DrawDown Measure in Portfolio Optimization

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Motivation

■ **Practical Portfolio Management:** Robust expected drawdown (maximal, average) estimation while building trading strategies (with proliferation of quantitative hedge funds). A very "natural" risk measure from an investor's standpoint: most of the current/past allocations of proprietary capital come with strict drawdown conditions (example: at 15% drawdown, 10% warning with risk reduction). Robust weight allocation between different strategies, markets, hedge funds, etc. – Markowitz mean-variance portfolio optimization does not work very well in practice.

■ **Academic Interest in Portfolio Optimization with Generalized Risk Measures.** The mathematics for this risk measure was not fully developed: Maximal DrawDown, Average DrawDown and Conditional DrawDown. Some mathematical properties of the drawdown measures. Transition from a historical (single-path) to stochastic (multi-path) formulation.

■ **Computational Efficiency:** Reduction of portfolio optimization problem to a linear optimization problem (Linear Programming Problem), which has a unique solution, and for which very efficient solvers exist.

Thinking of Risk in Terms of Worst DrawDowns: Analogy with Fluid Turbulence

- Similarities between securities price time-differences and velocity space-differences in turbulent fluid flow produced many interesting analogies, one example being *intermittency*. (D. Sornette, J.-P. Bouchaud)

- In application to finance it is hypothesized that the assumption of (daily) returns independency is indeed a good assumption which leads to an *exponential* (Poisson-like) *distribution* function of drawdowns with respect to their sizes. It was observed that such distribution works well across a wide range of drawdown sizes, *but a few largest ones*.

- If there is a range where this distribution law does not hold, this would imply presence of conditional serial correlations in returns. Empirically, deviations from exponential distribution was indeed found for the worst 10-20% of drawdowns (outliers). The presence of these *outliers* was registered for daily returns in virtually all major markets (D. Sornette).

- Illustrative examples here: S&P 500, Long/Short U.S. Equity, Diversified Global Futures daily returns.

Two Practical Examples Considered

- *A typical CTA/Global Macro example*. Daily returns for a portfolio of well-diversified long-term trend following futures trading systems. Most major asset classes tradable through futures and spot markets present (32 markets total): major currencies (spot and crosses), fixed-income (long- and short-term, U.S. and international), metals (non-precious and precious), equity indices. The system is trying to capture long-term trends of each of markets considered, and quality of the portfolio may be dramatically improved by high degree of diversification (some CTAs trade up to 70 markets simultaneously). Each market position is either long, short or flat.

- *A typical U.S. Long/Short Equity hedge fund example*. Daily returns for a sample portfolio of a long/short liquid U.S. equity trading system. Most major sectors of U.S. equity market covered: Financials, Health Care, Oil & Gas, Utilities, and Autos Transportation. A total of 250 equity tickers, with about 50 tickers in each sector. Each sector is traded in nearly market-neutral fashion, and sector-dependency is reduced by mixing the returns of different sectors together in one portfolio (some long/short managers trade up to 1,000 names simultaneously). The portfolio is typically re-balanced once a week with a turnover rate of about 40 times a year.
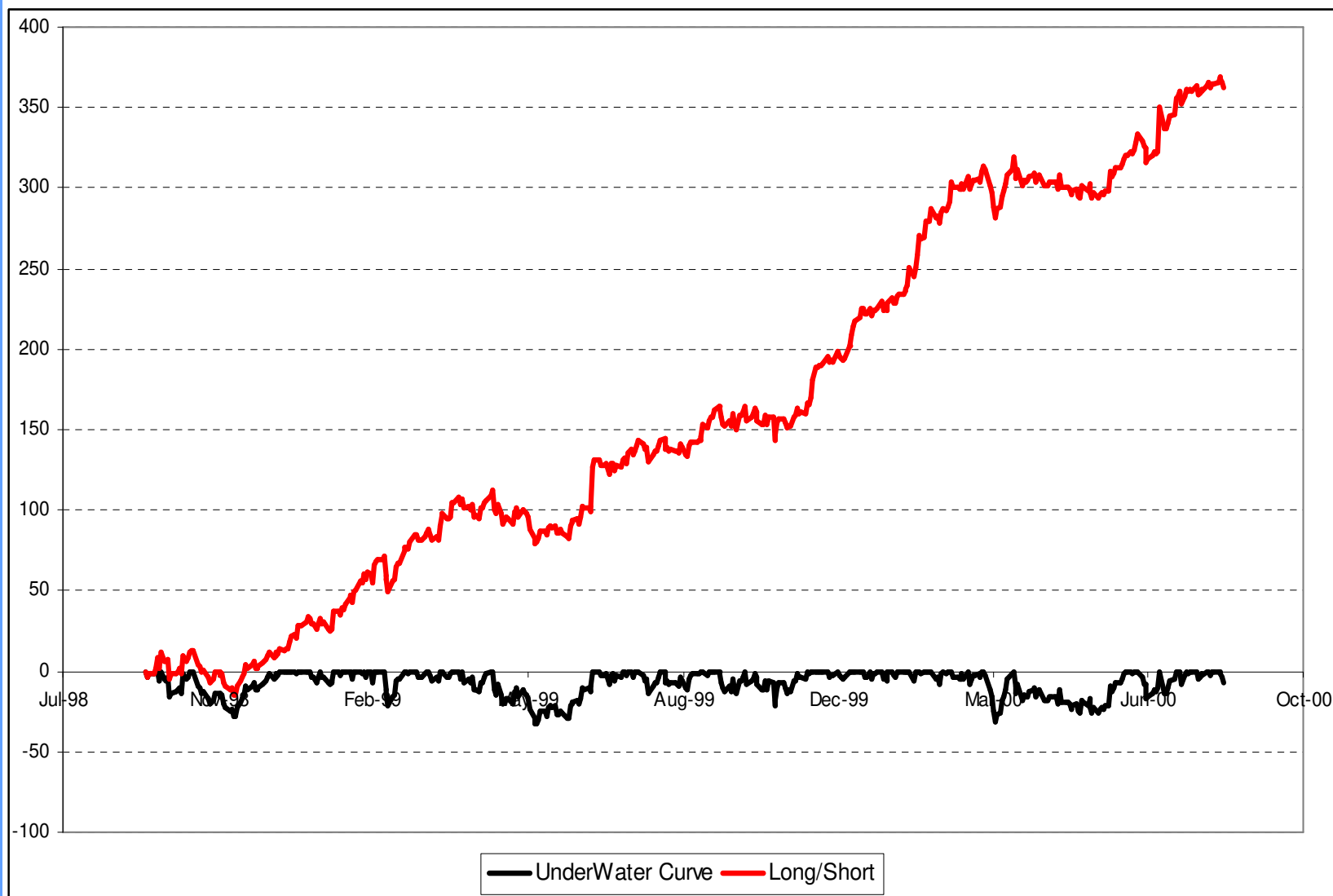
Historical Portfolio Equity and UnderWater Curve for Diversified Global Futures

