Math Methods – Financial Price Analysis

Mathematics GR5360

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Is There an Order in Constant π?

- π is a mathematical constant equal to the ***ratio of a circle's circumference to its diameter***, approximately (as a double) equal to 3.141592653589793.
- π is an ***irrational*** number: cannot be expressed exactly as a common fraction. Therefore, π decimal representation is infinitely long and never settles into a permanent repeating pattern.
- "***Feynman point***" is a sequence of decimal digits of π that starts at spot 762 and contains six 9's in a row.

3.14159265358979323846264338327950288419716939937510582097494459230781640628620899
8628034825342117067982148086513282306647093844609550582231725359408128481117450284
1027019385211055596446229489549303819644288109756659334461284756482337867831652712
0190914564856692346034861045432664821339360726024914127372458700660631558817488152
0920962829254091715364367892590360011330530548820466521384146951941511609433057270
3657595919530921861173819326117931051185480744623799627495673518857527248912279381
8301194912983367336244065664308602139494639522473719070217986094370277053921717629
3176752384674818467669405132000568127145263560827785771342757789609173637178721468
4409012249534301465495853710507922796892589235420199561121290219608640344181598136
2977477130996051870721134999999837297804995105973173281609631859502445945534690830
2642522308253344685035261931188171010003137838752886587533208381420617177669147303
5982534904287554687311595628638823537875937519577818577805321712268066130019278766
1119590921642019893809525720106548586327886593615338182796823030195203530185296899
5773622599413891249721775283479131515574857242454150695950829533116861727855889075
0983817546374649393192550604009277016711390098488240128583616035637076601047101819
4295559619894676783744944825537977472684710404753464620804668425906946912...

Is There an Order in Constant π?

- π ***digits*** seem to be ***randomly distributed***, although up until now no rigorous proof of this was discovered.
- π is a ***transcendental*** number, that is not a root of any non-zero polynomial having rational coefficients – it is impossible to square the circle with a compass and straight-edge.
- Currently over 10^{13} digits of π were computed. For all practical scientific applications ***40 digits of π*** or less ***are sufficient***. Humans were able to memorize up to 67,000 digits.
- Digits of π do not seem to have any apparent order or pattern. A number of infinite length is called ***normal*** when all possible sequences of digits of any given length appear equally often.
- Multiple series and integral ***representations of π*** are known.

$$\text{Gregory - Leibniz series}: \pi = 4 \cdot \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} = 4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots\right)$$

$$\text{Gaussian integral}: \pi = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx \right)^2 .$$

Is There an Order in Constant π?

- Using the file "*Pi1.25Million.txt*" of the first 1.25 Million digits of π we will count sequences of same digits of different lengths (1,2,3,4,5) and compare those frequencies to those generate totally randomly.
- At first look, we seem to see signs of "order" in π.

Total Count
1,254,540

Count of Sequences of Digits in Pi

	0	1	2	3	4	5	6	7	8	9	Random
1	125,505	125,083	125,594	125,793	125,372	125,880	124,796	125,452	125,376	125,689	125,454.0
2	12,514	12,596	12,698	12,642	12,405	12,876	12,249	12,407	12,685	12,656	12,545.4
3	1,201	1,282	1,268	1,265	1,174	1,318	1,164	1,256	1,239	1,273	1,254.5
4	108	141	126	132	106	142	121	136	124	144	125.5
5	6	19	11	14	6	27	16	17	10	14	12.5

Difference of Count of Sequences of Digits in Pi vs. Random

	0	1	2	3	4	5	6	7	8	9
1	51.0	-371.0	140.0	339.0	-82.0	426.0	-658.0	-2.0	-78.0	235.0
2	-31.4	50.6	152.6	96.6	-140.4	330.6	-296.4	-138.4	139.6	110.6
3	-53.5	27.5	13.5	10.5	-80.5	63.5	-90.5	1.5	-15.5	18.5
4	-17.5	15.5	0.5	6.5	-19.5	16.5	-4.5	10.5	-1.5	18.5
5	-6.5	6.5	-1.5	1.5	-6.5	14.5	3.5	4.5	-2.5	1.5

