LECTURE 13

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LOCAL VOLATILITY MODELS

- In a local volatility model, the instantaneous stock volatility $\sigma(S, t)$ is a function of stock price and future time.
- How to build and use a binomial tree with variable local volatility.
- The BSM implied volatility of a standard option in a local volatility model is approximately the average of the local volatilities between the initial stock price and the strike.

Where we're going with local volatility:

- Building a local volatility tree
- The implied volatility surface that results
- Calibrating a local volatility tree
- The Dupire equation
- The relation between local and implied vol
- Hedge ratios of vanilla options
- Exotics

  etc

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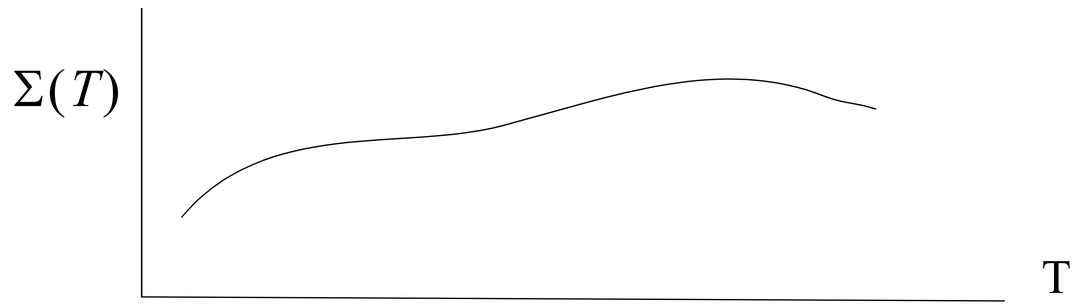
13.1 Recap: Extending Black-Scholes to Time-dependent Deterministic Volatility to Account for the Term Structure of Skew.



GBM is $$\frac{dS}{S} = \mu dt + \sigma(t)dZ,$$

Given implied volatilities $\Sigma(T)$, how do we find $\sigma(t)$ in the evolution?

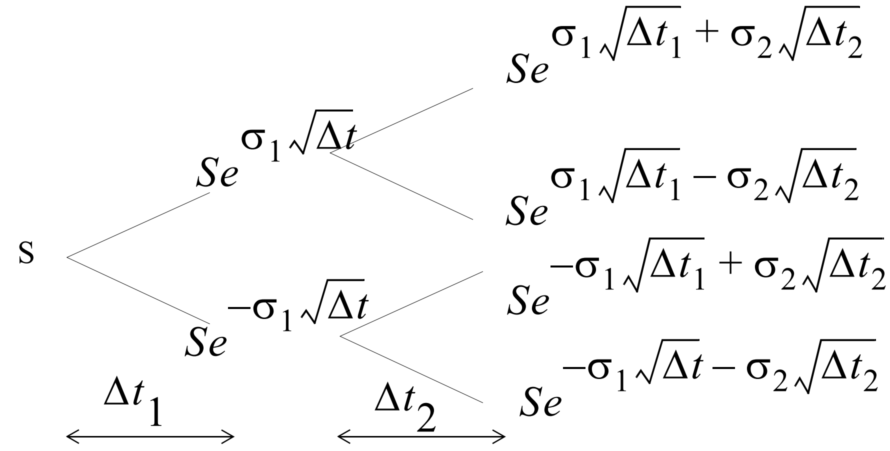
Variance in GBM is additive: $$\Sigma^2(T) = \frac{1}{T}\int_t^T \sigma^2(s)ds$$

Therefore $\sigma^2(T) = \frac{\partial}{\partial T}[\tau\Sigma^2(\tau)]$ gives the volatility at any future time from the term structure.

How do we build a tree to do American options? One method:

. To make the tree close

$$\sigma_1\sqrt{\Delta t_1} = \sigma_2\sqrt{\Delta t_2} = \ldots = \sigma_N\sqrt{\Delta t_N}$$



Constant volatility of 20%

sigma		0.2	0.2	0.2	0.2	0.2
delta t		0.25	0.25	0.25	0.25	0.25
Time	0	0.25	0.5	0.75	1	1.25
sig*sqrt(delta	0.1	0.1	0.1	0.1	0.1	0.1
u		1.105	1.105	1.105	1.105	1.105
r_annual	0.05					

risk neutral stock tree CRR-style with constant sigma(t)

					164.87
				149.18	
			134.99		134.99
		122.14		122.14	
	110.52		110.52		110.52
100.00		100.00		100.00	
	90.48		90.48		90.48
		81.87		81.87	
			74.08		74.08
				67.03	
					60.65

same tree in terms of prices, but different total variance and time

Time	0	0.25	0.5	0.75	1	1.25
					0.536	
				0.536		
			0.536		0.536	
		0.536		0.536		
p-tree	0.536		0.536		0.536	
		0.536		0.536		
			0.536		0.536	
				0.536		
					0.536	

First 3-months volatility is 10%

sigma		0.1	0.2	0.2	0.2	0.2
delta t		1	0.25	0.25	0.25	0.25
Time	0	1	1.25	1.5	1.75	2
sig*sqrt(delta	0.1	0.1	0.1	0.1	0.1	0.1
u		1.105	1.105	1.105	1.105	1.105
r_annual	0.05					

risk neutral stock tree CRR-style with variable sigma(t)

					164.87
				149.18	
			134.99		134.99
		122.14		122.14	
	110.52		110.52		110.52
100.00		100.00		100.00	
	90.48		90.48		90.48
		81.87		81.87	
			74.08		74.08
				67.03	
					60.65

Time	0	1	1.25	1.5	1.75	2
					0.536	
				0.536		
			0.536		0.536	
		0.536		0.536		
p-tree	0.725		0.536		0.536	
		0.536		0.536		
			0.536		0.536	
				0.536		
					0.536	

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How do we build a binomial tree to price options that's consistent with yield and vol term structures? This is important for American-style options and early exercise. We have to make sure to use the right forward rate and the right forward volatility at each node, with $\sigma\sqrt{\Delta t}$ constant.

Example:

	Year 1	Year 2	Year 3
Term structure of zero coupons:	5%	7.47%	9.92%
Forward rates:	5%	$10\% = \frac{(1.0747^2)}{1.05} - 1$	15%
Term structure of Implied vols:	Σ_1 20%	Σ_2 25.5%	Σ_3 31.1%
Forward vols:	Σ_1 20%	$\Sigma_{12} = \sqrt{2\Sigma_2^2 - \Sigma_1^2}$ 30%	$\Sigma_{23} = \sqrt{3\Sigma_3^2 - 2\Sigma_2^2}$ 40%

Note that the forward volatility rises twice as fast with future time as the implied volatility does with current time to expiration. And the forward rates rise twice as fast with future time as the yields do with current time to maturity.

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13.2 The Binomial Model for Options

Options Valuation in the q-measure for any local volatility $\sigma(S, t)$ as long as its Brownian

$$S = \frac{qS_u + (1-q)S_d}{R}$$

so that in this measure the expected future stock price is the forward price.

Any option C which pays C_u (C_d) in the up (down)-state is replicated by $C = C_u\Pi_u + C_d\Pi_d$ with

$$C = \frac{qC_u + (1-q)C_d}{R}$$

Regard the stock equation as *defining* the measure q, given the values of S, S_u and S_d; the second equation specifies the value C in terms of the option payoffs and the value of p. This is why probability theory seems to be important in options pricing, because of complete markets.

