Price Impact Models and Applications

Introduction to Algorithmic Trading

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Plan

Last Week
Application to optimal execution

For this Week
Waelbroeck's simulation engine.

(a) Detrending historical prices.
(b) Backtest new strategies.
(c) Application to statistical arbitrage.

Next Week
Applications to risk management

Last Week's Summary (1/2)

Communicate a pre-trade cost model
E.g., under the OW model,

$$TC(Q) = -\frac{\Lambda_T}{2} Q^2; \quad \mathbb{E}[Y_T] = \frac{\alpha}{2} Q$$

where

$$\Lambda_T = \frac{\sigma}{(2 + \beta T)\mathrm{adv}}.$$

Consider intraday signals for the trading schedule
Under the OW model, an intraday alpha signal α_t changes the order by

$$Q_T(\alpha) = \int_0^T \frac{\mathrm{adv}}{2\sigma} (\beta \alpha_t - \alpha'_t)\, dt.$$

Last Week's Summary (2/2)

Micro-alphas are de-centralized signals.
They lead to tactical deviations form the trading schedule. For the OW model,

$$\mathbb{E}\left[Y_T(Q^r)\right] - \mathbb{E}\left[Y_T(Q^*)\right] = -\mathbb{E}\left[\beta \int_0^T \frac{\text{adv}}{\sigma}\left(\delta I_t\right)^2 dt + \frac{\text{adv}}{2\sigma}\left(\delta I_T\right)^2\right].$$

For multiple orders, update the implied alpha.
The myopic relationship updates the target impact state.

Introduction

Refresher on Backtesting vs A-B Testing (1/2)

What is a back test?
Back tests simulate competing strategies on identical input data.

Simulations rely on a *price impact model* to simulate prices for novel strategies.

What is an A-B test?
A-B tests are *live trading experiments* where a coin flip determines the trading strategy.

For example, one could randomize the decision to use strategy A or B to statistically estimate which performs best.

Refresher on Backtesting vs A-B Testing (2/2)

When to use which test?

(a) Traders use back tests prior to deploying a strategy: they are forward-looking but entail model-risk.

(b) Traders use A-B tests after live trading: they are backward-looking and subject to trading biases.

This week we only cover back tests.
A-B tests require careful statistics to avoid common trading biases.
However, A-B tests are crucial to trading: please use with care.

Trading Algorithm Refresher (1/2)

The Research Schema

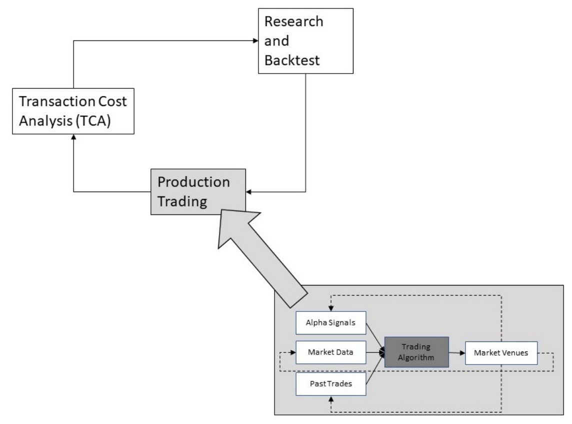


Figure 1: Research cycle schema.

Trading Algorithm Refresher (2/2)

