Chapter 17: Capital Asset Pricing Model (CAPM)

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- In this chapter, we assume an idealized framework for an open market place, where all the risky assets refer to (say) all the tradeable stocks available to all.
- In addition we have a risk-free asset (for borrowing and/or lending in unlimited quantities) with interest rate μ_f.
- We assume that all information is available to all such as covariances, variances, mean rates of return of stocks.
- We also assume that everyone is a risk-averse rational investor who uses the same financial engineering mean-variance portfolio theory from Markowitz.
- A little thought leads us to conclude that since everyone has the same assets to choose from, the same information about them, and the same decision methods, everyone has a portfolio on the same efficient frontier, and hence has a portfolio that is a mixture of the risk-free asset and a unique tangent portfolio T (of risky assets).
- In other words, everyone sets up the same optimization problem, does the same calculation, gets the same answer and chooses a portfolio accordingly.

- This tangency portfolio used by all is called the market portfolio and is denoted by M.
- The fact that it is the same for all leads us to conclude that it should be computable without using all the optimization methods from Markowitz:
- The market has already reached an equilibrium so that the weight for any asset in the market portfolio is given by its capital value (total worth of its shares) divided by the total capital value of the whole market (all assets together).

- For example, if asset i refers to shares of stock in Company A, and this company has $S_i = 10,000$ shares outstanding, each worth $P_i = \$20.00$, then its captital value for asset i is $V_i = S_i P_i = (10,000)(\$20) = \$200,000$.
- Computing this value for each asset and summing over all i for all the n companies in the market (total number of assets) yields the total capital value of the whole market, $V = V_1 + V_2 + \ldots + V_n$,
- and the weight

  $$\omega_i = \frac{S_i P_i}{\sum_{j=1}^{n} S_j P_j} = V_i / V$$

  is the weight used for asset i in the market portfolio.
- The general idea is that assets under high demand will fetch high prices and yield high expected rates of return (and vice versa); repeated trading of the assets over time adjusts the various prices yielding an equilibrium that reflects this; the optimal weights wi for the market portfolio are thus governed by supply and demand.
- In the end, we dont need to use the optimization methods nor any of the detailed data (covariances, variances, mean rates, nor even the risk-free rate μ_f) to determine the market portfolio; we only need all the V_i values.

- Note that all the weights $\omega_i > 0, \forall i$ (no shorting exists in the market portfolio).
- This implies that even if apriori we ruled out shorting of the assets in our framework, the same equilibrium market portfolio M would arise.
- Also note that since presumably all risky assets available on the open market have a non-zero capital value, all risky assets are included in the portfolio (although some have very small weights).
- The above equilibrium model for portfolio analysis is called the Capital Asset Pricing Model (CAPM).

The CAPM has a variety of uses:

- It provides a theoretical justification for the widespread practice of passive investing known as indexing. Indexing means holding a portfolio similar to a broad market index such as the S&P 500.
- Individual investors can just buy an index fund.
- CAPM can provide estimates of expected rates of return on individual investments. Estimates based on the mean of historic time series are known to be inaccurate. The problem is lack of enough data. So alternative estimates are useful

The Capital Market Line (CML)

- The capital market line (CML) relates the excess expected return on an efficient portfolio to its risk;

$$\text{excess expected return} = \text{expected return} - \text{risk-free rate}$$

- The CML is given by

$$\mu_R = \mu_f + \frac{\mu_M - \mu_f}{\sigma_M}\sigma_R \tag{1}$$

where

- R is the return on a given efficient portfolio
- R_M is the return on the market portfolio
- $\mu_R = E(R)$
- $\mu_M = E(R_M)$
- μ_f is the rate of return on the risk-free asset
- σ_R is the standard deviation of R
- σ_M is the standard deviation of R_M

- In
$$\mu_R = \mu_f + \frac{\mu_M - \mu_f}{\sigma_M}\sigma_R$$
μ_f, μ_M and σ_M are constant.
- What varies are σ_R and μ_R.
  - These vary as we change the efficient portfolio R.
  - Think of the CML as showing how μ_R depends on σ_R.

- The slope of the CML is

$$\frac{\mu_M - \mu_f}{\sigma_M}$$

can be interpreted as the ratio of the risk premium to the standard deviation of the market portfolio.

