Economics 361
Two Stage Least Squares (2SLS)

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Supply & Demand Model

Consider the Supply & Demand model discussed in class (in observation format)

$$
\begin{aligned}
\text{Supply}: &\quad P_t = \alpha_0 + \alpha_1 Q_t^s + \delta C_t + \eta_t \\
\text{Demand}: &\quad Q_t^d = \beta_0 + \beta_1 P_t + \gamma I_t + \nu_t \\
\text{Market Equilibrium}: &\quad Q_t^s = Q_t^d \\
\text{Assume} &\quad \eta_t \overset{i.i.d.}{\sim} N(0,\sigma_\eta^2) \quad \nu_t \overset{i.i.d.}{\sim} N(0,\sigma_\nu^2)
\end{aligned}
$$

t indexes the observations. P_t is the price of gasoline, (Q_t^s, Q_t^d) quantity supplied and quantity demanded, C_t the cost of providing gasoline, and I_t consumer income. In addition to the assumptions above, assume that I_t and C_t are **exogenous** variables. In other words, (I_t, C_t) are independent of (η_t, ν_t). Moreover, assume that η and ν are completely independent of each other.

Consistent estimates of the structural parameters and their corresponding asymptotically valid standard errors can be calculated using **Two Stage Least Squares** (2SLS). We know from the earlier analysis that the inclusion of the endogenous variables Q_t in the Supply Equation and P_t in the Demand Equation lead to violation of the standard OLS "unbiasedness" assumption in general (due to $Q_t^s = Q_t^d$):

$$
\begin{aligned}
E(P_t \mid Q_t^s, C_t) &\neq \alpha_0 + \alpha_1 Q_t^s + \delta C_t \\
E(Q_t^d \mid Pt, I_t) &\neq \beta_0 + \beta_1 P_t + \gamma I_t
\end{aligned}
$$

But consider instead the expectation of P_t and Q_t^d conditional only on the exogenous variables: $(1, I_t, C_t)$

$$
\begin{aligned}
E(P_t \mid I_t, C_t) &= E(\ \alpha_0 + \alpha_1 Q_t^s + \delta C_t + \eta_t \mid 1, I_t, C_t) \\
E(Q_t^d \mid I_t, C_t) &= E(\ \beta_0 + \beta_1 P_t + \gamma I_t + \nu_t \mid 1, I_t, C_t)
\end{aligned}
$$

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Note two things: First, the conditional expectation of a random variable conditional on itself is simply its conditioned value. So $E(C_t \mid 1, I_t, C_t) = C_t$. Second, as (I_t, C_t) are exogenous variables (see model description above) and a constant is always exogenous ...

$$
\begin{aligned}
E(\eta_t \mid 1, C_t, I_t) = E(\eta_t) = 0 \\
E(\nu_t \mid 1, C_t, I_t) = E(\nu_t) = 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
E(P_t \mid 1, I_t, C_t) &= \alpha_0 + \alpha_1 \underbrace{E(Q_t^s \mid 1, I_t, C_t)}_{\tilde{Q}_t} + \delta C_t \\
E(Q_t^d \mid 1, I_t, C_t) &= \beta_0 + \beta_1 \underbrace{E(P_t \mid 1, I_t, C_t)}_{\tilde{P}_t} + \gamma I_t
\end{aligned}
$$

Hence, if we observed $\tilde{Q}$ and $\tilde{P}$ we could regress P on $(1, \tilde{Q}, C)$ and Q^d on $(1, \tilde{P}, I)$ to obtain unbiased estimates of $(\alpha_0, \alpha_1, \delta)$ and $(\beta_0, \beta_1, \gamma)$, respectively. However, we do not observe $\tilde{Q}$ nor $\tilde{P}$ in general. Consequently, consider using estimates of $\tilde{Q}$ and $\tilde{P}$ instead.

More precisely, we will be using the OLS prediction of Q and the OLS prediction of P stemming from regressing Q on $(1, I, C)$ and P on $(1, I, C)$, respectively.

$$
\begin{aligned}
\text{First Stage Regression for Estimating Supply :} \quad & Q = \Pi_{11} + \Pi_{12} I + \Pi_{13} C + \epsilon_1 \\
\text{First Stage Regression for Estimating Demand :} \quad & P = \Pi_{21} + \Pi_{22} I + \Pi_{23} C + \epsilon_2
\end{aligned}
$$

But note that this is basically the same regression as the **reduced form** regression. So we approximate $\tilde{Q}$ and $\tilde{P}$ with $\hat{Q}$ and $\hat{P}$, respectively:

$$
\begin{aligned}
\hat{Q} &= p_{11} + p_{12} I + p_{13} C \\
\hat{P} &= p_{21} + p_{22} I + p_{23} C
\end{aligned}
$$

where p_{ij} is the OLS estimate of Π_{ij} from the first stage regression (reduced form regression).