Relative Excess of Count of Sequences of Digits in Pi vs. Random

	0	1	2	3	4	5	6	7	8	9
1	0.04%	-0.30%	0.1%	0.27%	-0.1%	0.34%	-0.52%	0.0%	-0.1%	0.2%
2	-0.25%	0.4%	1.2%	0.8%	-1.1%	2.6%	-2.4%	-1.1%	1.1%	0.9%
3	-4.27%	2.2%	1.1%	0.8%	-6.4%	5.1%	-7.2%	0.1%	-1.2%	1.5%
4	-13.91%	12.4%	0.4%	5.2%	-15.5%	13.2%	-3.6%	8.4%	-1.2%	14.8%
5	-52.17%	51.4%	-12.3%	11.6%	-52.2%	115.2%	27.5%	35.5%	-20.3%	11.6%

Is There an Order in Constant π?

- A more careful analysis shows that this seeming “order” is nothing more than statistical sample noise.

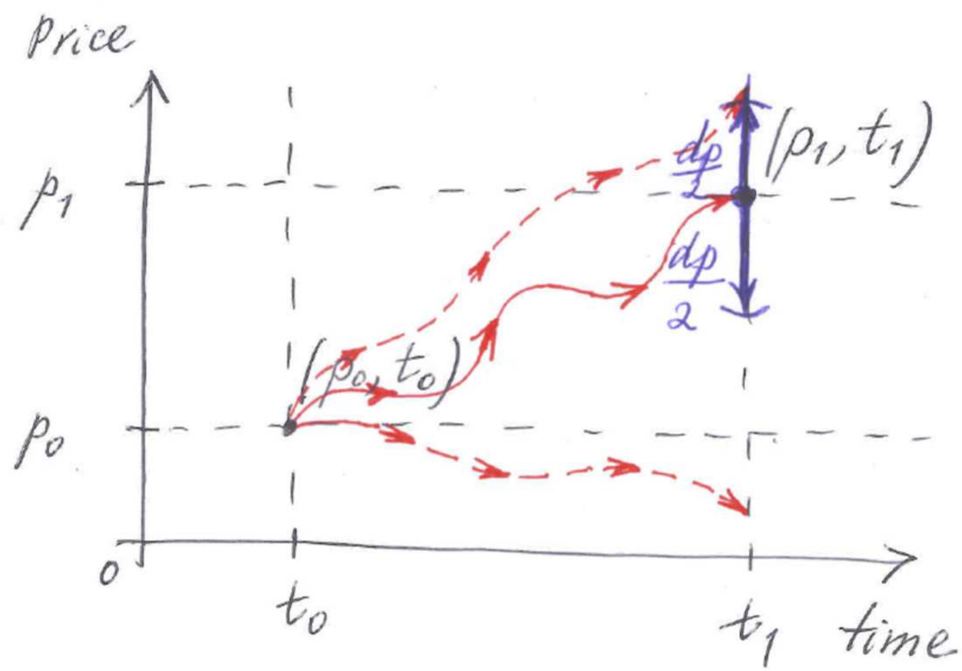
Chi-Squared

	Random	0	1	2	3	4	5	6	7	8	9
7.6	125,454.0	0.020733	1.097143	0.156233	0.916041	0.053597	1.446554	3.451177	3.19E-05	0.048496	0.440201
24.2	12,545.4	0.078591	0.204088	1.856199	0.743823	1.571266	8.712067	7.002803	1.526819	1.553411	0.975047
18.5	1,254.5	2.284926	0.601058	0.144413	0.087213	5.170574	3.210078	6.534261	0.001699	0.192494	0.271631
13.7	125.5	2.428317	1.926428	0.002376	0.34156	3.016708	2.182235	0.158131	0.886525	0.016852	2.741675
30.4	12.5	3.414978	3.320887	0.190369	0.168656	3.414978	16.65435	0.951286	1.581732	0.516449	0.168656
21.7	at 1% significance										
27.8	at 0.1%										

Elements of Statistics Relevant to Price Analysis*

- Two-point Probability Density Function

Let $p = p(t)$ be a random price process and we will assume that we can repeat the "experiment" from a slightly different "seed" as many times as necessary, thus allowing us to generate realizations. For any starting time t_0, ending time t_1 and the target size dp we can run such "experiment" N times and calculate the number of trajectories hitting the target/semi-interval $\left[p_1 - \frac{dp}{2}, p_1 + \frac{dp}{2}\right)$ to be: $N(p_0, t_0; p_1, t_1; dp)$. The normalization condition: $\sum_{p_1} N(p_0, t_0; p_1, t_1; dp) = N$ simply means that all possible final targets/semi-intervals are densly covering all real-valued p_1 and do not overlap.