- This is Sharpe's reward-to-risk ratio
- Equation (1) can be rewritten as

$$\frac{\mu_R - \mu_f}{\sigma_R} = \frac{\mu_M - \mu_f}{\sigma_M}$$

the reward-to-risk ratio for any efficient portfolio equals that of the market portfolio

- Suppose that $\mu_f = 0.06, \mu_M = .15$ and $\sigma_M = 0.22$, then the slope of the CML is

$$(0.15 - 0.06)/0.22 = 9/22.$$

- Derivation of the CML
  Consider an efficient portfolio that allocates w to the market portfolio, then

$$R = \omega R_M + (1-\omega)\mu_f = \mu_f + \omega(R_M - \mu_f)$$

Therefore,

$$\mu_R = \mu_f + \omega(\mu_M - \mu_f) \quad \text{and} \quad \sigma_R = \omega\sigma_M.$$

This implies that

$$\omega = \sigma_R/\sigma_M.$$

Consequently

$$\mu_R = \mu_f + \frac{\sigma_R}{\sigma_M}(\mu_M - \mu_f)$$

and we have the CML.

- CAPM suggests that the optimal way to invest is to:
  1. Decide on the risk σ_R that you can tolerate, $0 \leq \sigma_R \leq \sigma_M$. ($\sigma_R > \sigma_M$ is possible by borrowing money to buy risky assets on margin.)
  2. Calculate $\omega = \sigma_R/\sigma_M$.
  3. Invest proportion ω in an index fund.
  4. Invest proportion $1 - \omega$ in risk-free treasury bills.

  Alternatively,
  1. Choose the reward $\mu_R - \mu_f$ that you want.
  2. Calculate $\omega = \frac{\mu_R - \mu_f}{\mu_M - \mu_f}$
  3. Do steps 3 and 4 as above.

- The CAPM relates the excess return of an efficient portfolio to its risk
- The Security Market Line (SML), on the other hand, relates the excess return on an asset to its "β", the slope of its regression on the market portfolio.
- We will show using CAPM that

$$\mu_j - \mu_f = \beta_j(\mu_M - \mu_f).$$

- The SML says that the risk premium of an asset is the product of its beta and the risk premium of the market
- β_j measures both the riskiness of the *jth* asset and the reward for assuming that risk. It is a measure of how aggressive the *jth* asset is or how sensitive it is to market movements.
- Suppose securities are indexed by j and define σ_{jM} =covariance between the *jth* security and the market portfolio
- We will show that

$$\beta_j = \frac{\sigma_{jM}}{\sigma_M^2}$$

- By definition, $\beta_M = 1$. If
- $\beta_j = 1$, we have same risk as the market (average risk)
- $\beta_j > 1$, we have an aggressive asset
- $\beta_j < 1$, we a have non-agressive asset

- Another way to appreciate the significance of β is based on linear regression.
- The econometrics model corresponding to the SML is

$$R_{j,t} - \mu_{f,t} = \beta_j(R_{M,t} - \mu_{f,t}) + \epsilon_{j,t}$$

where $\epsilon_{j,t}$ is $N(0, \sigma^2_{\epsilon,j})$.

- Regression can now be used to estimate β_j
- Suppose that we have a bivariate time series $(R_{j,t}, R_{M,t})$, $t = 1, 2, \ldots, n$ of returns on the *jth* asset and the market portfolio. An estimate of β_j is given by

$$\hat{\beta}_j = \frac{\sum_{t=1}^{n}(R_{j,t} - \mu_{f,t})(R_{M,t} - \mu_{f,t})}{\sum_{t=1}^{n}(R_{M,t} - \mu_{f,t})^2}$$

Consider a portfolio P with weight ω given to the risky asset j with return R_j and weight $(1-\omega)$ given to the market portfolio.