Back tests simulate production trading

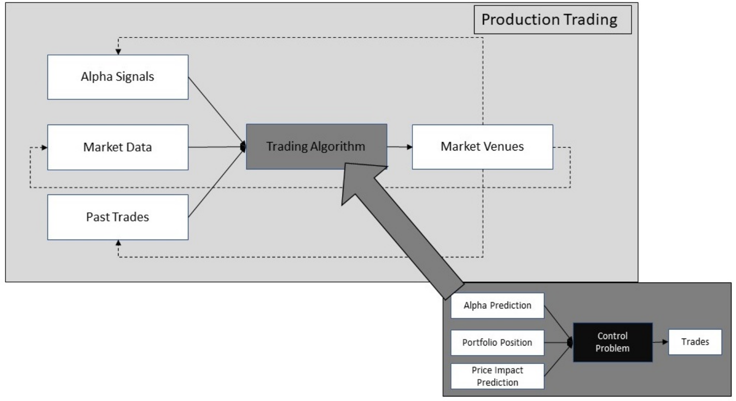


Figure 2: Schema in production trading: live trading connects inputs and outputs in a *feedback loop.* Back tests mimic this feedback loop via a *market simulator.*

Waelbroeck's Backtest Algorithm

Why is a Market Simulator so Important?

Powrie, Zhang, and Zohren (Man Group, 2021)

"The powerful aspect of having a market simulator is that it effectively enables the re-running of different realities where our interventions with the market either do, or do not, occur."

This relates back to the causality challenge from the first week.

Trading of stock cause price moves for the stock that otherwise would not have happened.

Waelbroeck's Backtest Algorithm (1/4)

> *"The customer chose a 20% participation rate, and one observes the P/(L) of [an execution with] 20% [participation rate]. The customer has a loss of 15bps.*
> *Would the customer have had a lower loss if he had picked a 10% participation rate? To answer that question entails simulating the P/(L) the customer would have gotten if the customer had picked the 10% participation rate."*

Example: simulating whether a slower order has higher P&L.
A participation rate is a trading speed measure. For instance, a 20% participation order trades 200 shares for every 1000 shares traded on the market.

Waelbroeck's Backtest Algorithm (2/4)

i) "take the observed prices [...]
ii) subtract from observed prices the impact of the execution at 20% to see what the impact-free price is [...]
iii) calculate the average impact-free price for the execution at 10% [...]
iv) to get the P/(L) at 10% one then needs to add to the impact-free price the impact of the execution at 10%

If one did not take impact into account, the only thing that one would notice is that 10% takes more time, and if the stock moves away the customer will incur more losses. If the impact is not taken into account, one will not see how much is saved in impact by lowering the speed."

Waelbroeck's Backtest Algorithm (3/4)

One sentence summary

> *"To make a good assessment of alternative strategies, one may wish to first subtract out the impact of those strategies to then be able to simulate accurately alternative strategies."*

Waelbroeck's Backtest Algorithm (4/4)

Assume given the following inputs:

(a) Realized data Q^r, P^r for the historical strategy and prices.
(b) A new strategy Q.
(c) A price impact model I.

Define the following outputs:

$$S_t = P_t^r - I_t(Q^r)$$

is the unperturbed (detrended) price.

$$P_t(Q) = P_t^r - I_t(Q^r) + I_t(Q)$$

is the simulated price under the new trading process Q.

Example

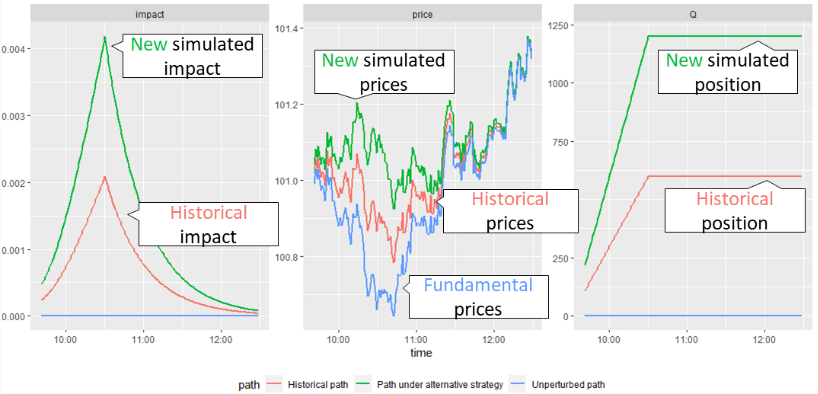


Figure 3: Simulated example of realized, unperturbed, and simulated prices.

