Lecture 11:

STATIC REPLICATION.

11.1 The Breeden-Litzenberger Formula: Finding the risk-neutral probability density from call prices:

$$\exp(r\tau) \times C(S, t, K, T) \equiv \int_K^{\infty} dS_T (S_T - K) p(S, t, S_T, T)$$

$$\equiv \int_0^{\infty} dS_T (S_T - K) \theta(S_T - K) p(S, t, S_T, T)$$

$$\frac{\partial C(S,t,K,T)}{\partial K} = -e^{-r\tau} \int_K^{\infty} p(S,t,S_T,T)\, dS_T$$

$$\frac{\partial^2 C(S,t,K,T)}{\partial K^2} = e^{-r\tau} p(S,t,K,T) \qquad p(S,t,K,T) = e^{r\tau} \frac{\partial^2 C(S,t,K,T)}{\partial K^2}$$

Breeden
Litzenberger
Formula

The second derivative with respect to K of call prices is the risk-neutral probability distribution.

Exactly the same formula holds for puts. **Note the dual role** of K in the LHS and RHS:.

$$p(S,t,K,T) = e^{r\tau} \frac{\partial^2 C(S,t,K,T)}{\partial K^2}$$

stock price   strike

11.2 Static Replication: Valuing Arbitrary European Payoffs at a Fixed Expiration Without Any Model

For any V(K,t):

$$V(S,t) = \int_0^{\infty} \frac{\partial^2 C(S,t,K,T)}{\partial K^2} V(K,T)\, dK \qquad V(S,t) = \int_0^{\infty} \frac{\partial^2 P(S,t,K,T)}{\partial K^2} V(K,T)\, dK$$

If we know call prices (or put prices) and their derivatives for all strikes at a fixed expiration, we can find the value of any other European-style derivative security at that expiration.

This **involves no use of option theory at all, and no use of the Black-Scholes equation**. It works even if there is a smile or skew or jumps. As long as the options' payoffs are honored.

Replicating by standard options

Integration by parts to get V as the sum of portfolios of zero coupon bonds, forwards, puts & calls.

European payoff $V(K, T)$. K represents **the terminal stock price**.
Use puts below strike A and calls above strike A.

$$V(S, t) = \int_0^A \frac{\partial^2}{\partial K^2} P(S, t, K, T) V(K, T) dK + \int_A^{\infty} \frac{\partial^2}{\partial K^2} C(S, t, K, T) V(K, T) dK$$

Integrate by parts twice to get

$$\begin{aligned}
V(S,t) &= \int_0^A \frac{\partial^2 P(S,t,K,T)}{\partial K^2} V(K,T)\,dK + \int_A^\infty \frac{\partial^2 C(S,t,K,T)}{\partial K^2} V(K,T)\,dK \\
&= \int_0^A \frac{\partial^2 V(K,T)}{\partial K^2} P(S,K)\,dK + \int_A^\infty \frac{\partial^2 V(K,T)}{\partial K^2} C(S,K)\,dK \\
&\quad + \left[V(K,T)\frac{\partial P(S,K)}{\partial K} - P(S,K)\frac{\partial V(K,T)}{\partial K} \right]_{K=0}^{K=A} \\
&\quad + \left[V(K,T)\frac{\partial C(S,K)}{\partial K} - C(S,K)\frac{\partial V(K,T)}{\partial K} \right]_{K=A}^{K=\infty}
\end{aligned}$$

where $P(S, K)$ and $C(S, K)$ are shorthand for the current values at time t and stock price S of a put and call with strike K and expiration T. Use the following conditions for the current call and put prices.

$$\begin{aligned}
&P(S,0) = 0 \\
&\frac{\partial P(S,0)}{\partial K} = 0 \\
&C(S,\infty) = 0 \\
&\frac{\partial C(S,\infty)}{dK} = 0 \\
&P(S,K) - C(S,K) = Ke^{-r(T-t)} - S \\
&\frac{\partial P(S,K)}{\partial K} - \frac{\partial C(S,K)}{\partial K} = e^{-r(T-t)}
\end{aligned}$$

Then

$$V(S,t) = V(A,T)e^{-r(T-t)} + \left.\frac{\partial V(K,T)}{\partial K}\right|_{K=A}(S - Ae^{-r(T-t)})$$
$$+ \int_0^A \frac{\partial^2 V(K,T)}{\partial K^2}P(S,K)\,dK + \int_A^\infty \frac{\partial^2 V(K,T)}{\partial K^2}C(S,K)\,dK \qquad \text{Eq 11.1}$$

