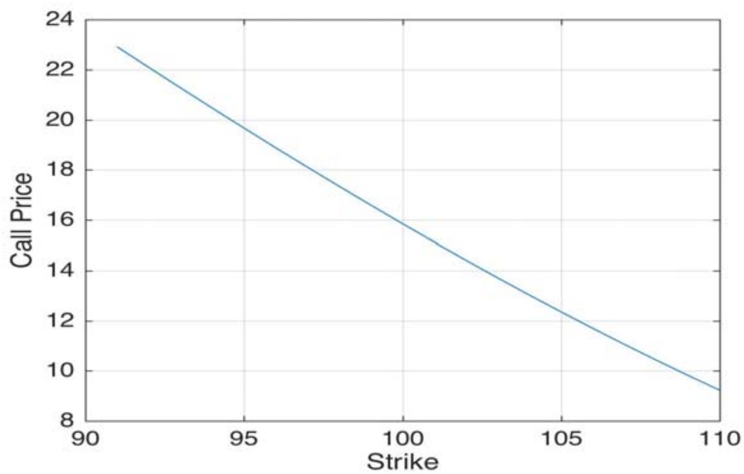
Homework 1: Wed Jan 18 2023

Due Monday January 30 2023 before midnight **Total 100 points**

Problem 1: The Smile **[20 points]**

The graph below shows call prices as a function of strike *K* for one-year call options on a stock whose price is 100. Interest rates and dividend yields are zero.



The data that corresponds to this graph is:

Strike: 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110

CallPrice = 22.91 22.09 21.27 20.47 19.67 18.88 18.11 17.34 16.59 15.85 15.12 14.41 13.71 13.02 12.35 11.69 11.05 10.43 9.82 9.23

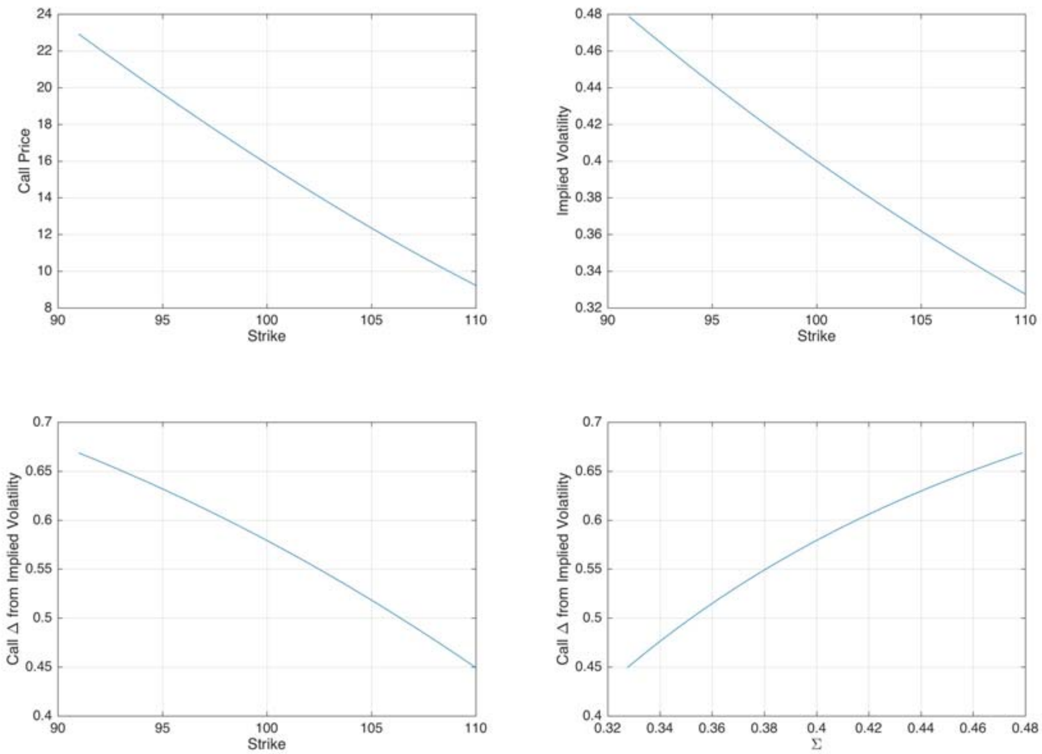
Produce three graphs of:

(i) Implied Black-Scholes call volatilities for each option as a function of strike; [5]

(ii) the Delta of the call options as a function of strike; and [10]

(iii) the Delta of the call options as a function of the implied volatility of the call. [5]

Problem 1. The Smile. Solution.