The Black-Scholes Partial Differential Equation and the Binomial Model for Any Local Volatility $\sigma = \sigma(S, t)$

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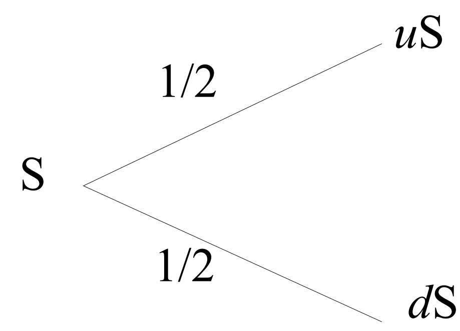
The BS PDE can be obtained by taking the limit of the binomial pricing *q*-measure valuation equation as $\Delta t \to 0$.

We'll use the JR choice of u & d and set $q = 1/2$ and $\mu = r - 0.5\sigma^2$ so that the stock price grows at the riskless rate *r,* as required for the q-measure. The log must grow at $\mu = r - 0.5\sigma^2$



$$u = e^{(r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} \qquad d = e^{(r - 0.5\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}$$

Since q = 1/2, option value is given by this **backward equation:**

$$e^{r\Delta t}C(S, t) = \frac{1}{2}C(S_u, t + \Delta t) + \frac{1}{2}C(S_d, t + \Delta t)$$

Expand in a Taylor series:

$$e^{r\Delta t}C = 0.5C\left(e^{(r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}}S, t + \Delta t\right) + 0.5C\left(e^{(r - 0.5\sigma^2)\Delta t - \sigma\sqrt{\Delta t}}S, t + \Delta t\right)$$

Now $e^{(r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}} \approx 1 + (r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t} + 0.5\sigma^2\Delta t \approx 1 + \sigma\sqrt{\Delta t} + r\Delta t$ so

$$e^{r\Delta t}C = 0.5C(\underbrace{S + S\sigma\sqrt{\Delta t} + rS\Delta t}_{dS}, t + \Delta t) + 0.5C(S - S\sigma\sqrt{\Delta t} + rS\Delta t, t + \Delta t)$$

$$(1 + r\Delta t)C = 0.5\left[C(S, t) + \frac{\partial C}{\partial S}S\{\sigma\sqrt{\Delta t} + r\Delta t\} + \frac{1}{2}\frac{\partial^2}{\partial S^2}C\left\{S^2\sigma^2\Delta t\right\} + \frac{\partial C}{\partial t}\Delta t\right] +$$

Thus to order Δt

$$0.5\left[C(S, t) + \frac{\partial C}{\partial S}S\{-\sigma\sqrt{\Delta t} + r\Delta t\} + \frac{1}{2}\frac{\partial^2}{\partial S^2}C\left\{S^2\sigma^2\Delta t\right\} + \frac{\partial C}{\partial t}\Delta t\right]$$

$$= C(S, t) + \frac{\partial C}{\partial S}S[r\Delta t] + \frac{1}{2}\frac{\partial^2}{\partial S^2}C\left\{S^2\sigma^2\Delta t\right\} + \frac{\partial C}{\partial t}\Delta t$$

and so

$$(\cancel{1} + r\Delta t)C = \cancel{C}(S, t) + \frac{\partial C}{\partial S}S[r\Delta t] + \frac{1}{2}\frac{\partial^2}{\partial S^2}C\left\{S^2\sigma^2\Delta t\right\} + \frac{\partial C}{\partial t}\Delta t$$

When you cancel the $C(S, t)$ term you obtain $Cr\Delta t = \frac{\partial C}{\partial S}\{rS\Delta t\} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\left\{S^2\sigma^2\Delta t\right\} + \frac{\partial C}{\partial t}\Delta t$ or

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2(S, t)S^2\frac{\partial^2 C}{\partial S^2} = rC$$

Same **Black-Scholes Equation**.

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13.3 Local Volatility Models

We just extended the constant-volatility geometric Brownian motion picture underlying the Black-Scholes model to account for a volatility $\sigma(t)$ that can vary with future time. and found a relation between forward and implied volatility..

$$\Sigma^2(t, T) = \frac{1}{T-t}\int_t^T \sigma^2(s)ds$$

How to make $\sigma = \sigma(S, t)$ a function of future stock price S and future time t? Why are we doing this?

Remember: Realized volatility does go up when the market goes down;

We want to see if this simple extension of Black-Scholes can then lead to an explanation of the smile.

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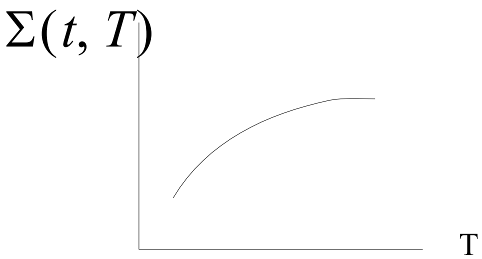
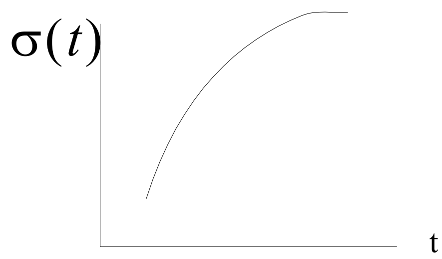
Modeling a stock with a variable volatility $\sigma(S, t)$: Intuition

Model a stock with a variable volatility $\sigma(S, t)$, value options, examine $\Sigma(S, t, K, T)$.

Pure term structure $\Sigma(t, T)$, calibrate the forward volatilities $\sigma(t)$, $\Sigma^2(t, T) = \frac{1}{T-t}\int_t^T \sigma^2(s)ds$.

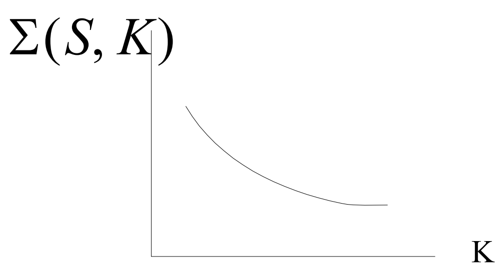
forward vol rises twice as fast with time as implied vol with expiration

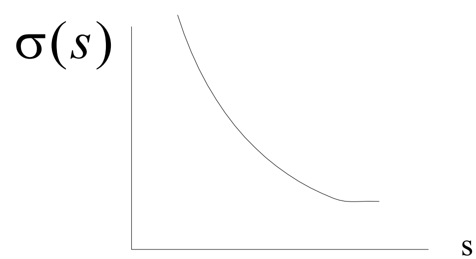




We will discover that for sideways" volatilities $\Sigma(S, t, K, T)$ to $\sigma(S, t)$ --

sideways vol rises approximately twice as fast with future stock price
as implied volatilities rise with current strike





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Questions

1. Can we find a unique local volatility function or surface $\sigma(S, t)$ to match the observed implied volatility surface $\Sigma(S, t, K, T)$? If we can, that means that we can explain the observed smile by means of a local volatility process for the stock.
2. But is the explanation meaningful? Does the stock actually evolve according to an observable local volatility function? There are, as we will see, many different models that can match the implied volatility surface, but achieving a match doesn't mean that model is "correct."
3. What does the local volatility model tell us about the hedge ratios of vanilla options and the values of exotic options? How do the results differ from those of the classic BSM model?