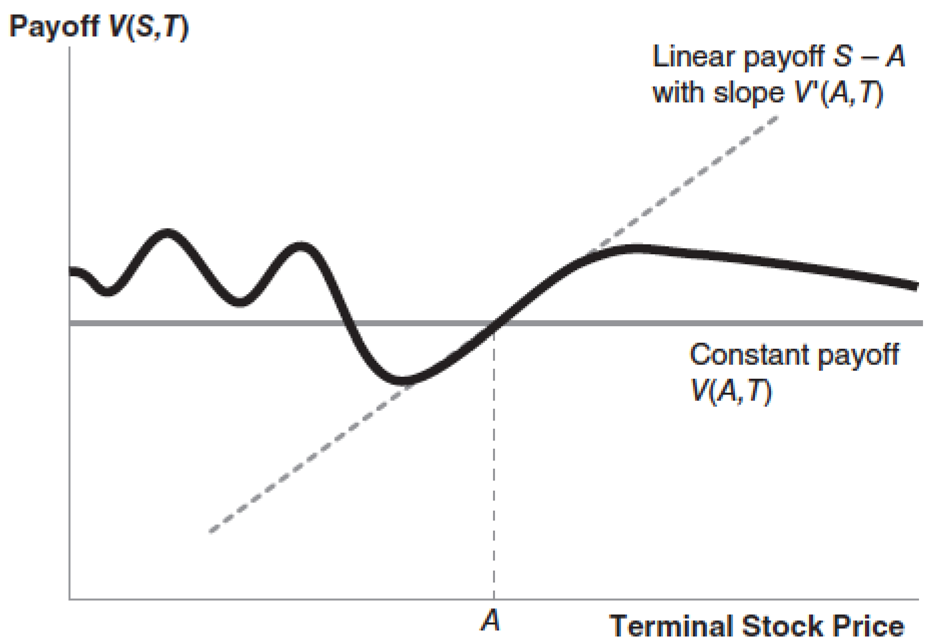


FIGURE 11.5 Replication of Exotic European Payoff

If we choose A to be the forward price $Se^{r(T-t)}$, the second term vanishes.

Two views of static replication.

- If you know the risk-neutral density ρ then you can write the value $V(S, t)$ today as a discounted integral over the terminal payoff $V(K, T)$ time the density ρ.

- Alternatively, if you know the derivatives up to $\frac{\partial^2}{\partial K^2}V(K, T)$ of the terminal payoff, you can write V(S,t) today as an integral over today's call and put prices with different strikes.

If you can buy every option in the continuum you need from someone who will never default on their payoff, then you have a perfect static replication. No math involved.

If you cannot buy every single option, then you have only an approximate replicating portfolio whose value will deviate from the value of the target option's payoff. Picking a reasonable or tolerable replicating portfolio is up to you.

This works even if there is a volatility skew.

Note: The Black-Scholes risk-neutral implied probability density corresponding to a flat skew

In the BS evolution, returns $\ln S_T/S_t$ are normal with a risk-neutral mean $r\tau - \frac{1}{2}\sigma^2\tau$ and a standard deviation $\sigma\sqrt{\tau}$, where $\tau = T - t$.

Therefore,

$$x = \frac{\ln S_T/S_t - (r\tau - \frac{1}{2}\sigma^2\tau)}{\sigma\sqrt{\tau}} \qquad \text{Eq.11.1}$$

is normally distributed with mean 0 and standard deviation 1, with a probability density $h(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$. The returns $\ln S_T/S_t$ can range from $-\infty$ to ∞. Differentiation Eq.11.1,

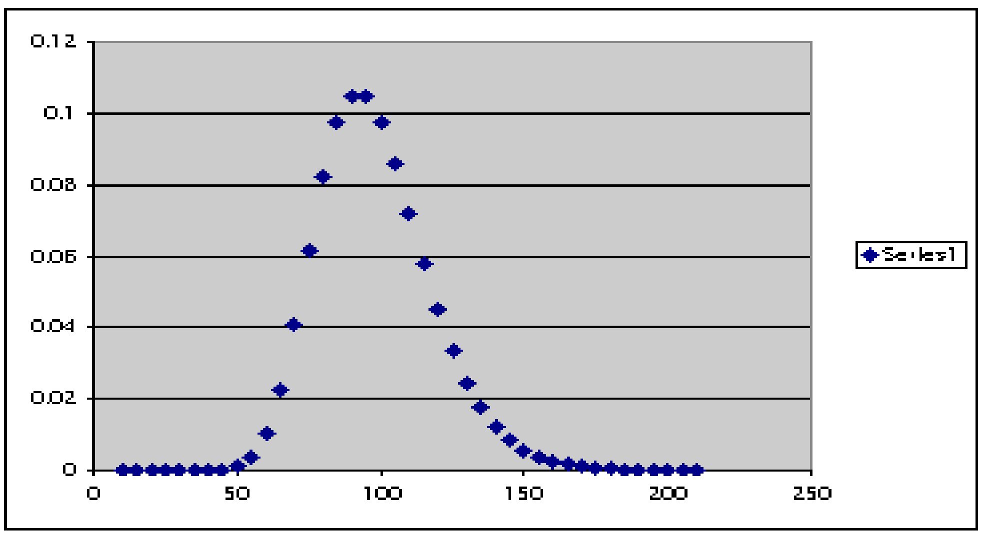
$$\frac{dS_T}{S_T} = \sigma\sqrt{\tau}dx$$

The risk-neutral value of the option is given by

$$e^{r\tau}C = \frac{1}{\sqrt{2\pi}}\int_{-d_2}^{\infty}(S_T - K)\exp\left(\frac{-x^2}{2}\right)dx = \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{K}^{\infty}(S_T - K)\exp\left(\frac{-x^2}{2}\right)\frac{dS_T}{S_T}$$

where
$$\frac{\exp\left(\frac{-x^2}{2}\right)}{\sqrt{2\pi\tau}\sigma S_T}$$

is the risk-neutral density function to be used in integrating payoffs over S_T, plotted below



11.3 A Static Replication Example in the Presence of a Skew

Consider an exotic option, strike B that pays one share of stock for every dollar in the money:

$$V(S_T) = S_T \times \max[S_T - B, 0] = S_T \times (S_T - B)H(S_T - B)$$

Quadratic curved payoff, which we must replicate.

