Class 7: Discrete random variables

Tuesday, September 24, 2019 11:39 AM

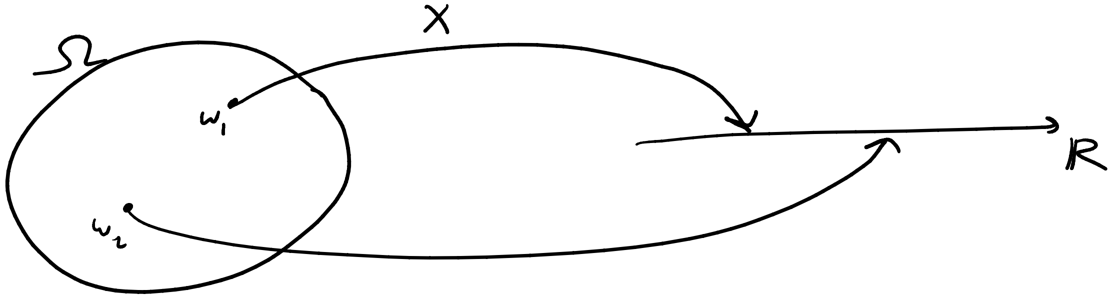
- HW 3 due Thurs, 1pm, on Gradescope
- Midterm 1 on Oct 8 in class (2 weeks)

Random Variables (discrete)

Sample space Ω.

Def: A random variable is a function from Ω to $\mathbb{R}$,

That is, to every possible outcome it assigns a number.



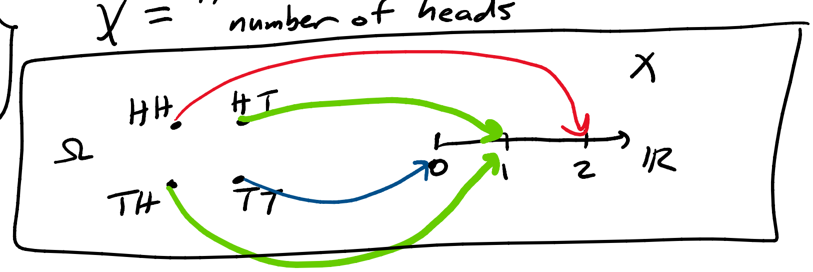
$$X : \Omega \to \mathbb{R}$$

Ex 1: $\Omega = \{H,T\}^2$ 2 coin flips

Define $X : \Omega \to \mathbb{R}$ by

$$\left.\begin{aligned} X(HH) &= 2 \\ X(TH) = X(HT) &= 1 \\ X(TT) &= 0 \end{aligned}\right\}$$

X = "number of heads"



Define Y by

$$\left.\begin{aligned} Y(HH) &= 0 \\ Y(TH) = Y(HT) &= 1 \\ Y(TT) &= 2 \end{aligned}\right\}$$

Y = "number of tails"

Events involving random variables

Events involving random variables

Notation: Let $X : \Omega \to \mathbb{R}$.

Write $\{X = 1\} = \{\omega \in \Omega : X(\omega) = 1\}$

$=$ "the event that $X = 1$".

Similarly, $\{X \geq 2\} = \{\omega \in \Omega : X(\omega) \geq 2\}$

$=$ "the event that $X \geq 2$".

More generally, if $A \subseteq \mathbb{R}$, write

$$\{X \in A\} = \{\omega \in \Omega : X(\omega) \in A\}$$

$=$ "event that X belongs to A".

Ex 1: $\Omega = \{H, T\}^2$, $X =$ "# heads".

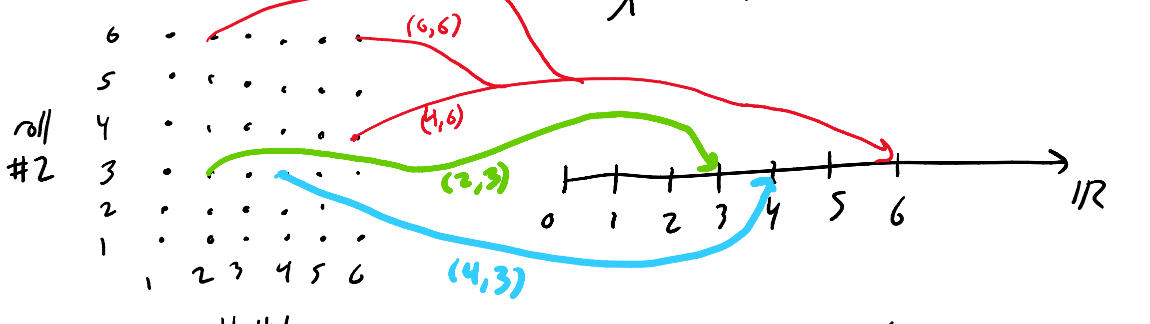
$$\{X = 1\} = \{\omega \in \Omega : X(\omega) = 1\}$$
$$= \{HT, TH\}$$

