# Price Impact Models and Applications

## Introduction to Algorithmic Trading

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# Plan

**Last Week**
Empirical estimates of price impact.

**For this Week**

(a) Extend the solution to consider dynamic liquidity

(b) This generalization comes at a cost: potential for price manipulation

(c) Compute no-price manipulation conditions based on volume curves

**Next Week**
Application to optimal execution

# Last Week's Summary

**Price impact is universal**
Researchers have fit price impact models across public and proprietary data, across asset classes, and across time scales with broadly consistent results.

**Common sanity checks**

(a) Using the public trading tape, $R^2$ is in the double digits for short time scales.
(b) Magnitude of price impact is 30bps on liquid stocks for a typical day order.
(c) Price impact exhibits a time of day effect.
(d) Price impact is concave for sizable orders.

# Introduction

# What is Price Manipulation? (1/4)

**Definition from** `investor.gov`

> *"Market manipulation is when someone artificially affects the supply or demand for a security (for example, causing stock prices to rise or to fall dramatically)."*

**SEC's market manipulation case studies (2019)**
The SEC defines a common thread among manipulative trading patterns the agency looks out for as

> *""Bizarre Trading" - is there any sense in the trading you see?"*

Trading firms must be able to explain *why* their algorithms are trading.

# What is Price Manipulation? (2/4)

**Example of a prosecution**
Bill Hwang, Head of Archegos Capital, was indicted for manipulative trading.

**United States Attorney for the Southern District of New York**

> *"We allege that the defendants and their co-conspirators lied to banks to obtain billions of dollars, that they then used to inflate the stock price of a number of publicly traded companies."*

**Consequences for the defendants**
Bill Hwang could face up to 380 years in prison.

# What is Price Manipulation? (3/4)

**Manipulation steps**
"For example, as alleged, by March 24, 2021, Hwang effectively controlled more than 50% of the freely trading shares of Viacom [...] because, as alleged, by using various banks and brokerages for his swaps, Hwang made sure that no single institution would have any idea that he was behind all of this trading."

**Consequences for the market**
"Last year, when the prices fell, Hwang's positions were sold off and he could no longer manipulate the prices, and billions of dollars of capital evaporated nearly overnight."

# What is Price Manipulation? (4/4)

**There are many price manipulation strategies**
each leading to a different mathematical definition. Three examples follow.

**Round-trip price manipulation ("pump and dump")**
A round-trip trade whose expected profit is positive due to price impact alone.

**Transaction-triggered price manipulation ("printing")**
A buy trade whose impact costs is decreased through intermediate selling to increase perceived liquidity.

**Mark-to-market price manipulation ("pump only")**
A position that is only profitable mark-to-market due to price impact but cannot be profitably closed. Usually, these paper profits are used to commit other fraud (e.g., secure loans under false pretenses).

# Focus on Round-Trip Price Manipulation (1/2)

**Mathematical definition**
Let be $I$ an impact model with $I_0 = 0$. Then, a round-trip price manipulation strategy $Q$ is defined as a strategy such that

$$\mathbb{E}\left[\int_0^T I_t dQ_t + \frac{1}{2}[I, Q]_T\right] < 0$$

and $Q_0 = Q_T = 0$.

**In words,**
you shouldn't profitably trade unless you have alpha (or non-zero initial impact).

# Focus on Round-Trip Price Manipulation (2/2)

**A convenient lemma**
If the general trading problem

$$\sup_Q \mathbb{E}\left[\int_0^T (\alpha_t - I_t)\, dQ_t - \frac{1}{2}[I, Q]_T\right]$$

is strictly concave, then there is no round-trip price manipulation strategy.

**Why to avoid models with price manipulation:**

(a) They could be wrong: one loses money thinking they can "arbitrage" the market at any time. Algorithms accentuate the problem, by repeating the "arbitrage" continuously.