* - with some changes from “Statistical Hydrodynamics” by Monin and Yaglom, ref. B6.

Elements of Statistics Relevant to Price Analysis

It is intuitevely clear that for larger dp's the $N(p_0,t_0;p_1,t_1;dp)$ will be proportionately larger, therefore:

$$\frac{N(p_0,t_0;p_1,t_1;dp)}{N}\cdot\frac{1}{dp}\rightarrow \text{finite limit for } N\rightarrow+\infty, \text{ namely} =$$

$$= P(p_0,t_0;p_1,t_1).$$

In other words, $P(p_0,t_0;p_1,t_1)\cdot dp =$ probability to find a price trajectory starting at (p_0,t_0) ending within the semi-interval $\left[p_1-\frac{dp}{2},p_1+\frac{dp}{2}\right)$. In the continuous limit, the normalization condition is: $\int_{-\infty}^{+\infty}P(p_0,t_0;p_1,t_1)dp_1=1$.

$P(p_0,t_0;p_1,t_1)$ is called a two-point probability density fucntion (PDF).

Elements of Statistics Relevant to Price Analysis

For a stationary process:

$P(p_0, t_0; p_1, t_1) \equiv \tilde{P}(p_1 - p_0, t_1 - t_0) \equiv \tilde{P}(\Delta p, \Delta t)$ and

introducing zero-mean price changes or fluctuations

$x = \Delta p - \overline{\Delta p}$ and time change $\tau = \Delta t$, we get:

$\bar{x} = \int_{-\infty}^{+\infty} x \cdot P(x,\tau) \cdot dx$ is the mean or the 1st-order moment or

structure function,

$\overline{x^2} = \int_{-\infty}^{+\infty} x^2 \cdot P(x,\tau) \cdot dx$ is the 2nd-order moment or

structure function.

Generalization to all positive (real-valued) orders

for a symmetric $P(x,\tau) = P(-x,\tau)$ leads to ν-th order

moment or structure function:

$$\overline{|x|^\nu} = \int_{-\infty}^{+\infty} |x|^\nu \cdot P(x,\tau) \cdot dx = 2 \cdot \int_{0}^{+\infty} x^\nu \cdot P(x,\tau) \cdot dx.$$

Auto-correlation function for a stationary process $x(t)$ is:

$C(\tau) \equiv \overline{x(t) \cdot x(t+\tau)}$.

