Lecture 6:

P&L of Options Trading When:

Hedging continuously

Hedging discretely

Transactions costs:

Conclusion:

Accurate Replication is Very Difficult

We are going to see how one replicates an option via various hedge ratios, and then see what happens to the P&L along the way to expiration where there is uncertainty, even if the final payoff is guaranteed.

Then we'll look at what happens if we can't implement the hedge ratios continuously in real life, and if there are transactions costs.

6.1 The P&L of Any Hedged/Replicated Trading Strategy

Consider an initial position at time t_0 when the stock price is S_0 and an option C that is bought and hedged with borrowed money which earns interest r, and then rehedged in discrete steps at times t_i and stock prices S_i. We do accounting again.

Notation: $C_n = C(S_n, t_n)$ $\quad\quad \Delta_n = \Delta(S_n, t_n)$

Δ can be **any function**, not BSM value. Interest rates non-zero.

t_n,S_n	Hedging action	No. Shares	Share Value	Dollars Received From Shares and Options Traded	Net Value of Position: Option + Stock + Cash
t_0,S_0	Buy C_0, short Δ_0 shares	$-\Delta_0$	$-\Delta_0 S_0$	$\Delta_0 S_0 - C_0$	0
t_1,S_1	none	$-\Delta_0$	$-\Delta_0 S_1$	$(\Delta_0 S_0 - C_0)e^{r\Delta t}$	$C_1 - \Delta_0 S_1 + (\Delta_0 S_0 - C_0)e^{r\Delta t}$
	get short Δ_1 shares by shorting $\Delta_1 - \Delta_0$ shares	$-\Delta_1$	$-\Delta_1 S_1$	$(\Delta_0 S_0 - C_0)e^{r\Delta t}$ $+ (\Delta_1 - \Delta_0)S_1$	$C_1 - \Delta_1 S_1 + (\Delta_0 S_0 - C_0)e^{r\Delta t}$ $+ (\Delta_1 - \Delta_0)S_1$
t_2,S_2	none	$-\Delta_1$	$-\Delta_1 S_2$	$(\Delta_0 S_0 - C_0)e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0)S_1 e^{r\Delta t}$	$C_2 - \Delta_1 S_2 + (\Delta_0 S_0 - C_0)e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0)S_1 e^{r\Delta t}$

t_n, S_n	Hedging action	No. Shares	Share Value	Dollars Received From Shares and Options Traded	Net Value of Position: Option + Stock + Cash
t_2, S_2	get short Δ_2 shares by shorting $\Delta_2 - \Delta_1$ shares	$-\Delta_2$	$-\Delta_2 S_2$	$(\Delta_0 S_0 - C_0)e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0)S_1 e^{r\Delta t}$ $+ (\Delta_2 - \Delta_1)S_2$	$C_2 - \Delta_2 S_2 + (\Delta_0 S_0 - C_0)e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0)S_1 e^{r\Delta t}$ $+ (\Delta_2 - \Delta_1)S_2$
etc.					
t_n, S_n	get short Δ_n shares by shorting $\Delta_n - \Delta_{n-1}$ shares	$-\Delta_n$	$-\Delta_n S_n$	$(\Delta_0 S_0 - C_0)e^{nr\Delta t}$ $+ (\Delta_1 - \Delta_0)S_1 e^{(n-1)r\Delta t}$ $+ (\Delta_2 - \Delta_1)S_2 e^{(n-2)r\Delta t}$ $\ldots + (\Delta_n - \Delta_{n-1})S_n$	$C_n - \Delta_n S_n + (\Delta_0 S_0 - C_0)e^{nr\Delta t}$ $+ (\Delta_1 - \Delta_0)S_1 e^{(n-1)r\Delta t}$ $+ (\Delta_2 - \Delta_1)S_2 e^{(n-2)r\Delta t}$ $\ldots + (\Delta_n - \Delta_{n-1})S_n$

The initial value of the positions was 0 and that cash invested would have generated zero in future.

The final value is $C_T - \Delta_T S_T + (\Delta_0 S_0 - C_0)e^{r(T-t_0)} + \int_{t_0}^{T} e^{r(T-x)} S_x [d\Delta_x]_b$

where x is the intermediate time and the subscript b at the end of the formula denotes **a backwards Ito integral**. C_T is known as a function of the terminal stock price.

Write $T - t = \tau$.

$$\text{final value} = C_T - \Delta_T S_T + (\Delta_0 S_0 - C_0)e^{r\tau} + \int_0^{\tau} e^{r(\tau - x)} S_x [d\Delta_x]_b \qquad \text{Eq 6.1}$$

If there is a perfect BSM Δ hedge at each step, then the growth of the portfolio at each step is riskless, and since we start from a zero value, the final value of Equation 6.1 must be zero.

In that case only we get the VERY IMPORTANT REPLICATION FORMULA for C_0:

$$(C_0 - \Delta_0 S_0)e^{r\tau} = (C_T - \Delta_T S_T) + \int_0^{\tau} e^{r(\tau - x)} S_x [d\Delta_x]_b \qquad \text{Eq 6.2}$$

Simply stated: The future value of the initial continuously perfectly hedged portfolio is equal to the final value of the hedged portfolio plus the future cost of all the incremental re-hedges. If we know Δ we can use this to calculate the final value of the option via Monte Carlo.

If it is not perfectly hedged *a la* BSM, there is a distribution of final values. It is path dependent, and so not unique in general. $\Delta_x(S, x, \sigma)$ can depend on the volatility you use.

6.2 Different Hedging Strategies In The GBM BSM World: The P&L When Hedging with Realized (Imagined Known) Volatility

Realized volatility is a noisy statistic. Implied volatility is a parameter reflecting fear, hedging costs, the inability to hedge perfectly, the uncertainty of future volatility, the chance to make a profit, etc., and is therefore usually greater than recent realized volatility.

Future evolution is always at realized volatility.

Purchase/traded price is always at implied volatility, by definition.

Δ-Hedging can be **at any volatility**, since we don't know what future volatility will actually be.

But **let's pretend** we know the future volatility with perfect forecasting accuracy.

We buy the option at its implied volatility and then hedge it at the (assumed known future) realized volatility to replicate the option perfectly at the BS value assuming that we have perfect knowledge of future realized volatility. (GS story)

In the end, when we hedge all the way to expiration, the total P&L will be the value gained from replication MINUS Black-Scholes implied value:

$$\text{Total PV[P\&L(I,R)]} = V(S, \tau, \sigma) - V(S, \tau, \Sigma) = V_R - V_I$$

Notation:

"I = bought at implied, known"

"R = hedged at realized, assumed known"

We know the final answer but we're going to examine how this evolves now, with better notation.

Notation and Remarks About Changes in the P&L

To look at the evolution of the P&L we will
(1) use accounting to put together all the bits of profit and expense, and then
(2) use Ito's Lemma for GBM to understand the evolution of the profit.

Notation: A portfolio is bought at an implied volatility I, and then evolves at a realized volatility R, and is hedged at a hedge volatility H.

We call such a hedged portfolio $\pi[I, R, H] = \boldsymbol{V}_I - \Delta_H \boldsymbol{S}$, where the I means the option is bought at implied volatility, and the H means it's hedged using the hedge ratio of some other volatility H, and R means that meanwhile the stock evolves at the realized volatility. And we assume we borrowed the money to set up the portfolio, so the initial value of the portfolio is always zero.

One thing we will have to make use of is our knowledge of Black-Scholes for GBM, namely that if you buy an option at some implied volatility I and if the realized volatility R = I, and if hedge it at that same I, then it is riskless and if you spent no money on the portfolio, i.e. if you borrowed the money to set up that portfolio then there can be no change in its value. So initially

$\pi[I, I, I] = \boldsymbol{V}_I - \Delta_I \boldsymbol{S} - [\text{cash needed to set up the position}] = 0$ and

$d\pi[I, I, I] = 0$ over a short time dt as the stock price moves at the implied volatility.

