Week 5 HW - Dhyey Mavani

4.12 A branching process has offspring distribution $\boldsymbol{a} = (1/4, 1/4, 1/2)$. Find the following:

(a) μ.

(b) $G(s)$.

(c) The extinction probability. $(= e)$

(d) $G_2(s)$.

(e) $P(Z_2 = 0)$.

Sol

(a) $\mu = 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{4}\right) + 2\left(\frac{1}{2}\right) = \frac{1}{4} + 1 = \boxed{\frac{5}{4}}$ Ans

(b) $G(s) = \frac{1}{4} + \frac{s}{4} + \frac{s^2}{2}$

(c) to find the extinction probability we need to solve $G(s) = s$

$\Rightarrow 4s = 1 + s + 2s^2 \Rightarrow 2s^2 - 3s + 1 = 0$

$$s = \frac{3 \pm \sqrt{9-8}}{4} = \boxed{1, \frac{1}{2}}$$

The extinction probability is the lower of the two values of s which means $\boxed{e = \frac{1}{2}}$ Ans

(d) $G_2(s) = G(G(s)) = \frac{1}{4} + \frac{G(s)}{4} + \frac{(G(s))^2}{2}$

$$= \frac{1}{4} + \frac{1}{4}\left(\frac{1}{4} + \frac{s}{4} + \frac{s^2}{2}\right) + \frac{1}{2}\left(\frac{1}{4} + \frac{s}{4} + \frac{s^2}{2}\right)^2$$

$$= \left(\frac{1}{4} + \frac{1}{16}\right) + \frac{s}{16} + \frac{s^2}{8} + \frac{1}{2}\left[\left(\frac{1}{4} + \frac{s}{4}\right)^2 + \frac{s^4}{4} + 2\left(\frac{1}{4} + \frac{s}{4}\right)\left(\frac{s^2}{2}\right)\right]$$

$$= \boxed{\frac{1}{32}\left(4s^4 + 4s^3 + 9s^2 + 4s + 11\right)}$$ Ans

(e) $P(Z_2 = 0) = G(G(0)) = G_2(0) = \boxed{\frac{11}{32}}$ Ans

4.14 A branching process has offspring distribution with $a_0 = p$, $a_1 = 1 - p - q$, and $a_2 = q$. For what values of p and q is the process supercritical? In the supercritical case, find the extinction probability.

Sol

$$\left.\begin{aligned} a_0 &= p \\ a_1 &= 1-p-q \\ a_2 &= q \end{aligned}\right\} \quad \mu = 0(p) + 1(1-p-q) + 2(q) = \boxed{1+q-p}$$

The process is supercritical iff $\mu > 1$.

$$\Rightarrow \quad 1+q-p > 0 \quad \Rightarrow \quad \boxed{p<q}$$

Extinction Probability $(=e)$ can be found by solving $s = G(s)$

$$G(s) = a_0 + a_1 s + a_2 s^2 = p + (1-p-q)s + qs^2$$

$$G(s) = s \Rightarrow qs^2 - (p+q)s + p = 0 \Rightarrow s = \frac{(p+q) \pm \sqrt{(p+q)^2 - 4pq}}{2q}$$

$$\boxed{e = \frac{p}{q}} \Leftarrow \quad s = \frac{(p+q) \pm (p-q)}{2q} = 1 \text{ or } \frac{p}{q}$$

Ans

Because,

The extinction probability is the lower of the two values of s.

4.15 Assume that the offspring distribution is uniform on $\{0, 1, 2, 3, 4\}$. Find the extinction probability.

Sol uniform distribution implies each probability is $\boxed{\frac{1}{5}}$.

$G(s) = \frac{1}{5} + \frac{s}{5} + \frac{s^2}{5} + \frac{s^3}{5} + \frac{s^4}{5} \Rightarrow$ e can be obtained by looking for the smallest +ve root of $G(s) = s$

$\Rightarrow 5s = 1 + s + s^2 + s^3 + s^4$

$\Rightarrow s^4 + s^3 + s^2 - 4s + 1 = 0$

$\Rightarrow s = 1$ satisfies so,

$$
\begin{array}{r|l}
 & s^3 + 2s^2 + 3s - 1 \\
\hline
(s-1) & s^4 + s^3 + s^2 - 4s + 1 \\
 & -s^4 - s^3 \\
\hline
 & 2s^3 + s^2 - 4s + 1 \\
 & -\ 2s^3 - 2s^2 \\
\hline
 & 3s^2 - 4s + 1 \\
 & -\ 3s^2 - 3s \\
\hline
 & -s + 1 \\
 & -\ -s + 1 \\
\hline
 & 0
\end{array}
$$

$\Rightarrow s^3 + 2s^2 + 3s - 1 = 0$

$\Downarrow$

has one real solution of $\boxed{0.27568}$ which is indeed the extinction probability

Ans

4.17 For $0 < p < 1$, let $\boldsymbol{a} = (1-p, 0, p)$ be the offspring distribution of a branching process. Each individual in the population can have either two or no offspring. Assume that the process starts with two individuals.