Elements of Statistics Relevant to Price Analysis

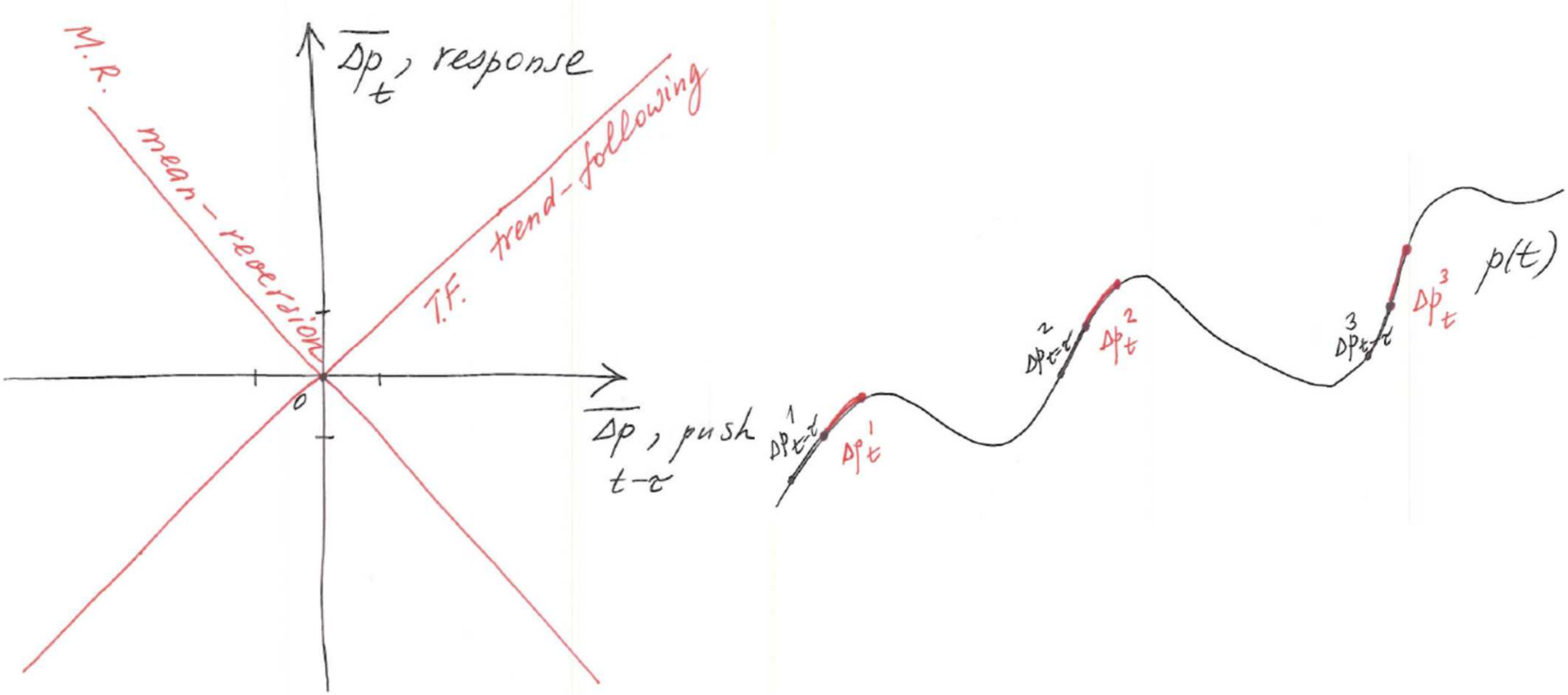
In the context of the current course, what does it mean, "to study a financial price process $p(t)$"?
By that we will mean to statistically study its fluctuations $x = \Delta p - \overline{\Delta p}$ by at least experimentally aswering the following questions:

1. What is (the shape of) the PDF of x : $P(x,\tau)$?
2. What are the structure functions : $S_\nu(\tau) = \overline{|\mathrm{x}|^\nu}$, in particular, what is the variance of fluctuations, $S_2(\tau) = \overline{\mathrm{x}^2}$?
3. What is the auto-correlation function $C(\tau)$?
4. What are the conditional response functions : $\mathrm{R} = \overline{\left(\Delta p|_{condition}\right)}$?

Push-Response Functions

A particular case of Conditional response function $R = \overline{\left(\Delta p|_{condition}\right)}$ is a Push - Response function. For a particular choice of time shift τ ($=1$, for example), for every "push" $x = p(t) - p(t-\tau)$, given a financial time - series $p(t)$, we can find a conditional average "response"

$$y = \overline{\left(p(t+\tau) - p(t)\right)}\Big|_{p(t)-p(t-\tau)=x}.$$



More on Push-Response Functions*

Let x - be a price change "push" over Δt_1, y - a price change "response" over Δt_2, where Δt_1 and Δt_2 are two non-overlapping joined time-intervals, then $P(x, y)$ is a bi-variate PDF.

