# Economics 361
# Instrumental Variables

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First, some preliminaries. Under the mean squared error (MSE) criterion, the best predictor of $Y$ given $X$ is simply $E[Y|X]$. So, when $E[Y|X] = E[Y]$, $X$ is said **not** to be predictive of $Y$. Similarly, when $E[Y|X]$ varies with $X$, $X$ is said to be predictive of $Y$. So, one could say that **instruments** are random variables that are predictive of the *observable* component of some empirical model but not of the *unobservable* component of the empirical model.

Consider the usual regression model

$$
\begin{aligned}
Y_i &= X_i'\beta + \epsilon_i \quad \text{for } i = 1 \dots N \\
\underbrace{\begin{pmatrix} Y_1 \\ \vdots \\ Y_N \end{pmatrix}}_{Y} &= \underbrace{\begin{pmatrix} 1 & X_{11} & \dots & X_{k-11} \\ \vdots & \vdots & & \vdots \\ 1 & X_{1N} & \dots & X_{k-1N} \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} \beta_0 \\ \vdots \\ \beta_{k-1} \end{pmatrix}}_{\beta} + \underbrace{\begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_N \end{pmatrix}}_{\epsilon} \\
Y &= X\beta + \epsilon
\end{aligned}
$$

where $\epsilon \equiv Y - X\beta$ by definition

Consider a matrix of instruments $Z$

$$E[Y|Z] = E[X|Z]\beta + E[\epsilon|Z]$$

As a matrix of instruments, $Z$ is predictive of X and, as such, $E[X|Z]$ varies with $Z$. But $Z$ is not predictive of $\epsilon$ and, as such, $E[\epsilon|Z] = E[\epsilon]$ which does not vary with $Z$. $E[\epsilon] = 0$ without loss of generality as the regression model includes a non-specific intercept.

Instrumental variables (IV) estimator is simply the method of moments (MoM) estimator that is based on the moment condition associated with the instrument not being predictive of the unobserved component of the model: $E[\epsilon|Z] = E[\epsilon] = 0$.

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This moment condition may be re-expressed using the Law of Iterated Expectations

$$E[Z'\epsilon] \;=\; E_Z\left[\; E[Z'\epsilon|Z] \;\right] \;=\; E_Z\left[\; Z' \underbrace{E[\epsilon|Z]}_{=0} \;\right] \;=\; 0$$

To get the sample analog to the above moment condition, simply replace $\beta$ with the estimator of $\beta$

$$\epsilon \;=\; Y - X\beta \quad\Longrightarrow\quad e \;=\; Y - Xb^{IV}$$

So the sample analog to the moment condition is simply

$$\frac{1}{N}\sum_{i=1}^{N} Z_i' e_i \;=\; \frac{1}{N} Z'e \;=\; 0$$

Substituting $e = Y - Xb^{IV}$ into $Z'e = 0$

$$\begin{aligned} Z'(Y - Xb^{IV}) &= 0 \\ b^{IV} &= (Z'X)^{-1}Z'Y \end{aligned}$$

Note that $b^{IV}$ may be re-expressed as

$$b^{IV} \;=\; \left(\; (Z'Z)^{-1}Z'X \;\right)^{-1} \;\left(\; (Z'Z)^{-1}Z'Y \;\right)$$

Using the heuristic interpretation of OLS coefficients as marginal effects

- $(Z'Z)^{-1}Z'X$ is analogous to $\frac{\partial X}{\partial Z}$
- $(Z'Z)^{-1}Z'Y$ is analogous to $\frac{\partial Y}{\partial Z}$
- and therefore $b^{IV}$ is analogous to $\frac{\frac{\partial Y}{\partial Z}}{\frac{\partial X}{\partial Z}} = \frac{\partial Y}{\partial X} = \beta$

**Ordinary Least Squares (OLS) as an Instrumental Variables Estimator**

Using $X$ as the matrix of instruments, $Z = X$ and $b^{IV} = (X'X)^{-1}X'Y = b^{ols}$

**Two Stage Least Squares (2SLS) as an Instrumental Variables Estimator**

Suppose that, instead of using $Z$ directly, we used $Z$ to arrive at a linear estimator of $X$ via OLS

$$\hat{X} \;=\; Z\,(Z'Z)^{-1}Z'X$$

Note that: $\hat{X}'\hat{X} = X'Z(Z'Z)^{-1}Z'Z(Z'Z)^{-1}Z'X = X'Z(Z'Z)^{-1}Z'X = \hat{X}'X$

Using $\hat{X}$ as the matrix of instruments, $Z = \hat{X}$ and $b^{IV} = (\hat{X}'X)^{-1}\hat{X}'Y = (\hat{X}'\hat{X})^{-1}\hat{X}'Y = b^{2sls}$

So, both OLS and 2SLS estimators are instrumental variables estimators. But not all instrumental variables estimators are OLS or 2SLS.