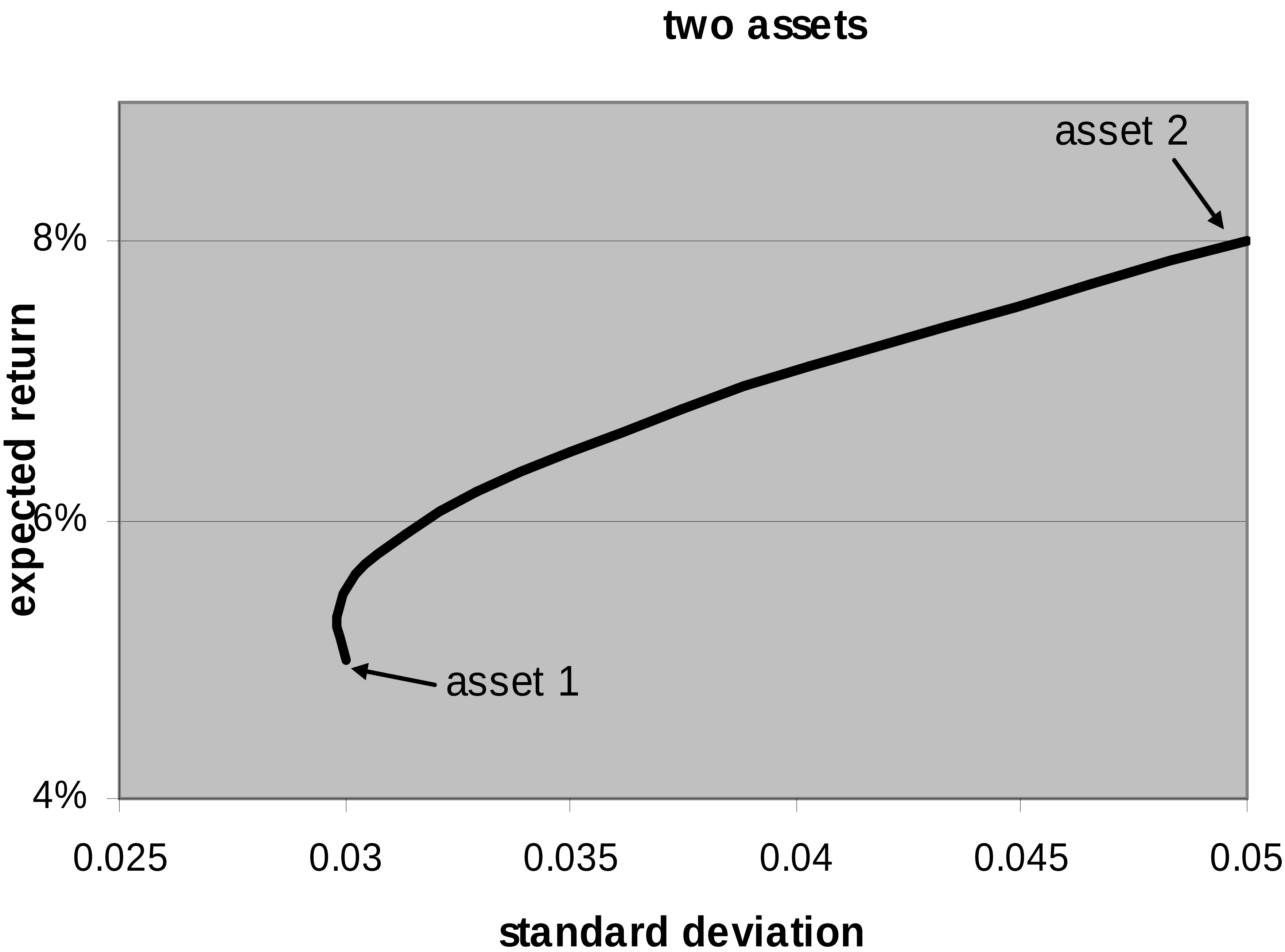
Introduction to Portfolio Theory

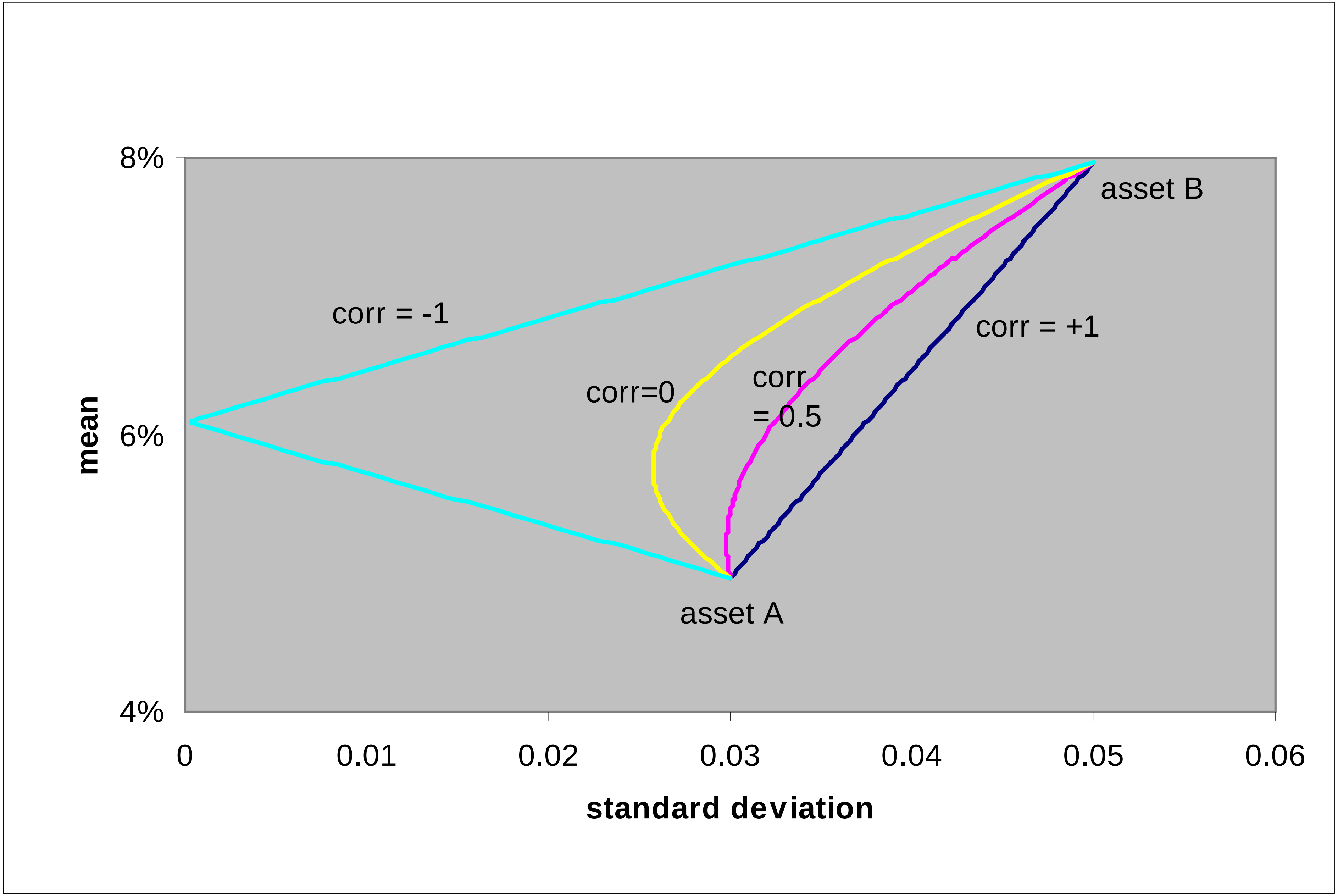

two assets
asset 2
asset 1
8%
6%
4%
expected return
0.025
0.03
0.035
0.04
0.045
0.05
standard deviation

