Math 280, Spring 2022

Solutions to Midterm Exam 1

1. **(18 points)** Let $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, and let G be the graph with adjacency matrix A.

   1a. Compute A^3.

   1b. How many walks are there on G from vertex 1 to vertex 4 of length exactly 3?

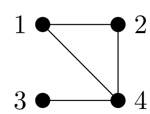
   1c. Draw the graph G, with the vertices appropriately labelled.

Solutions. (a): $A^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix}$

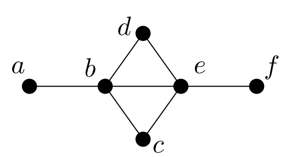
So $A^3 = AA^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 3 & 1 & 4 \\ 3 & 2 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 4 & 4 & 3 & 2 \end{bmatrix}}$

(b): This is the $(1, 4)$ entry of A^3, which is $\boxed{4}$

(c): According to the matrix A, here is the graph G:



2. **(22 points)** Consider the following graph G, which has 6 vertices and 7 edges:



   (a) Determine the degree sequence of G.

   (b) Find a trail in G of maximum length, and briefly explain why no longer trail is possible.

   (c) Find the radius, diameter, and center of G.

   (d) What is the connectivity $\kappa(G)$? Why?

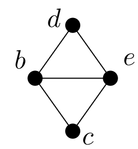
Solutions. (a): in alphabetical order, the vertices have degrees 1,4,2,2,4,1,
so the degree sequence is $\boxed{4,4,2,2,1,1}$

(b): The walk $\boxed{a, b, c, e, b, d, e, f}$ is a trail that uses all 7 edges, each exactly once.
No longer trail is possible because a trail cannot use an edge more than once, so nothing beyond length 7 can be a trail.

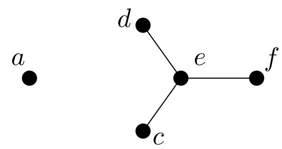
(c): Vertices a and f are distance 3 from each other, but every other vertex is distance at most 2 from another. But no vertex is distance only 1 from every other vertex. Thus, the eccentricities of the vertices are:

$$\mathrm{ecc}(a) = \mathrm{ecc}(f) = 3, \quad \text{and} \quad \mathrm{ecc}(b) = \mathrm{ecc}(c) = \mathrm{ecc}(d) = \mathrm{ecc}(e) = 2.$$

Thus, the radius is $\boxed{\mathrm{rad}(G) = 2}$, the diameter is $\boxed{\mathrm{diam}(G) = 3}$, and the center is the subgraph induced by $\{b, c, d, e\}$, which is:



(d): The graph G is connected (since every vertex has finite eccentrity, as noted in (c)), so $\kappa(G) \geq 1$. However, removing vertex b leaves the subgraph $G - b$, which is disconnected:



Since one vertex must be removed to disconnect G, we have $\boxed{\kappa(G) = 1}$

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3. **(16 points.)** Let G be a graph of order $n \geq 1$ and of size m. Prove that

$$\delta(G) \leq \frac{2m}{n} \leq \Delta(G).$$

Proof. For every $v \in V(G)$, we have $\deg(v) \geq \delta(G)$, and hence

$$2m = 2|E(G)| = \sum_{v \in V(G)} \deg(v) \geq \sum_{v \in V(G)} \delta(G) = n\delta(G),$$

so dividing by $n \geq 1$ gives $\delta(G) \leq 2m/n$, i.e., the first desired inequality.
For every $v \in V(G)$, we have $\deg(v) \leq \Delta(G)$, and hence

$$2m = 2|E(G)| = \sum_{v \in V(G)} \deg(v) \leq \sum_{v \in V(G)} \Delta(G) = n\Delta(G),$$

so dividing by $n \geq 1$ gives $\Delta(G) \geq 2m/n$, i.e., the second desired inequality. QED

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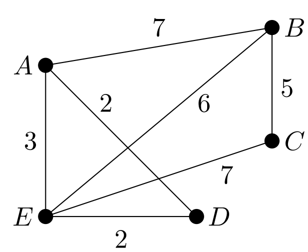
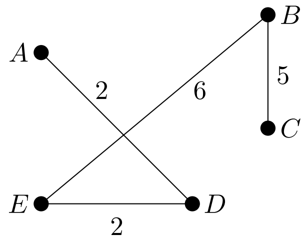
4. **(16 points)** Use Kruskal's algorithm to find a minimum spanning tree on the following weighted graph. As always, show your work.



Solution.
Step 1: Add the shortest edge, AD [or DE, which has the same length] (length 2)
Step 2: Add the remaining shortest edge, DE [or AD, if we had added DE first] (length 2)
Step 3: The next shortest edge, AE, would form a cycle; instead add *next* shortest: BC (length 5)

Step 4: Add the next shortest edge: BE (length 6)
Stop; we have added 4 edges, attaining a spanning tree on our 5 vertices:



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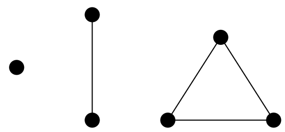
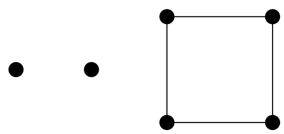
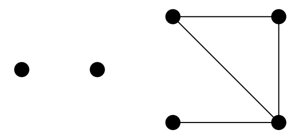
5. **(12 points)** Give an example of a graph with 6 vertices and 4 edges that has three connected components.

