STAT GU4221/GR5221 Homework 2 [100 pts]
Due: Thursday, March 2nd at 11:59pm (ET)

Problem 1

Let $\{X_t : t \in \mathbb{Z}\}$ be a time series defined by the *First-Order Autoregressive Model* (AR(1)):

$$X_t = \phi X_{t-1} + Z_t, \quad t \in \mathbb{Z}, \tag{1}$$

where $Z_t \sim WN(0, \sigma^2)$.

1.i Derive the unique stationary solution of the above AR(1) for $|\phi| > 1$, i.e., derive the linear process that satisfies equation (1) for $|\phi| > 1$. Uniqueness will be established in part 1.iii.

1.ii Briefly describe (in one or two sentences) why the solution obtained in part 1.i is not very useful in practice.

1.iii Show that the stationary solution derived in part 1.i is unique. To solve this problem, suppose that $\{Y_t\}$ is another stationary solution that satisfies (1), and show that the solution in (i), which can be written as an infinite sum, converges in mean square to Y_t. See the lecture notes for more guidance.

Problem 2

Let $\{Y_t : t \in \mathbb{Z}\}$ be a time series defined by the *First-Order Autoregressive Model with Non-zero Mean*:

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + Z_t, \quad t \in \mathbb{Z}, \tag{2}$$

where $Z_t \sim WN(0, \sigma^2)$ and $|\phi| < 1$. Note that $E[Y_t] = \mu$ and hence, $E[Y_t - \mu] = 0$.

2.i Using properties of the prediction operator $P(\cdot|Y_n, Y_{n-1}, \ldots, Y_1)$, derive the h-step ahead forecast, i.e., derive

$$P(Y_{n+h}|Y_n, Y_{n-1}, \ldots, Y_1), \quad h > 0.$$

Note: you don't have to solve the matrix inverse $\mathbf{\Gamma a} = \boldsymbol{\gamma}$ for this problem. Simply use properties of the prediction operator P.

2.ii Compute the *mean square prediction error* using the formula

$$E[(Y_{n+h} - P(Y_{n+h}|Y_n, Y_{n-1}, \ldots, Y_1))^2] = \gamma(0) - \mathbf{a}^T \boldsymbol{\gamma}.$$

Problem 3

Let $\{X_t : t \in \mathbb{Z}\}$ be a time series defined by the *First-Order Moving Average Model* (MA(1)):

$$X_t = Z_t + \theta Z_{t-1}, \quad t \in \mathbb{Z}, \tag{3}$$

where $Z_t \sim WN(0, \sigma^2)$.

3.i Derive the *one-step ahead forecast* X_3, based on X_2, X_1. More specifically, derive $P(X_3|X_2, X_1)$, assuming the $MA(1)$ process (3). To solve this problem, find coefficients $\mathbf{a}$ that satisfy $\mathbf{\Gamma a} = \boldsymbol{\gamma}$. See lecture notes for more details.

3.ii Derive the *expected prediction error* of $P(X_3|X_2, X_1)$.

Problem 4

Let $\{X_t : t \in \mathbb{Z}\}$ be a time series defined by:

$$X_t = Z_t + 0.5 Z_{t-1}, \quad t \in \mathbb{Z}, \tag{4}$$

where $Z_t \sim WN(0, 2^2)$.

4.i Consider using $n = 10$ cases $\{X_1, \ldots, X_{10}\}$ to perform a one-step ahead forecast $P(X_{11}|X_{10}, \ldots, X_1)$. Find the coefficients $\mathbf{a}$ that satisfy $\mathbf{\Gamma a} = \boldsymbol{\gamma}$. For full credit:

4.i.a. Display the (10×10) matrix $\mathbf{\Gamma}$ and the (10×1) vector $\boldsymbol{\gamma}$. You will have to write down each entry manually based on the true ACVF of the MA(1) process (4).

4.i.b. Solve the problem numerically, i.e., use `R` or similar to solve $\mathbf{a} = (\mathbf{\Gamma})^{-1}\boldsymbol{\gamma}$.

4.ii Compute the *expected prediction error* of $P(X_{11}|X_{10}, \ldots, X_1)$. This will be a numeric answer.

4.iii Run a simulation to check if the *theoretical expected prediction error* from part 4.ii matches the *empirical prediction error*. To solve this problem, write a loop using `10000` iterations:

`for k in 1:10000` $\{$

- `simulate time series` X_t `(MA(1)) with` $n = 11$ `observations`
- `forecast the 11th observation using` $X_{10}, X_9, \ldots, X_1$`, i.e., compute` $P(X_{11}|X_{10}, X_9, \ldots, X_1)$
- `store both the forecasted 11th case and the simulated` X_{11} `for each iteration` k

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Use the stored values to compute the *empirical prediction error* and compare this result with the *expected prediction error* from 4.ii.