Problem 2. Another Smile **[20 points]**

Consider the smile for one-year European options on a stock with price 100, given by the Black-Scholes implied volatility formula:

$$\Sigma(K) = 0.3\exp\left[-5\left(\frac{K}{100}-1\right)\right]$$

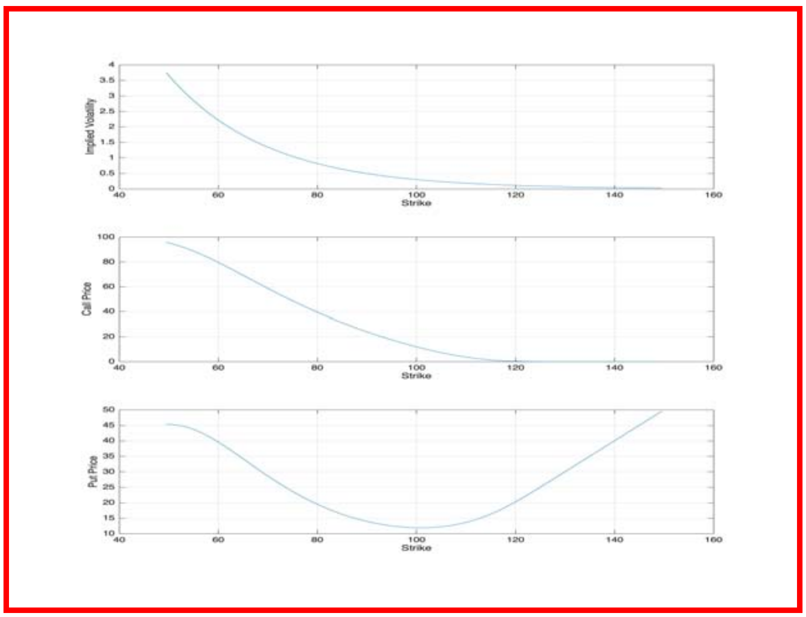
Assume interest rates and dividend yields are zero.

(a) Calculate and plot the values of call options and put options on this stock for strikes prices between 50 and 150. [10]

(b) What would you do if you saw these prices in the market? [10]

Solution 2:

(a)



(b) The volatility increases quite sharply as the strike drops. Thus puts with strike lower than 100 increase in value. But a put with a lower strike should always be worth less than a put with a higher strike. Thus, if this phenomenon existed in the market, you could buy the higher strike put, sell the lower strike put, pocket the difference in price and be assured that at expiration you would not have to pay any money since your high strike put would have at least the payoff of the lower strike put. This is of course a violation of the principle of no riskless arbitrage.

Problem 3: **SVI Parameterization** **[10 points]**

A commonly used parameterization of the implied volatility smile for a fixed expiration T is given by the following expression for the variance:

$$\sigma_I^2(m, T) = a + b[\rho(m-c) + \sqrt{(m-c)^2 + \theta^2}]$$

where σ_I is the Black-Scholes implied volatility function, $m = \ln\frac{K}{F}$ is the forward log moneyness, K is the strike, $F = Se^{r\tau}$ is the forward price of the underlying stock with zero dividend yield, and τ is the time to expiration of the option. The interest rate is r.