Similarly, $d\pi[R, R, R] = 0$.

If implied, realized, and hedged vol are identical, then BS tells you you're perfectly hedged and zero value remains zero.

Now, How Do We Capture $V_R - V_I$ Over Time?

In our notation, the Hedged Portfolio is $\pi[I, R, R] = \boldsymbol{V}_I - \Delta_R \boldsymbol{S}$, bought with borrowed money, so that there is zero initial value of the portfolio, and then it evolve and is hedged at realized volatility.

We assume the usual GBM Evolution of the stock w**ith realized vol:** $dS = \mu S dt + \sigma_R S dZ$ and we assume it has a dividend yield = D, and the riskless rate = r.

Now we look at changes from accounting rules:

interest owed dividend owed

$$dP\&\mathrm{L(I,R,R)} = d\boldsymbol{V}_I - \Delta_R dS - r dt(\boldsymbol{V}_I - \Delta_R S) - \Delta_R D S dt$$

accounting

$$= d V_I - r V_I dt - \Delta_R [dS - (r - D) S dt]$$

But dP&L[R,R,R] hedged *and bought* at realized volatility is zero:

$$dP\&\mathrm{L(R,R,R)} = d\boldsymbol{V}_R - r dt \boldsymbol{V}_R - \Delta_R [dS - (r - D) S dt] = 0$$

$$\Delta_R [dS - (r - D) S dt] = d\boldsymbol{V}_R - r dt \boldsymbol{V}_R$$

Therefore combining the

$$dP\&\mathrm{L(I,R,R)} = d\boldsymbol{V}_I - r d\boldsymbol{V}_I dt - \Delta_R [dS - (r - D) S dt] = d\boldsymbol{V}_I - r d\boldsymbol{V}_I dt - (d\boldsymbol{V}_R - r dt \boldsymbol{V}_R)$$

two

$$= d\boldsymbol{V}_I - d\boldsymbol{V}_R - r dt(\boldsymbol{V}_I - \boldsymbol{V}_R) = e^{rt} d\left[e^{-rt}(V_I - V_R)\right]$$

$$PV[d(\text{P\&L(I,R.R)})] = e^{-r(t-t_0)}e^{rt}d\left[e^{-rt}(V_I - V_R)\right] = e^{rt_0}d\left[e^{-rt}(V_I - V_R)\right]$$

$$\text{PV[P\&L(I,R,R)]} = e^{rt_0}\int_{t_0}^{T} d\left[e^{-rt}(V_I - V_R)\right]$$

$$= 0 - (V_{I,\,t_0} - V_{R,\,t_0}) = V_{R,\,t_0} - V_{I,\,t_0} \qquad \text{if T is expiration}$$

At expiration $V_{I,\,T} - V_{R,\,T} = 0$ because terminal value is independent of volatility.

The final P&L at the expiration of the option is known and deterministic, provided that we know the realized volatility and that we can hedge continuously.

How is this deterministic P&L realized over time? Stochastically.

It's a bit like a zero coupon bond whose final principal is known in advance but whose present value varies with interest rates, and the dependence on rates and time decreases as we approach maturity.

We now use Ito calculus to see how this happens:

$$\text{dP\&L[I,R,R]}) = dV_I - \Delta_R dS - \Delta_R SDdt - (V_I - \Delta_R S)rdt$$

Using Ito

$$\text{dP\&L[I,R,R]} = \left[\Theta_I dt + \Delta_I dS + \frac{1}{2}\Gamma_I S^2 \sigma_R^2 dt\right] - \Delta_R dS - \Delta_R SDdt$$
$$-(V_I - \Delta_R S)rdt$$
$$= \left[\Theta_I + \frac{1}{2}\Gamma_I S^2 \sigma_R^2\right] dt + (\Delta_I - \Delta_R)dS - \Delta_R SDdt$$
$$-(V_I - \Delta_R S)rdt \qquad \text{Eq 6.3}$$

But d[P&L(I,I,I)] = 0 $\quad d\text{P\&L}[I, I, I] = \left[\Theta_I + \frac{1}{2}\Gamma_I S^2 \Sigma^2\right]dt - \Delta_I SDdt - (V_I - \Delta_I S)rdt \equiv 0$

or $\boxed{\Theta_I = -\frac{1}{2}\Gamma_I S^2 \Sigma^2 + rV_I - (r - D)S\Delta_I}$ which is just the BS PDE $\qquad$ Eq 6.4

and so substituting Equation 6.4 into Equation 6.3 and setting $dS = \mu S dt + \sigma_R S dZ$ we get

$$dP\&L[I,R,R] = \left[\Theta_I + \frac{1}{2}\Gamma_I S^2 \sigma_R^2\right]dt + (\Delta_I - \Delta_R)dS - \Delta_R SDdt - (V_I - \Delta_R S)rdt$$

$$= \frac{1}{2}\Gamma_I S^2\left(\sigma_R^2 - \Sigma^2\right)dt + (\Delta_I - \Delta_R)(dS - (r-D)dt)$$

$$= \frac{1}{2}\Gamma_I S^2\left(\sigma_R^2 - \Sigma^2\right)dt + (\Delta_I - \Delta_R)\{((\mu - r + D)dt + \sigma_R S dZ)\}$$

We saw the total integrated P&L was deterministic but the increments have a random component dZ because the implied hedge ratio and the realized hedge ratio differ.

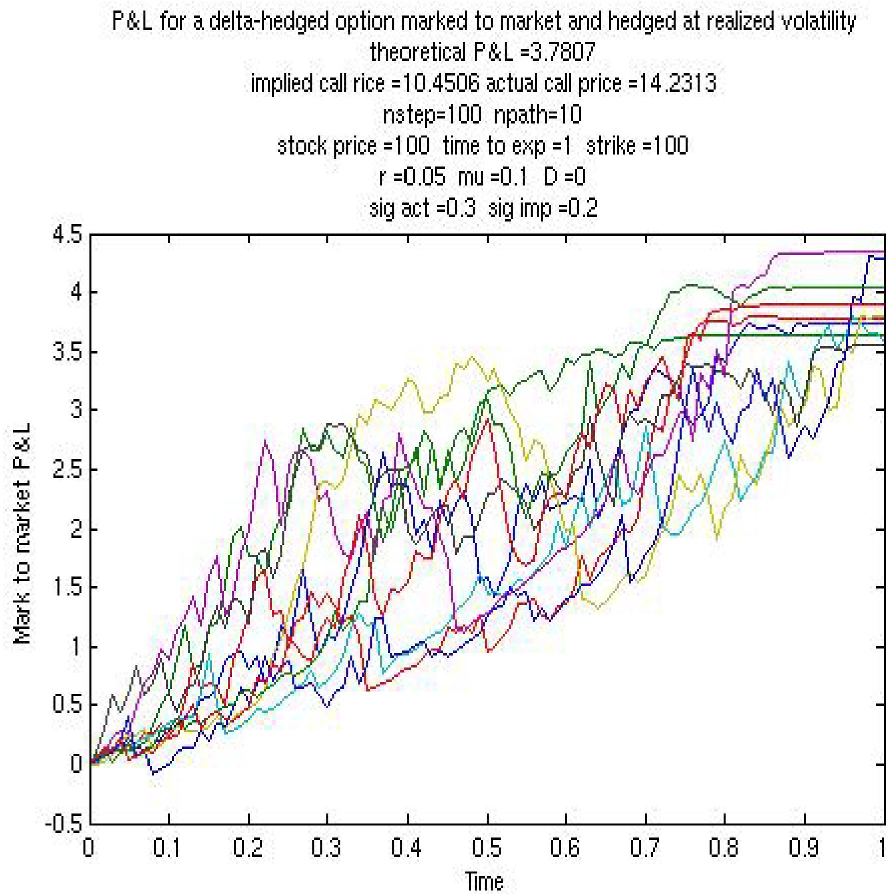
Code used to get graph on next page:

```
PandL(i,:) =  PandL(i-1,:) + ...
                  exp(-rate*time_elapsed) .* (
                    0.5*((StockpriceIncrement.^2) - sig_imp^2 * (Stockprice(i-
1,:).^2)) .*Gamma_imp(i-1,:) * dt + ...
                    ( Delta_imp(i-1,:) - Delta_real(i-1,:) ) .* ...

                    (   (mu - rate + div_rate) .* Stockprice(i-1,:) * dt +
sig_real .* Stockprice(i-1,:) .*Z(i-1,:) *sqrt(dt)    ) ...
```

To illustrate this, plot cumulative **discounted P&L(I, R,R)** along ten random stock paths with100 steps

P&L starts at zero because initial position is totally financed



$$\sigma_r = 0.3$$

$$\sigma_i = 0.2$$

/Users/emanuelderman/Dropbox/Vol Smile David Park Book copy/Matlab Code for Hedging/BSDiscreteHedgingAtRealized_PandLactualStocl

The **final P&L** is almost path-independent – almost, because 100 rehedgings per year is not quite the same as continuous hedging.