Intuitively: We can replicate by adding together a collection of vanilla calls with strikes starting at B, and then adding successively more of them to create a quadratic payoff, as illustrated below.

$$\mathrm{V(S)} = \int_0^{\infty} q(K)\theta(K-B)\mathrm{C(S,K)dK}$$

where $q(K)$ is the unknown density of calls with strike K.

More formally, we can choose A in Equations 11.1 to be zero.

$$q(K) = \frac{\partial^2}{\partial K^2}V(K, T)$$

$$
\begin{aligned}
\frac{\partial V}{\partial K}(K) &= \frac{\partial}{\partial K}[K \times (K-B)\theta(K-B)] \\
&= (K-B)\theta(K-B) + K\theta(K-B) + K(K-B)\delta(K-B) \\
&= (K-B)\theta(K-B) + K\theta(K-B)
\end{aligned}
$$

Second derivative:

$$
\begin{aligned}
\frac{\partial^2 V}{\partial K^2} &= (K-B)\delta(K-B) + 2\theta(K-B) + K\delta(K-B) \\
&= 2\theta(K-B) + K\delta(K-B)
\end{aligned}
$$

Integrate over calls with this density: $V(S, t) = \int_B^{\infty} \frac{\partial^2}{\partial K^2} V(K, T) C(S, K) dK$

$$
\begin{aligned}
V(S,t) &= \int_B^{\infty} \frac{\partial^2 V(K,T)}{\partial K^2} C(S,K)\, dK \\
&= \int_B^{\infty} K \times \delta(K-B) C(S,K)\, dK + 2\int_B^{\infty} H(K-B) C(S,K)\, dK \\
&= BC(S,B) + 2\int_B^{\infty} C(S,K)\, dK \qquad (11.26)
\end{aligned}
$$

S***ecurity*** **V** in terms of call options $C(K)$ of various strikes K:

Security: $\boldsymbol{V} = B\mathbf{C}(\mathbf{B}) + \int_{B}^{\infty} 2\mathbf{C}(\mathrm{K})\mathrm{dK}$

Value: $V(S, t) = BC(S, t, B, T) + 2\int_{B}^{\infty} C(S, t, K, T)dK$

Payoff of 50 calls with strikes equally spaced and $1 apart between 100 and 150.

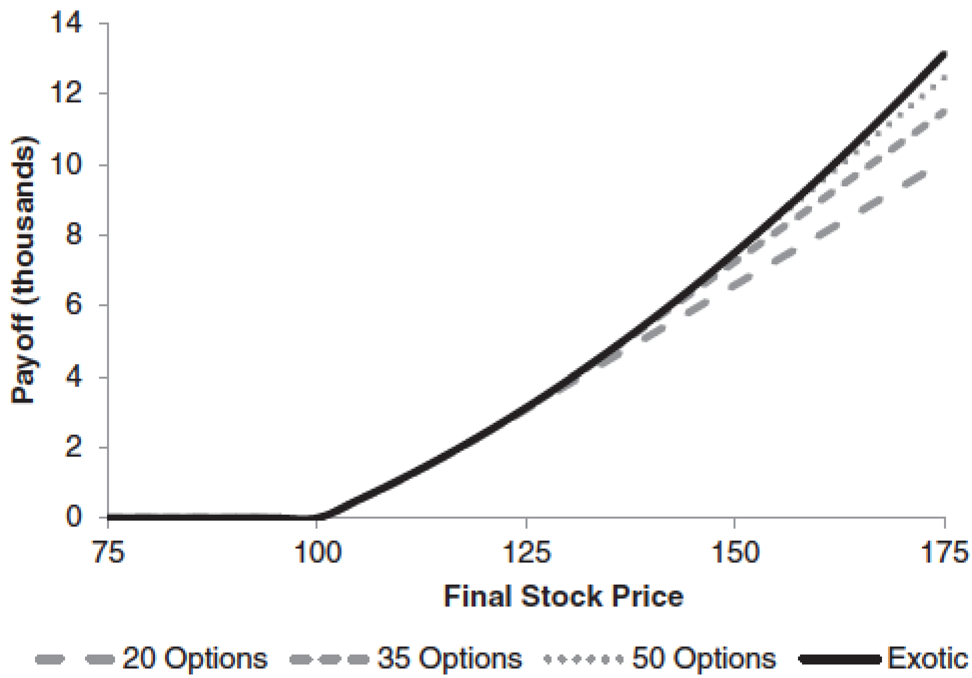


FIGURE 11.7 Approximation to Quadratic Payoff with Calls Spaced $1 Apart

Convergence of the value of the replicating formula to the correct no-arbitrage value for two different smiles.

$$\Sigma(K) = 0.2\left(\frac{K}{100}\right)^{\beta}$$

$\beta = -0.5$ “negative” skew. Implied volatility increases with decreasing strike.

