STOCHASTIC VOLATILITY

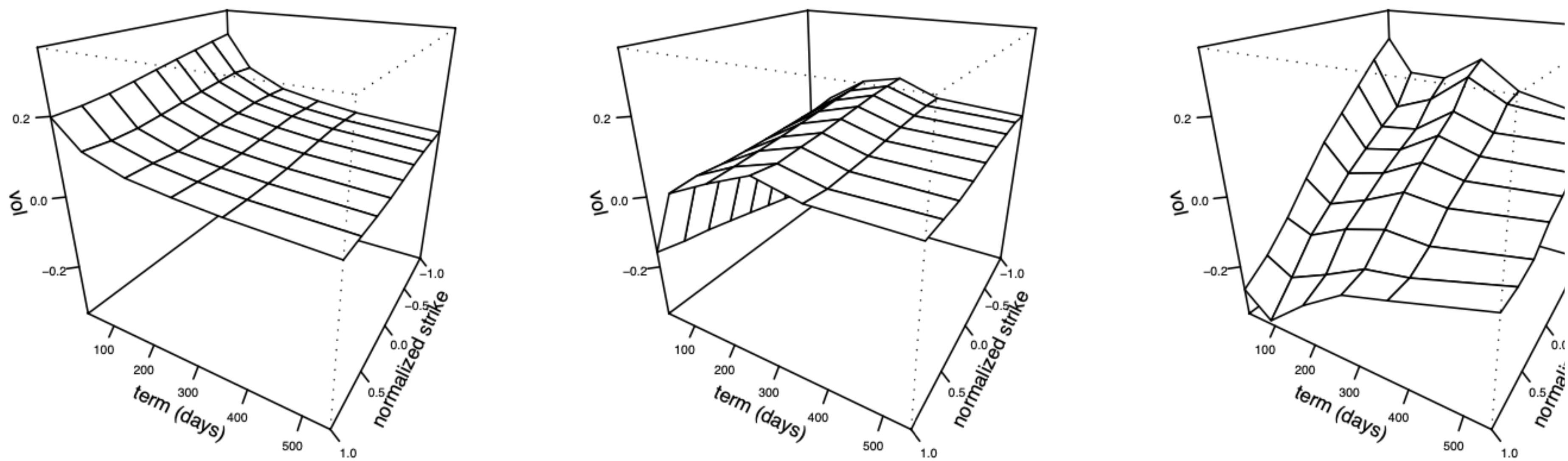
8 CLASSES LEFT:

(Weighted variance swaps)
(Programming local volatility trees)

local binomial.pdf in Matlab

PCA Modes of the Index Volatility Surface

Figure 3: PCA Modes for Merrill Lynch S&P500 Volatility Surfaces: (a) Level; (b) Term-structure; (c) Skew



(a) (b) (c)

19.2 Replicating Weighted Variance Swaps ($r = q = 0$ for simplicity)

An ordinary variance swaps pays $\frac{1}{T}\int_0^T \sigma^2(t)dt \equiv \frac{1}{T}\int_0^T \frac{(dS_t)^2}{S_t^2}$. Eq 19.1

Think of it as a $\frac{1}{S^2}$ weighting of quadratic variation $(dS)^2$ because $(dS)^2 = \sigma(t)^2 S^2(t)dt$.

Let's re-derive the standard variance swap replication using the static replication formula:

Choose some payoff $W(S)$ to replicate and apply Ito's Lemma to it to generate the weighted variance and a total derivative dW:

$$\frac{dS}{S} = \mu dt + \sigma dZ \qquad dW = \frac{\partial W}{\partial S}dS + \frac{1}{2}\frac{\partial^2 W}{\partial S^2}(dS)^2 = \frac{\partial W}{\partial S}dS + \frac{1}{2}\frac{\partial^2 W}{\partial S^2}S^2\sigma^2 dt$$

Then integrating

$$W(S_T) - W(S_0) = \int_0^T \frac{1}{2}\frac{\partial^2 W}{\partial S^2} S^2 \sigma^2 dt + \int_0^T \frac{\partial W}{\partial S} dS$$

$$\underbrace{\int_0^T \frac{1}{2}\frac{\partial^2 W}{\partial S^2} S^2 \sigma^2 dt}_{\text{swap}} = \underbrace{W(S_T) - W(S_0)}_{\text{static replicate}} - \underbrace{\int_0^T \frac{\partial W}{\partial S} dS}_{\text{dynamic rebalance}}$$

0

You can replicate a European payoff $W(S)$ at expiration time T as a function of the terminal stock price S_T by means of the decomposition

$$W(S_T) = W(A) + W'(A)[S_T - A] + \int_0^A P(S_T, K) W''(K) dK + \int_A^{\infty} C(S_T, K) W''(K) dK$$

nice if this is zero at A and choose A = S_0 for simplicity

where A is an arbitrary positive number, $P(S_T, K)$ is the terminal payoff of a standard put and $C(S_T, K)$ is the terminal payoff of a standard cal.

There are two ways to look at $W(S)$:
(i) replication of this payoff at expiration produces calls and puts with weight W'';
(ii) Ito on $W(S)$ produces the weighted variance as $W(\;\;)$ evolves.