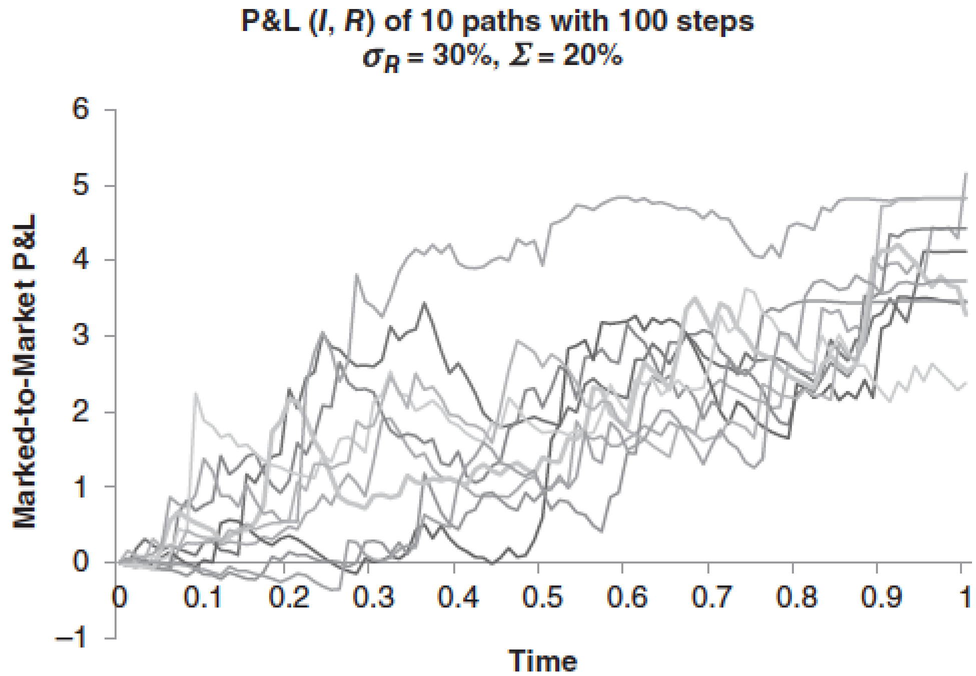
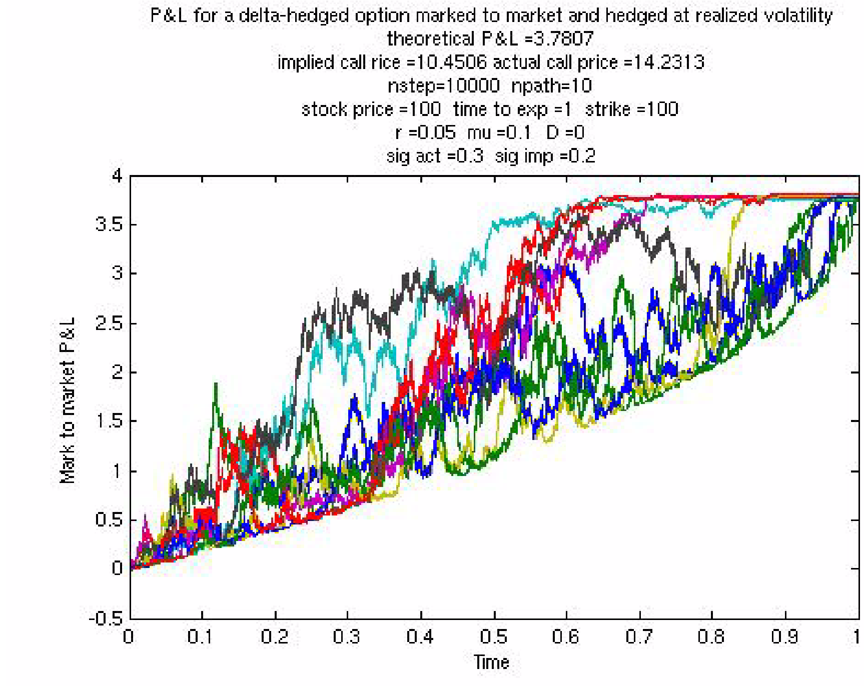


FIGURE 5.1 Hedging with Realized Volatility: Cumulative Discounted P&L of a Call with One Year to Expiration Simulated with 100 Steps

Rehedge 10,000 times, almost path-independent **final P&L**:



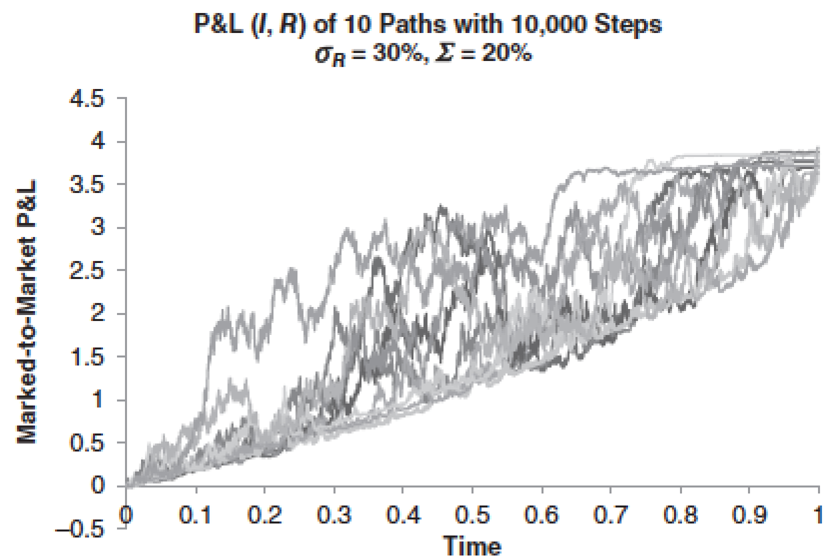


FIGURE 5.2 Hedging with Realized Volatility: Cumulative Discounted P&L of a Call with One Year to Expiration Simulated with 10,000 Steps