- The return on this portfolio is

$$R_P = \omega R_j + (1-\omega)R_M$$

- The expected return is

$$\mu_P = \omega\mu_j + (1-\omega)\mu_M$$

- the risk is

$$\sigma_P = \sqrt{\omega^2\sigma_j^2 + (1-\omega)^2\sigma_M^2 + 2\omega(1-\omega)\sigma_{jM}}$$

- We have

$$\frac{d\mu_P}{d\sigma_P}|_{\omega=0} = \frac{\mu_M - \mu_f}{\sigma_M}$$

- $$\frac{d\mu_P}{d\sigma_P} = \frac{d\mu_P/d\omega}{d\sigma_P/d\omega}$$

  but

$$\frac{d\mu_P}{d\omega} = \mu_j - \mu_M$$

  and

$$\frac{d\sigma_P}{d\omega} = \frac{\omega\sigma_j^2 - (1-\omega)\sigma_M^2 + (1-2\omega)\sigma_{jM}}{\sigma_P}$$

- therefore

$$\frac{d\mu_P}{d\sigma_P} = \frac{(\mu_j - \mu_M)\sigma_P}{\omega\sigma_j^2 - (1-\omega)\sigma_M^2 + (1-2\omega)\sigma_{jM}}$$

and

$$\frac{d\mu_P}{d\sigma_P}\Big|_{\omega=0} = \frac{(\mu_j - \mu_M)\sigma_M}{\sigma_{jM} - \sigma_M^2}$$

- Consequently

$$\frac{(\mu_j - \mu_M)\sigma_M}{\sigma_{jM} - \sigma_M^2} = \frac{\mu_P - \mu_f}{\sigma_M}$$

which is equivalent to

$$\begin{aligned} \mu_j - \mu_f &= \frac{\sigma_{jM}}{\sigma_M^2}(\mu_M - \mu_f) \\ &= \beta_j(\mu_M - \mu_f) \end{aligned}$$

- Another way to view the security market line is as follows.
- Notice that the expected return of a portfolio is

$$\begin{aligned}\mu_p &= (1 - \sum_{i=1}^{N} \omega_i)\mu_f + \sum_{i=1}^{N} \omega_i \mu_i \\ &= \mu_f + \sum_{i=1}^{N} \omega_i(\mu_i - \mu_f)\end{aligned}$$

- The marginal contribution of risky asset i to the expected portfolio return is

$$\frac{\partial \mu_p}{\partial w_i} = \mu_i - \mu_f$$

(Marginal contribution of x to A means the increment changes of A when x changes by a small amount)

- Notice also that the marginal contribution of risky asset i to the portfolio volatility is

$$\begin{aligned}\frac{\partial \sigma_p}{\partial w_i} &= \frac{1}{2\sigma_p}\frac{\partial \sigma_p^2}{\partial w_i} \\ &= \frac{Cov(R_i, R_p)}{\sigma_p} \\ &= \frac{\sigma_{ip}}{\sigma_p}\end{aligned}$$

- The (marginal) return to risk ratio (RRR) of the risky asset i is a portfolio P is defined as

$$RRR_i = \frac{\text{marginal return}}{\text{marginal risk}} = \frac{\partial \mu_p / \partial w_i}{\partial \sigma_/ \partial w_i}$$

- Claim: For the market (tangent) portfolio, the return-risk ratio of all risky assets must be the same That is

$$RRR_i = \frac{\mu_i - \mu_f}{(\sigma_{iM}/\sigma_M)} = RRR_M = \frac{\mu_M - \mu_f}{\sigma_M}$$

- Note that

$$\frac{a}{b} = \frac{c}{d} \Longrightarrow \frac{a}{b} = \frac{c}{d} = \frac{a+b}{c+d}$$

- Because it has to be the case in the market portfolio that

$$\frac{\mu_1 - \mu_f}{(\sigma_{1M}/\sigma_M)} = \frac{\mu_2 - \mu_f}{(\sigma_{2M}/\sigma_M)} = \ldots = \frac{\mu_N - \mu_f}{(\sigma_{NM}/\sigma_M)}$$

then

$$\frac{\omega_1(\mu_1 - \mu_f)}{\omega_1(\sigma_{1M}/\sigma_M)} = \frac{\omega_2(\mu_2 - \mu_f)}{\omega_2(\sigma_{2M}/\sigma_M)} = \ldots = \frac{\omega_N(\mu_N - \mu_f)}{\omega_N(\sigma_{NM}/\sigma_M)} = \frac{\sum_{i=1}^N w_i(\mu_i - \mu_f)}{\sum_{i=1}^N w_i(\sigma_{iM}/\sigma_M)} = \frac{\mu_M - \mu_f}{\sigma_M}$$