Fully invested portfolio of two assets R_1 σ_1 and R_2 σ_2 with weights x_1 and $(1 - x_1)$

$$E(R_p) = \quad x_1 E(R_1) + (1 - x_1) E(R_2)$$

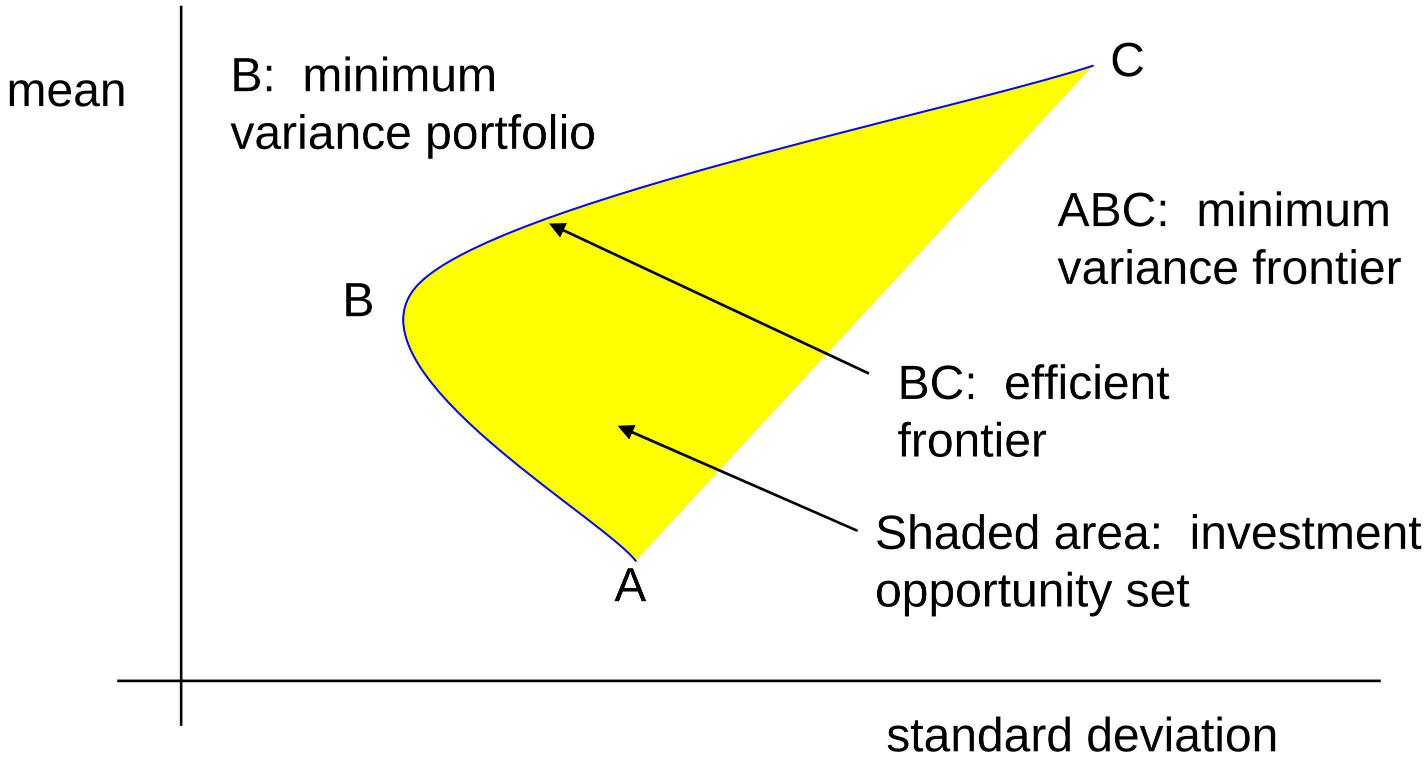
$$\sigma_p^2 = x_1^2 \sigma_1^2 + (1 - x_1)^2 \sigma_2^2 + 2x_1(1 - x_1) \rho_{12} \sigma_1 \sigma_2$$

$$\rho_{12} \equiv \text{Corr}\ (R_1, R_2) = \text{cov}\ (R_1, R_2)/(\sigma_1 \sigma_2).$$



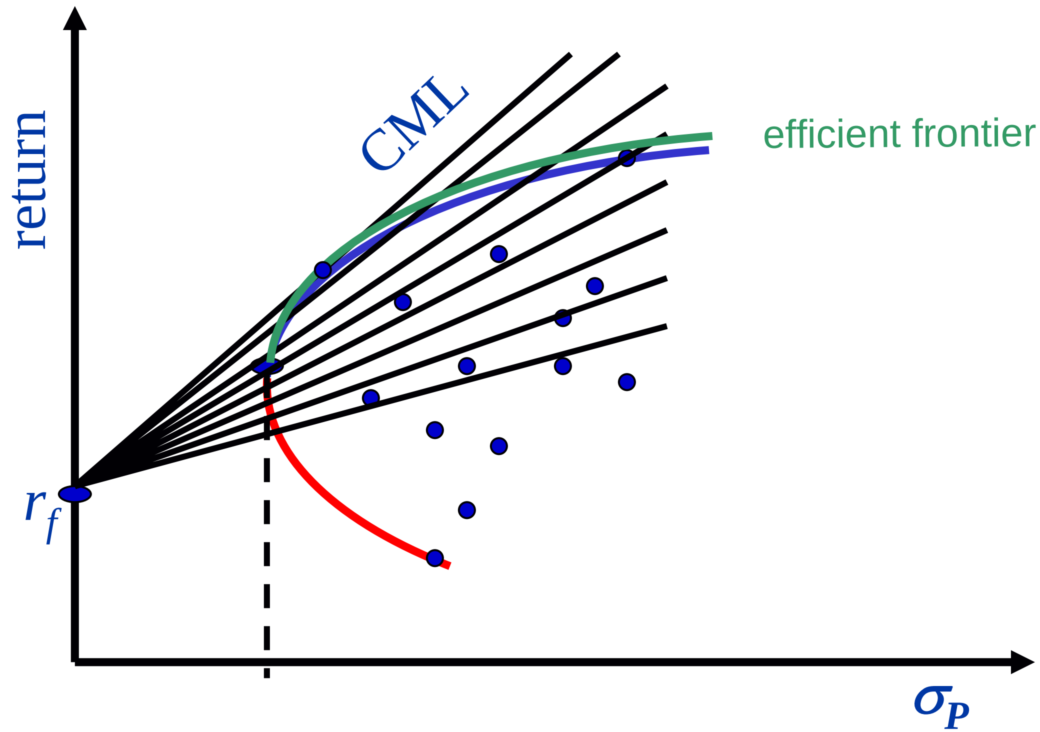
8%
6%
4%
mean
0
0.01
0.02
0.03
0.04
0.05
0.06
standard deviation
corr = -1
corr=0
corr
= 0.5
corr = +1
asset A
asset B

