E4718 Midterm Examination 2
March 22, 2023

70 minutes, 75 points total
Formula sheet is on the last page.

Write your name: ______________________________

Write your UNI: _______________________________

Please sign the honor pledge:

I pledge that I have neither given nor received unauthorized aid during this examination.

Signature:______________________________________

Write your name and UNI on this examination sheet and on the blue book. Write your answers in the blue books and then hand back this entire examination with your blue books.

Note: Even if you cannot prove some result required in the first part of some problem, you can still use those results in subsequent parts of the problem.

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Problem 1: **[25 points]**

A stock S undergoes geometric Brownian motion with constant volatility. You can assume all option prices satisfy the Black-Scholes formula.

(i) Consider a derivative security V whose sensitivity to volatility is proportional to the square of the stock price. You can replicate it by means of a portfolio V consisting of European puts and calls of all strikes with expiration T:

$$V(S, t, A, \sigma, T) = \int_0^A \rho(K) P_K dK + \int_A^\infty \rho(K) C_K dK$$

where P_K are puts with strikes K below A and C_K are calls with strikes K above A.

Show that in order that this portfolio's volatility sensitivity $\frac{\partial V}{\partial \sigma}$ be proportional to the square of the stock price S, you must choose options with a strike density $\rho(K)$ independent of K.

[15 points]

(ii) At expiration when $t = T$ and the terminal stock price is S_T, use the payoffs of puts and calls to find the payoff of the replicating portfolio in terms of S_T and A. **[10 points]**

Solution 1.

(i) $\boxed{V(S, t, S_0, \sigma, T) = \int_0^A \rho(K)P_K dK + \int_A^\infty \rho(K)C_K dK}$

Note that $\boxed{\frac{\partial C_K}{\partial \sigma} = \frac{S\exp(-d_1^2)}{\sqrt{2\pi}}\sqrt{T-t} \equiv Sf\left(\frac{K}{S}, \sigma, t, T\right)}$ where f(.) is a function of the ratio of K/S only, not K and S separately, and the same formula holds for puts.

$$\boxed{\begin{aligned}\frac{\partial V}{\partial \sigma} &= \int_0^A \rho(K)\frac{\partial P_K}{\partial \sigma}dK + \int_A^\infty \rho(K)\frac{\partial C_K}{\partial \sigma}dK \\ &= \int_0^A \rho(K)Sf\left(\frac{K}{S}, \sigma, t, T\right)dK + \int_A^\infty \rho(K)Sf\left(\frac{K}{S}, \sigma, t, T\right)dK \\ &= \int_0^\infty \rho(K)Sf\left(\frac{K}{S}, \sigma, t, T\right)dK\end{aligned}}$$

Change variables of integration to $\boxed{x = \frac{K}{S}}$ so that $\boxed{dK = Sdx}$

Then $\boxed{\frac{\partial V}{\partial \sigma} = S^2\int_0^\infty \rho(xS)f(x, \sigma, t, T)dx}$ which we want to be proportional to $\boxed{S^2}$.

This means that $\boxed{S^2\rho(xS) \sim S^2}$ and so $\boxed{\rho(xS) \sim 1}$ and therefore $\boxed{\rho(xS) \sim 1}$ or $\boxed{\rho(K) \sim 1}$

(ii) $\boxed{V(S, t, S_0, \sigma, T) = \int_0^{S_0} \rho(K)P_K dK + \int_{S_0}^\infty \rho(K)C_K dK}$ and so at expiration when $\boxed{t = T}$

$$V(S_T, T) = \int_0^A [K - S_T]\theta(K - S_T)dK + \int_A^\infty [S_T - K]\theta(S_T - K)dK$$

$$= \int_{S_T}^A [K - S_T]dK \quad \text{or} \quad \int_A^{S_T} [S_T - K]dK$$

for A > S_T for S_T > A

$$= \left(\frac{A^2 - S_T^{\,2}}{2} - S_T(A - S_T)\right) or \left(S_T(S_T - A) - \frac{S_T^{\,2} - A^2}{2}\right)$$

$$= \frac{(A - S_T)^2}{2} \quad \text{in either case}$$

Problem 2: Weak Static Hedging **[25 points]**

Assume that a stock price S satisfies arithmetic Brownian motion with zero risk-neutral interest rates and zero dividends:

