Lecture 5:

The VIX.
P&L of Options Trading
When:
Hedging continuously
Hedging discretely
Transactions costs:

Conclusion:
Accurate Replication is Very Difficult

5.1 Replication of Variance when Volatility is Stochastic

As long as there is continuous diffusion (**no jumps**), the log contract still captures realized volatility even if the volatility is time dependent.

$$\frac{dS_t}{S_t} = \mu dt + \sigma(t) dZ_t$$

$$d\ln S_t = \left(\mu - \frac{\sigma(t)^2}{2}\right)dt + \sigma dZ_t$$

$$\frac{dS_t}{S_t} - d\ln S_t = \frac{1}{2}\sigma(t)^2 dt \qquad \text{Eq 5.1}$$

$$\text{average total variance} = \frac{1}{T}\int_0^T \sigma(t)^2 dt = \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_T}{S_0}\right]$$

cost of rebalancing hedge $1/S$ shares dynamically by borrowing

static short log contract

$$\frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[rT - \ln\frac{S_*}{S_0} - \left(\frac{S_0 e^{rT}}{S_*} - 1\right) + e^{rT}\int_{S^*}^{\infty} C(S_0, K, 0)\frac{dK}{K^2} + e^{rT}\int_0^{S^*} P(S_0, K, 0)\frac{dK}{K^2}\right]$$

Every option's price can be taken from the marketplace, even with a skew, and we can value the variance (almost) independent of theory.

If $S_* = S_0$ then we get the simpler formula

$$\frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[rT - (e^{rT} - 1) + e^{rT}\int_{S_0}^{\infty} C(S_0, K, 0)\frac{dK}{K^2} + e^{rT}\int_0^{S_0} P(S_0, K, 0)\frac{dK}{K^2}\right]$$

$$\approx \frac{2}{T}\left[e^{rT}\int_{S_0}^{\infty} C(S_0, K, 0)\frac{dK}{K^2} + e^{rT}\int_0^{S_0} P(S_0, K, 0)\frac{dK}{K^2}\right]$$

The initial value of the portfolio of puts and calls is the value of the time-averaged total variance.

5.2 Imperfections in Valuation by Replication

- **Discrete strikes** with a limited range capture less variance than the true variance. You gamble by omitting some strikes because when/if the stock price gets to those strikes, you have no options strikes and therefore no option gamma to capture the variance.
- **Effect of jumps**
  The log contract doesn't capture the true variance if jumps occur, for two reasons.

1. Jumps can move the stock price out of the range of strikes you use for replication.
2. A jump contributes to the rigorous definition of the realized variance which depends on the second moment of the return, i.e. J^2, but jumps contribute to the Taylor series expansion of the log contract with a J^3 term too, plus higher orders.

"

The log contract hedging strategy

$$\text{total variance} = \frac{1}{T}\int_0^T \sigma^2 dt = \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \ln\frac{S_T}{S_0}\right] \text{captures}$$

$$\sum \frac{\Delta S_i}{Si} - \log\frac{S_T}{S_0} = \sum\left[\frac{\Delta S_i}{Si} - \log\frac{S_{i+1}}{S_i}\right] = \sum\left[\frac{\Delta S_i}{Si} - \log\left(1+\frac{\Delta S_i}{S_i}\right)\right] = \quad \approx \sum \frac{1}{2}\left(\frac{\Delta S_i}{S_i}\right)^2 - \frac{1}{3}\left(\frac{\Delta S_i}{S_i}\right)^3 + \ldots$$

5.3 The first term is the true realized variance contribution; the second is normally negligible, but for a large jump $(\Delta S_i)/S_i = J$ will add an asymmetric term to the P&L that is absent from the true variance.

Valuing Volatility Swaps: Negative Convexity

Volatility is a derivative, the square root of variance which we know how to replicate with a combination of calls and puts. You can replicate volatility using the continuous dynamic trading of portfolios of variance swaps, just as you can replicate $\sqrt{S}$ by trading S.

To estimate the effect, expand about V_E, the expected value of the distribution of variance:

$$\sigma = \sqrt{\sigma^2} = \sqrt{V} \equiv \sqrt{V_E + \{V - V_E\}}$$

$$= \sqrt{V_E}\left(1 + \frac{V - V_E}{V_E}\right)^{1/2}$$

$$\approx \sqrt{V_E}\left[1 + \frac{V - V_E}{2V_E} - \frac{1}{8}\left(\frac{V - V_E}{V_E}\right)^2 + \dots\right]$$ The square root has negative convexity therefore worth less.

$$\approx \sqrt{V_E} + \frac{V - V_E}{2\sqrt{V_E}} - \frac{1}{8}\frac{(V - V_E)^2}{V_E^{3/2}}$$

Taking risk-neutral expectation to value the volatility: $E(\sigma) \approx \sqrt{V_E} - \frac{1}{8}\frac{E\left[(V - V_E)^2\right]}{V_E^{3/2}}$

Thus the fair volatility is smaller than the square root of the variance, and depends on the volatility of variance, like an option on variance.

Fair Volatility Must Be Smaller Than Square Root Of Variance

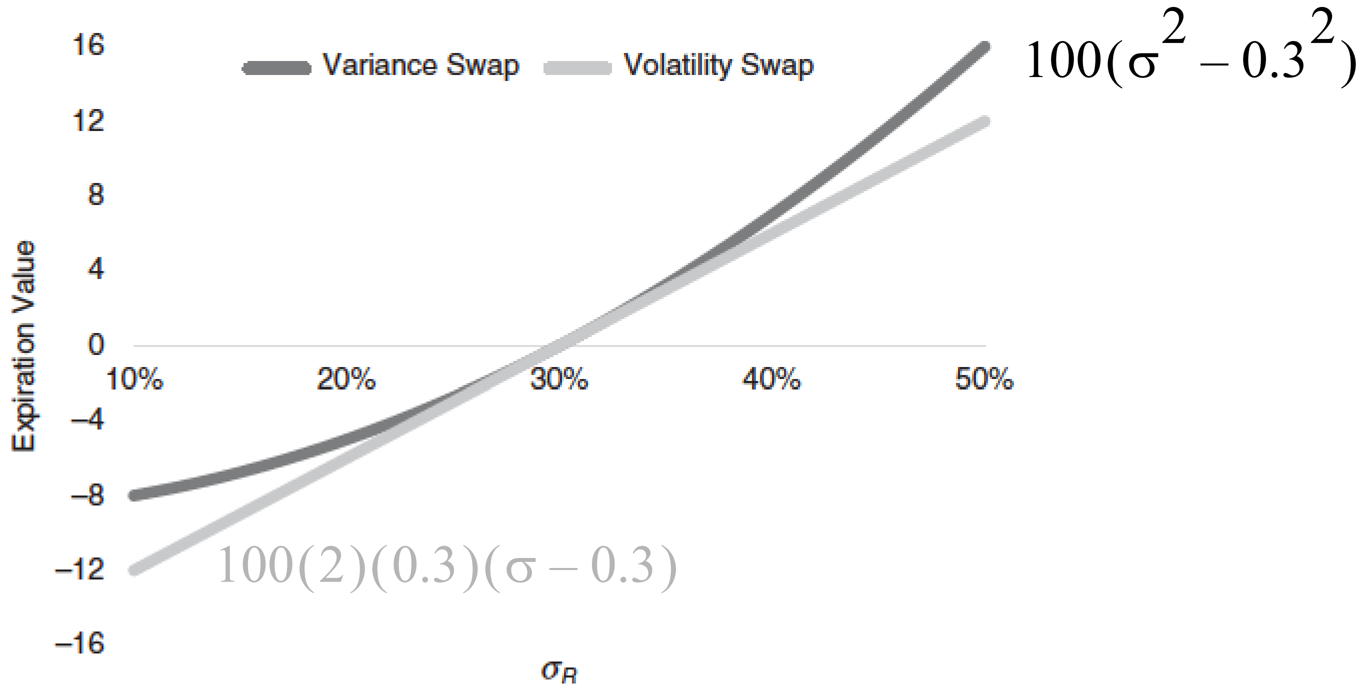


FIGURE 4.3 Comparison of a Volatility Swap with a Variance Swap

We are comparing a variance swap with notional 100 and a vol swap with equivalent exposure at the strike. $\sigma^2 - 0.3^2 = (\sigma - 0.3)(\sigma + 0.3) \approx (2)(0.3)(\sigma - 0.3)$

The variance swap dominates the vol swap and must be worth more and would generate an arbitrage-free profit if we bought the variance swap and sold the vol swap at the same delivery price. This cannot be.