This implies that

$$\mu_j - \mu_f = \frac{\sigma_{iM}}{\sigma_M^2}(\mu_M - \mu_f)$$

A model equivalent to SML

$$\frac{\mu_j - \mu_f}{\sigma_j} = \rho_{jM} \frac{\mu_M - \mu_f}{\sigma_M}$$

Since ρ_{jM} is always smaller than 1, the above equation means that in theory, your portfolio cannot beat market portfolio in terms of Sharpe's ratio or reward-to-risk ratio; the Sharpe's ratio of your portfolio is equal to the market one if and only if its return is perfectly linear correlated with market return or you are replicating market portfolio. This theoretical result supports the investment in those index funds which are trying to replicate the market portfolio

Below is an example of an oil drilling company. It comes from David Luenberger's book.
Example: An oil drilling company is currently priced at \$875 a share. Suppose that the expected share price after one year is \$1,000 and that the standard deviation of the return is oil $= 40\%$. Furthermore, the risk-free rate is 10% and the market portfolio has an expected return 17% with a standard deviation 12%.

1. is the investment in the oil drilling company a good one?
2. is the current price for the oil drilling company a fair price if $\beta_{oil} = 0.6$?

1.
$$\mu_{oil} = \frac{1000 - 875}{875} = 14\%$$

Since $\sigma_{oil} = 40\%$, the SML imples that

$$\mu = 0.10 + \frac{0.17 - 0.10}{012} \times 0.40 = 33\%.$$

Thus, the expected return of the oil drilling company is well below the CML so that the investment is not a good one.

2. We are going to use the SML as a pricing tool. Suppose we know that the beta of the oil drilling company is 0.6. Suppose that instead of $875 the fair price P_0 i is to be determined. The SML implies

$$\frac{1000 - P_0}{P_0} = 0.10 + 0.6 \times (0.17 - 0.10) = 0.142$$

Therefore $P_0 = 875.66$. So the price is correct even though itself is not a good investment.
Remark: note that here the fair price is determined by the beta.

- The CML applies only to the return R of an efficient portfolio and says that:

$$\mu_R - \mu_f = \frac{\sigma_R}{\sigma_M}(\mu_M - \mu_F)$$

- The SML applies to any asset and says that:

$$\mu_j - \mu_f = \beta_j(\mu_M - \mu_F)$$

- For an efficient portfolio with return R

$$\beta_R = \frac{\sigma_R}{\sigma_M}$$

- Let $\mu_{j,t} = E(R_{j,t})$ and $\mu_{M,t} = E(R_{M,t})$. Then

$$\mu_{j,t} = \mu_{f,t} + \beta_j(\mu_{M,t} - \mu_{f,t})$$

- In addition

$$\begin{aligned} \sigma_j^2 &= \beta_j^2 \sigma_M^2 + \sigma_{\epsilon,t}^2 \\ \sigma_{jj'} &= \beta_j \beta_{j'} \sigma_M^2, \quad j \neq j' \\ \sigma_{Mj} &= \beta_j \sigma_M^2 \end{aligned}$$

- The total risk of the *jth* asset is

$$\sigma_j = \sqrt{\beta_j^2 \sigma_M^2 + \sigma_{\epsilon,j}^2}$$

$\beta_j^2 \sigma_M^2$ is called the market risk or systematic risk component of risk and $\sigma_{\epsilon,j}^2$ is called the unique, non market or unsystematic component of risk,

- The market component cannot be reduced by diversification.
- The unique component can be reduced by diversification.
- Suppose that there are N assets with returns $R_{1,t}, R_{2,t}, \ldots, R_{N,t}$. If we form a portfolio with weights $\omega_1, \omega_2, \ldots, \omega_N$, then the return of this portfolio is

$$R_{P,t} = \omega_1 R_{1,t} + \omega_2 R_{2,t} + \ldots + \omega_N R_{N,t}.$$

- Let $R_{M,t}$ be the return on the market portfolio. According to the characteristic line model

$$R_{j,t} = \mu_{f,t} + \beta_j (R_{M,t} - \mu_{f,t}) + \epsilon_{j,t}$$

so that

$$R_{P,t} = \mu_{f,t} + \left(\sum_{i=1}^{N} \beta_j \omega_j \right) (R_{M,t} - \mu_{f,t}) + \sum_{i=1}^{N} \omega_j \epsilon_{j,t}$$