Portfolio Equity & Underwater Curve

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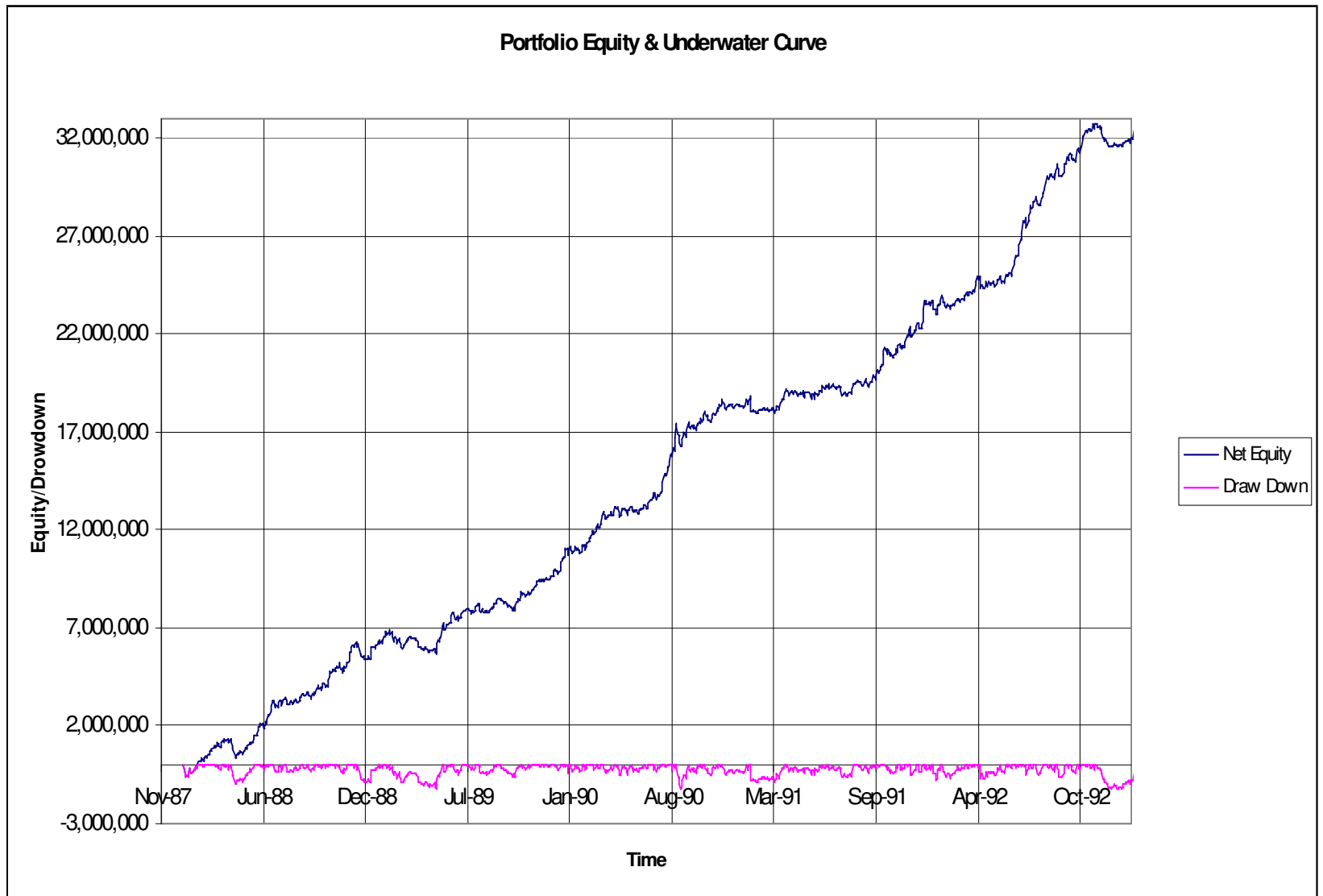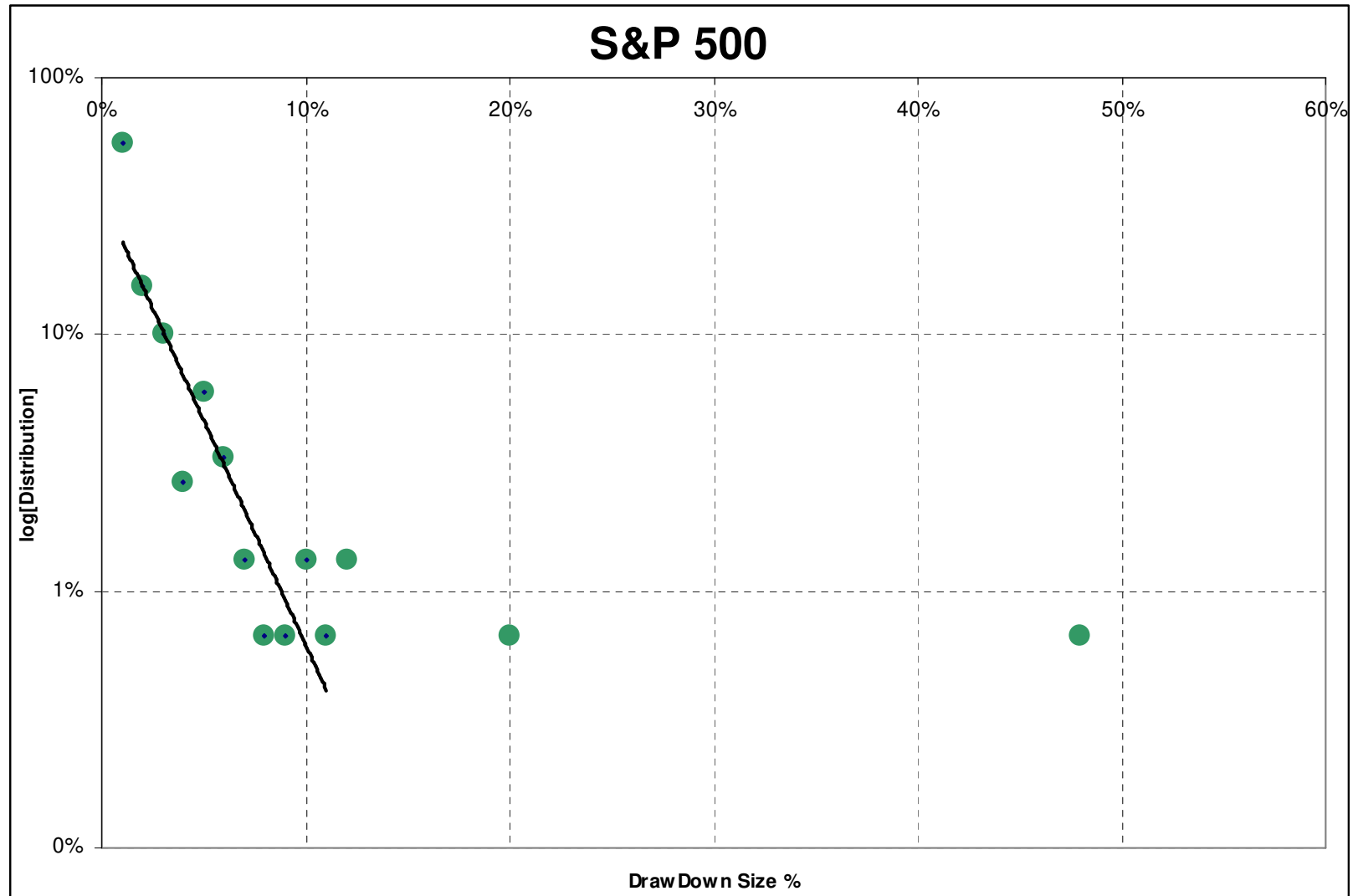
Historical Portfolio Equity and Underwater Curve for Long/Short U.S. Equity

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S&P 500 Total Return Daily DrawDown Analysis (1992-present)

DD Start	DD Peak	DD Size Pct
9/4/00	10/9/02	47.52%
7/17/98	8/31/98	19.19%
7/16/99	10/15/99	11.81%
3/24/00	4/14/00	11.13%
10/7/97	10/27/97	10.79%
2/18/97	4/11/97	9.41%
12/31/99	2/25/00	9.11%
2/2/94	4/4/94	8.49%
6/5/96	7/24/96	7.43%
5/13/99	5/27/99	6.24%
8/6/97	8/29/97	6.22%
8/30/94	12/8/94	5.74%
9/14/92	10/9/92	5.11%
4/12/99	4/19/99	5.09%
1/5/98	1/9/98	5.04%
2/12/92	4/8/92	5.01%
1/29/99	2/9/99	4.93%
1/8/99	1/14/99	4.92%
3/10/93	4/26/93	4.81%
11/29/96	12/16/96	4.67%

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S&P 500 Total Return Daily DrawDown Analysis (1992-present)

