LECTURE 23

JUMP DIFFUSION

Please Attend For

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23.1 Jumps

- Why are we interested in jump models? Because stocks and indexes don't diffuse smoothly, and do seem to jump. Even currencies sometimes jump. It's one of the things that happen.

- How big is a jump vs. a diffusive move? Does it make sense to think of two different things going on with one underlyer? One wouldn't do that in physics. to describe, say, phase transitions.

- Jumps provide an easy way to produce the steep short-term skew that persists in equity index markets, and that indeed appeared soon after the jump/crash of 1987. They seem to play a part, behaviorally.

- Jumps are unattractive from a theoretical point of view because you cannot continuously hedge a distribution of finite-size jumps, and so risk-neutral arbitrage-free pricing isn't possible.

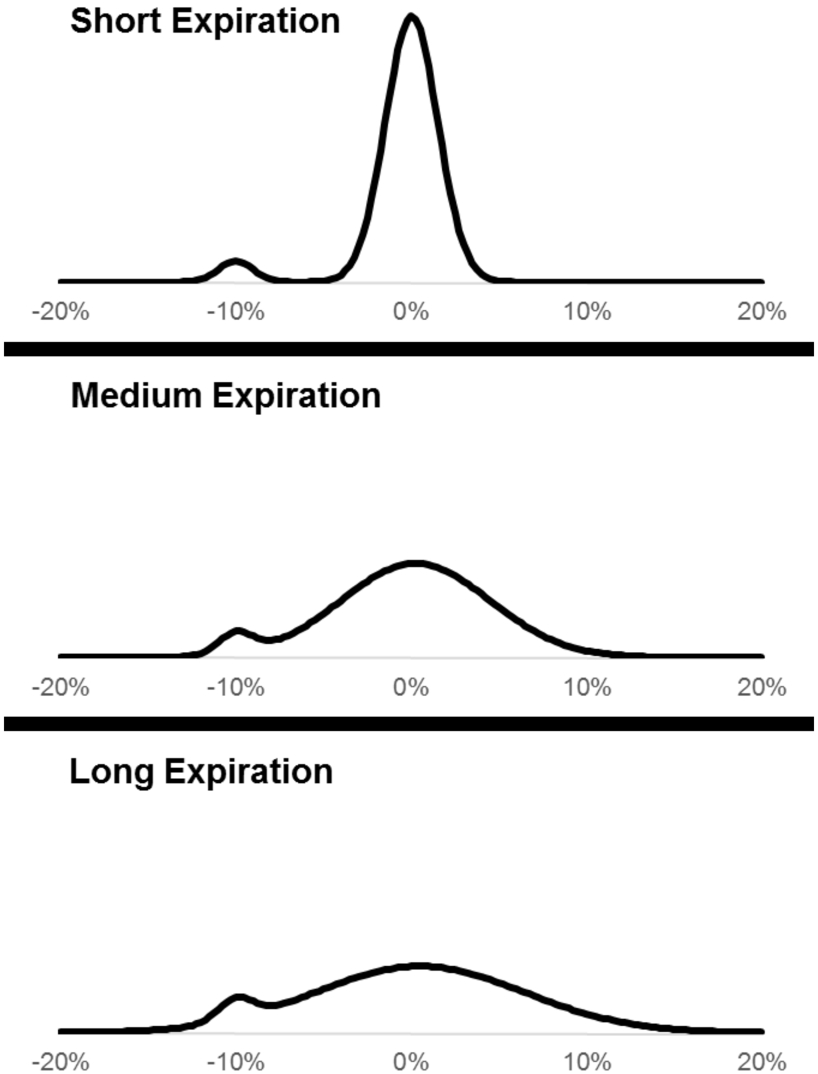
  As a result, most jump-diffusion models simply assume risk-neutral pricing without a thorough justification. It may make sense to think of the implied volatility skew in jump models as simply representing what sellers of options will charge to provide protection from crashes, actuarially, forgetting about hedging.