For a regular variance swap choose $W(S) = \left(\frac{S-S_0}{S_0} - \ln\frac{S}{S_0}\right)$ and note

$$W(S_0) = 0 \qquad W'(S_0) = \frac{1}{S_0} - \frac{1}{S}\bigg|_{S = S_0} = 0 \qquad W''(S) = S^{-2}$$

$$\int_0^T \frac{1}{2}\sigma^2 dt = W(S_T) + \int_0^T \left(\frac{1}{S} - \frac{1}{S_0}\right) dS = \frac{1}{K^2}[\text{calls} + \text{puts}] + \int_0^T \left(\frac{1}{S} - \frac{1}{S_0}\right) dS$$

from the static replication of the payoff with calls and puts

We can replicate $W(S_T)$ with calls and puts with weights K^{-2} because $W''(K) = K^{-2}$ from the replication general formula, and since its value and derivative vanishes at *A,* there is no need for zero coupon bonds or forwards.

$$\cdot\frac{1}{2}\int_0^T \sigma^2 dt = \underbrace{\frac{1}{K^2}[\text{calls} + \text{puts}]}_{\text{replicate}} + \underbrace{\int_0^T \left(\frac{1}{S} - \frac{1}{S_0}\right) dS}_{\text{dynamic rebalance}}$$

The cost of replicating the variance is only the cost of the weighted puts and calls, and the rebalancing generates the variance.

Note the serendipity that the density of calls and puts is an inverse square, and the weighting of $(dS)^2$ in the definition of the variance swap in Equations 19.1 $\frac{1}{T}\int_0^T \sigma^2(t)dt \equiv \frac{1}{T}\int_0^T \frac{(dS)^2}{S^2}$ is an inverse square in S. **It's because replication of a function involves the second derivative density and Ito term relating the function to variance involves second derivative too.**

Now let's use this to look at weighted variance swaps:

To replicate $\int_0^T f(S_t)\sigma^2(t)dt \equiv \frac{1}{T}\int_0^T f(S_t)\frac{(dS_t)}{S_t^2}$ as the **weighted variance** for some desired weighting function $f(S_u)$ choose the function $W(S_u)$ to replicate such that when you apply Ito's Lemma to it you get $S^2 W''(S) \;=\;$ desired weighting $f(S_u)$.

$$dW \;=\; W'dS + \frac{1}{2}W''\sigma^2 S^2 dt$$

$$W''\sigma^2 S^2 dt \;=\; 2[dW - W'dS]$$

$$\int_0^T W''\sigma^2 S^2 dt \;=\; 2\left[W(S_T) - W(S_0) - \int_0^T\left(\frac{\partial W}{\partial S}\right)dS\right]$$

weighted variance
$W''S^2 = f$

replicate
with puts and calls

dynamic rebalancing

1. $W'' = S^{-2}$ gives standard variance capture and $W''(K) = K^{-2}$ is also weight of calls and puts.

2. $W'' = S^{-1}$ and $\int_0^T \sigma^2 S dt$ gives less variance for low S, sometimes called a gamma swap, and note that the weight of calls and puts needed is K^{-1}.

$$W(S) = \frac{S}{S_0}\ln\frac{S}{S_0} - \frac{(S-S_0)}{S_0} \qquad W(S_0) = 0$$

$$W'(S) = \frac{1}{S_0}\ln\frac{S}{S_0} \qquad W'(S_0) = 0 \qquad .$$

$$W''(S) = \frac{1}{S_0}\frac{1}{S}$$

Replicated by 1/K weighted sum of puts and calls.

3. $W'' = 1$ and $\int_0^T \sigma^2 S^2 dt$ gives even less variance for low S, sometimes called an **arithmetic variance swap**, and note that the weight of calls and puts to replicate the swap is 1.

$$W(S) = \frac{1}{2}(S-S_0)^2 \qquad W' = (S-S_0) \qquad W'' = 1$$

4. Corridor variance swap -- choose a clever W to capture variance only in a corridor of the stock price by using indicator functions to constrain it. Maybe later …

Stochastic Volatility

19.3 Introduction to Stochastic Volatility

Approaches to Stochastic Volatility Modeling

We want to find the BS implied volatilities that correspond to a stochastic volatility option value.

Several approaches to introducing a stochastic volatility:

• Allow the instantaneous stock volatility σ itself to be truly stochastic:

(i) **Start with Black-Scholes, no skew**, and obtain a skew by making σ a stochastic volatility correlated with changes in the stock price.

(ii) **Start with local volatility** $\sigma = \sigma(S, t)$ and a skew, *and then add uncorrelated volatility* to that skew. (SABR-type models)

(iii) **Stochastic local volatility**. Stochastic vol for realistic behavior of volatility, then add non-parametric local vol to calibrate to the current skew.

• BGM-type market models. Let the Black-Scholes implied volatilities $\Sigma(K, t)$ be stochastic. There are then strong constraints on the evolution of the B-S implied volatilities in order to avoid arbitrage. (Analogous to letting zero-coupon yields vary independently.) Not used much.

Comment: Modeling stochastic volatility is much more complex than modeling local volatility. We will develop models and study the character of the solutions and their smile.

Our elliptical path:

For risk management with a stochastic volatility model, we will see that it's hard to understand really well the connection between **observed changes in implied volatility** Σ and **changes in the stochastic parameters** σ, ξ... **of the model**. In the local volatility model, in contrast, there was a simple (averaging) relation and the Dupire equation. So we'll explore from several angles.