$\beta = 0$ corresponds to no skew at all.

$\beta = 0.5$ corresponds to a positive skew.

For $\beta = 0$ the fair value of V when replicated by an infinite number of calls is 1033. With 10 strikes the value has almost converged.

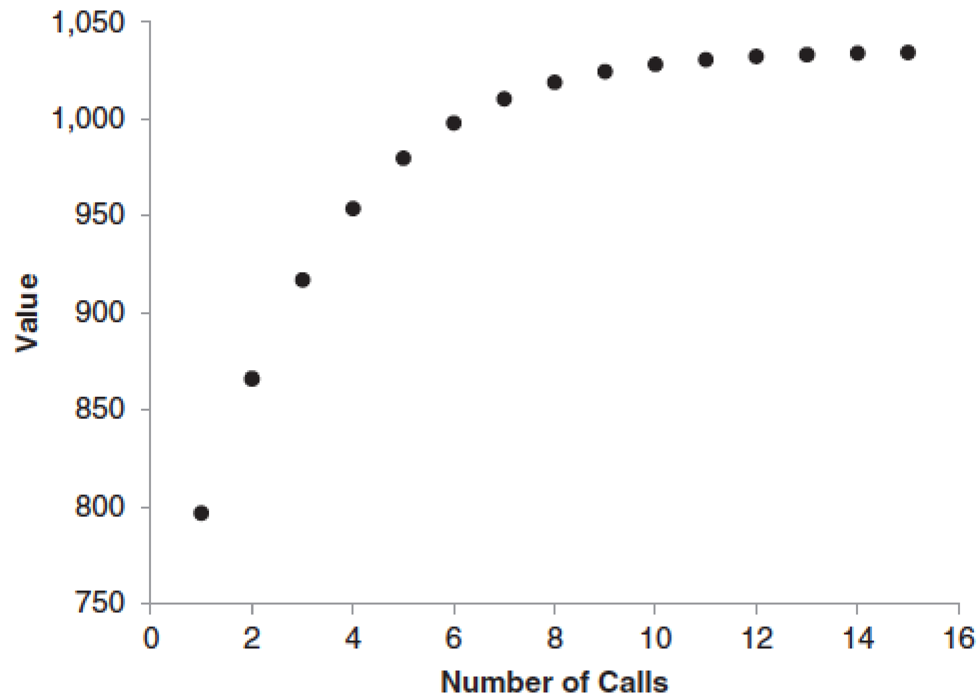


FIGURE 11.8 Convergence for No Skew, $\beta = 0$

Positive skew $\beta = 0.5$: Convergence for a positive skew to a fair value of 1100 is slower and requires more strikes