S&P 500

Diversified Futures Daily DrawDown Analysis (1988-present)

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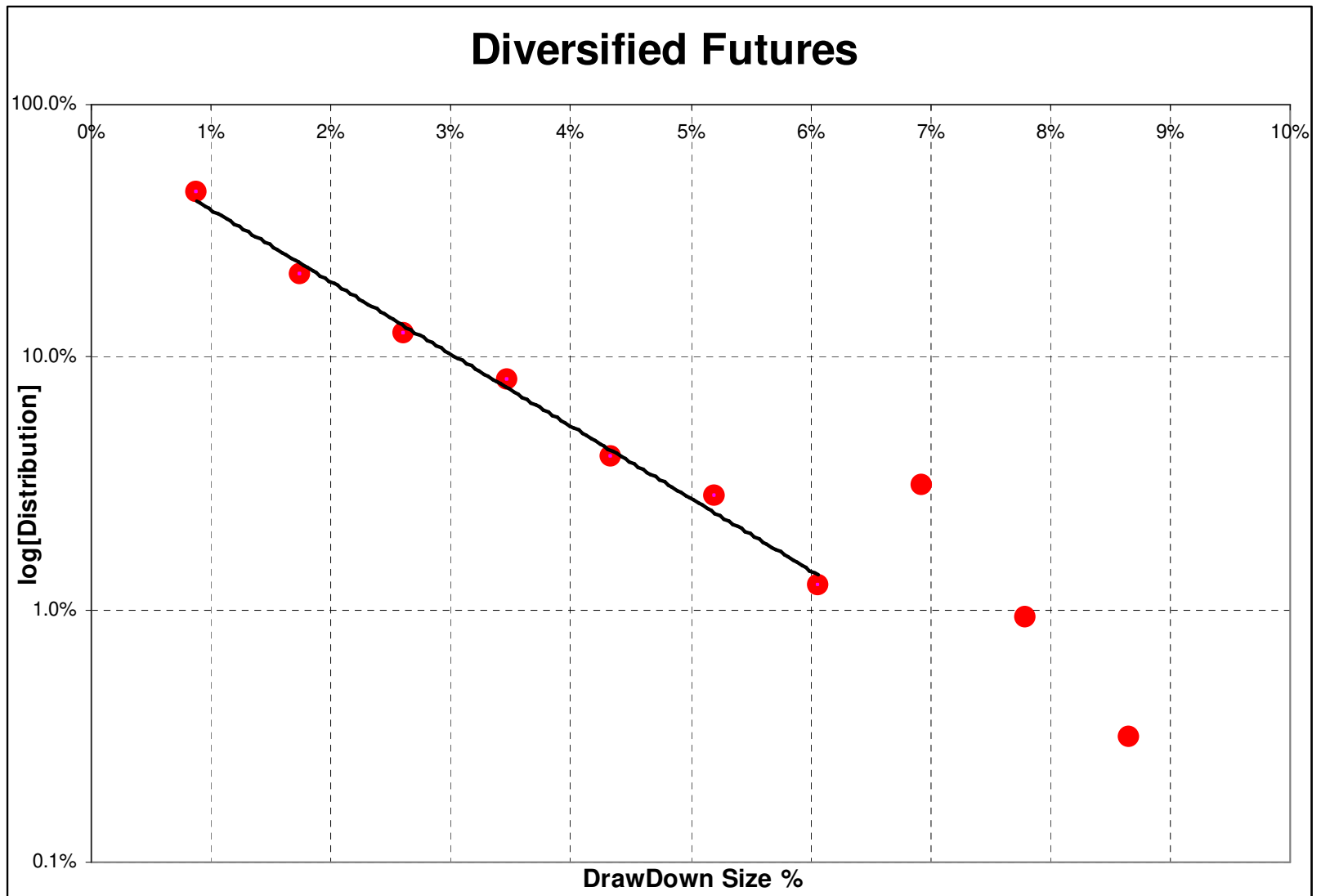
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Long/Short Equity Daily DrawDown Analysis (1998-present)
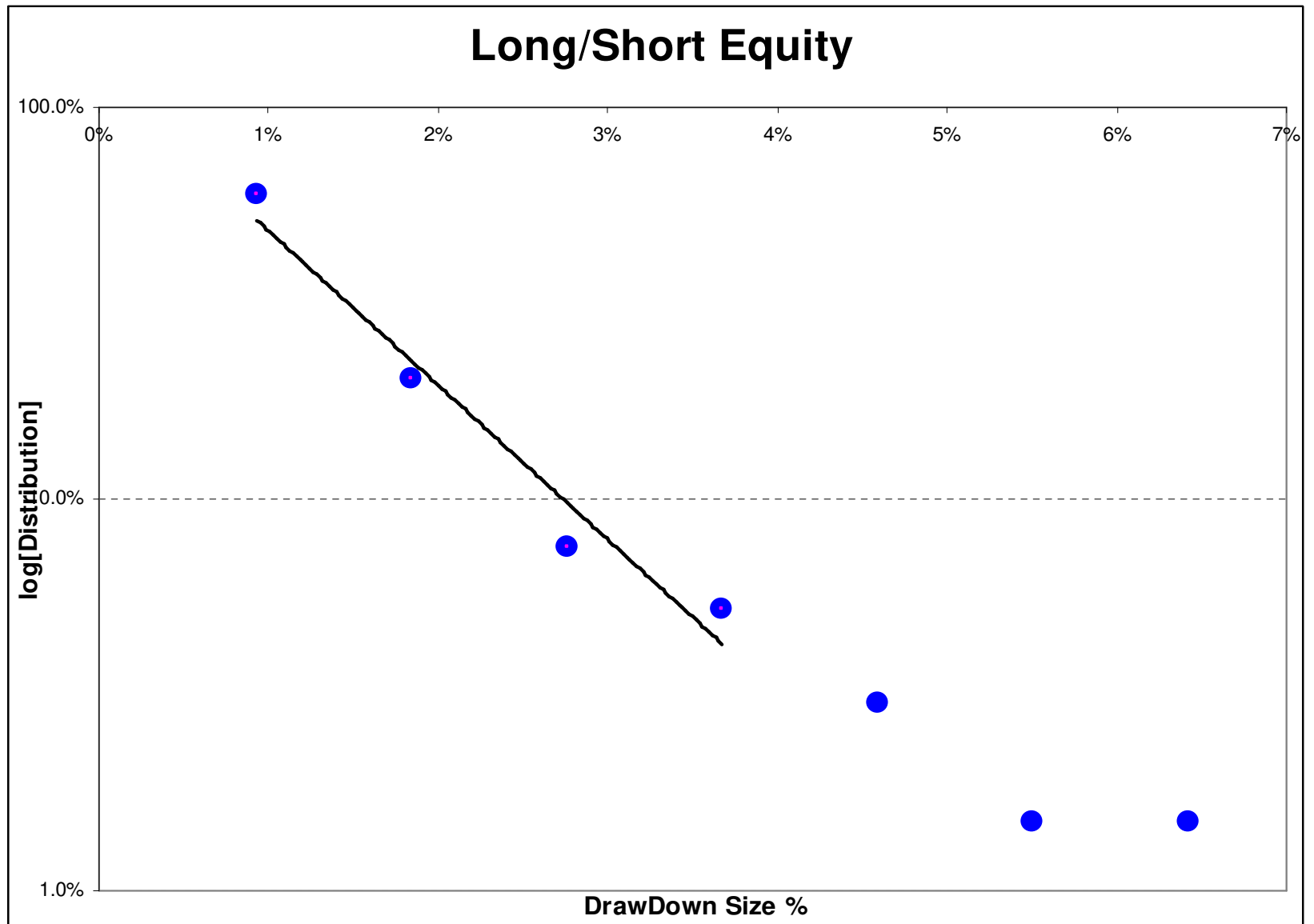
Long/Short Equity

Portfolio Optimization with Drawdown Constraints
(Chekhlov, Uryasev, Zabarankin 2000)

Traditional Portfolio Optimization:

$$\max_{\vec{x} \in X} E(w)$$

$$Risk(w) \leq Const$$

$$w = \sum_{i=1}^{m} r_i x_i$$

with drawdown risk measures:

$$M(\vec{x}) = \max_{t \in [0,T]} \{ D(\vec{x}, t) \},$$

$$A(\vec{x}) = \frac{1}{T} \int_{t=0}^{T} D(\vec{x}, t) dt,$$

$$\Delta_{\alpha}(\vec{x}) = \frac{1}{(1-\alpha)T} \int_{t \in \Omega_{\alpha}} D(\vec{x}, t) dt,$$

$$\Omega_{\alpha} = \{ t \in [0,T], D(\vec{x}, t) \leq \zeta_{\alpha}(\vec{x}) \}.$$

was introduced and solved in historical (1 sample path) formulation.

