Solutions Assignment 3

Part 2

7.2.10

In this exercise

$$f(x|\theta) = \begin{cases} 1 & \text{for } \theta - \dfrac{1}{2} < x < \theta + \dfrac{1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\xi(\theta) = \begin{cases} \dfrac{1}{10} & \text{for } 10 < \theta < 20, \\ 0 & \text{otherwise.} \end{cases}$$

The condition that $\theta - 1/2 < x < \theta + 1/2$ is the same as the condition that $x - 1/2 < \theta < x + 1/2$. Therefore, $f(x \mid \theta)\xi(\theta)$ will be positive only for values of θ which satisfy both the requirement that $x - 1/2 < \theta < x + 1/2$ and the requirement that $10 < \theta < 20$. Since $X = 12$ in this exercise, $f(x \mid \theta)\xi(\theta)$ is positive only for $11.5 < \theta < 12.5$. Furthermore, since $f(x \mid \theta)\xi(\theta)$ is constant over this interval, the posterior p.d.f. $\xi(\theta \mid x)$ will also be constant over this interval. In other words, the posterior distribution of θ must be a uniform distribution on this interval.

7.3.15b)

$$\xi(\theta \mid \underline{x}_n) = \frac{\prod_{i=1}^{n} f(x_i \mid \theta)\ \xi(\theta)}{g_n(\underline{x}_n)} = \frac{(2\pi\theta)^{-\frac{n}{2}} e^{-\frac{1}{2\theta}\sum_{i=1}^{n}(x_i-\mu)^2} C_{\alpha,\beta}\, \theta^{-(\alpha+1)} e^{-\beta/\theta}}{g_n(\underline{x}_n)}$$

$$= C_{n,x,\alpha,\beta}\ \theta^{-\left(\frac{n}{2}+\alpha+1\right)} e^{-\frac{1}{\theta}\left(\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2 + \beta\right)}$$

$\Rightarrow$ Posterior distribution = Inverse Gamma$\left(\frac{n}{2}+\alpha,\ \beta + \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right)$

7.3.10

In this exercise, $\sigma^2 = 4$ and $\upsilon^2 = 1$. Therefore, by Eq. (7.3.2)

$$\upsilon_1^2 = \frac{4}{4+n}.$$

It follows that $\upsilon_1^2 \leq 0.01$ if and only if $n \geq 396$.

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7.3.18

Suppose that the prior distribution of θ is the Pareto distribution with parameters x_0 and α ($x_0 > 0$ and $\alpha > 0$). Then the prior p.d.f. $\xi(\theta)$ has the form

$$\xi(\theta) \propto 1/\theta^{\alpha+1} \qquad \text{for} \quad \theta \geq x_0.$$

If $X_1, \ldots, X_n$ form a random sample from a uniform distribution on the interval $[0, \theta]$, then

$$f_n(\boldsymbol{x} \mid \theta) \propto 1/\theta^n \qquad \text{for} \quad \theta > \max\{x_1, \ldots, x_n\}.$$

Hence, the posterior p.d.f. of θ has the form

$$\xi(\theta \mid \boldsymbol{x}) \propto \xi(\theta) f_n(\boldsymbol{x} \mid \theta) \propto 1/\theta^{\alpha+n+1},$$

for $\theta > \max\{x_0, x_1, \ldots, x_n\}$, and $\xi(\theta \mid \boldsymbol{x}) = 0$ for $\theta \leq \max\{x_0, x_1, \ldots, x_n\}$. This posterior p.d.f. can now be recognized as also being the Pareto distribution with parameters $\alpha + n$ and $\max\{x_0, x_1, \ldots, x_n\}$.