Thus the fair strike of the volatility swap, the value of the volatility for which the swap is worth zero, must be lower. How much depends on the vol of vol.

5.4 The VIX Volatility Index

The VIX, from 1993 - 2003, was defined as the weighted average of various atm and otm implied volatilities of the S&P. This was a rather arbitrary average of parameters. In 2003 the CBOE changed the definition of the VIX to be the square root of the fair delivery price of variance as captured by a variance swap, using the formula from our paper, extended to account for stock dividends.

$$\int_0^T \sigma^2 dt = \frac{2}{T}\left[(r-d)T - \ln\frac{S_*}{S_0} - \left(\frac{S_0 e^{(r-d)T}}{S_*} - 1\right) + e^{rT}\left[\int_{(K>S^*)} C(S,K,0)\frac{dK}{K^2} + e^{rT}\int_{(K<S^*)} P(S,K,0)\frac{dK}{K^2}\right]\right]$$

Writing F as the forward price of the S&P, $F = S_0 e^{(r-d)T}$, the RHS is

$$\frac{2}{T}\left\{\ln\frac{F}{S_0} - \ln\frac{S_*}{S_0} - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls above S* plus puts below S*}]\right\}$$

$$= \frac{2}{T}\left\{\ln\frac{F}{S^*} - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls and puts}]\right\}$$

$$= \frac{2}{T}\left\{\ln\left(1 + \frac{F}{S^*} - 1\right) - \left(\frac{F}{S_*} - 1\right) + e^{rT}[\text{sum of calls and puts}]\right\}$$

$$\approx \frac{2}{T}\left\{e^{rT}[\text{sum of calls and puts}] - \frac{1}{2}\left(\frac{F}{S_*} - 1\right)^2\right\}$$

The CBOE uses a finite sum over traded options at two expirations near 30 days, and then interpolates/extrapolates to thirty day volatility.

Some advantages of the "new VIX":

- The VIX is an estimate of one-month future realized volatility based on listed options prices.
- The estimate is independent of volatility at one particular market level because it involves the sum of different options prices.
- It is *relatively* insensitive to model issues, because it assumes only continuous underlier movement, but doesn't assume Black-Scholes no-smile.
- It can be replicated and hedged because it involves a portfolio of listed options.
- The VIX is the most liquid measure of short-term implied volatility. People tend to regard it as an indicator of future realized volatility. "The fear index"

Future Extensions

Many variance swaps are capped and implicitly contain embedded volatility options.

Valuing options on volatility is the big challenge. More on volatility of volatility later.

With a model for volatility of the VIX, one can price futures, forwards and options on the VIX. The CBOE offers listed futures and options on the VIX.

5.5 Perspective: The Black-Scholes Equation is Also a Statement about Sharpe Ratios

Valuation by perfect replication. We assume

- Continuous stock price movements (one-factor geometric Brownian motion) with constant volatility; no other sources of randomness; no jumps.
- Ability to hedge continuously by taking on arbitrarily large long or short positions.
- No bid-ask spreads.
- No transactions costs.
- No forced unwinding of positions.

$$dS = \mu_S S dt + \sigma_S S dZ$$
$$dB = Brdt$$

The option price $C(S_t, t)$ whose evolution is given by

$$\begin{aligned} dC &= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\sigma_S S)^2 dt \\ &= \left\{\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu_S S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\sigma_S S)^2\right\}dt + \frac{\partial C}{\partial S}\sigma_S S dZ \\ &= \mu_C C dt + \sigma_C C dZ \end{aligned}$$

where by definition

$$\mu_C = \frac{1}{C}\left\{\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu_S S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\sigma_S S)^2\right\}$$
$$\sigma_C = \frac{S}{C}\frac{\partial C}{\partial S}\sigma_S = \frac{\partial \ln C}{\partial \ln S}\sigma_S \qquad \text{Eq 5.2}$$

Riskless portfolio $\pi = \alpha S + C$

Then

$$d\pi = \alpha(\mu_S S dt + \sigma_S S dZ) + (\mu_C C dt + \sigma_C C dZ)$$
$$= (\alpha\mu_S S + \mu_C C)dt + (\alpha\sigma_S S + \sigma_C C)dZ \qquad \textit{Eq 5.3}$$

Riskless necessitates dZ coefficient must be zero:

$$\alpha\sigma_S S + \sigma_C C = 0$$
$$\alpha = -\frac{\sigma_C C}{\sigma_S S} \qquad \text{Eq 5.4}$$

Therefore

$$d\pi = (\alpha\mu_S S + \mu_C C)dt$$

No riskless arbitrage: $d\pi = \pi r dt$.

Requires

$$\alpha\mu_S S + \mu_C C = (\alpha S + C)r$$

Rearranging the terms we obtain

$$\alpha S(\mu_S - r) = -C(\mu_C - r)$$

Substituting for α from Equation 5.4 leads to the relation

$$\frac{(\mu_C - r)}{\sigma_C} = \frac{(\mu_S - r)}{\sigma_S}$$ Equal Sharpe Ratios of Stock and Option Eq 5.5

This is the argument by which Black originally derived the Black-Scholes equation.

Substituting from Equation 5.2 into Equation 5.5 for μ_C and σ_C we obtain

$$\frac{\frac{1}{C}\left\{\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}\mu_S S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\sigma_S S)^2\right\} - r}{\frac{1}{C}\frac{\partial C}{\partial S}\sigma_S S} = \frac{(\mu_S - r)}{\sigma_S}$$

which leads to

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma_S^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$ Black-Scholes equation, no drift Eq 5.6

The solution, the Black-Scholes formula and its implied volatility, is the quoting currency for trading prices of vanilla options.

You can get a great deal of insight into more complex models by regarding them as perturbations or mixtures of different Black-Scholes solutions.

$$C(S,K,t,T,\sigma,r) = e^{-r(T-t)}[S_F N(d_1) - KN(d_2)]$$

$$S_F = e^{r(T-t)}S$$

$$d_1 = \frac{\ln\left(\frac{S_F}{K}\right) + \left(\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \quad d_2 = \frac{\ln\left(\frac{S_F}{K}\right) - \left(\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}y^2}\,dy$$ Eq 5.7

Notice that except for the $r(T-t)$ term, time to expiration and volatility always appear together in the combination $\sigma^2(T-t)$. If you rewrite the formula in terms of the prices of traded securities –

the present value of the bond K_{PV} and the stock price S – then indeed time and volatility always appear together:

$$C(S, K, t, T, \sigma) = [SN(d_1) - K_{PV}N(d_2)]$$

$$K_{PV} = e^{-r(T-t)}K$$

$$d_{1,2} = \frac{\ln(S/K_{PV}) \pm 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} \qquad \text{Eq 5.8}$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-y^2/2}\, dy$$

Note that $\sigma^2(T-t)$ is the total future variance. Smart users of the formula can enter their estimates of the total variance, which on average may be smaller on weekends than on weekdays, for example.

The Effective Instantaneous Volatility of a Call -- Example

According to analysts at your firm, the expected return on Microsoft (MSFT) is 11%. MSFT is currently trading at \$50. 3-month at-the-money calls on MSFT have a delta of 0.52 and trade at \$2.00. The volatility of MSFT is 15%. What is current volatility of the call options?

$$\begin{aligned}\sigma_C &= \frac{S}{C}\frac{\partial C}{\partial S}\sigma_S \\ &= \frac{50}{2}\times 0.52\times 0.15 \\ &= 1.95\end{aligned}$$

The instantaneous current volatility of the option is 195%, much riskier than the stock itself.

5.6 The P&L of Any Hedged/Replicated Trading Strategy

Consider an initial position at time t_0 when the stock price is S_0 and an option C that is bought and hedged with borrowed money which earns interest r, and then rehedged in discrete steps at times t_i and stock prices S_i. We do accounting again.

