Economics 361
Change of Variables

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1 Overview

Often, what we desire is not probability statements about X but that of some transformation of X, $Y = g(X)$. Under certain conditions, the probability distribution of Y can be derived from that of X. This derivation is usually an application of the calculus "change of variables" technique. Here, we review the technique.

1.1 Monotonic Transformations

(This section borrows from Casella & Berger (1990) Ch 2.1)

A function $g(X)$ is considered a **monotonic** transformation if

(a) $u > v$ necessarily implies $g(u) > g(v)$
– or –
(b) $u > v$ necessarily implies $g(u) < g(v)$

Monotonic transformations that satisfy (a) are considered **increasing** and those that satisfy (b) are considered **decreasing**.

Let $\mathcal{X}$ be the set of possible values of X and $\mathcal{Y}$ the set of possible values of $Y = g(X)$. When $g(X)$ is a monotonic transformation, $g(X)$ is a one-to-one mapping from $\mathcal{X}$ to $\mathcal{Y}$. That is, $g(\cdot)$ uniquely pairs the x's and y's. This means the intervals of y values map to specific intervals of x values.

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If $g(X)$ is monotonic and increasing

$$
\begin{aligned}
\{x \in \mathcal{X} : g(x) \leq y\} &= \{x \in \mathcal{X} : g^{-1}(g(x)) \leq g^{-1}(y)\} \\
&= \{x \in \mathcal{X} : x \leq g^{-1}(y)\}
\end{aligned}
$$

$$
\begin{aligned}
F_Y(y) &= P_X(\{x \in \mathcal{X} : g(x) \leq y\}) \\
&= P_X(\{x \in \mathcal{X} : x \leq g^{-1}(y)\}) \\
&= F_X(g^{-1}(y))
\end{aligned}
$$

If $g(X)$ is monotonic and decreasing

$$
\begin{aligned}
\{x \in \mathcal{X} : g(x) \leq y\} &= \{x \in \mathcal{X} : g^{-1}(g(x)) \geq g^{-1}(y)\} \\
&= \{x \in \mathcal{X} : x \geq g^{-1}(y)\}
\end{aligned}
$$

$$
\begin{aligned}
F_Y(y) &= P_X(\{x \in \mathcal{X} : g(x) \leq y\}) \\
&= P_X(\{x \in \mathcal{X} : x \geq g^{-1}(y)\}) \\
&= 1 - P_X(\{x \in \mathcal{X} : x < g^{-1}(y)\}) \\
&= 1 - F_X(g^{-1}(y)) \\
&\quad \text{if } X \text{ is continuous ... so we can “ignore” } \{x = g^{-1}(y)\}
\end{aligned}
$$

When $g(X)$ is a monotonic transformation of X, deriving the cdf of $Y = g(X)$, F_Y, from the cdf of X, F_X, is fairly straight-forward. Similarly, the pmf/pdf of Y, $f(y)$, can be derived from the pmf/pdf of X, $f(x)$. But deriving the pmf/pdf is a bit trickier.

1.2 Discrete Random Variables

When the random variable X and the transformed random variable Y are both discrete, deriving $f(y)$ from $f(x)$ is usually simple but possibly tedious, depending on the number of possible realizations.

Suppose random variable X can realize one of k mutually exclusive values and that each of these k values correspond to only one value of the random variable Y. Then the probability distribution of Y can be derived via arithmetic sum

$$
P_Y(Y = y) = \sum_{i \in \mathcal{X}_y} P_X(X = x_i)
$$

where $\mathcal{X}_y$ is the set of realizations of X that correspond to the event $Y = y$.

The above class of transformations include but do not entail monotonic transformations.