Some references on Local Volatility Models (there are many more).

- *The Volatility Smile*, Derman and Miller (2016)

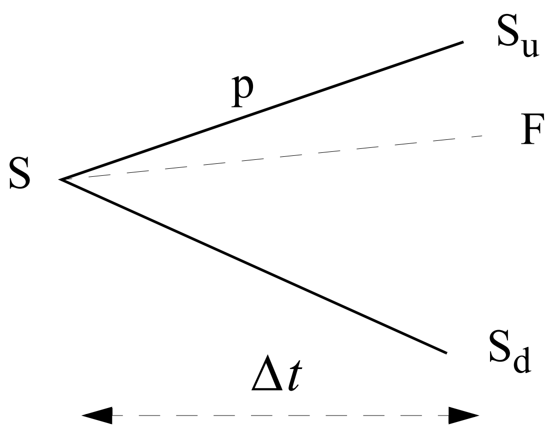
- *The Volatility Smile and Its Implied Tree*, Derman and Kani, RISK, 7-2 Feb.1994, pp. 139-145, pp. 32-39 https://emanuelderman.com/the-volatility-smile-and-its-implied-tree/

- *The Local Volatility Surface by Derman, Kani and Zou, Financial Analysts Journal, (July-Aug 1996), pp. 25-36 h*ttps://emanuelderman.com/the-local-volatility-surface/

Local volatility models are widely used because they are so easy to understand as an extension of Black-Scholes, contain some measure of reality, and can be modified to add stochastic volatility as well.

13.4 Binomial Local Volatility Modeling assuming known $\sigma(S, t)$

How do we modify the usual binomial model to build a binomial tree with $\sigma(S, t)$ that closes (in order to avoid computational complexity)? Here we keep Δt constant via another approach:



p is here the **risk-neutral** no-arbitrage probability

This must reduce at $\Delta t \to 0$ to
$$\frac{dS}{S} = (r-d)dt + \sigma(S, t)dZ$$

How do we find p, S_u and S_d?

We know:

Expected value of S is the **forward price** $F = Se^{(r-d)\Delta t}$ or $F = Se^{r\Delta t} - D$

Furthermore, the SDE implies that $\mathrm{var}[S] = (dS)^2 = \sigma^2(S, t)S^2\Delta t$

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On the binomial tree:

$$F = pS_u + (1-p)S_d \qquad \text{the mean of } S$$

$$S^2\sigma^2(S,t)\Delta t = p(S_u - F)^2 + (1-p)(S_d - F)^2 \qquad \text{the variance}$$

Solve:

$$p = \frac{F - S_d}{S_u - S_d}$$

$$(F - S_d)(S_u - F) = S^2\sigma^2(S,t)\Delta t$$

$$S_u = F + \frac{S^2\sigma^2(S,t)\Delta t}{F - S_d} \qquad \text{or} \qquad S_d = F - \frac{S^2\sigma^2(S,t)\Delta t}{S_u - F}$$

If you know S_d you can calculate S_u and vice versa.

Reference: *The Volatility Smile and Its Implied Tree*, by Derman and Kani.
http://emanuelderman.com/wp-content/uploads/1994/01/gs-volatility_smile.pdf
Derman and Miller textbook

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Building a local-volatility tree

- Build out the tree at any time level by starting **from the middle node** and then moving up or down to successive nodes at that level.

- If we know the **local (forward and sideways) volatilities** $\sigma(S, t)$ and the f**orward interest rates** at each future period, we can determine the stock prices all the up nodes and down nodes from equations.

- Given all the nodes in the tree, we can then use equation for p to compute the risk-neutral probabilities at each node.

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There are many ways to choose the central spine of a binomial tree:

- For every level with an odd number of nodes (1,3,5, etc.) choose the central node to be S. (CRR)
- For every period with even nodes (2,4,6 etc.) choose the two central nodes in those periods to lie above and below the initial stock price *S* exactly as in the CRR tree, given by

$$S_u = Se^{\sigma(S,t)\sqrt{dt}}$$

$$S_d = Se^{-\sigma(S,t)\sqrt{dt}}$$

Once you have the central nodes, you can generate the up and down nodes relative to the central node at each level of the tree by

$$S_u = F + \frac{S^2\sigma^2(S,t)dt}{F - S_d}$$

$$S_d = F - \frac{S^2\sigma^2(S,t)dt}{S_u - F}$$

You could equally well choose a tree whose spine corresponds to the forward price *F* of the stock, growing from level to level. Or anything else.

Toy Example with the local volatility a function only of the stock price *S*:

$$\sigma(S) = max\left[0.1 + \left(\frac{S}{100} - 1\right), 0\right] \qquad \text{independent of } t$$

$$S = 100$$

$$\Delta t = 0.01;\ d = 0,\ r = 0;\ F/S = 1; \sqrt{\Delta t} = 0.1;\ e^{\sigma(S)\sqrt{\Delta t}} = e^{\sigma(S)0.1}$$

The local stock volatility starts out at 10% and increases/falls by 1 percentage point for every 1 point rise/drop in the stock price, but never goes below zero.

$$\sigma(100) = 0.1 \text{ and } \sigma(101) = 0.11$$

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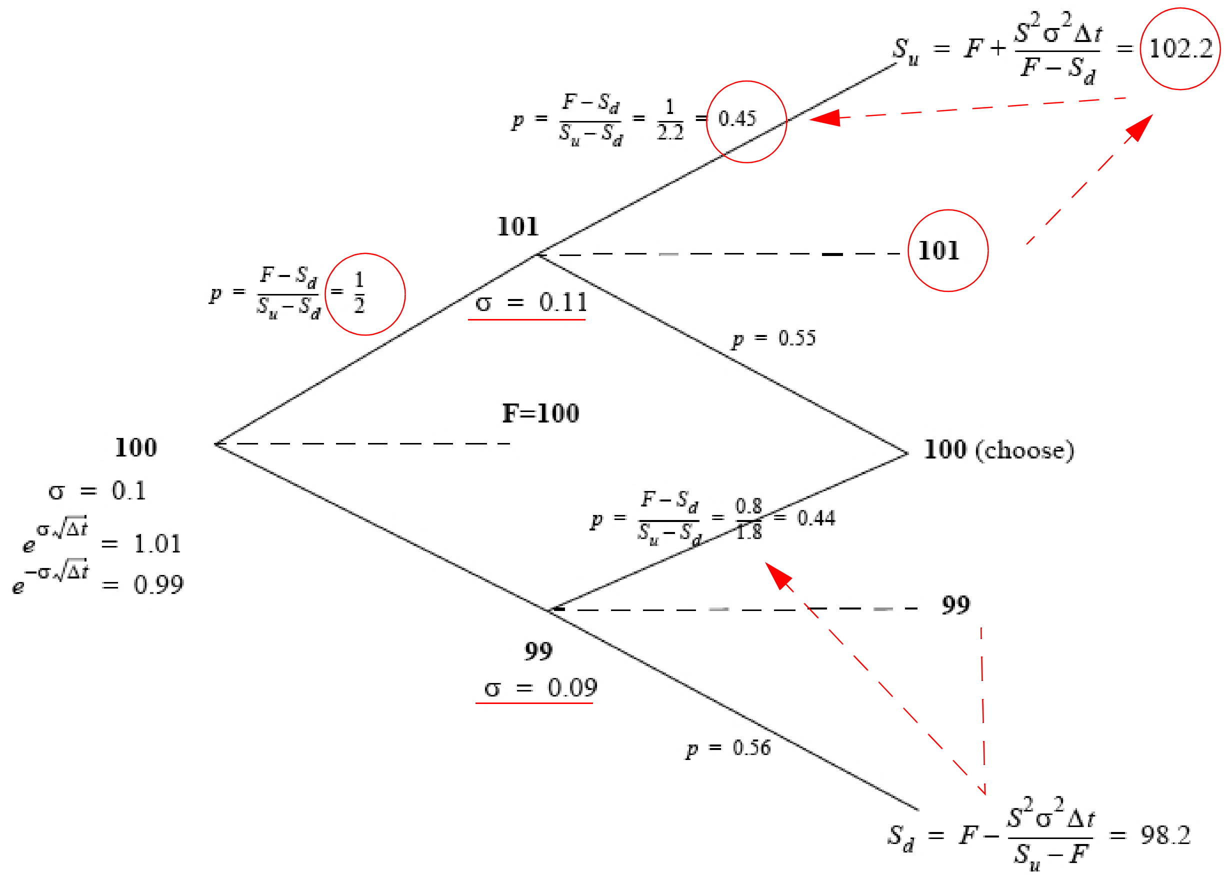