Advantages and Disadvantages

Advantages

(a) Applies to any trading strategy, including discretionary trades.

(b) Applies to any price impact model.

(c) Full market simulation: produces P&L *and* auxiliary market metrics (e.g. price dislocations).

Disadvantages

(a) A two-step computation: one simulate trades, then P&L.

(b) Carries model risk.

(c) Can lead to over-confidence and over-fitting (Lopez de Prado, ADIA, 2014, 2016).

Applications of Detrended Prices

Alpha research
Alpha signals predict price moves *not caused* by one's trading.

Distinguish the formulas

$$\alpha_t = \mathbb{E}\left[\, S_T - S_t \middle|\, \mathcal{F}_t\right]$$

and

$$\mathbb{E}\left[\, P_T - P_t \middle|\, \mathcal{F}_t\right] = \alpha_t + \mathbb{E}\left[\, I_T - I_t \middle|\, \mathcal{F}_t\right].$$

Application to risk management
Next week covers liquidity risk models. Detrended prices play a crucial role.

Statistical Arbitrage (Theory)

Liquidity Signal using a Volume Model

Price impact with dynamic liquidity

$$dI_t = -\beta I_t dt + \lambda_t dQ_t$$

with

$$\lambda_t = e^{\gamma_t}; \quad d\gamma_t = \gamma_t' dt.$$

Example: reduced form model

$$\lambda_t = \frac{\lambda}{\sqrt{v_t}}$$

where v_t are observed market volumes. Furthermore,

$$\gamma_t' = -\frac{v_t'}{2v_t}$$

is proportional to percentage changes in market volumes.

General Alpha Signal

Using your favorite machine learning model,
you estimate

$$\alpha_t = \mathbb{E}\left[S_T - S_t \middle| \mathcal{F}_t \right]$$

and its decay $-\mu_t$ where

$$d\alpha_t = \mu_t dt + \sigma_t dW_t.$$

Standard signal examples from Almgren (Quantitative Brokers, 2018)

(*) "Prices overshoot and relax
(*) Related assets tend to move together
(*) Presence of imbalance in the market"

Statistical Arbitrage Problem Setup

Objective function
The trading algorithm maximizes its expected wealth,

$$\sup_{Q} \mathbb{E}\left[\int_0^T (\alpha_t - I_t)\, dQ_t + [\alpha, Q]_T - \frac{1}{2}[I, Q]_T\right].$$

A daily strategy
For simplicity, we simulate the strategy daily, resetting all variables the next day.

(a) Let T be the end of the trading day, also called the *close*.

(b) Let the impact model's half-life be $\log(2)/\beta = 1$ hour.

(c) The alpha's prediction horizon $T - t$ *shrinks* as the day goes on.

Optimal Trading Strategy

In impact space

$$I_t^* = \frac{\beta + \gamma_t'}{2\beta + \gamma_t'}\alpha_t - \frac{1}{2\beta + \gamma_t'}\mu_t$$

Gârleanu and Pedersen (AQR, 2013)

> *"The alpha decay is important because it determines how long time the investor can enjoy high expected returns and, therefore, affects the trade-off between returns and transactions costs."*

Consider alpha decay early in the research process
For instance, one may use the same features to jointly fit (α_t, μ_t): α_t predicts the close, and μ_t a short horizon (e.g., 5 minutes).

Expected Profits

For the optimal strategy I^*,

$$\mathbb{E}\left[Y_T(I^*)\right] = \mathbb{E}\left[\int_0^T \frac{2\beta_t + \gamma_t'}{2\lambda_t}\left(I_t^*\right)^2 dt\right].$$

For any other strategy I,

$$\mathbb{E}\left[Y_T(I)\right] = \mathbb{E}\left[Y_T(I^*)\right] - \mathbb{E}\left[\int_0^T \frac{2\beta_t + \gamma_t'}{2\lambda_t}\left(I_t^* - I_t\right)^2 dt + \frac{1}{2\lambda_T} I_T^2\right].$$

Implications for Alpha Research (2/3)

Naive regression
applies a mean-squared error (MSE) for alpha

$$\sum_t w_t \left(\alpha_t^r - \alpha_t\right)^2$$

where w_t is a weighting scheme (e.g., equal-weighted).