Notation: $C_n = C(S_n, t_n) \qquad \Delta_n = \Delta(S_n, t_n)$

Δ can be **any function**, not BSM value. Interest rates non-zero.

t_n,S_n	Hedging action	No. Shares	Share Value	Dollars Received From Shares and Options Traded	Net Value of Position: Option + Stock + Cash
t_0,S_0	Buy C_0, short Δ_0 shares	$-\Delta_0$	$-\Delta_0 S_0$	$\Delta_0 S_0 - C_0$	0
t_1,S_1	none	$-\Delta_0$	$-\Delta_0 S_1$	$(\Delta_0 S_0 - C_0)e^{r\Delta t}$	$C_1 - \Delta_0 S_1 + (\Delta_0 S_0 - C_0)e^{r\Delta t}$
	get short Δ_1 shares by shorting $\Delta_1 - \Delta_0$ shares	$-\Delta_1$	$-\Delta_1 S_1$	$(\Delta_0 S_0 - C_0)e^{r\Delta t} + (\Delta_1 - \Delta_0)S_1$	$C_1 - \Delta_1 S_1 + (\Delta_0 S_0 - C_0)e^{r\Delta t} + (\Delta_1 - \Delta_0)S_1$
t_2,S_2	none	$-\Delta_1$	$-\Delta_1 S_2$	$(\Delta_0 S_0 - C_0)e^{2r\Delta t} + (\Delta_1 - \Delta_0)S_1 e^{r\Delta t}$	$C_2 - \Delta_1 S_2 + (\Delta_0 S_0 - C_0)e^{2r\Delta t} + (\Delta_1 - \Delta_0)S_1 e^{r\Delta t}$

t_n, S_n	Hedging action	No. Shares	Share Value	Dollars Received From Shares and Options Traded	Net Value of Position: Option + Stock + Cash
t_2, S_2	get short Δ_2 shares by shorting $\Delta_2 - \Delta_1$ shares	$-\Delta_2$	$-\Delta_2 S_2$	$(\Delta_0 S_0 - C_0)e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0)S_1 e^{r\Delta t}$ $+ (\Delta_2 - \Delta_1)S_2$	$C_2 - \Delta_2 S_2 + (\Delta_0 S_0 - C_0)e^{2r\Delta t}$ $+ (\Delta_1 - \Delta_0)S_1 e^{r\Delta t}$ $+ (\Delta_2 - \Delta_1)S_2$
etc.					
t_n, S_n	get short Δ_n shares by shorting $\Delta_n - \Delta_{n-1}$ shares	$-\Delta_n$	$-\Delta_n S_n$	$(\Delta_0 S_0 - C_0)e^{nr\Delta t}$ $+ (\Delta_1 - \Delta_0)S_1 e^{(n-1)r\Delta t}$ $+ (\Delta_2 - \Delta_1)S_2 e^{(n-2)r\Delta t}$ $\ldots + (\Delta_n - \Delta_{n-1})S_n$	$C_n - \Delta_n S_n + (\Delta_0 S_0 - C_0)e^{nr\Delta t}$ $+ (\Delta_1 - \Delta_0)S_1 e^{(n-1)r\Delta t}$ $+ (\Delta_2 - \Delta_1)S_2 e^{(n-2)r\Delta t}$ $\ldots + (\Delta_n - \Delta_{n-1})S_n$

The initial value of the positions was 0 and that cash invested would have generated zero in future.

The final value is $C_T - \Delta_T S_T + (\Delta_0 S_0 - C_0)e^{r(T - t_0)} + \int_{t_0}^{T} e^{r(T-x)} S_x [d\Delta_x]_b$

where x is the intermediate time and the subscript b at the end of the formula denotes **a backwards Ito integral**. C_T is known as a function of the terminal stock price.

Write $T - t = \tau$.

$$\text{final value} = C_T - \Delta_T S_T + (\Delta_0 S_0 - C_0)e^{r\tau} + \int_0^{\tau} e^{r(\tau - x)} S_x [d\Delta_x]_b \qquad \text{Eq 5.9}$$

If there is a perfect BSM Δ hedge at each step, then the growth of the portfolio at each step is riskless, and since we start from a zero value, the final value of Equation 5.9 must be zero.

In that case only we get the VERY IMPORTANT REPLICATION FORMULA for C_0:

$$(C_0 - \Delta_0 S_0)e^{r\tau} = (C_T - \Delta_T S_T) + \int_0^{\tau} e^{r(\tau - x)} S_x [d\Delta_x]_b \qquad \text{Eq 5.10}$$

Simply stated: The future value of the initial continuously perfectly hedged portfolio is equal to the final value of the hedged portfolio plus the future cost of all the incremental re-hedges. This will be used to calculate the effect of discrete hedging on the option price via Monte Carlo.

If it is not perfectly hedged *a la* BSM, there is a distribution of final values. It is path dependent, and so not unique in general. $\Delta_x(S, x, \sigma)$ can depend on the volatility you use.

In general, the P&L depends upon the path taken to expiration. You can do Monte Carlo simulation to find the distribution. You get a unique value for the P&L and C_0, independent of the stock path to expiration, *only if the hedge is riskles*s, so that there is a unique value for the portfolio at every instant. Else the investment leads to a distribution.

You can integrate by parts using the relation

$$d\left[e^{r(\tau-x)}S_x\Delta_x\right] = -re^{r(\tau-x)}\Delta_x S_x dx + e^{r(\tau-x)}\Delta_x dS_x + e^{r(\tau-x)}S_x[d\Delta_x]_b$$ (see next section for backward Ito)

to obtain

Eq 5.11

$$(C_0-\Delta_0 S_0)e^{r\tau} = (C_T-\Delta_T S_T) + \int_0^\tau d\left[e^{r(\tau-x)}S_x\Delta_x\right] + \int_0^\tau re^{r(\tau-x)}\Delta_x S_x dx - \int_0^\tau e^{r(\tau-x)}\Delta_x dS_x$$

$$(C_0-\cancel{\Delta_0 S_0})e^{r\tau} = (C_T-\cancel{\Delta_T S_T}) + \left[\cancel{\Delta_T S_T}-\cancel{\Delta_0 S_0 e^{r\tau}}\right] + e^{r\tau}\int_0^\tau e^{-rx}\Delta(S_x, x)[dS_x - rS_x dx]$$

Eq 5.12

$$C_0 = C_T e^{-r\tau} - \int_0^\tau \Delta(S_x, x)[dS_x - rS_x dx]e^{-rx}$$

PV of payoff

PV of change in value of the hedged shares funded at the riskless rate

Equation 5.10 and Equation 5.12 provide a way to calculate the value of the option C_0 in terms of its final payoff and the hedging strategy. If you hedge perfectly and continuously to get a riskless position, Black-Scholes tells you that this value will be independent of the path the stock takes to expiration. Else the fair value of C_0 has a spread-out distribution. There is homework on this.

Note: So far we've made no assumptions about dynamics. Suppose that the stock really satisfies GBM with a drift equal to the riskless interest rate so that

$$dS - Srdt = \sigma SdZ.$$

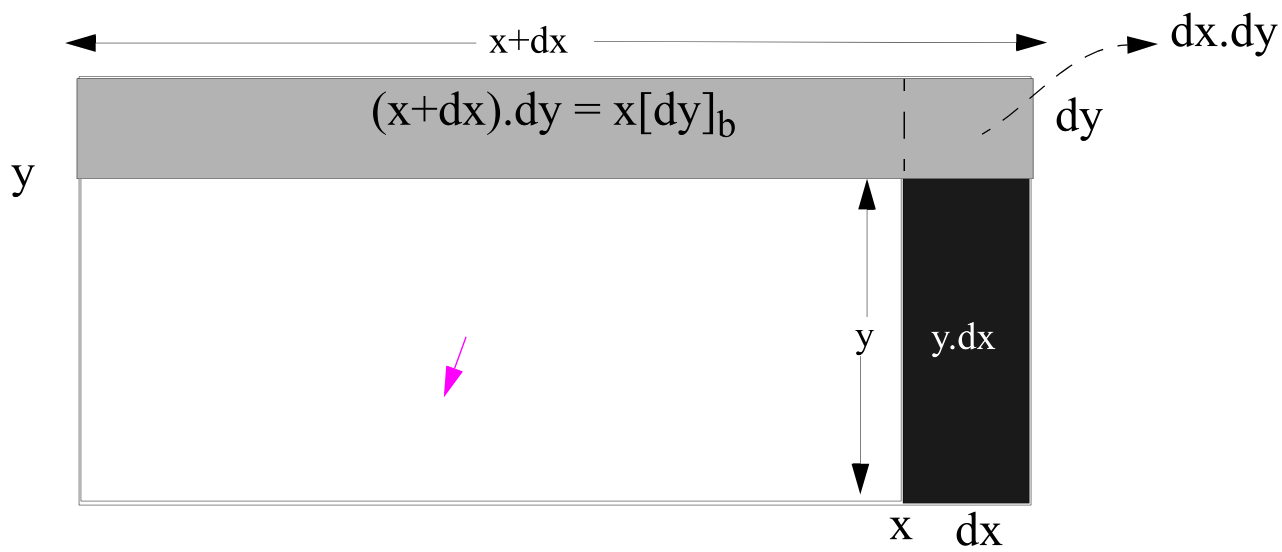
Then
$$C_0 = C_T e^{-r\tau} - \int_0^{\tau'} \Delta(S_x, x)\sigma S_x e^{-rx} dZ_x$$

$$E[C_0] = E[C_T]e^{-r\tau}$$

independent of the hedging strategy Δ, i.e. even if you don't hedge at all with $\Delta = 0$.