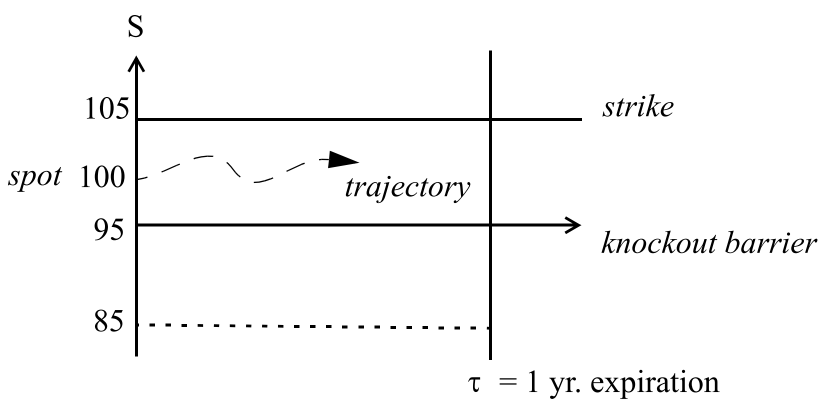
$$dS = \sigma dZ$$
$$S_t = S_0 + \sigma Z$$

where Z is a Brownian motion and $\sigma = 10$ is the annualized stock volatility.

(i) Explain briefly why $C(S, S+h) = P(S, S-h)$, where $C(S, K)$ and $P(S, K)$ are the values of a call and respectively, both with the same expiration. **[10 points]**

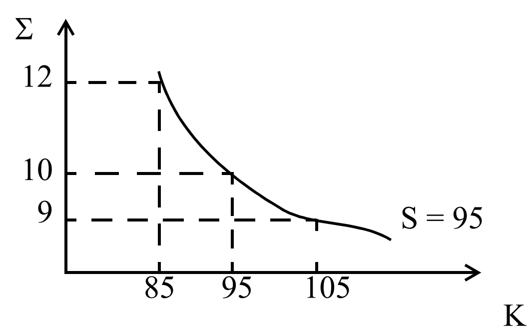
(ii) You are risk manager for the options desk. A trader on your desk sells a **down-and-out barrier call option C_{BK} with strike $K = 105$ and knockout barrier $B = 95$, with zero rebate.** The initial stock price is $S = 100$. There is an active market in **one-year standard calls C_{105} with strike 105 and one-year standard puts P_{85} with strike 85**. All options are and will be traded at their theoretical arithmetic-Brownian-motion value.

Find a replicating portfolio Π made out of standard calls and puts to replicate the payoff of the down and out barrier call C_{BK} above. **[5 points]**



(iii) Explain how you can use the portfolio Π to replicate the barrier option. **[5 points]**

(iv) Suppose initially you sold the barrier option and hedged it by purchasing the portfolio Π for zero net cost. Now after six months the stock hits the barrier at 95 and there is a volatility skew as shown at right. Do you make or lose money on the trade? **[5 points]**



Solution 2:

(i) In arithmetic Brownian motion there is equal probability of moving up to S+h and down to S-h from S. They payoffs of the call on the upside is symmetric with the put on the downside of S. Therefore the expected risk-neutral values are equal too.

(ii) The portfolio is $\boxed{\Pi = C_{105} - P_{85}}$.

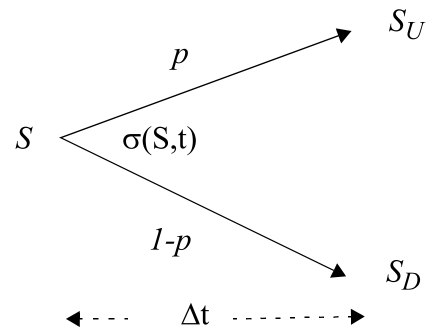
(iii) It has the payoff of the down and out call as long as stock never hits the barrier, because the call has the same strike as the down and out call, and the put will expire out of the money. If the stock hits the barrier, then by the symmetry of (i) above, the call and put have equal values anywhere on the barrier, and so the value of the portfolio is zero. As soon as the stock hits the barrier, you unwind the portfolio and are left with no position just when the barrier options knocks out.

(iv) $\boxed{\Pi = C_{105} - P_{85}}$ is the value of the portfolio on the barrier, when the down and out call knocks out at six months. Because of the skew, the call struck at 105 is now worth less than the put struck at 85, and so you lose money when you unwind the trade by selling the call and buying back the put.

Problem 3: **[25 points]**

A stock S that pays no dividends evolves according to
$\frac{dS}{S} = rdt + \sigma(S, t)dZ$ where r is the riskless interest rate.

Shown at right is a one-period binomial tree for the stock S.