1. Heuristic vanna-volga approach to skew by perturbing the Black-Scholes solution with a stochastic volatility.
2. The rigorous SDEs for stochastic volatility and stochastic stock price
3. Mean reversion of volatility
4. Another approach: Stochastic Local Volatility. The parametric SABR model that begins with a local volatility $\sigma = \sigma(S)$ and then adds stochasticity to the evolution of the local volatility
5. Risk-neutral valuation: The riskless hedge and the resultant PDE for the value of the option
6. The Hull-White expected discounted value solution to stochastic volatility when the correlation is zero
7. Monte Carlo solutions to the PDE more generally
8. Semi-analytic solutions and the asymptotic properties of the smile with stochastic volatility.
9. Hedging in a stochastic volatility model

19.4 A HEURISTIC VANNA-VOLGA/TRADER APPROACH FOR INTRODUCING STOCHASTIC VOLATILITY INTO THE BLACK-SCHOLES MODEL

Assume rates and dividends are zero for simplicity. Add stochastic volatility to BS:

$$\begin{aligned} dC &= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dS d\sigma \\ &= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dS d\sigma \\ &= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \cancel{\frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma} + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dS d\sigma \end{aligned}$$

Now suppose that we constructed a riskless hedge $\pi = C - \alpha S - \beta C'$ that is long the call C and short just enough stock and enough other options dependent on volatility σ so that the hedged portfolio is instantaneously riskless. Then all dS and $d\sigma$ terms in $d\pi$ will vanish.

change in vega as stock moves

The resultant changes in the call price are

$$dC = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dS d\sigma$$

We don't know the value of the partial derivatives in the above equation, since we haven't applied the methods of risk-neutral valuation to determine the partial differential equation for the value of the option with both stochastic volatility and stochastic stock price.

In order to proceed further we will replace the unknown partial derivatives by their values in the Black-Scholes model, C_{BSM}, hoping that these capture the approximate contribution to the P&L from the stochastic volatility.

Then for zero rates and dividends

$$\frac{\partial C_{\mathrm{BSM}}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{\mathrm{BSM}}}{\partial S^2}\sigma^2 S^2 = 0$$

The **expected change** in the value of the hedged portfolio from stochastic volatility over and above Black-Scholes is approximately

$$dC \cong \frac{1}{2}\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2} E\left[d\sigma^2\right] + \frac{\partial^2 C_{\mathrm{BSM}}}{\partial S \partial \sigma} E\left[dS d\sigma\right]$$

volga, vol butterfly spread | vanna risk reversal

Approximate by using the BS derivatives of $C_{BS}(S, t, K, T, r, \sigma)$ in the Ito expansion

$$V = \frac{\partial C_{\mathrm{BSM}}}{\partial \sigma} = \frac{\sqrt{\tau}}{\sqrt{2\pi}} S e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right)^2}$$ Vega is always positive

Vega decreases rapidly as ln(S/K) gets more negative or more positive.

$$\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2} = \frac{V}{\sigma}\left[\frac{\ln^2\left(\frac{S}{K}\right)}{\sigma^2 \tau} - \frac{\sigma^2 \tau}{4}\right]$$

Volga is mostly positive except atm

Thus Volga has two peaks

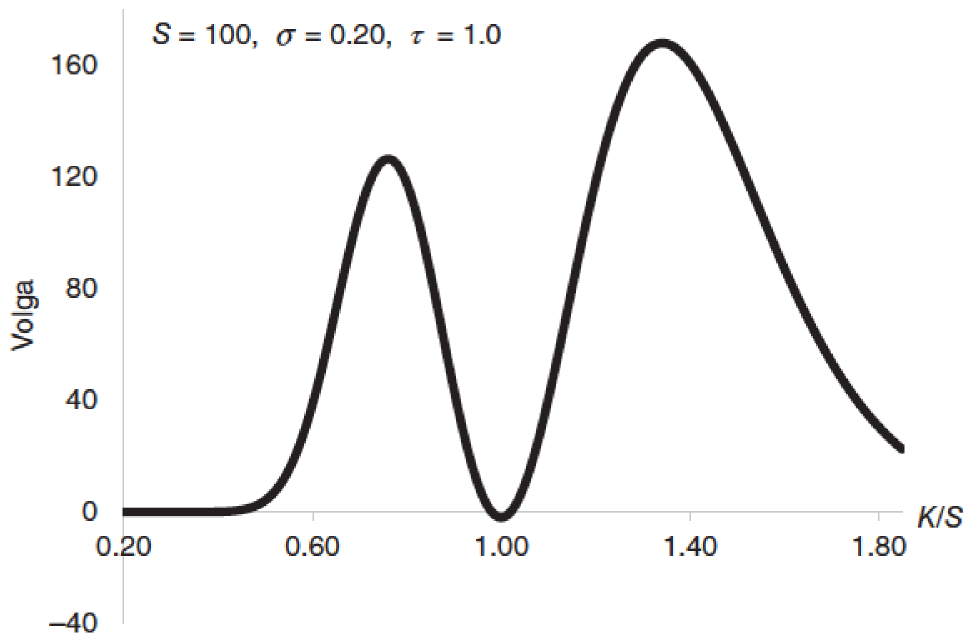


FIGURE 19.1 BSM Volga of a Standard Call Option

A BS option is long volatility because Gamma is positive and so moves in the stock price in either direction are good for you, it's convex in *S*. .Similarly is it long volatility of volatility because Volga is positive and moves in volatility in either direction are good for you, it's convex in σ.

Mostly positive convexity, with peaks on either side

$$dC = \frac{1}{2}\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2} E\left[d\sigma^2\right] + \frac{\partial^2 C_{\mathrm{BSM}}}{\partial S \partial \sigma} E\left[dS d\sigma\right]$$

A hedged option is long gamma $\frac{\partial^2 C}{\partial S^2}$, long volatility $\frac{\partial C}{\partial \sigma}$ and **long volatility of volatility** $\frac{\partial^2 C}{\partial \sigma^2}$, esp out of money or deep in the money. Check that C_{BSM}(high vol) + C_{BSM}(low vol) > C_{BSM}(av vol) because the second derivative w.r.t volatility is positive -- convexity in volatility.

If volatility is volatile, then the convexity in volatility adds value to the option away from at-the-money, and adds value to out-of-the-money options relative to at-the-money options in a U-shaped smile.