Example: Let X be a standard six-sided die roll. Let Y be an indicator variable that takes on the value of 1 when X realizes an even value and 0 otherwise.

$$
P_Y(Y = y) = \begin{cases} P_X(X = 2) + P_X(X = 4) + P_X(X = 6) & \text{for y = 1} \\ P_X(X = 1) + P_X(X = 3) + P_X(X = 5) & \text{for y = 0} \end{cases}
$$

1.3 Continuous Random Variable

For monotonic transformations, the calculus technique of "**change of variables**" may be used to derive $f(y)$ from $f(x)$.[1]

When $g(\cdot)$ is monotonic and $g^{-1}(\cdot)$ differentiable

$$f(y) \quad = \quad f_X(g^{-1}(y)) \underbrace{\left| \frac{d}{dy}\, g^{-1}(y) \right|}_{\text{“Jacobian”}} \qquad \text{for } y \in \mathcal{Y}$$

Note: $g^{-1}(\cdot)$ refers to the inverse function of $g(\cdot)$, **not** $\frac{1}{g(\cdot)}$. The monotone nature of $g(\cdot)$ ensures the existence of $g^{-1}(\cdot)$.

The change of variables technique also allows us to derive the cdf of Y when $g(X)$ is not strictly monotonic but is piecewise monotonic. Under piecewise monotone, $\{-\infty < x \leq g^{-1}(y)\}$ can be partitioned into k disjoint intervals where, for each interval, $g(X)$ is monotonic: $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k$

$$\begin{aligned} F_Y(y) \quad = \quad & \int_{\mathcal{Y}_1} f_X(g^{-1}(y)) \left| \frac{d}{dy}\, g^{-1}(y) \right| \, dy \\ & + \int_{\mathcal{Y}_2} f_X(g^{-1}(y)) \left| \frac{d}{dy}\, g^{-1}(y) \right| \, dy \\ & \vdots \\ & + \int_{\mathcal{Y}_k} f_X(g^{-1}(y)) \left| \frac{d}{dy}\, g^{-1}(y) \right| \, dy \end{aligned}$$

where $\mathcal{Y}_i = \{y \in Y : g^{-1}(y) \in \mathcal{X}_i\}$

1.3.1 Continuous Bivariate Random Variables

The change of variables technique applies for the multivariate case, as well.

Let $U = g(X)$ and $V = h(Y)$. When $g(\cdot)$ and $h(\cdot)$ are monotonic and their inverse functions differentiable

$$f_{UV}(u,v) \quad = \quad f_{XY}(g^{-1}(x), h^{-1}(y)) \underbrace{\left\| \frac{d(x,y)}{d(u,v)} \right\|}_{\text{“Jacobian”}} \qquad \text{for } y \in \mathcal{Y}$$

[1] The change of variables technique is an application of the "chain rule" for differentiation.

2 Applications

2.1 Gaussian Integral

A random variable X distributed normally with mean μ and variance σ^2, namely $X \sim N(\mu, \sigma^2)$, has the following pdf for $X = x$

$$f(x) \quad = \quad \frac{1}{\sqrt{2\pi}\sigma}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \ \text{ for } \ x \in (-\infty, +\infty)$$

As a proper pdf, we know that

$$\int_{-\infty}^{+\infty} f(x)dx \quad = \quad \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \quad = \quad 1$$

Consider the following monotonic transformation of X, $Z = \frac{X-\mu}{\sigma}$. Note that for the case with $\mu = 0$ and $\sigma = 1$, $X = Z$. Z is referred to as the "standardized normal" – the normal random variable once its mean has been standardized to zero and its variance to one.

Question: what is the pdf of Z?

Applying change of variable to the regular normal pdf ...

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{2\pi}\sigma}\ e^{-\frac{1}{2}(z)^2}\ |\text{Jacobian}| \\ &= \frac{1}{\sqrt{2\pi}\sigma}\ e^{-\frac{1}{2}(z)^2}\ \left|\frac{dx(z)}{dz}\right| \\ &= \frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}(z)^2} \end{aligned}$$

Note that $x(z) = \sigma z + \mu$ and $\frac{dx(z)}{dz} = \sigma$. The Jacobian cancels out the $\frac{1}{\sigma}$ in the regular normal pdf.

Why do we need the Jacobian in the change of variable? If we ignore the Jacobian, the "pdf" of Z would not properly sum up to one

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\ e^{-\frac{1}{2}(z)^2} dz \quad \neq \quad 1$$

even if $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}\ e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1$

To demonstrate this, let us show that the proper pdf (with Jacobian) does sum up to one.