Solution. Here is such a graph:



Note: There are others, namely these two:





[*Note*: Up to isomorphism, it turns out that these are the only examples of such a graph.]

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6. **(16 points.)** Let T be a tree, and let $e \in E(T)$ be an edge. Prove that e is a bridge, i.e., prove that $T - e$ is disconnected.

Proof. (**Method 1**): Let $a, b \in V(T)$ be the two distinct vertices incident with e. Suppose (towards a contradiction) that $T - e$ is connected. Then there is a path W

$$a = x_1, x_2, \ldots, x_k = b$$

in $T - e$. Since $a \neq b$, we have $k \geq 2$. Moreover, since W is a path in $T - e$, it cannot use the edge e, and hence the path is not simply a, b, and therefore $k \neq 2$. Thus, $k \geq 3$, i.e., W has length at least 2. Define a walk W' in T by appending the edge e to W, i.e.,

$$a = x_1, x_2, \ldots, x_k = b, a.$$

Clearly W' starts and ends at the same vertex a. Because W is a path, the walk W' has no other repeated vertices. And because W has length at least 2, the new walk W' has length at least 3. Therefore, by definition, W' is a cycle in T, contradicting the fact that T is a tree.
By this contradiction, it follows that $T - e$ must be disconnected. QED

(**Method 2**): Let $n = |V(T)|$ be the order of T, so that the size of T is $|E(T)| = n - 1$ because T is a tree.
The graph $T - e$ has the same vertex set $|V(T - e)| = |V(T)|$, and hence $T - e$ also has order n. In addition, $T - e$ is a subgraph of the acyclic graph T, and hence $T - e$ is also acyclic. However, we have $E(T - e) = E(T) \smallsetminus \{e\}$, and hence

$$|E(T - e)| = |E(T)| - 1 = n - 2 \leq n - 2 = |V(T)| - 2.$$

Thus, $T - e$ is an acyclic graph with *two* fewer edges than vertices, so by a theorem from class, $T - e$ must be disconnected. QED

OPTIONAL BONUS. (2 points.) Let G be a connected graph such that $\operatorname{diam}(G) \geq 3$. Prove that the complement graph $\overline{G}$ is also connected, and that $\operatorname{diam}(\overline{G}) \leq 3$.

Proof. By hypothesis, there are two vertices $a, b \in V(G)$ such that the shortest path in G from a to b is of length $m \geq 3$. Because every walk between a and b contains a path from a to b, it follows that there is no walk from a to b of length 2 or shorter in G. In particular, ab cannot be an edge of G, because if it were, then a, b would be a walk of length 1 from a to b.
Note that $a \neq b$, and that ab cannot be an edge of G, because otherwise, a and b would be distance either 0 or 1 apart, a contradiction. Therefore, ab is an edge of $\overline{G}$.
Let $d(\cdot, \cdot)$ denote the distance in the graph $\overline{G}$. Given any $v, w \in V(\overline{G}) = V(G)$, we wish to show that there is a walk in $\overline{G}$ from v to w of length less than or equal to 3, i.e., that $d(v, w) \leq 3$.

Claim. Either $d(a, v) \leq 1$ or $d(b, v) \leq 1$. Similarly, either $d(a, w) \leq 1$ or $d(b, w) \leq 1$.

Proof of Claim: We prove the first statement; the second is similar, with w in place of v.
If $a = v$, then $d(a, v) = 0 \leq 1$, and if $b = v$, then $d(b, v) = 0 \leq 1$. Thus, we may assume $a \neq v$ and $b \neq v$.
Suppose neither av nor bv is an edge of $\overline{G}$. Then both av and bv are edges of G, and hence a, v, b is a walk from a to b in G. But that is a walk of length 2, a contradiction. Thus, either av or bv belongs to $E(\overline{G})$, and hence either $d(a, v) = 1$ or $d(b, v) = 1$. QED Claim

If $d(a, v), d(a, w) \leq 1$, then by the triangle inequality applied to the distance d on $\overline{G}$, we have

$$d(v, w) \leq d(v, a) + d(a, w) \leq 1 + 1 < 3,$$

and similarly if $d(b, v), d(b, w) \leq 1$.
Recalling that ab is an edge of $\overline{G}$, we have $d(a, b) = 1$. Thus, if $d(a, v), d(b, w) \leq 1$, the triangle inequality again gives

$$d(v, w) \leq d(v, a) + d(a, b) + d(b, w) \leq 1 + 1 + 1 = 3.$$

By the Claim, the only other possibility is that $d(b, v), d(a, w) \leq 1$, in which case

$$d(v, w) \leq d(v, b) + d(b, a) + d(a, w) \leq 1 + 1 + 1 = 3,$$

as desired. QED