P&L-aware regression minimizes P&L regret

$$\mathbb{E}\left[Y_T(I(\alpha^r))\right] - \mathbb{E}\left[Y_T(I(\alpha))\right] =$$
$$\mathbb{E}\left[\int_0^T \frac{2\beta_t + \gamma_t'}{2\lambda_t}\left(I_t(\alpha^r) - I_t(\alpha)\right)^2 dt + \frac{1}{2\lambda_T} I_T^2(\alpha)\right].$$

Therefore, maximizing P&L is the same as minimizing MSE for impact.

Implications for Alpha Research (3/3)

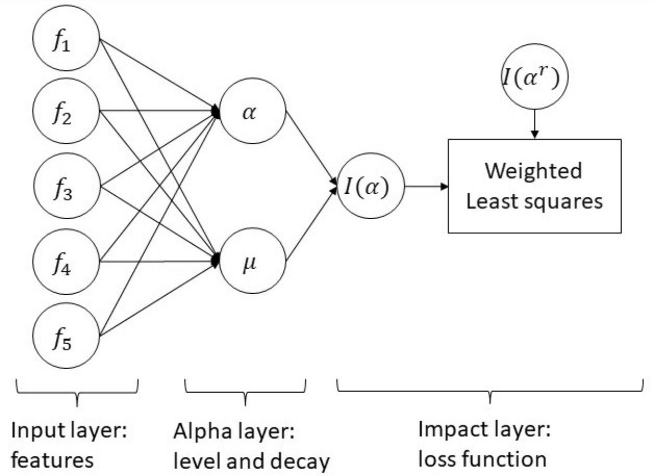


Figure 4: Implementation of the anticipated P&L regret function in a layered machine learning architecture allowing for backpropagation.

Statistical Arbitrage (Implementation)

Discretizing Equations

General principle
There are multiple ways to choose time-scales to discretize variables and equations (e.g. the timestep $\delta t = 10$ seconds). Two ways to pick:

(a) Pick the time-scale where the sensitivity is lowest.

(b) Pick the time-scale where simulated and live trading data match best.

Example
For $\Delta t = 5$ minutes, one can use

$$\mu_t = \frac{1}{\Delta t}\mathbb{E}\left[\left.\frac{S_{t+\Delta t} - S_t}{S_t}\right| \mathcal{F}_t\right].$$

5 minutes is long enough to ignore microstructure noise, but short enough to capture short-term decay for an alpha until the close. This choice probably fails in the last 20 minutes of the day!

Alpha Predictions

Use two prediction horizons
For instance, if T is the close and $\Delta t = 5$ minutes, one can use

$$\alpha_t = \mathbb{E}\left[\left. \frac{S_T - S_t}{S_t} \right| \mathcal{F}_t \right]$$

and

$$\mu_t = \frac{1}{\Delta t}\mathbb{E}\left[\left. \frac{S_{t+\Delta t} - S_t}{S_t} \right| \mathcal{F}_t \right]$$

from a synthetic look-ahead alpha like in the homework.

Liquidity Signals

Volume model
For instance, one can use ($\delta t = 10$ seconds)

$$v_t = \sum_{s=t-\Delta t}^{t} |q_s|$$

as market volume estimates.

