is obtained by solving the diffusion equation under appropriate boundary conditions and finding the rms of the solution. This shows that the displacement varies as the square root of the time (not linearly), which explains why previous experimental results concerning the velocity of Brownian particles gave nonsensical results. A linear time dependence was incorrectly assumed.

At very short time scales, however, the motion of a particle is dominated by its inertia and its displacement will be linearly dependent on time: $\Delta x = v\Delta t$. So the instantaneous velocity of the Brownian motion can be measured as $v = \Delta x / \Delta t$, when $\Delta t \ll \tau$, where τ is the momentum relaxation time. In 2010, the instantaneous velocity of a Brownian particle (a glass microsphere trapped in air with an optical tweezer) was measured successfully.^[19] The velocity data verified the Maxwell–Boltzmann velocity distribution, and the equipartition theorem for a Brownian particle.

Astrophysics: Star Motion within Galaxies

In stellar dynamics, a massive body (star, black hole, etc.) can experience Brownian motion as it responds to gravitational forces from surrounding stars.^[20] The rms velocity V of the massive object, of mass M , is related to the rms velocity v_\star of the background stars by

$$MV^2 \approx mv_\star^2$$

where $m \ll M$ is the mass of the background stars. The gravitational force from the massive object causes nearby stars to move faster than they otherwise would, increasing both v_\star and V .^[20] The Brownian velocity of Sgr A*, the supermassive black hole at the center of the Milky Way galaxy, is predicted from this formula to be less than 1 km s^{-1} .^[21]

Mathematics

In mathematics, Brownian motion is described by the **Wiener process**; a continuous-time stochastic process named in honor of Norbert Wiener. It is one of the best known Lévy processes (càdlàg stochastic processes with stationary independent increments) and occurs frequently in pure and applied mathematics, economics and physics.

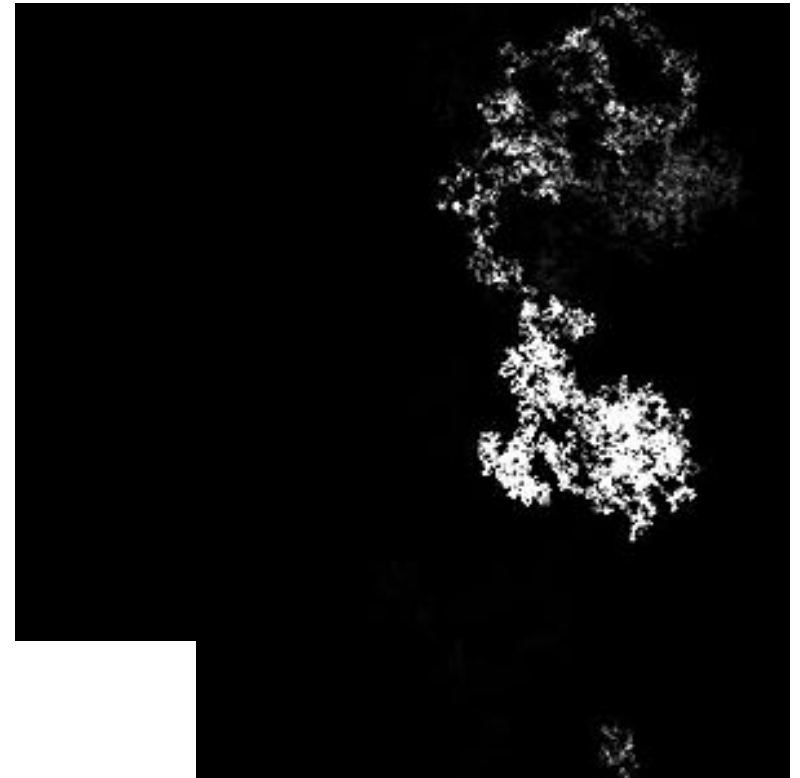
The Wiener process W_t is characterised by four facts:

1. $W_0 = 0$
2. W_t is almost surely continuous
3. W_t has independent increments
4. $W_t - W_s \sim \mathcal{N}(0, t - s)$ (for $0 \leq s \leq t$).