- Therefore, the portfolio beta is

$$\beta_P = \sum_{i=1}^{N} \omega_j \beta_j$$

and its ϵ is

$$\epsilon_{P,t} = \sum_{i=1}^{N} \omega_j \epsilon_{j,t}$$

- Assuming $\epsilon_{1,t}, \epsilon_{2,t}, \ldots, \epsilon_{N,t}$ uncorrelated, we get

$$\sigma^2_{\epsilon,P} = \sum_{i=1}^{N} \omega_j^2 \sigma^2_{\epsilon,j}.$$

and

$$\sigma_P = \sqrt{\beta_P^2 \sigma_M^2 + \sigma^2_{\epsilon,P}}$$

- Example: Suppose $w_j = 1/N$ for all j. Then

$$\beta_P = \frac{\sum_{j=1}^{N} \beta_j}{N}$$

and

$$\sigma^2_{\epsilon,P} = \frac{N^{-1}\sum_{j=1}^{N} \sigma^2_{\epsilon,j}}{N} = \frac{\overline{\sigma}^2_{\epsilon}}{N}$$

where $\overline{\sigma}^2_{\epsilon}$ is the average of the $\sigma^2_{\epsilon,j}$. If $\sigma_{\epsilon,j} = \sigma^2_{\epsilon}$ for all j, then

$$\sigma_{\epsilon,P} = \frac{\sigma_{\epsilon}}{\sqrt{N}}.$$

For example, suppose that if $\sigma_{\epsilon} = 5\%$ then
  - if $N = 20$, then $\sigma_{\epsilon,P} = \frac{0.05}{\sqrt{20}} = 1.12$ %
  - if $N = 100$, then $\sigma_{\epsilon,P} = \frac{0.05}{\sqrt{100}} = 0.5$ %
  - There are approximately 1600 stocks on the NYSE. If $N = 1600$, then $\sigma_{\epsilon,P} = 0.0125\%$

- The security characteristic line

$$R_{j,t} = \mu_{f,t} + \beta_j(R_{M,t} - \mu_{f,t}) + \epsilon_{j,t},$$

Let $R^*_{j,t} = R_{j,t} - \mu_{f,t}$ (excess expected return on jth asset) and let $R^*_{M,t} = R_{M,t} - \mu_{f,t}$ (excess expected return on market portfolio), then the security characteristic line can be written as

$$R^*_{j,t} = \beta_j R^*_{M,t} + \epsilon_{j,t}.$$

This equation is a regression model without an intercept.

1. If asset A has a beta of 1.50 and $\mu_A = 14\%$ and $\mu_M = 10\%$ and $\mu_f = 6\%$, is this asset appropriately priced? If not, is it underpriced or overpriced?
2. Suppose that a security A in an efficient market is such that $\sigma_A = 0.25, \rho_{AM} = 0.75$. If $\sigma_M = 0.20$, compute β_A.
3. Assume that the risk free rate $\mu_f = 6\%$ and the average return of the market is $\mu_M = 8\%$. If $\mu_A = 8\%$ and $\mu_B = 8.2\%$ and $\mu_C = 8.5\%$ and $\beta_A = 0.9$, $\beta_B = 1.3$ and $\beta_C = 1.25$, which asset has the best performance?
4. Assume that an efficient market consists of two risky assets A and B with the following β's and $\sigma^2_{\epsilon_j}$

	β_j	σ_{ϵ_j}
Asset A	0.95	10%
Asset B	1.05	15%

Assume that the pairwise correlations among the ϵ_js are all equal to zero and that $\sigma_M = 15\%$ and that you hold a portfolio with weights $\omega_A = 0.30$ and $\omega_B = 0.70$ on these assets.

   1. If β_P is the beta of your portfolio, what is it equal to? What is the risk of your portfolio equal to?
   2. What proportion of the total risk of your portfolio is due to the market risk?