Bounds on the P&L When Hedging at the Realized Volatility

We had $\sigma > \Sigma$. Notice the upper and lower bounds. Why? We had

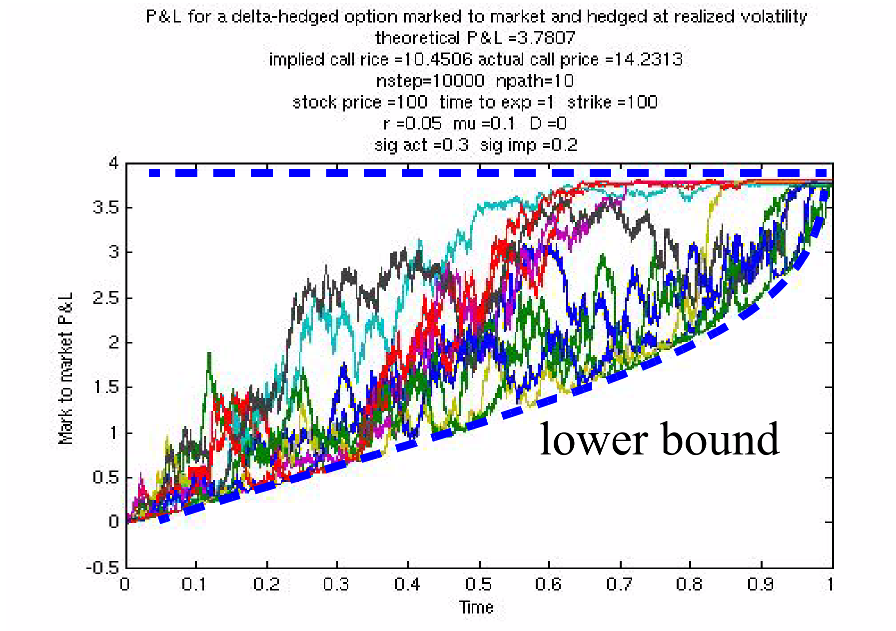
$$dPV[P\&L(I,R,R)] = e^{-r(t-t_0)}e^{rt}d\left[e^{-rt}(V_I - V_R)\right] = e^{rt_0}d\left[e^{-rt}(V_I - V_R)\right]$$

Integrate from t_0, S_0 to intermediate nonterminal time m when the stock price is S, to obtain

$$\begin{aligned}
PV(P\&L[I,R,R]) &= e^{rt_0}\int_{t_0}^{m} d\left[e^{-rt}(V_I - V_R)\right] \\
&= e^{rt_0}\left[e^{-rt}(V_I - V_R)\right]_{t_0}^{m} \\
&= e^{rt_0}\left[e^{-rm}(V_{I,m} - V_{R,m}) - e^{-rt_0}(V_{I,0} - V_{R,0})\right] \\
&= \underbrace{(V_{R,0} - V_{I,0})}_{\text{value at inception} > 0} - \underbrace{e^{-r(m-t_0)}(V_{R,m} - V_{I,m})}_{\text{value along way}}
\end{aligned}$$

Both terms in the brackets are positive.

Upper bound $(V_{R,0} - V_{I,0})$ occurs when the second term is zero, when the option value is independent of volatility, which occurs at $S = 0$ or $S = \infty$, and the gamma of the option is zero.



The lower bound to the P&L occurs when second term $[V(\sigma, S, m) - V(\Sigma, S, m)] \sim \frac{\partial V}{\partial \sigma}(\Sigma - \sigma)$ is a maximum, i.e. when vega is largest, close to at-the money, which turns out to be at $S = Ke^{-(r - 0.5\sigma\Sigma)(T - t)}$

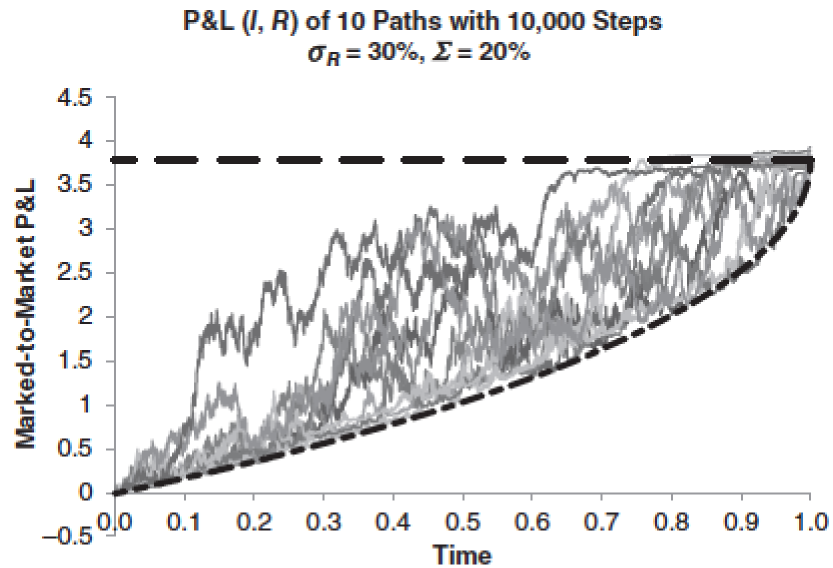


FIGURE 5.3 Hedging with Realized Volatility: Cumulative Discounted P&L with 10,000 Steps and Upper and Lower Bounds

6.3 P&L When Hedging with Implied Volatility

When you hedge with implied, the final value of the P&L(I,R,I) depends on the path taken, and is not deterministic, but there is **no random mishedging component at each instant**. At each instant we know the incremental P&L irrespective of stock price move,, because we are hedged, but the next contribution after that will depend on whether the stock will have moved up or down.

Table 1: Position Values when Hedging with Implied Volatility

Time	Option Position, Value	Stock Position, Value	Value of Cash Position	Net Position Value
t	$\vec{V}_i, V_i$	$-\Delta_i\vec{S}, -\Delta_i S$	$\Delta_i S - V_i$	0
t + dt	$\vec{V}_i, V_i + dV_i$	$-\Delta_i\vec{S}, -\Delta_i(S + dS)$	$(\Delta_i S - V_i)(1 + rdt)$ $-\Delta_i DSdt$	$(V_i + dV_i - \Delta_i(S + dS))$ $(\Delta_i S - V_i)(1 + rdt)$ $-\Delta_i DSdt$

$$
\begin{aligned}
d\text{P\&L(I,R,I)} &= [V_i + dV_i - \Delta_i(S + dS)] + (\Delta_i S - V_i)(1 + rdt) - \Delta_i DSdt \\
&= dV_i - \Delta_i dS - r(V_i - \Delta_i S)dt - \Delta_i DSdt
\end{aligned}
$$

$$dP\&L(I,R,I) = \left[\Theta_i dt + \cancel{\Delta_i dS} + \frac{1}{2}\Gamma_i S^2 \sigma^2 dt\right] - \cancel{\Delta_i dS} - r(V_i - \Delta_i S)dt - \Delta_i DSdt$$

Using Ito:

$$= \left\{\Theta_i + \frac{1}{2}\Gamma_i S^2 \sigma^2 + (r-D)\Delta_i S - rV_i\right\}dt$$

The Black-Scholes equation when hedging and realized are both equal to σ_i, is dP&L[I,I,I]=0

$$\Theta_i + \frac{1}{2}\Gamma_i S^2 \Sigma^2 + (r-D)\Delta_i S - rV_i = 0$$

So

$$dP\&L(I,R,I) = \frac{1}{2}\Gamma_i S^2(\sigma^2 - \Sigma^2)dt \qquad \text{Eq.6.1}$$

$$PV[P\&L(I,R,I)] = \frac{1}{2}\int_{t_0}^{T} \Gamma_i S^2(\sigma^2 - \Sigma^2)e^{-r(t-t_0)}dt \qquad \text{Eq.6.2}$$

The P&L is highly path-dependent. Although the hedging strategy captures a value proportional to $(\sigma^2 - \Sigma^2)$, it depends strongly on moneyness. We saw this before.