Liquidity signal

$$\lambda_t = \frac{\lambda}{\sqrt{v_t}}; \quad \gamma_t' = \frac{|q_t| - |q_{t-\Delta t}|}{2\delta t \cdot v_t}.$$

Translating Signals into a Target

Target in impact space

$$I_t^* = \frac{\beta + \gamma_t'}{2\beta + \gamma_t'}\alpha_t - \frac{1}{2\beta + \gamma_t'}\mu_t.$$

A helpful heuristic
Empirically, one often observes $\gamma_t' \ll \beta$. This "slow-moving liquidity" assumption simplifies the target impact formula:

$$I_t^* = \frac{1}{2}\left(\alpha_t - \beta^{-1}\mu_t\right).$$

The optimal impact state doesn't depend on γ_t' anymore!
But one still needs λ_t to translate the target impact I^* into a trading strategy Q^*.

Example

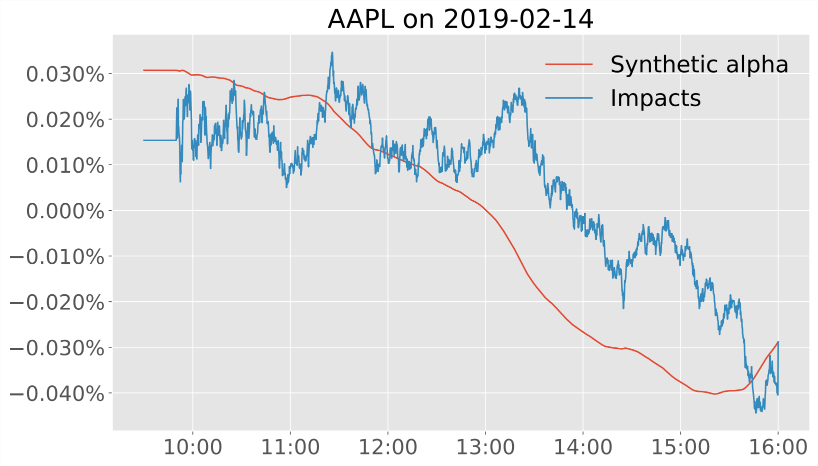


Figure 5: Sample path of a smooth alpha signal and a target impact state under dynamic liquidity.

Translating a Target into Trades

Use the inverse map (discretized SDE)

$$\Delta_t Q = \frac{1}{\lambda_t}\left(\beta I_t \delta t + \Delta_t I\right).$$

Simplified heuristic formula
If $\gamma_t' \ll \beta$, the position formula is more robust:

$$Q_t = \frac{1}{\lambda_t} I_t + \sum_{s \le t} \frac{\beta}{\lambda_s} I_s \delta t.$$

Example

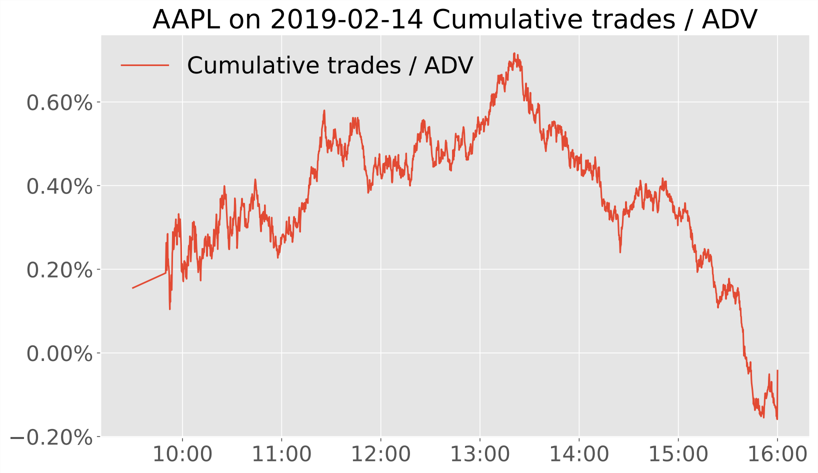


Figure 6: Sample optimal trade path.