Efficient frontier with N assets



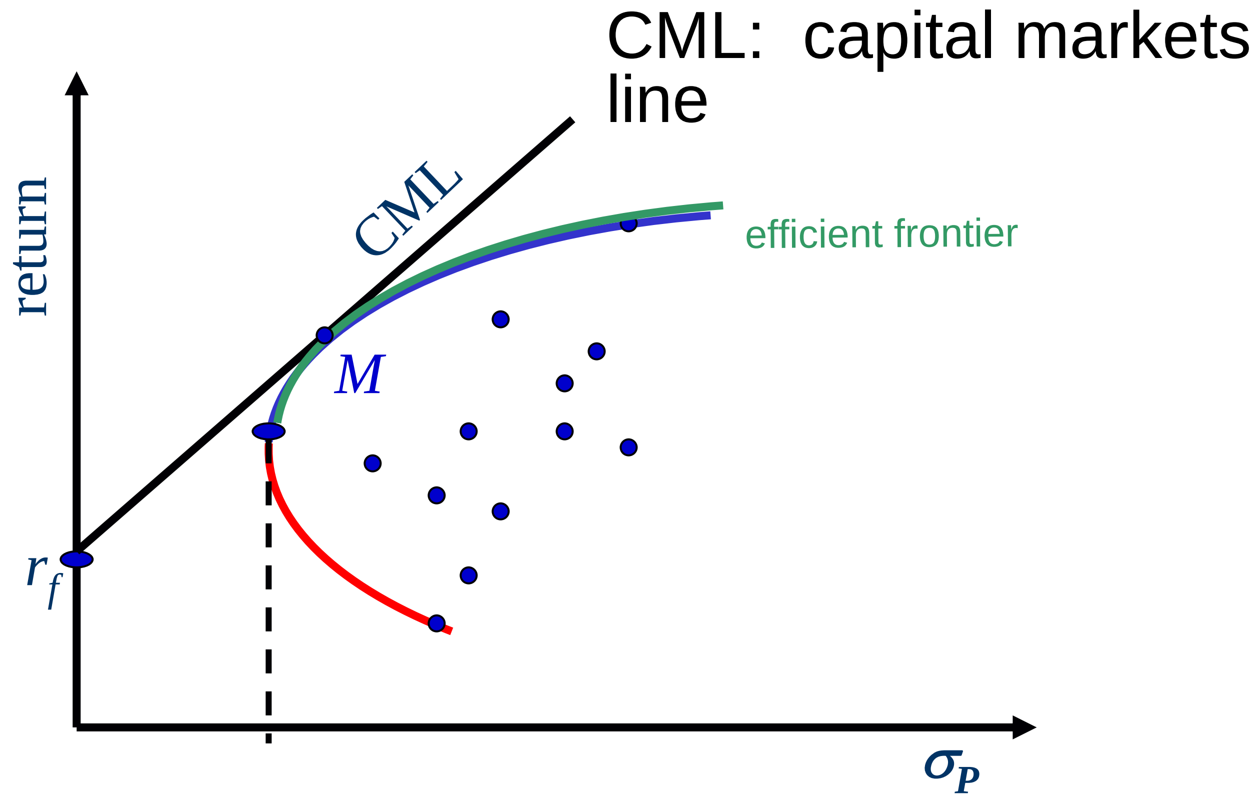
Diversification with N assets: covariance among assets affects the variance of a portfolio with N assets.

Riskless Borrowing and Lending



If a risk-free asset available and the efficient frontier identified, an investor chooses the capital allocation line with the steepest slope.

Riskless Borrowing and Lending



With the capital allocation line identified, an investor chooses a point along the line—some combination of the risk-free asset and a risky portfolio *M*.

Sharpe ratio was introduced in 1966 by William Sharpe (1990 Nobel Prize in Economics for the capital asset pricing model). It is a risk/return measure and one of the simplest of such measures.

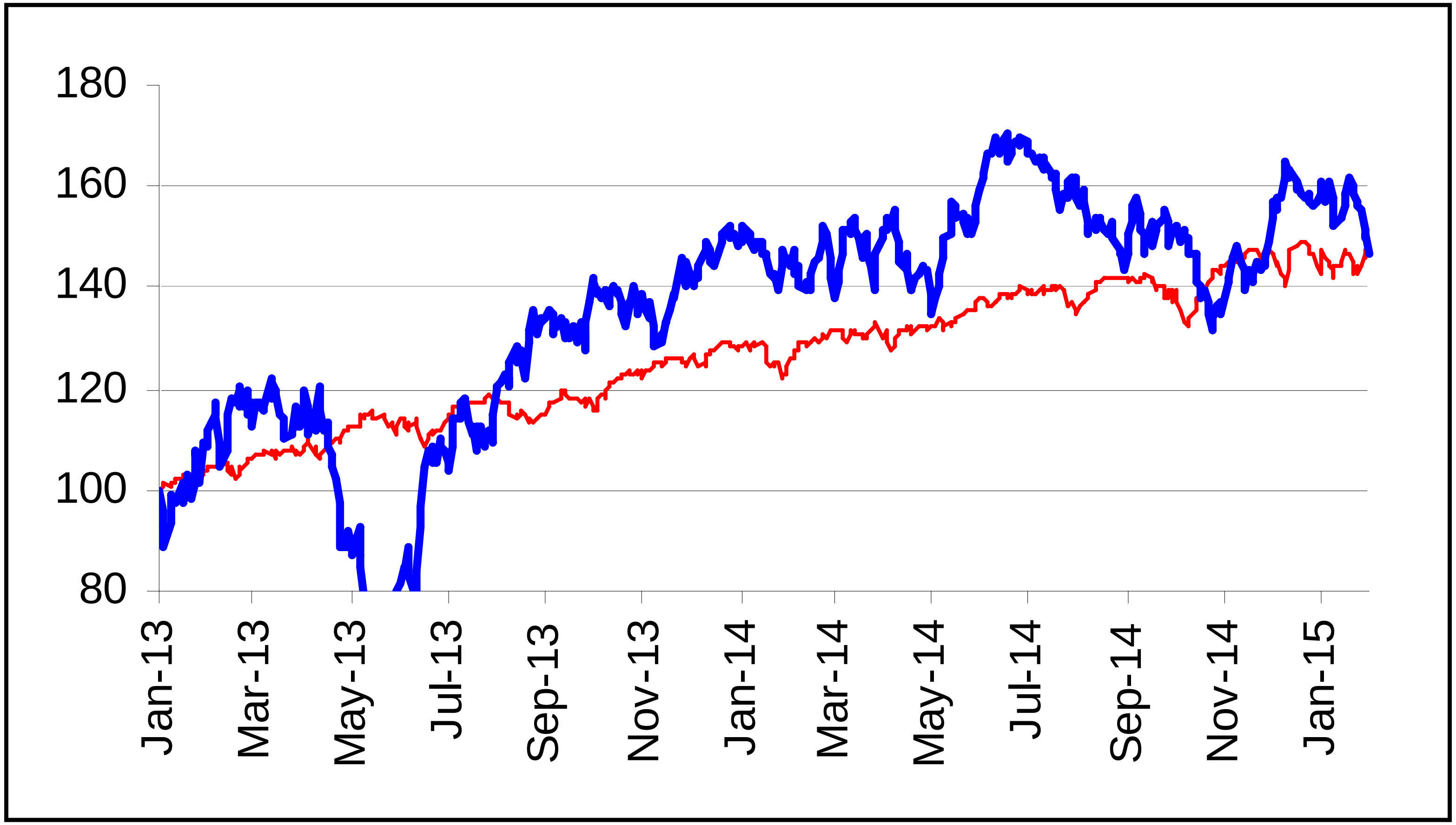
$$\text{Sharpe Ratio} = \frac{\text{(Annual Expected Portfolio Return} - \text{Ann Risk Free Rate)}}{\text{Annual Volatility of Excess Returns over Risk Free rate}}$$

Sharpe ratio measures fund’s risk-adjusted performance.