Of course in real life the stock does not satisfy GBM with a drift equal to the riskless rate.

Backward Ito Integral for Brownian Motion



backward or forward differential makes a difference

$$d[xy] = ydx + x[dy]_b$$

$dx.dy \sim dt$ is not negligible when you have Ito processes.

5.7 Different Hedging Strategies In The GBM BSM World: The P&L When Hedging with Realized (Imagined Known) Volatility

Realized volatility is a noisy statistic. Implied volatility is a parameter reflecting fear, hedging costs, the inability to hedge perfectly, the uncertainty of future volatility, the chance to make a profit, etc., and is therefore usually greater than recent realized volatility.

Future evolution is always at realized volatility.

Purchase/traded price is always at implied volatility, by definition.

Δ-Hedging can be **at any volatility**, since we don't know what future volatility will actually be.

But **let's pretend** we know the future volatility with perfect forecasting accuracy.

We buy the option at its implied volatility and then hedge it at the (assumed known future) realized volatility to replicate the option perfectly at the BS value assuming that we have perfect knowledge of future realized volatility. (GS story)

In the end, when we hedge all the way to expiration, the total P&L will be the value gained from replication MINUS Black-Scholes implied value:

$$\text{Total PV[P\&L(I,R)]} = V(S, \tau, \sigma) - V(S, \tau, \Sigma) = V_R - V_I$$

Notation:

"I = bought at implied, known"

"R = hedged at realized, assumed known"

We know the final answer but we're going to examine how this evolves now, with better notation.

Notation and Remarks About Changes in the P&L

To look at the evolution of the P&L we will
(1) use accounting to put together all the bits of profit and expense, and then
(2) use Ito's Lemma for GBM to understand the evolution of the profit.

Notation: A portfolio is bought at an implied volatility I, and then evolves at a realized volatility R, and is hedged at a hedge volatility H.

We call such a hedged portfolio $\pi[I, R, H] = \boldsymbol{V}_I - \Delta_H \boldsymbol{S}$, where the I means the option is bought at implied volatility, and the H means it's hedged using the hedge ratio of some other volatility H, and R means that meanwhile the stock evolves at the realized volatility. And we assume we borrowed the money to set up the portfolio, so the initial value of the portfolio is always zero.

One thing we will have to make use of is our knowledge of Black-Scholes for GBM, namely that if you buy an option at some implied volatility I and if the realized volatility R = I, and if hedge it at that same I, then it is riskless and if you spent no money on the portfolio, i.e. if you borrowed the money to set up that portfolio then there can be no change in its value. So initially

$\pi[I, I, I] = \boldsymbol{V}_I - \Delta_I \boldsymbol{S} - [\text{cash needed to set up the position}] = 0$ and

$d\pi[I, I, I] = 0$ over a short time dt as the stock price moves at the implied volatility.

Similarly, $d\pi[R, R, R] = 0$.

If implied, realized, and hedged vol are identical, then BS tells you you're perfectly hedged and zero value remains zero.

Now, How Do We Capture $V_R - V_I$ Over Time?

In our notation, the Hedged Portfolio is $\pi[I, R, R] = \boldsymbol{V}_I - \Delta_R \boldsymbol{S}$, bought with borrowed money, so that there is zero initial value of the portfolio, and then it evolve and is hedged at realized volatility.

We assume the usual GBM Evolution of the stock w**ith realized vol:** $dS = \mu S dt + \sigma_R S dZ$ and we assume it has a dividend yield = D, and the riskless rate = r.

Now we look at changes from accounting rules:

interest owed dividend owed

$$dP\&\mathrm{L(I,R,R)} = d\boldsymbol{V}_I - \Delta_R dS - rdt(\boldsymbol{V}_I - \Delta_R S) - \Delta_R DSdt$$

accounting

$$= dV_I - rV_I dt - \Delta_R[dS - (r - D)Sdt]$$

But dP&L[R,R,R] hedged *and bought* at realized volatility is zero:

$$dP\&\mathrm{L(R,R,R)} = d\boldsymbol{V}_R - rdt\boldsymbol{V}_R - \Delta_R[dS - (r - D)Sdt] = 0$$

$$\Delta_R[dS - (r - D)Sdt] = d\boldsymbol{V}_R - rdt\boldsymbol{V}_R$$

Therefore combining the two

$$d\mathrm{P\&L(I,R,R)} = d\boldsymbol{V}_I - rd\boldsymbol{V}_I dt - \Delta_R[dS - (r - D)Sdt] = d\boldsymbol{V}_I - rd\boldsymbol{V}_I dt - (d\boldsymbol{V}_R - rdt\boldsymbol{V}_R)$$

$$= d\boldsymbol{V}_I - d\boldsymbol{V}_R - rdt(\boldsymbol{V}_I - \boldsymbol{V}_R) = e^{rt} d\left[e^{-rt}(\boldsymbol{V}_I - \boldsymbol{V}_R)\right]$$

$$PV[d(\text{P\&L(I,R.R)})] = e^{-r(t-t_0)} e^{rt} d\left[e^{-rt}(V_I - V_R)\right] = e^{rt_0} d\left[e^{-rt}(V_I - V_R)\right]$$

$$\text{PV[P\&L(I,R,R)]} = e^{rt_0} \int_{t_0}^{T} d\left[e^{-rt}(V_I - V_R)\right]$$

$$= 0 - (V_{I,\,t_0} - V_{R,\,t_0}) = V_{R,\,t_0} - V_{I,\,t_0} \quad \text{if T is expiration}$$

At expiration $V_{I,\,T} - V_{R,\,T} = 0$ because terminal value is independent of volatility.

The final P&L at the expiration of the option is known and deterministic, provided that we know the realized volatility and that we can hedge continuously.

How is this deterministic P&L realized over time? Stochastically.

It's a bit like a zero coupon bond whose final principal is known in advance but whose present value varies with interest rates, and the dependence on rates and time decreases as we approach maturity.

We now use Ito calculus to see how this happens:

$$\text{dP\&L[I,R,R]}) = dV_I - \Delta_R dS - \Delta_R SDdt - (V_I - \Delta_R S)rdt$$

Using Ito

$$\begin{aligned}\text{dP\&L[I,R,R]} &= \left[\Theta_I dt + \Delta_I dS + \frac{1}{2}\Gamma_I S^2 \sigma_R^2 dt\right] - \Delta_R dS - \Delta_R SDdt \\ &\quad -(V_I - \Delta_R S)rdt \\ &= \left[\Theta_I + \frac{1}{2}\Gamma_I S^2 \sigma_R^2\right] dt + (\Delta_I - \Delta_R)dS - \Delta_R SDdt \\ &\quad -(V_I - \Delta_R S)rdt\end{aligned}$$

Eq 5.13

But d[P&L(I,I,I)] = 0 $\quad d\text{P\&L}[I, I, I] = \left[\Theta_I + \frac{1}{2}\Gamma_I S^2 \Sigma^2\right]dt - \Delta_I SDdt - (V_I - \Delta_I S)rdt \equiv 0$

or

$$\boxed{\Theta_I = -\frac{1}{2}\Gamma_I S^2 \Sigma^2 + rV_I - (r - D)S\Delta_I}$$

which is just the BS PDE $\qquad$ Eq 5.14

and so substituting Equation 5.14 into Equation 5.13 and setting $dS = \mu Sdt + \sigma_R SdZ$ we get

$$dP\&L[I,R,R] = \left[\Theta_I + \frac{1}{2}\Gamma_I S^2 \sigma_R^2\right] dt + (\Delta_I - \Delta_R) dS - \Delta_R S D dt - (V_I - \Delta_R S) r dt$$