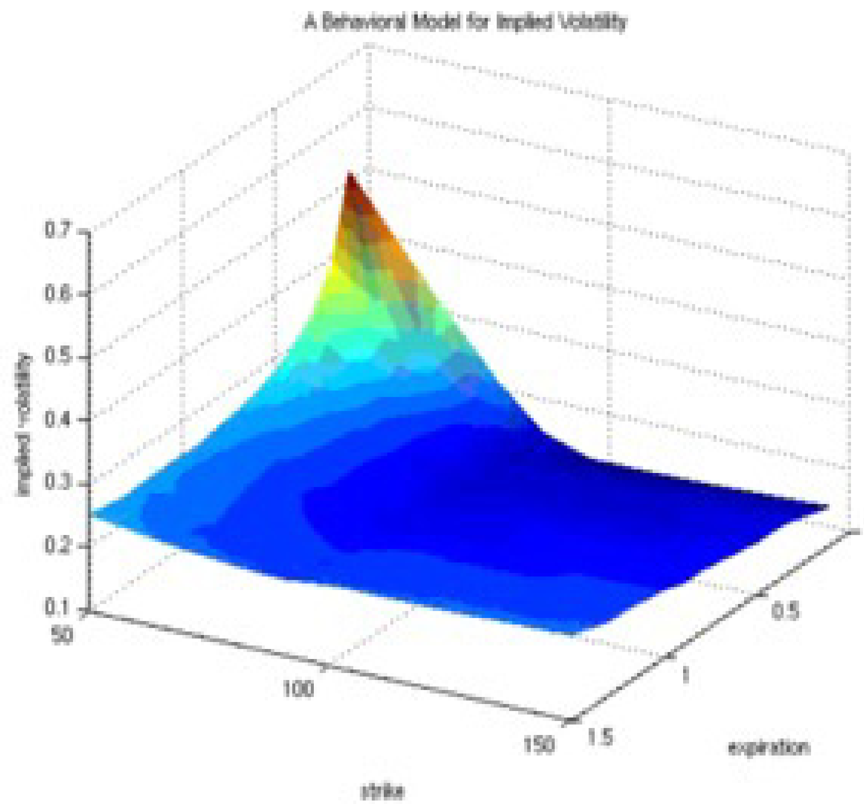
- Whatever the case, there have been and will be jumps in asset prices, and even if you can't hedge them, we are still interested in seeing what sort of skew they produce.

- What characterizes a jump?
  jump process (Poisson), jump size, jump probability or frequency,
  jump size uncertainty (Gaussian)…

The Effect of Jumps

Just one jump





Although jumps are not diffusive, we want to know what μ and σ their distribution has from a diffusion point of view. How do they modify the diffusion process?

Another Intuitive, Expectations View of the Skew Arising from Jumps

Assume:

Probability p_K that **a single jump** will occur taking the market from S to K sometime before option expiration T in N days, and will then diffuse at $\sigma(T)$

Without that jump the future diffusion volatility of the index would have been $\sigma(T)$. Now with the jump the standard deviation/volatility is larger.

Assume: The implied variance for strike K is the expected value of the realized variance as the stock gets to K via diffusion plus a jump.

Realized daily volatility σ_d of an index S_i over a period of N days to expiration is the square root of the variance of the daily log returns r_i.

Without the jump, $\sigma_d^2 \approx \frac{1}{N}\sum_{i=1}^{N} r_i^2$

With one jump from S to K, $\sigma_d^2 \approx \frac{1}{N}\left\{\sum_{i=1}^{N-1} r_i^2 + \left(\ln\frac{S}{K}\right)^2\right\} \approx \sigma_d^2(T) + \frac{\left(\ln\frac{S}{K}\right)^2}{N}$

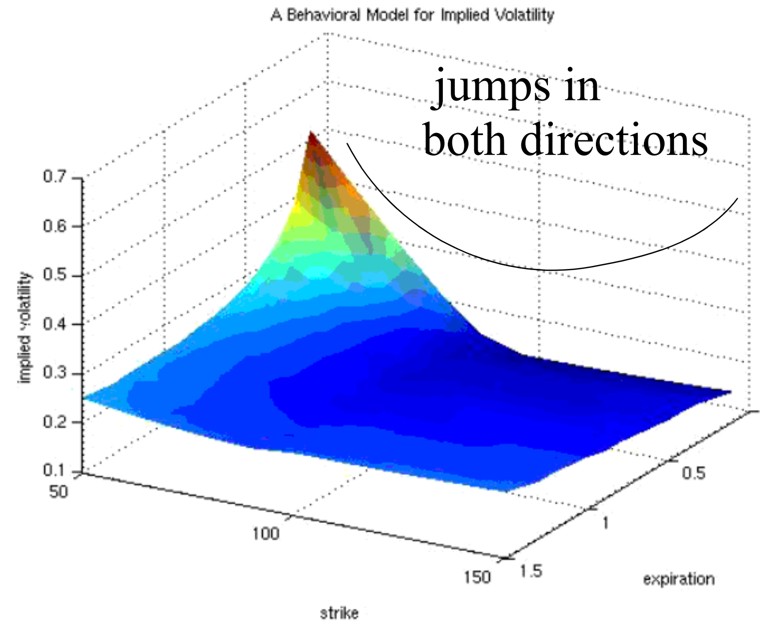
Expected daily variance is approx $p_K\left(\sigma_d^2(T) + \frac{\left(\ln\frac{S}{K}\right)^2}{N}\right) + (1-p_K)\sigma_d^2(T) = \sigma_d^2(T) + p_K\frac{\left(\ln\frac{S}{K}\right)^2}{N}$

Expected annualized variance is $365\left(\sigma_d^2 + p_K\frac{\ln^2\left(\frac{S}{K}\right)}{N}\right) = \sigma_a^2(T) + p_K\ln^2\left(\frac{S}{K}\right)\frac{365}{N}$

where $\sigma_a(T)$ is the annualized diffusion volatility and $\frac{N}{365} = T$ the time to expiration in years.