The higher Sharpe ratio is the better is the fund manager.

Typical range 0.6 to 2.5. Sharpe 2.5 is VERY VERY good.

Negative Sharpe ratio implies that a cash would perform better than the fund. Cash would have no volatility.



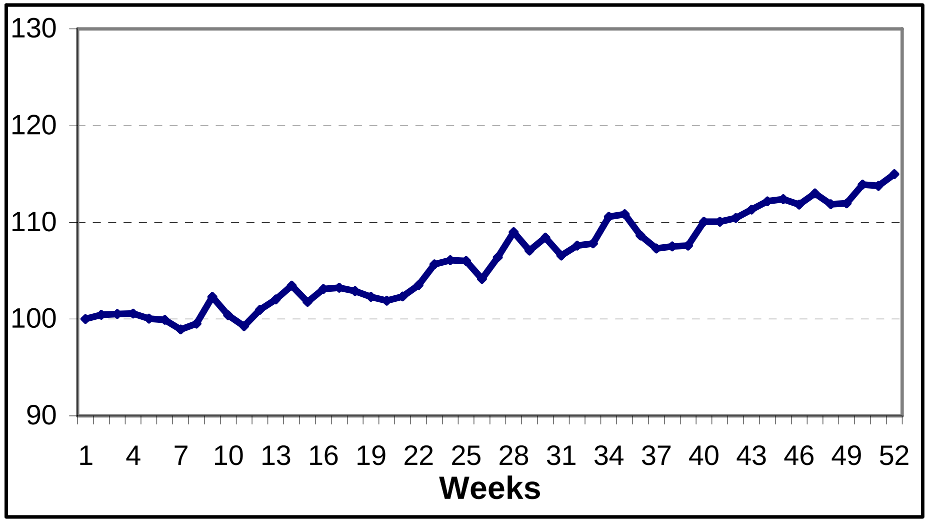
High Sharpe ratio fund (red thin line) Sharpe=1.6

Low Sharpe ratio fund (blue thick line) Sharpe=0.7

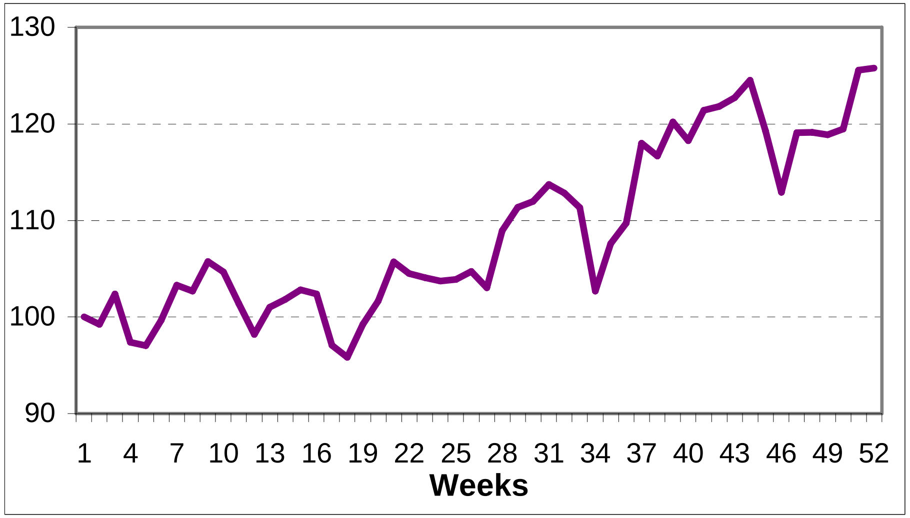
Measuring of Performance of Leveraged Investment

Example.

Fund A, Annual Return 15%



Fund B, Annual Return 25%



Which one is better to choose if one can leverage the investment?

Postulate:

If two funds X and Y have the same return

then the fund that has less risk as measured by “oscillation” or standard deviation of returns is better.

Note that risk can be measured by other measures like drawdown, downside deviation etc.

Answer. The one with higher Sharpe ratio.

Suppose that the risk free rate is 5% (one can borrow and lend money at 5%). Define Sharpe Ratio, named after economist W.Sharpe as

$$\textbf{Sharpe Ratio} = \frac{\textbf{(Annual Return} - \textbf{Risk Free Rate)}}{\textbf{Annual Volatility of Excess Returns over Risk Free rate}}$$

Fund A.

Annual Return	15%,	
Annual Volatility	10%	$\text{Sharpe Ratio} = \dfrac{(15\% - 5\%\text{Risk Free Rate})}{10\%} = \mathbf{1}$
(of returns over 5% risk free)		

Fund B.

Annual Return	25%,	
Annual Volatility	30%	$\text{Sharpe Ratio} = \dfrac{(25\% - 5\%\text{Risk Free Rate})}{30\%} = \mathbf{0.66}$
(of returns over 5% risk free)		

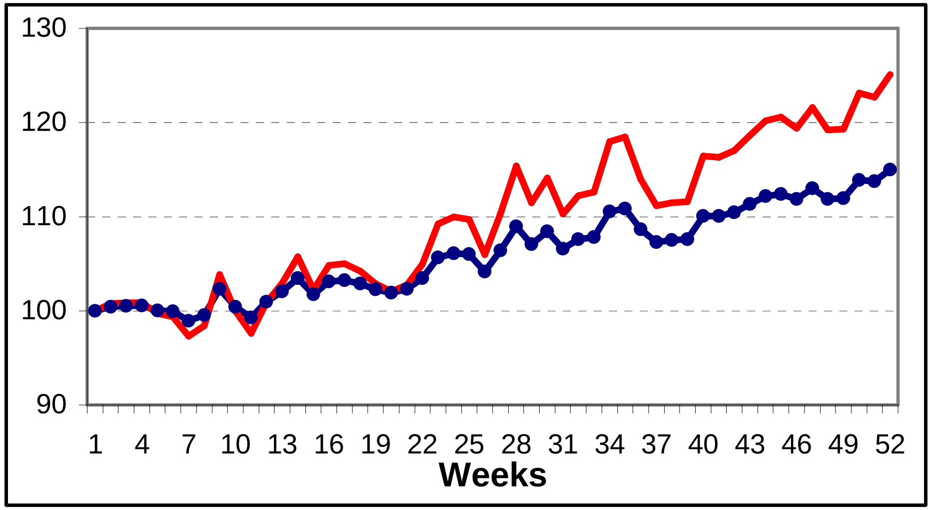
Fund A Leveraged 2 to 1.