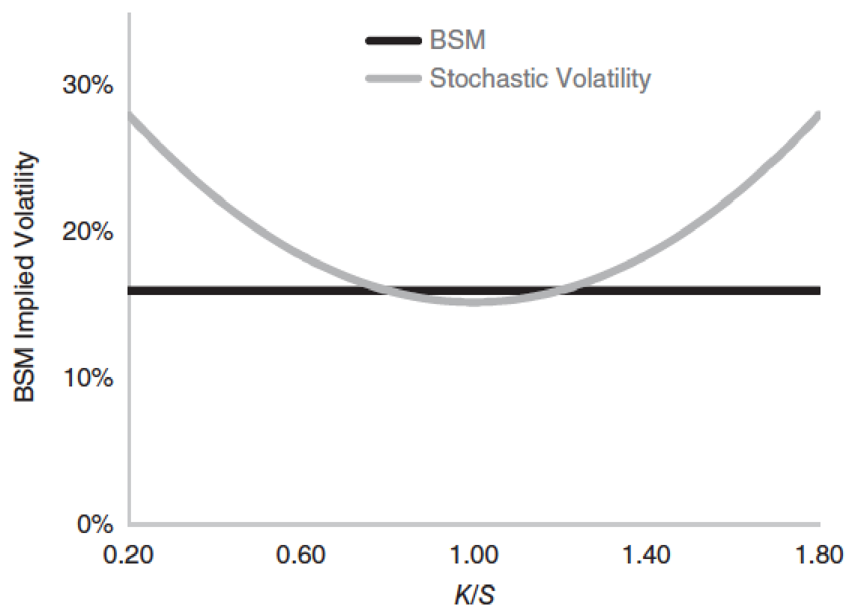


FIGURE 19.2 BSM Implied Volatility versus Moneyness

Similarly, Vanna

$$\frac{\partial^2 C_{BSM}}{\partial S \partial \sigma} = \frac{V}{S}\left(\frac{1}{2} - \frac{1}{\sigma^2 \tau}\ln\left(\frac{S}{K}\right)\right)$$

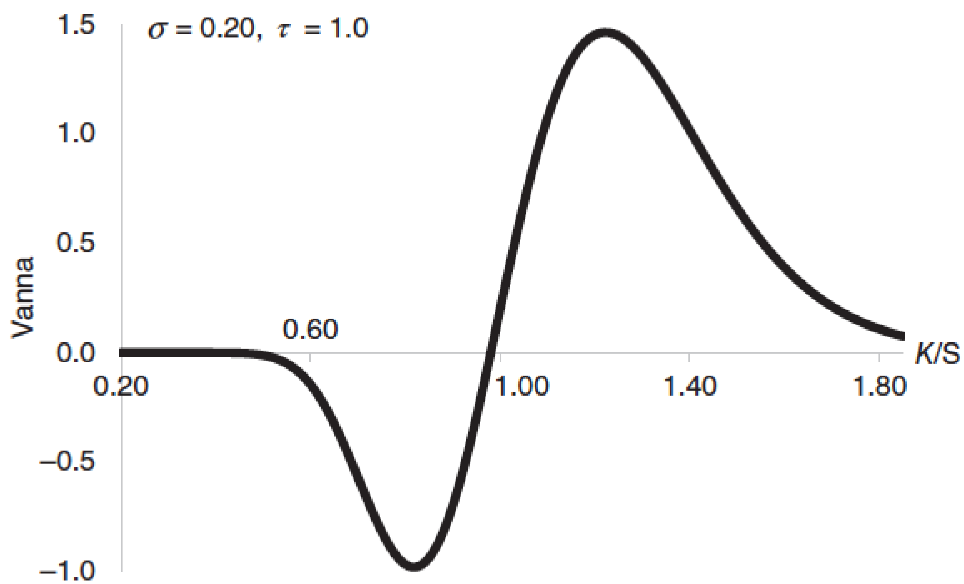


FIGURE 19.3 BSM Vanna, DvegaDspot, or Ddelta/Dsigma

This adds value to one out-of-the-money wing of the implied volatility and subtracts from the opposite one. For typical values of σ and τ, vanna will be positive when the call option is out of the money ($K > S$) and negative when the call option is in the money ($K < S$). If E[dSdσ] is positive (if the stock price and its volatility are positively correlated), the vanna term will enhance the P&L and hence value of a Black-Scholes option at high strikes and reduce it at low strikes. The opposite is the case if the correlation is negative. Since the equity index skew is typically negative, with low strikes carrying greater implied volatility than high ones, we can guess that in a stochastic volatility model we will require a negative correlation between the index and its volatility in order to reflect the skew.

extra terms from stochastic vol - we'll estimate them from BS below

Crude usefully intuitive ways to understand the effect of stochastic volatility on the smile.

19.1 Aside: Vanna Volga Models for Exotics in a Skew. A Trick

One good reference is Castagna and Mercurio, Risk Magazine 2007: *The Vanna Volga Method for Implied Volatilities*. You can find it on the internet.

For risk management with a stochastic volatility model, we will see that it's hard to understand really well the connection between **observed changes in implied volatility** Σ and **changes in the stochastic parameters** σ, ξ... **of the model**. In the local volatility model, in contrast, there was a simple (averaging) relation and the Dupire equation.