(b) **They could be right: one goes to jail.**

# Does the OW Model Admit Round-Trip Price Manipulation?

**Straightforward proof in impact space**
In impact space, the control problem is

$$\sup_{I} \mathbb{E}\left[\frac{\beta}{\lambda}\int_0^T \left(\alpha_t I_t - \beta^{-1}\alpha_t' I_t - I_t^2\right) dt + \frac{1}{\lambda}\left(\alpha_T I_T - \frac{1}{2} I_T^2\right)\right].$$

The pointwise optimization problem is

$$\alpha_t I_t - \beta^{-1}\alpha_t' I_t - I_t^2,$$

which is strictly concave in $I$. Same for the terminal problem

$$\alpha_T I_T - \frac{1}{2} I_T^2.$$

Therefore, there are no round-trip price manipulation opportunities in the OW model.

# Motivating Stochastic Liquidity

# Intuition: Time of Day Effect (1/3)

**Fruth, Schöneborn, and Urusov (BNP, 2013)**

> *"In financial markets, liquidity is not constant over time but exhibits strong seasonal patterns."*

**Cont et al. (2014)**

> *"Since the market depth follows a predictable pattern of intraday seasonality, the price impact coefficient must also have a predictable intraday pattern."*

**Therefore, trading algorithms behave differently at different times of the day.**
For instance, trading near the close *differs* from trading at noon: There is a lot of liquidity near the close as index funds want to "hit the close". Conversely, lunch time is when manual trading slows down.

# Intuition: Time of Day Effect (2/3)

**Banerji (2020, WSJ)**

*"Closing auctions have grown in volume over the past decade, in part because of the rising popularity of index funds, whose managers passively track indexes like the S&P 500 rather than actively seeking to pick stocks. These types of investments often use closing prices as a benchmark, leading their managers to execute trades at the end of the session.*

*As index funds have fueled a frenzy of trading at the close, other big investors have shifted much of their trading to the end of the day, taking advantage of the growing presence of big market participants."*

**Bernstone (2020, Credit Suisse)**

"People can bring [order imbalance data] into their models and be predictive about where they think the market is going to close."

# Intuition: Time of Day Effect (3/3)

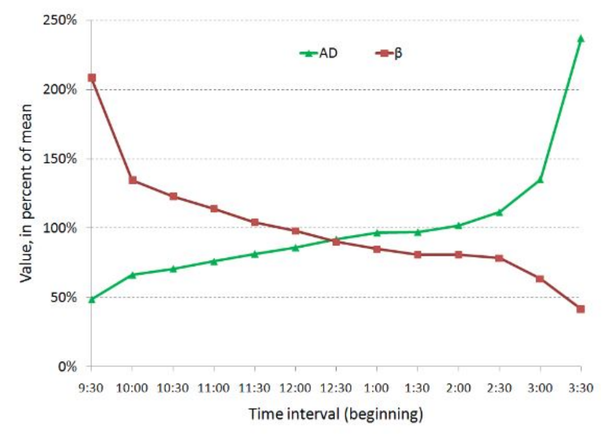


**Figure 1:** Order book depth (AD) and price impact coefficient ($\beta$) as a function of time of day. Source: Cont et al. (2014).

# Intuition: Stochastic Liquidity

**Fruth, Schöneborn, and Urusov (BNP, 2019)**

> *"In addition, there exist random changes in liquidity such as liquidity shocks that superimpose the deterministic evolution. To benefit from times when trading is cheap, institutional investors continuously monitor the available liquidity and schedule their order flow accordingly."*

**Therefore, trading algorithms react to liquidity surprises.**
An algo accelerates when liquidity floods the market. Conversely, it slows down during liquidity droughts.

# TWAP vs VWAP

**Time Weighted Average Price (TWAP) algorithm**
A TWAP targets an average price over an interval (e.g., the day). To achieve that price, a TWAP trades at a *constant speed* over the interval.

**Vime Weighted Average Price (TWAP) algorithm**
A VWAP targets a volume-weighted average price over an interval (e.g., the day). To achieve that price, a VWAP trades at a *speed proportional to market volumes.*

**VWAP is an example of a rule based algorithm.**
It intuitively assumes that price impact is inversely proportional to market volumes. Therefore, many trading teams spend sizable resources predicting future volumes.

# Volume Curves (1/2)

**Volume curves are trading signals**

(a) Non-directional

(b) Multi-time scale, ranging from milliseconds to days.

(c) Regular trigger times

(d) Often assumed exogenous, but could be endogenous ("Did we cause that volume? Are people copying our trades?")