Annual Return 25%=2×15%–5%(borrowing),

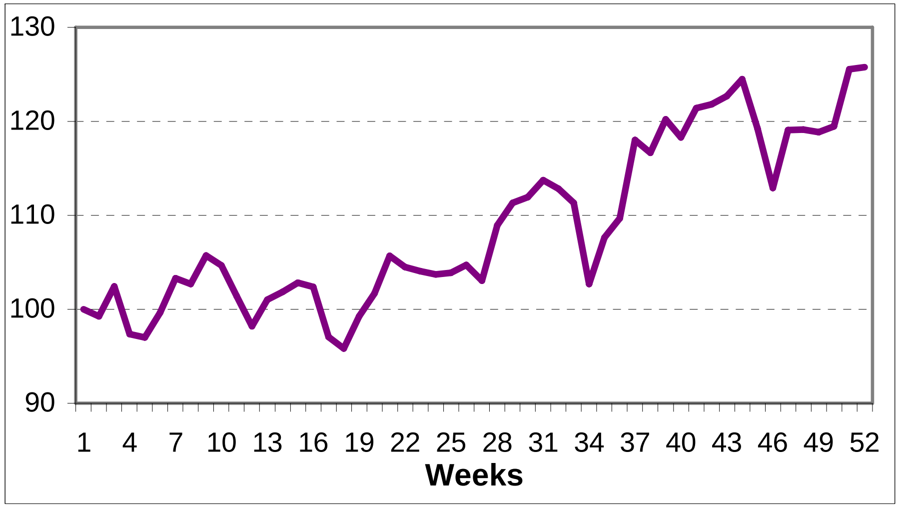
Annual Volatility 20%=2×10%
(of returns over 5% risk free)

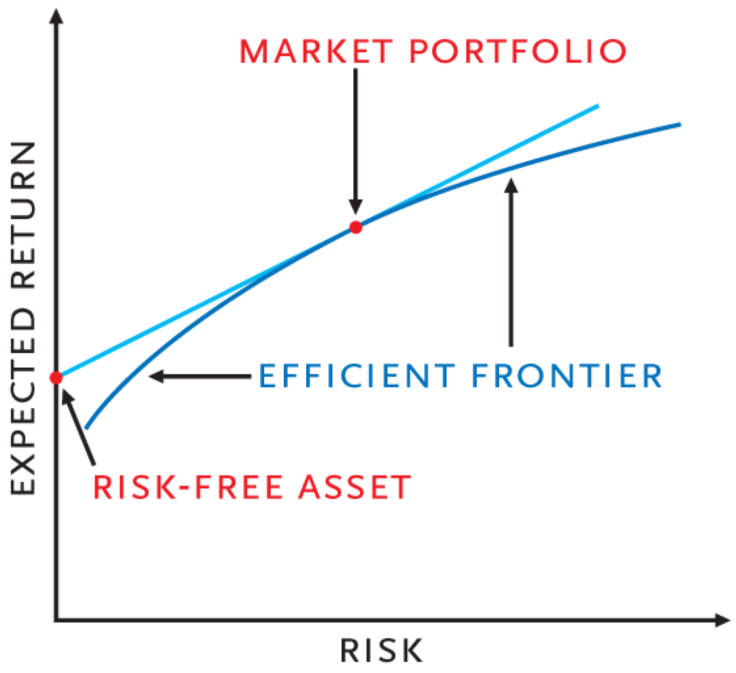
$$\text{Sharpe Ratio} = \frac{(25\% - 5\%\text{Risk Free Rate})}{20\%} = 1$$

Fund A Leveraged 2 to1, Return 25%, Vol.20%



Fund B, Return 25%, Volatility 30%



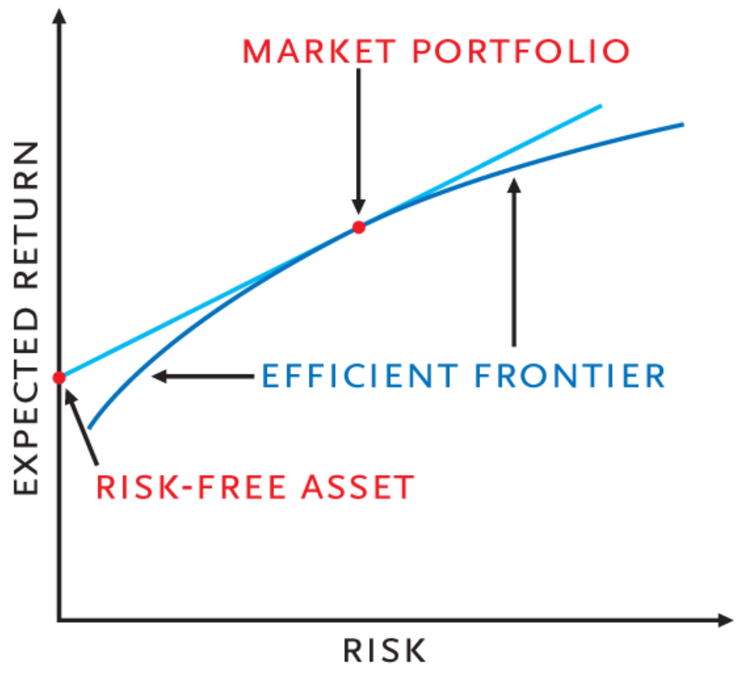


Portfolio theory models the return of an asset as a random variable and a portfolio as a weighted combination of assets.

Return of a portfolio is also a random variable and consequently has an expected value and a variance. Risk in model is usually identified with the standard deviation of portfolio return.

Rational investor choosing between several portfolios with identical expected returns, will prefer that portfolio which minimizes risk

The market portfolio

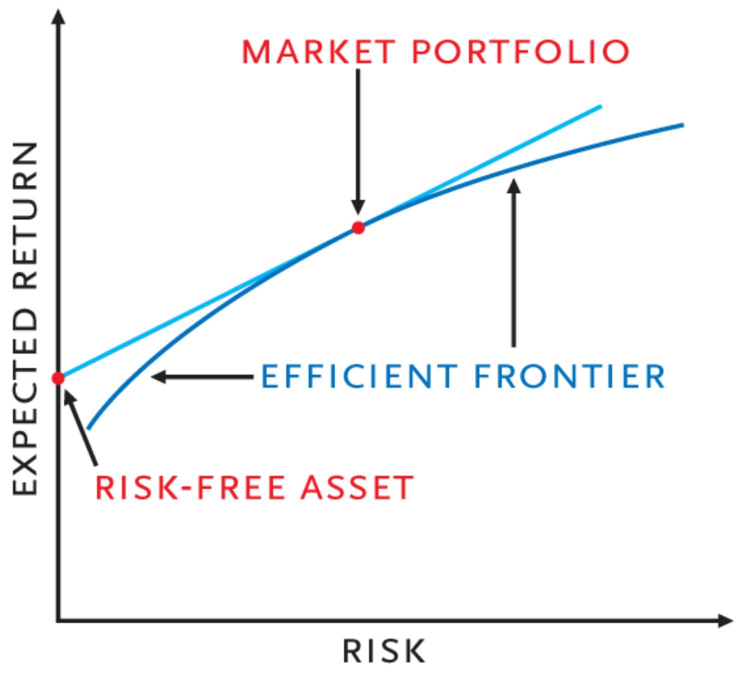


The efficient frontier is a collection of portfolios, each optimal for a given amount of risk.

Sharpe ratio represents a measure of the amount of additional return above the risk-free rate a portfolio provides compared to the risk it carries. (Sharpe ratio is the slope of the line)

