Problem 1

a) $Var(X_{ij} \mid \mu, \tau^2, \sigma^2)$ is larger since θ_j provides more information than μ, τ^2.

b) $Cov(X_{i_1 j}, X_{i_2 j} \mid \theta_j, \sigma^2) = 0$ since they are independent.

$Cov(X_{i_1 j}, X_{i_2 j} \mid \mu, \tau^2, \sigma^2) > 0$ since they are from the same distribution.

c) $Var(X_{ij} \mid \theta_j, \sigma^2) = \sigma^2$

$$Var(X_{ij} \mid \mu, \tau^2, \sigma^2) = E[\sigma^2 \mid \mu, \tau^2, \sigma^2] + Var(\theta_j \mid \mu, \tau^2, \sigma^2)$$
$$= \sigma^2 + \tau^2$$

$$Cov(X_{i_1 j}, X_{i_2 j} \mid \mu, \tau^2, \sigma^2) = E[X_{i_1 j} X_{i_2 j} \mid \mu, \tau^2, \sigma^2] - [E(X_{ij} \mid \mu, \tau^2, \sigma^2)]^2$$
$$= E[\theta_j^2 \mid \mu, \tau^2, \sigma^2] - [E(\theta_j \mid \mu, \tau^2, \sigma^2)]^2$$
$$= \mu^2 + \tau^2 - \mu^2 = \tau^2$$

d) $\mu \to \vec{\theta}$, $\tau^2 \to \vec{\theta}$, $\vec{\theta} \to \vec{x}$, $\sigma^2 \to \vec{x}$ given $\vec{\theta}$, $\vec{x}, \sigma^2$ are independent of μ

$$f(\mu \mid \vec{\theta}, \sigma^2, \tau^2, \vec{x}) \propto f(\sigma^2 \mid \mu, \tau^2, \vec{\theta}, \vec{x}) \cdot f(\vec{x} \mid \mu, \tau^2, \vec{\theta}) \cdot f(\mu \mid \tau^2, \vec{\theta})$$
$$= f(\sigma^2 \mid \vec{x}) \cdot f(\vec{x} \mid \vec{\theta}, \sigma^2) \, f(\mu \mid \tau^2, \vec{\theta})$$
$$\propto f(\mu \mid \tau^2, \vec{\theta})$$

Problem 2

a) likelihood $= \prod_{i=1}^{n} p_i^{Y_i} (1-p_i)^{1-Y_i} = \prod_{i=1}^{n} \frac{e^{(\alpha+\beta x_i) Y_i}}{1+e^{\alpha+\beta x_i}}$