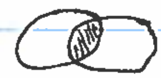
Section 2.5 Relations to convex geometry

Background on convex sets

<u>Def'n</u> A set C is said to be <u>convex</u> if for every $x_1, x_2 \in C$ and every λ with $0<\lambda<1$, $\lambda x_1 + (1-\lambda)x_2 \in C$.

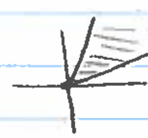
- intersection of convex sets is convex (follows quickly from def'n)
  but not the union



<u>Def'n</u> The <u>convex hull</u> of a set S, denoted $co(S)$, is the intersection of all convex sets containing S.
Can also think of as the set of all convex combinations of points in S: $co(S) = \{x = \sum_{i=1}^{m} \lambda_i x_i : \lambda_i \geq 0, \sum \lambda_i = 1, x_i \in S\}$

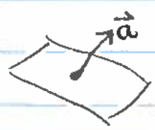
<u>Def'n</u> A set C is a <u>cone</u> if $x \in C$ implies $\alpha x \in C$ for all $\alpha > 0$.
A cone that is also convex is called a <u>convex cone</u>.

Examples: line through origin

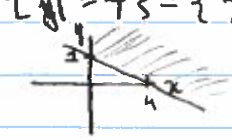


<u>Def'n</u> $\mathring{H}_+ = \{\vec{x} : \vec{a}^T\vec{x} > c\}$ (open) closed H_+ $\geq$

$\mathring{H}_-$ $<$ H_- $\leq$



Example: $\vec{a} = \begin{bmatrix}1\\2\end{bmatrix}, c = 4$ $H_+ = \{\begin{bmatrix}1\\2\end{bmatrix} \cdot \begin{bmatrix}x\\y\end{bmatrix} \geq 4\} = \{x + 2y \geq 4\}$



<u>Def'n</u> A set that is the intersection of a finite # of closed half-spaces is called a <u>convex polytope</u>.
(set of sol'ns to system of inequalities $\vec{a}_1^T\vec{x} \leq b_1, \cdots, a_m^T \vec{x} \leq b_m$)

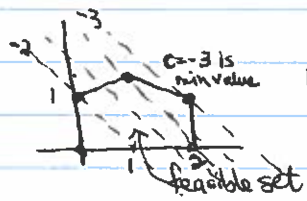
Worksheet on convex sets

Def'n An <u>extreme point</u> x of a convex set C is such that there are no two distinct pts x_1 and x_2 in C s.t.
$x = \alpha x_1 + (1-\alpha)x_2$ for some α with $0<\alpha<1$

Examples: corners of polytopes 
boundary of disk 
a line has no extreme pts 

Thm: Let A be an $m\times n$ matrix of rank m and $b\in\mathbb{R}^m$.
Let K be the convex polytope consisting of all $x\in\mathbb{R}^n$ satisfying $A\vec{x}=\vec{b}$, $\vec{x}\geq 0$.
Then a vector $\vec{x}$ is an extreme pt of K <u>iff</u> $\vec{x}$ is a basic feasible sol'n.

* Reduces solving LPs to looking at corner pts in typical problems.

~~Proof~~: Idea: 

level curves of objective fn $-x-y$
are lines $-x-y=c$, $c=$ value of obj fn at (x,y)
$y=-x-c$

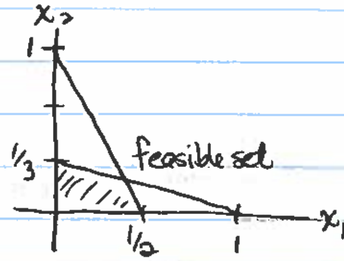
Proof: "$\Leftarrow$" Suppose $\vec{x}=\begin{bmatrix}x_1\\ \vdots\\ x_m\\ 0\\ \vdots\\ 0\end{bmatrix}$ is a basic feasible sol'n to $A\vec{x}=\vec{b}, \vec{x}\geq 0$, with 1^{st} m columns of A lin. ind.
Then $x_1\vec{a}_1+\cdots+x_m\vec{a}_m=\vec{b}$. We need to show $\vec{x}$ is an extreme pt.
Suppose it is not: $\vec{x}=\alpha\vec{y}+(1-\alpha)\vec{z}$, $0<\alpha<1$, $\vec{y}\neq\vec{z}$ in K. All components of $\vec{x}$, $\vec{y}$, and $\vec{z}$ are nonneg, as is α, the last $n-m$ components of $\vec{y}$ & $\vec{z}$ are zero (as in $\vec{x}$). So $y_1\vec{a}_1+\cdots+y_m\vec{a}_m=\vec{b}=z_1\vec{a}_1+\cdots+z_m\vec{a}_m$.
Because $\vec{a}_1,\dots,\vec{a}_m$ are lin. ind., $\vec{x}=\vec{y}=\vec{z}$, contradicting our assumption.
Hence $\vec{x}$ is an extreme pt.