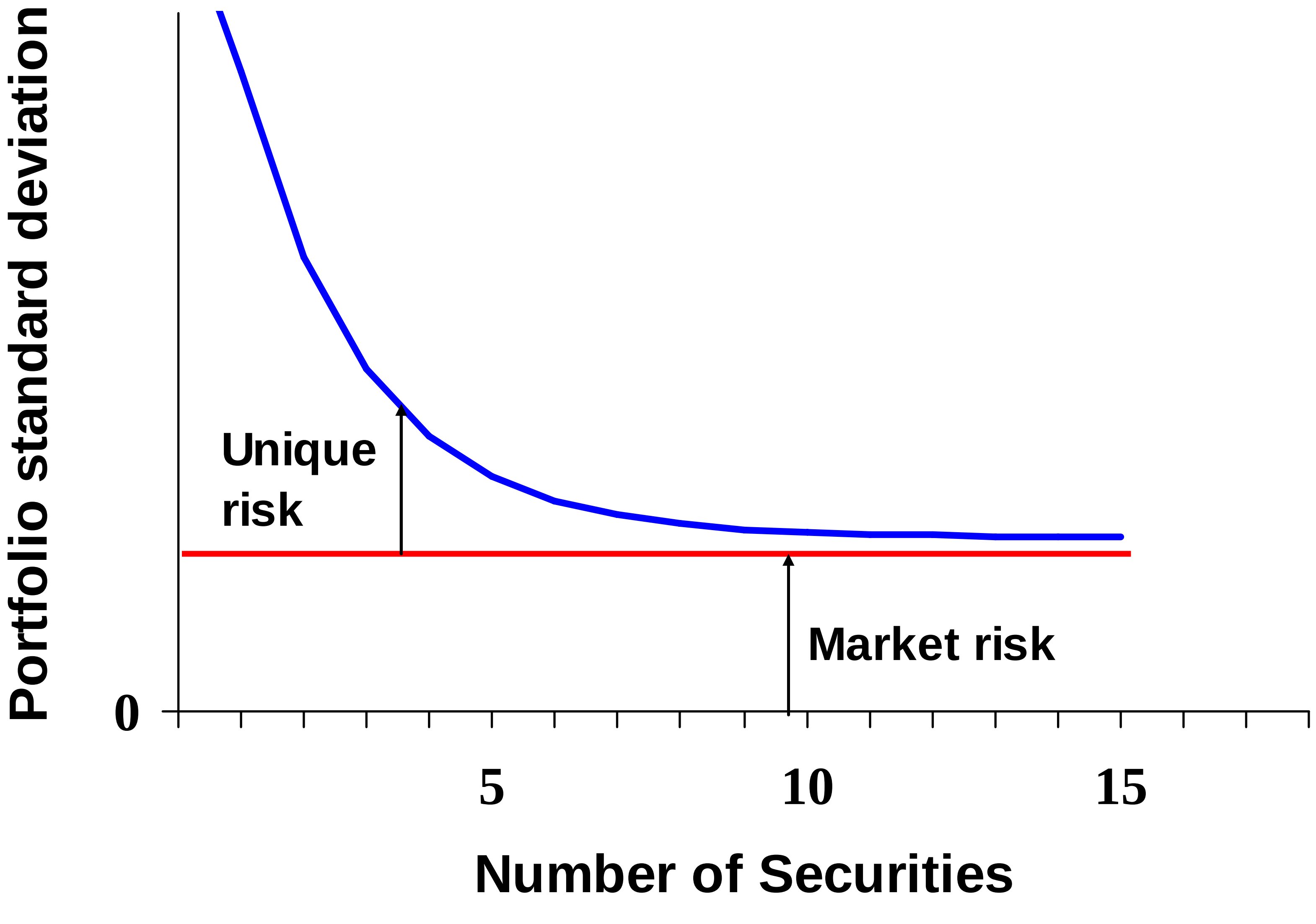
The portfolio on the efficient frontier with the highest Sharpe Ratio is known as the Market portfolio, or super-efficient portfolio. It is the tangency-portfolio in the above diagram.

This portfolio has the property that any combination of it and the risk-free asset will produce a return that is above the efficient frontier



When the market portfolio is combined with the risk-free asset, the result is the Capital Market Line. All points along the CML have superior risk-return profiles to any portfolio on the efficient frontier.

A position with zero cash weighting is on the efficient frontier –the market portfolio.



Portfolio standard deviation
Unique
risk
Market risk
0
5
10
15
Number of Securities

Firm specific risk - affecting only that firm, diversifiable risk.

Market Risk - economy-wide sources of risk that affect the overall stock market, systematic risk.

Different models of source of return and risk

One factor model with Market return as a main factor
(all returns are excess returns above risk free rate)

$$r_{it} = \alpha_i + \beta_i \cdot r_{Mt} + \varepsilon_{it}$$

Multi-Factor Models

$$r_{it} = \alpha_i + \beta_{1i} \cdot F_{1t} + \beta_{2i} \cdot F_{2t} + ... + \beta_{ni} \cdot F_{nt} + \varepsilon_{it}$$

Types of factors

Market
Fundametal P/E P/B
Industry
Macroeconomic

Methods of getting factor exposures
Z-score

Principles in choosing factors

- We want as few factors as possible with as much ability to explain returns as possible; too many factors does not simplify the model;
- We want these factors explain as much of the risk as possible, so that investors are sufficiently concerned about exposure to them.

Chen, Roll, and Ross Model

1. IP = % change in industrial production;
2. EI = % change in expected inflation;
3. UI = % change in unanticipated inflation;
4. CG = excess return of long-term corporate bonds over long-term government bonds;
5. GB = excess return of long-term government bonds over T-Bills.

- This leads to the following 5-factor model:

$$r_{it} = \alpha_i + \beta_{iIP} \cdot IP_t + \beta_{iEI} \cdot EI_t + \beta_{iUI} \cdot UI_t + \beta_{iCG} \cdot CG_t + \beta_{iGB} \cdot GB_t + \varepsilon_{it}.$$

Fama and French Three-Factor Model

- Systematic risk factors proposed by Fama and French

$$r_{it} = \alpha_i + \beta_{iM} \cdot R_{Mt} + \beta_{iSMB} \cdot SMB_t + \beta_{iHML} \cdot HML_t + \varepsilon_{it}, \text{ where:}$$

R - market index return;

SMB - Small Minus Big, i.e., the return of a portfolio of small stocks in excess of the return on a portfolio of large stocks;

HML - High Minus Low, i.e., the return of a portfolio of stocks with high book-to-market ratio in excess of the return on a portfolio of stocks with a low book-to-market ratio.

- The two variables are chosen based on observations of pricing anomalies. These variables may proxy for yet-unknown more fundamental variables.

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Classical mean-variance optimization

1.Max Return when Risk<GivenRiskLevel

2.Min Risk when Return=GivenReturnLevel

3.Max (Return-λ*Risk) usually (Return-λ*Variance)

Tradeoff risk-return with another Return, Risk function

4.Maximize Sharpe Ratio

Need model for predicted returns of assets

$\vec{R}_i$ series of i-th asset (Security) daily returns

r_i predicted return of asset for next period.

x_i is percentage of portfolio in the i-th asset.

Sum of all x_i is 1 (100%)

Each x_i is between 0 and $\mathrm{M_i}$ (maximum for i-th asset.)

Need model for Risk (Covariance Matrix). The variance is:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \rho_{ij} \sigma_i \sigma_j$$

where $x_i =$ proportion of total investment in Security i

$\rho_{ij} =$ correlation coefficient between Security i and Security j

Classical Mean-Variance Optimization I

$$\text{Optimization Problem: find } x_i \text{ such that}$$

$$1)\ \text{Return} = (x_1 r_1 + \ldots + x_n\ r_n) \to Max$$

$$\text{Risk} = \sigma < Given\ Level$$

2) Min Risk when Return=GivenReturnLevel

3) Max $\text{Return} - \lambda\sigma^2$ **where λ is risk aversion coefficient.**
Tradeoff between risk-return with another Return, Risk function

4.Maximize Sharpe Ratio

Classical Mean-Variance Optimization II

$\vec{R}_i$ series of i-th asset daily returns

r_i predicted return of asset for next period.

x_i is percentage of portfolio in the i-th asset.

Sum of all x_i is 1 (100%)

Each x_i is between 0 and $\mathrm{M_i}$ (maximum for i-th substr.)