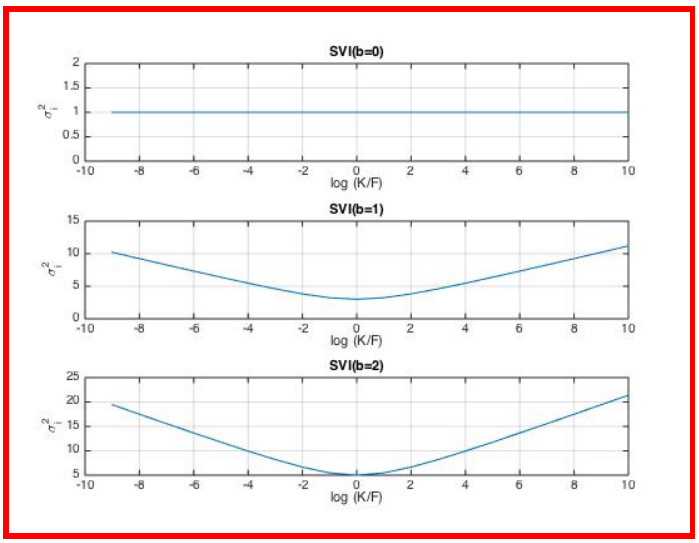
Explore the formula as a function of m in the range $(-\infty, \infty)$ by picking a few values for a,b,c,ρ, θ and plotting some graphs, or by examining the formula. You will see that, for example,

a determines the general level of implied volatilities.

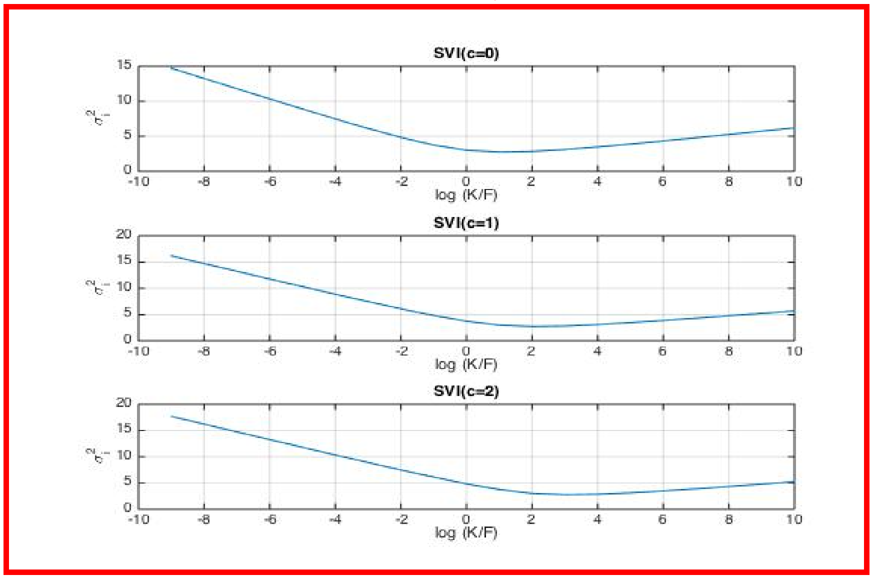
Write four more similar qualitative sentences, one for each of the variables b,c,ρ, θ, explaining very roughly what effect the variation of that parameter has on the shape of the smile. The variable ρ lies in [-1,1].

Solution 3.

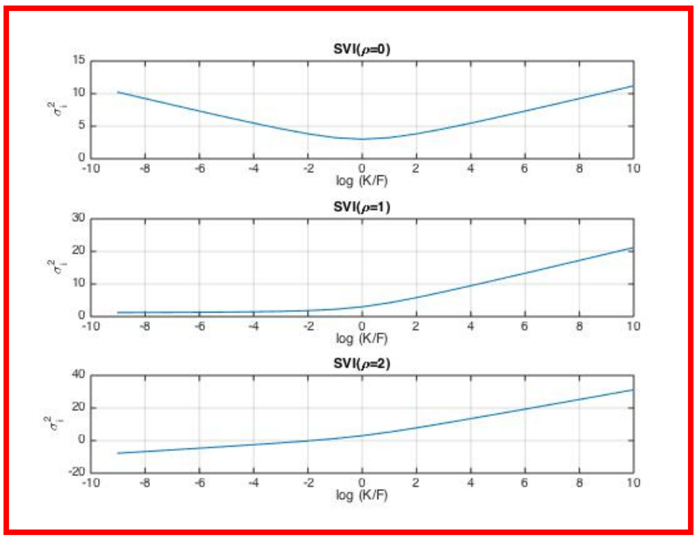
b approximately determines the magnitude of the smile effect, the deviation from flatness.