(e) Usually cross-sectional (e.g. volume on a stock predicts volume on the sector)

# Volume Curves (2/2)

**Some approaches**
First, because volumes are always positive and have heavy tails, one usually predicts log-volumes. Four basic approaches are:

(a) Cluster stocks (e.g. sector, tick-size) together and use average intraday curves per cluster

(b) Predict future volumes based on recent volume surprises (e.g. ARMA or LSTM model)

(c) Predict future volumes based on recent cross-sectional volumes (reuse the clusters!)

(d) Use forward-looking data (e.g. order book, implied volatility, news, and stock events)

# Changing Clock

**A parsimonious way to model stochastic liquidity**
By default, algorithms measure time using the standard clock. One can instead use

(a) the *volume clock*, where one measures time by the cumulative notional traded on the public tape.
(b) the *trade clock*, where one measures time by the cumulative number of trades on the public tape.
(c) the *tick clock*, where one measures time by the cumulative number of ticks on the limit order book.

All three clocks express the same economic intuition:

*Price impact decays faster with market activity.*

# The Generalized OW Model

# Model Definition

**Fruth et al. (2019)**

$$dI_t = -\beta_t I_t dt + \lambda_t dQ_t.$$

**Example: Cont et al. (2013)**
Price impact is inversely proportional to order book depth $D_t$.

$$\beta_t = \beta; \quad \lambda_t = \frac{\lambda}{D_t}$$

# More Parametric Model Examples

**Example: Bussetti and Lillo (2012)**
Price impact is inversely proportional to market volumes $v_t$. Price impact decays in volume-time.

$$\beta_t = \beta v_t; \quad \lambda_t = \frac{\lambda}{v_t}$$

**Example: Muhle-Karbe, Wang, and Webster (2022)**
Price impact is inversely proportional to square-root of market volumes $v_t$.

$$\beta_t = \beta; \quad \lambda_t = \frac{\lambda}{\sqrt{v_t}}$$

# Control Problem

**Summary of the generalized OW model**
Given the price impact model

$$dI_t = -\beta_t I_t dt + \lambda_t dQ_t$$

a trader optimizes

$$\max_Q \mathbb{E}\left[\int_0^T (\alpha_t - I_t)\, dQ_t - \frac{1}{2}[I, Q]_T + [\alpha, Q]_T\right]$$

where

$$\alpha_t = \mathbb{E}\left[S_\tau - S_t \middle| \mathcal{F}_t\right]$$

is their alpha over horizon $\tau \geq T$.

**A small simplification**
Assume $\lambda_t = e^{\gamma_t}$ with $d\gamma_t = \gamma_t' dt$.

# Control Problem in Impact Space

**The problem can be mapped in impact space**
When $\alpha \in C^1$, the following myopic problem is equivalent to the original trading algo problem:

$$\sup_I \mathbb{E}\left[\int_0^T \frac{1}{\lambda_t}\left((\beta_t + \gamma_t')\,\alpha_t I_t - \alpha_t' I_t - \left(\beta_t + \frac{1}{2}\gamma_t'\right) I_t^2\right) dt - \frac{1}{2\lambda_T} I_T^2\right]$$

**Proof on the blackboard**

# Optimal trading strategy

**Target impact state**

$$I_t^* = \frac{\beta_t + \gamma_t'}{2\beta_t + \gamma_t'}\alpha_t - \frac{1}{2\beta_t + \gamma_t'}\alpha_t'.$$

There is still a myopic, linear relationship between impact, alpha, and alpha decay. The derivative $\gamma_t'$ of log-liquidity plays an essential role.

**Recovering the trading strategy**
Let $Q$ be a candidate trading process and $I$ its impact. Then,

$$Q_t = \frac{1}{\lambda_t} I_t + \int_0^t \frac{\beta_s + \gamma_s'}{\lambda_s} I_s ds,$$

## Implication for Alpha Research

**Regret Function with Stochastic Liquidity**
Let $Q$ be a candidate trading process and $I$ its impact. Let $Q^*$ and $I^*$ be the corresponding optima. Then,

$$\mathbb{E}\left[Y_T(Q^*)\right] = \mathbb{E}\left[\int_0^T \frac{2\beta_t + \gamma_t'}{2\lambda_t}\left(I_t^*\right)^2 dt\right],$$

and

$$\mathbb{E}\left[Y_T(Q)\right] = \mathbb{E}\left[Y_T(Q^*)\right] - \mathbb{E}\left[\int_0^T \frac{2\beta_t + \gamma_t'}{2\lambda_t}\left(I_t - I_t^*\right)^2 dt + \frac{1}{2\lambda_T} I_T^2\right].$$