Optimization Problem: find x_i such that

$$\text{Return} = (x_1 r_1 + \ldots + x_n\ r_n) \rightarrow Max$$

$$\text{Risk} = \text{Stdev}(x_1 \vec{R}_1 + \ldots + x_n\ \vec{R}_n) < Given\ Level$$

Non-Classical Mean - Linear Risk Measure Optimization

$\vec{R}_i$ series of i - th asset daily returns

r_i predicted return of asset for next period.

x_i is percentage of portfolio in the i - th asset.

Sum of all x_i is 1 (100%)

Each x_i is between 0 and $\mathrm{M_i}$ (maximum for i - th substr.)

Optimization Problem : find x_i such that

$$\text{Return} = (x_1 r_1 + ... + x_n\ r_n) \to Max$$

$$\text{Risk} = \text{RiskMeasure}(x_1 \vec{R}_1 + ... + x_n\ \vec{R}_n) < Given\ Level$$

Drawbacks of Classic Mean-Variance optimization

1. **Estimation of Covariance Matrix**
2. **Quadratic problem**

Alternative Mean-Linear risk Measure (for example Average of Negative Returns, Expected Shortfall etc.)

1. **Use different risk measure (average negative return, expected shortfall exceeding 60%VAR, peak to through drawdown based measures, etc)**
2. **Becomes linear problem. EASY AND FAST with good efficient frontier.**

OPTIMIZATION

Using predicted forward looking returns

Using historical daily data for risk, choosing appropriate periods.

Using risk-return function that includes several risk measures.

Creating several historical in-sample out-of-sample optimizations and comparing results with actual resuts.

Perturbing data and averaging

DRAWDOWN AS ANOTHER MEASURE OF RISK

$P(t)$ be a portfolio value at time t.

$M(t)$ be a maximal value of a portfolio prior to t.

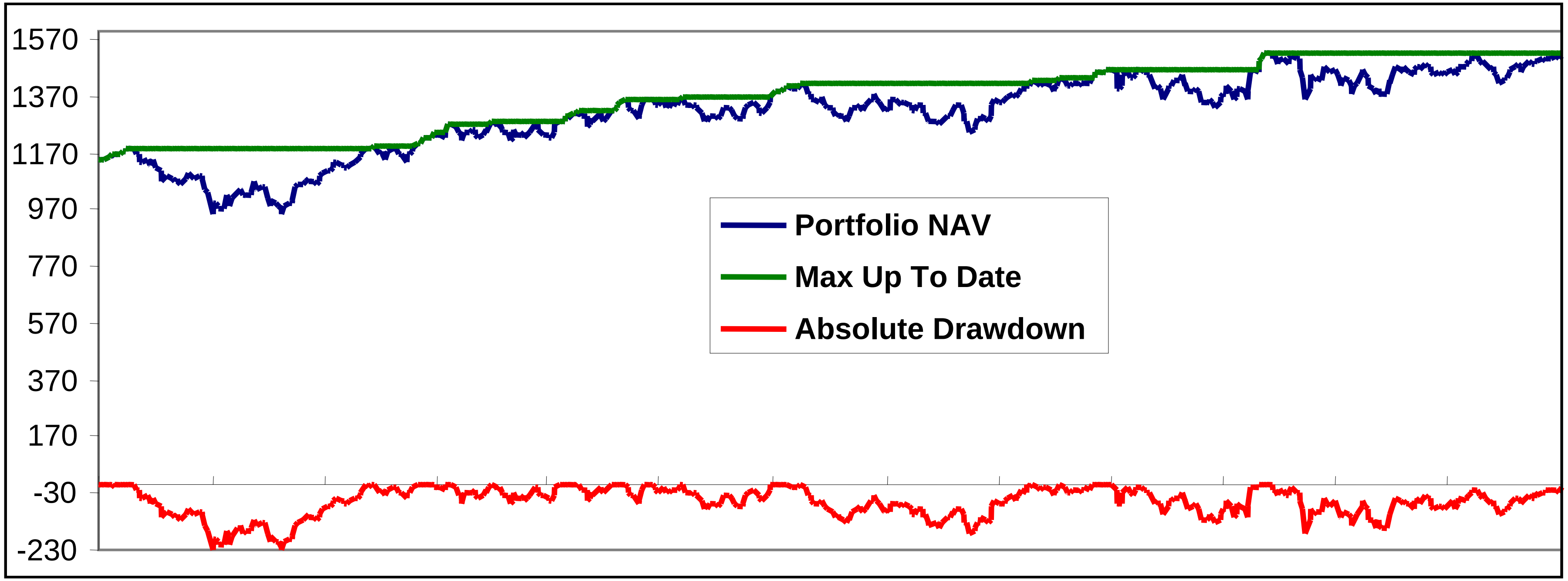
If portfolio value is below its maximal prior value, portfolio is said to be in ***drawdown.***

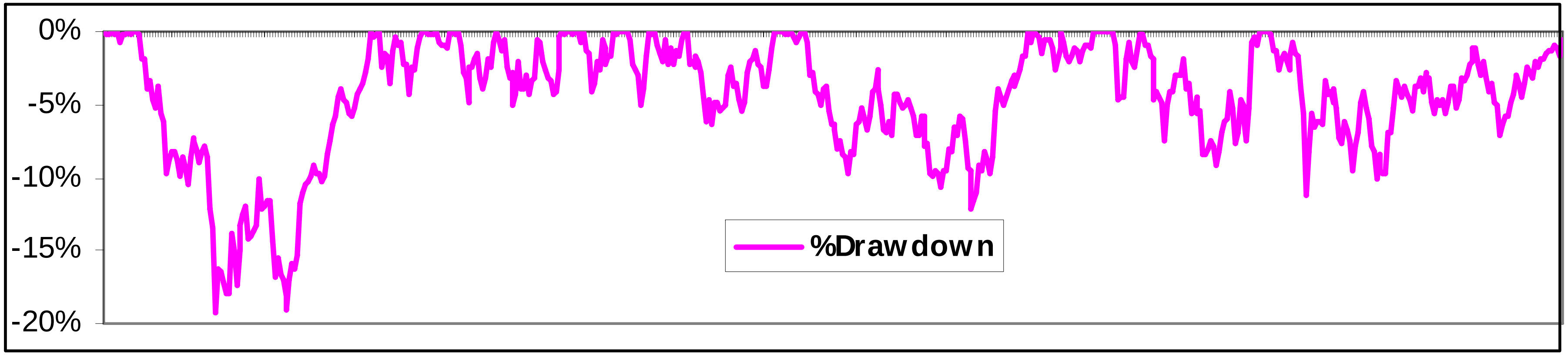
Unless portfolio is at its up to date peak it is in drawdown.

Absolute drawdown $ADD(t)=P(t)-M(t)$ where $M(t)=Max(P(s), s<t)$.

Percentage Drawdown $PDD(t)$ is absolute drawdown as percentage of $M(t)$.

Percentage and Absolute Drawdowns of a Sample Portfolio





We will take as measures of risk maximal of percentage drawdowns MaxDD and Average percentage drawdown AvDD.

Comparing to Standard deviation MaxDD captures extreme events and suited for skewed distributions.
CAPM boundary can be not smooth.

AvDD captures averages of negative events.
Averaging is smoothing Efficient Frontier Boundary.

Investment managers and Hedge Funds

- Losing client’s accounts is equivalent to death of investment management business;
- Highly unlikely to hold an account which was in a drawdown for 2 years;
- Highly unlikely to be permitted to have a 50% drawdown;
- Shutdown condition: 20% drawdown;
- Warning condition: 15% drawdown;
- Longest time to get out of a drawdown – 1-2 years.