Matlab+Bad hedging/BSDiscreteHedgingAtImplied_PandL.m

```
PandL(i,:) = PandL(i-1,:) + 0.5*(sig_act^2 - sig_imp^2)*exp(-rate*time_elapsed).*(Stockprice(i-1,:).^2).*Gamma(i-
1,:)*dt;
```

Cumulative P&L along 10 random stock paths, 100 hedging steps to expiration

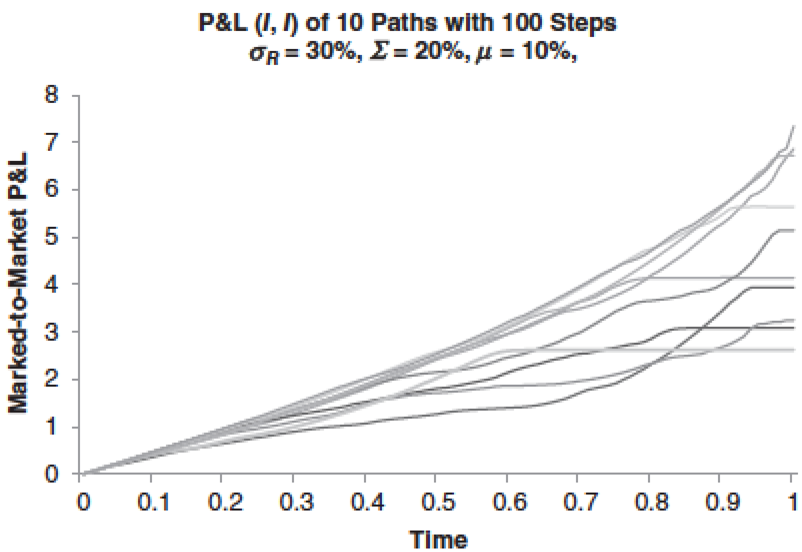


FIGURE 5.4 Hedging with Implied Volatility: Cumulative Discounted P&L with 100 Steps, Drift = 10%, Time to Expiration = 1 Year

most P&L when gamma stays large

with high drift the gamma quickly becomes close to zero

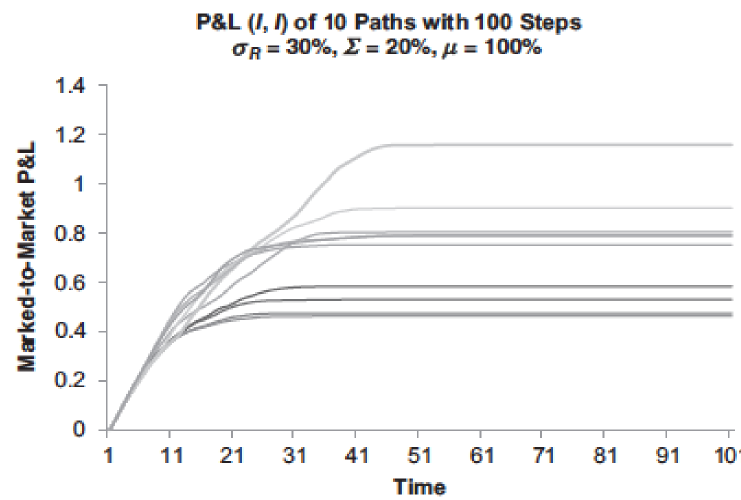


FIGURE 5.5 Hedging with Implied Volatility: Cumulative Discounted P&L with 100 Steps, Drift = 100%, Time to Expiration = 1 Year

In practice, realized volatility isn't known in advance. A trading desk would most likely hedge at the constantly varying implied volatility which would move in synchronization with the recent realized volatility.

Summary

- In our BS laboratory, we assumed that we could know future realized volatility with certainty.
- In fact, you know the implied volatility from the market price of the option, but you can only try to predict future volatility. Therefore, when you hedge an option, you usually have to choose between hedging at implied volatility and hedging using a guess for the future realized volatility.
- If you estimate future realized volatility correctly and hedge (or replicate) continuously at that volatility, your P&L will eventually capture the exact value of the option.
- But along the way your P&L will have a random component.
- Suppose you have lost money randomly by guessing a future realized volatility. Should you be allowed to continue? Who knows if you have guessed the realized volatility correctly.

- If implied volatility is not equal to realized volatility and you hedge continuously at implied volatility, your P&L will be path-dependent and unpredictable. The P&L will be a maximum when the gamma of the option is a maximum, which occurs when the stock price stays close to the strike price on its path to expiration.

6.4 Hedging at an Arbitrary Constant Volatility

We don't of course know the future volatility. Suppose we just choose some hedging volatility. PV(I,R,H)

Buy an option at implied vol Σ, hedge it to expiration at volatility σ_h, while realized volatility σ_r

Table 2: Position Values when Hedging with an Arbitrary Volatility

Time	Option Position, Value	Stock Position, Value	Value of Cash Position	Net Position Value
t	$\vec{V}_i, V_i$	$-\Delta_h \vec{S}, -\Delta_h S$	$\Delta_h S - V_i = (\Delta_h S - V_h) + (V_h - V_i)$	0
t + dt	$\vec{V}_i, V_i + dV_i$	$-\Delta_h \vec{S}, -\Delta_h (S + dS)$	$(\Delta_h S - V_i)(1 + rdt) - \Delta_h DSdt$	$(V_i + dV_i - \Delta_h (S + dS)) + (\Delta_h S - V_i)(1 + rdt) - \Delta_h DSdt$

P&L:

$$dP\&L(I,R,H) = dV_i - \Delta_h dS - \Delta_h SDdt + \{(\Delta_h S - V_h) + (V_h - V_i)\}rdt$$

$$= dV_h - \Delta_h dS - \Delta_h SDdt + (dV_i - dV_h) + \{(\Delta_h S - V_h) + (V_h - V_i)\}rdt$$

$$= \underbrace{\left\{\Theta_h + \frac{1}{2}\Gamma_h S^2 \sigma_r^2 + (r-D)S\Delta_h - rV_h\right\}dt}_{\text{all the H terms}} + \underbrace{(dV_i - dV_h) + (V_h - V_i)rdt}_{\text{the leftovers}}$$

Now the BS solution with σ_h satisfies the p.d.e $dP\&L[H,H,H] = 0$

$$\Theta_h + (r-D)S\Delta_h + \frac{1}{2}\Gamma_h S^2 \sigma_h^2 - rV_h = 0$$

Substituting this last equation into the previous one

$$dP\&L(I,R,H) = \frac{1}{2}\Gamma_h S^2\left(\sigma_r^2 - \sigma_h^2\right)dt + (dV_i - dV_h) + (V_h - V_i)rdt$$

$$= \frac{1}{2}\Gamma_h S^2\left(\sigma_r^2 - \sigma_h^2\right)dt + e^{rt}d\left\{e^{-rt}(V_i - V_h)\right\}$$

Taking present values $e^{-r(t-t_0)}$ to time t_0 leads to

$$dPV(P\&L(I,R,H)) = e^{-r(t-t_0)}\frac{1}{2}\Gamma_h S^2\left(\sigma_r^2 - \sigma_h^2\right)dt + e^{rt_0}d\left\{e^{-rt}(V_i - V_h)\right\}$$

Integrate:

$$PV[\text{P\&L(I,H)}] = V_h - V_i + \frac{1}{2}\int_{t_0}^{T} e^{-r(t-t_0)}\Gamma_h S^2\left(\sigma_r^2 - \sigma_h^2\right)dt \qquad \text{Eq.6.3}$$

Note that $V_h = V_i$ have equal values at expiration. When σ_h is set equal to either σ_r or σ_i, Equation 6.3 reduces to our previous results.

Summary:

Hedge at implied: stochastic path-dependent P&L because of Γ, but at each instant it's deterministic because there is no dZ in the evolution. But depending on whether the stock goes up or down, the next increment to the P&L will depend upon the Γ at that new stock price. So it's locally deterministic but long-run uncertain.

Hedging at realized: the P&L is locally stochastic because of the dZ, but long run deterministic (assuming you know the realized volatility).

6.5 Hedging Errors from Discrete Hedging at $\sigma_h = \sigma_r$

- Hedging perfectly and continuously at no cost is a Platonic ideal with no error term dZ.
- In real life, you can rebalance the hedge only a finite number of times.
- You are mishedged in the intervals, and the P&L picks up a random component.
- The more often you hedge, the smaller the deviation from perfection.
- Transaction costs affect things, too, but that's later.