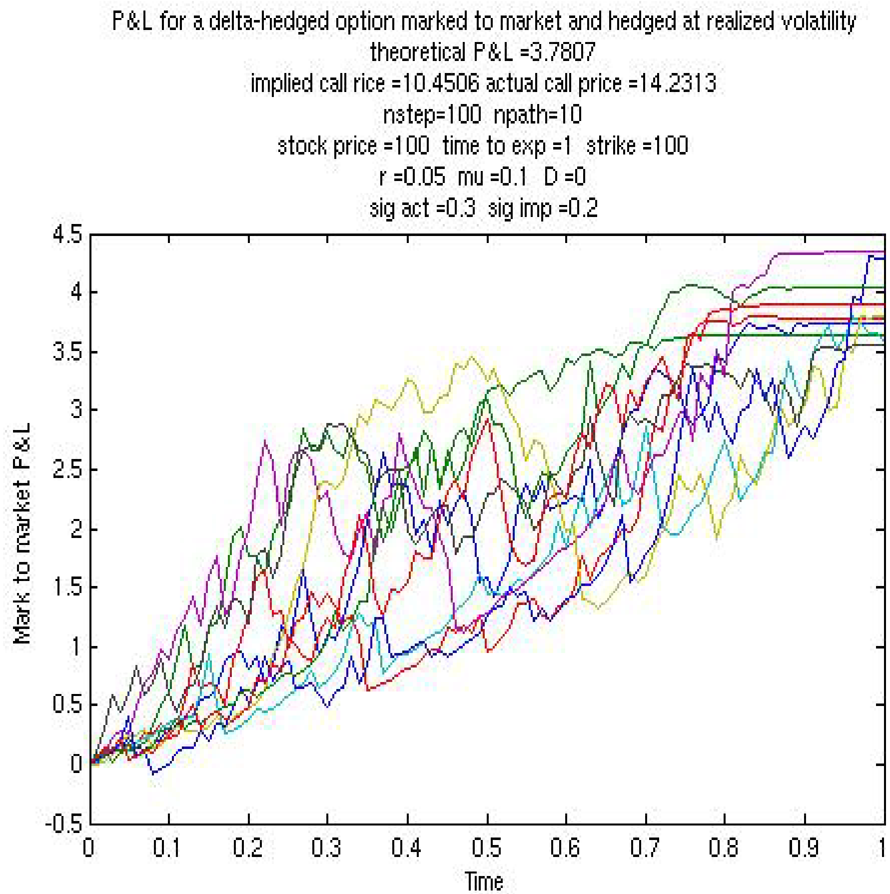
$$= \frac{1}{2}\Gamma_I S^2 \left(\sigma_R^2 - \Sigma^2\right) dt + (\Delta_I - \Delta_R)(dS - (r - D) dt)$$

$$= \frac{1}{2}\Gamma_I S^2 \left(\sigma_R^2 - \Sigma^2\right) dt + (\Delta_I - \Delta_R)\{((\mu - r + D) dt + \sigma_R S dZ)\}$$

We saw the total integrated P&L was deterministic but the increments have a random component dZ because the implied hedge ratio and the realized hedge ratio differ.

To illustrate this, plot cumulative **discounted P&L(I, R,R)** along ten random stock paths with100 steps

P&L starts at zero because initial position is totally financed



$$\sigma_r = 0.3$$

$$\sigma_i = 0.2$$

The **final P&L** is almost path-independent – almost, because 100 rehedgings per year is not quite the same as continuous hedging.

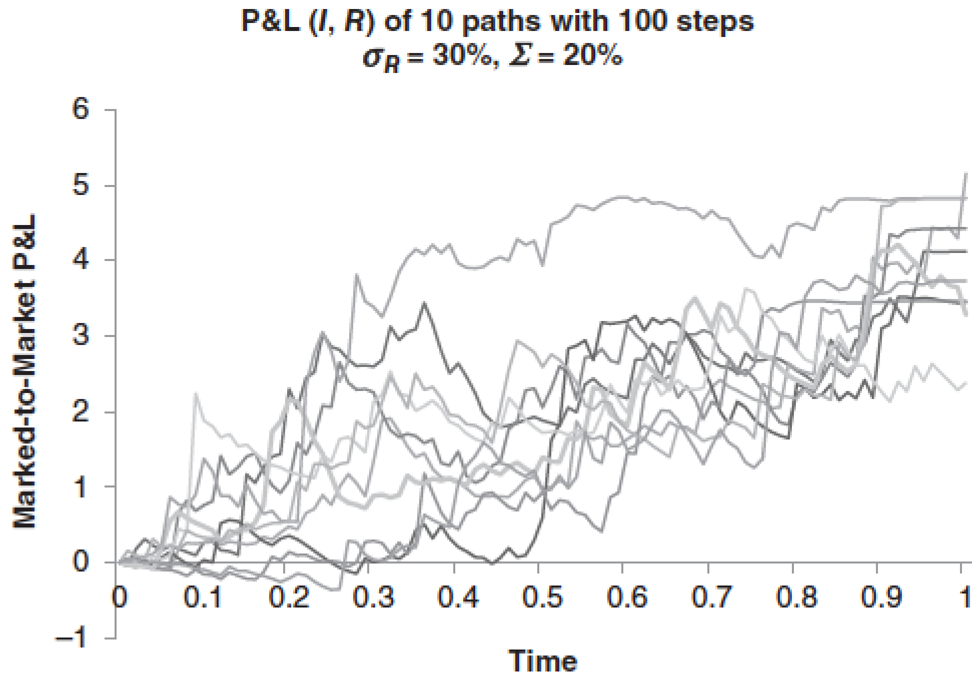
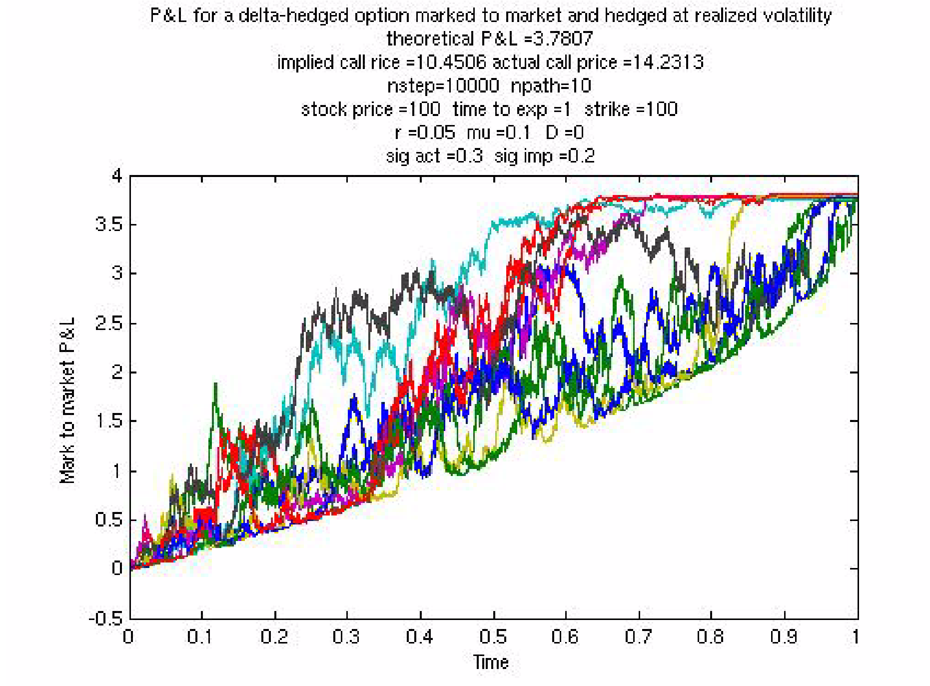


FIGURE 5.1 Hedging with Realized Volatility: Cumulative Discounted P&L of a Call with One Year to Expiration Simulated with 100 Steps

Rehedge 10,000 times, almost path-independent **final P&L**:



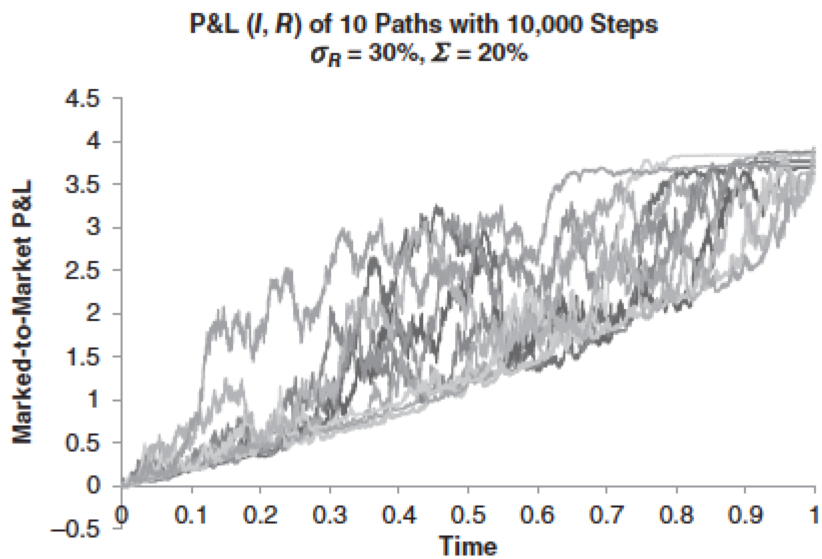


FIGURE 5.2 Hedging with Realized Volatility: Cumulative Discounted P&L of a Call with One Year to Expiration Simulated with 10,000 Steps