Drawdown Optimization:

$w(\vec{x},t)$ - portfolio value at time t;

$\vec{x} = (x_1, x_2, \ldots, x_m)$ - set of unknown weights;

$f(\vec{x},t) = \max_{0 \le \tau \le t} \{ w(\vec{x},\tau) \} - w(\vec{x},t)$ - Absolute drawdown function.

ConsiderThree Measures of Risk:

•Maximum drawdown (MaxDD): $\mathbf{M}(\vec{x}) = \max_{0 \le t \le T} \{ f(\vec{x},t) \}$

•Average drawdown (AvDD): $\mathbf{A}(\vec{x}) = \dfrac{1}{T} \int_0^T f(\vec{x},t)\,dt$

•Conditional drawdown-at-risk (CDaR) CDaR is the average of the worst (1- β)*100% drawdowns.

$$\Delta_\beta(\vec{x}) = \frac{1}{(1-\beta)T} \int_{t \in [0,T],\, f(\vec{x},t) \ge \alpha(\vec{x},\beta)} f(\vec{x},t)\,dt.$$

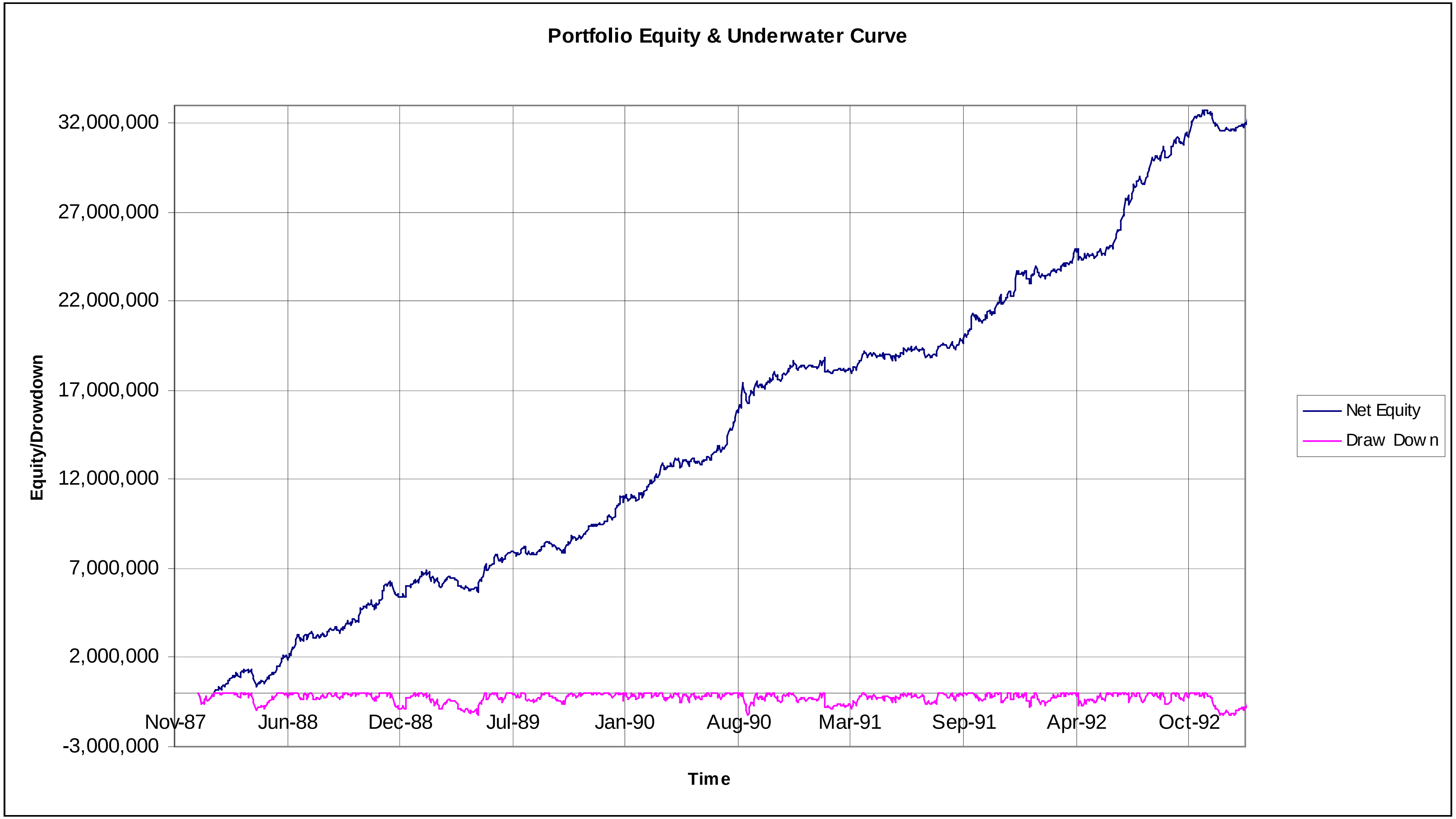
For some value of the parameter β in [0,1], the CDaR, is defined as the mean of the worst (1- β)*100% drawdowns.

For example, if β=0, then CDaR is the average drawdown, and
if β=0.95, CDaR is the average of the worst 5% drawdowns.

Let α(*x*, β) be a threshold such that (1-β)*100% of drawdowns exceed this threshold.

CDaR with level β is $$\Delta_\beta(\vec{x}) = \frac{1}{(1-\beta)T} \int_{t\in[0,T],\, f(\vec{x},t)\ge\alpha(\vec{x},\beta)} f(\vec{x},t)\,dt.$$

When β tends to 1, CDaR tends to the maximum drawdown


Portfolio Equity & Underwater Curve
Equity/Drowdown
32,000,000
27,000,000
22,000,000
17,000,000
12,000,000
7,000,000
2,000,000
-3,000,000
Nov-87
Jun-88
Dec-88
Jul-89
Jan-90
Aug-90
Mar-91
Sep-91
Apr-92
Oct-92
Time
Net Equity
Draw Down

Maximize Return Limiting the risk:

- MaxDD: $\mathbf{M}(\vec{x}) \le \nu_1 C$
- AvDD: $\mathbf{A}(\vec{x}) \le \nu_2 C$
- CDaR: $\Delta_\beta(\vec{x}) \le \nu_3 C$
- Combination: $\mathbf{M}(\vec{x}) \le \nu_1 C,\ \mathbf{A}(\vec{x}) \le \nu_2 C,\ \Delta_\beta(\vec{x}) \le \nu_3 C$

$$\textit{for some}\ \ 0 \le \nu_1, \nu_2, \nu_3 \le 1$$

Continuous Optimization Problems:

MaxDD:

$$\max_{\vec{x}} R(\vec{x})$$

subject to

$$\begin{cases} \mathbf{M}(\vec{x}) \le \nu_1 C \\ \vec{x} \in X \end{cases}$$

AvDD:

$$\max_{\vec{x}} R(\vec{x})$$

subject to

$$\begin{cases} \mathbf{A}(\vec{x}) \le \nu_2 C \\ \vec{x} \in X \end{cases}$$

CDaR:

$$\max_{\vec{x}} R(\vec{x})$$

subject to

$$\begin{cases} \Delta_\beta(\vec{x}) \le \nu_3 C \\ \vec{x} \in X \end{cases}$$

Weight constraints:

$$X = \left\{ \vec{x} : x_{\min} \le x_k \le x_{\max}, \, all \;\, k = \overline{1, m} \right\}$$

We will take as a new measure of risk CDAR and its limiting cases MaxDD and AvDD.

Comparing to Standard deviation MaxDD captures extreme negative events and suited for skewed distributions. Efficient frontier can be not smooth.

AvDD captures averages of negative events. Averaging is smoothing efficient frontier.