But this is our intuitive estimate for the total implied variance for strikes accessible by jumps in just one direction.

$$\Sigma^2(S, K, T) = \sigma_a^2(T) + \frac{p_K}{T}\ln^2\left(\frac{S}{K}\right)$$



23.2 Recall: Modeling Diffusion and the Logarithmic Drift

Discrete binomial approximation to a diffusion process for $\log(S/S_0)$ over time Δt:

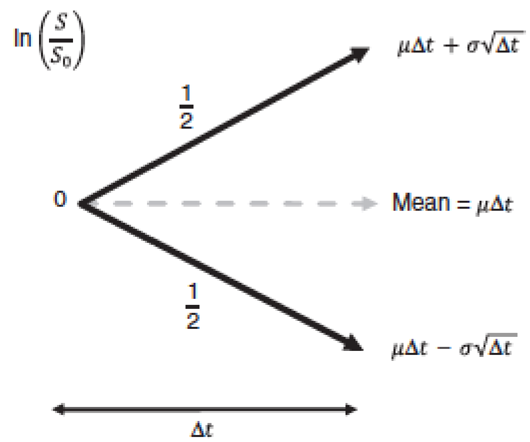


FIGURE 23.3 Binomial Diffusion-Only Model

For diffusion, **probabilities of both up and down** moves are **finite**, but the **moves** themselves are **small**, of order $\sqrt{\Delta t}$.

The net variance is $\sigma^2\Delta t$ and the logarithmic drift is μ. In continuous time this represents the process $d\ln S = \mu dt + \sigma dZ$ $\quad \frac{dS}{S} = \left(\mu + \frac{1}{2}\sigma^2\right)dt + \sigma dZ$

For risk neutrality, if the stock grows at r, we need $\mu = r - \frac{1}{2}\sigma^2$ as the **calibrated drift of the log diffusion** in order to get the expected value of the stock to grow at the riskless rate.

23.3 Modeling Jumps and their Calibrated Log Drift

With jumps, the **probability of a jump J is small**, of order Δt, but the **jump itself is finite**.

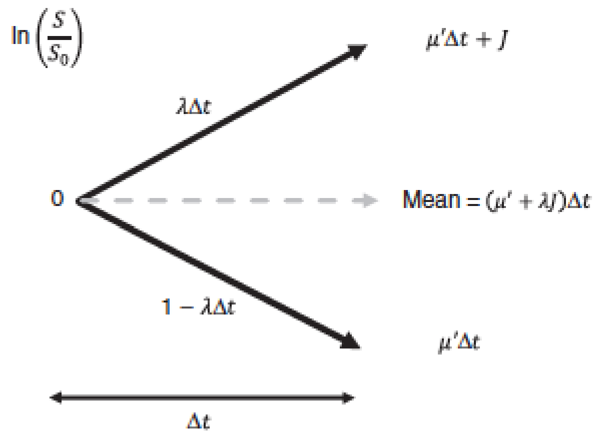


FIGURE 23.4 Binomial Jump Model

3 parameters μ', J, λ The average:

$$
\begin{aligned}
E[\ln S] &= \lambda\Delta t[\mu'\Delta t + J] + (1-\lambda\Delta t)\mu'\Delta t \\
&= (\mu' + \lambda J)\Delta t
\end{aligned}
$$

Variance

$$
\begin{aligned}
\text{var} &= \lambda\Delta t[J(1-\lambda\Delta t)]^2 + (1-\lambda\Delta t)[J\lambda\Delta t]^2 \\
&= (1-\lambda\Delta t)J^2\lambda\Delta t[1-\lambda\Delta t + \lambda\Delta t] \\
&= (1-\lambda\Delta t)J^2\lambda\Delta t \\
&\to J^2\lambda\Delta t \qquad \text{as} \quad \Delta t \to 0
\end{aligned}
$$

$$\textit{Observed}\text{ drift }\mu = (\mu' + \lambda J)$$

$$\textit{Observed}\text{ instantaneous volatility }\sigma = J\sqrt{\lambda}.$$

How do we choose the jump process to match what we observe continuously for a stock's log returns?

Calibration to Observations: If we *observe* a drift μ and a volatility σ, we must calibrate the jump process so that

$$J = \frac{\sigma}{\sqrt{\lambda}}$$

$$\mu' = \mu - \sqrt{\lambda}\sigma$$

The one unknown is λ which is the probability of a jump of size J in log return $\ln S$ per unit time.