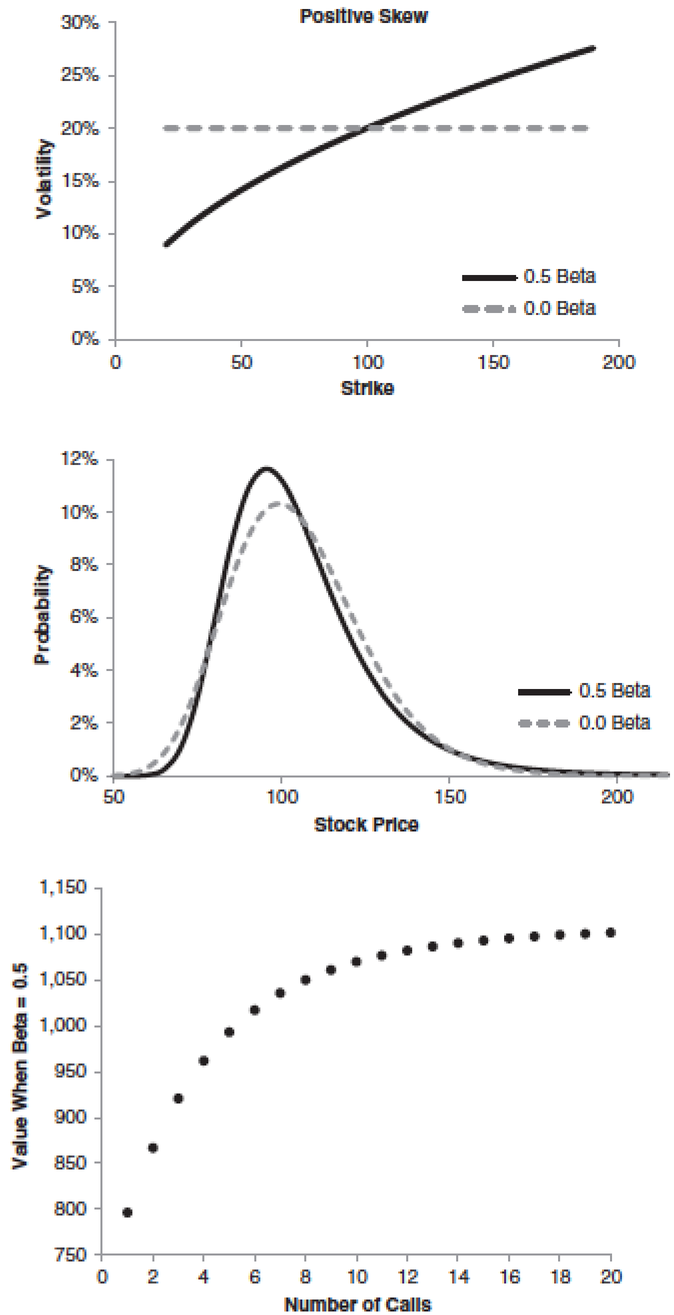
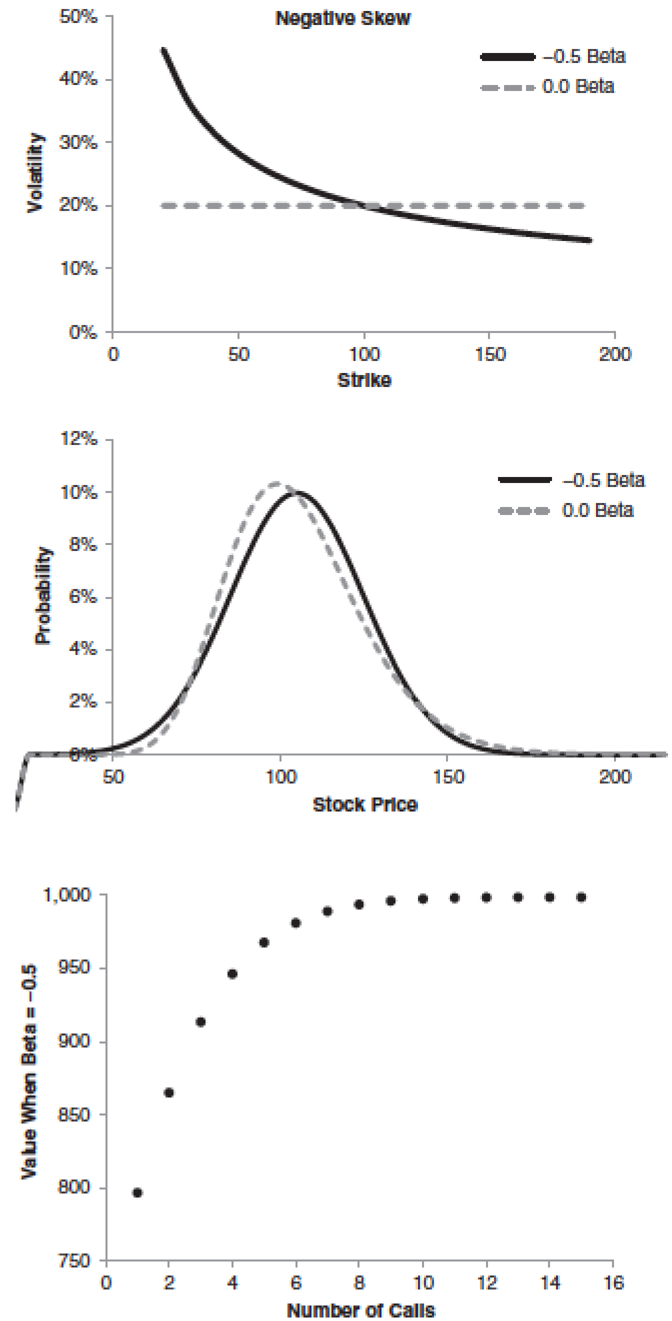


FIGURE 11.9 Convergence for Negative and Positive Skews

Negative skew $\beta = -0.5$ **:** Convergence for a positive skew to a fair value of 996 is faster and requires less strike



Weak Static Replication and Non-European Options: Some Tricks …

- Dynamic replication of exotic options requires frequent and sometimes expensive rebalancing.
- Weak static replication tries to match the payoffs of an exotic option on all its boundaries using portfolios of standard options.
- The weights of the static replication portfolio depend on the model used (as does the hedge ratio in dynamic replication).
- The portfolio often has to be unwound as the option approaches a barrier.
- There is no unique static replication portfolio. It takes art and a knowledge of valuation to find a good one.

What We've Learned

1. The most reliable way to value a security is to replicate it, and static replication is best. If you cannot find a static replicating portfolio, use dynamic replication.

2. The Black-Scholes-Merton (BSM) model relies on continuous dynamic replication. Even if the model were correct in principle, hedging errors and transaction costs limit its practical implementation unless we can eliminate most of the hedging in the portfolio.

3. Even within the scope of the BSM model, we still need to pick a volatility to use for hedging. Hedging with implied volatility leads to an uncertain path-dependent total profit and loss (P&L); hedging with future realized volatility leads to a theoretically deterministic final P&L, but might involve large fluctuations in the P&L along the way to expiration. In practice, since future volatility cannot be known, significant P&L losses along the way might make it necessary to unwind the hedge before expiration in order to limit potential future losses.