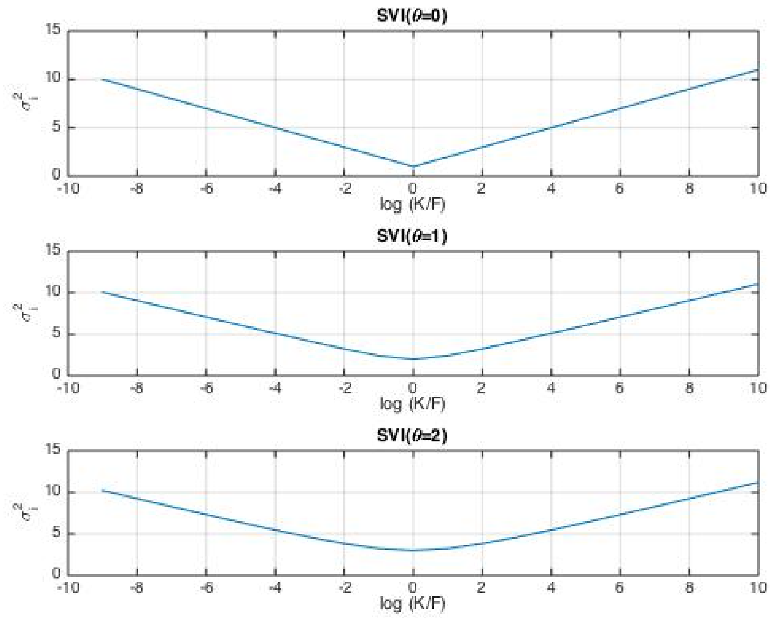
c moves the minimum/vertex of the smile horizontally.



ρ affects the angle between the left and right branches.



$\boxed{\theta}$ affects the convexity or curvature of the smile near the vertex.



Problem 4. Uncertain Volatility **[20 points]**

Suppose we know that the volatility of a Black-Scholes call will be either 15% or 25% over the next year, with equal probability for each possibility.

A reasonable model-guess for the value of the option today could be the average

$$C_{guess}(S, K, \tau) = 0.5C_{BS}(S, K, \tau, 0.25) + 0.5C_{BS}(S, K, \tau, 0.15)$$

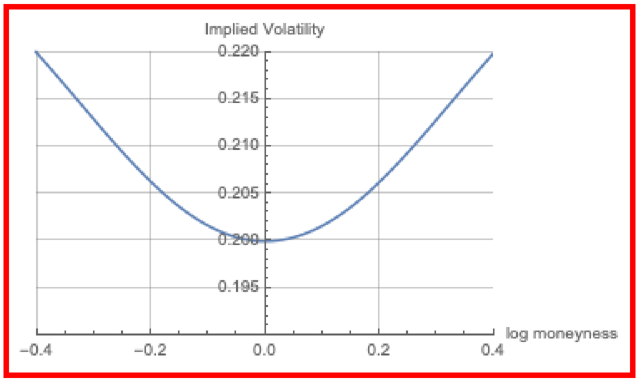
where S is the stock price, K the strike, and τ the time to expiration, and $C_{BS}(\ )$ is the usual Black-Scholes function for a call option.

Assume a one-year option with zero dividends and zero interest rates. Plot the Black-Scholes implied volatility of $C_{guess}(\ )$ for $S = 100$ as a function of the log moneyness $m = \ln\frac{K}{100}$ for the range $m = [-0.4, 0.4]$.