$\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2 . The condition that it has independent increments means that if $0 \leq s_1 < t_1 \leq s_2 < t_2$ then $W_{t_1} - W_{s_1}$ and $W_{t_2} - W_{s_2}$ are independent random variables.

An alternative characterisation of the Wiener process is the so-called *Lévy characterisation* that says that the Wiener process is an almost surely continuous martingale with $W_0 = 0$ and quadratic variation $[W_t, W_t] = t$.

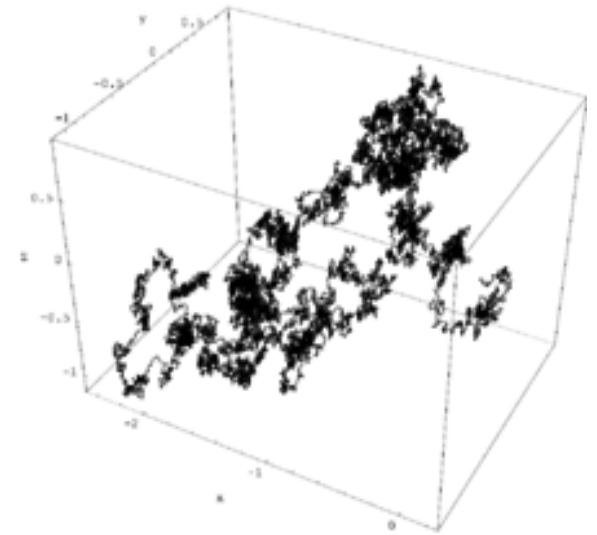
A third characterisation is that the Wiener process has a spectral representation as a sine series whose coefficients are independent $\mathcal{N}(0, 1)$ random variables. This representation can be obtained using the Karhunen–Loève theorem.



An animated example of a Brownian motion-like random walk on a torus. In the scaling limit, random walk approaches the Wiener process according to Donsker's theorem.

The Wiener process can be constructed as the scaling limit of a random walk, or other discrete-time stochastic processes with stationary independent increments. This is known as Donsker's theorem. Like the random walk, the Wiener process is recurrent in one or two dimensions (meaning that it returns almost surely to any fixed neighborhood of the origin infinitely often) whereas it is not recurrent in dimensions three and higher. Unlike the random walk, it is scale invariant.

The time evolution of the position of the Brownian particle itself can be described approximately by a Langevin equation, an equation which involves a random force field representing the effect of the thermal fluctuations of the solvent on the Brownian particle. On long timescales, the mathematical Brownian motion is well described by a Langevin equation. On small timescales, inertial effects are prevalent in the Langevin equation. However the mathematical *Brownian motion* is exempt of such inertial effects. Note that inertial effects have to be considered in the Langevin equation, otherwise the equation becomes singular. so that simply removing the inertia term from this equation would not yield an exact description, but rather a singular behavior in which the particle doesn't move at all.



A single realisation of three-dimensional Brownian motion for times $0 \leq t \leq 2$

Statistics

The Brownian motion can be modeled by a random walk.^[22] Random walks in porous media or fractals are anomalous.^[23]

In the general case, Brownian motion is a non-Markov random process and described by stochastic integral equations.^[24]

Lévy characterisation

The French mathematician Paul Lévy proved the following theorem, which gives a necessary and sufficient condition for a continuous \mathbf{R}^n -valued stochastic process X to actually be n -dimensional Brownian motion. Hence, Lévy's condition can actually be used as an alternative definition of Brownian motion.

Let $X = (X_1, \dots, X_n)$ be a continuous stochastic process on a probability space $(\Omega, \Sigma, \mathbf{P})$ taking values in \mathbf{R}^n . Then the following are equivalent:

1. X is a Brownian motion with respect to \mathbf{P} , i.e., the law of X with respect to \mathbf{P} is the same as the law of an n -dimensional Brownian motion, i.e., the push-forward measure $X_*(\mathbf{P})$ is classical Wiener measure on $C_0([0, +\infty); \mathbf{R}^n)$.
2. both
 1. X is a martingale with respect to \mathbf{P} (and its own natural filtration); and
 2. for all $1 \leq i, j \leq n$, $X_i(t)X_j(t) - \delta_{ij}t$ is a martingale with respect to \mathbf{P} (and its own natural filtration), where δ_{ij} denotes the Kronecker delta.