Problem 5

Let $\{Y_t : t \in \mathbb{Z}\}$ be a time series defined by the *Second-Order Moving Average Model with Non-zero Mean*:

$$Y_t - \mu = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, \quad t \in \mathbb{Z}, \tag{5}$$

where $Z_t \sim WN(0, \sigma^2)$. Also assume that $\{Y_t\}$ is invertible.

5.i Show that $\{Y_t\}$ is a linear process of the form

$$Y_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$

i.e., identify the coefficients ψ_j. **Hint: this is very easy.. don't overcomplicate this problem!**

5.ii Is the process $\{Y_t\}$ causal? Explain your reasoning in one or two sentences.

5.iii Derive the covariance function $\gamma_Y(h)$. Note that you can use the following formula which computes the covariance function $\gamma(h)$ for a generic $MA(q)$ model:

$$\gamma(h) = \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}, & \text{if} \quad |h| \le q, \\ \\ 0, & \text{if} \quad |h| > q \end{cases}$$

5.iv Compute the *Long-Run Variance* of $\{Y_t\}$. Simplify the result as much as possible.

5.v Consider the sample average

$$\bar{Y}_n = \frac{1}{n}\sum_{t=1}^{n} Y_t,$$

where the Y_t's come from the MA(2) model defined in equation (5). Further, assume that the noise structure is IID, i.e., $Z_t \sim IID(0, \sigma^2)$. Identify the limiting distribution of

$$\sqrt{n}(\bar{Y}_n - \mu).$$

Note that you don't have to prove this result, you can simply reference the appropriate theorem and compute the limiting distribution's mean and variance.

Problem 6

In this problem students will prove property (4) of the *Second-order Prediction Operator* $P(\cdot|\mathbf{W})$.

Statement: Suppose that U and V are random variables such that $E[U^2] < \infty$ and $E[V^2] < \infty$. Also suppose that $\mathbf{\Gamma} = cov(\mathbf{W}, \mathbf{W})$ and that $\beta, \alpha_1, \alpha_2$ are constants. Prove that

$$P(\alpha_1 U + \alpha_2 V + \beta|\mathbf{W}) = \alpha_1 P(U|\mathbf{W}) + \alpha_2 P(V|\mathbf{W}) + \beta.$$

To prove this result, follow the steps shown below

- Assume the defining properties of the projection operator for $P(U|\mathbf{W})$ and $P(V|\mathbf{W})$. More specifically, you can assume

  $$P(U|\mathbf{W}) = E[U] + \mathbf{a}_1^T(\mathbf{W} - E\mathbf{W}), \quad \text{where} \quad \mathbf{\Gamma}\mathbf{a}_1 = cov(U, \mathbf{W}),$$

  and

  $$P(V|\mathbf{W}) = E[V] + \mathbf{a}_2^T(\mathbf{W} - E\mathbf{W}), \quad \text{where} \quad \mathbf{\Gamma}\mathbf{a}_2 = cov(V, \mathbf{W}).$$

- Note that $\mathbf{a}_1 = \mathbf{\Gamma}^{-1}cov(U, \mathbf{W})$ and $\mathbf{a}_2 = \mathbf{\Gamma}^{-1}cov(V, \mathbf{W})$, which can be substituted in the above expressions $P(U|\mathbf{W})$ and $P(V|\mathbf{W})$.
- Use the defining formula of the projection operator to simplify $P(\alpha_1 U + \alpha_2 V + \beta|\mathbf{W})$. The first step follows:

  $$P(\alpha_1 U + \alpha_2 V + \beta|\mathbf{W}) = E[\alpha_1 U + \alpha_2 V + \beta] + \mathbf{a}^T(\mathbf{W} - E\mathbf{W}),$$

  where $\mathbf{\Gamma}\mathbf{a} = cov(\alpha_1 U + \alpha_2 V + \beta, \mathbf{W})$.
- Solve for $\mathbf{a}$ and apply linearity of covariance $cov(\alpha_1 U + \alpha_2 V + \beta, \mathbf{W})$.
- Put everything together, i.e.,

  $$\begin{aligned} P(\alpha_1 U + \alpha_2 V + \beta|\mathbf{W}) &= E[\alpha_1 U + \alpha_2 V + \beta] + \mathbf{a}^T(\mathbf{W} - E\mathbf{W}) \\ &= \cdots \text{ fill in missing gaps } \cdots \\ &= \alpha_1 P(U|\mathbf{W}) + \alpha_2 P(V|\mathbf{W}) + \beta. \end{aligned}$$

- QED