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}(z)^2} dz \quad = \quad 1$$

Note that

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\ e^{-\frac{1}{2}(z)^2} dz \quad = \quad \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(z)^2} dz}_{(*)}$$

(*) is referred to as the Gaussian Integral.[2]

Evaluating this integral directly is difficult as the integrand lacks a well-defined anti-derivative. There are several ways to evaluate this integral. Below, I will adopt the clever approach credited to the great French mathematician, Pierre-Simon Laplace. It involves change of variable and polar coordinates!

Laplace noticed that while $\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(z)^2} dz$ was difficult to manage, $I = \left(\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(z)^2} dz \right)^2$ was not. Moreover, he noted that $e^{-\frac{1}{2}(z)^2} \geq 0$ for any real-valued z. So one could solve for $\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(z)^2} dz$ by first solving for I and then simply taking the square root of I.

I is easier to evaluate as it can be represented by the following double integral

$$\begin{aligned} \left(\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(z)^2} dz \right)^2 &= \left(\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(x)^2} dx \right) \left(\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(y)^2} dy \right) \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(x^2+y^2\right)} \, dx \, dy \end{aligned}$$

Note that the set of values (x, y) span the entire two dimensional Euclidean space. Laplace recognized that any point in two dimensional Euclidean space can be represented by either its rectangular coordinates, (x, y), or its polar coordinates, (r, θ). A change of variable from rectangular to polar coordinates makes the above double integral simple to evaluate.

To do the change of variable, we need to figure out [1] the mapping from (x, y) to (r, θ) [2] the relevant range of values for (r, θ) – the new domain of integration [3] the Jacobian of the transformation.

[1] From trigonometry, recall that $x = r\cos(\theta)$ and $y = r\sin(\theta)$. So $x^2+y^2 = r^2\cos^2(\theta)+r^2\sin^2(\theta) = r^2(\cos^2(\theta) + \sin^2(\theta)) = r^2$.

[2] The original domain of integration is the entire two dimensional Euclidean space. So the new domain of integration is $r \in [0, +\infty)$ and $\theta \in [0, 2\Pi]$

[3] The Jacobian is

$$\frac{d(x, y)}{d(r, \theta)} = \left\| \begin{matrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{matrix} \right\| = \left\| \begin{matrix} \cos(\theta) & -r \ \sin(\theta) \\ \sin(\theta) & r \ \cos(\theta) \end{matrix} \right\| = r\cos^2(\theta) + r\sin^2(\theta) = r$$

Therefore

$$I = \int_0^{2\Pi} \int_0^{+\infty} e^{-\frac{1}{2}r^2} \, r \, dr \, d\theta$$

Note that $e^{-\frac{1}{2}r^2}$ r does have a well defined anti-derivative when r is strictly non-negative. The anti-derivative is simply $-e^{\frac{1}{2}r^2}$.

$$I = \int_0^{2\Pi} \lim_{a \to +\infty} \left(-e^{-\frac{1}{2}a^2} + 1 \right) d\theta = \int_0^{2\Pi} d\theta = 2\Pi$$

[2]It is sometimes represented without the coefficient $\frac{1}{2}$ before z^2.

So $\int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2} dz = \sqrt{I} = \sqrt{2\Pi}$ and

$$\frac{1}{\sqrt{2\Pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2} dz \quad = \quad \frac{1}{\sqrt{2\Pi}} \times \sqrt{2\Pi} \quad = \quad 1$$

Note: $\int_{-\infty}^{+\infty} e^{z^2} dz = \sqrt{\Pi}$ can be shown using similar steps.

2.2 MGF of Chi-squared Random Variable

A random variable X distributed Chi-squared with r degrees of freedom, namely $X \sim \chi^2_r$, has the following pdf for $X = x$

$$f(x) \quad = \quad \frac{1}{\Gamma(r/2)2^{r/2}} \; x^{r/2-1} \; e^{-x/2} \quad \text{for} \quad x \in (0, +\infty)$$

where $\Gamma(\cdot)$ is the famed "Gamma" function

$$\Gamma(\alpha) \quad = \quad \int_0^{+\infty} y^{\alpha-1} e^{-y} dy$$

Note that $\Gamma(1) = 1$ and $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ for positive integers $\alpha > 1$. The latter can be shown using integration by parts.