When we do option pricing we will impose risk neutrality, which means that the expected value of the stock grows at the riskless rate, so we need to figure out how to choose the jump parameters to produce this.

23.4 Calibration of the Logarithmic Drift of the Jump Process to the Riskless Growth Rate of the Stock (Risk Neutrality)

We described how $\ln(S)$ evolves. How does S evolve? The effect of convexity:

$$
\begin{aligned}
E[S] &= (1-\lambda\Delta t)S\exp(\mu'\Delta t) + \lambda\Delta t S\exp(\mu'\Delta t + J) \\
&= S\exp(\mu'\Delta t)[1+\lambda\Delta t(e^{J}-1)] \\
&\approx S\exp\left[\left\{\mu' + \lambda(e^{J}-1)\right\}\Delta t\right]
\end{aligned}
$$

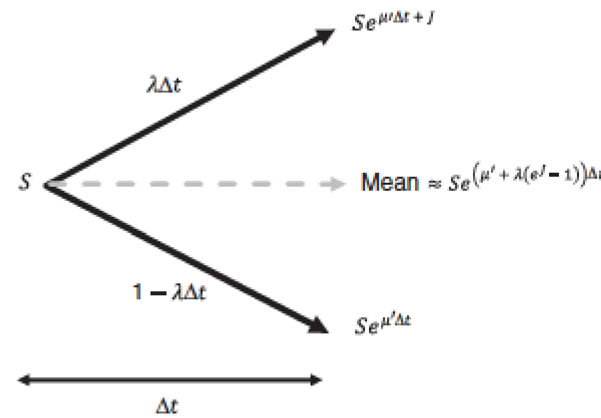


FIGURE 23.5 Binomial Jump Model for Price

A positive jump adds to the drift, a negative jumps lowers the drift.

Thus a calibrated risk-neutral growth would mean

$$
\begin{aligned}
r &= \mu' + \lambda(e^{J}-1) \\
\mu' &= r - \lambda(e^{J}-1)
\end{aligned}
$$

To maintain risk neutrality and achieve an expected return of r **for the stock**, we have to **compensate the logarithmic drift for the convexity of the jump contribution.**

In continuous-time notation the jump can be written as a Poisson process

$$d\ln S = \mu' dt + J dq$$

Here dq is a jump / Poisson process that is modeled as follows:

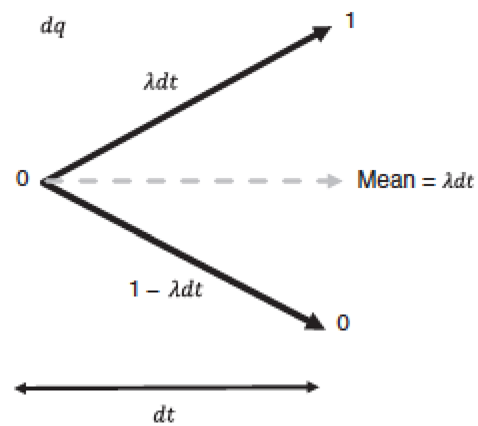


FIGURE 23.6 Binomial Poisson Process

The increment dq takes the values:

1 with probability λdt if a jump occurs
0 with probability $1 - \lambda dt$ if no jump occurs
expected value $E[dq] = \lambda dt$

variance $var(dq) = \lambda dt(1-\lambda dt)^2 + (1-\lambda dt)(\lambda dt)^2 = \lambda dt(1-\lambda dt) \to \lambda dt$

23.5 The Poisson Distribution of Jumps

λ = the constant probability per unit time of a jump J occurring in the logarithm of the stock price.

Let $P(n, t)$ be the probability of n jumps occurring during time where $dt = t/N$ for N periods.

$$P[0, t] = (1-\lambda dt)^{\frac{t}{dt}} = \left(1-\lambda t\frac{dt}{t}\right)^{\frac{t}{dt}} = \left(1-\frac{\lambda t}{N}\right)^{N} \to e^{-\lambda t} \quad \text{as} \quad N \to \infty$$

$$\begin{aligned}
P(n, t) &= \frac{N!}{n!(N-n)!}(\lambda dt)^{n}(1-\lambda dt)^{N-n} \\
&= \frac{N!}{n!(N-n)!}\left(\frac{\lambda t}{N}\right)^{n}\left(1-\frac{\lambda t}{N}\right)^{N-n} \\
&= \frac{N!}{N^{n}(N-n)!}\frac{(\lambda t)^{n}}{n!}\left(1-\frac{\lambda t}{N}\right)^{N-n} \\
\lim(N \to \infty) \text{ for fixed } n &= \frac{(\lambda t)^{n}}{n!}e^{-\lambda t}
\end{aligned}$$

as $N \to \infty$ for fixed n. Note that $\sum_{n=0}^{\infty} P(n,t) = 1$

The mean number of jumps $\overline{n(t)}$ during time t is λt.

