Problem 5 : Way 1

$$P\Big(\sum_{i=1}^{n} \alpha_i W_i + \beta \,\Big|\, W\Big) \overset{\text{1st property}}{=} \mathbb{E}\Big(\sum_{i=1}^{n} \alpha_i W_i + \beta\Big) + a^T\big(W - \mathbb{E}(W)\big) \quad \text{—①}$$

where $\Gamma a = \mathrm{Cov}\Big(\sum_{i=1}^{n} \alpha_i W_i + \beta, W\Big)$

Solve for a:

$$\text{LHS} = \Gamma a = \mathrm{Cov}(W, W)\, a = \begin{bmatrix} \mathrm{Cov}(W_1, W_1) & \cdots & \mathrm{Cov}(W_1, W_n) \\ \vdots & & \vdots \\ \mathrm{Cov}(W_n, W_1) & \cdots & \mathrm{Cov}(W_n, W_n) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{n} a_i \mathrm{Cov}(W_1, W_i) \\ \vdots \\ \sum_{i=1}^{n} a_i \mathrm{Cov}(W_n, W_i) \end{bmatrix}$$

$$\text{RHS} = \mathrm{Cov}\Big(\sum_{i=1}^{n} \alpha_i W_i + \beta, W\Big) \underset{\substack{\uparrow \\ \text{linearity} \\ \text{of cov.}}}{=} \sum_{i=1}^{n} \alpha_i \mathrm{Cov}(W_i, W) = \begin{bmatrix} \sum_{i=1}^{n} \alpha_i \mathrm{Cov}(W_i, W_1) \\ \vdots \\ \sum_{i=1}^{n} \alpha_i \mathrm{Cov}(W_i, W_n) \end{bmatrix}$$

$\because \mathrm{Cov}(W_s, W_t) = \mathrm{Cov}(W_t, W_s) \quad \forall s, t$

$\therefore a_i = \alpha_i \quad \forall i = 1, \dots, n$ —②

So, $$P\Big(\sum_{i=1}^{n} \alpha_i W_i + \beta \,\Big|\, W\Big) \overset{①}{=} \sum_{i=1}^{n} \alpha_i \mathbb{E}(W_i) + \beta + \sum_{i=1}^{n} a_i W_i - \sum_{i=1}^{n} a_i \mathbb{E}(W_i)$$

$$\overset{②}{=} \cancel{\sum_{i=1}^{n} \alpha_i \mathbb{E}(W_i)} + \beta + \sum_{i=1}^{n} \alpha_i W_i - \cancel{\sum_{i=1}^{n} \alpha_i \mathbb{E}(W_i)} = \sum_{i=1}^{n} \alpha_i W_i + \beta$$

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Way 2