(i) Show that, in order to match the stochastic differential equation, we must require that

$$p = \frac{F - S_D}{S_U - S_D} \qquad \text{and}$$

$$(S_U - F)(F - S_D) \approx \{S\sigma(S, t)\}^2 \Delta t$$

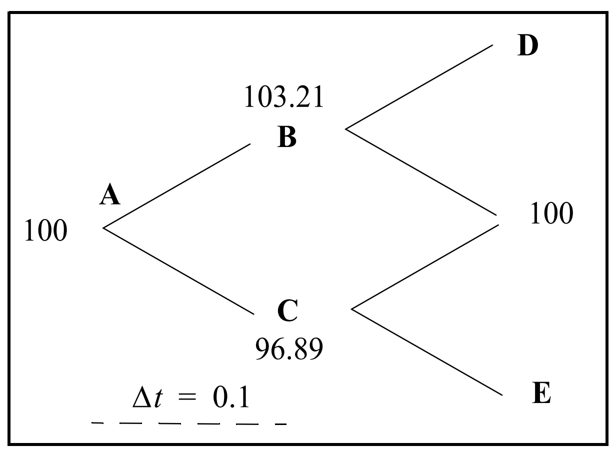
where F is the forward price of S at time Δt in the future.[[10 points]

(ii) A stock has a variable volatility $\sigma(S)$ that depends on stock price alone, and is given by

$$\sigma(S) = \max\left[0.1 + 2\left(\frac{S}{100} - 1\right), 0\right]$$

Suppose interest rates and dividends are zero, and that S = 100 initially. Shown at right is a two-period binomial tree with stock prices chosen so that the volatility at the central nodes of the tree (when S = 100) is 0.1. Each period Δt is 0.1 years.

Find the stock price at node D and node E such that the local volatility at each period in the tree is consistent with the stock-dependent volatility specified above.



[15 points]

Suggested Solution to Problem1:

(i): Risk-neutral condition on returns:

$$\boxed{F = pS_u + (1-p)S_d}$$

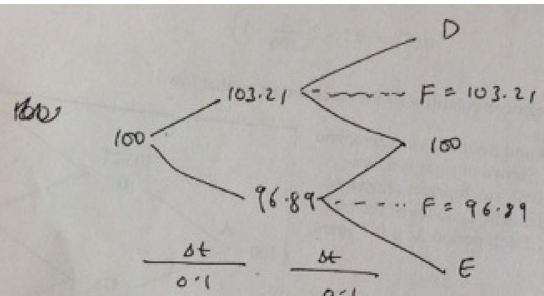
From Ito, the variance of the stock over time $\boxed{dt}$ is $\boxed{(dS)^2 = \sigma^2(S,t)S^2 dt}$, so that we must require *approximately*, to leading order in $\boxed{\Delta t}$, that

$$\boxed{S^2\sigma^2\Delta t = p(S_u - F)^2 + (1-p)(S_d - F)^2}.$$

Solving we obtain

$$\boxed{\begin{aligned} p &= \frac{F - S_d}{S_u - S_d} \\ (F - S_d)(S_u - F) &= S^2\sigma^2(S,t)\Delta t \end{aligned}}$$

(ii) Using the formulas



$\sigma(103.21) = 0.1 + 2(1.0321 - 1) = .1642$ (16.4%)

$\sigma(96.89) = 0.1 + 2(.9689 - 1) = .0378$ (3.8%)

$$S_D = F + \frac{(103.21)^2(.164)^2(0.1)}{F - S_D} = 103.21 + \frac{(103.21)^2(.164)^2(0.1)}{3.21} = 103.21 + 8.92 = \underline{112.14}$$

$$S_E = F - \frac{(96.89)^2(.0378)^2(0.1)}{(100 - 96.89)} = 96.89 - .43 = \underline{96.46}$$

Formulas

The **Black-Scholes formula** for a European call option at time t with strike K expiring at time T on a non-dividend-paying stock of price S with future volatility σ and a continuously compounded riskless rate r is given by

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

$$C_{BS}(S, t, K, T, r, \sigma) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$d_{1,2} = \frac{\ln(S_F/K) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$S_F = e^{r(T-t)}S$$

$$N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \exp\left(\frac{-y^2}{2}\right)dy$$

You may find these formulas useful too:

$$S_F N'(d_1) = KN'(d_2) \text{ where } N'(x) \equiv \frac{dN}{dx}.$$

$$\frac{\partial C_{BS}}{\partial K} = -e^{-r(T-t)}N(d_2) \qquad \frac{\partial C_{BS}}{\partial S} = N(d_1)$$

$$\frac{\partial C_{BS}}{\partial \sigma} = SN'(d_1)\sqrt{T-t}$$

When you hedge a long position in an option V at implied volatility Σ, the instantaneous P&L during time dt is given by

$$d\text{P\&L} = \frac{1}{2}\Gamma S^2(\sigma^2 - \Sigma^2)dt$$

where σ is the realized volatility of the stock S and $\Gamma = \frac{\partial^2 V}{\partial S^2}$ evaluated at the implied volatility.