Pure jump risk-neutral option pricing

Risk-neutral means the expected value of the stock (with no dividends) grows at the riskless rate.

We can value a standard call option (**assuming risk-neutrality**) for a **pure** jump model:

$$C = e^{-r\tau} \sum_{n=0}^{\infty} max[Se^{\mu'\tau + nJ} - K, 0] \frac{(\lambda\tau)^n}{n!} e^{-\lambda t}$$

where $Se^{\mu'\tau + nJ}$ is the final stock price after n Poisson jumps, and the payoff of the call is multiplied by the probability of the jump occurring, and the log growth rate of the jump process is

$$\mu' = r - \lambda(e^J - 1)$$

We choose this value so that the stock grows at the riskless rate, and so that a call with zero strike is worth the stock price,

23.6 Jumps plus Diffusion

Some comments

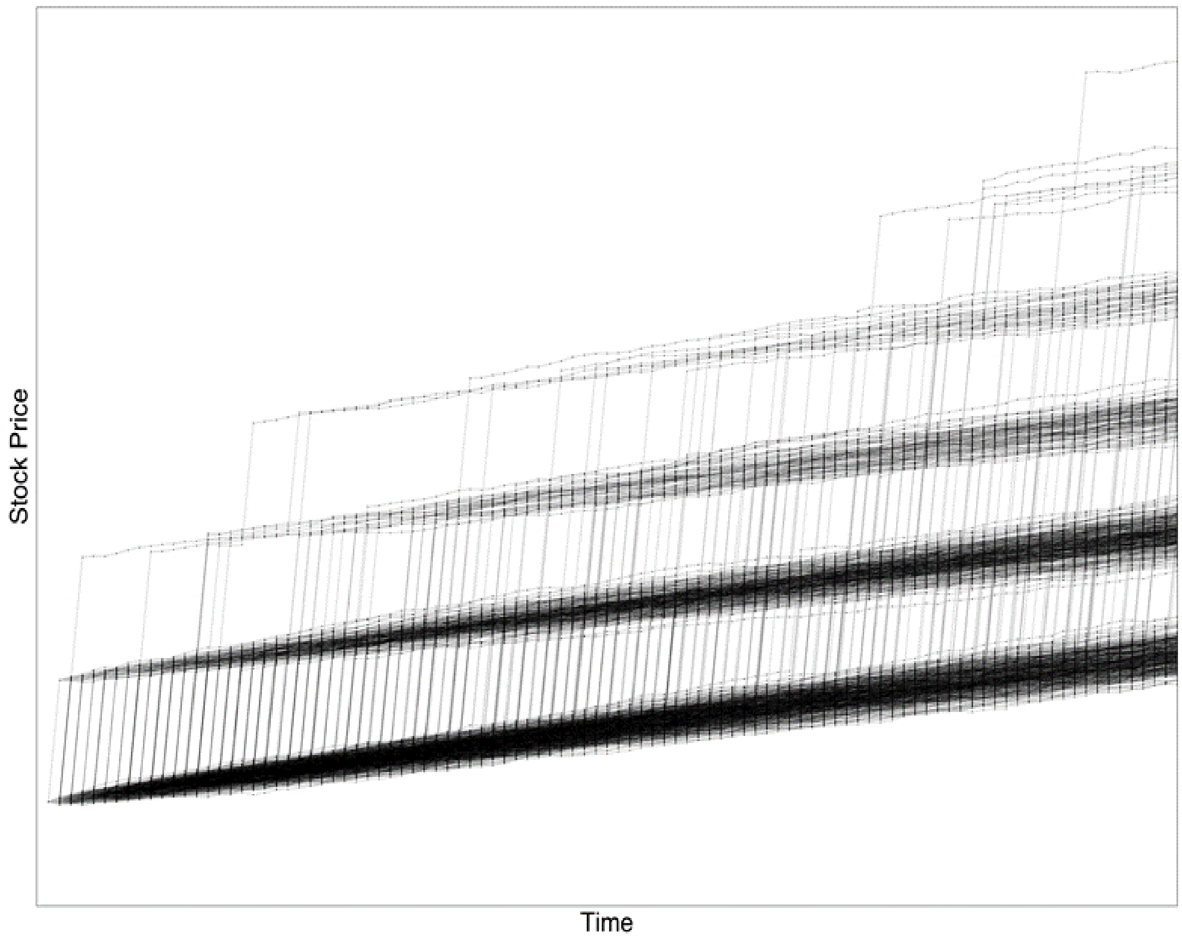
- Hedging an option with stochastic volatility requires a stock and another option. Similarly you can hedge an option exactly with stock and a finite number of options for a finite number of jumps of known size,.