Bounds on the P&L When Hedging at the Realized Volatility

We had $\sigma > \Sigma$. Notice the upper and lower bounds. Why? We had

$$d\mathrm{PV}[\mathrm{P\&L(I,R,R)}] = e^{-r(t-t_0)} e^{rt} d\left[e^{-rt}(V_I - V_R)\right] = e^{rt_0} d\left[e^{-rt}(V_I - V_R)\right]$$

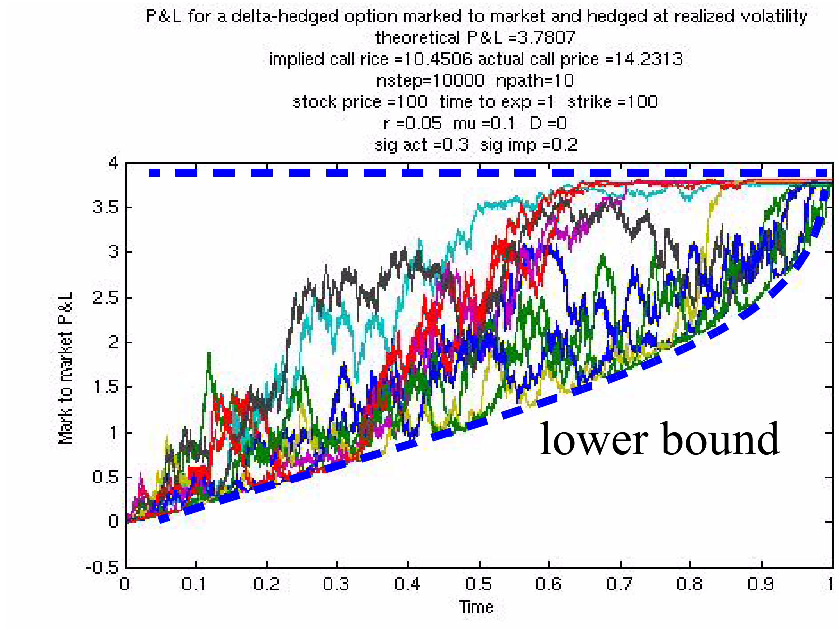
Integrate from t_0, S_0 to intermediate nonterminal time m when the stock price is S, to obtain

$$\begin{aligned}
\mathrm{PV(P\&L[I,R,R])} &= e^{rt_0}\int_{t_0}^{m} d\left[e^{-rt}(V_I - V_R)\right] \\
&= e^{rt_0}\left[e^{-rt}(V_I - V_R)\right]_{t_0}^{m} \\
&= e^{rt_0}\left[e^{-rm}(V_{I,m} - V_{R,m}) - e^{-rt_0}(V_{I,0} - V_{R,0})\right] \\
&= (V_{R,0} - V_{I,0}) - e^{-r(m-t_0)}(V_{R,m} - V_{I,m})
\end{aligned}$$

value at inception > 0 value along way

Both terms in the brackets are positive.

Upper bound $(V_{R,0} - V_{I,0})$ occurs when the second term is zero, when the option value is independent of volatility, which occurs at $S = 0$ or $S = \infty$, and the gamma of the option is zero.



The lower bound to the P&L occurs when second term $[V(\sigma, S, m) - V(\Sigma, S, m)] \sim \frac{\partial V}{\partial \sigma}(\Sigma - \sigma)$ is a maximum, i.e. when vega is largest, close to at-the money, which turns out to be at

$$S = Ke^{-(r - 0.5\sigma\Sigma)(T - t)}$$

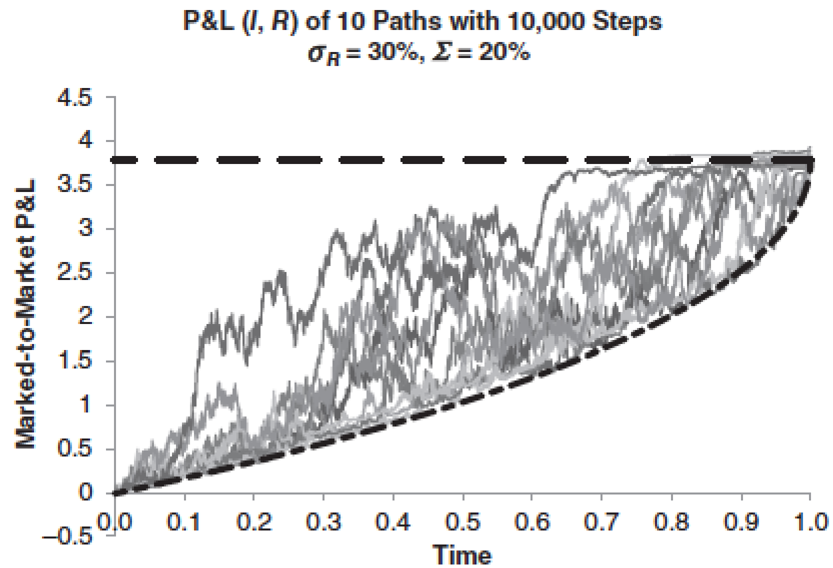


FIGURE 5.3 Hedging with Realized Volatility: Cumulative Discounted P&L with 10,000 Steps and Upper and Lower Bounds

5.8 P&L When Hedging with Implied Volatility

When you hedge with implied, the final value of the P&L(I,R,I) depends on the path taken, and is not deterministic, but there is **no random mishedging component at each instant**. At each instant we know the incremental P&L irrespective of stock price move,, because we are hedged, but the next contribution after that will depend on whether the stock will have moved up or down.

Table 1: Position Values when Hedging with Implied Volatility

Time	Option Position, Value	Stock Position, Value	Value of Cash Position	Net Position Value
t	$\vec{V}_i, V_i$	$-\Delta_i\vec{S}, -\Delta_i S$	$\Delta_i S - V_i$	0
t + dt	$\vec{V}_i, V_i + dV_i$	$-\Delta_i\vec{S}, -\Delta_i(S + dS)$	$(\Delta_i S - V_i)(1 + rdt)$ $-\Delta_i DSdt$	$(V_i + dV_i - \Delta_i(S + dS))$ $(\Delta_i S - V_i)(1 + rdt)$ $-\Delta_i DSdt$

$$
\begin{aligned}
d\text{P\&L(I,R,I)} &= [V_i + dV_i - \Delta_i(S + dS)] + (\Delta_i S - V_i)(1 + rdt) - \Delta_i DSdt \\
&= dV_i - \Delta_i dS - r(V_i - \Delta_i S)dt - \Delta_i DSdt
\end{aligned}
$$

Using Ito:

$$d\text{P\&L(I,R,I)} = \left[\Theta_i dt + \cancel{\Delta_i dS} + \frac{1}{2}\Gamma_i S^2\sigma^2 dt\right] - \cancel{\Delta_i dS} - r(V_i - \Delta_i S)dt - \Delta_i DSdt$$

$$= \left\{\Theta_i + \frac{1}{2}\Gamma_i S^2\sigma^2 + (r-D)\Delta_i S - rV_i\right\}dt$$

The Black-Scholes equation when hedging and realized are both equal to σ_i, is dP&L[I,I,I]=0

$$\Theta_i + \frac{1}{2}\Gamma_i S^2\Sigma^2 + (r-D)\Delta_i S - rV_i = 0$$

So

$$d\text{P\&L(I,R,I)} = \frac{1}{2}\Gamma_i S^2(\sigma^2 - \Sigma^2)dt \qquad \text{Eq.5.1}$$

$$PV[\text{P\&L(I,R,I)}] = \frac{1}{2}\int_{t_0}^{T}\Gamma_i S^2(\sigma^2 - \Sigma^2)e^{-r(t-t_0)}dt \qquad \text{Eq.5.2}$$

The P&L is highly path-dependent. Although the hedging strategy captures a value proportional to $(\sigma^2 - \Sigma^2)$, it depends strongly on moneyness. We saw this before.