7.3.17

The joint p.d.f. of the three observations is

$$f(x_1, x_2, x_3 \mid \theta) = \begin{cases} 1/\theta^3 & \text{for } 0 < x_i < \theta \ (i = 1, 2, 3), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the posterior p.d.f. $\xi(\theta \mid x_1, x_2, x_3)$ will be positive only if $\theta \geq 4$, as required by the prior p.d.f., and also $\theta > 8$, the largest of the three observed values. Hence, for $\theta > 8$,

$$\xi(\theta \mid x_1, x_2, x_3) \propto \xi(\theta) f(x_1, x_2, x_3 \mid \theta) \propto 1/\theta^7.$$

Since

$$\int_8^\infty \frac{1}{\theta^7} d\theta = \frac{1}{6(8)^6},$$

it follows that

$$\xi(\theta \mid x_1, x_2, x_3) = \begin{cases} 6(8^6)/\theta^7 & \text{for } \theta > 8 \\ 0 & \text{for } \theta \leq 8. \end{cases}$$

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7.4.2

The posterior distribution of θ is the beta distribution with parameters $5 + 1 = 6$ and $10 + 19 = 29$. The mean of this distribution is $6/(6 + 29) = 6/35$. Therefore, the Bayes estimate of θ is $6/35$.

For stating Bayes estimate = mean of posterior distribution

Interpretation: When n is small, the prior distribution (μ_0; = your initial belief) has a large impact on the Bayes estimate (= posterior mean). As n increases, the influence of the data ($\overline{X}_n$) on your estimate becomes stronger and the influence of the prior becomes weaker.

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7.4.6

Suppose that the parameters of the prior gamma distribution of θ are α and β. Then $\mu_0 = \alpha/\beta$. The posterior distribution of θ was given in Theorem 7.3.2. The mean of this posterior distribution is

$$\frac{\alpha + \sum_{i=1}^n X_i}{\beta + n} = \frac{\beta}{\beta + n}\mu_0 + \frac{n}{\beta + n}\overline{X}_n.$$

Hence, $\gamma_n = n/(\beta + n)$ and $\gamma_n \to 1$ *as* $n \to \infty$.

Posterior pdf:

$$\xi(\theta \mid \underline{x}_n) = \frac{\prod_{i=1}^n f(x_i \mid \theta)\, \xi(\theta)}{g_n(\underline{x}_n)} = \frac{\prod_{i=1}^n \frac{e^{-\theta}\theta^{x_i}}{x_i!}\, C_{\alpha,\beta}\, \theta^{\alpha-1} e^{-\beta\theta}}{g_n(\underline{x}_n)}$$

$$= C_{n,x,\alpha,\beta}\; e^{-(n+\beta)\theta}\; \theta^{\sum x_i + \alpha - 1}$$

$\Longrightarrow$ Posterior distribution is Gamma$(\sum x_i + \alpha,\ n+\beta)$

7.4.12

(a) A's prior distribution for θ is the beta distribution with parameters $\alpha = 2$ and $\beta = 1$. Therefore, A's posterior distribution for θ is the beta distribution with parameters $2+710 = 712$ and $1+290 = 291$. B's prior distribution for θ is a beta distribution with parameters $\alpha = 4$ and $\beta = 1$. Therefore, B's posterior distribution for θ is the beta distribution with parameters $4 + 710 = 714$ and $1 + 290 = 291$.

(b) A's Bayes estimate of θ is $712/(712+291) = 712/1003$. B's Bayes estimate of θ is $714/(714+291) = 714/1005$.

(c) If y denotes the number in the sample who were in favor of the proposition, then A's posterior distribution for θ will be the beta distribution with parameters $2 + y$ and $1 + 1000 - y = 1001 - y$, and B's posterior distribution will be a beta distribution with parameters $4+y$ and $1+1000-y = 1001 - y$. Therefore, A's Bayes estimate of θ will be $(2 + y)/1003$ and B's Bayes estimate of θ will be $(4 + y)/1005$. But

$$\left|\frac{4+y}{1005} - \frac{2+y}{1003}\right| = \frac{2(1001 - y)}{(1005)(1003)}.$$

This difference is a maximum when $y = 0$, but even then its value is only

$$\frac{2(1001)}{(1005)(1003)} < \frac{2}{1000}.$$