The Vanna Volga method for exotics tries to heuristically cut out the model parameters and deal directly with the implied volatilities. In its simplest form, it uses the Black-Scholes values for three different strikes as "control variates" for a stochastic volatility world to then estimate all other values for different strikes relative to them. Given the price of three different strike options of a given expiration, it lets us value any other option in terms of them. Here is a sketch of a proof. P is an exotic option with stochastic volatility, e.g. a barrier or average option.

$$dP = \frac{\partial P}{\partial t} + dS\frac{\partial P}{\partial S} + d\sigma\frac{\partial P}{\partial \sigma} + \frac{1}{2}\frac{\partial^2 P}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 P}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 P}{\partial S\partial \sigma}dSd\sigma$$

We first go into a BS world with no stochastic volatility where we can value the exotic option as $P_{BS}(T;\sigma)$ in a GBM BS world, where σ is the flat implied volatility of all options with expiration T. Then, knowing its value, we find three combinations of vanilla options $C_{BS}(K_i, T;\sigma)$, for three strikes K_i and the same expiration, so that the exotic has the same Vega, Volga, Vanna as the combination:

$$\frac{\partial}{\partial\sigma}P_{BS}(T;\sigma) = \sum_{i=1}^{3} w_i\left[\frac{\partial}{\partial\sigma}C_{BS}(K_i, T;\sigma)\right]$$

$$\frac{\partial^2}{\partial\sigma^2}P_{BS}(T;\sigma) = \sum_{i=1}^{3} w_i\left[\frac{\partial^2}{\partial\sigma^2}C_{BS}(K_i, T;\sigma)\right] \quad \sigma \text{ is atm volatility of vanilla options w maturity } T$$

$$\frac{\partial^2}{\partial\sigma\partial S}P_{BS}(T;\sigma) = \sum_{i=1}^{3} w_i\left[\frac{\partial^2}{\partial\sigma\partial S}C_{BS}(K_i, T;\sigma)\right]$$

To fix the weights w_i of the replicating portfolio, we require that the Vega, Volga and Vanna of the exotic option in a BS world, which we know, equals the vega, volga and vanna of the three replicating options at the current time and stock price. This gives three equations for three unknowns in a BS world.

Now we turn on the skew and depart from the "flat-smile" BS world so that the three $C_{BS}(K_i, T;\sigma)$ each move away from their flat implied volatility σ to an implied volatility taken from the current skew. We "hope" that to first order the volatility derivatives in the BS worlds tell us how much the price will change as the implied volatility changes. The new exotic option price is estimated by adding to the "flat-smile" BS exotic price the price difference between market price and BS theory price for each vanilla option.

$$P_{VannaVolga}(\ ) = P_{BS}(\ ) + \sum_{i=1}^{3} w_i[C_{MKT}(K_i) - C_{BS}(K_i, \sigma)]$$

When $P(\ )$ is a vanilla option with one of the three strikes K_i, $w_i = 1$ for just that option, and we reproduce the market price of the three vanilla options. This lets us estimate the price and hence the implied vol in a smile of any exotic option, or vanilla option with a different strike, from the implied vols of three strikes, by a rule for extrapolation and interpolation.

This formula lets us estimate how the observed change in one option's implied volatility will affect the value of an option of any strike, and in particular the value of an exotic option such as a barrier option.

We are using the familiar BS for three strikes as a control variate.

We'll cover this again towards the end of the stochastic volatility section.

19.5 Next: Extending Black-Scholes to Stochastic Volatility: The Stochastic Differential Equation for Volatility

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dZ_t$$

$$d\sigma = \text{several possibilities discussed below}$$

The Hull-White (1987) stochastic volatility model with GBM

$$\frac{dV}{V} = \alpha dt + \xi dW \text{ where } V = \sigma^2$$

ξ is the volatility of variance; typical fluctuations of volatility can be very large. How large?

Realized and implied volatilities, like interest rates and credit spreads, are parameters rather than prices **and are range-bound**. For example, between 2005 and 2015 the 30-day realized volatility of the S&P 500 lay between 5% and 82%.

We therefore want to model volatility or variance as a mean-reverting variable, else the range will keep increasing with the square root of the time elapsed.

Stochastic Mean Reversion and its Qualities

Ornstein-Uhlenbeck models:

$$dY = \alpha(m - Y)dt + \beta dW \qquad \text{Ornstein Uhlenbeck}$$

First let's solve it for $\beta = 0$ with zero volatility.

$$dY = \alpha(m - Y)dt \qquad Y(t) = m + (Y_0 - m)e^{-\alpha t}$$

When Y is large, dY is negative, and vice versa. That's mean reversion via the drift.

As t gets large, the initial position Y_0 becomes irrelevant.

$$Y(t) - m = (Y_0 - m)e^{-\alpha t}$$

$$Y(0) - m = (Y_0 - m)$$

Half life $t_{1/2}$:

half life is when displacement from m halves

$$Y(t_{1/2}) - m \equiv \frac{1}{2}(Y_0 - m) = (Y_0 - m)e^{-\alpha t_{1/2}}$$

$$t_{1/2} = \frac{\ln(2)}{\alpha}$$

greater alpha, shorter half life

The half life of volatility seems to be of the order of weeks, perhaps months sometimes. And volatility in reality tends to jump up suddenly, which we're not modeling here. That takes more.

For non-zero volatility this is the solution:

$$Y(t) = m + (Y_0 - m)e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)} dW_s \qquad \text{Eq.19.1}$$

The contribution of random previous moves to the long-term value of Y(t) damps out exponentially.

This solution satisfies the stochastic differential equation, as shown below.

$$\begin{aligned} dY(t) &= -\alpha(Y_0 - m)e^{-\alpha t} + \beta dW_t - \beta\alpha \int_0^t e^{-\alpha(t-s)} dW_s \\ &= -\alpha\left[Y(t) - m - \cancel{\beta \int_0^t e^{-\alpha(t-s)} dW_s} \right] + \beta dW_t \cancel{- \beta\alpha \int_0^t e^{-\alpha(t-s)} dW_s} \\ &= \alpha[m - Y(t)] + \beta dW_t \end{aligned}$$

The cross-sectional mean $\overline{Y(t)}$ of Y(t) at time *t*, averaged over all increments dW_s.

$$\overline{Y(t)} = m + (Y_0 - m)e^{-\alpha t}$$

so that the average displacement at time *t* is just the deterministic one.

