PROBABILITY THEORY

Mathematics GU4155 SPRING 2023

SYLLABUS

This Syllabus is very tentative. You should consult the site at *CourseWorks* for the course on a regular basis, for the latest available information. Lecture notes, as well as homework assignments and solutions to the problems in them, will be posted on *CourseWorks* every week.

Probability spaces. Random Variables, their distribution functions and modes of convergence. Conditional Expectations, Independence. The laws of large numbers and iterated logarithm, the central limit theorem, large deviations. Random Walk. Markov Chains. Introduction to Martingales. Discussion of "paradoxes". Examples from analysis, queueing, finance, biology, combinatorics, information theory.

Prerequisites: A course on "Modern Analysis" at the level of W. RUDIN's "Principles of Mathematical Analysis" (McGraw Hill). Some exposure to an undergraduate course in Probability, say at the level, of R.B. ASH's "Basic Probability Theory" (Dover), is desirable but not necessary.

Required Textbook:

J.B. WALSH (2012) *Knowing the Odds: An Introduction to Probability*. Graduate Texts in Mathematics Vol. 139. American Mathematical Society. ISBN 9780821885321.

Recommended Textbook:

D. STIRZAKER (2003) *Elementary Probability*. Second Edition, Cambridge University Press. ISBN 9780521534284.

Jan. 17, 19: Walsh Chapters 1 and 2
Stirzaker Chapters 1 and 2 (and Chapter 0 if you're interested)

Historical overview of the subject. Preview of the course.
Sample space, sigma algebra of events. Definition and
properties of measure. Random Variables. Elementary examples.
Definition of Conditional Probability and of Independence.
Sigma Algebras generated by classes of sets. The Borel sets of the real line.

Homework #1: Solutions.

Jan. 24, 26: Walsh Chapter 3
Stirzaker sections 4.1-4.3

Measurable functions and their properties. Integrals of simple
functions. Distributions of random variables. Properties of
distribution functions. Examples of distributions: Bernoulli, Binomial,
Geometric, Multinomial, Gaussian, the DeMoivre-Laplace limit theorem.

Homework #2: Solutions.

Jan. 31, Feb. 2: Walsh Chapters 1-3
Stirzaker sections 7.1-7.2, 7.4-7.5

The Weak Law of Large Numbers for Binomial Random Variables.
The Poisson Distribution. The uniform and exponential distributions.
The Lebesgue Integral for measurable functions. Properties.
Composition, Change of Variable. Fatou's Lemma.
The Monotone and Dominated Convergence Theorems.

Homework #3: Solutions.

Feb. 7, 9: Walsh Chapters 4 and 5
Stirzaker sections 5.2-5.4, 6.4, 7.5

Comparison of Lebesgue and Riemann Integrals.
Independence and Conditioning, again. Law of total
Probability, Bayes Rule. Examples and Paradoxes.
Borel-Cantelli Lemmata, Applications.
Basic inequalities (Cebysev, Holder, Minkowski, Jensen).

Homework #4: Solutions.

Feb. 14, 16: Walsh Chapter 6
Stirzaker Chapter 6

Product Measure. Tonelli-Fubini Theorems.
Product form of expectation; additivity of variance under independence.
Convolution. Applications.
The Bernstein approach to the Weierstrass approximation theorem.

Homework #5: Solutions.

Feb. 21, 23: Walsh Chapter 6
Stirzaker Sections 5.6, 6.8
Billingsley "Probability and Measure", Chapter 1.

Construction of a Sequence of Independent Random Variables.
Rademacher Functions. Instances of Independence. The Inspection Paradox.
Simple Random Walk: Recurrence, Last Visits, Arc-Sine Distribution.

Homework #6: Solutions.

Feb. 28: Walsh Section 1.7, Chapter 4
Stirzaker Sections 5.6, 6.8

Simple Random Walk: Arc-Sine Distribution, The Gambler's Ruin Problem.
Modes of Convergence for Random Variables: Almost Sure, In Probability, In Distribution.

Homework #7: Solutions.

March 2: Mid-Term Examination

March 7, 9: Walsh: Chapter 6.

The Simple Random Walk. Basic properties.
Combinatorial Approach. Proof of the Strong Law of Large Numbers.
Vague Convergence of Probability Measures.
Proof of the Central Limit Theorem.

Homework #8: Solutions.

March 14, 16: Spring Break

March 21, 23:

Completion of the Proof of the Central Limit Theorem.
Construction of the Lebesgue-Stieltjes measure induced on the Borel subsets of the real line by a given distribution function.
Lebesgue measure and its properties.
Statement of the Caratheodory extension theorem.

Homework #9: Solutions.

March 28, 30: Stirzaker: Sections 9.1 – 9.3 on Markov Chains.

The Skorohod Construction of Random Variables with given probability distribution function, from the Lebesgue measure on the unit interval.
Equivalence of distributional and vague convergence, revisited.
Laplace and Fourier transforms of distribution functions.

Homework #10: Solutions.

April 4, 6: Walsh Chapter 9.

Markov Chains: basic notions, transition probability matrices,
First passage times. The Markov property.

April 11, 13: Walsh Chapter 9.

Discrete-Parameter Martingales.
Filtrations. Stopping Times.
Optional Sampling and Convergence Theorems for Martingales.

Homework #11: Solutions.

April 18, 20: Walsh Chapter 10.

Ergodic Theory for Markov Chains.
Introduction to Brownian Motion.

April 25, 27: Walsh Chapter 10.

Topics in Brownian Motion and Diffusion.

MAY 5 – 12: FINAL EXAM PERIOD