With a particular case of $\Delta t_1 = \Delta t_2 = \tau$ (measured in, say, minutes), we have the following probabilistic shape of response for a given push:

$$P(x|y) = \frac{P(x, y)}{P(x)}.$$

One can de-compose bi-variate PDF into a symmetric and asymmetric parts: $P(x, y) = P^s(x, y) + P^a(x, y)$, where:

$$P^s(x, y) = \frac{P(x, y) + P(x, -y)}{2} \text{ and } P^a(x, y) = \frac{P(x, y) - P(x, -y)}{2}.$$

Then we have for the conditional mean response:

$$\overline{y}\Big|_x = \int_{-\infty}^{+\infty} y \cdot P(x|y) \cdot dy = \int_{-\infty}^{+\infty} y \cdot P^a(x|y) \cdot dy.$$

* - from papers by V. Trainin et. al., refs. A43-46.

More on Push-Response Functions*

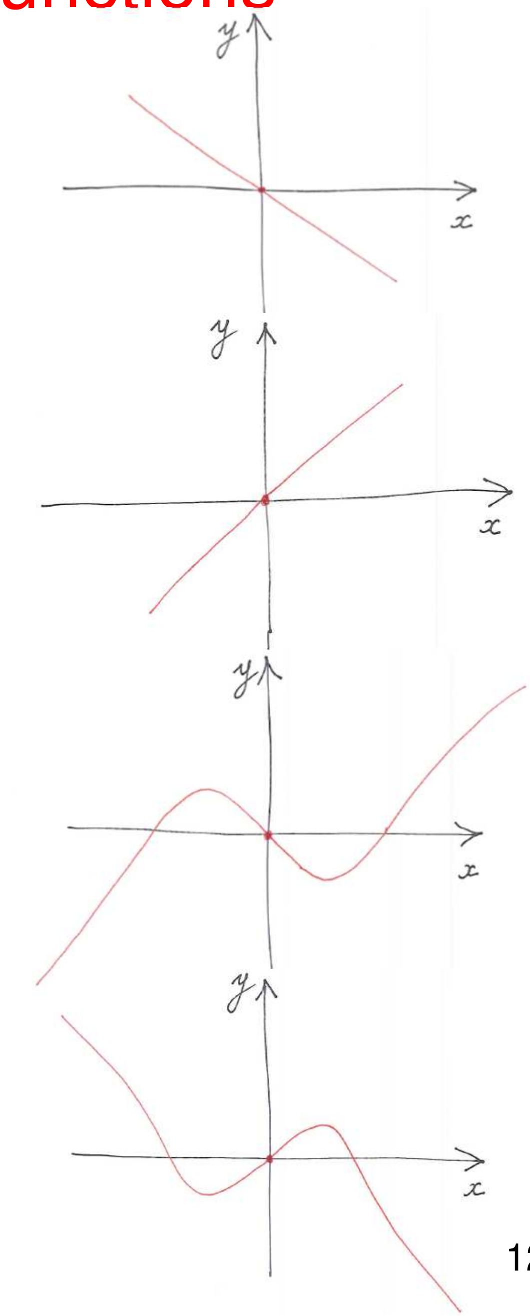
Using this de-composition, for a price-series with predictabilities we can identify the following four basic shapes of push-response diagrams:

1. mean-reversion

2. trend-following

3. short-term mean-reversion and long-term trend-following

4. short-term trend-following and long-term mean-reversion



Naturally, more complex shapes are possible, as well as the price-series may have no predictabilities (Random Walk).

* - from papers by V. Trainin et. al., refs. A43-46.

Fourier Expansion and Fourier Transform*

Harmonic analysis, or representation of functions by Fourier series or integrals is widely used in mathematical physics:

- For ordinary (non-random) functions, representation by Fourier series is only possible for periodic functions;
- For ordinary (non-random) non-periodic functions representation by Fourier integrals is only possible for functions which decay to 0 "fast enough" at $\pm\infty$;
- For random functions a Fourier expansion is possible for any stationary random process which, be definition, do not decay to 0 at $\pm\infty$ and are not periodic.

Let $u(t)$ be a stationary random process with $\overline{u(t)} = 0$ (otherwise we can re-define $u(t) \to u(t) - \overline{u(t)}$).

* - from Monin & Yaglom, vol. II, ref. B6.