General Applications of a market simulator

Quantitative Brokers (2016)
Almgren highlights four applications in *Using a Simulator to Develop Execution Algorithms.*

(*) "Historical

- rerun scenarios for algo improvement
- backtest for potential clients

(*) Real-time

- clients can connect to "test-drive" algos

(*) Algorithm development

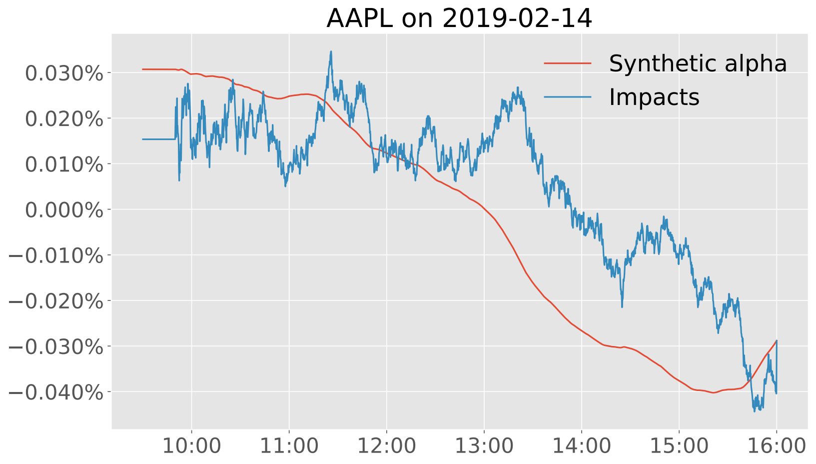
- test new signals on historical orders
- multi-market legging trades

(*) Real-time splitting for testing

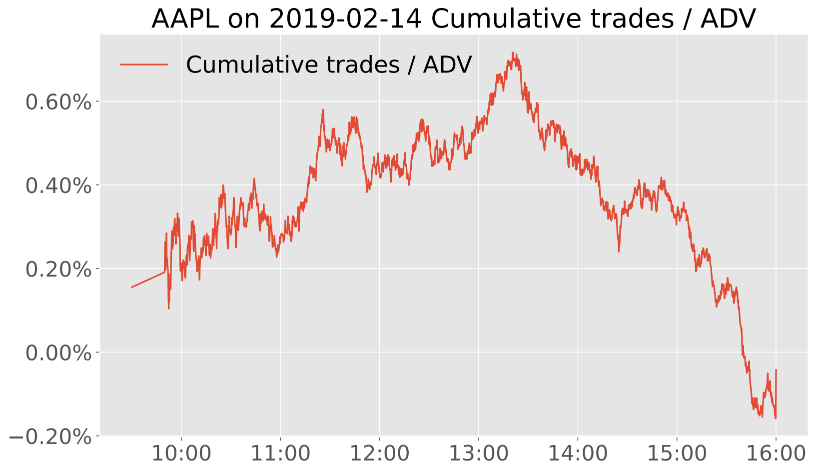
- compare simulator executions with real"