- But with an infinite number of possible jumps, you cannot replicate; you can only minimize the variance of the P&L.

- Merton's model of jump-diffusion regards jumps as "abnormal" market events that have to be superimposed upon "normal" diffusion.
  Mandelbrot, and Eugene Stanley and his econophysics collaborators prefer a single model, rather than a "normal" and "abnormal" model.

- There are some so-called Variance-Gamma models of the smile in which **all** stock price movements are jumps of various sizes.

A Monte Carlo Simulation of Stock Prices in a Jump-Diffusion Model



23.7 Merton's Jump-diffusion Model And Its PDE

Poisson jumps + GBM diffusion, $\frac{dS}{S} = \mu dt + \sigma dZ + Jdq$ - combination of two processes

$$E[dq] = \lambda dt$$
$$var[dq] = \lambda dt$$

J is like a random percentage dividend that lowers or raises the stock price, but it is **not** paid to the stockholder. Later we'll make J a **normal random** variable.

You can derive a partial differential equation for option valuation: option $C(S, t)$ and usual hedged portfolio $\pi = C - nS$ (bad notation -- here n is number of shares shorted, not number of jumps)

$$\begin{aligned} ndS &= nS(\mu dt + \sigma dZ + Jdq) \\ &= n(\mu Sdt + \sigma SdZ) + (nJS)dq \end{aligned}$$

$$\begin{aligned} dC = &\left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt \\ &+ \frac{\partial C}{\partial S}(\mu Sdt + \sigma SdZ) + [C(S+JS,t) - C(S,t)]\, dq \end{aligned}$$

Expected value only if there is a distribution of jumps

if diffuses **if jumps**

Keeping terms up to order dt

$$
\begin{aligned}
d\pi &= dC - ndS \\
&= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right) dt + \left(\frac{\partial C}{\partial S} - n\right)(\mu S dt + \sigma S dZ) \\
&\quad + [C(S+JS,t) - C(S,t) - nJS]\, dq
\end{aligned}
$$

Choose number of shares *n* to hedge the diffusion: $n = C_S$. We can't hedge everything with it.

$$
d\pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \left[C(S+JS,t) - C(S,t) - \frac{\partial C}{\partial S}JS\right]dq
$$

The partially hedged portfolio is still risky because of the possibility of jumps.

$$
E[d\pi] = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + E\left[C(S+JS,t) - C(S,t) - \frac{\partial C}{\partial S}JS\right]E[dq]
$$

Imagine that we can diversify our portfolio over many different stocks and their options, where the stocks have uncorrelated jumps, so that jump risk becomes diversifiable and can be eliminated. Or suppose simply that even though there is some risk, we expect roughly the riskless return if we average over all jumps:

$$E[d\pi] = r\pi dt = r(C - SC_S)dt \qquad E[dq] = \lambda dt$$

So averaging over all jump sizes gives

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + E\left[C(S+JS,t) - C(S,t) - \frac{\partial C}{\partial S}JS\right]\lambda dt$$
$$= r\left(C - S\frac{\partial C}{\partial S}\right)dt$$

Or

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 + r\left(S\frac{\partial C}{\partial S} - C\right) + E\left[C(S+JS,t) - C(S,t) - \frac{\partial C}{\partial S}JS\right]\lambda = 0 \tag{24.10}$$

This is a mixed difference/partial-differential equation for a standard call with terminal payoff $C_T = max(S_T - K, 0)$. For $\lambda = 0$ it reduces to the Black-Scholes equation. We will solve it a little later by the Feynman-Kaç method as an expected discounted value of the payoffs.

I don't find the diversification argument very compelling, but we will continue with this logic anyhow.