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Multi-Path or Multi-Scenario Formulation

- *Portfolio optimization is based on generation of sample paths for the assets' rates of return*

- *Portfolio optimization uses uncompounded cumulative portfolio rate of return* **w** *rather than compounded portfolio rate of return.*

$$\Omega = \left\{ w_j, j = \overline{1,k} \right\}, \text{ with } p_j \text{ - probabilities of } w_j$$

$$\forall j : p_j > 0, \sum_{j=1}^{K} p_j = 1.$$

Stochastic risk measures:

$$M\left(w(\vec{x})\right) = \max_{1 \le k \le N} \max_{1 \le j \le K} \left\{ \zeta_{jk} \right\},$$

$$A\left(w(\vec{x})\right) = \frac{1}{N} \sum_{k=1}^{N} \sum_{j=1}^{K} p_j \cdot \zeta_{jk},$$

with a similar expression for $\Delta_\alpha \left(w(\vec{x}) \right)$.

Axioms of Measure

1. Non - negativity :

$$\Delta_\alpha(w) \geq 0, \forall \alpha \in [0,1]$$

2. Insensitivity to Const shift :

$$\Delta_\alpha(w + Const) = \Delta_\alpha(w),$$

3. Positive Homogeneity :

$$\Delta_\alpha(\lambda \cdot w) = \lambda \cdot \Delta_\alpha(w), \forall \lambda \in [0,1],$$

4. Convexity : $\forall \lambda \in [0,1]$:

$$\Delta_\alpha\left(\lambda \cdot w_1 + (1 - \lambda) \cdot w_2\right) \leq \lambda \cdot \Delta_\alpha(w_1) + (1 - \lambda) \cdot \Delta_\alpha(w_2)$$

Weights Allocation for a Diversified Global Futures Example

1. AAO - The Australian All Ordinaries Index (OTC)

2. AD - Australian Dollar Currency Futures (CME)

3. AXB - Australian 10-Year Bond Futures (SFE)

4. BD - U.S. Long (30-Year) Treasury Bond Futures (CBT)

5. BP - British Pound Sterling Currency Futures (CME)

6. CD - Canadian Dollar Currency Futures (CME)

7. CP - Copper Futures (COMEX)

8. DGB - German 10-Year Bond (Bund) Futures (LIFFE)

9. DX - U.S. Dollar Index Currency Futures (FNX)

10. ED - 90-Day Euro Dollar Futures (CME)

11. EU - Euro Currency Futures (CME)

12. FV - U.S. 5-Year Treasury Note Futures (CBT)

13. FXADJY - Australian Dollar vs. Japanese Yen Cross Currency Forward (OTC)

14. FXBPJY - British Pound Sterling vs. Japanese Yen Cross Currency Forward (OTC)

15. FXEUBP - Euro vs. British Pound Sterling Cross Currency Forward (OTC)

16. FXEUJY - Euro vs. Japanese Yen Cross Currency Forward (OTC)

17. FXEUSF - Euro vs. Swiss Franc Cross Currency Forward (OTC)

18. FXNZUS - New Zealand Dollar Currency Forward (OTC)

19. FXUSSG - Singaporean Dollar Currency Forward (OTC)

20. FXUSSK - Swedish Krona Currency Forward (OTC)

21. GC - Gold 100 Oz. Futures (COMEX)

22. JY - Japanese Yen Currency Futures (CME)

23. LBT - Italian 10-Year Bond Forward (OTC)

24. LFT - FTSE-100 Index Futures (LIFFE)

25. LGL -Long Gilt (U.K. 10-Year Bond) Futures (LIFFE)

26. LML - Aluminum Futures (COMEX)

27. MNN - French National Bond Futures ()

28. SF - Swiss Franc Currency Futures (CME)

29. SI - Silver Futures (COMEX)

30. SJB - JGB (Japanese 10-Year Government Bond) Futures (TSE)

31. SNI - NIKKEI-225 Index Futures (SIMEX)

32. TY - 10-Year U.S. Government Bond Futures (CBT)

A set of 32 time series with daily rates of return covers a period of time between 6/12/1995 and 12/13/1999. A set of additional ("technological") box constraints on portfolio weights $0.2 <= x <= 0.8$, $i = 1,32$.

Weights Allocation for a Long/Short U.S. Equity Example

A characteristic nearly market-neutral Long/Short U.S Equity trading system, employing 250 individual equities from Russell-1000's 5 sectors:

1. Financial Services

2. Utilities

3. Oil & Gas (Integrated Oil & Other Energy)

4. Auto & Transportation

5. Health Care

A set of 5 individual sector time series with daily rates of return covers a period of time between 9/16/1998 and 1/9/2004. A set of additional constraints on portfolio weights was Sum[x_i] =1.

Random Paths Generation by Block-Bootstrap-Resampling, Futures Case

1. We empirically <u>study the correlation properties</u> of all time series involved. For all data series, we have numerically calculated their auto-correlation coefficients $C(n)$ for the separation period of up to 200 days.

2. Reducing separation period n from 200 to 0, for all the time series, we found the 1-st value of n that violates condition $C(n) < 2.5\%$. In this case, we determined that the largest <u>statistically significant correlation length</u> for all considered time series, <u>is 100 trading days</u>.

3. Instead of randomly picking an individual daily return from the original data series, <u>we pick un-interchanged blocks of daily returns of length 100 trading days</u>, <u>starting from a random starting point</u>. To ensure consistency across all time series, and preserve the cross-market correlation structure, we choose the same starting point for all 32 time series.