Examples

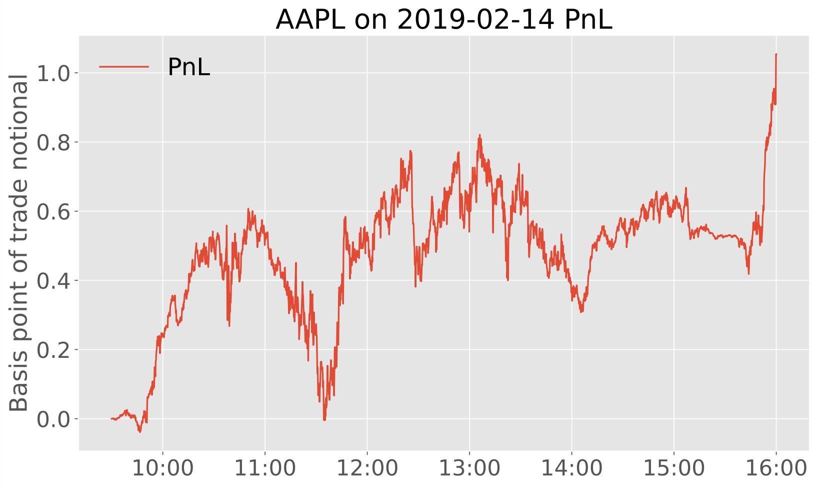
Alphas and impact



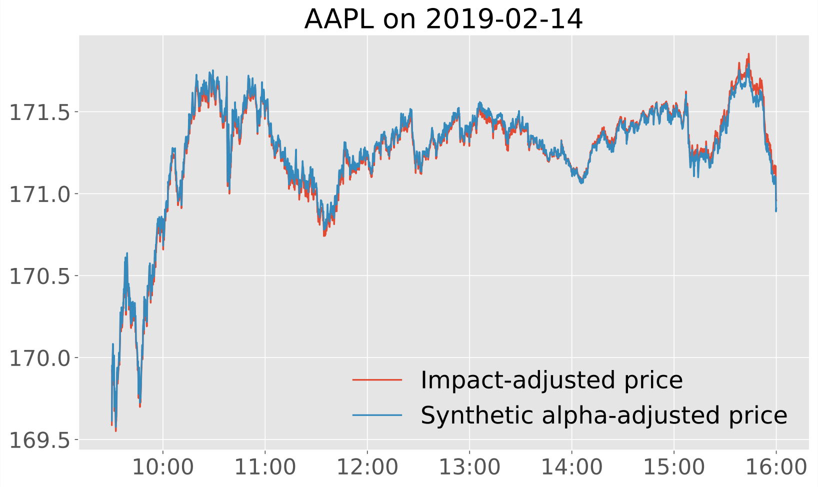
Trades


AAPL on 2019-02-14 Cumulative trades / ADV
Cumulative trades / ADV
0.60%
0.40%
0.20%
0.00%
−0.20%
10:00
11:00
12:00
13:00
14:00
15:00
16:00

P&L


AAPL on 2019-02-14 PnL
PnL
Basis point of trade notional
1.0
0.8
0.6
0.4
0.2
0.0
10:00
11:00
12:00
13:00
14:00
15:00
16:00

Comparison to unperturbed and VWAP prices



Daily S&L 500 simulation (1/2)

$\tilde{\alpha}$	strategy	size (% of ADV)	impact (bps)	slippage (bps)
50bps	TWAP	14	37	37
50bps	no alpha	15	27	26
50bps	with alpha; $\rho = 1\%$	15	27	25
50bps	with alpha; $\rho = 5\%$	15	27	24
50bps	heuristic; $\rho = 5\%$	15	27	24
50bps	with alpha; $\rho = 10\%$	15	27	23

Table 1: Average performance metrics.

Daily S&L 500 simulation (2/2)

$\tilde{\alpha}$	strategy	size (% of ADV)	impact (bps)	slippage (bps)
10bps	TWAP	3	7.5	7.0
10bps	no alpha	3	5.3	4.4
10bps	with alpha; $\rho = 5\%$	3	5.6	-0.1
10bps	heuristic; $\rho = 5\%$	3	5.6	1.4
100bps	TWAP	27	75	74
100bps	no alpha	30	53	52
100bps	with alpha; $\rho = 5\%$	30	52	52
100bps	heuristic; $\rho = 5\%$	30	52	52

Table 2: Average performance metrics.

Weekly Summary

Waelbroeck's backtest algorithm

> *"To make a good assessment of alternative strategies, one may wish to first subtract out the impact of those strategies to then be able to simulate accurately alternative strategies."*

Statistical arbitrage theory
Consider alpha decay in alpha research and trading strategies.

Statistical arbitrage implementation

(a) Given alpha level and decay, implement target impact state.

(b) Invert the map to translate impact into trades.

Questions?

Next week
Applying price impact to risk management.

(a) Distinguishing between position entering and exiting trades.
(b) How price impact distorts mark-to-market P&L.
(c) Application to portfolio scaling.
(d) Simulating fire sales.