23.8 Trinomial Jump-Diffusion and Compensation

Diffusion can be modeled binomially, as in

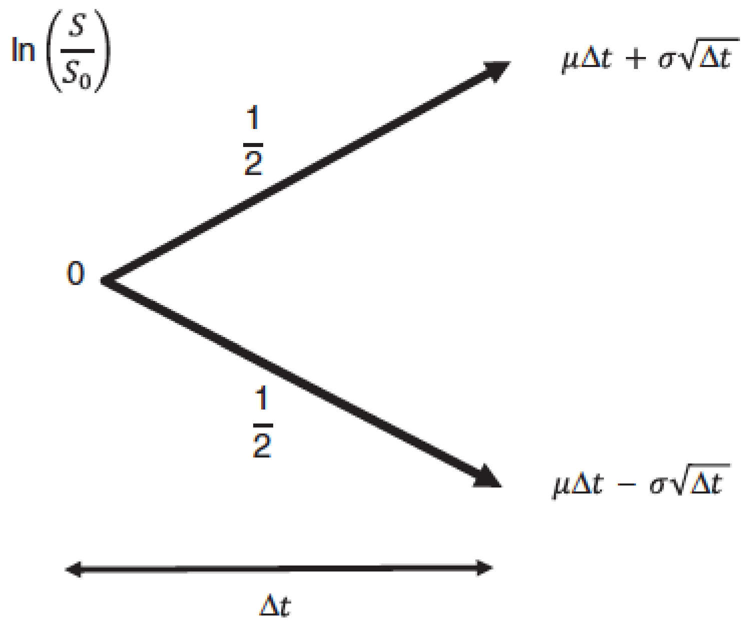


FIGURE 24.1 Binomial Model of Diffusion

The volatility σ of the log returns adds an Ito $\sigma^2/2$ term to the drift of the stock price S itself, so that for pure risk-neutral diffusion one must choose the log drift of the diffusion process to be $\mu = r - \sigma^2/2$.

To add jumps one J needs a third, trinomial, leg in the tree: Just as diffusion modifies the drift of the

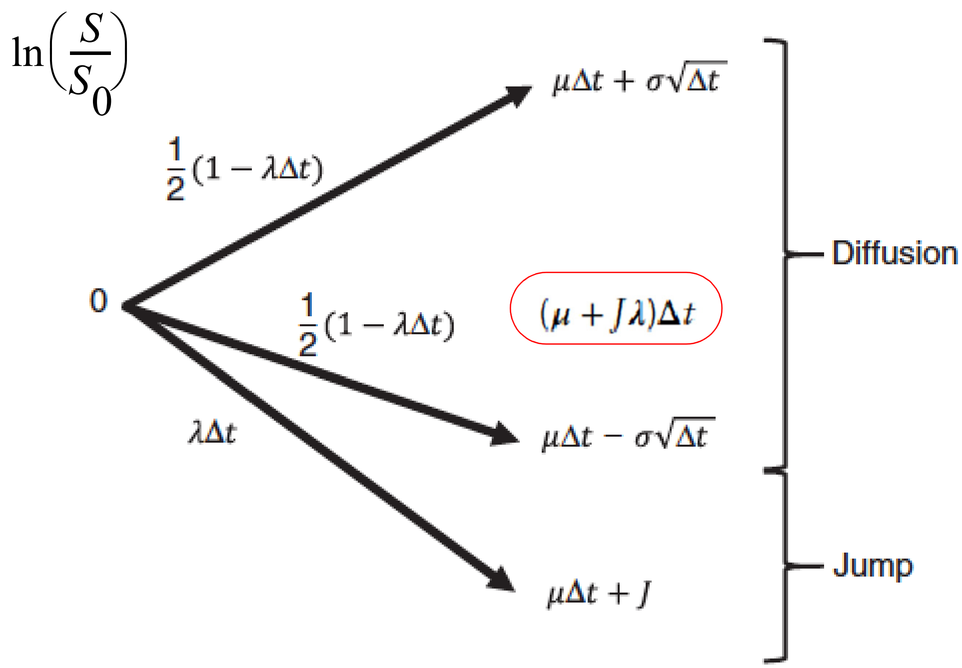


FIGURE 24.2 Trinomial Tree with One Jump

stock price, so do jumps.

The expected log return after time Δt:

$$
\begin{aligned}
E\left[\ln\left(\frac{S}{S_0}\right)\right] &= \frac{1}{2}(1-\lambda\Delta t)\left(\mu\Delta t + \sigma\sqrt{\Delta t}\right) + \frac{1}{2}(1-\lambda\Delta t)\left(\mu\Delta t - \sigma\sqrt{\Delta t}\right) \\
&\quad + \lambda\Delta t(\mu\Delta t + J) \qquad (24.11) \\
&= (\mu + J\lambda)\Delta t
\end{aligned}
$$

Thus the effective **log drift** of the jump-diffusion process will be $\mu_{JD} = \mu + J\lambda$.

$$var \equiv \left(\frac{1-\lambda\Delta t}{2}\right)[\sigma\sqrt{\Delta t} - J\lambda\Delta t]^2 + \left(\frac{1-\lambda\Delta t}{2}\right)[\sigma\sqrt{\Delta t} + J\lambda\Delta t]^2$$

$$+ \lambda\Delta t[J(1-\lambda\Delta t])]^2$$

The variance at time Δt is

$$= \left(\frac{1-\lambda\Delta t}{2}\right)[2\sigma^2\Delta t + 2J^2\lambda^2(\Delta t)^2] + \lambda\Delta t J^2(1-\lambda\Delta t)^2$$

$$= (1-\lambda\Delta t)[\sigma^2\Delta t] + (1-\lambda\Delta t)J^2\lambda\Delta t(\lambda\Delta t + 1 - \lambda\Delta t)$$

$$= (1-\lambda\Delta t)[\sigma^2 + J^2\lambda]\Delta t$$

so that, as $\Delta t \to 0$, the variance of the log jump diffusion process is ${\sigma_{JD}}^2 = [\sigma^2 + J^2\lambda]$,

the sum of the diffusion variance plus the expected jump variance. The drift and variance are both affected by the fractional jump J and its probability λ of occurring per unit time.