We can find the variance of the displacements at time *t* by making use of the fact that the dW_s are independent:

$$E[dW_s dW_u] = du ds \delta(u - s)$$

So

$$\begin{aligned}
E[Y(t) - \overline{Y(t)}]^2 &= \beta^2 \int_0^t \int_0^t e^{-\alpha(t-s)} e^{-\alpha(t-u)} E[dW_s dW_u] \\
&= \beta^2 \int_0^t \int_0^t e^{-2\alpha t} e^{\alpha(s+u)} du ds \delta(u-s) \\
&= \beta^2 \int_0^t ds e^{-2\alpha t} e^{2\alpha s} = \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t})
\end{aligned}$$

For small times t, variance behaves like $\beta^2 t$, which is like standard Brownian motion.

As $t \to \infty$ the limiting variance is $\lim_{t\to\infty} \mathrm{Var}[Y_t] = \frac{\beta^2}{2\alpha}$.

As α gets larger, the variance gets smaller. Here is a rough sketch of the distribution of the process over time.

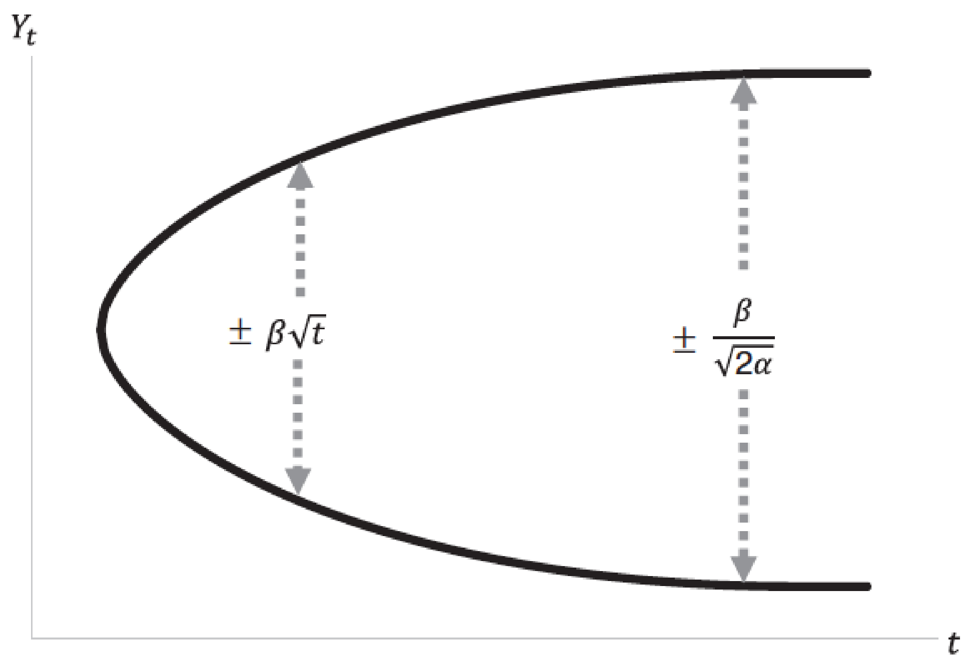


FIGURE 19.4 Schematic Illustration of the Standard Deviation of Y_t

At time $t \approx 1/(2\alpha)$ the variance grows no larger. In contrast, for regular Brownian motion, the linear dependence of the variance on t continues for all time.

Rough Sketch: $E[dW_s dW_u] \equiv \delta(u-s)duds$

Approximating by the discrete sum:

$$E\left[\left(\int_{T_1}^{T_2} a(t)dW(t)\right)^2\right] \approx E\left[\sum a_i dW_i \sum a_j dW_j\right] = E\left[\sum_{i,j} a_i a_j dW_i dW_j\right]$$

$$= \sum_{i,j} a_i a_j E[dW_i dW_j] = \sum_{i,j} a_i a_j \delta_{ij} dt = \sum_i a_i^2 dt$$

$$\approx \int_{T_1}^{T_2} a(t)^2 dt$$

This is equivalent to $E[dW_s dW_u] \equiv \delta(u-s)duds$:

$$E\left[\left(\int_{T_1}^{T_2} a(t)dW(t)\right)^2\right] \approx E\left[\int_{T_1}^{T_2}\int_{T_1}^{T_2} a(u)a(s)dW(u)dW(s)\right]$$

$$= \int_{T_1}^{T_2}\int_{T_1}^{T_2} a(u)a(s)E[dW(u)dW(s)] = \int_{T_1}^{T_2}\int_{T_1}^{T_2} a(u)a(s)\delta(u-s)duds$$

$$= \int_{T_1}^{T_2} a(u)^2 du$$

19.6 SABR - A Particular Case of Stochastic Local Vol

Add a stochastic element to a local volatility model.

$$\frac{dS}{S} = \alpha S^{\beta - 1} dW \equiv \sigma(S) dW \qquad \sigma(S) = \frac{\alpha}{S^{1-\beta}}$$

SABR model

$$d\alpha = \xi \alpha dZ$$
$$dZdW = \rho dt$$

Assume $\rho = 0$ since we already have a skew, and β close to 1 with $1 - \beta > 0$ (**small skew**). Then estimate the skew using our knowledge of local volatility.

One can solve the model analytically but we'll work perturbatively by assuming vol of vol is small, that we are close to at the money, and that skew slope is small, just to get an idea of what happens.