Cumulative P&L along 10 random stock paths, 100 hedging steps to expiration

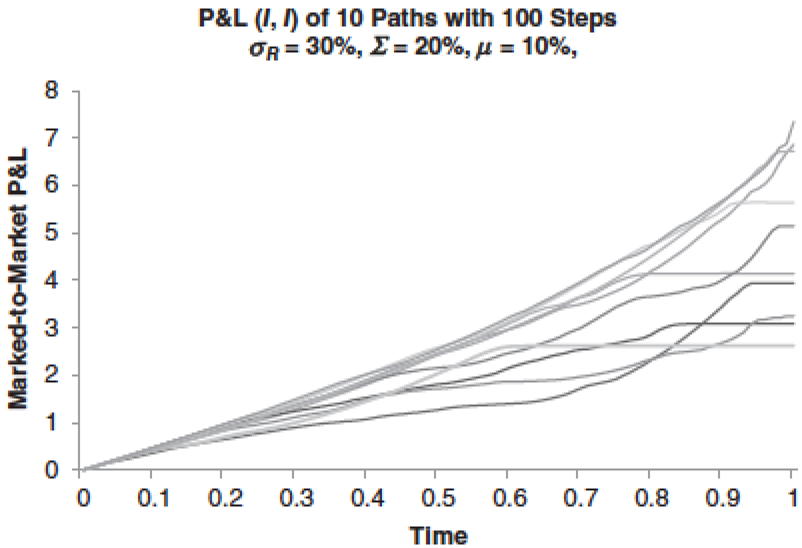


FIGURE 5.4 Hedging with Implied Volatility: Cumulative Discounted P&L with 100 Steps, Drift = 10%, Time to Expiration = 1 Year

most P&L when gamma stays large

with high drift the gamma quickly becomes close to zero

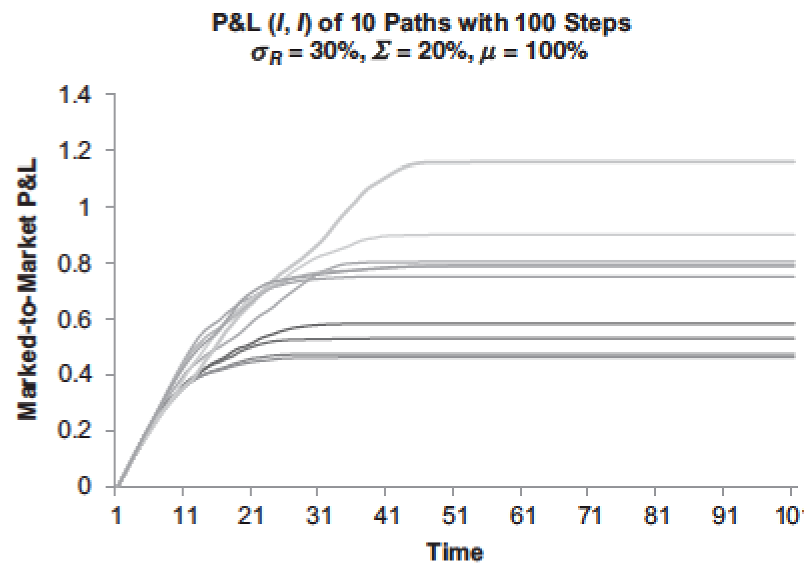


FIGURE 5.5 Hedging with Implied Volatility: Cumulative Discounted P&L with 100 Steps, Drift = 100%, Time to Expiration = 1 Year

In practice, realized volatility isn't known in advance. A trading desk would most likely hedge at the constantly varying implied volatility which would move in synchronization with the recent realized volatility.

Summary

- In our BSM laboratory, we assumed that we could know future realized volatility with certainty.
- In fact, you know the implied volatility from the market price of the option, but you can only try to predict future volatility. Therefore, when you hedge an option, you usually have to choose between hedging at implied volatility and hedging using a guess for the future realized volatility.
- If you estimate future realized volatility correctly and hedge (or replicate) continuously at that volatility, your P&L will eventually capture the exact value of the option.
- But along the way your P&L will have a random component.
- Suppose you have lost money randomly by guessing a future realized volatility. Should you be allowed to continue? Who knows if you have guessed the realized volatility correctly.

- If implied volatility is not equal to realized volatility and you hedge continuously at implied volatility, your P&L will be path-dependent and unpredictable. The P&L will be a maximum when the gamma of the option is a maximum, which occurs when the stock price stays close to the strike price on its path to expiration.

5.9 Hedging at an Arbitrary Constant Volatility

We don't of course know the future volatility. Suppose we just choose some hedging volatility.
PV(I,R,H)

Buy an option at implied vol Σ, hedge it to expiration at volatility σ_h, while realized volatility σ_r

Table 2: Position Values when Hedging with an Arbitrary Volatility

Time	Option Position, Value	Stock Position, Value	Value of Cash Position	Net Position Value
t	$\vec{V}_i, V_i$	$-\Delta_h \vec{S}, -\Delta_h S$	$\Delta_h S - V_i = (\Delta_h S - V_h) + (V_h - V_i)$	0
t + dt	$\vec{V}_i, V_i + dV_i$	$-\Delta_h \vec{S}, -\Delta_h (S + dS)$	$(\Delta_h S - V_i)(1 + r dt) - \Delta_h D S dt$	$(V_i + dV_i - \Delta_h (S + dS)) + (\Delta_h S - V_i)(1 + r dt) - \Delta_h D S dt$

P&L:

$$dP\&L(I,R,H) = dV_i - \Delta_h dS - \Delta_h SDdt + \{(\Delta_h S - V_h) + (V_h - V_i)\}rdt$$

$$= dV_h - \Delta_h dS - \Delta_h SDdt + (dV_i - dV_h) + \{(\Delta_h S - V_h) + (V_h - V_i)\}rdt$$

$$= \left\{\Theta_h + \frac{1}{2}\Gamma_h S^2 \sigma_r^{\ 2} + (r-D)S\Delta_h - rV_h\right\}dt + (dV_i - dV_h) + (V_h - V_i)rdt$$

all the H terms　　　　　　　　　　the leftovers

Now the BS solution with σ_h satisfies the p.d.e $dP\&L[H,H,H] = 0$

$$\Theta_h + (r-D)S\Delta_h + \frac{1}{2}\Gamma_h S^2 \sigma_h^2 - rV_h = 0$$

Substituting this last equation into the previous one

$$dP\&L(I,R,H) = \frac{1}{2}\Gamma_h S^2\left(\sigma_r^{\ 2} - \sigma_h^2\right)dt + (dV_i - dV_h) + (V_h - V_i)rdt$$

$$= \frac{1}{2}\Gamma_h S^2\left(\sigma_r^{\ 2} - \sigma_h^2\right)dt + e^{rt}d\left\{e^{-rt}(V_i - V_h)\right\}$$

Delta is calculated at the "hedging" or "replication" volatility

Taking present values $e^{-r(t-t_0)}$ to time t_0 leads to

$$dPV(P\&L(I,R,H)) = e^{-r(t-t_0)}\frac{1}{2}\Gamma_h S^2\left(\sigma_r^{\ 2} - \sigma_h^2\right)dt + e^{rt_0}d\left\{e^{-rt}(V_i - V_h)\right\}$$

Integrate:

$$PV[\text{P\&L(I,H)}] = V_h - V_i + \frac{1}{2}\int_{t_0}^{T} e^{-r(t-t_0)}\Gamma_h S^2\left(\sigma_r^2 - \sigma_h^2\right)dt \qquad \text{Eq.5.3}$$

Note that $V_h = V_i$ have equal values at expiration. When σ_h is set equal to either σ_r or σ_i, Equation 5.3 reduces to our previous results.

Summary:

Hedge at implied: stochastic path-dependent P&L because of Γ, but at each instant it's deterministic because there is no dZ in the evolution. But depending on whether the stock goes up or down, the next increment to the P&L will depend upon the Γ at that new stock price. So it's locally deterministic but long-run uncertain.

Hedging at realized: the P&L is locally stochastic because of the dZ, but long run deterministic (assuming you know the realized volatility).

5.10 Hedging Errors from Discrete Hedging at $\Sigma = \sigma$

- Hedging perfectly and continuously at no cost is a Platonic ideal.
- In real life, you can rebalance the hedge only a finite number of times.
- You are mishedged in the intervals, and the P&L picks up a random component.
- The more often you hedge, the smaller the deviation from perfection.
- Transaction costs affect things, too, but that's later.

