COPYRIGHT EMANUEL DERMAN 2022

Homework 3:February 6, 2023
Due Wednesday February 15, 2022.

Total: **[100]**

Problem 1: Expected P&L when Hedging at Implied Volatility **[20 points]**

We showed that the present value at time t_0 of the P&L when hedging an option at implied volatility until it matures at time T is given by:

$$\mathrm{PV[P\&L]} = \frac{1}{2}\int_{t_0}^{T} \Gamma_i S^2(\sigma_r^2 - \sigma_i^2)e^{-r(t-t_0)}dt$$

where Γ_i is evaluated at each time t at the prevailing stock price using the implied volatility σ_i.

Write a Monte Carlo program to simulate the stock evolution in discrete steps Δt with the formula $\Delta S = \mu S \Delta t + \sigma_r S dZ(t)$ with initial price 100, 5% annual continuously compounded dividend yield, a 25% drift (total continuously compounded growth rate including dividends), a 10% riskless continually compounded discount rate, an implied annual volatility of 15% and a realized annual volatility of 30%, with one year to expiration. As the stock evolves, evaluate the incremental realized P&L from that stock move, and sum over all of increments as in the integral above. Since the P&L is path dependent, choose 100 time steps and 10,000 stock paths for the simulation and calculate the P&L for each path.

Find the value of the expected P&L for an option with strike equal to 99% of spot and a time to expiration of 12 months. (Not the present value of the expected P&L, just the expected P&L).

Solution 1: Expected P&L when Hedging at Implied Volatility.

The expected P&L for $\boxed{\mu = 0.25}$ is about 4.9% of spot

Problem 2: Hedging with an Arbitrary Volatility **[20 points]**

Suppose we buy an option at an implied volatility Σ and hedge it to expiration at a volatility σ_h, the hedge volatility, when the realized volatility is σ_r. Assume all volatilities are constant through the life of the option that expires at time T. Extend the argument in Lecture 6 to show that in this more general case the total present value of the P&L is given by

$$PV(\text{P\&L}) = V_h - V_i + \frac{1}{2}\int_{t_0}^{T} e^{-r(t-t_0)}\Gamma_h S^2(\sigma_r^{\ 2} - \sigma_h^2)dt$$

Solution 2: Hedging with an Arbitrary Volatility

Suppose, for the more general case, we buy an option at an implied volatility $\boxed{\Sigma}$ and hedge it to expiration at a volatility $\boxed{\sigma_h}$, the hedge volatility, when realized volatility is $\boxed{\sigma_r}$. Here is the computation of the change in P&L over a time dt:

Position Values when Hedging with an Arbitrary Volatility

Time	Option Position, Value	Stock Position, Value	Value of Cash Position	Net Position Value
t	$\vec{V}_i$, V_i	$-\Delta_h\vec{S}$	$\Delta_h S - V_i = (\Delta_h S - V_h) + (V_h - V_{ip})$	0
t + dt	$\vec{V}_i$, $V_i + dV_i$	$-\Delta_h\vec{S}$, $-\Delta_h(S+dS)$	$(\Delta_h S - V_i)(1 + rdt)$ $-\Delta_h DSdt$	$(V_i + dV_i - \Delta_h(S+dS))$ $(\Delta_h S - V_i)(1 + rdt)$ $-\Delta_h DSdt$

The P&L is given by

$$\begin{aligned} d\text{P\&L} &= dV_i - \Delta_h dS - \Delta_h SDdt + \{(\Delta_h S - V_h) + (V_h - V_i)\}rdt \\ &= dV_h - \Delta_h dS - \Delta_h SDdt + (dV_i - dV_h) + \{(\Delta_h S - V_h) + (V_h - V_i)\}rdt \\ &= \left\{\Theta_h + \frac{1}{2}\Gamma_h S^2\sigma_r^{\ 2} + (r-D)S\Delta_h - rV_h\right\}dt + (dV_i - dV_h) + (V_h - V_i)rdt \end{aligned}$$

Now the Black-Scholes solution with the hedge volatility satisfies the p.d.e.

$$\Theta_h + (r-D)S\Delta_h + \frac{1}{2}\Gamma_h S^2\sigma_h^2 - rV_h = 0$$

Substituting this last equation into the previous one, we obtain

$$\begin{aligned} d\text{P\&L} &= \frac{1}{2}\Gamma_h S^2(\sigma_r^{\ 2} - \sigma_h^2)dt + (dV_i - dV_h) + (V_h - V_i)rdt \\ &= \frac{1}{2}\Gamma_h S^2(\sigma_r^2 - \sigma_h^2)dt - e^{rt}d\{e^{-rt}(V_h - V_i)\} \end{aligned}$$

COPYRIGHT EMANUEL DERMAN 2022

Taking present values leads to

$$\boxed{dPV(\text{P\&L}) = e^{-r(t-t_0)}\frac{1}{2}\Gamma_h S^2(\sigma_r^2 - \sigma_h^2)dt - e^{rt_0}d\{e^{-rt}(V_h - V_i)\}}$$

and so, integrating over the life of the option from $\boxed{t_0}$ to *T,* we get

$$\boxed{PV(\text{P\&L}) = V_h - V_i + \frac{1}{2}\int_{t_0}^{T} e^{-r(t-t_0)}\Gamma_h S^2(\sigma_r^2 - \sigma_h^2)dt}$$ At expiration the terminal values of the option *V* are independent of the volatility and so they cancel.

