astic processes S_n. Both andom variables x_i and licated by their starting n increases, both pdfs attractor $P_G(S_\infty)$. The ce of $P(S_n)$ to $P_G(S_\infty)$ of the two families of ost important attractor exist, and we consider

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4

Lévy stochastic processes and limit theorems

In Chapter 3, we briefly introduced the concept of stable distribution, namely a specific type of distribution encountered in the sum of n i.i.d. random variables that has the property that it does not change its functional form for different values of n. In this chapter we consider the entire class of stable distributions and we discuss their principal properties.

4.1 Stable distributions

In §3.2 we stated that the Lorentzian and Gaussian distributions are stable. Here we provide a formal proof of this statement.

For Lorentzian random variables, the probability density function is

$$P(x) = \frac{\gamma}{\pi} \frac{1}{\gamma^2 + x^2}. \tag{4.1}$$

The Fourier transform of the pdf

$$\varphi(q) \equiv \int_{-\infty}^{+\infty} P(x) e^{iqx} dx \tag{4.2}$$

is called the characteristic function of the stochastic process. For the Lorentzian distribution, the integral is elementary. Substituting (4.1) into (4.2), we have

$$\varphi(q) = e^{-\gamma|q|}. \tag{4.3}$$

The convolution theorem states that the Fourier transform of a convolution of two functions is the product of the Fourier transforms of the two functions,

$$\mathscr{F}\left[f(x) \bigotimes g(x)\right] = \mathscr{F}[f(x)]\mathscr{F}[g(x)] = F(q)G(q). \tag{4.4}$$

For i.i.d. random variables,

$$S_2 = x_1 + x_2. \tag{4.5}$$

The pdf $P_2(S_2)$ of the sum of two i.i.d. random variables is given by the convolution of the two pdfs of each random variable

$$P_2(S_2) = P(x_1) \bigotimes P(x_2), \tag{4.6}$$

The convolution theorem then implies that the characteristic function $\varphi_2(q)$ of S_2 is given by

$$\varphi_2(q) = [\varphi(q)]^2. \tag{4.7}$$

In the general case,

$$P_n(S_n) = P(x_1) \bigotimes P(x_2) \bigotimes \cdots \bigotimes P(x_n), \tag{4.8}$$

where S_n is defined by (3.1). Hence

$$\varphi_n(q) = [\varphi(q)]^n. \tag{4.9}$$

The utility of the characteristic function approach can be illustrated by obtaining the pdf for the sum S_2 of two i.i.d. random variables, each of which obeys (4.1). Applying (4.6) would be cumbersome, while the characteristic function approach is quite direct, since for the Lorentzian distribution,

$$\varphi_2(q) = e^{-2|q|\gamma}. \tag{4.10}$$

By performing the inverse Fourier transform

$$P(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(q) e^{-iqx} dq, \tag{4.11}$$

we obtain the probability density function

$$P_2(S_2) = \frac{2\gamma}{\pi} \frac{1}{4\gamma^2 + x^2}. \tag{4.12}$$

The functional form of $P_2(S_2)$, and more generally of $P_n(S_n)$, is Lorentzian. Hence a Lorentzian distribution is a stable distribution.

For Gaussian random variables, the analog of (4.1) is the pdf

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}. \tag{4.13}$$

The characteristic function is

$$\varphi(q) = e^{-(\sigma^2/2)q^2} = e^{-\gamma q^2}, \tag{4.14}$$

where $\gamma \equiv \sigma^2/2$. Hence from (4.7),

$$\varphi_2(q) = e^{-2\gamma q^2}. \tag{4.15}$$

By performing the inverse Fourier transform, we obtain

$$P_2(S_2) = \frac{1}{\sqrt{8\pi\gamma}} e^{-x^2/8\gamma}. \tag{4.16}$$

Thus the Gaussian distribution is also a stable distribution. Writing (4.16) in the form

$$P_2(S_2) = \frac{1}{\sqrt{2\pi}(\sqrt{2}\sigma)} e^{-x^2/2(\sqrt{2}\sigma)^2}, \tag{4.17}$$

we find

$$\sigma_2 = \sqrt{2}\sigma. \tag{4.18}$$

We have verified that two stable stochastic processes exist: Lorentzian and Gaussian. The characteristic functions of both processes have the same functional form

$$\varphi(q) = e^{-\gamma|q|^\alpha}, \tag{4.19}$$

where $\alpha = 1$ for the Lorentzian from (4.3), and $\alpha = 2$ for the Gaussian from (4.15).

