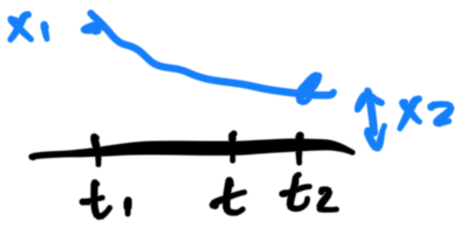
log-linear interpolation

$$\ln DF(t) = \ln \underbrace{DF(t_1)}_{DF_1} + (\ln \underbrace{DF(t_2)}_{DF_2} - \ln DF(t_1)) \times \frac{t-t_1}{t_2-t_1}$$



$$x(t) = x_1 + (x_2 - x_1) \times \frac{t-t_1}{t_2-t_1} \quad \text{linear}$$

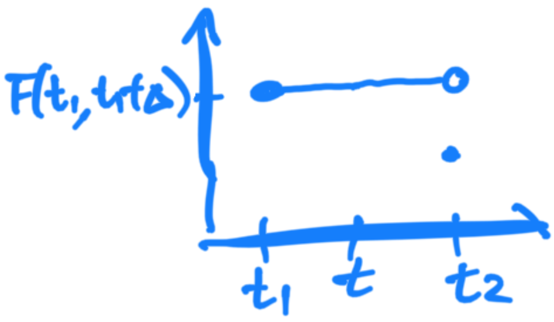
(*) $$DF(t) = (DF_1)^{\frac{t_2-t}{t_2-t_1}} (DF_2)^{\frac{t-t_1}{t_2-t_1}}$$ (Exercise)

Daily forward

$$F(t, t+\Delta) = \left(\frac{DF(t)}{DF(t+\Delta)} - 1 \right) \frac{1}{\Delta} \quad , \quad \Delta = \frac{1}{365}$$

$$= \left(\frac{(DF_1)^{\frac{t_2-t}{t_2-t_1}} (DF_2)^{\frac{t-t_1}{t_2-t_1}}}{(DF_1)^{\frac{t_2-t-\Delta}{t_2-t_1}} (DF_2)^{\frac{t+\Delta-t_1}{t_2-t_1}}} - 1 \right) \frac{1}{\Delta}$$

$$= \left(\left(\frac{DF_1}{DF_2} \right)^{\frac{\Delta}{t_2-t_1}} - 1 \right) \frac{1}{\Delta}$$



Monte Carlo simulations

- pricing financial instruments
- future value of portfolios
  - potential future exposures, risk
- market risk / credit risk
- model validation
- simulate data to test trading strategies
- generate synthetic data to train ML models

References :

- Hull's book 11th Ed. Chap 21.5
- Glasserman Chap 1, Appendix A

Monte Carlo integration

Let Z be r.v.

$f: \mathbb{R} \to \mathbb{R}$ integrable

$$V = \mathbb{E}[f(Z)]$$

① Draw n samples of Z

$z_1, z_2, \dots, z_n$

② $f(z_1), f(z_2), \dots, f(z_n)$

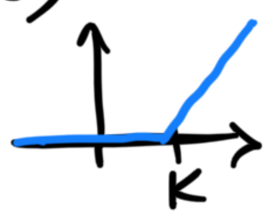
$$V_n = \frac{1}{n} \sum_{i=1}^{n} f(z_i) \quad \text{estimator of } V$$

unbiased

$$\mathbb{E}[V_n] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[f(z_i)] = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[f(Z)] = \mathbb{E}[f(Z)] = V$$

Example: European call option

option payoff at expiry $T = \max(S_T - K, 0)$



option value at time 0

$$= \mathbb{E}[\underbrace{e^{-rT} \max(S_T - K, 0)}]$$

Let's suppose S_t follows GBM.