Estimation Strategy:

1. Implied Volatility is Approximate Average of Local Volatilities
2. For Stochastic Implied Volatility, Option Value is Approximate Average of Option Prices with different volatility

So:

For $\xi = 0$, pure local volatility, we know that the implied volatility is roughly the average of the local volatilities from S to K:

$$\Sigma_{\mathrm{LV}}(S,t,K,T,\alpha,\beta) \approx \frac{1}{2}\left(\alpha S^{\beta-1} + \alpha K^{\beta-1}\right)$$
$$\approx \alpha S^{\beta-1}\frac{1}{2}\left[1 + \left(\frac{K}{S}\right)^{\beta-1}\right]$$

Taylor expansion in K for β close to 1:

$$\left(\frac{K}{S}\right)^{\beta-1} = e^{(\beta-1)\ln\left(\frac{K}{S}\right)} \approx 1 + (\beta-1)\ln\left(\frac{K}{S}\right)$$

So

$$\Sigma_{\mathrm{LV}}(S,t,K,T,\alpha,\beta) \approx \alpha S^{\beta-1}\frac{1}{2}\left[1 + 1 + \ln\left(\frac{K}{S}\right)(\beta-1)\right]$$
$$\approx \frac{\alpha}{S^{1-\beta}}\left[1 - \frac{(1-\beta)}{2}\ln\left(\frac{K}{S}\right)\right]$$

So for zero volatility of volatility we have a linear skew with negative slope, $\frac{\partial\Sigma}{\partial K} \approx \frac{\partial\Sigma}{\partial S}$ **for at-the-money options K = S**, as is true in local vol.

Now switch on the stochastic volatility $\xi \neq 0$ for small ξ. There is a range of possible α values. The skew becomes stochastic and the implied volatility Σ_{LV} has a range of random values.

Estimate C_{SLV} in this Stochastic Local Vol model as average of the BS prices over the range of α with prob density $f(\alpha)$ and $\int f(\alpha)d\alpha = 1$:

$$C_{\mathrm{SLV}} \approx \int C_{\mathrm{BSM}}(\Sigma_{\mathrm{LV}}(S,t,K,T,\alpha,\beta))f(\alpha)d\alpha$$

Taylor expand this about the mean $\bar{\alpha}$ for small volatility of volatility:

$$\begin{aligned}
C_{\mathrm{SLV}} &= \int C_{\mathrm{BSM}}(\Sigma_{\mathrm{LV}}(S,t,K,T,\bar{\alpha}+(\alpha-\bar{\alpha}),\beta))f(\alpha)d\alpha \\
&\approx \int \left[C_{\mathrm{BSM}}(\Sigma_{\mathrm{LV}}(S,t,K,T,\bar{\alpha},\beta)) + \left.\frac{\partial C_{\mathrm{BSM}}}{\partial \alpha}\right|_{\bar{\alpha}} (\alpha-\bar{\alpha}) \right. \\
&\quad \left. + \frac{1}{2}\left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \alpha^2}\right|_{\bar{\alpha}} (\alpha-\bar{\alpha})^2 \right] f(\alpha)d\alpha \\
&\approx C_{\mathrm{BSM}}(\bar{\alpha}) + \frac{1}{2}\left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \alpha^2}\right|_{\bar{\alpha}} \mathrm{var}(\alpha)
\end{aligned}$$

The implied volatility Σ_{SLV} in this model is given by $C_{SLV} \equiv C_{BSM}(\Sigma_{SLV})$.

Because the volatility of volatility has been assumed to be small, α stays close to $\bar{\alpha}$ and thus Σ_{SLV} should not differ by much from $\Sigma_{LV}(S, t, K, T, \bar{\alpha}, \beta) \equiv \Sigma_{LV}(\bar{\alpha})$ evaluated at $\bar{\alpha}$.

Because the volatility of volatility is small, we write

$$\Sigma_{SLV} \equiv \Sigma_{LV}(\bar{\alpha}) + (\Sigma_{SLV} - \Sigma_{LV}(\bar{\alpha}))$$

and then

$$\begin{aligned} C_{SLV} &= C_{BSM}(\Sigma_{LV}(\bar{\alpha}) + (\Sigma_{SLV} - \Sigma_{LV}(\bar{\alpha}))) \\ &\approx C_{BSM}(\bar{\alpha}) + \frac{\partial C_{BSM}}{\partial \Sigma_{LV}}(\Sigma_{SLV} - \Sigma_{LV}(\bar{\alpha})) \end{aligned}$$

Comparing the above two equations for C_{SLV}, we obtain

$$\Sigma_{SLV} \approx \Sigma_{LV}(\bar{\alpha}) + \frac{\frac{1}{2} \left. \frac{\partial^2 C_{BSM}}{\partial \alpha^2} \right|_{\bar{\alpha}} \mathrm{var}(\alpha)}{\frac{\partial C_{BSM}}{\partial \Sigma_{LV}}}$$

Evaluate the BS derivatives above **for small total variance** $\Sigma^2\tau$ **and close to at-the-money** S = K

$$\Sigma_{\mathrm{LV}}(\bar{\alpha}) \approx \bar{\alpha}/S^{1-\beta}.$$

Then from the chain rule

$$\begin{aligned} \left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \alpha^2}\right|_{\bar{\alpha}} &\approx \left(\frac{1}{S^{1-\beta}}\right)^2 \left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2}\right|_{\sigma=\Sigma_{\mathrm{LV}}} \\ &\approx \left(\frac{\Sigma_{\mathrm{LV}}}{\bar{\alpha}}\right)^2 \left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2}\right|_{\sigma=\Sigma_{\mathrm{LV}}} \end{aligned}$$