4. We showed that you can statically replicate any European payoff with a portfolio of standard puts and calls, independent of any valuation model. This is called strong replication, because it involves no assumptions about the behavior of assets or markets except the absence of credit risk. While such perfect strong replication is possible in theory, it may require an infinite number of options. In practice, therefore, one can create only approximate replicating portfolios whose mismatch with the payoff of the actual security will lead to basis risk.

11.4 Weak Static Replication of Non-European Options

- **Strong static replication:** Replication is independent of model.

  That's what we just did.

- **Weak static replication:** Weak replication needs a model and an assumption about the future evolution of the stock and its volatility etc., i.e. about the future smile. The method relies on the assumptions behind the Black-Scholes theory, or any other theory you used to replace it.

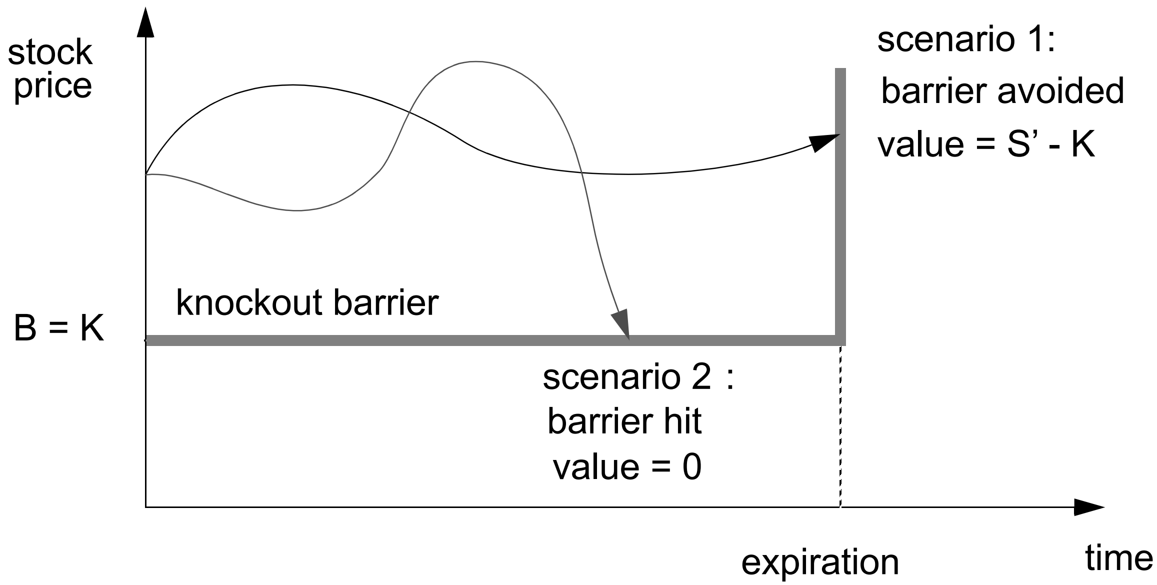
- In both cases, the costs of replication and transaction are embedded in the market prices of the standard options employed in the replication.

11.5 First Steps: Some Exact Static Hedges in Simple Cases

Sometimes you can statically replicate a barrier option with a position in stocks and bonds alone.

European Down-and-Out Call with Barrier at Strike

Scenario 1 in which the barrier is avoided and the option finishes in-the-money;
Scenario 2 in which the barrier is hit before expiration and the option expires worthless.

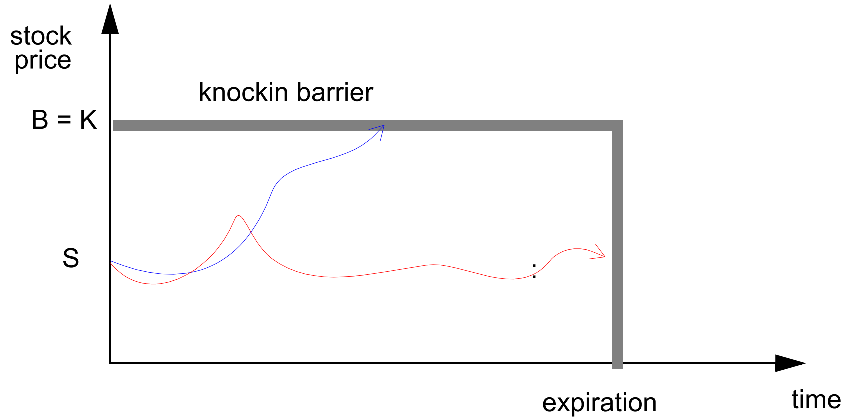


Replicate scenario 1 with a forward $F = Se^{-dt} - Ke^{-rt}$

If $d = r$ we also have perfect replication on the barrier, *if the stock moves continuously*.

An up-and-in European put with the strike K equal to the barrier. Assume r = d = 0.

Now consider an up-and-in put with strike K equal to the barrier B, as illustrated below.



Blue trajectories that hit the barrier generate a standard put $P(S{=}K, K, \sigma, \tau)$
Red trajectories that avoid the barrier expire worthless.