# How Much Does Stochastic Liquidity Matter?

| $\alpha$ | strategy | order size (% of ADV) | impact (bps) | slippage (bps) |
|---|---|---|---|---|
| 10bps | OW | 3 | 7.0 | 6.1 |
| 10bps | gen. OW | 3 | 5.3 | 4.4 |
| 50bps | OW | 15 | 32 | 32 |
| 50bps | gen. OW | 15 | 27 | 26 |
| 100bps | OW | 30 | 64 | 63 |
| 100bps | gen. OW | 30 | 53 | 52 |

**Table 1:** Backtest of different alpha signal strengths and trading strategies when the true impact model is a generalized OW model.

# No Price Manipulation

# Can Dynamic Liquidity Trigger Price Manipulation?

**Fruth, Schöneborn, and Urusov (BNP, 2013)**

> *"Time-dependent liquidity can potentially lead to price manipulation. In periods of low liquidity, a trader could buy the asset and push market prices up significantly; in a subsequent period of higher liquidity, he might be able to unwind this long position without depressing market prices to their original level, leaving the trader with a profit after such a round trip trade."*

# Fruth's price manipulation scenario.

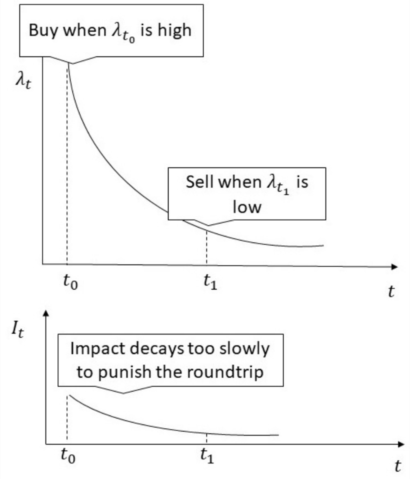


**Figure 2:** A race between increasing liquidity and impact reversion.

# Example: What if there is no Reversion?

**A stochastic Almgren and Chriss model**
The price impact model

$$dI_t = \lambda_t dQ_t$$

admits price manipulation opportunities whenever $\lambda_t$ decreases.

**Proof**
Suppose the trader buys $\Delta_t$ shares in a jump trade at time $t$ and sells them in another jump trade at time $t' > t$. Then, the corresponding trading profits are

$$\begin{aligned} &-\left(S_t + \frac{1}{2}\lambda_t\Delta_t\right)\Delta_t + \left(S_{t'} + \lambda_t\Delta_t - \frac{1}{2}\lambda_{t'}\Delta_t\right)\Delta_t \\ &= (S_{t'} - S_t)\Delta_t + \frac{1}{2}(\lambda_t - \lambda_{t'})\Delta_t^2. \end{aligned}$$

Without alpha, the first term has zero expectation. If $\lambda_t > \lambda_{t'}$, the second term is positive and yields a round-trip price manipulation trade.

# Example: Fruth et al. (2013)

**Case for the generalized OW model**

$$dI_t = -\beta_t I_t dt + \lambda_t dQ_t$$

with $\lambda_t = e^{\gamma_t}$ and $d\gamma_t = \gamma_t' dt$ admits price manipulation when $2\beta_t + \gamma_t' < 0$.

**Proof**
Suppose the trader buys $\Delta_t$ shares in a jump trade at time $t$ and sells them in another jump trade at time $t' > t$. Then, the corresponding trading profits are

$$\begin{aligned}
&-\left(S_t + \frac{1}{2}\lambda_t\Delta_t\right)\Delta_t + \left(S_{t'} + e^{-\int_t^{t'}\beta_s ds}\lambda_t\Delta_t - \frac{1}{2}\lambda_{t'}\Delta_t\right)\Delta_t \\
&= (S_{t'} - S_t)\Delta_t + \frac{1}{2}\lambda_t\left(2e^{-\int_t^{t'}\beta_s ds} - 1 - e^{\int_t^{t'}\gamma_s' ds}\right)\Delta_t^2 \\
&= (S_{t'} - S_t)\Delta_t - \frac{1}{2}\lambda_t\left(2\beta_t + \gamma_t'\right)(t' - t)\Delta_t^2 + o(t' - t).
\end{aligned}$$

# Fruth et al.'s No Price Manipulation Condition

$$2\beta_t + \gamma_t' > 0$$

The condition guarantees a strictly concave control problem and rules out price manipulation strategies.