Numerical Stochastic Convergence: Average DrawDown, Futures Case
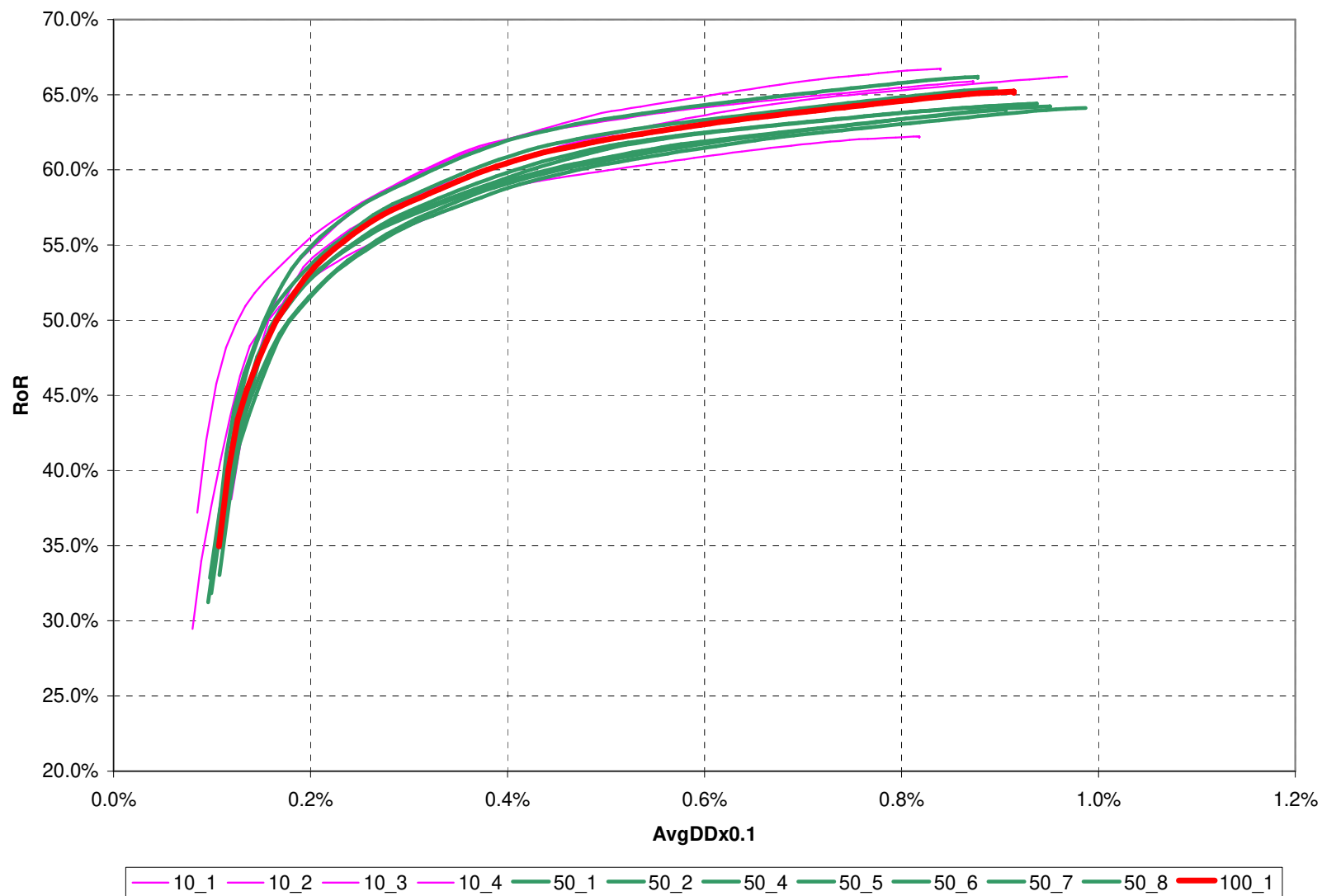
Numerical Stochastic Convergence: Average DrawDown, L/S Equity Case
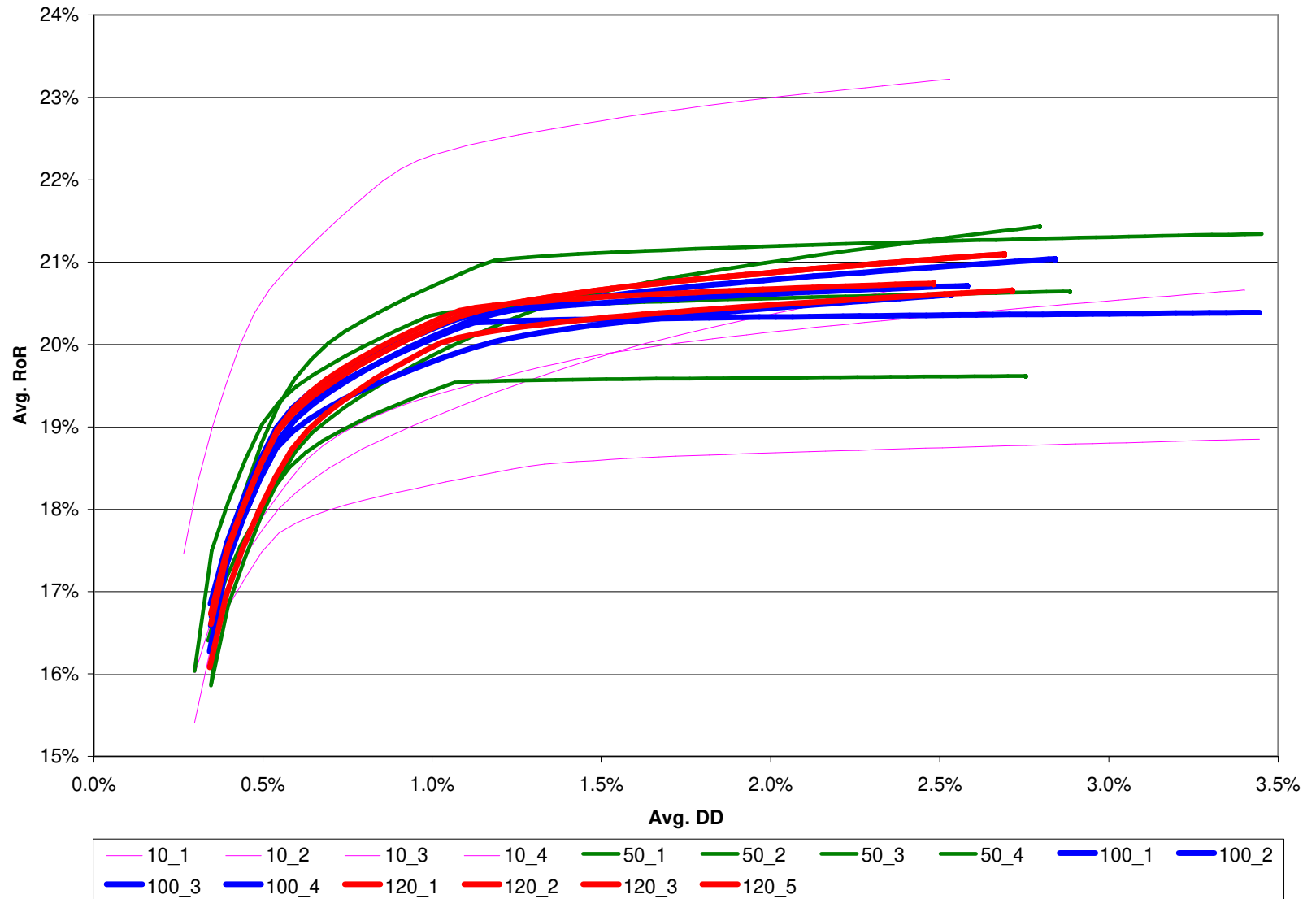
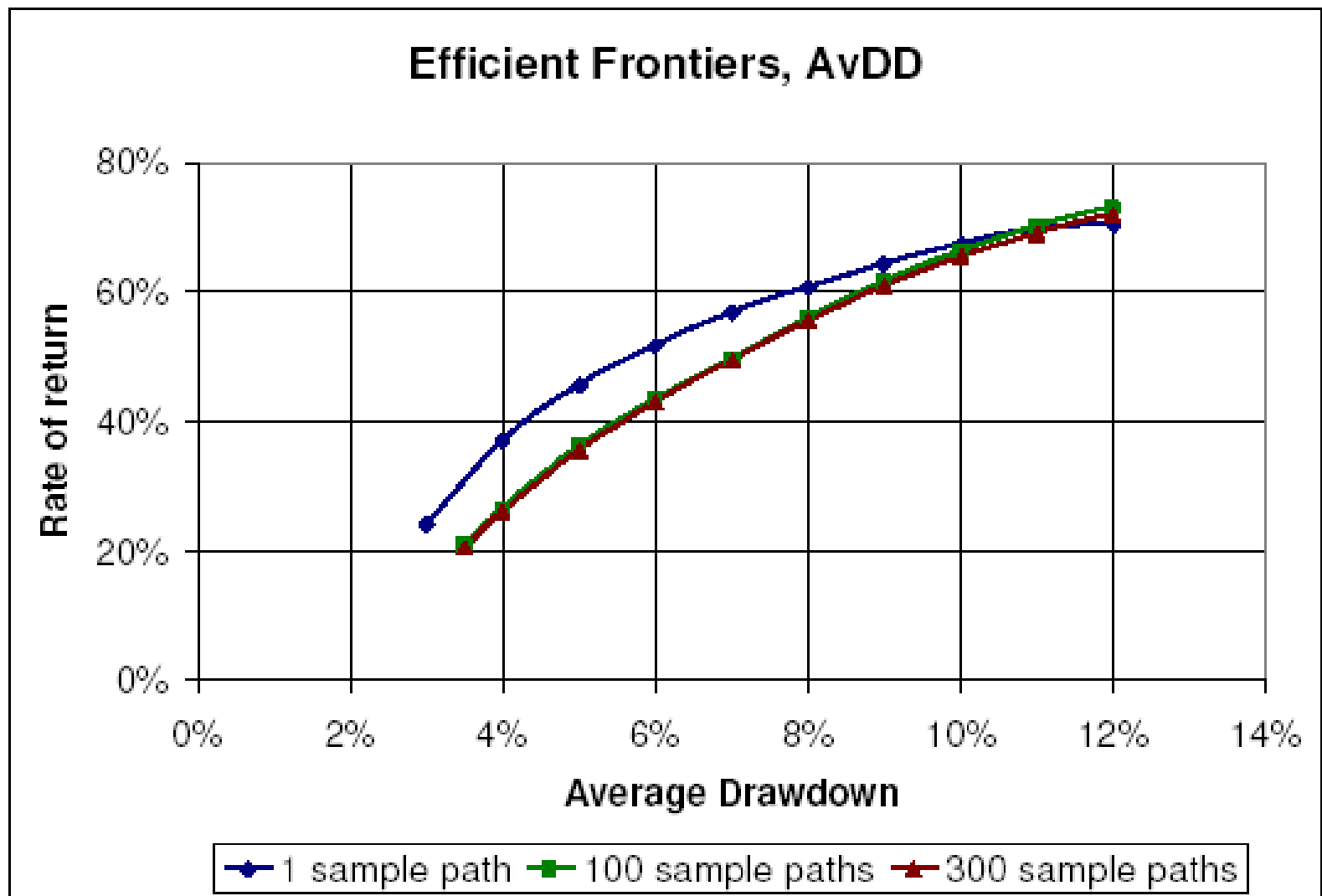# Stochastic vs. Historical Efficient Frontiers: Average DrawDown, Futures Case