Fourier Expansion and Fourier Transform*

Consider, for generality, a random complex function $u(t)$ to be represented as follows:

$$u(t) = \sum_{k=1}^{n} Z_k \cdot e^{i\omega_k t},$$

where a set $\omega_1,...,\omega_n$ are given numbers and $Z_1,...,Z_n$ are complex random variables such that:

$$\overline{Z_k} = 0, \overline{Z_k^* Z_l} = 0 \text{ for all } k \neq l.$$

A representation of stationary random process as a superposition of components of a given functional form with random and mutually uncorrelated coefficients is possible under some very general conditions.

* - from Monin & Yaglom, vol. II, ref. B6.

Fourier Expansion and Fourier Transform*

Using the above, we get for the correlation function:

$$B(t_1,t_2)=\overline{u^*(t_1)\cdot u(t_2)}=\sum_{k=1}^{n}F_k\cdot e^{i\omega_k(t_2-t_1)},\ \text{where}$$

we re-defined $F_k=\overline{\left|Z_k\right|^2}\geq 0.$

We see that the correlation function depends only on (t_2-t_1)

as it should for a stationary random process.

If, additionally, $\mathrm{Z_k}$ are having Gaussian probability distributions,

all moments and probability distrubution for $u(t)$ will also

only depend only on (t_2-t_1).

For real-valued processes $u(t)$ the number of terms is even $n=2m$ and

$$u(t)=\sum_{k=1}^{m}\left(Z_k^1\cos\left(\omega_k t\right)+Z_k^2\sin\left(\omega_k t\right)\right)=\sum_{k=1}^{m}W_k\cdot\cos(\omega_k t-\varphi_k),\ \text{where}$$

$$Z_k^1=Z_k+Z_k^*,\ Z_k^2=i(Z_k-Z_k^*),\ W_k=2\left|Z_k\right|,\ \varphi_\mathrm{k}=arctg\left(\frac{Z_k^2}{Z_k^1}\right).$$

* - from Monin & Yaglom, vol. II, ref. B6.

Fourier Expansion and Fourier Transform*

We also have:

$$\overline{Z_k^i Z_l^j} = \delta_{ij}\delta_{kl}E_k\,,\ E_k = \frac{\overline{W_k^2}}{2}.$$

Therefore, it is clear that a real-valued random process is a superposition of uncorrelated harmonic oscillations with random amplitudes and phases.

Its correlation function is given by:

$$B(\tau) = \sum_{k=1}^{m} E_k \cdot \cos(\omega_k \tau).$$

It depends on the mean squares of aplitudes W_k and does not depend on the statistical characteristics of phases φ_{k}.

* - from Monin & Yaglom, vol. II, ref. B6.

Fourier Expansion and Fourier Transform*

An arbitrary stationary random process $u(t)$ also has a spectral representation similar to above in the continuous limit of $n \to +\infty$. For that we will need to assume that frequencies $\omega_1, \ldots, \omega_n$ can approach each other without limit, such that the sum of amplitudes Z_k remains finite.

If we denote $Z(\omega) = \sum_{\omega_k < \omega} Z_k$, a random complex function such that

$\overline{Z(\omega)} = 0$ and for $\omega^1 < \omega^2 \leq \omega^3 < \omega^4$:

$$\overline{\left[Z^*(\omega^2) - Z^*(\omega^1)\right]\left[Z(\omega^4) - Z(\omega^3)\right]} = 0$$

because of properties of Z_k described before.

This can be re-written in the differential form as :

$\overline{dZ^*(\omega) \cdot dZ(\omega_1)}$ when $\omega \neq \omega_1$.

* - from Monin & Yaglom, vol. II, ref. B6.

Fourier Expansion and Fourier Transform*

Taking such limit $n \to +\infty$ will transform the Fourier series expansion into:

$$u(t) = \lim_{\Omega \to +\infty} \left\{ \lim_{\omega_{k+1} - \omega_k \to 0+} \sum_{k=0}^{n-1} \left[Z(\omega_{k+1}) - Z(\omega_k) \right] \cdot e^{i\omega_k' t} \right\},$$

where: $-\Omega = \omega_0 < \omega_1 < \ldots < \omega_n = \Omega$ and $\omega_k < \omega_k' < \omega_{k+1}$

and the limits are understood as mean-square limits:

$$W = \lim_{n \to +\infty} W_n \text{ if } \lim_{n \to +\infty} \overline{\left| W - W_n \right|^2} = 0.$$

Such limits are improper Stieltes integrals symbolically written as:

$$u(t) = \int_{-\infty}^{\infty} e^{i\omega t} \cdot dZ(\omega),$$

which is the general Fourier integral representation of a stationary process $u(t)$ first derived by Kolmogorov.