A Simulation Approach

You cannot hedge continuously, and therefore it is important to understand the errors that creep into your P&L when you hedge at discrete intervals. Some traders hedge at regularly spaced time intervals; others hedge whenever the delta changes by more than a certain amount. In what follows here we will discuss replication at regular time intervals as the stock evolves at σ_r.

$$(C_0 - \Delta_0 S_0)e^{r\tau} = (C_T - \Delta_T S_T) + \int_0^\tau e^{r(\tau - x)} S_x [d\Delta_x]_b$$

$\Delta(\sigma_h)$ is the replication/hedging volatility.

Sample code: Evolving the stock through time on each path that corresponds to given drift and vol:

```
Stockprice(i,:) = Stockprice(i-1,:).*exp((mu-div_rate-0.5*sig_act^2)*dt + sig_r*sqrt(dt)*Z(i-1,:))
```

Summing up all the gamma contributions along each path:

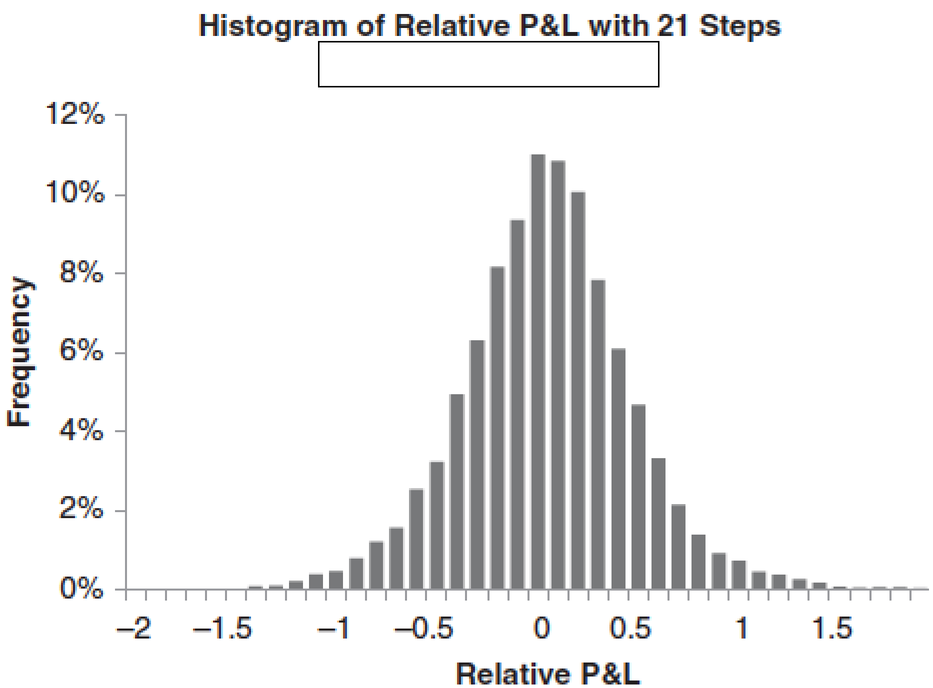
```
Time_integral = Time_integral + Stockprice(i,:).*(Delta(i,:) - Delta(i-1,:))*exp(rate *(time_to_exp))
```

Calculating initial call value from the path integral in Equations 6.2:

Because we hedge or replicate discretely, different paths will give different values and so we'll get a histogram of call values, and an average with a standard deviation, the hedging error.

Monte Carlo: ATM option, expiration 1 month, the realized volatility is 20%, $\mu = r = 0$, hedged or replicated at an implied volatility of 20% equal to the realized volatility. $\sigma_r = \sigma_h = 0.2$

Plot: Relative P&L = Present Value of Payoff – BSM Fair Value.



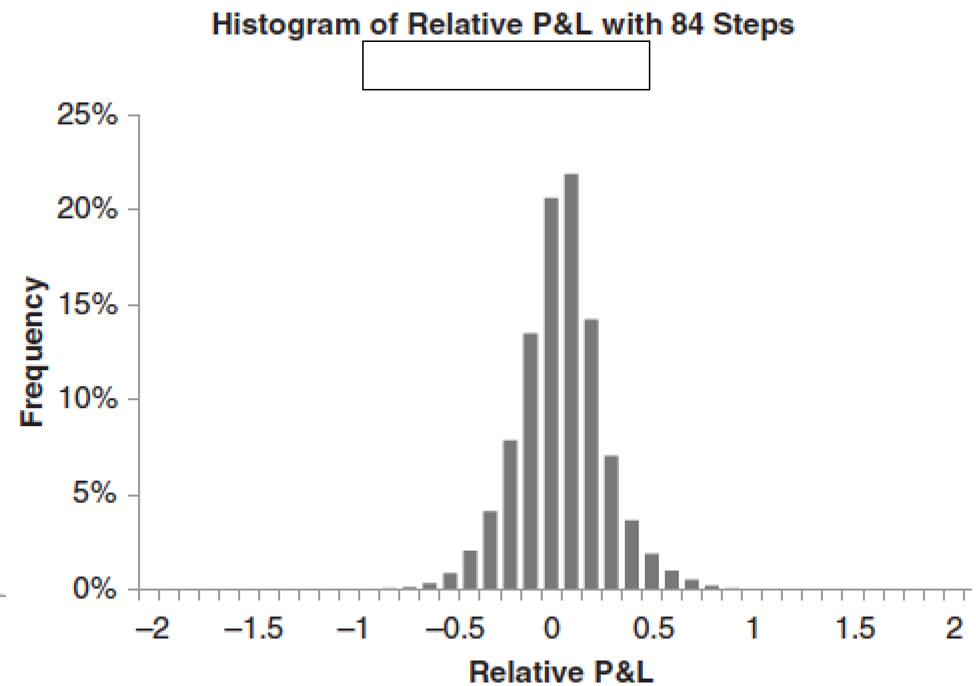


FIGURE 6.1 Distribution of Relative P&L for a One-Month At-the-Money Call Option When Hedging Volatility = Realized Volatility, $\mu = r$ (Relative P&L = Present Value of Payoff – BSM Fair Value)

The mean deviation from BS is zero; When we quadruple the number of hedgings, the standard deviation of the P&L halves. We do better by hedging/replicating more frequently.

Now let's see what happens $\sigma_h \neq \sigma_r$ and we hedge at realized. Choose a replication volatility of 40%.

P&L is relative to BSM valued at realized.

There is no longer the same reduction in standard deviation when the number of rebalancings quadruple. Both distributions are more or less symmetric though.

This is the Matlab file: /Users/emanuelderman/Library/Mobile Documents/com~apple~CloudDocs/mm/Columbia/E4718 Skew/2006 suggestions stochastic vol etc/Matlab+Bad hedging/BSDiscreteHedging_histPandL.m