Figure 6: Efficient frontiers: Average Drawdown

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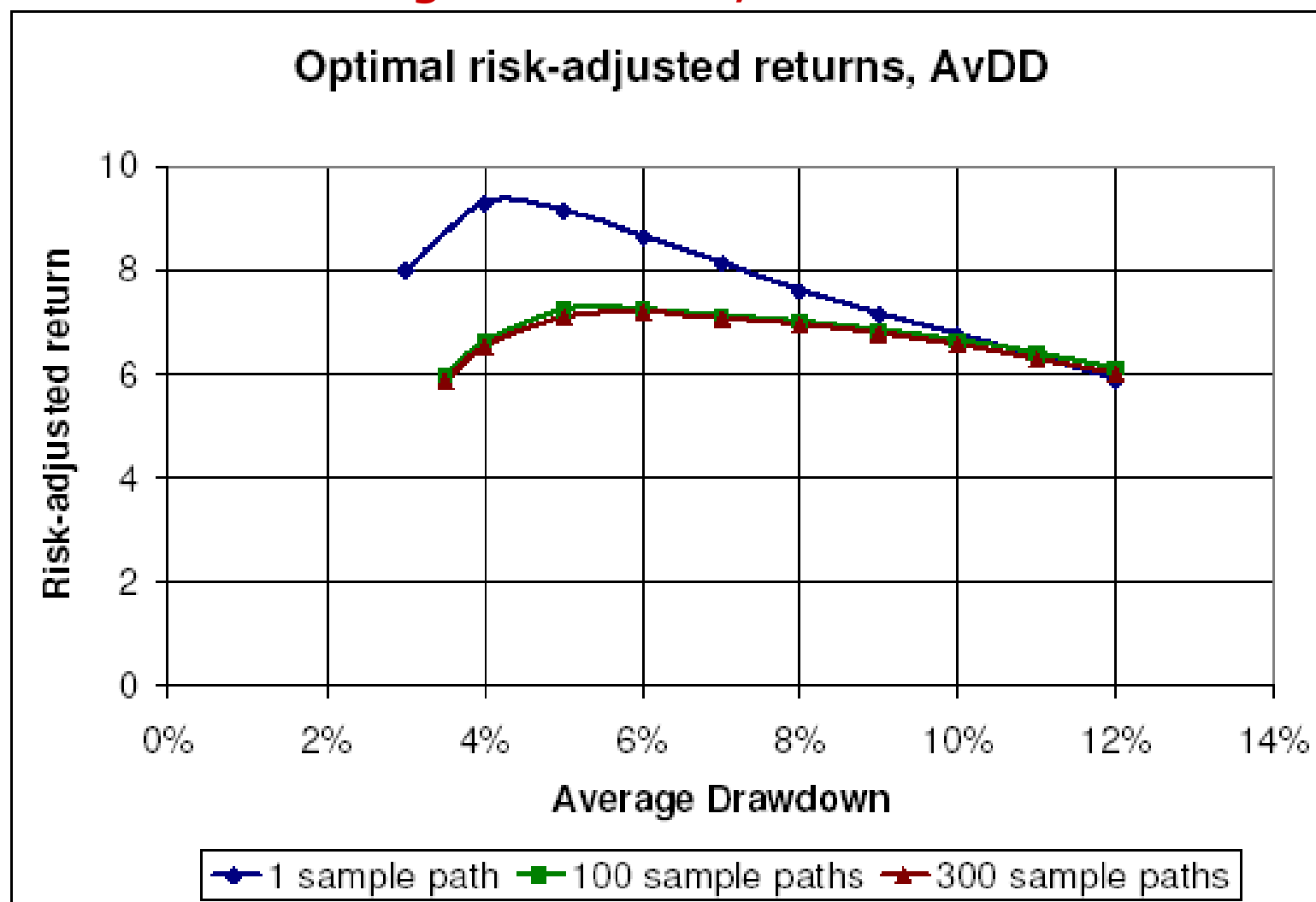# Stochastic vs. Historical Risk-Adjusted Returns: Average DrawDown, Futures Case

