1. a) $p(\theta|y) \propto p(y|\theta)p(\theta) \propto \theta^{2a} e^{-y^2\theta^2} p(\theta)$

5+5+5 +5+5

A class of conjugate prior is $p(\theta) = c(\alpha,\beta)\, \theta^{\alpha} e^{-\beta\theta^2}$ which is the Galenshore distribution.

b) $p(\theta) = \frac{2}{\Gamma(\alpha)} \beta^{2\alpha} \theta^{2\alpha-1} e^{-\beta^2\theta^2}$

$p(\theta|y_1 \cdots y_n) \propto \prod_{i=1}^{n} p(y_i|\theta)p(\theta) \propto \theta^{2na} e^{-\theta^2 \sum y_i^2} \cdot \theta^{2\alpha-1} \cdot e^{-\beta^2\theta^2} = \theta^{2na+2\alpha-1} e^{-(\beta^2+\sum y_i^2)\theta^2}$

which is Galenshore $(na+\alpha, \sqrt{\beta^2+\sum y_i^2})$

c) $\frac{p(\theta_a|y_1\cdots y_n)}{p(\theta_b|y_1\cdots y_n)} = \left(\frac{\theta_a}{\theta_b}\right)^{2na+2\alpha-1} e^{-(\beta^2+\sum y_i^2)(\theta_a^2-\theta_b^2)}$

$\sum Y_i^2$ is a sufficient statistic.

d) $E[\theta|y_1\cdots y_n] = \frac{\Gamma(na+\alpha+\frac{1}{2})}{\sqrt{\beta^2+\sum y_i^2}\;\Gamma(na+\alpha)}$

e) $p(\tilde{y}|y_1\cdots y_n) = \int p(\tilde{y}|\theta)p(\theta|y_1\cdots y_n)d\theta \propto \int \theta^{2a}\tilde{y}^{2a-1} e^{-\theta^2\tilde{y}^2} \cdot \theta^{2an+2\alpha-1} e^{-(\beta^2+\sum y_i^2)\theta^2} d\theta$

$= \tilde{y}^{2a-1} \int \theta^{2(n+1)a+2\alpha-1} e^{-(\beta^2+\tilde{y}^2+\sum y_i^2)\theta^2} d\theta$

$\propto \tilde{y}^{2a-1} (\beta^2+\sum y_i^2+\tilde{y}^2)^{-((n+1)a+\alpha)}$

The integral can be derived from the density function that $\int y^{2a-1} e^{-\theta^2 y^2} dy = \frac{\Gamma(a)}{2\theta^{2a}}$

2. a) $\theta_A | x_A \sim \text{Gamma}(237, 20)$ Posterior mean is $\frac{237}{20}$, variance $\frac{237}{400}$

10+10+15 +15+10

$\theta_B | x_B \sim \text{Gamma}(118, 13)$ Posterior mean is $\frac{118}{13}$, variance $\frac{118}{169}$

$P(\theta_A > \theta_B | x_A, x_B) = F_{Beta(118,237)}\left(\frac{13}{20+13}\right) \approx 0.992$

3. $P(X=1) = \int p(x=1|p)\pi(p)dp = \frac{a}{a+b}$

10

so the variance of prior predictive distribution is $EX^2-(EX)^2 = \frac{ab}{(a+b)^2}$

$P(X=1|x_1\cdots x_n) = \int p(x=1|p)\pi(p|x_1\cdots x_n)dp = \frac{a+\sum x_i}{a+b+n}$

so the variance of posterior predictive distribution is $\frac{(a+\sum x_i)(b+n-\sum x_i)}{(a+b+n)^2}$

Note that the first variance depends on the ratio of $\frac{a}{b}$ while the other is $\frac{a+\sum x_i}{b+n-\sum x_i}$

4. a) $\int_0^1 4\sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} 4\cos y\, d\sin y = \int_0^{\frac{\pi}{2}} 4\cos^2 y\, dy = \int_0^{\frac{\pi}{2}} 2\cos 2y + 2\, dy = \pi$

5+5+5