Note that when the hedge volatility equals the realized volatility we obtain the result derived in class for that case; similarly, when the hedge volatility equals the implied volatility, we obtain the result obtained in class for that case.

Problem 3. **[20 points]**

(i) You live in a world where you assume the Black-Scholes model holds exactly and interest rates and dividends are zero. S = 100 and annual implied volatility is 30%. Use some simple analytic approximations, assuming that $\sigma\sqrt{\text{time to expiration}} \ll 1$, to find the values of the following Greeks for a standard call option C struck **at the money** with one-year expiration:

$$\frac{\partial C}{\partial S}$$

$$\frac{\partial C}{\partial \sigma}$$

$$\frac{\partial^2 C}{\partial S^2}$$

$$\frac{\partial C}{\partial t}$$

[8 points]

(ii) Find the same Greeks for a put P with the same strike and expiration. [2 points]

(iii) You sell three one-year puts P from part (ii) above struck at the money. Suppose you delta-hedge your position with the appropriate number of shares of the stock. Now suppose that the stock price instantaneously jumps up to 110, violating the geometric Brownian motion assumption of Black-Scholes. Using a Taylor series and the values of the Greeks, **estimate** the break-even value for the subsequent implied volatility of the put P so that the hedged position has neither lost nor made money.

[10 points]

COPYRIGHT EMANUEL DERMAN 2022

Solution to Problem 3. **[20 points]**

(i) Proof for S = K = 100 and r = 0:

$$\boxed{\begin{aligned}
\frac{\partial C}{\partial S} &= N(d_1) = N\left(\frac{\sigma\sqrt{T-t}}{2}\right) = N\left(\frac{0.3\times\sqrt{1}}{2}\right) = N(0.15)\approx 0.5+(0.4\times 0.15)\approx 0.56 \\
\frac{\partial C}{\partial \sigma} &= SN'(d_1)\frac{\partial d_1}{\partial \sigma} - KN'(d_2)\frac{\partial d_2}{\partial \sigma} = SN'(d_1)\left[\frac{\partial d_1}{\partial \sigma} - \frac{\partial d_2}{\partial \sigma}\right] \\
&\approx SN'(d_1)\sqrt{T-t}\approx 100\times\frac{1}{\sqrt{2\pi}}\times\sqrt{1}\approx 40 \\
\frac{\partial^2 C}{\partial S^2} &= N'(d_1)\frac{\partial d_1}{\partial S} = N'(d_1)\frac{1}{S\sigma\sqrt{T-t}}\approx\frac{1}{\sqrt{2\pi}}\frac{1}{100(0.3)\sqrt{1}}\approx\frac{0.4}{30}\approx 0.013 \\
\frac{\partial C}{\partial t} &= -\frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} = -\frac{(30)^2}{2}\times 0.013\approx -5.9
\end{aligned}}$$

(iii) By put call parity, the Greeks are respectively -0.44, 40, 0,013, -5.9.

(ii) The value of the delta-hedged portfolio is

$$\boxed{\pi = -3\left[P(S) - S\left(\frac{\partial P}{\partial S}\right)\right]}$$

The approximate change in value is

$$\boxed{\begin{aligned}
d\pi &= -3\left[P(S+\delta S, \sigma+\delta\sigma) - P(S,\sigma) - (\delta S)\frac{\partial P}{\partial S}\right] \\
&\approx -3\left[\cancel{\frac{\partial P}{\partial S}(\delta S)} + \frac{1}{2}\frac{\partial^2}{\partial S^2}(P)(\delta S)^2 + \frac{\partial P}{\partial \sigma} - \cancel{(\delta S)\frac{\partial P}{\partial S}}\right] \\
&\approx -3\left[\left(\frac{1}{2}\frac{\partial^2}{\partial S^2}P\right)(\delta S)^2 + \frac{\partial P}{\partial \sigma}\delta\sigma\right] \\
&\approx -3\left[\frac{0.013}{2}\times(10)^2 + 40(\delta\sigma)\right]\approx -2[0.007\times 100 + 40(\delta\sigma)]
\end{aligned}}$$

The P&L change is zero when $\boxed{\delta\sigma\approx-\frac{0.007\times 100}{40}\approx -0.018}$ which is about 1.8 volatility points, so the break-even final volatility is about $\boxed{30-1.8 = 28.2}$ percent annualized volatility.