$$\{X = 2\} = \{HH\}$$

$$\{X \text{ is even}\} = \{HH, TT\}$$

Ex 2: $\Omega = \{1, 2, 3, 4, 5, 6\}^2$ 2 dice

$X =$ "maximum of two rolls"



$X(4,3) = 4$ $X(2,1) = 2$

1 2 3 4 5 6 $(4,3)$

roll #1

$$X(4,3)=4, \quad X(2,1)=2$$

$$\ldots \quad X(i,j)=\max(i,j).$$

Events: $\{X=2\}$ = "event that $X=2$"

$= \{(1\,2),(2\,2),(2\,1)\}$

$\{2 \le X \le 3\}$ = "event that $2 \le X \le 3$"

$= \{(1\,2),(2\,2),(2\,1),(1\,3),(2\,3),(3\,3),$
$(3\,2),(3\,1)\}$

$\{X=0\} = \emptyset$

$\{X \le 6\} = \Omega$

$\{X^2=4\}$ = "event that $X^2=4$"

$= \{X=2\}$ = see above

Probabilities involving random variables

Notation: If X is a random variable:

$$\mathbb{P}(X=1) = \mathbb{P}(\{X=1\})$$

= "probability that $X=1$"

$$\mathbb{P}(X \le 4) = \mathbb{P}(\{X \le 4\})$$

= "probability that $X \le 4$"

Ex 3: $\Omega = \{H,T\}^3$ 3 coins (fair)

X = # heads.

$$\{X=1\} = \{HTT, THT, TTH\}$$

$$\mathbb{P}(X=1) = \mathbb{P}(\{HTT, THT, TTH\})$$

$$= \frac{3}{8} = \frac{\#\text{ outcomes in event}}{\text{total }\#\text{ outcomes}}.$$

$$\mathbb{P}(X \geq 2) = \mathbb{P}(\{HHH, HHT, HTH, THH\}) = \frac{4}{8} = \frac{1}{2}.$$

Notation:

Write $\mathbb{P}(X \geq 2, X \leq 4) = \underline{\mathbb{P}(\{X \geq 2\} \cap \{X \leq 4\})}$

$$= \mathbb{P}(X \geq 2 \text{ and } X \leq 4) = \mathbb{P}(2 \leq X \leq 4).$$

Read "comma" as "and".

No analogous notation for union/or. Just write

$$\mathbb{P}(X \geq 2 \text{ or } X \leq 4) = \underline{\mathbb{P}(\{X \geq 2\} \cup \{X \leq 4\})}.$$

(Need curly brackets because $\cap$ and $\cup$ are <u>set</u> operations. Do not write $X \geq 2 \cup X \leq 4$.)

Discrete Random Variables

Def: A discrete random variable is a random variable which takes only finitely many or countably many possible values.

Ex: $X : \Omega \to \{0, 1, \ldots, n\}$ for some fixed n.

$X : \Omega \to \mathbb{N} = \{1, 2, 3, \ldots\}$.

<u>Note</u>: Sample should be "big enough" for what you're trying to model. <u>Ex 1 revisited:</u> $\Omega = \{H,T\}^2$, $X = \#$ heads.

<u>Another way to model this "same" random variable:</u>

$\tilde{\Omega} = \{0, 1, 2\}$, $\mathbb{P}(0) = \mathbb{P}(2) = \frac{1}{4}$

heads

$\mathbb{P}(1) = \frac{1}{2}$.

Let $\tilde{X} : \tilde{\Omega} \to \mathbb{R}$ be defined by

$\tilde{X}(0) = 0$, $\tilde{X}(1) = 1$, $\tilde{X}(2) = 2$.

This $\tilde{X}$ models the same thing as X, but on a different sample space.

All probabilities involving $\tilde{X}$ are the same as for X.

<u>Careful</u>: Suppose we want to model Y as the number of Tails we get <u>before</u> any Heads appear.