23.9 The Compensated Logarithmic Drift of the Diffusion Process for Risk Neutrality

How must we choose/calibrate the diffusion log drift μ so that $E[dS] = Srdt$?

First let's compute the stock growth rate under jump diffusion.

$$E\left[\frac{S}{S_0}\right] = \frac{(1-\lambda\Delta t)}{2}e^{\mu\Delta t + \sigma\sqrt{\Delta t}} + \frac{(1-\lambda\Delta t)}{2}e^{\mu\Delta t - \sigma\sqrt{\Delta t}} + \lambda\Delta t e^{\mu\Delta t + J}$$

$$= e^{\mu\Delta t}\left[\frac{(1-\lambda\Delta t)}{2}\left(e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}}\right) + \lambda\Delta t e^{J}\right]$$

One can show by expanding this to keep terms of order Δt that

$$E\left[\frac{S}{S_0}\right] = \exp\left(\left\{\mu + \frac{\sigma^2}{2} + \lambda(e^J - 1)\right\}\Delta t\right) + \text{higher order terms}$$

so that, if we want the stock to grow risk-neutrally, we must set $r = \mu + \frac{\sigma^2}{2} + \lambda(e^J - 1)$

$$\mu_{JD} = r - \frac{\sigma^2}{2} - \lambda(e^J - 1)$$

the log drift of the diffusion process

diffusion compensation

jump compensation

The option value is $C_{\mathrm{JD}} = e^{-r\tau}\sum_{n=0}^{\infty}\frac{(\lambda\tau)^n}{n!}e^{-\lambda\tau}E\left[\max\left(S_T^n - K, 0\right)\right]$ and we'll show that

$$C_{JD} = e^{-\bar{\lambda}\tau}\sum_{n=0}^{\infty}\frac{(\bar{\lambda}\tau)^n}{n!}C_{BS}\left(S, K, \tau, \sqrt{\sigma^2 + \frac{n\sigma_J^2}{\tau}}, r + \frac{n\left(\bar{J} + \frac{1}{2}\sigma_J^2\right)}{\tau} - \lambda\left(e^{\bar{J} + \frac{1}{2}\sigma_J^2} - 1\right)\right)$$

where $\bar{\lambda} = \lambda e^{\bar{J} + \frac{1}{2}\sigma_J^2}$

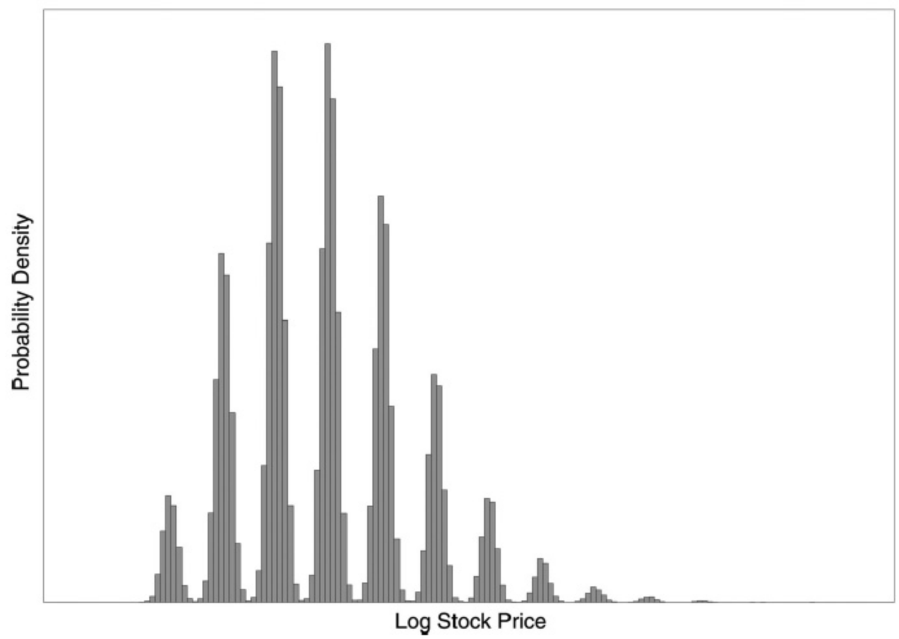


FIGURE 24.4 Multimodal Probability Density Function