COPYRIGHT EMANUEL DERMAN 2022

COPYRIGHT EMANUEL DERMAN 2022

Problem4 **[20 points]**

We showed in class that

$$e^{r(T-t)}C_0 = C_T - \Delta_T S_T + \Delta_0 S_0 e^{r(T-t)} + \int_t^T e^{r(T-\tau)} S_\tau [d\Delta_\tau]_b \qquad \text{Eq. A}$$

This indicates how to find the fair value of an option given a definite rehedging/replication strategy between inception and expiration.

Consider a one-month-expiration call with S = 100, K = 100, r =0%, d = 0%, $\mu = 0\%$ and realized and implied volatilities both 30% annually. **Suppose also** that each time you hedge the call, **you pay** a transactions cost of 0.2% of the dollar amount of the shares traded, whether bought or sold. This means that in addition to the rehedging cost in A, there is an additional term.

(i) Write a Monte Carlo program to simulate Eq. A above over 10,000 paths to expiration, **assuming also** that it is modified because there is a transactions cost as indicated. Suppose you simulate with 40 discrete time steps at equal time intervals over the life of the option, and hedge after each time step. Find the expected value of the call with transactions costs, and the standard deviation of the call prices. [10]

(ii) Suppose you hedge the same option 160 times in 160 steps over the life of the options. Simulate again to find the expected value of the call and its standard deviation including transactions costs. [10]

Solution 4.

Value of call with zero transactions costs = 3.44

(i) Expected call value = 2.93. Std Dev = 0.47

(ii) Expected value = 2.53 Std Dev = 0.38; smaller standard deviation as you hedge more, but option is worth less.

Problem 5: Portfolio Insurance and Static Hedging **[10 points]**

In the 1980s the idea of portfolio insurance – paying for protection against a market decline – was very popular, and this strategy played a large part in the market crash of 1987.

One way of insuring an index S, whose current value is *S*, against a future market decline that occurs between the current time and time *T* is to own a security PI, a derivative of S, with the payoff at expiration time *T*

$$PI(S_T, T) = max[K, S + \beta(S_T - S)]$$

COPYRIGHT EMANUEL DERMAN 2022

where K is the floor (the minimum level of S you can tolerate), S_T is the future stock price at time T, and β, positive and usually less than or equal to 1, is sometimes referred to as the upside gain.The value of β can be chosen by the client who buys the insurance. A security with this payoff still profits or loses as S or falls, but will never fall below the level K.

Suppose that $Z(K, T)$ is the current market price of a riskless zero-coupon bond with face value K that matures at time T, and that $C(S,K,T)$ is the current market price of a European call option with expiration T and strike K. Show that the current value of the security $PI(S,T)$ is given by

$$PI(S, T) = Z(K, T) + \beta C\left(S, \frac{K - S(1-\beta)}{\beta}, T\right)$$

Solution 5: Portfolio Insurance and Static Hedging

The payoff at time T is: **[10]**

$$\boxed{PI(S_T, T) = max[K, S + \beta(S_T - S)]}$$

We can use algebra to write the payoff as a function of the terminal stock price S_T as

$$\boxed{\begin{aligned} max[K, S+\beta(S_T - S)] &= K + max[0, S - K + \beta(S_T - S)] \\ &= K + max[0, \beta S_T + S(1-\beta) - K] \\ &= K + \beta max\left[0, S_T - \left\{\frac{K - S(1-\beta)}{\beta}\right\}\right] \end{aligned}}$$

This is exactly the payoff at time T of a zero coupon bond with face value K plus the payoff of a call option on S_T with strike $\boxed{\left\{\frac{(K - S(1-\beta))}{\beta}\right\}}$, and therefore must have the same value as the portfolio of these securities today.

So

$$\boxed{PI(S, T) = Z(K, T) + \beta C\left(S, \frac{K - S(1-\beta)}{\beta}, T\right)}$$

Problem 6. **[10 points]**

Economists at your firm believe that the expected return for the Hang Seng Index (HSI) will be 12% over the coming year. What is the expected return for a one-year call with a delta of 0.60? Assume the riskless rate is 2.0%. The Hang Seng is currently at 25,000 and the price of the call is 2,500 HKD. Make the usual assumptions about geometric Brownian motion such that the Black-Scholes model holds. You can use the Black-Scholes PDE or the Sharpe ratios of the index and the call to relate the behavior of the call to the behavior of the index.

Solution 6. Using Sharpe ratios and the stochastic calculus relations between GBM for the stock and for the call

$$\frac{(\mu_C - r)}{\sigma_C} = \frac{(\mu_S - r)}{\sigma_S}$$

$$\mu_C = (\mu_S - r)\frac{\sigma_C}{\sigma_S} + r$$

$$= (\mu_S - r)\frac{S}{C}\frac{|\Delta|\sigma_S}{\sigma_S} + r$$

$$= (\mu_S - r)\frac{S}{C}|\Delta| + r$$

$$= (12\% - 2\%)\frac{25{,}000}{2{,}500}|0.60| + 2\%$$

$$= 62\%$$

The expected return of the option is 62%.

COPYRIGHT EMANUEL DERMAN 2022