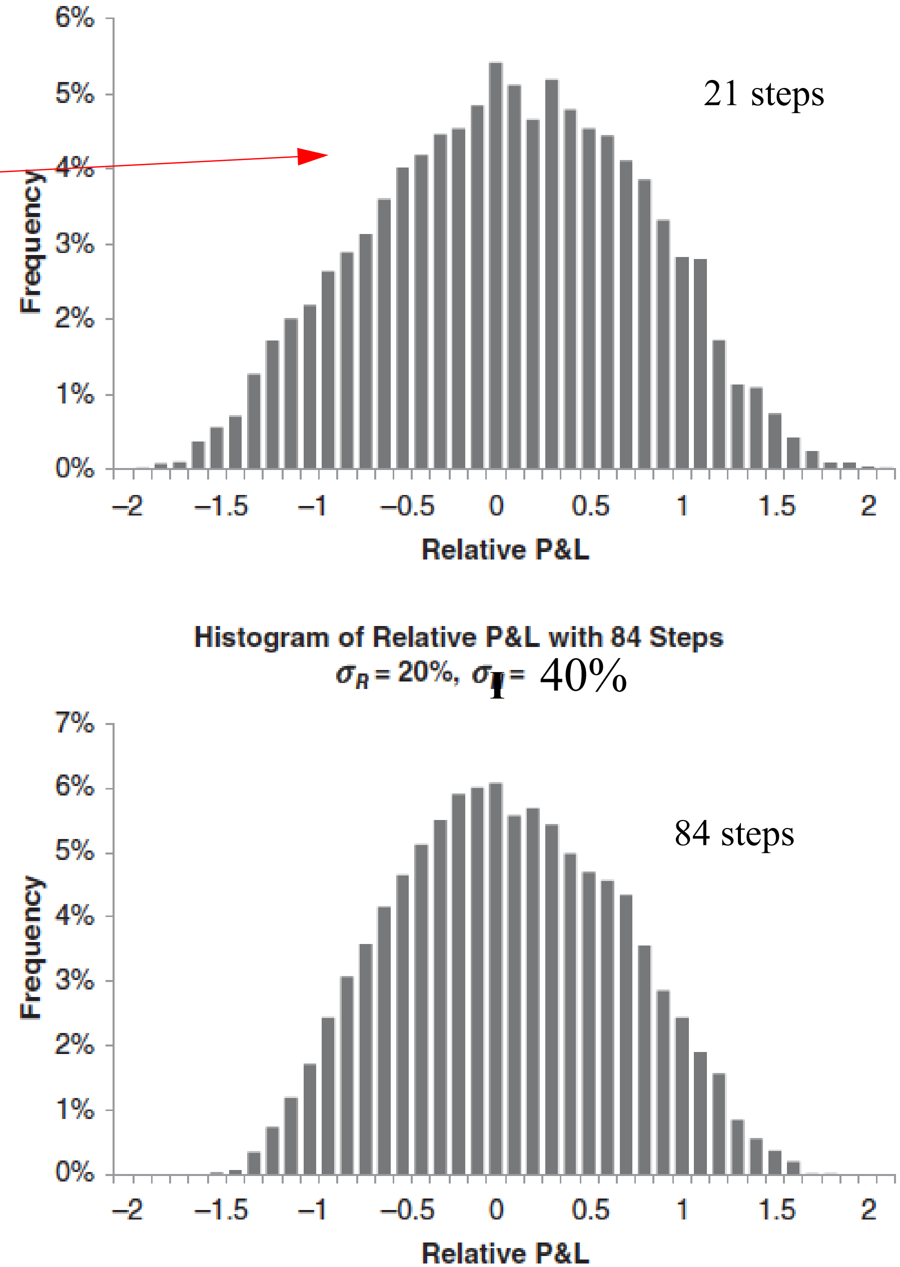


FIGURE 6.2 Distribution of Relative P&L for a One-Month At-the-Money Call Option When Hedging Volatility ≠ Implied Volatility, $\mu = r$ (Relative P&L = Present Value of Payoff – BSM Fair Value)

6.6 Understanding ATM Discrete Hedging Error Analytically when $\sigma_i = \sigma_r \equiv \sigma$. (Assuming we know the future volatility)

Discrete time dt is larger than infinitesimal, the stock goes through a finite move.

$$\frac{dS}{S} = \mu dt + \sigma Z \sqrt{dt}$$

$$Z \sim N(0, 1)$$

Hedged portfolio $\pi = C - \left(\frac{\partial C}{\partial S}\right) S$; Initial long π bought with borrowed money. If we hedged continuously the P&L would be zero.

Hedging error owing to mismatch between a **continuous** hedge ratio and a **discrete** time step:

$$\begin{aligned} HE_{dt} &= \pi + d\pi - \pi e^{rdt} \\ &\approx d\pi - r\pi dt \\ &\approx \left[\frac{\partial C}{dt} dt + \cancel{\frac{\partial C}{\partial S} dS} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 Z^2 dt - \cancel{\frac{\partial C}{\partial S} dS}\right] - rdt\left[C - \frac{\partial C}{\partial S} S\right] \\ &\approx \left[\frac{\partial C}{dt} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 Z^2 - r\left(C - \frac{\partial C}{\partial S} S\right)\right] dt \end{aligned}$$

discrete continuous

Now from Black-Scholes

$$r\left(C - \frac{\partial C}{\partial S}S\right) = \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2$$

$$HE_{dt} \approx \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 \left(Z^2 - 1\right) dt$$

Gamma distribution Eq.6.4

Over n steps to expiration, the total HE is

$$HE = \sum_{i=1}^{n-1} \frac{1}{2}\Gamma_{i-1} S_{i-1}^2 \sigma^2 \left(Z_i^2 - 1\right) dt \qquad \text{Eq.6.5}$$

Z is normal with $E(Z^2) = 1$ and $E(Z^4) = 3$ so $E[HE] = 0$ with a χ^2 distribution.

The variance of the hedging error is

$$E\left[\sum_{i=1}^{n-1} \frac{1}{2}\Gamma_{i-1} S_{i-1}^2 \sigma^2 \left(Z_i^2 - 1\right) dt\right]^2 = E\left[\sum_{i=1}^{n-1} \left(\frac{1}{2}\Gamma_{i-1} S_{i-1}^2 \sigma^2 dt\right)^2 \left(Z_i^2 - 1\right)^2\right]$$

$$\sigma_{HE}^2 = E\left[\sum_{i=1}^{n} \frac{1}{2}\left[\Gamma_{i-1} S_{i-1}^2\right]^2 (\sigma^2 dt)^2\right] \text{ over all paths} \qquad \text{Eq.6.6}$$

Integrating over all returns starting from S_0 **for an single atm option**

$$E\left[\Gamma_i S_i^2\right]^2 = S_0^4 \Gamma_0^2 \sqrt{\frac{T^2}{T^2 - t_i^2}}$$

Thus for constant volatility

$$\begin{aligned}
\sigma_{HE}^2 &\approx \sum_{i=1}^{n} \frac{1}{2} S_0^4 \Gamma_0^2 \sqrt{\frac{T^2}{T^2 - t_i^2}} \left(\sigma^2 dt\right)^2 \\
&\approx \frac{1}{2} S_0^4 \Gamma_0^2 \left(\sigma^2 dt\right)^2 \sum_{i=1}^{n} \sqrt{\frac{T^2}{T^2 - t_i^2}} \\
&\approx \frac{1}{2} S_0^4 \Gamma_0^2 \left(\sigma^2 dt\right)^2 \frac{1}{dt} \int_t^T \sqrt{\frac{T^2}{T^2 - \tau^2}} d\tau \\
&\approx S_0^4 \Gamma_0^2 \left(\sigma^2 dt\right)^2 \frac{\pi (T - t)}{4dt} \\
&\approx \frac{\pi}{4} n \left(S_0^2 \Gamma_0 \sigma^2 dt\right)^2
\end{aligned}$$

$$n = \frac{T-t}{dt}. \quad \text{\# of steps}$$

From BS we can interpret $S_0^2\Gamma_0 = \frac{1}{\sigma(T-t)}\frac{\partial C}{\partial\sigma}$

$$\sigma_{HE}^2 \approx \frac{\pi}{4}n\left(\frac{1}{\sigma(T-t)}\frac{\partial C}{\partial\sigma}\sigma^2 dt\right)^2$$

$$\approx \frac{\pi}{4}n\left(\sigma\frac{1}{n}\frac{\partial C}{\partial\sigma}\right)^2 \qquad \frac{dt}{T-t} = n$$

$$\approx \frac{\pi}{4n}\left(\sigma\frac{\partial C}{\partial\sigma}\right)^2$$

$$\sigma_{HE} \approx \sqrt{\frac{\pi}{4}}\frac{\partial C}{\partial\sigma}\frac{\sigma}{\sqrt{n}} \qquad \text{Eq.6.7}$$

Thus, the hedging error is approximately $\frac{\partial C}{\partial\sigma} \times \frac{\sigma}{\sqrt{n}}$. In order to halve the error we must quadruple the number of rehedgings. What does this mean?

