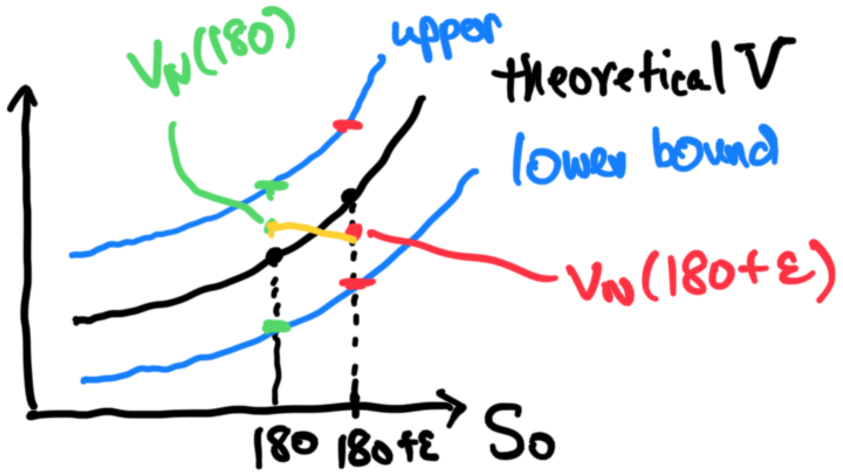
HW2 greeks



$$\Delta_1 = \frac{V_N(S_0+\varepsilon) - V_N(S_0)}{\varepsilon} \neq \frac{\partial V}{\partial S_0}$$

$$V = \mathbb{E}\Big[\underbrace{e^{-rT}\max(S_T-K,\,0)}_{f(S_0,\,z)}\Big]$$

$$S_T(z) = S_0 \exp\Big(\big(r-q-\tfrac{\sigma^2}{2}\big)T + \sigma\sqrt{T}\,z\Big)$$

$$V = \mathbb{E}[f(S_0, z)]$$

$$\frac{\partial V}{\partial S_0} = \mathbb{E}\Big[\frac{\partial}{\partial S_0} f(S_0, z)\Big]$$

$$\Delta_2(\varepsilon) = \frac{1}{N}\sum_{i=1}^{N} \frac{f(S_0+\varepsilon, z_i) - f(S_0, z_i)}{\varepsilon}$$

$$\lim_{\varepsilon\to 0} \Delta_2(\varepsilon) = \frac{1}{N}\sum_{i=1}^{N} \frac{\partial f}{\partial S_0}(S_0, z_i) \longrightarrow \mathbb{E}\Big[\frac{\partial f}{\partial S_0}\Big]$$

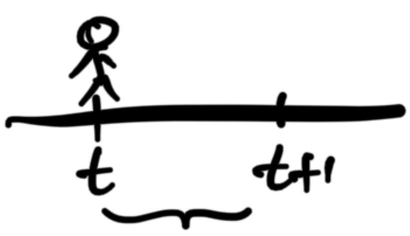
theoretical delta.

---

volatility forecast

Ref : Hull Chap 22

RiskMetrics Chap 5.2



- historical data
- portfolio risk
- RiskMetrics (1996)

given asset price series $\{P_i\}$ by date

want: estimate 1-day volatility.

= std dev. of daily price return

Relative price return $= \dfrac{P_i - P_{i-1}}{P_{i-1}} = R(i)$

* Log price return $r_i = \log P_i - \log P_{i-1}$

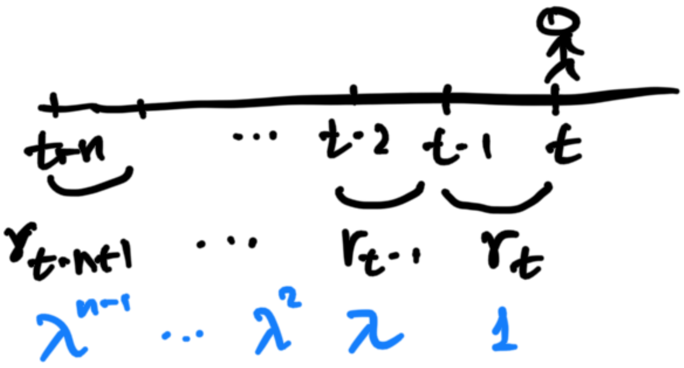
$$P_i = P_{i-1} \exp(r_i)$$

cont. compd. rate of return

① Simple moving average over n days.