The moment generating function (MGF) of X is

$$M(t) \;=\; E(e^{tX}) \;=\; \int_0^{\infty} e^{tx} \frac{1}{\Gamma(r/2)2^{r/2}} \; x^{r/2-1} \; e^{-x/2} dx \;=\; \int_0^{\infty} \frac{1}{\Gamma(r/2)2^{r/2}} \; x^{r/2-1} \; e^{-(1-2t)x/2} dx$$

for $t < \frac{1}{2}$

Consider the following monotonic transformation of X, $Z \equiv (1 - 2t)X$. Recall that $t < \frac{1}{2}$; so $(1 - 2t) > 0$

Apply change of variables, from X to Z, to M(t)

$$\begin{aligned} M(t) &= \int_0^{+\infty} \frac{1}{\Gamma(r/2)2^{r/2}} \; (\frac{1}{1-2t} z)^{r/2-1} \; e^{-z/2} \; |\frac{dx(z)}{dz}| \; dz \\ &= \int_0^{+\infty} \frac{1}{\Gamma(r/2)2^{r/2}} \; (1-2t)(1-2t)^{-r/2} \; z^{r/2-1} \; e^{-z/2} \; |\frac{1}{1-2t}| \; dz \\ &= (1-2t)^{-r/2} \int_0^{+\infty} \underbrace{\frac{1}{\Gamma(r/2)2^{r/2}} \; z^{r/2-1} \; e^{-z/2}}_{\chi^2_r \text{ pdf}} dz \\ &= (1-2t)^{-r/2} \end{aligned}$$

for $t < \frac{1}{2}$

The mean of X can be found using M(t)

$$E(X) \;=\; \frac{\partial M(t)}{\partial t}\bigg|_{t=0} \;=\; (-r/2)(-2)(1-2t)^{-r/2-1}\Big|_{t=0} \;=\; r$$

and the variance

$$E(X^2) - (E(X))^2 \;=\; (-r/2)(-2)(-r/2-1)(-2)(1-2t)^{-r/2-2}\Big|_{t=0} - r^2 \;=\; 2r$$

2.3 Relationship between Standard Normal and Chi-squared

Recall the standard normal, $Z \sim N(0,1)$. Change of variables can be used to show that $Y \equiv Z^2$ is distributed Chi-squared with 1 degree of freedom, $Z \sim \chi_1^2$

Let us start with the Gaussian integral associated with Z

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\Pi}} e^{-\frac{1}{2}z^2} dz \;=\; \int_{-\infty}^{0} \frac{1}{\sqrt{2\Pi}} e^{-\frac{1}{2}z^2} dz + \int_{0}^{+\infty} \frac{1}{\sqrt{2\Pi}} e^{-\frac{1}{2}z^2} dz$$

Note that Y is a monotonic transformation of Z only for non-negative or non-positive domains of Z. So we split the Gaussian integral into the two appropriate halves.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\Pi}}\; e^{-\frac{1}{2}z^2} dz \;&=\; \int_{-\infty}^{0} \frac{1}{\sqrt{2\Pi}}\; e^{-\frac{1}{2}(\sqrt{y})^2}\; \left|\frac{dz(y)}{dy}\right|\; dy \;+\; \int_{0}^{+\infty} \frac{1}{\sqrt{2\Pi}}\; e^{-\frac{1}{2}(\sqrt{y})^2}\; \left|\frac{dz(y)}{dy}\right|\; dy \\
&=\; 2\; \times\; \int_{0}^{+\infty} \frac{1}{\sqrt{2\Pi}}\; e^{-\frac{1}{2}(\sqrt{y})^2}\; \frac{1}{2\sqrt{y}}\; dy \;\;\text{(by symmetry)} \\
&=\; \int_{0}^{+\infty} \frac{1}{\sqrt{2\Pi}}\; e^{-\frac{1}{2}y}\; \frac{1}{\sqrt{y}}\; dy
\end{aligned}$$

The pdf of a χ_1^2 random variable is

$$f(y) \;=\; \frac{1}{\Gamma(1/2)2^{1/2}}\; y^{1/2-1}\; e^{-y/2} \quad \text{for} \;\; y \in (0, +\infty)$$

We can show that $\Gamma(1/2) = \sqrt{\Pi}$.[3] So

$$f(y) \;=\; \frac{1}{\sqrt{2\Pi}}\; e^{-\frac{1}{2}y}\; \frac{1}{\sqrt{y}} \quad \text{for} \;\; y \in (0, +\infty)$$

which is the integrand from the above change of variables from Z to Y.