Figure 7: Optimal risk-adjusted returns: Average Drawdown

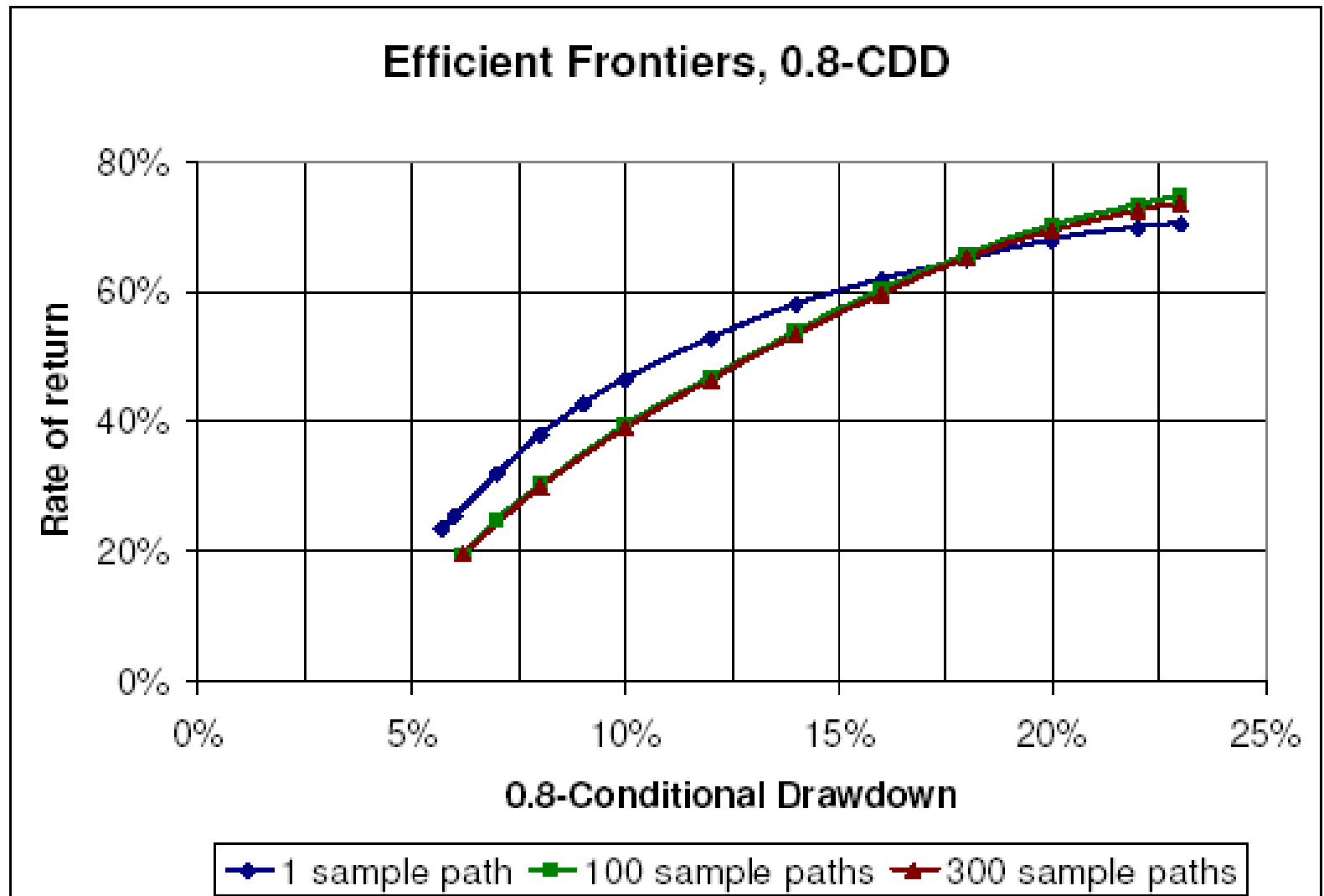# Stochastic vs. Historical Efficient Frontiers: 20% of the Worst DrawDowns, Futures Case

Figure 8: Efficient frontiers: 0.8-Conditional Drawdown

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Figure 9: Optimal risk-adjusted returns: 0.8-Conditional Drawdown

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Stochastic vs. Historical Efficient Frontiers: Worst DrawDown, Futures Case
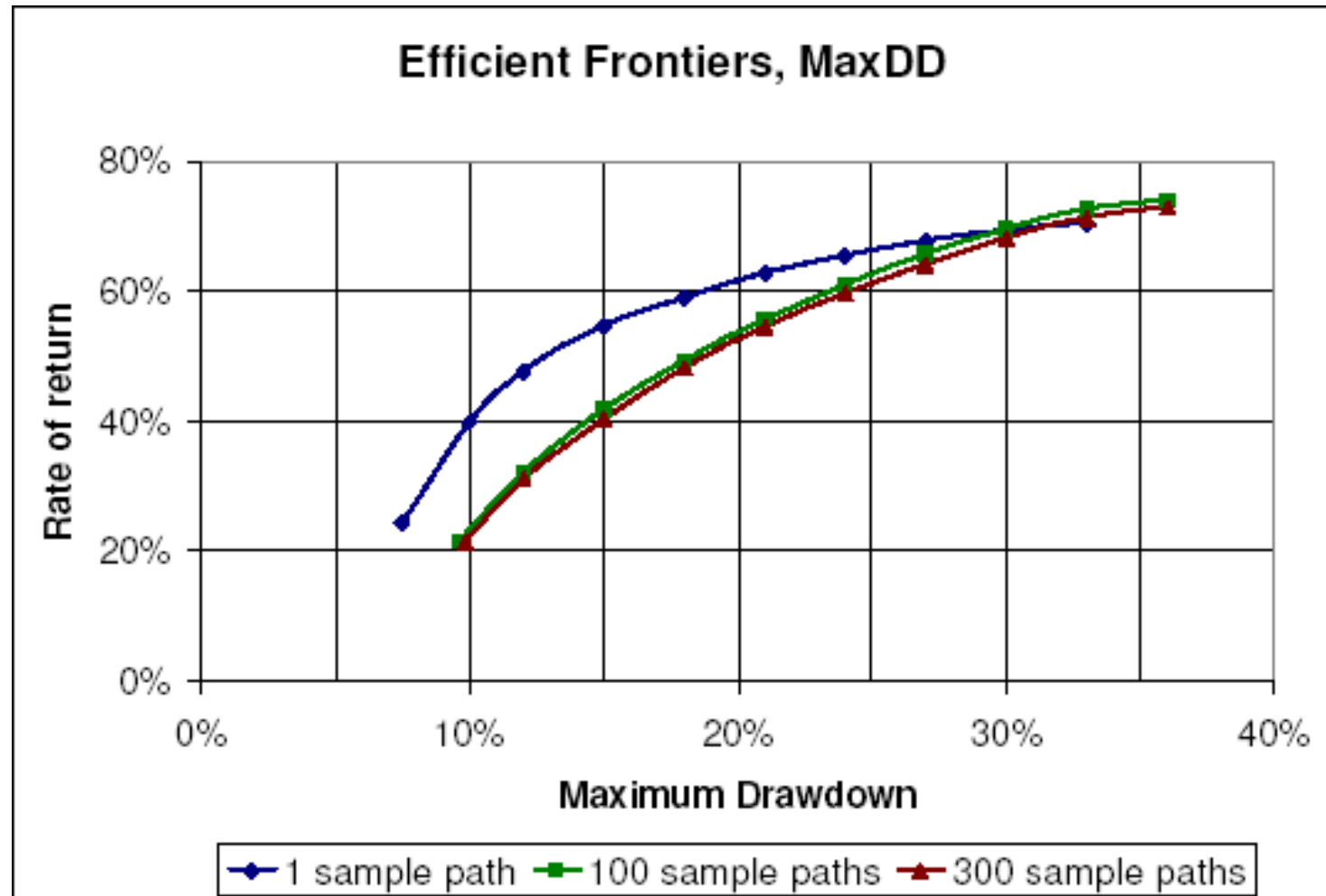
Figure 10: Efficient frontiers: Maximum Drawdown