Using $(\hat{Q}, \hat{P})$, we apply OLS on the second stage regression suggested by $E(P_t \mid 1, I_t, C_t)$ and $E(Q_t \mid 1, I_t, C_t)$ above.

$$
\begin{aligned}
\text{Second Stage Regression for Estimating Supply :} \quad & P = \alpha_0 + \alpha_1 \hat{Q} + \delta C + \tilde{\eta} \\
\text{Second Stage Regression for Estimating Demand :} \quad & Q = \beta_0 + \beta_1 \hat{P} + \gamma I + \tilde{\nu}
\end{aligned}
$$

Note that $(\tilde{\eta}, \tilde{\nu})$ are, in general, **not** the same random variables as (η, ν). This is because we are using $\hat{Q}$ and $\hat{P}$ instead of $\tilde{Q}$ and $\tilde{P}$.

The estimated OLS coefficients from the second stage regressions yields us the 2SLS estimates of the structural parameters

$$
\begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\delta} \end{pmatrix} = (\hat{Z}_1' \hat{Z}_1)^{-1} \hat{Z}_1 P \qquad \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\gamma} \end{pmatrix} = (\hat{Z}_2' \hat{Z}_2)^{-1} \hat{Z}_2 Q
$$

$$
\hat{Z}_1 = \begin{pmatrix} 1 & \hat{Q} & C \end{pmatrix} \qquad \hat{Z}_2 = \begin{pmatrix} 1 & \hat{P} & I \end{pmatrix}
$$

For notational simplicity, let us define a few more matrices

$$B_1 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \delta \end{pmatrix} \qquad B_2 = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \gamma \end{pmatrix}$$

$$Z_1 = \begin{pmatrix} 1 & Q & C \end{pmatrix} \qquad Z_2 = \begin{pmatrix} 1 & P & I \end{pmatrix} \qquad X = \begin{pmatrix} 1 & I & C \end{pmatrix}$$

Moreover, note the following about the relationship between $(\hat{Z}_1, \hat{Z}_2)$ and (Z_1, Z_2)

$$\begin{aligned} \hat{Z}_1 &= \underbrace{X(X'X)^{-1}X'}_{N_x} Z_1 &= N_x Z_1 = N_x' N_x Z_1 \\ \hat{Z}_2 &= \underbrace{X(X'X)^{-1}X'}_{N_x} Z_2 &= N_x Z_2 = N_x' N_x Z_2 \end{aligned}$$

Note: $N_x X = X$ (we use this in the first step of the equality equations given above).

Therefore, we can re-express the 2SLS estimates as

$$\begin{aligned} \begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\delta} \end{pmatrix} &= (\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1' P \\ &= (\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1'(Z_1 B_1 + \eta) \\ &= (\hat{Z}_1'\hat{Z}_1)^{-1}(N_x' N_x Z_1)' Z_1 B_1 + (\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1'\eta \\ &= (\hat{Z}_1'\hat{Z}_1)^{-1}(Z_1' N_x' N_x Z_1) B_1 + (\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1'\eta \\ &= (\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1'\hat{Z}_1 B_1 + (\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1'\eta \\ &= B_1 + (\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1'\eta \\ \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\gamma} \end{pmatrix} &= (\hat{Z}_2'\hat{Z}_2)^{-1}\hat{Z}_2' Q \\ &= (\hat{Z}_2'\hat{Z}_2)^{-1}\hat{Z}_2'(Z_2 B_2 + \nu) \\ &= B_2 + (\hat{Z}_2'\hat{Z}_2)^{-1}\hat{Z}_2'\nu \end{aligned}$$

So the statistical properties of the 2SLS estimators depends on the statistical properties of $(\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1'\eta$ and $(\hat{Z}_2'\hat{Z}_2)^{-1}\hat{Z}_2'\nu$. Recall that $(\hat{Z}_1, \hat{Z}_2)$ are functions of both exogenous variables X and endogenous variables (P, Q). This makes it difficult to take expectations as conditioning on $\hat{Z}_1$ implies conditioning on η and, similarly, conditioning on $\hat{Z}_2$ implies conditioning on ν: remember from the reduced form equations that (P, Q) are implicitly functions of (η, ν) as well as X.