* - from Monin & Yaglom, vol. II, ref. B6.

Fourier Expansion and Fourier Transform*

From that, the following inverse Fourier transform can be obtained:

$$Z(\omega) = \lim_{T \to +\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-i\omega t} - 1}{-it} u(t)dt + const,$$

so that:

$$Z(\omega_2) - Z(\omega_1) = Z([\omega_1, \omega_2]) = \lim_{T \to +\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-i\omega_2 t} - e^{-i\omega_1 t}}{-it} u(t)dt.$$

From this follows that: if $u(t)$ is Gaussian, then $Z(\omega)$ will be Gaussian, if $u(t)$ is real-valued process, then the spectral representation can be re-written through real-valued Fourier transform.

The real physical significance to the Fourier transform is in the fact that individual short spectral ranges can be isolated experimentally by means of properly chosen filters:

$$u(\Delta\omega, t) = \int_{\omega_1}^{\omega_2} e^{i\omega t} dZ(\omega) + \int_{-\omega_2}^{-\omega_1} e^{i\omega t} dZ(\omega) = 2 \cdot \mathrm{Re}\left\{ \int_{\Delta\omega} e^{i\omega t} dZ(\omega) \right\}.$$

This expression gives the spectral component of the process $u(t)$ corresponding to the frequency interval $\Delta\omega$.

* - from Monin & Yaglom, vol. II, ref. B6.

Fourier Expansion and Fourier Transform*

The fact that spectral component $u(\Delta\omega, t)$ can be isolated experimentally gives real significance to the frequency distribution of the (mean) energy of the process $u(t)$.

In simple physics applications, the energy of a process $u(t)$ (velocity) is usually proportional to $|u(t)|^2$ (kinetic energy).

From that follows that, for a stationary random process $u(t)$, the quantity $B(0) = \overline{|u(t)|^2}$ plays a role of the mean energy.

Then, the mean energy corresponding to the harmonic oscillations with frequencies in the range $\Delta\omega = [\omega_1, \omega_2]$ is given by:

$$\overline{[\mathrm{u}(\Delta\omega, \mathrm{t})]^2} = \overline{|Z(\Delta\omega)|^2} + \overline{|Z(-\Delta\omega)|^2} = 2 \cdot \overline{|Z(\Delta\omega)|^2},$$

where $Z(\Delta\omega) = Z([\omega_1, \omega_2]) = Z(\omega_2) - Z(\omega_1)$.

* - from Monin & Yaglom, vol. II, ref. B6.

Fourier Expansion and Fourier Transform*

We thus see that non - random non - negative function $\overline{|Z(\Delta\omega)|^2}$ describes the distribution of the energy of the process $u(t)$ over the frequency ω spectrum.

We can introduce the spectral density function (spectrum) $F(\omega)$ of the process $u(t)$:

$$\overline{|Z(\Delta\omega)|^2} = \int_{\Delta\omega} F(\omega)d\omega.$$

The energy spectrum $E(\omega) = 2 \cdot F(\omega)$ is often used for $0 \le \omega < +\infty$:

$$\overline{|dZ(\omega)|^2} = F(\omega)d\omega = \frac{1}{2}E(\omega)d\omega.$$

For the above to be true, the following symbolic equation should be correct:

$$\overline{dZ^*(\omega) \cdot dZ(\omega_1)} = \delta(\omega - \omega_1) \cdot F(\omega) \cdot d\omega \cdot d\omega_1 =$$

$= \frac{1}{2}\delta(\omega - \omega_1) \cdot E(\omega) \cdot d\omega \cdot d\omega_1$, where δ is a Dirac delta function.

* - from Monin & Yaglom, vol. II, ref. B6.