A Simulation Approach

You cannot hedge continuously, and therefore it is important to understand the errors that creep into your P&L when you hedge at discrete intervals. Some traders hedge at regularly spaced time intervals; others hedge whenever the delta changes by more than a certain amount. In what follows here we will discuss replication at regular time intervals.

Sample code: Evolving the stock through time on each path that corresponds to given drift and vol:

```
Stockprice(i,:) = Stockprice(i-1,:).*exp((mu-div_rate-0.5*sig_act^2)*dt + sig_act*sqrt(dt)*Z(i-1,:))
```

Summing up all the gamma contributions along each path:

```
Time_integral = Time_integral + Stockprice(i,:).*(Delta(i,:) - Delta(i-1,:))*exp(rate *(time_to_exp))
```

Calculating initial call value from the path integral in Equations 5.10:

$$(C_0 - \Delta_0 S_0)e^{r\tau} = (C_T - \Delta_T S_T) + \text{Time\_integral}$$

Because we hedge or replicate discretely, different paths will give different values and so we'll get a histogram of call values, and an average with a standard deviation, the hedging error.

Monte Carlo: ATM option, expiration 1 month, the realized volatility is 20%, $\mu = r = 0$, hedged or replicated at an implied volatility of 20% equal to the realized volatility. $\sigma_r = \sigma_i = 0.2$

Plot: Relative P&L = Present Value of Payoff – BSM Fair Value.

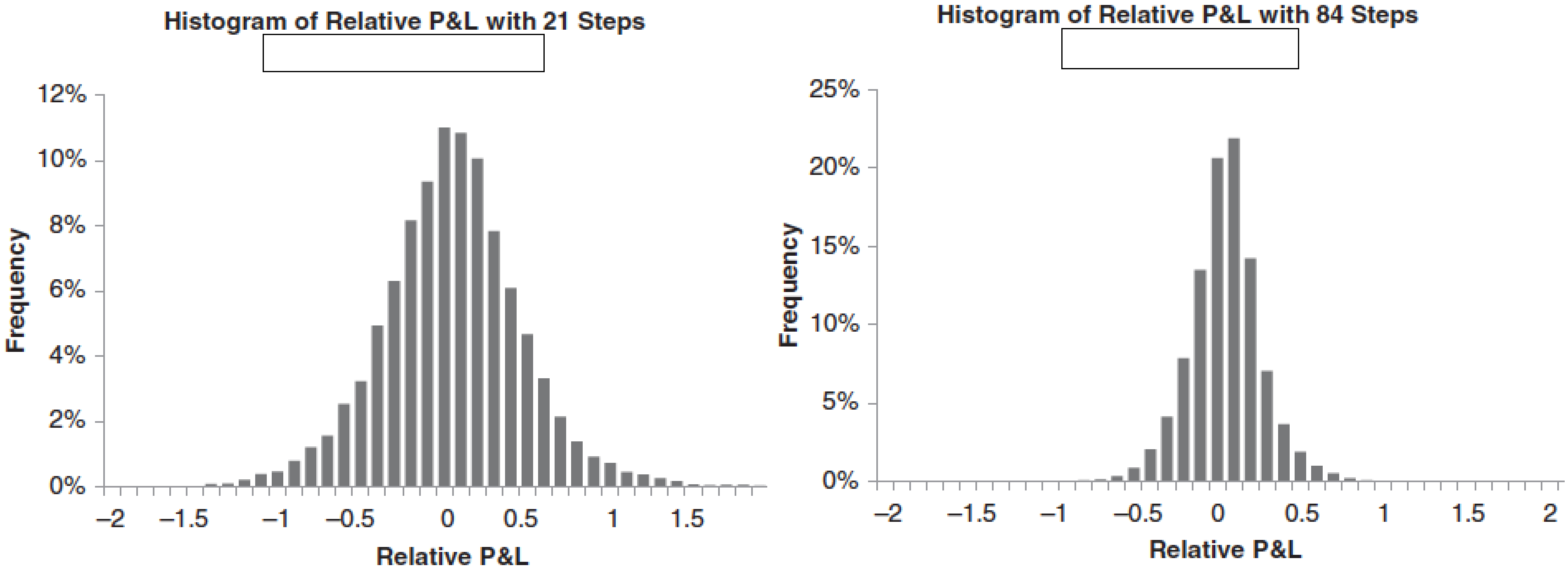


FIGURE 6.1 Distribution of Relative P&L for a One-Month At-the-Money Call Option When Hedging Volatility = Realized Volatility, $\mu = r$ (Relative P&L = Present Value of Payoff – BSM Fair Value)

The mean P&L is zero; When we quadruple the number of hedgings, the standard deviation of the P&L halves. We do better by hedging more frequently.

Now let's see what happens $\sigma_i \neq \sigma_r$.

Choose a realized volatility of 20% and replication volatility of 40% as the hedging volatility, that is, as the volatility used to calculate the value of Δ.

No longer the same reduction in standard deviation when the number of rebalancings quadruple. Both distributions are more or less symmetric though.

Hedging more often doesn't help because I am valuing and hedging at implied and so it's already path dependent and number of steps don't really matter.)

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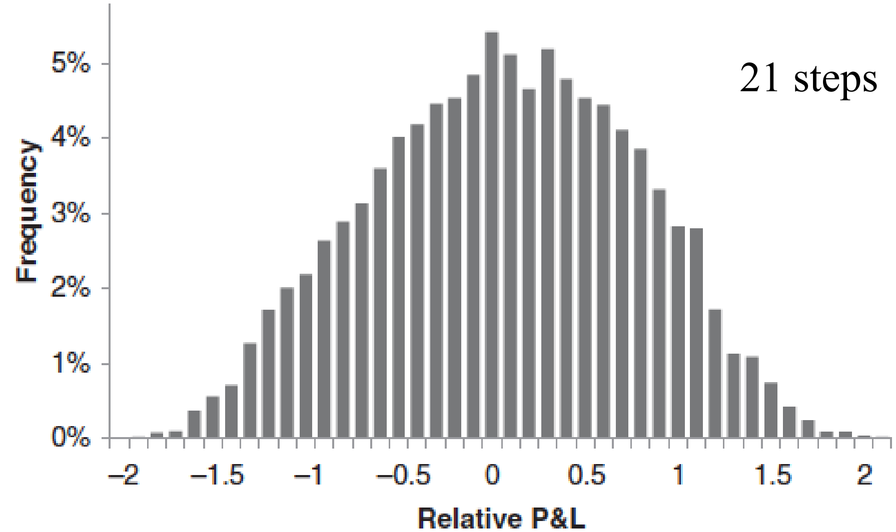

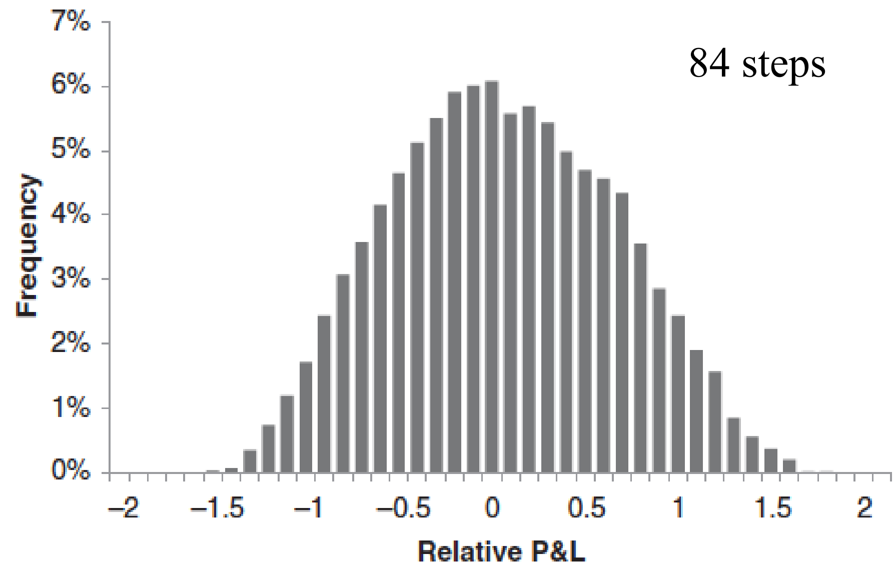

FIGURE 6.2 Distribution of Relative P&L for a One-Month At-the-Money Call Option When Hedging Volatility ≠ Realized Volatility, $\mu = r$ (Relative P&L = Present Value of Payoff – BSM Fair Value)