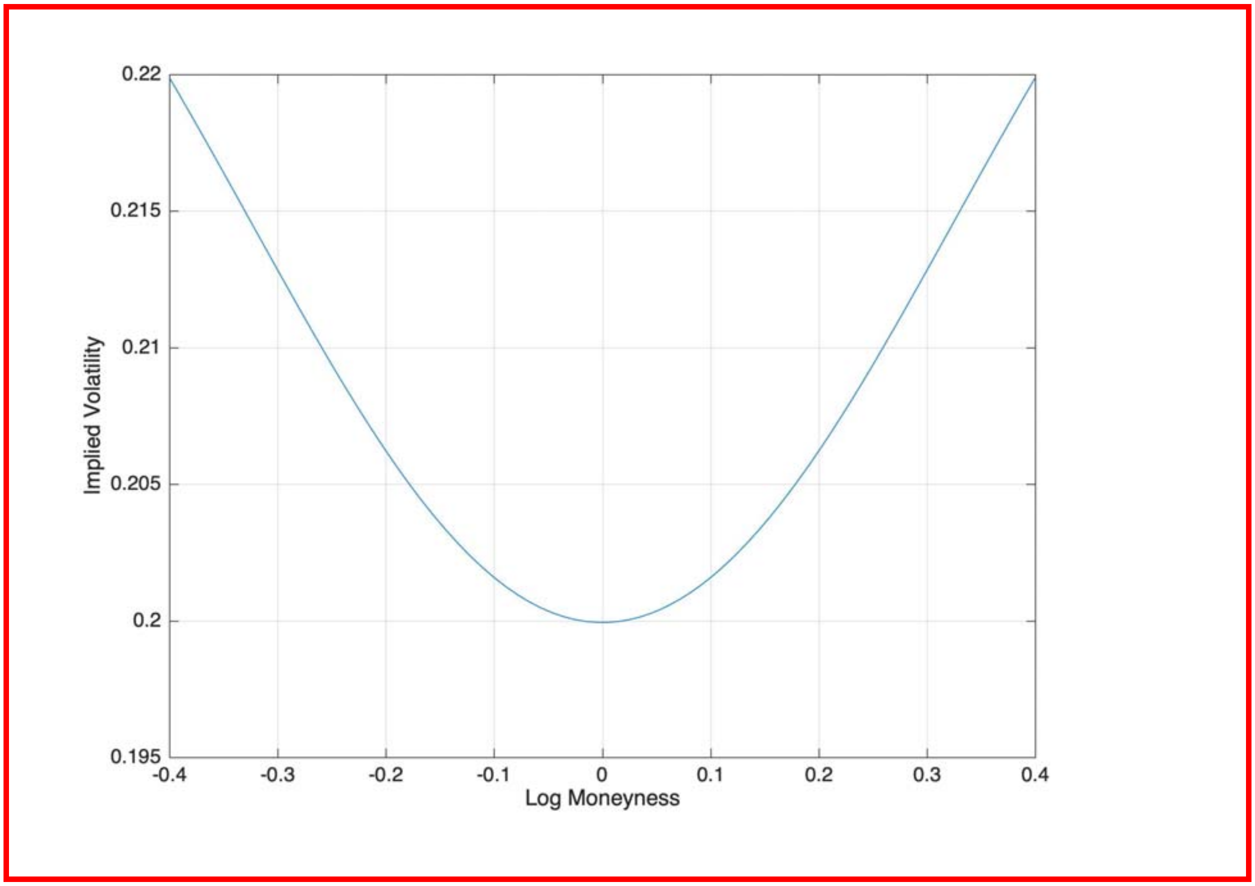
Compare your plot of the implied volatilities of $C_{guess}(\ )$ for different strikes to the average volatility $0.2 = 0.5(0.25) + 0.5(0.15)$.

Suggested Solution

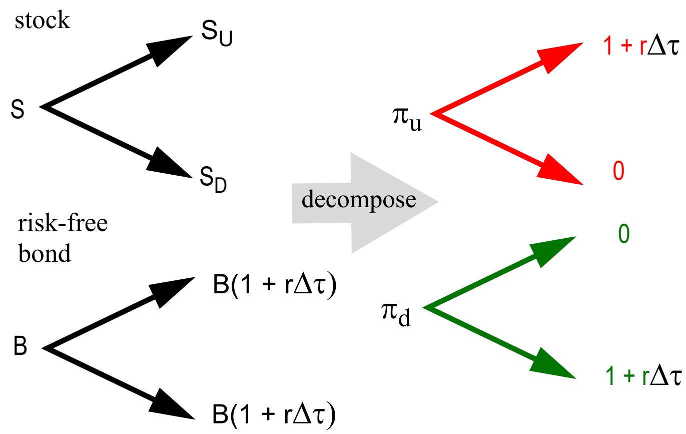
From Mathematica



From Matlab



Problem 5. State securities in a binomial model **[20 points]**



Shown above are simple binomial trees for a stock and a riskless bond over one period Δt. Find the current linear combination π_u of the stock S and the bond B whose value at a time Δt later is $(1 + r\Delta t)$ if the stock moves up and zero if it moves down.