Understanding The Results Intuitively

Hedging discretely introduces uncertainty in the hedging outcome but no bias: $E[HE] = 0$

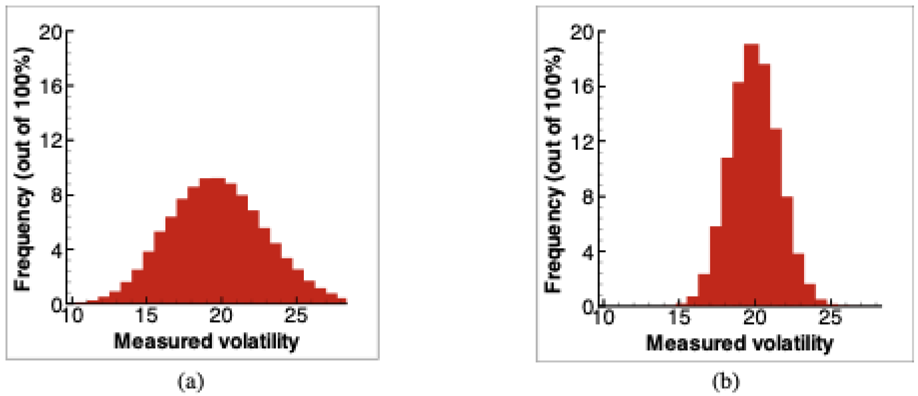
Simple analytic rule $$\sigma_{HE} \sim \sqrt{\frac{\pi}{4}}\frac{\partial C}{\partial \sigma} \times \frac{\sigma}{\sqrt{n}}$$

For $S \sim K$, more simply $$.C \approx \frac{S\sigma\sqrt{\tau}}{\sqrt{2\pi}} \qquad \frac{\partial C}{\partial \sigma} \approx \frac{S\sqrt{\tau}}{\sqrt{2\pi}}$$

Therefore $$\frac{\sigma_{HE}}{C} \approx \sqrt{\frac{\pi}{4n}} = \frac{0.89}{\sqrt{n}}:$$ 9% error for 100 rehedges

Think of this as statistical sampling error: discrete hedging samples/estimates the volatility discretely and is therefore subject to error owing to only n observations. The standard deviation or standard error in a Monte Carlo simulation using constant volatility σ but sampled with discrete steps is $\frac{\sigma}{\sqrt{2n}}$. Do a simulation and measure the volatility on each path with n steps per path:

Exhibit 2: Histograms showing the volatilities estimated from (a) 21 and (b) 84 simulated returns.



The error in the observed call price owing to the error in the volatility estimate is

$$\sigma_{HE} \approx dC \approx \frac{\partial C}{\partial \sigma} d\sigma \approx \frac{\sigma}{\sqrt{n}} \frac{\partial C}{\partial \sigma}$$

This is quite a large error even assuming we know the future volatility with certainty. In real life your hedge ratio is incorrect not just because hedging is discrete, but because you don't actually know the appropriate volatility to use.

What happens when the hedging volatility and the realized volatility are different for discrete hedging? More frequent hedging doesn't reduce the error.

Example for $r = d = 0$ and $\sigma_R = 30\%$, one-month atm option.

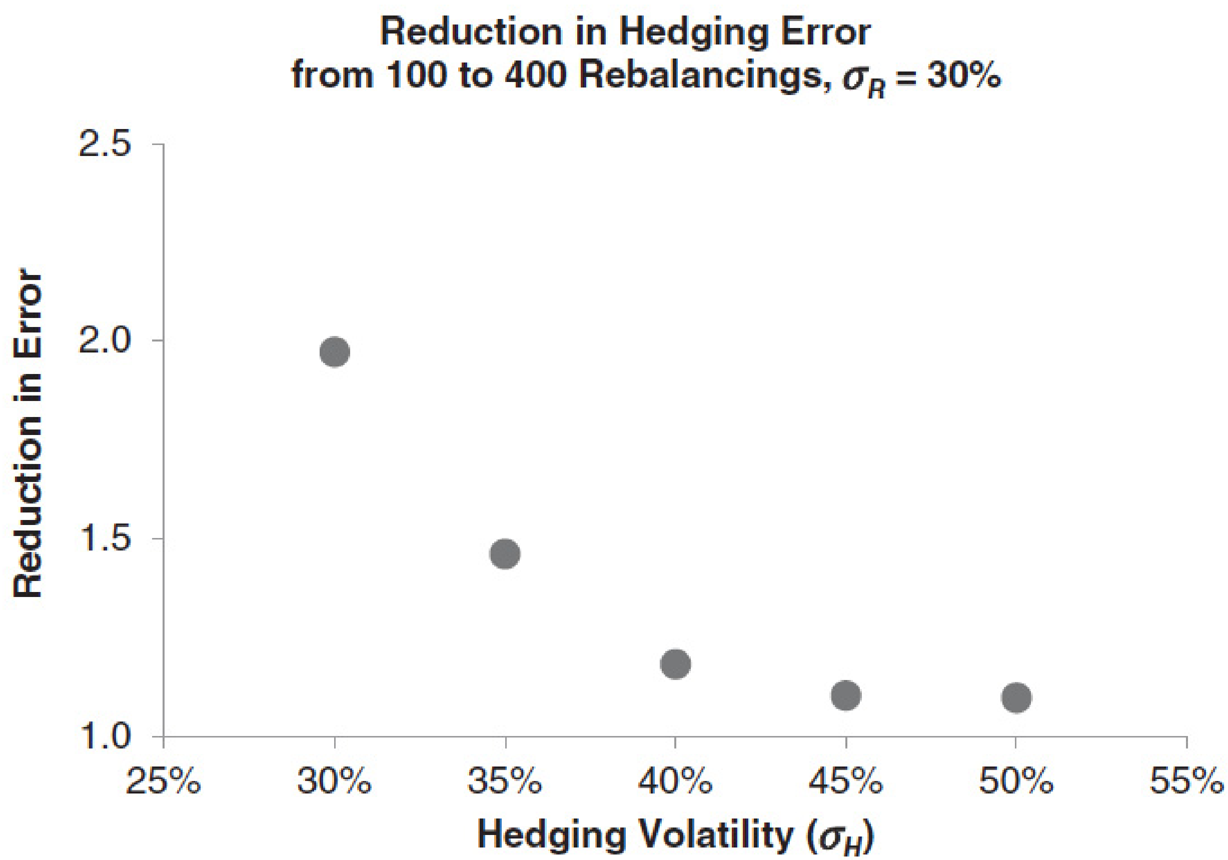


FIGURE 6.5 Reduction in Replication Error with Fourfold Increase in the Number of Rebalancings

Conclusion: Accurate Replication and Hedging are Very Difficult

- In our BSM laboratory, we assumed that we could know future realized volatility with certainty.
- In fact, you know the implied volatility from the market price of the option, but you can only try to predict future volatility. Therefore, when you hedge an option, you usually have to choose between hedging at implied volatility and hedging using a guess for the future realized volatility.
- If you estimate future realized volatility correctly and hedge (or replicate) **continuously** at that volatility, your P&L will capture the exact value of the option.
- If you hedge discretely at the realized volatility, your P&L will have a random component. You will get closer and closer to the exact BSM value the more often you hedge, with the discrepancy decreasing proportional to $1/\sqrt{n}$ where n is the number of rehedgings.
- If implied volatility is not equal to realized volatility and you hedge continuously at implied volatility, your P&L will be path-dependent and unpredictable. The P&L will be a maximum when the gamma of the option is a maximum, which occurs when the stock price stays close to the strike price on its path to expiration.
- If you hedge discretely at implied volatility, not only will your P&L be path-dependent and unpredictable, but in addition your P&L will pick up a random component that occurs because the hedge is accurate only instantaneously, but not during the intervals between rebalancing.
- In practice, traders are most likely to hedge at implied volatility. The more implied volatility differs from the realized volatility, the more they will lose the benefit of increasing the number of rehedgings.