$(x_1-y_1)\vec{a}_1+\cdots+(x_m-y_m)\vec{a}_m=\vec{0}$
$(y_1-z_1)\vec{a}_1+\cdots+(y_m-z_m)\vec{a}_m=\vec{0}$
$\Rightarrow x_k-y_k=0,\ y_k-z_k=0$

"$\Rightarrow$" Suppose $\vec{x}$ is an extreme pt of K. WLOG assume the (positive) nonzero components of $\vec{x}$ are $x_1,\dots,x_k$, so $x_1\vec{a}_1+\cdots+x_k\vec{a}_k=\vec{b}$. We want to show $\vec{x}$ is a basic feasible sol'n, which means we need to show $\{\vec{a}_1,\dots,\vec{a}_k\}$ lin. ind.
Suppose not: $\exists y_1,\dots,y_k$ not all zero s.t. $y_1\vec{a}_1+\cdots+y_k\vec{a}_k=\vec{0}$.
Let $\vec{y}=\begin{bmatrix}y_1\\ \vdots\\ y_k\\ 0\\ \vdots\end{bmatrix}$. Because $x_1,\dots,x_k>0$, we can find $\varepsilon>0$ s.t. $\vec{x}\pm\varepsilon\vec{y}\geq 0$.
Then $\vec{x}=\frac{1}{2}(\vec{x}+\varepsilon\vec{y})+\frac{1}{2}(\vec{x}-\varepsilon\vec{y})$. $A(\vec{x}\pm\varepsilon\vec{y})=\vec{b}+\varepsilon\vec{0}=\vec{b}$, so $\vec{x}\pm\varepsilon\vec{y}\in K$, making $\vec{x}$ not an extreme pt, a contradiction.
Hence $\vec{x}$ is a basic feasible sol'n. $\square$

Exploratory example:

$$\min_{\vec{x}\in\mathbb{R}^2} \vec{c}^T\vec{x} \quad \text{subject to} \quad \begin{aligned} 2x_1+x_2&\le 1\\ x_1+3x_2&\le 1\\ x_1\ge 0,\ x_2&\ge 0\end{aligned}$$



Standard form: $2x_1+x_2+x_3=1$
$x_1+3x_2+x_4=1$
$x_1,x_2,x_3,x_4\ge 0$

$A=\begin{bmatrix}2&1&1&0\\1&3&0&1\end{bmatrix}^*$ rank $A=2$
$\vec{b}=\begin{bmatrix}1\\1\end{bmatrix}$

Basic feasible sol'ns are those with at least 2 zeros
- rearrange columns of A to have different pairs of lin ind

$\binom{4}{2}=6$ ways columns as first two columns, row reduce $[A|\vec{b}]$ to solve, set free variables to zero (and put in original order)

$\boxed{\text{R script}}$ $\begin{bmatrix}2/5\\1/5\\0\\0\end{bmatrix}$ $\cancel{\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}}$ $\begin{bmatrix}1/2\\0\\0\\1/2\end{bmatrix}$ $\begin{bmatrix}0\\1/3\\2/3\\0\end{bmatrix}$ $\cancel{\begin{bmatrix}0\\1\\0\\-2\end{bmatrix}}$ $\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$ Correspond to 4 extreme pts

$x_3<0$ $\quad$ $x_4<0$

$\vec{c}=\begin{bmatrix}1\\1\end{bmatrix}$: $\min x_1+x_2$ sol'n $\vec{x}=\begin{bmatrix}0\\0\end{bmatrix}$, min value 0

$\vec{c}=\begin{bmatrix}-1\\-1\end{bmatrix}$: $\min -x_1-x_2$ $\vec{x}=\begin{bmatrix}2/5\\1/5\end{bmatrix}$, min value $-3/5$

$\vec{c}=\begin{bmatrix}-1\\0\end{bmatrix}$: $\min -x_1$ $\vec{x}=\begin{bmatrix}1/2\\0\end{bmatrix}$, min value $-1/2$

$\vec{c}=\begin{bmatrix}0\\-1\end{bmatrix}$: $\min -x_2$ $\vec{x}=\begin{bmatrix}0\\1/3\end{bmatrix}$, min value $-1/3$

$\vec{c}=\begin{bmatrix}1\\0\end{bmatrix}$: $\min x_1$ $\vec{x}=\begin{bmatrix}0\\x_2\end{bmatrix}$, $0\le x_2\le 1/3$ entire line segment with min value 0

If flip inequalities to $\ge$, have unbounded region for which there may be no sol'n, e.g., $\vec{c}=\begin{bmatrix}-1\\-1\end{bmatrix}$. Or the ~~[illegible]~~ feasible set could be empty, so no sol'n possible

How to discern these different cases?