These are the nodes and probabilities that produce the correct discrete version of the desired diffusion. There are other examples in my textbook in chapter 14.

For a two-period call struck at 101, the payoff at the top node is 1.2 with a risk-neutral probability of (0.5)(0.45) for a value of 0.27.

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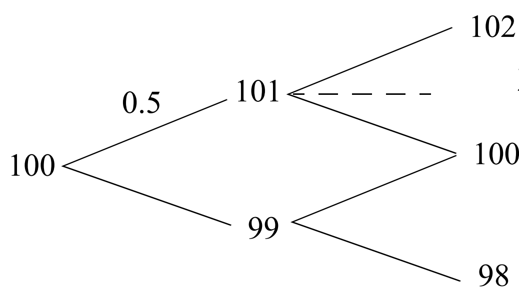
Comparison to flat vol

With local vol the call was worth 0.27.

Compare to value of a similar call on a CRR tree with a flat 10% volatility everywhere.



2-period 101 call $= 0.5 \times 0.5 \times 1 = 0.25$

In the local volatility tree with increasing $\sigma(S)$ there are larger moves up and smaller moves down in the stock price. It produces an upward sloping skew.

Building a binomial tree with variable volatility is in principle possible.

In practice, one may get better trees (i.e. easier to calibrate, more efficient to price with, converging more rapidly as $\Delta t \to 0$,etc.) by using trinomial trees or other finite difference PDE approximations. Nevertheless, we will stick to binomial trees in most of our examples here because of the clarity of the intuition they provide.

You can find more references to trinomial trees with variable volatility in Derman, Kani and Chriss, *Implied Trinomial Trees of the Volatility Smile*, The Journal of Derivatives, 3(4) (Summer 1996), pp. 7-22, and also for example in Peter James' (2003) book on Option Theory which is a good general reference on this topic.

13.5 Investigation of The Relation Between Local and Implied Volatilities.

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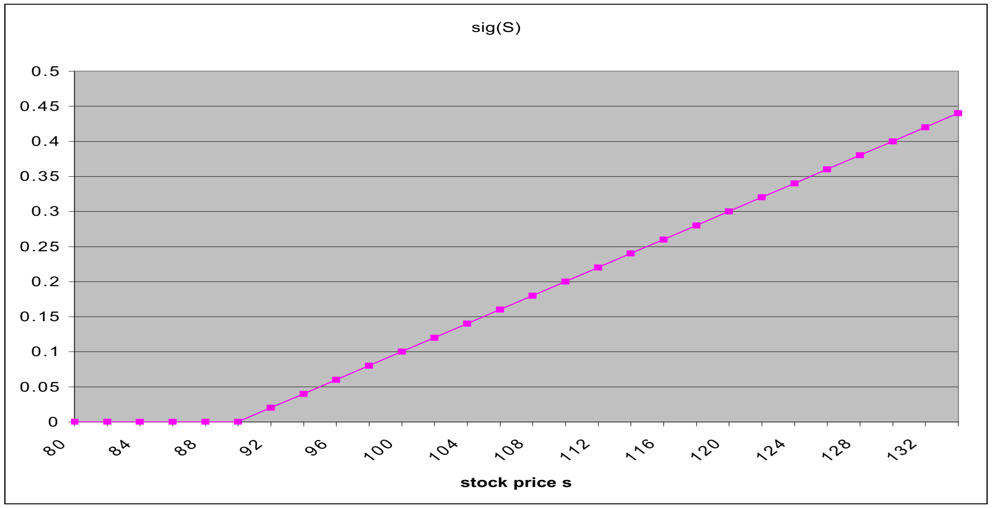
Calibration: The inverse scattering problem: How to build a local volatility tree that matches the smile?

What is the relation between local volatilities as a function of S and implieds as a function of K?

Intuition: Here is a graph of local volatilities that satisfy a positive skew:

$$\sigma(S) = Max[0.1 + (S/100 - 1), 0]$$



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Here is the binomial local-volatility tree for the stock price, assuming $\Delta t = 0.01$, $S = 100$, $r = 0$.

This is a tree with flat volatility 0.1, usual CRR type

stock tree sig = 0.1, p = 0.5

				104.08
			103.05	
		102.02		102.02
	101.01		101.01	
100		100		100
	99		99	
		98.02		98.02
			97.04	
				96.08

$$\sigma^2 = \frac{(F-S_d)(S_u-F)}{S^2\Delta t} = 0.01$$

$$\sigma = 0.1$$

stock tree sigma(s,t)

This is a tree with variable local volatility

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

$$\sigma^2 = \frac{(F-S_d)(S_u-F)}{S^2\Delta t}$$

$$= \frac{(1.28)(1.53)}{(103.5^2)(0.01)} = 0.018$$

$$\sigma = 0.135$$

p-tree

			0.456
		0.488	
	0.453		0.453
0.498		0.498	
	0.441		0.441
		0.509	
			0.434

$$\frac{F-S_d}{S_u-S_d}$$

vol(stock)

			0.135
		0.122	
	0.110		0.110
0.100		0.100	
	0.090		0.090
		0.082	
			0.074

vol ranges from 13.5 to 7.5
as stock ranges from 103.5 to 97.4

13.5.1 An option with strike 102 after 4 periods.

CRR Implied Volatility: What constant volatility CRR tree has the same price for the option? Call with strike 102 has the same value on the *local volatility tree* as it does on a *fixed-volatility CRR tree* with a volatility of 11%. (Analog of BS implied vol for discrete steps)/

NUMERICAL ILLUSTRATION OF RELATION BETWEEN LOCAL AND IMPLIED VOL

local vol tree

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

LOCAL VOL TREE CALL STRUCK AT 102

				3.040
			1.510	
		0.790		0.230
	0.386		0.104	
0.204		0.052		0.000
	0.023		0.000	
		0.000		0.000
			0.000	
				0.000

stock tree with 11% vol

				104.50
			103.36	
		102.22		102.22
	101.11		101.11	
100.00		100.00		100.00
	98.91		98.91	
		97.82		97.82
			96.75	
				95.70

CALL TREE FOR STOCK TREE ON RIGHT STRIKE = 102

				2.498
			1.355	
		0.730		0.224
	0.391		0.112	
0.208		0.055		0.000
	0.028		0.000	
		0.000		0.000
			0.000	
				0.000

Note: 11% is the average of the local volatilities between 10% at S=100 and 12% at K=102.