Furthermore in the SABR model α undergoes GBM with variance $\mathrm{var}(\alpha) \approx \bar{\alpha}^2 \xi^2 \tau$. So

$$\frac{\frac{1}{2}\left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \alpha^2}\right|_{\bar{\alpha}} \mathrm{var}(\alpha)}{\frac{\partial C_{\mathrm{BSM}}}{\partial \Sigma_{\mathrm{LV}}}} \approx \frac{1}{2}\left[\left(\frac{\Sigma_{\mathrm{LV}}}{\bar{\alpha}}\right)^2 \left.\frac{\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2}}{\frac{\partial C_{\mathrm{BSM}}}{\partial \sigma}}\right|_{\sigma=\Sigma_{\mathrm{LV}}} (\bar{\alpha}\xi)^2 \tau\right]$$

$$\approx \frac{1}{2}\Sigma_{\mathrm{LV}}^2 \boxed{\left.\frac{\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2}}{\frac{\partial C_{\mathrm{BSM}}}{\partial \sigma}}\right|_{\sigma=\Sigma_{\mathrm{LV}}}} \xi^2 \tau \qquad \text{(B)}$$

$$V = \frac{\partial C_{\mathrm{BSM}}}{\partial \sigma} = \frac{\sqrt{\tau}}{\sqrt{2\pi}} S e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right)^2}$$

$$\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2} = \frac{V}{\sigma}\left[\frac{\ln^2\left(\frac{S}{K}\right)}{\sigma^2 \tau} - \frac{\sigma^2 \tau}{4}\right]$$

Using our volga and vanna formulas for small total variance $\sigma^2 \tau$ and close to at-the-money

$$\boxed{\frac{\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2}}{\frac{\partial C_{\mathrm{BSM}}}{\partial \sigma}}} = \frac{1}{\sigma}\left[\frac{1}{\sigma^2 \tau}\left(\ln\left(\frac{S}{K}\right)\right)^2 - \frac{\sigma^2 \tau}{4}\right] \approx \frac{1}{\sigma^3 \tau}\left(\ln\left(\frac{S}{K}\right)\right)^2$$

So from (B) above, setting $\sigma = \Sigma_{LV}$

$$\frac{1}{2}\frac{\left.\frac{\partial^2 C_{\mathrm{BSM}}}{\partial\alpha^2}\right|_{\bar{\alpha}} \mathrm{var}(\alpha)}{\frac{\partial C_{\mathrm{BSM}}}{\partial\Sigma_{\mathrm{LV}}}} \approx \frac{1}{2}\frac{\xi^2}{\Sigma_{\mathrm{LV}}(\bar{\alpha})}\left(\ln\left(\frac{S}{K}\right)\right)^2$$

and substituting this into Equation (A) we get

$$\Sigma_{SLV} \approx \Sigma_{LV}(\bar{\alpha})\left\{1 + \frac{1}{2}\left[\frac{\xi}{\Sigma_{LV}(\bar{\alpha})}\right]^2\left[\ln\frac{S}{K}\right]^2\right\} \qquad \text{Eq.19.2}$$

The local volatility smile $\Sigma_{LV}(\bar{\alpha})$ is altered by the addition of a quadratic term in $\ln\frac{S}{K}$

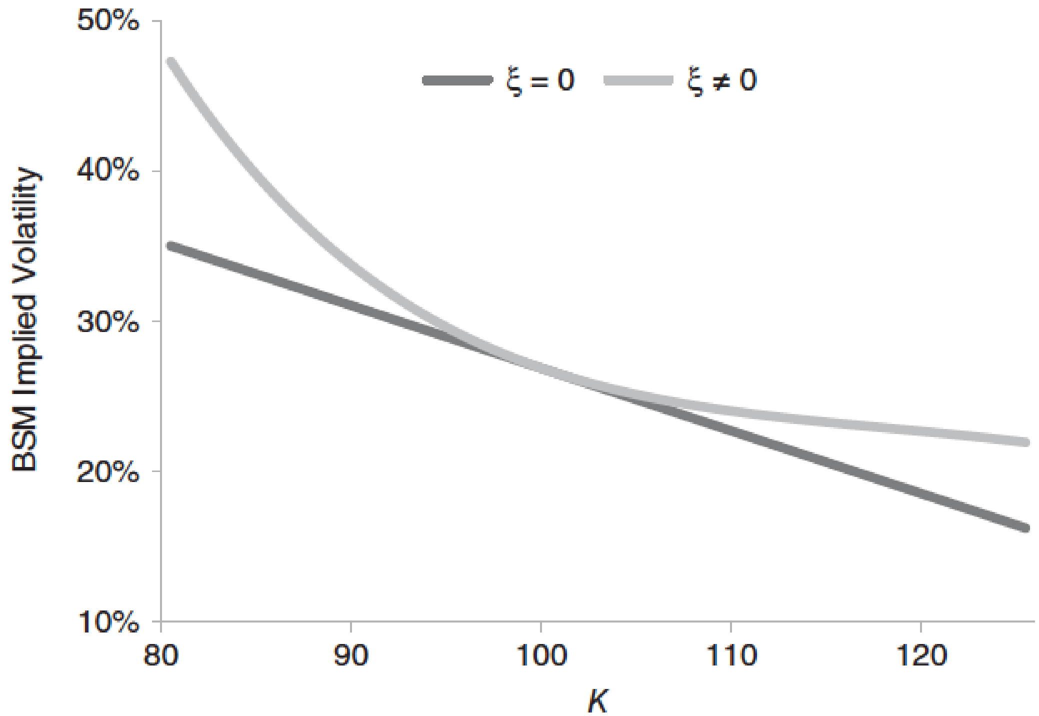


FIGURE 20.1 The Impact of Stochastic Volatility of Volatility on the Smile in the SABR Model

No need for correlation between volatility and stock price in order to obtain a smile if we start from local volatility.