- A more general model is

$$R^*_{j,t} = \alpha_j + \beta_j R^*_{M,t} + \epsilon_{j,t}$$

  The CAPM says that $\alpha_j = 0$ but by allowing $\alpha_j \neq 0$ we recognize the possibility of mispricing.
- Given series $R_{j,t}, R_{M,t}$ and $\mu_{f,t}$ for $t = 1, 2, \dots, n$,
  - we can calculate R^*_{jt} and $R^*_{M,t}$ and
  - regress $R^*_{j,t}$ on $R^*_{j,M}$ to estimate α_j, β_j and $\sigma^2_{\epsilon,j}$.
  - By testing the null hypothesis that $\alpha_j = 0$, we are testing whether the *jth* asset is misspriced according to the CAPM

- Consider an asset j with price P_0 at time $t = 0$, payoff (random) P_1 and expected payoff $E(P_1)$ at time $t = 1$.
- By definition

$$\mu_j = \frac{E(P_1) - P_0}{P_0}$$

- Solving for P_0 yields

$$P_0 = \frac{E(P_1)}{1 + \mu_j}$$

  which expresses the price as the discounted payoff (present value) if using μ_j as the discount rate.
- But

$$\mu_j = \mu_f + \beta_j(\mu_M - \mu_f)$$

  from the SML formula, we conclude therefore that

$$P_0 = \frac{E(P_1)}{1 + \mu_f + \beta_j(\mu_M - \mu_f)}$$

Market Portfolio

- In the real open market place where the number of assets is enormous, trying to actually construct the market portfolio would be an awsome and unrealistic task for any financial analyst.
- Thus so-called index funds (or mutual funds) have been created as an attempt to approximate the market portfolio.
- Such an index is a smaller portfolio made up of what are viewed as the markets most dominant assets, that captures the essence of M.
- The most well-known such index is the Standard & Poors 500-stock index (S&P), made up of 500 stocks
- A beta for a given asset is then estimated by using the S&P in replace of M, and then collecting past data for both rates of return

- The security characteristic line is a regression model:

$$R_{j,t} = \mu_{f,t} + \beta_j(R_{M,t} - \mu_{f,t}) + \epsilon_{j,t}$$

  The variable $R_{M,t} - \mu_{f,t}$ is sometimes called a factor and it is the sole source of correlation between asset returns.

- A multi-factor model is

$$R_{j,t} - \mu_{f,t} = \beta_{0,j} + \beta_{1,j}F_{1,t} + \ldots + \beta_{p,j}F_{j,t} + \epsilon_{j,t}$$

- $F_{1,t}, F_{2,t}, \ldots, F_{pt}$ are the values of p factors at time t. A factor can be anything that can be measured and is thought to affect assets returns.

- Examples:
  - return on market portfolio (market model, e.g., CAPM)
  - growth rate of GDP
  - interests rate on short term Treasury bills
- With enough factors the ϵ_{jt} s should be uncorrelated across j (across assets).

We start with two factors for simplicity, i.e. $p = 2$, in which case

$$R_{j,t} - \mu_{f,t} = \beta_{0,j} + \beta_{1,j} F_{1,j} + \beta_{2,j} F_{2,j} + \epsilon_{j,t}$$

Therefore,

- $$E(R_{j,t}) = \mu_f + \beta_{0,j} + \beta_{1,j} E(F_1) + \beta_{2,j} E(F_2)$$

- $$\text{Var}(R_{j,t}) = \beta_{1,j}^2 \text{Var}(F_1) + \beta_{2,j}^2 \text{Var}(F_2) + 2\beta_{1,j}\beta_{2,j}\text{Cov}(F_1, F_2) + \sigma_\epsilon^2$$

  and

- $$\begin{aligned} \text{Cov}(R_{j,t}, R_{j',t}) &= \beta_{1,j}\beta_{j',t}\text{Var}(F_1) + \beta_{2,j}\beta_{2,j'}\text{Var}(F_2) \\ &+ (\beta_{1,j}\beta_{2,j'} + \beta_{2,j}\beta_{1,j'})\text{Cov}(F_1, F_2) + \sigma_\epsilon^2 \end{aligned}$$