The CRR implied volatility for a given strike is roughly the average of the local volatilities from spot to that strike.

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Call with strike 103 on the same tree.

local vol tree

				105.04
			103.51	
		102.23		102.23
	101.01		101.01	
100		100		100
	99		99	
		98.21		98.21
			97.39	
				96.76

LOCAL VOL TREE CALL STRUCK AT 103 (sig = 13%)

				2.040
			0.929	
		0.453		0.000
	0.205		0.000	
0.102		0.000		0.000
	0.000		0.000	
		0.000		0.000
			0.000	
				0.000

stock tree with 11.5% vol

				104.71
			103.51	
		102.33		102.33
	101.16		101.16	
100.00		100.00		100.00
	98.86		98.86	
		97.73		97.73
			96.61	
				95.50

CALL TREE FOR STOCK TREE ON RIGHT STRIKE = 103

				1.707
			0.849	
		0.422		0.000
	0.210		0.000	
0.104		0.000		0.000
	0.000		0.000	
		0.000		0.000
			0.000	
				0.000

Implied volatility is about 11.5%, the average of the local volatilities between S = 100 and K = 103.

What have we learned?

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13.5.2 The Rule of 2: Understanding The Relation Between Local and Implied Vols

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We see that: Implied volatility $\Sigma(S, K)$ of an option is approximately the average of the expected local volatilities $\sigma(S)$ encountered over the life of the option between spot S and strike K.

Cf: yields to maturity for zero-coupon bonds are an average over future short-term rates over the life of the bond.

Forward short-term rates grow twice as fast with future time as yields to maturity grow with time to maturity.

Local volatilities grow approximately twice as fast with stock price as implied volatilities grow with strike.

Illustration/"proof" from *The Local Volatility Surface*. Later we'll prove it more rigorously.

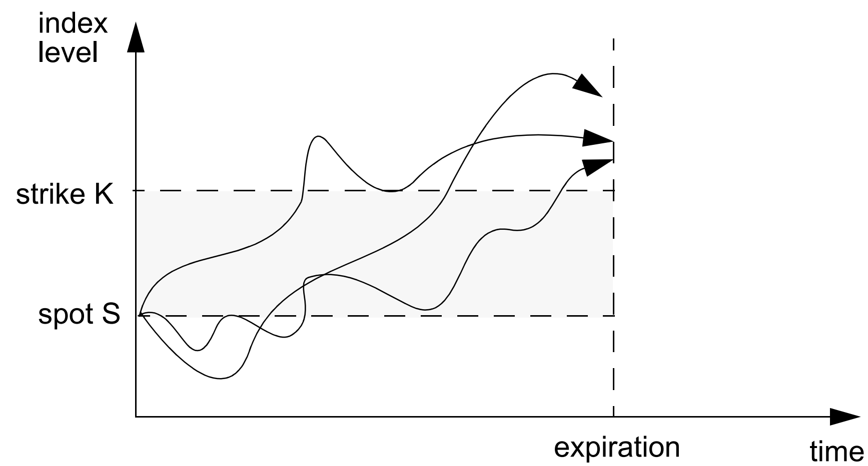
Simple "sideways" linear vol case: $\sigma(S) = \sigma_0 + \beta S$ for all time t

$\Sigma(S, K)$: Any paths that contribute to the option value must pass between S and K

Implied volatility for the option of strike K when the index is at S ~ average of the local volatilities

$$\Sigma(S, K) \approx \frac{1}{K-S}\int_S^K \sigma(S')dS'$$

FIGURE 13.1. Index evolution paths that finish in the money for a call option with strike K when the index is at S. The shaded region is the volatility domain whose local volatilities contribute most to the value of the call option.



$$\Sigma(S, K) \approx \sigma_0 + \frac{\beta}{2}(S + K)$$

Therefore: If (implied volatility varies linearly with strike K at a fixed market level S) then (it also varies linearly at the same rate with the index level S itself).
Local volatility varies with S at twice that rate.

Let's see how well it works with actual local volatilities and implied volatilities in the model.

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13.5.3 Testing Some Examples of Local and Implied Volatilities.

$\sigma(S, t) = 0.1\exp(-[S/100 - 1])$ inserted into a binomial model **with many periods**. Calculate options prices from the binomial local vol model, and then convert the prices to BS implied vols. Note the slopes.

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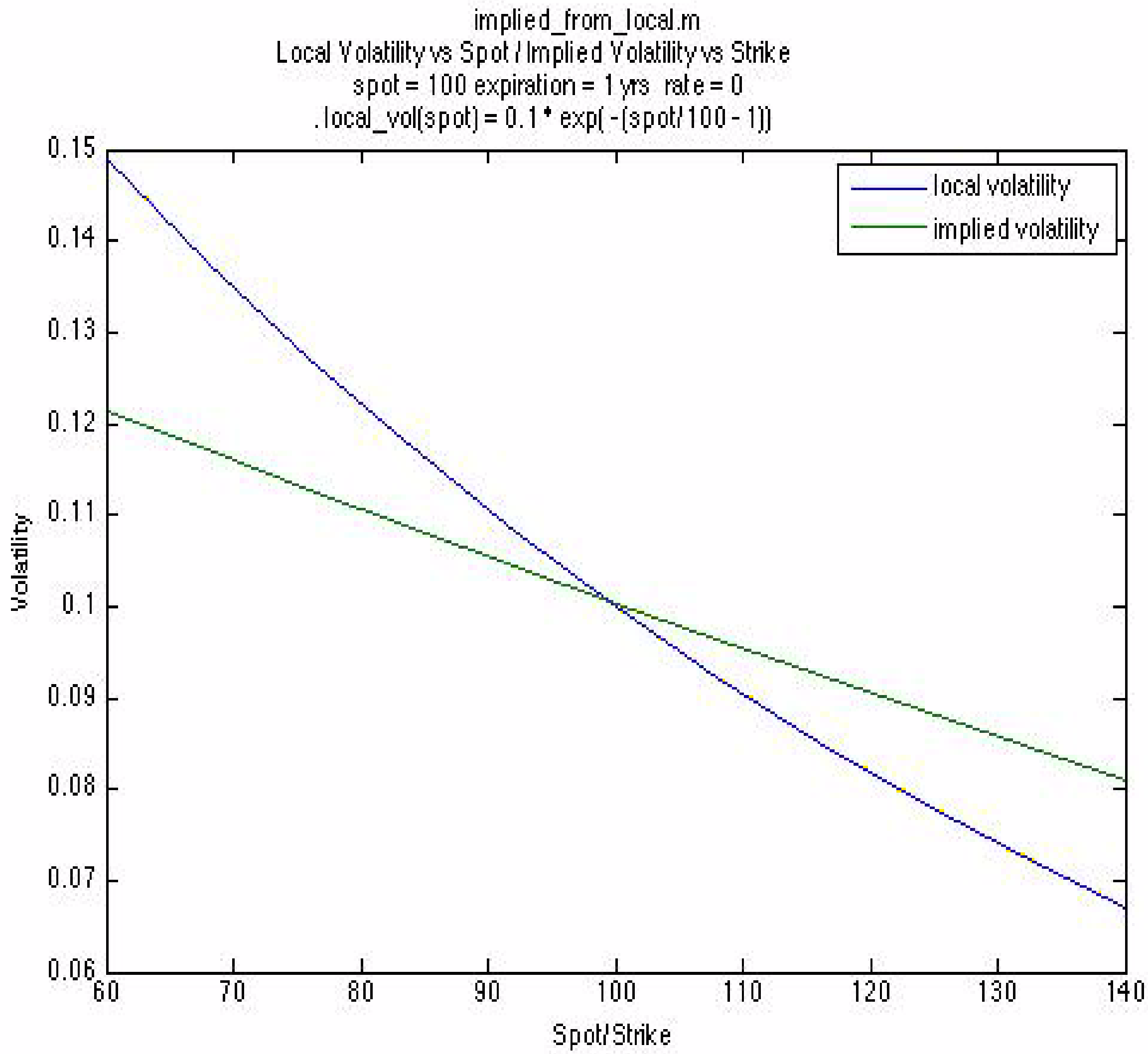
There can be no arbitrage violations of implied vol when you choose positive local vols.