$SMA(r_t)$

$$= \frac{1}{n} \sum_{i=0}^{n-1} r_{t-i} = \bar{r}$$



variance estimator

$$v_t^{SMA} = \frac{1}{n-1} \sum_{i=0}^{n-1} (r_{t-i} - \bar{r})^2$$

λ = decay factor

$0 < \lambda < 1$

vol estimator :

$$\sigma_t^{SMA} = \sqrt{v_t^{SMA}}$$

② Exponentially weighted moving avg. (EWMA)

$$EWM(r_t) = \gamma (r_t + \lambda r_{t-1} + \lambda^2 r_{t-2} + \dots + \lambda^{n-1} r_{t-n+1}) = \hat{r}_{t,n}$$

make sum of wts = 1

$$\gamma = \frac{1-\lambda}{1-\lambda^n}$$

$$v_{t,n}^{EWA} = \gamma \sum_{i=0}^{n-1} \lambda^i (r_{t-i} - \hat{r}_{t,n})^2$$

$$\sigma_{t,n}^{EWA} = \sqrt{v_{t,n}^{EWA}}$$

Remarks

① $v_{t,n}^{EWA} = EWM(r_t^2) - EWM(r_t)^2$

② As $n \to \infty$.

$$\hat{r}_{t,n} \to (1-\lambda) \sum_{i=0}^{\infty} \lambda^i r_{t-i} = \tilde{\mu}_t^{EWA}$$

$$v_{t,n}^{EWA} \to (1-\lambda) \sum_{i=0}^{\infty} \lambda^i (r_{t-i} - \tilde{\mu}^{EWA})^2$$

$$= \tilde{v}_t^{EWA}$$

Alternate def. of EWMA

③ $\tilde{\mu}_t^{EWM}$, $\tilde{v}_t^{EWM}$ recursive relation.

$$\begin{cases} \tilde{\mu}_t = (1-\lambda) r_{t-1} + \lambda \tilde{\mu}_{t-1} \\ \underline{\underline{\tilde{v}_t}} = (1-\lambda)(\underline{\underline{r_{t-1}}} - \tilde{\mu}_{t-1})^2 + \lambda \tilde{v}_{t-1} \end{cases}$$

Special case of IGARCH(1,1)

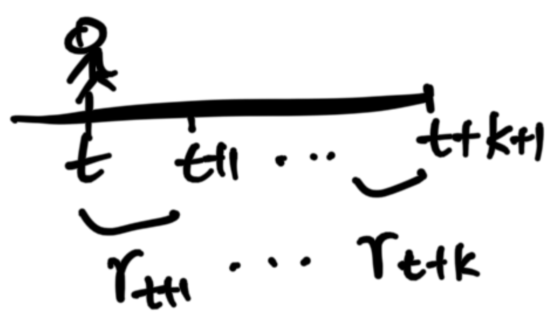
Assume $r_{t+1} | \mathcal{F}_t \sim N(\mu_t, \underline{\underline{\sigma_{t+1}^2}})$

Let $a_t = r_t - \mu_t$

$$\begin{cases} a_t = \underline{\underline{\sigma_{t+1}}} \varepsilon_{t+1} & \varepsilon_{t+1} \sim N(0,1) \\ \sigma_{t+1}^2 = (1-\lambda) a_t^2 + \lambda \sigma_t^2 \end{cases}$$

k-day vol estimate

Define k-day price return



$$r_t(k) = \ln P_{t+k} - \ln P_k$$

$$= r_{t+1} + r_{t+2} + \cdots + r_{t+k}$$

$$var_t(r_{t+1}) = var_t(r_{t+2}) = \cdots$$

$$\mathrm{var}_t(r_t(k)) = k\,\mathrm{var}_t(r_{t+1})$$

$$\boxed{k\text{-day vol} = \sqrt{k}\ \text{1-day vol}}$$

Estimating covariance

given two log return series $\{r_t\}$, $\{u_t\}$

$$\mathrm{cov}_{t,n}^{EWM} = \gamma \sum_{i=0}^{n-1} \lambda^i (r_t - \underbrace{\hat{r}_{t,n}}_{\to\,\tilde{r}_t})(u_t - \underbrace{\hat{u}_{t,n}}_{\to\,\hat{u}_t})$$

$n \to \infty$

$$\widetilde{\mathrm{cov}}_t^{EWM} = (1-\lambda) \sum_{i=0}^{n-1} \lambda^i (r_t - \hat{r}_t)(u_t - \tilde{u}_t)$$

satisrifies recursive relation.

$\lambda = 0.94 \leftarrow$ RiskMetrics.