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dW_t, \quad W_t = \text{standard BM.}$$

$$d(\ln S_t) = \left(r - q - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

$$\ln S_T - \ln S_0 = \left(r - q - \frac{\sigma^2}{2}\right)T + \sigma W_T, \quad W_T \sim N(0, T)$$

$$* \quad S_T = S_0 \exp\left[\left(r - q - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T} Z\right], \quad Z \sim N(0,1)$$

$$= S_T(Z)$$

Call option price

$$C = \mathbb{E}[\underbrace{e^{-rT} \max(S_T(Z) - K, 0)}_{f(Z)}] \quad Z \sim N(0,1)$$

Steps

- Draw random samples from $N(0,1)$

	simulated price	discounted payoff	
$z_1 \longrightarrow$	$S_T(z_1)$	$f(z_1)$	take simple avg
$z_2 \longrightarrow$	$S_T(z_2)$	$f(z_2)$	$\rightarrow C_n$
$\vdots$	$\vdots$	$\vdots$	
z_n		$f(z_n)$	

eg. $S_0 = 70$, $K = 72$

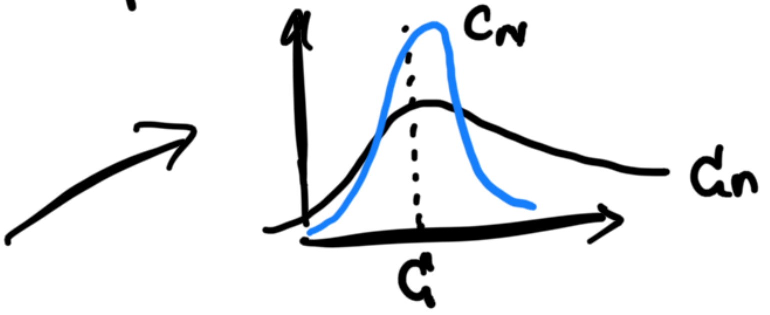
$z_1 = 0.5 \rightarrow S_T(0.5) = 75 \rightarrow f(z_1) = 0.99 \times 3$

$z_2 = -1 \rightarrow S_T(-1) = 68 \rightarrow f(z_2) = 0.99 \times 0 = 0$

simulated option value $C_n = \frac{1}{n} \sum_{i=1}^{n} f(z_i)$

Trial #1	$n = 10000$	C^1	different
#2	$n = 10000$	C^2	
		$\vdots$	

empirical distribution. of C_n



$N > n$

Central limit Thm

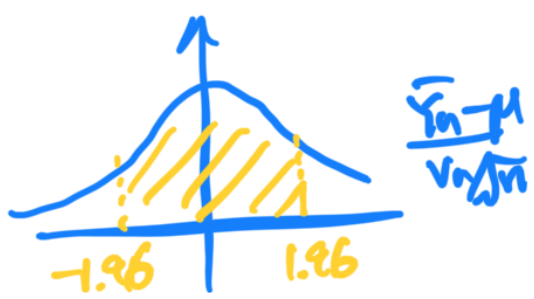
Let $Y_1, \ldots, Y_n$ r.v. iid.
with same mean μ and variance $= V_y^2 < \infty$

Denote $\overline{Y_n}$ = sample mean

Then $\quad \dfrac{\overline{Y_n} - \mu}{V_y / \sqrt{n}} \xrightarrow{\text{dist.}} N(0,1)$

$Y_i = f(z_i)$

$\overline{Y_n} = V_n = \frac{1}{n} \sum_{i=1}^{n} f(z_i)$



With 95% confidence,

$$\text{true value} \in \left(\bar{Y}_n - 1.96 \frac{V_Y}{\sqrt{n}}, \; \bar{Y}_n + 1.96 \frac{V_Y}{\sqrt{n}} \right)$$

unknown.

Let s_n^2 = sample variance of $f(z_1), \cdots, f(z_n)$

$$\text{true value} \in \left(\bar{Y}_n - 1.96 \frac{s_n}{\sqrt{n}}, \; \bar{Y}_n + 1.96 \frac{s_n}{\sqrt{n}} \right)$$

with 95% confidence