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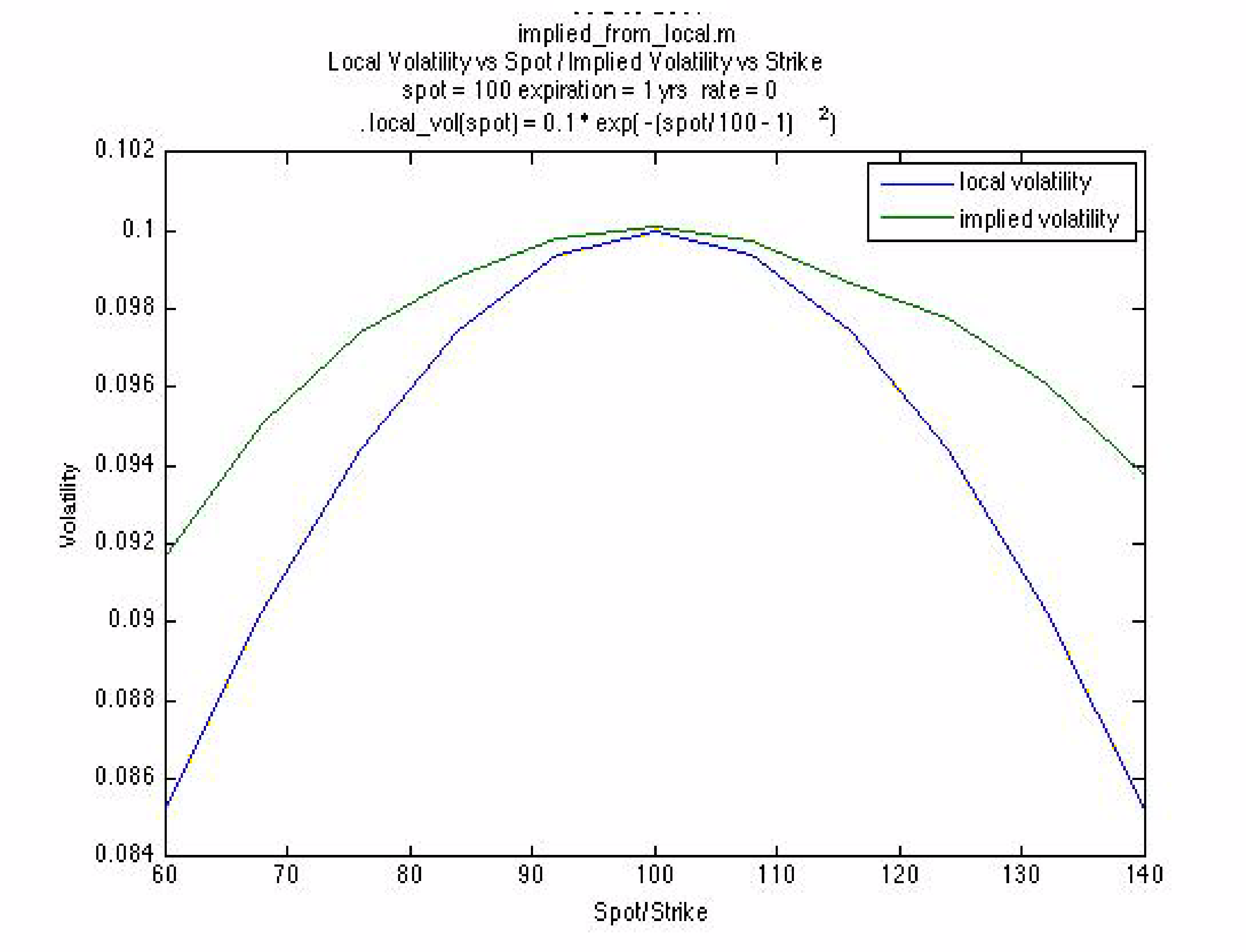
$$\sigma(S, t) = 0.1 \exp(-[S/100 - 1]^2)$$