To replicate we need a security that expires worthless if the barrier is avoided and has the value of the put $P(K, K, \sigma, \tau)$ on the barrier, so that we can buy the put.

A standard call $C(S, K, \sigma, \tau)$ bought at the beginning will expire worthless for all values of the stock price below K at expiration. And, on the boundary $S = K$, the value
$C(S{=}K, K, \sigma, \tau) = P(S{=}K, K, \sigma, \tau)$ *if Black-Scholes with interest rates and dividend yields zero*.
At the barrier, you must *sell* the standard call and *immediately buy* a standard put. This assumes the stock moves continuously, else it could jump across the barrier before you trade.

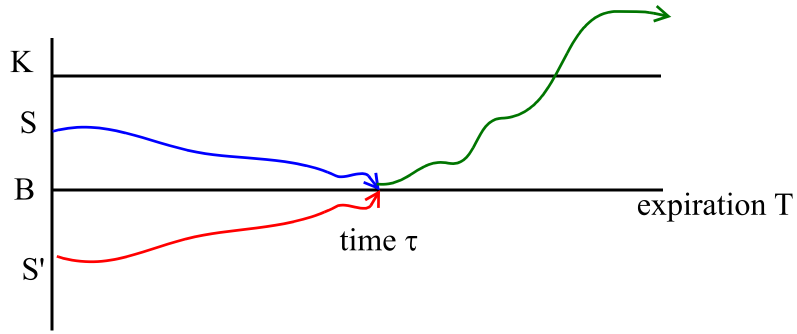
This is weak replication because it depends on the dynamics of the model (BSM with zero rates and dividends). If, for example, there is a smile when the stock touches the barrier, put-call symmetry will fail and you will not be able to exchange the call for the put at zero cost.

11.6 Valuing Barrier Options with K != B, assuming GBM

This is important because **the valuation under GBM suggests a method of replication**.

Valuing a Barrier Option for GBM with Zero Risk-Neutral Stock Drift

A **down-and-out** option with strike K and barrier B.



- **The Method of Images: cf. Electric Potentials or Mirror Images**
- Choose a "reflected" imaginary stock S' that evolves like S: The blue trajectory from S and the red trajectory from S' have equal probability to get to any point on B at time τ,.
- From there, they have equal probability of taking the future green trajectory that finishes in the money.
- For any green trajectory finishing in the money, the paths beginning at S and S' have the same probability of producing the green trajectory.
- Subtract the two probability densities to get a new density, and then above the barrier B, the contribution from every path emanating from S that touched the barrier at any time τ will be cancelled by a similar path emanating from S'. This is appropriate distribution for evaluating the payoff of a down-and-out knockout option.

Where is S'?

In GBM, the log of the stock price undergoes arithmetic Brownian motion. We saw that the returns $\ln S_T / S_t$ are normally distributed with a risk-neutral mean $r\tau - \frac{1}{2}\sigma^2\tau$ and a standard deviation $\sigma\sqrt{\tau}$, where $\tau = T - t$. The variable $x = \dfrac{\ln S_T / S_t - (r\tau - \frac{1}{2}\sigma^2\tau)mp}{\sigma\sqrt{\tau}}$ is normally distributed with mean 0, variance 1 with a probability density $N'(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$

The probability to get from S to B in a GBM world depends only on $\ln S / B$; The probability to get from S' to B depends on $\ln S' / B$. So we need the log distances from the higher to the lower to be the same:

$$\ln\frac{S}{B} = \ln\frac{B}{S'} \text{ or } S' = \frac{B^2}{S}$$

Now, if B = 100 and S = 120, then S′ = 83.33.

Thus assuming GBM, the correct density for getting from S to S_τ a time τ later, for $r = 0$, is something like

$$N'_{\mathrm{DO}}(S_\tau) = N'\left(\frac{\ln\left(\frac{S_\tau}{S}\right) + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right) - \alpha N'\left(\frac{\ln\left(\frac{S_\tau S}{B^2}\right) + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right)$$

for some coefficient α, where $N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$.

We want this density to vanish when $S_\tau = B$, which requires $\alpha = \left(\frac{S}{B}\right)$ independent of τ.

Thus the down-and-out option price under GBM is

$$C_{DO}(S, K, \sigma, \tau) = C_{BS}(S, K, \sigma, \tau) - \frac{S}{B}C_{BS}\left(\frac{B^2}{S}, K, \sigma, \tau\right)$$

Eq.11.2

Is this a reasonable answer in a BS world?

Yes. Because solutions to backward PDEs depend on the PDE and the boundary conditions.

1. One can show that C_{DO} satisfies the Black-Scholes PDE.(This is a HW problem).

2. C_{DO} has the correct boundary conditions for a down-and-out option if S > K > B.

It vanishes on boundary S = B at any time τ.

It is worth an ordinary call at expiration. If S > K > B at expiration, the first term has the payoff of a call, and since $B^2/S < K^2/S < K$ and second option finishes out of the money.

Thus C_{DO} has the correct boundary condition for an ordinary call if it hasn't knocked out along the way.