$$\begin{aligned} E[\ (\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1'\eta \mid \hat{Z}_1\ ] &= (\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1' E[\ \eta \mid \hat{Z}_1] = ? \\ \mathrm{Var}[\ (\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1'\eta \mid \hat{Z}_1\ ] &= (\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1' \underbrace{\mathrm{Var}(\eta \mid \hat{Z}_1)}_{=?} \hat{Z}_1 (\hat{Z}_1'\hat{Z}_1)^{-1} \\ E[\ (\hat{Z}_2'\hat{Z}_2)^{-1}\hat{Z}_2'\nu \mid \hat{Z}_2\ ] &= (\hat{Z}_2'\hat{Z}_2)^{-1}\hat{Z}_2' E[\ \nu \mid \hat{Z}_2] = ? \\ \mathrm{Var}[\ (\hat{Z}_2'\hat{Z}_2)^{-1}\hat{Z}_2'\eta \mid \hat{Z}_2\ ] &= (\hat{Z}_2'\hat{Z}_2)^{-1}\hat{Z}_2' \underbrace{\mathrm{Var}(\nu \mid \hat{Z}_2)}_{=?} \hat{Z}_2 (\hat{Z}_2'\hat{Z}_2)^{-1} \end{aligned}$$

So consider instead the asymptotic properties of the 2SLS estimator. As will be demonstrated, the following exercise works whether we assume (η, ν) is distributed normally. As long as we are willing to assume that (η, ν) are drawn *i.i.d.* from some distribution with mean zero and finite variance, the results of the exercise holds. This is because we will be relying upon the Central Limit Theorem. For now, just consider the 2SLS estimator for the supply equation.

$$\begin{aligned}
\begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\delta} \end{pmatrix} &= B_1 + (\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1'\eta \\
&= B_1 + \underbrace{\left(\frac{\hat{Z}_1'\hat{Z}_1}{N}\right)^{-1}}_{(1)} \underbrace{\left(\frac{\hat{Z}_1'\eta}{N}\right)}_{(2)} \\
N &\equiv \text{number of observations in sample}
\end{aligned}$$

Given that (η, ν) have finite variance and assuming further that so do the exogenous variables X, we can show that the term (1) converges in probability to some finite constant term. Let us call this term $(Q_1)^{-1}$. Similarly, under the same assumptions and the fact that (η, ν) are drawn *i.i.d.*, we can use the Central Limit Theorem[1] to show that the term (2) has an asymptotic distribution of Normal $(0, \sigma_\eta^2 Q_1)$. Therefore, using the theorem provided after Slutsky (S1-S4) on Goldberger p.102, we have that

$$\underbrace{\left(\frac{\hat{Z}_1'\hat{Z}_1}{N}\right)^{-1}}_{(1)} \xrightarrow{p} (Q_1)^{-1} \qquad \sqrt{N}\underbrace{\left(\frac{\hat{Z}_1'\eta}{N}\right)}_{(2)} \overset{d}{\sim} N(0, \sigma_\eta^2 Q_1)$$

$$\begin{aligned}
\sqrt{N}(\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1'\eta &\overset{d}{\sim} N(0, \underbrace{(Q_1)^{-1}\sigma_\eta^2 Q_1 (Q_1)^{-1}}_{\sigma_\eta^2 (Q_1)^{-1}}) \\
\sqrt{N}\left[\begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\delta} \end{pmatrix} - B_1\right] &\overset{d}{\sim} N(0, \sigma_\eta^2 (Q_1)^{-1})
\end{aligned}$$

Applying similar arguments to the 2SLS estimator for the demand equation and using $\left(\frac{\hat{Z}_1'\hat{Z}_1}{N}\right)^{-1}$ to approximate $(Q_1)^{-1}$ and similarly $\left(\frac{\hat{Z}_2'\hat{Z}_2}{N}\right)^{-1}$ for $(Q_2)^{-1}$, we argue that the asymptotic distribution of the 2SLS estimators are

$$\begin{aligned}
\begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\delta} \end{pmatrix} &\overset{a}{\sim} N(\ B_1\ ,\ \sigma_\eta^2 (\hat{Z}_1'\hat{Z}_1)^{-1}\ ) \\
\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\gamma} \end{pmatrix} &\overset{a}{\sim} N(\ B_2\ ,\ \sigma_\nu^2 (\hat{Z}_2'\hat{Z}_2)^{-1}\ )
\end{aligned}$$

[1] Either Lindeberg-Levy or Liapounov, depending on the exact distributional assumption of (η, ν)

Note that the asymptotic distribution is a function of σ_η^2 and σ_ν^2 which may or may not be known. In the case that they are unknown, we can approximate each with s_η^2 and s_ν^2 where (s_η^2, s_ν^2) are defined in a similar manner as in the OLS case:

$$
\begin{aligned}
s_\eta^2 &= \frac{1}{N-3}\underbrace{(P - Z_1\theta_1)'(P - Z_1\theta_1)}_{n'n} \\
s_\nu^2 &= \frac{1}{N-3}\underbrace{(Q - Z_2\theta_2)'(Q - Z_2\theta_2)}_{v'v}
\end{aligned}
$$

where

$$
\theta_1 = \begin{pmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\delta} \end{pmatrix} \qquad \theta_2 = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\gamma} \end{pmatrix}
$$