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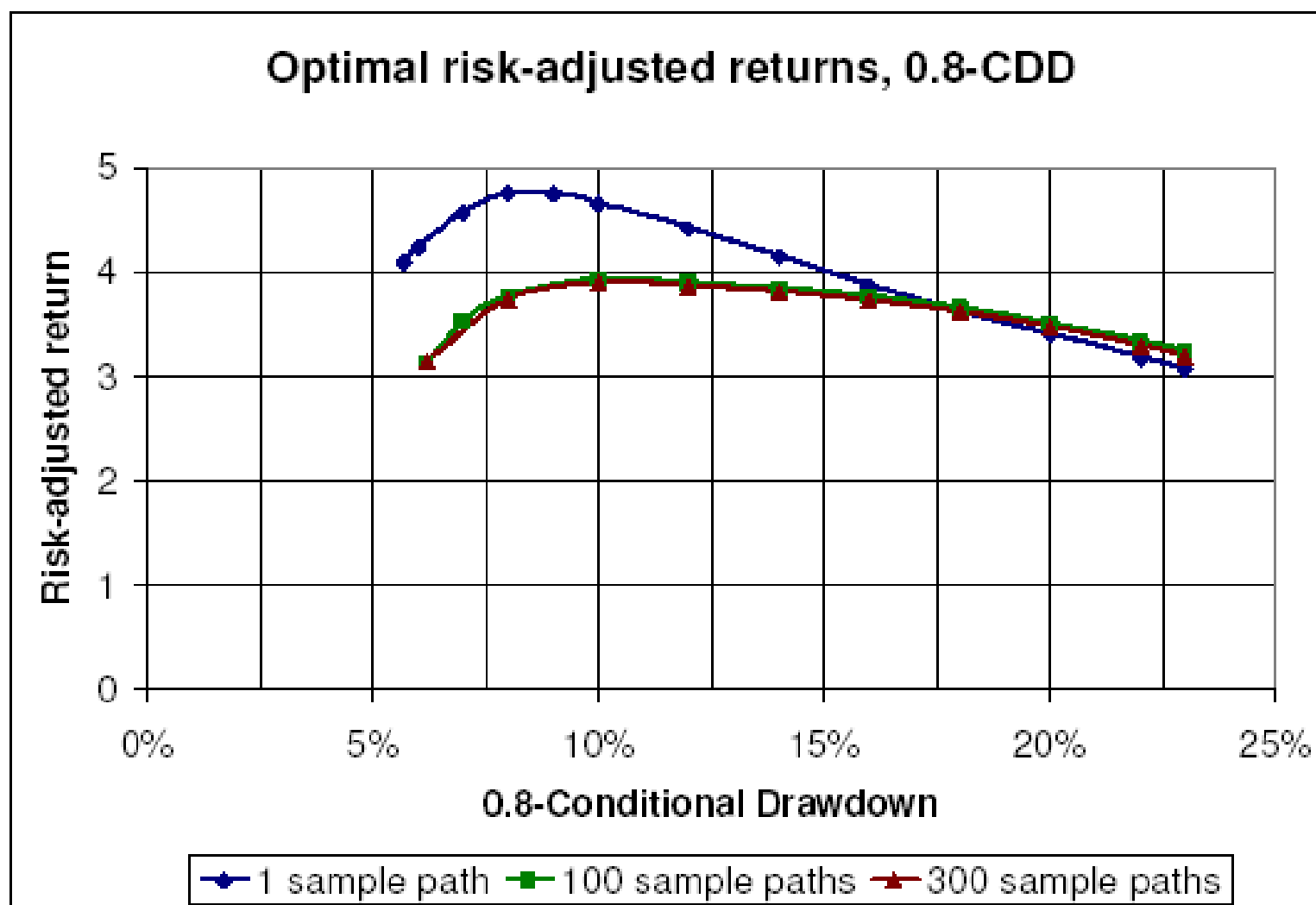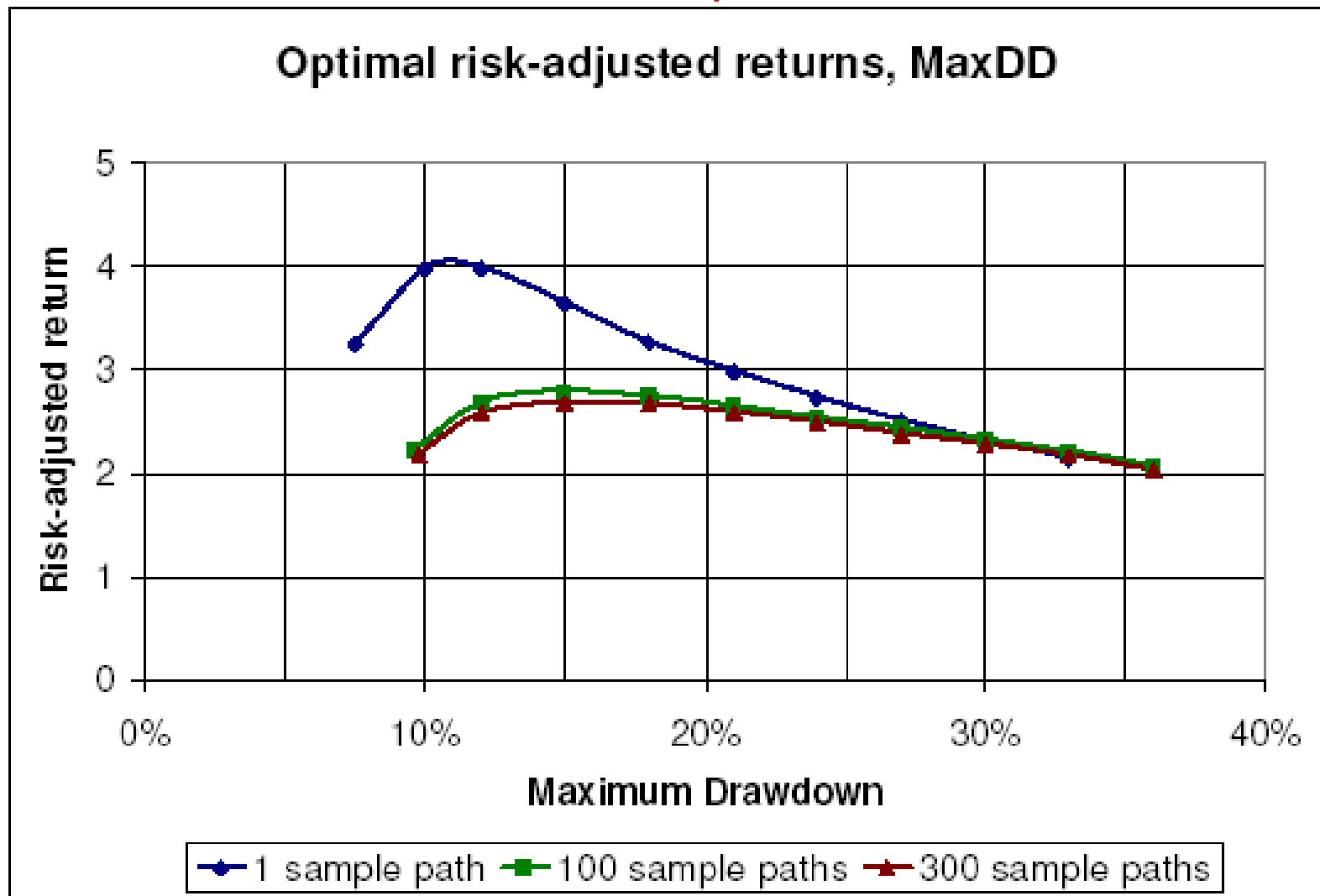# Stochastic vs. Historical Risk-Adjusted Returns : the Worst DrawDown, Futures Case

Figure 11: Optimal risk-adjusted returns: Maximum Drawdown

Analysis of Numerical Results

- The statistical accuracy is already sufficient for the 100-sample path case: 300-sample path case leads to a miniscule accuracy improvement.

- For most allowable risk values the efficient frontier for stochastic (re-sampled) solutions lies below and is less concave than the historical (1-scenario) efficient frontier.

- Optimal stochastic risk-adjusted returns are uniformly smaller than historical (20%-30% smaller in this case).

- As a 32-dimensional vector of instruments weights, we found the stochastic solutions substantially different from the historical ones: Euclidian norm is 50% smaller and they have a large angle (50 degrees) between them (Futures Case).

Advantages of Conditional DrawDown (CDD) Risk Measure

- This is arguably *the only risk measure* which *captures risk on long time intervals* (including possible crisis auto-correlation), without mixing the period returns.

- This measure of risk *can be measured* (pro-rated) *per time interval*, allowing one to compare a Hedge Fund with a 3-year track record to a Hedge Fund with 10-year track record.

- CDD is *convex*, that is, *diversification reduces risk*.

- Stochastic Optimization problems using CDD risk measure *can be mapped onto an highly-efficiently soluble LP-problems.*

- It is a multi-scenario measure, which can lead to *more robust out-of-sample* solutions for asset weights.

- CDD risk measure and its associated portfolio optimization problems are very *relevant in the alternative investments* arena.