Section 2.6 Farkas' Lemma

- way to check whether a feasible sol'n exists for an LP

Thm (Farkas' Lemma)

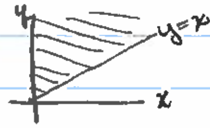
Let A be an $m\times n$ matrix and $\vec{b}\in\mathbb{R}^m$.

Then $A\vec{x}=\vec{b},\ \vec{x}\geq 0$ has a feasible sol'n $\Leftrightarrow$ $-\vec{y}^TA\geq\vec{0},\ \vec{y}^Tb=1$ has no feasible sol'n $\vec{y}$.

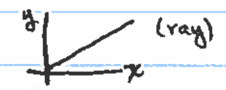
(so finding such a $\vec{y}$ for the "alternative" system means the original system is infeasible)

Lemma: Let C be the cone generated by the columns of A: $C=\{A\vec{x} : \vec{x}\geq\vec{0}\}$. Then C is a closed and convex set.

Examples: $A=\begin{bmatrix}1&0\\1&1\end{bmatrix}$ $A\vec{x}=\begin{bmatrix}x_1\\x_1+x_2\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$
$y=x_1+x_2\geq x$



$A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$ $A\vec{x}=\begin{bmatrix}x_1+x_2\\x_1+x_2\end{bmatrix}$ so $x=y$

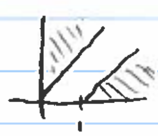


$A=\begin{bmatrix}2&0\\1&-1\end{bmatrix}$ $A\vec{x}=\begin{bmatrix}2x_1\\x_1-x_2\end{bmatrix}$ so $y\leq x/2$

Proof of thm: read pages 33-34.

Geometric interpretation: if $\vec{b}$ is not in this cone C, then there must be a hyperplane separating $\vec{b}$ and the cone C, where sol'n $\vec{y}$ to the alternative system is ~~the normal~~ a slope the normal vector to the hyperplane.

Example: $x_1-x_2\leq 0$
$-x_1+x_2\leq -1$
$x_1,x_2\geq 0$

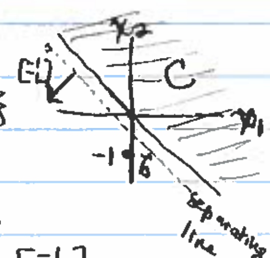


no intersection, empty feasible set (constraints are infeasible)

$A=\begin{bmatrix}1&-1&1&0\\-1&1&0&1\end{bmatrix}$
$\vec{b}=\begin{bmatrix}0\\-1\end{bmatrix}$

Standard form: $x_1-x_2+x_3=0$, $-x_1+x_2+x_4=-1$, $x_1,x_2,x_3,x_4\geq 0$

Cone $C=\{\underbrace{c_1\begin{bmatrix}1\\-1\end{bmatrix}+c_2\begin{bmatrix}-1\\1\end{bmatrix}}_{\text{any}\pm\text{multiple of }\begin{bmatrix}1\\-1\end{bmatrix}}+\underbrace{c_3\begin{bmatrix}1\\0\end{bmatrix}+c_4\begin{bmatrix}0\\1\end{bmatrix}}_{\text{can increase }x_1\text{ and/or }x_2} : c_1,c_2,c_3,c_4\geq 0\}$



Alt. system $[y_1, y_2]\begin{bmatrix}-1&1&-1&0\\1&-1&0&-1\end{bmatrix}=\begin{bmatrix}-y_1+y_2\\y_1-y_2\\-y_1\\-y_2\end{bmatrix}\geq\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\Rightarrow y_1=y_2$ so $\vec{y}=\begin{bmatrix}-1\\-1\end{bmatrix}$

and $[y_1,y_2]\begin{bmatrix}0\\-1\end{bmatrix}=1$
$-y_2=1\Rightarrow\boxed{y_2=-1}$