**In words**
The condition states that liquidity can't increase too fast compare to the model's mean-reversion speed.

*Liquidity floods are the problem, not liquidity droughts!*

Therefore, parametric and non-parametric models for $(\beta_t, \lambda_t)$ must satisfy this condition to be implemented in live trading.

# Implications for Parametric Models

**Example: Cont et al. (2013)**

$$\beta_t = \beta; \quad \lambda_t = \frac{\lambda}{D_t}$$

Order book signals cannot predict $D_t$ increasing too fast relative to $\beta^{-1}$.

**Example: Muhle-Karbe, Wang, and Webster (2022)**

$$\beta_t = \beta; \quad \lambda_t = \frac{\lambda}{\sqrt{v_t}}$$

Volume signals cannot predict $v_t$ increasing too fast relative to $\beta^{-1}$.

# Application (1/2)

**Some parameter values**
For intraday trades, a good model is a halflife $\log(2)/\beta$ of one hour and

$$\lambda_t \propto \frac{1}{\sqrt{v_t}}.$$

**Rule for the above model class**

> *All else being equal, trading activity cannot double in less than fifteen minutes without introducing a price manipulation opportunity.*

# Application (2/2)

**For a trading team,**
this constrains volume signals used in trading strategies. Ignoring this constraint leads trading algorithms to submit nonsensical round-trip trades.

**For a regulator,**
this can be run on realized market volumes to find suspicious trades. It doesn't *prove* price manipulation, as the guilty trader would have had to

(a) successfully predict or caused the sudden liquidity flood (e.g., with misleading tweets).
(b) buy the stock prior to the liquidity flood and exit the trade during the flood.

But the criterion can help narrow the number of stocks and periods to investigate.

# Recent Example (1/2)

**SEC says social media influencers used Twitter and Discord to manipulate stocks**

The regulatory agency charged them in what it says was a $100 million securities fraud scheme run by people who portrayed themselves as successful stock traders.

**Figure 3:** Source: nbcnews.com on December 14, 2022.

**Allegations**

(a) Defenders bought small-cap, illiquid stocks (large $\lambda_t$).

(b) Stock price goes up due to their price impact.

(c) Defenders hyped up those stocks due to the recent gains.

(d) Followers buy (or sell) the stock. The manufactured trading activity massively increases the stock's liquidity.

(e) Defenders exit their trade during manufactured liquidity flood (small $\lambda_t$).

# Recent Example (2/2)

Throughout the round-trip trades, the defendents hid or lied about their actual positions and trades. For example, they claimed to still be holding and losing money when the stock reverted.

**Crucially, the SEC has a recording of a voice call between the defendents allegedly discussing the pump and dump strategy.**

> *"We're robbing f—— idiots of their money," Knight allegedly said.*

# Weekly Summary

**Liquidity is dynamic**
it exhibits time-of-day patterns, as well as stochastic behavior. Trading strategies react strongly to these liquidity fluctations.

**Beware of price manipulation**
Trading algorithms, especially complicated black-box algorithms, can easily be tricked into thinking there is a price manipulation opportunity.

**Round-trip trade manipulation**
happens when liquidity increases too fast. For the generalized OW model

$$dI_t = -\beta_t I_t dt + e^{\gamma_t} dQ_t$$

the no-price manipulation condition is

$$2\beta_t + \gamma_t' > 0.$$

**Questions?**

**Next week**
Applications to optimal execution include four steps:

(a) Communicate a pre-trade cost model to the portfolio team.
(b) Establish a strategic trading schedule based on alpha signals and liquidity models.
(c) Implement microstructure tactics as deviations from the strategic schedule.
(d) Update the schedule when a new order arrives.