We can model this on $\Omega = \{H,T\}^2$ by

$Y(HH) = 0$, $Y(HT) = 0$, $Y(TH) = 1$, $Y(TT) = 2$.

We <u>cannot</u> model this on $\tilde{\Omega}$, because the <u>order</u> of T and H matters for Y.

<u>Def</u>: The <u>probability mass function</u> (PMF) of a random variable X is the function $p : \mathbb{R} \to \mathbb{R}$ defined by $p(k) = \mathbb{P}(X = k)$.

<u>Ex</u>: $\Omega = \{H,T\}^3$, $X = \#$ heads out of 3 coin flips

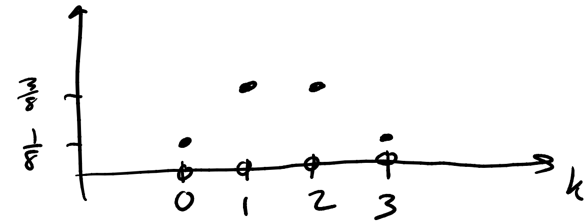
$p(0) = \frac{1}{8} = \mathbb{P}(X=0)$

$p(1) = \frac{3}{8} = \mathbb{P}(X=1)$

$p(2) = \frac{3}{8} = \mathbb{P}(X=2)$

$p(3) = \frac{1}{8} = \mathbb{P}(X=3)$

$p(k) = 0$ for all $k \notin \{0,1,2,3\}$



Might write this as

$$p(k) = \begin{cases} \frac{1}{8} & \text{if } k = 0, 3 \\ \frac{3}{8} & \text{if } k = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

In the $\Omega = \{H, T\}^2$ examples above, check that X and $\tilde{X}$ have the same PMF.

Notation / terminology:

- We may define a random variable in terms of its PMF. i.e. "Let X be a random variable with PMF $p(k) = \ldots$"
- If we are only interested in probabilities involving X, then all we need is the PMF, and the sample space Ω does not really matter.

Note: A PMF must have non-negative values adding up to 1.

Common Discrete Random Variables

(1) Bernoulli: X takes values 0 or 1

① **Bernoulli** : X takes values 0 or 1

- **Parameter** : $p \in [0,1]$. $(0 \le p \le 1)$ ("Bernoulli trials")

- **Interpretation**: 1 means success / true
  0 means failure / false

- **Write**: "$X \sim \mathrm{Ber}(p)$" or "$X \sim \mathrm{Bernoulli}(p)$"
  to mean "X has the Bernoulli distribution with parameter p".

- **PMF**:
$$\mathbb{P}(X=1) = p$$
$$\mathbb{P}(X=0) = 1-p$$
$$\mathbb{P}(X=x) = 0 \quad \text{for all } x \notin \{0,1\}$$

$$p_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \\ 0 & \text{otherwise.} \end{cases}$$

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② **Binomial Distribution** :

X takes values $0, 1, 2, \ldots, n-1, n$

- **Parameters**: $n \ge 1$ an integer = "# trials"
  $p \in [0,1]$ = "success probability"

- **Interpretation**: X is the number of successes out of n independent trials, each with success probability p.

- Write $X \sim \text{Bin}(n,p)$ or $\text{Binom}(n,p)$

read $\sim$ as "has the distribution"

- PMF:

$$p(0) = \mathbb{P}(X=0) = (1-p)^n$$

need all n to fail

$$p(1) = \mathbb{P}(X=1) = \underbrace{n}_{\#\text{ positions for the 1 success}} \underbrace{p}_{1\text{ success}} \underbrace{(1-p)^{n-1}}_{n-1\text{ failures}}$$

$$p(2) = \mathbb{P}(X=2) = \underbrace{\binom{n}{2}}_{\#\text{ positions or places for the 2 successes}} \underbrace{p^2}_{2\text{ successes}} \underbrace{(1-p)^{n-2}}_{n-2\text{ failures}}$$

$$\vdots$$

$$p(k) = \mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

for $k = 0, 1, \ldots, n$

↳ PMF of $\text{Bin}(n,p)$ is $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

for $k = 0, 1, \ldots, n$.

Notes:

- $\binom{n}{k} = \binom{n}{n-k}$
- If $p = \frac{1}{2}$, $1-p = \frac{1}{2}$ $\quad p(k) = \left(\frac{1}{2}\right)^n \binom{n}{k} = 2^{-n}\binom{n}{k}$.