(a) Find the extinction probability.

(b) Write down the general term P_{ij} for the Markov transition matrix of the branching process.

Sol

$$\bar{a} = (1-p, 0, p) \Rightarrow \begin{matrix} a_0 = 1-p \\ a_1 = 0 \\ a_2 = p \end{matrix}$$

(a)

$$\mu = (1-p)(0) + 0(1) + p(2) = 2p$$

$$\mu \leq 1 \Rightarrow \text{extinction probability} = 1$$

$$\Rightarrow 2p \leq 1 \Rightarrow \boxed{p \leq \frac{1}{2}}$$

for $p > \frac{1}{2}$, we need to solve $G(s) = s$...

$$G(s) = (1-p) + 0(s) + ps^2 = s$$

$$\Rightarrow ps^2 - s + (1-p) = 0$$

$$\Rightarrow s = \frac{1 \pm \sqrt{1 - 4p(1-p)}}{2p}$$

$$s = \frac{1 \pm \sqrt{4p^2 - 4p + 1}}{2p}$$

$$s = \frac{1 \pm (2p-1)}{2p} = \frac{2-2p}{2p} \text{ or } 1$$

$$= \frac{1-p}{p} \text{ or } 1$$

extinction probability

$$\Rightarrow \boxed{\underset{\text{(extinction probability)}}{e} = \begin{cases} \frac{1-p}{p} & , \forall p > \frac{1}{2} \\ 1 & , \forall p \leq \frac{1}{2} \end{cases}}$$ ANS

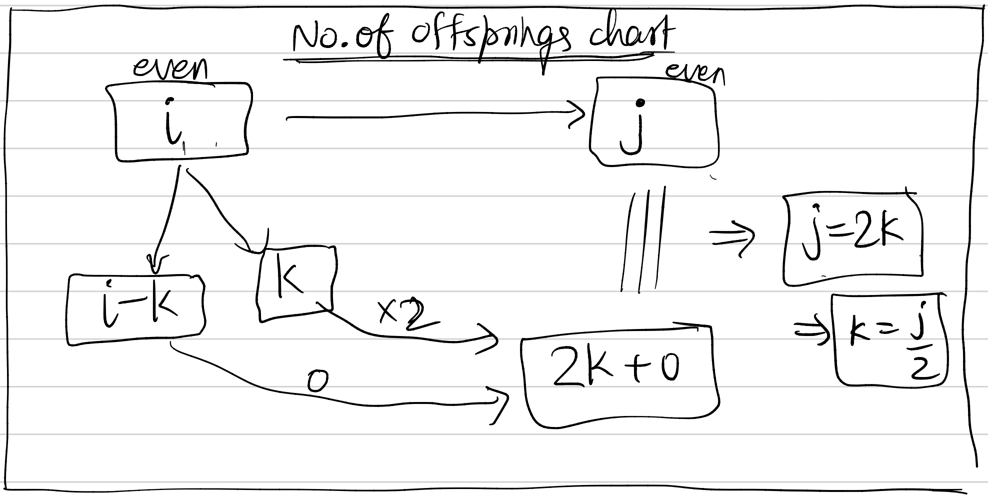
continued...

4.17 (b)

Firstly, we can say that we cannot have odd no. of offsprings in any generation because either 2 offsprings are born or no new offsprings are born.

$$\boxed{P_{i,j} = 0 \quad \text{for} \begin{cases} i = \text{even} \ \& \ j = \text{odd} \\ i = \text{odd} \ \& \ j = \text{even} \\ i = \text{odd} \ \& \ j = \text{odd} \end{cases}}$$ Ans

Moreover, in the case of i and j are even...



$$P_{i,j} \ (\forall i,j \in \text{even}) = \binom{i}{k} \cdot p^{k} \cdot (1-p)^{i-k}$$

$$\boxed{P_{ij} \ (\forall i,j \in \text{even}) = \binom{i}{j/2} \cdot p^{j/2} \cdot (1-p)^{i-(j/2)}}$$ Ans

But there is a special case of $\boxed{\begin{array}{c} P_{00} = 1 \\ \text{and} \\ P_{0j} \ (\forall j \in \mathbb{N}) = 0 \end{array}}$ Ans

4.31 R: Simulating a branching process whose offspring distribution is uniformly distributed on $\{0, 1, 2, 3, 4\}$.

(a) Use your simulation to estimate the probability that the process goes extinct by the third generation. Compare with the exact result obtained by numerical methods.

(b) See Exercise 4.15. Use your simulation to estimate the extinction probability e. Assume that if the process goes extinct it will do so by the 10th generation with high probability.

Sol// The simulations start on next page along with code to Calculate $G_3(0) = pgf(pgf(pgf(0)))$

(b) $\Rightarrow$ From exercise 4.15 the extinction probablity (e) would theoretically be 0.27568

(a) Moreover, we can see from the R code on next page that mean(sim1) $\approx$ $G_3(0)$], Hence Proved.