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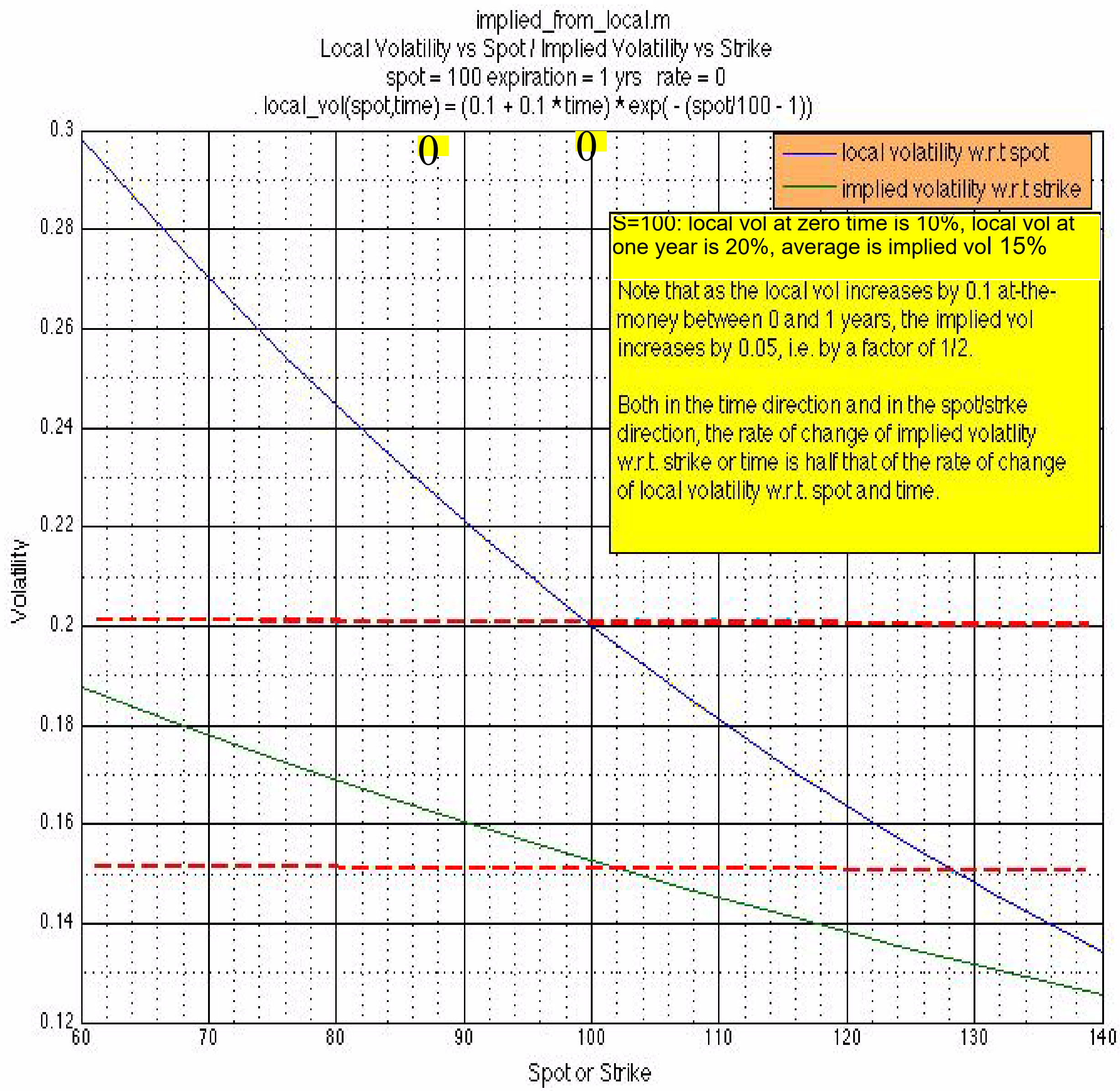
$\sigma(S, t) = (0.1 + 0.1t)\exp(-[S/100 - 1])$ Look at one-year implied volatility skew:



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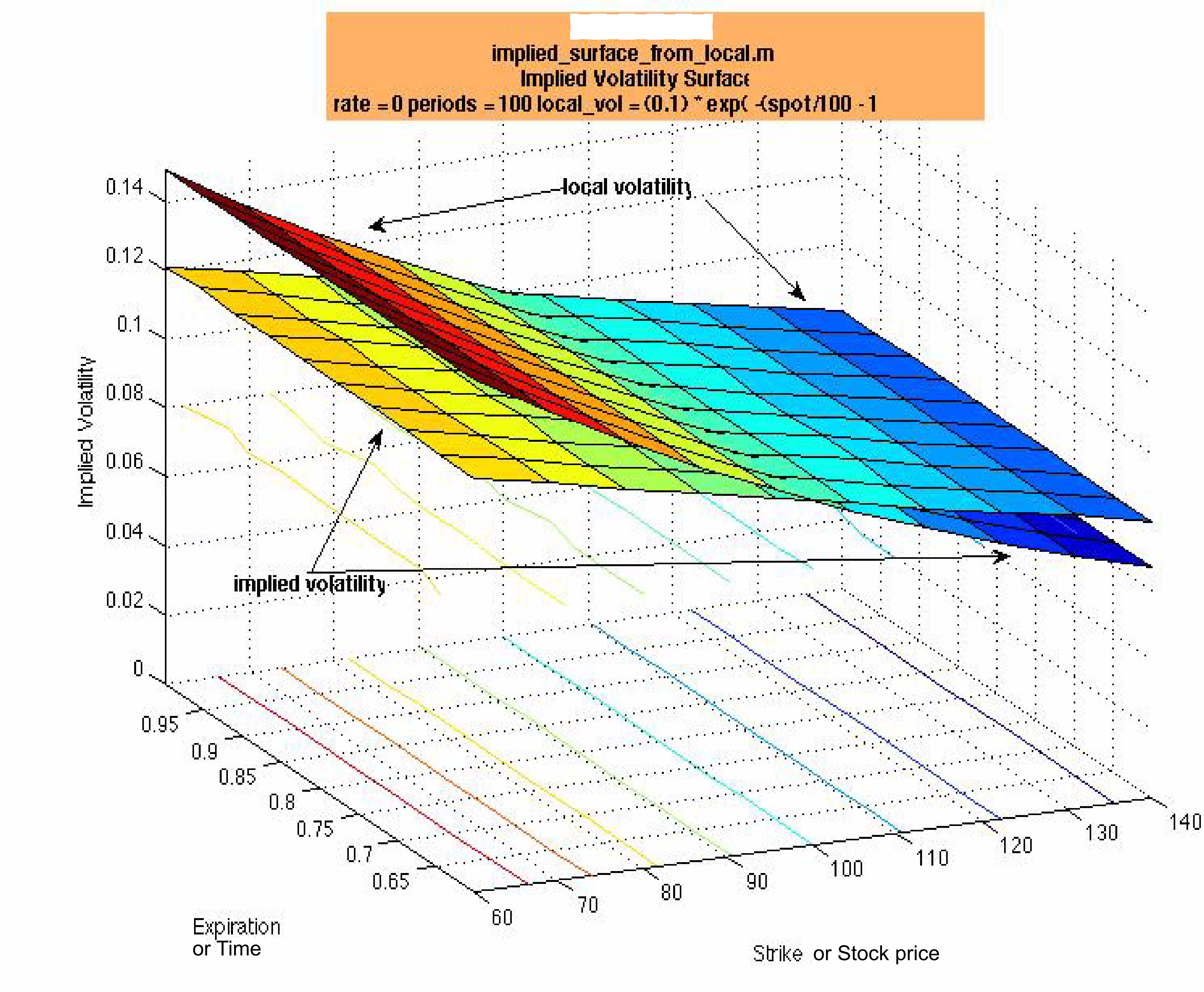
Dependent only on *S*: $\sigma(S, t) = 0.1\exp(-[S/100 - 1])$: Plot surface