Valuation for non-zero risk-neutral drift $\mu = r - 0.5\sigma^2$

When the drift is non-zero then probabilities for reaching B from both S and S' differ, since the drift distorts the symmetry. Try a superposition of densities and S and the same reflection $S' = B^2/S$.

Trial down-and-out density for reaching a stock price S_τ a time τ later is

$$N'_{DO} = N'\left(\frac{\ln\left(\frac{S_\tau}{S}\right) - \mu\tau}{\sigma\sqrt{\tau}}\right) - \alpha N'\left(\frac{\ln\left(\frac{S_\tau S}{B^2}\right) - \mu\tau}{\sigma\sqrt{\tau}}\right)$$

We want density to vanish on the barrier at any time:

$$N'\left(\frac{\ln\left(\frac{B}{S}\right) - \mu\tau}{\sigma\sqrt{\tau}}\right) - \alpha N'\left(\frac{\ln\left(\frac{S}{B}\right) - \mu\tau}{\sigma\sqrt{\tau}}\right) = 0$$

leads to $\alpha = \left(\frac{B}{S}\right)^{\frac{2\mu}{\sigma^2}} = \left(\frac{B}{S}\right)^{\frac{2r}{\sigma^2}-1}$ independent of τ, as we would like, where $\mu = r - 0.5\sigma^2$.

Integrating over the terminal stock price, we obtain

$$C_{DO}(S,K) = C_{BS}(S,K) - \left(\frac{B}{S}\right)^{\frac{2\mu}{\sigma^2}} C_{BS}\left(\frac{B^2}{S}, K\right)$$

11.7 Insight into Static Replication from the GBM Valuation Formula for Barrier Options in the Previous Section

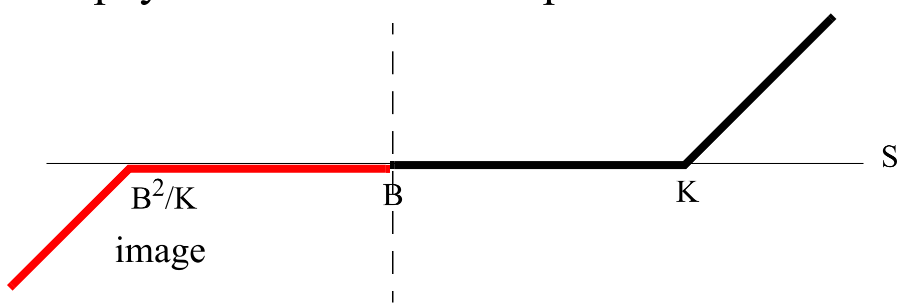
We showed above that, in a Black-Scholes world with zero rates, the fair value for a down-and-out call with strike K and barrier B is given by

$$C_{DO}(S, K) = C_{BS}(S, K) - \frac{S}{B} C_{BS}\left(\frac{B^2}{S}, K\right) \qquad \text{Eq.11.3}$$

Payoff of first term at expiration is: $\theta(S-K)(S-K)$

Payoff of $\frac{S}{B} C_{BS}\left(\frac{B^2}{S}, K\right)$ $\quad \frac{S}{B}\left(\frac{B^2}{S} - K\right)\theta\left(\frac{B^2}{S} - K\right) = \left(B - \frac{KS}{B}\right)\theta\left(\frac{B^2}{K} - S\right) = \frac{K}{B}\theta\left(\frac{B^2}{K} - S\right)\left(\frac{B^2}{K} - S\right)$

This second term represents the payoff of *K/B* standard puts with strike B^2/K.



Roughly speaking in GBM the payoff of a down-and-out-call is that of

1. long an ordinary call with strike K, and
2. short a put with its strike the image of K image as reflected (in log space rather than linear space) in the barrier.

$$\frac{K}{B} = \frac{B}{\text{image}} \qquad \text{image} = \frac{B^2}{K}$$

Think of replicating a down-and-out call by going long a call with with the same strike and positive expected value, and short the right amount of puts (negative expected value) with reflected strike. The weighting must be such that the expected value on the barrier must be zero. In GBM the weighting is $\frac{K}{B}$.

This is weak replication because

you need a model to find the value of a standard option on the barrier;

you need to close out the position if the barrier is touched

Think: what strike-volatilities are you sensitive to in this portfolio? Although this insight was derived from the formula for valuation in a BSM world, this is a sensible way to think about replicating a down-and-out barrier option in general.

If you can go long a call with strike above the barrier and short the right amount of puts with strike below the barrier, you will have the correct payoff both at expiration and on the barrier:

- At expiration if the stock has never touched the barrier, the call with strike K will have the correct payoff of a down-and-out call, and the put will expire out-of-the-money.
- If the stock S does touch the barrier at B before expiration, then the net value of the long call and short put positions will be close to zero if you can short the correct amount of puts. At that point, you *must* close out the position to replicate the extinguishing of the down-and-out call option.
- The number of puts required is K /B only if the stock price undergoes geometric Brownian motion with constant volatility. More generally, the number will depend on how you model the smile, but the general strategy still has validity even if we depart from a BSM world.
- It suggests model independence for the barrier option since it can be decomposed into vanilla calls and puts that depend only on the skew.