Alternatively, we can start from the CDF, $F(y) = Pr(Y \leq y)$

$$\begin{aligned}
F_Y(y) \;&=\; P_Y(Y \leq y) \\
&=\; P_X(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&=\; \int_{-\sqrt{y}}^{0} \frac{1}{\sqrt{2\Pi}}\; e^{-\frac{1}{2}x^2}\; dx \;+\; \int_{0}^{\sqrt{y}} \frac{1}{\sqrt{2\Pi}}\; e^{-\frac{1}{2}x^2}\; dx \\
&=\; 2\; \times\; \int_{0}^{\sqrt{y}} \frac{1}{\sqrt{2\Pi}}\; e^{-\frac{1}{2}x^2}\; dx \;\;\text{(by symmetry)}
\end{aligned}$$

[3] Apply change of variables, $u = 2\sqrt{y}$, on the Gamma function and use the Gaussian integral result: $\Gamma(1/2) = \sqrt{2}\; \int_0^{+\infty} e^{-\frac{1}{2}u^2} du = \sqrt{2}\; \frac{\sqrt{2\Pi}}{2} = \sqrt{\Pi}$

The pdf $f(y)$ can be obtained by differentiating the CDF $F(y)$

$$\begin{aligned}
f(y) &= \frac{d}{dy} F(y) \\
&= 2 \times \frac{d}{dy} \int_0^{\sqrt{y}} \frac{1}{\sqrt{2\Pi}}\ e^{-\frac{1}{2}x^2}\ dx \\
&= 2 \times \left(\frac{1}{\sqrt{2\Pi}}\ e^{-\frac{1}{2}y}\ \frac{1}{2\sqrt{y}} \right) \quad \text{(using Leibnitz's Rule)} \\
&= \frac{1}{\sqrt{2\Pi}}\ e^{-\frac{1}{2}y}\ \frac{1}{\sqrt{y}} \quad \text{for } y \in (0, +\infty)
\end{aligned}$$

Note: Leibnitz's Rule states that

$$\frac{d}{d\theta} \int_{a(\theta)}^{b(\theta)} f(x,\theta)\ dx = f(b(\theta),\theta)\frac{d}{d\theta}b(\theta) - f(a(\theta),\theta)\frac{d}{d\theta}a(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial\theta} f(x,\theta)\ dx$$

if $f(x,\theta), a(\theta), b(\theta)$ are differentiable with respect θ and $\infty < \{a(\theta), b(\theta)\} < +\infty$ for all θ

Even more formally, we can show that the mgf of $Y = Z^2$ is the mgf of χ_1^2.

$$\begin{aligned}
M(t) &= E(e^{tY}) \\
&= E(e^{tX^2}) \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\Pi}}\ e^{-\frac{1}{2}x^2}\ e^{tx^2}\ dx \\
&= 2 \times \int_0^{+\infty} \frac{1}{\sqrt{2\Pi}}\ e^{-\frac{1}{2}x^2}\ e^{tx^2}\ dx \quad \text{(by symmetry)}
\end{aligned}$$

Applying change of variables, $Y = X^2$

$$\begin{aligned}
E(e^{tY}) &= 2 \times \int_0^{+\infty} \frac{1}{\sqrt{2\Pi}}\ e^{-\frac{1}{2}y}\ e^{ty} \left| \frac{dx(y)}{dy} \right| dy \\
&= 2 \times \int_0^{+\infty} \frac{1}{\sqrt{2\Pi}}\ e^{-\frac{1}{2}y}\ e^{ty}\ \frac{1}{2\sqrt{y}}\ dy \\
&= \int_0^{+\infty} \frac{1}{\sqrt{2\Pi}}\ e^{-\frac{1}{2}y}\ e^{ty}\ \frac{1}{\sqrt{y}}\ dy \\
&= \int_0^{+\infty} \left(e^{ty}\right) \underbrace{\left(\frac{1}{\sqrt{2\Pi}}\ e^{-\frac{1}{2}y}\ \frac{1}{\sqrt{y}} \right)}_{=f(y)} dy
\end{aligned}$$