Replacing $(\sigma_\eta^2, \sigma_\nu^2)$ with (s_η^2, s_ν^2) provides us with an asymptotically valid standard error and the test statistic constructed using (s_η^2, s_ν^2) instead of $(\sigma_\eta^2, \sigma_\nu^2)$ will still be distributed asymptotically normal. This is because $\theta_1 \xrightarrow{p} B_1$ and $\theta_2 \xrightarrow{p} B_2$ implying that $n \xrightarrow{p} \eta$ and $v \xrightarrow{p} \nu$

Note: As we are working with asymptotic properties, we could have used N instead of $N-k$ as the denominators in (s_η^2, s_ν^2). While using N in the denominator can lead to biased estimates in general, such estimates are still **consistent**, which is all we need. Think about it: as N grows to infinity, the difference between N and $N-k$ becomes infinitesimal.

Key Point: You can show that $\hat{Z}_1'\hat{Z}_1 = \hat{Z}_1'Z_1$ and $\hat{Z}_2'\hat{Z}_2 = \hat{Z}_2'Z_2$. Therefore the variance (or asymptotic variance) of the 2SLS estimators can be re-expressed as $\sigma_\eta^2(\hat{Z}_1'Z_1)^{-1}$ and $\sigma_\nu^2(\hat{Z}_2'Z_2)^{-1}$. This implies that the 2SLS estimator for the structural supply equation parameters is more precise (lower variance) when $\hat{Z}_1$ and Z_1 are more correlated with each other (larger $\hat{Z}_1'Z_1$) and similarly, for the structural demand equation parameters, when $\hat{Z}_2$ and Z_2 are more correlated with each other (larger $\hat{Z}_2'Z_2$). But note

$$
\begin{aligned}
\hat{Z}_1'Z_1 &= Z_1'X(X'X)^{-1}X'Z_1 \\
\hat{Z}_2'Z_2 &= Z_2'X(X'X)^{-1}X'Z_2
\end{aligned}
$$

Therefore, we can go further and say that the estimators have lower variance when X and Z_1 and X and Z_2 are more correlated with each other (larger $X'Z_1$ and $X'Z_2$ values). This just means that the exogenous variables in X that are not included in Z must be highly correlated with the endogenous variables in Z. The idea is that you want to replace the **endogenous** variables in Z with **exogenous** variables that are **highly correlated** with the endogenous variables they are "replacing" !!!

In General ...

Above, we derive the 2SLS estimator within the Supply & Demand framework. Consider the general formulation of the 2SLS estimation problem

$$\begin{aligned} Y &= Z\beta + \epsilon \\ E(\epsilon) &= 0 \\ \mathrm{Var}(\epsilon) &= \sigma^2 I \\ E(\epsilon \mid Z) &\neq 0 \quad \text{because } Z \text{ includes endogenous variables} \end{aligned}$$

Let X represent the matrix of observed variables that are **exogenous** to ϵ. In addition to the variables in Z that are **exogenous**, X must also include some **exogenous** variables not included in ("excluded from") Z. Otherwise, β may not be fully identified. Given this X, we can show that

$$\begin{aligned} E(Y|X) &= E(Z|X)\beta + E(\epsilon|X) \\ &= E(Z|X)\beta \end{aligned}$$

The first stage regression consists of regressing Z on X to get $\hat{Z}$, the OLS prediction of Z

$$\hat{Z} = X(X'X)^{-1}X'Z$$

The second stage consists of regressing Y on $\hat{Z}$, yielding us the following 2SLS estimator

$$\hat{\beta}_{2SLS} = (\hat{Z}'\hat{Z})^{-1}\hat{Z}'Y$$

Note that $\hat{Z}'\hat{Z} = \hat{Z}'Z$.

The statistical properties of the 2SLS estimator are

$$\begin{aligned} \hat{\beta}_{2SLS} &= (\hat{Z}'\hat{Z})^{-1}\hat{Z}'Y \\ &= (Z'X(X'X)^{-1}X'X(X'X)^{-1}X'Z)^{-1}\hat{Z}'Y \\ &= (Z'X(X'X)^{-1}X'Z)^{-1}\hat{Z}'Y \\ &= (\hat{Z}'Z)^{-1}\hat{Z}'Y \\ &= (\hat{Z}'Z)^{-1}\hat{Z}'(Z\beta + \epsilon) \\ &= \beta + (\hat{Z}'Z)^{-1}\hat{Z}'\epsilon \\ \hat{\beta}_{2SLS} &\xrightarrow{p} \beta \\ \sigma^2(\hat{Z}'\hat{Z})^{-1} &\xrightarrow{p} \mathrm{Var}(\hat{\beta}_{2SLS}) \end{aligned}$$