Fourier Expansion and Fourier Transform*

Lastly, using the definition of Fourier intagral and the above we can now show the following important result known as Khinchin theorem (1934):

The correlation function $B(\tau) = \overline{u^*(t)u(t+\tau)}$ is the Fourier transform of the corresponding spectral density:

$$B(\tau) = \int_{-\infty}^{\infty} e^{i\omega t} F(\omega) d\omega = \int_{0}^{\infty} \cos(\omega\tau) E(\omega) d\omega, \text{ and}$$

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} B(\tau) d\tau, \; E(\omega) = \frac{2}{\pi} \int_{0}^{\infty} \cos(\omega\tau) B(\tau) d\tau.$$

The special case corresponding to $\tau = 0$:

$$B(0) = \int_{-\infty}^{\infty} F(\omega) d\omega = \int_{0}^{\infty} E(\omega) d\omega$$

has a particularly simple physical explanation: it shows that the total energy of the process $u(t)$ is the sum of the energies of the individual spectrum components.

* - from Monin & Yaglom, vol. II, ref. B6.

Fourier Expansion and Fourier Transform

It is clear that the Fourier transform of the correlation function of a stationary process should be a non-negative function. This constitutes Khinchin theorem on the Fourier expansion of correlation functions. Khinchin also showed that each function which has a non-negative Fourier transform is the correlation function of some stationary random process. Therefore, in order to verify whether a given function is the correlation function of a stationary random process, we must find its Fourier transform and establish if it is always non-negative.

Some notable examples of the correlation functions:

$$B(\tau) = C \cdot e^{-\alpha|\tau|},$$

$$B(\tau) = C \cdot e^{-\alpha\tau^2},$$

$$B(\tau) = \begin{cases} C \cdot \left(1 - \alpha|\tau|\right), \text{ when } |\tau| \le \dfrac{1}{\alpha}, \\ 0, \text{ when } |\tau| > \dfrac{1}{\alpha}, \end{cases}$$

that have the following Fourier transforms:

$$E(\omega) = \frac{2C\alpha}{\pi\left(\alpha^2 + \omega^2\right)},$$

$$E(\omega) = \frac{C}{\sqrt{\alpha\pi}} e^{-\omega^2/(4\alpha)},$$

$$E(\omega) = \frac{4C\alpha}{\pi} \frac{\sin^2\left(\omega/(2\alpha)\right)}{\omega^2}.$$

Algebraic Scaling Laws

It is clear that the Fourier transform of the correlation function of a stationary process should be a non-negative function. This constiututes Khinchin theorem on the Fourier expansion of correlation functions. A very important class of correlation functions and their corresponding energy spectrum is the case of algebraic scaling laws:

$$B(\tau) = A \cdot \tau^{\gamma}, \text{ for } A > 0, \text{ and } 0 < \gamma < 2,$$

has the following Fourier transform:

$$E(\omega) = \frac{C}{\omega^{1+\gamma}}.$$

For such processes a set of re-scaling transformations:

$$\begin{cases} \mathrm{t} \to T \cdot \mathrm{t} \\ u \to U \cdot u \end{cases}$$

does not change any of the governing statistical laws, only "zooms in and out the microscope". Such processes are called self-similar. They have no characteristic scale.

A Reminder on Gaussian Distribution Properties

Understanding Gaussian (Normal) distribution properties is very important as they are used almost everywhere in science. Its importance is related to Central Limit Theorem, which states that, under some mild conditions, the mean of many random variables independently drawn from the same distribution is distributed approximately normally, irrespectively of the original distribution.

A normal distribution probability density is :

$$P(x;m,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}},$$

where parameter m is the mean of the distribution (also its median and mode), parameter σ is its standard deviation (its variance is σ^2).

Its moments :

$$\overline{x} = m;$$

$$\overline{x^2} = \sigma^2;$$

$$\overline{x^n} = \begin{cases} \overline{x^{2k+1}} = 0, \\ \overline{x^{2k}} = (2k-1)!! \cdot \sigma^{2k} \end{cases}.$$

Its characteristic function :

$$\chi(\text{q}) = \hat{P}(x;m,\sigma) = e^{-\frac{\sigma^2 q^2}{2} + imq},$$

and, if $m = 0$, both characteristic function χ and the distribution P are "Gaussian".