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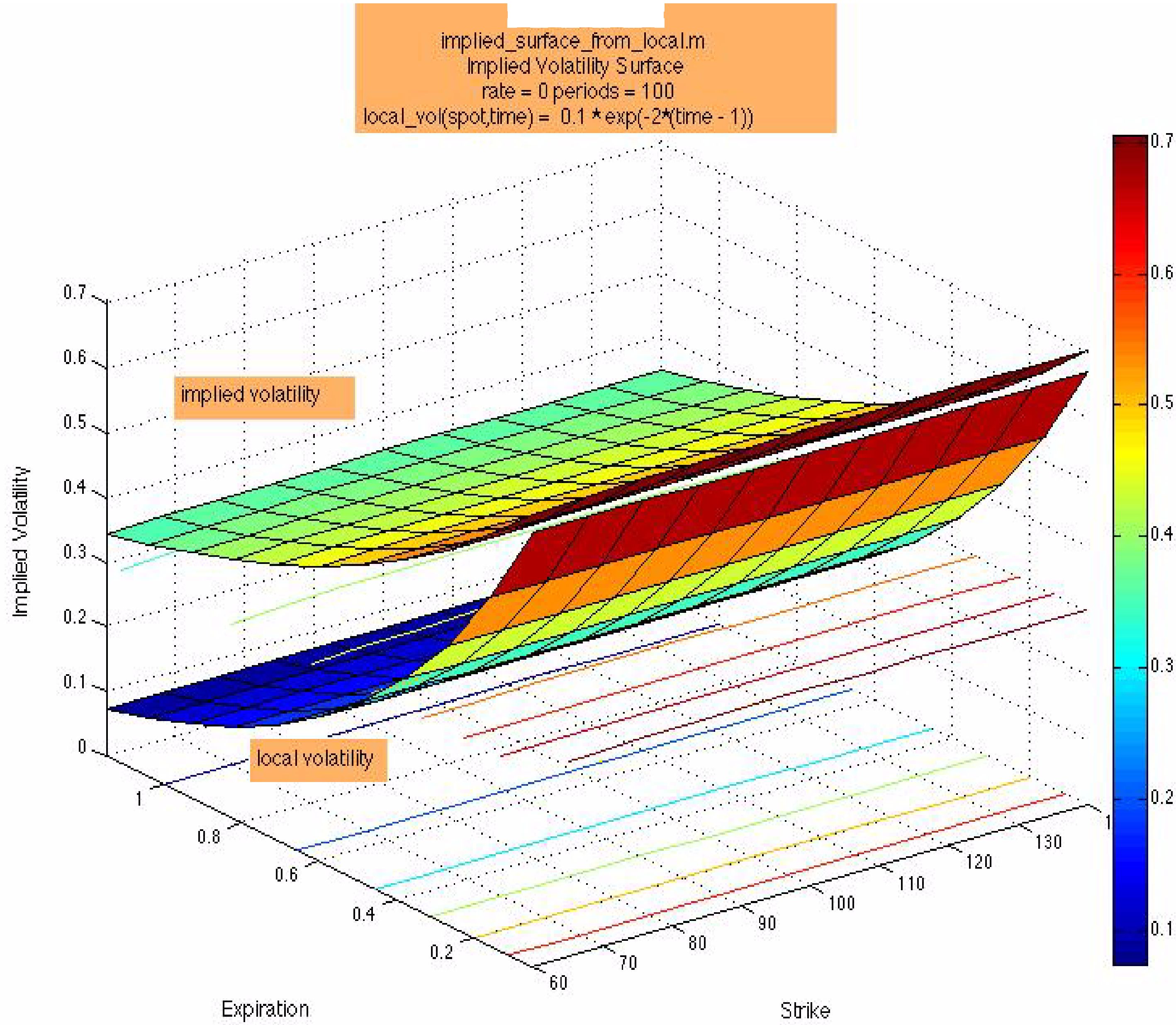
Dependent only on *t*: $\sigma(S, t) = 0.1\exp(-2[t-1])$: Plot Surface



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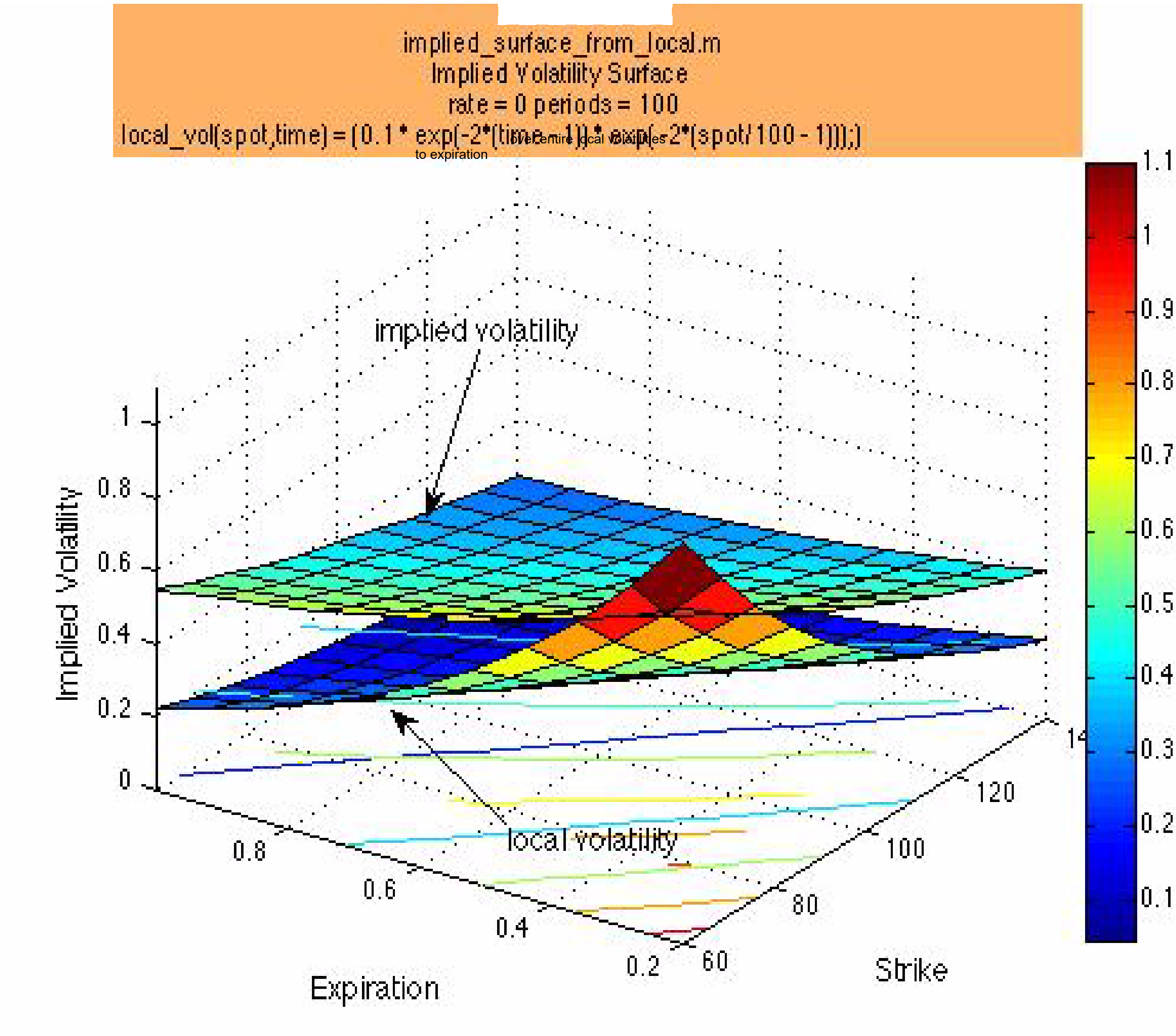
Dependent on *S* and *t*: $\sigma(S, t) = 0.1\exp(-2[t-1])\exp(-2[S/100-1])$**. Local vol is 10% at t = 1 and S = 100.**

Local volatilities stay constant along lines where S increases and t decreases. Similarly for implieds



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