Riemannian manifold

The infinitesimal generator (and hence characteristic operator) of a Brownian motion on \mathbf{R}^n is easily calculated to be $\frac{1}{2}\Delta$, where Δ denotes the Laplace operator. In image processing and computer vision, the Laplacian operator has been used for various tasks such as blob and edge detection. This observation is useful in defining Brownian motion on an m -dimensional Riemannian manifold (M, g) : a **Brownian motion on M** is defined to be a diffusion on M whose characteristic operator \mathcal{A} in local coordinates x_i , $1 \leq i \leq m$, is given by $\frac{1}{2}\Delta_{\text{LB}}$, where Δ_{LB} is the Laplace–Beltrami operator given in local coordinates by

$$\Delta_{\text{LB}} = \frac{1}{\sqrt{\det(g)}} \sum_{i=1}^m \frac{\partial}{\partial x_i} \left(\sqrt{\det(g)} \sum_{j=1}^m g^{ij} \frac{\partial}{\partial x_j} \right),$$

where $[g^{ij}] = [g_{ij}]^{-1}$ in the sense of the inverse of a square matrix.

Narrow escape

The **narrow escape problem** is a ubiquitous problem in biology, biophysics and cellular biology which has the following formulation: a Brownian particle (ion, molecule, or protein) is confined to a bounded domain (a compartment or a cell) by a reflecting boundary, except for a small window through which it can escape. The narrow escape problem is that of calculating the mean escape time. This time diverges as the window shrinks, thus rendering the calculation a singular perturbation problem.

See also

- Brownian bridge: a Brownian motion that is required to "bridge" specified values at specified times
- Brownian covariance
- Brownian dynamics
- Brownian motion of sol particles
- Brownian motor
- Brownian noise (Martin Gardner proposed this name for sound generated with random intervals. It is a pun on Brownian motion and white noise.)
- Brownian ratchet
- Brownian surface
- Brownian tree
- Brownian web
- Rotational Brownian motion
- Clinamen
- Complex system




Brownian motion on a 2-sphere

- Continuity equation
- Diffusion equation
- Geometric Brownian motion
- Itô diffusion: a generalisation of Brownian motion
- Langevin equation
- Lévy arcsine law
- Local time (mathematics)
- Many-body problem
- Marangoni effect
- Mark G. Raizen
- Nanoparticle tracking analysis
- Narrow escape problem
- Osmosis
- Random walk
- Schramm–Loewner evolution
- Single particle tracking
- Statistical mechanics
- Surface diffusion: a type of constrained Brownian motion.
- Thermal equilibrium
- Thermodynamic equilibrium
- Tyndall effect: physical chemistry phenomenon where particles are involved; used to differentiate between the different types of mixtures.
- Ultramicroscope

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External links

- Brownian motion java simulation (http://galileo.phys.virginia.edu/classes/109N/more_stuff/Applets/brownian/applet.html)
- A single-molecule brownian motion diffusion simulator (<https://tritemio.github.io/PyBroMo/>)
- Article for the school-going child (<http://xxx.imsc.res.in/abs/physics/0412132>)
- Einstein on Brownian Motion (<https://web.archive.org/web/20010222031055/http://www.bun.kyoto-u.ac.jp/~suchii/einsteinBM.html>)
- Brownian Motion, "Diverse and Undulating" (<https://arxiv.org/abs/0705.1951>)
- Discusses history, botany and physics of Brown's original observations, with videos (<http://physerver.hamilton.edu/Research/Brownian/index.html>)
- "Einstein's prediction finally witnessed one century later" (<http://www.gizmag.com/einsteins-prediction-finally-witnessed/16212/>) : a test to observe the velocity of Brownian motion
- "Café math : brownian motion (Part 1)" (<http://cafemath.fr/mathblog/article.php?page=BrownianMotion.php>) : A blog article describing brownian motion (definition, symmetries, simulation)

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