(i) Show that the initial value of π_u is $\dfrac{(1 + r\Delta t)S - S_D}{S_U - S_D}$. **[8]**

(ii) Find the value of the security π_d whose value a time Δt later is zero if the stock moves up and $(1 + r\Delta t)$ if it moves down. **[8]**

(iii) Give a simple explanation of the value of $\pi_u + \pi_d$. **[4]**

Solution 5. State securities

(i) Write the security π_u as the linear combination of the securities S and B, the stock and the bond, so that $\boxed{\pi_u = \alpha S + \beta B}$ where α and β are coefficients to be solved for.

Then, in the up state, we require that $\boxed{\pi_u = \alpha S_u + \beta B(1 + r\Delta t) \equiv 1 + r\Delta t}$ and in the down state we have that $\boxed{\pi_d = \alpha S_d + \beta B(1 + r\Delta t) \equiv 0}$. Solve these two equations for α and β to get the result you were asked to derive.(ii) Similarly you can show that the initial value of $\boxed{\pi_d}$ is $\boxed{\dfrac{S_U - (1 + r\Delta t)S}{S_U - S_D}}$.

(iii) $\boxed{\pi_u + \pi_d = 1}$ which must follow from the fact that the portfolio of the two securities has the payoff of a riskless bond worth \$1 today.

Problem 6. Black-Scholes Call Value Approximation **[10 points]**

The Black-Scholes call price for a stock with zero dividend yield is given by

$$C = SN(d_1) - Ke^{-r(T-t)}N(d_2) \text{ where } \qquad d_{1,2} = \frac{\ln(S/(Ke^{-r(T-t)})) \pm 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}} \quad .$$

Show that when the strike is equal to the stock's forward price (an at-the-money forward option), and when $\sigma\sqrt{(T-t)} \ll 1$. then you can approximate the call price by $C \approx 0.4S\sigma\sqrt{T-t}$.

Solution 2. Black-Scholes Call Value

The Black-Scholes call price is given by

$$\boxed{C = SN(d_1) - Ke^{-r(T-t)}N(d_2)}$$

where $\boxed{d_{1,2} = \frac{\ln(S/(Ke^{-rt})) \pm 0.5\sigma^2(T-t)}{\sigma\sqrt{T-t}}}$

Show that when the strike is equal to the stock's forward price (an at-the-money forward option), and when $\boxed{\sigma\sqrt{(T-t)} \ll 1}$. then you can approximate the call price by

$$\boxed{C \approx \frac{S\sigma\sqrt{T-t}}{\sqrt{2\pi}}}.$$

When the strike is at-the-money forward, $\boxed{\ln S/(Ke^{-r(T-t)}) = 0}$, and so $\boxed{d_{1,2} = \pm(0.5\sigma)\sqrt{T-t}}$ which is small.

Then
$$\boxed{\begin{aligned} N(d_1) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(0.5\sigma)\sqrt{T-t}} e^{-y^2/2}\,dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{-y^2/2}\,dy + \frac{1}{\sqrt{2\pi}}\int_{0}^{(0.5\sigma)\sqrt{T-t}} e^{-y^2/2}\,dy \\ &\approx \frac{1}{2} + \frac{1}{\sqrt{2\pi}}(0.5\sigma)\sqrt{T-t} + \text{higher orders} \end{aligned}}$$

and similarly $\boxed{N(d_2) \approx \frac{1}{2} - \frac{1}{\sqrt{2\pi}}(0.5\sigma)\sqrt{T-t} + \text{higher orders}}$

so that $\boxed{C_{atm-fwd} = S([N(d_1)] - N(d_2)) \approx \frac{S\sigma\sqrt{T-t}}{\sqrt{2\pi}} \approx 0.4S\sigma\sqrt{T-t}}$