Lévy [92] and Khintchine [80] solved the general problem of determining the entire class of stable distributions. They found that the most general form of a characteristic function of a stable process is

$$\ln \varphi(q) = \begin{cases} i\mu q - \gamma|q|^\alpha \left[1 - i\beta \frac{q}{|q|} \tan\left(\frac{\pi}{2}\alpha\right)\right] & [\alpha \neq 1] \\ i\mu q - \gamma|q| \left[1 + i\beta \frac{q}{|q|} \frac{2}{\pi} \ln|q|\right] & [\alpha = 1] \end{cases}, \tag{4.20}$$

where $0 < \alpha \leq 2$, γ is a positive scale factor, μ is any real number, and β is an asymmetry parameter ranging from -1 to 1.

The analytical form of the Lévy stable distribution is known only for a few values of α and β:

- $\alpha = 1/2$, $\beta = 1$ (Lévy–Smirnov)
- $\alpha = 1$, $\beta = 0$ (Lorentzian)
- $\alpha = 2$ (Gaussian)

Henceforth we consider here only the symmetric stable distribution ($\beta = 0$) with a zero mean ($\mu = 0$). Under these assumptions, the characteristic function assumes the form of Eq. (4.19). The symmetric stable distribution of index α and scale factor γ is, from (4.20) and (4.11),

$$P_{\mathrm{L}}(x) \equiv \frac{1}{\pi} \int_0^\infty e^{-\gamma|q|^\alpha} \cos(qx)dq. \tag{4.21}$$

For $\gamma = 1$, a series expansion valid for large arguments ($|x| \gg 0$) is [16]

$$P_{\mathrm{L}}(|x|) = -\frac{1}{\pi} \sum_{k=1}^{n} \frac{(-1)^k}{k!} \frac{\Gamma(\alpha k + 1)}{|x|^{\alpha k+1}} \sin\left[\frac{k\pi\alpha}{2}\right] + R(|x|), \tag{4.22}$$

where $\Gamma(x)$ is the Euler Γ function and

$$R(|x|) = \mathcal{O}(|x|^{-\alpha(n+1)-1}). \tag{4.23}$$

From (4.22) we find the asymptotic approximation of a stable distribution of index α valid for large values of $|x|$,

$$P_{\mathrm{L}}(|x|) \sim \frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi|x|^{1+\alpha}} \sim |x|^{-(1+\alpha)}. \tag{4.24}$$

The asymptotic behavior for large values of x is a power-law behavior, a property with deep consequences for the moments of the distribution. Specifically, $E\{|x|^n\}$ diverges for $n \geq \alpha$ when $\alpha < 2$. In particular, all Lévy stable processes with $\alpha < 2$ have *infinite* variance. Thus non-Gaussian stable stochastic processes do not have a characteristic scale – the variance is infinite!

4.2 Scaling and self-similarity

We have seen that Lévy distributions are stable. In this section, we will argue that these stable distributions are also self-similar. How do we rescale a non-Gaussian stable distribution to reveal its self-similarity? One way is to consider the 'probability of return to the origin' $P(S_n = 0)$, which we obtain by starting from the characteristic function

$$\varphi_n(q) = e^{-n\gamma|q|^\alpha}. \tag{4.25}$$

From (4.11),

$$P(S_n) = \frac{1}{\pi} \int_0^\infty e^{-n\gamma|q|^\alpha} \cos(qS_n)dq. \tag{4.26}$$

Hence

$$P(S_n = 0) = \frac{1}{\pi} \int_0^\infty e^{-n\gamma|q|^\alpha} dq = \frac{\Gamma(1/\alpha)}{\pi\alpha(\gamma n)^{1/\alpha}}. \tag{4.27}$$

The $P(S_n)$ distribution is properly rescaled by defining

$$\tilde{P}(\tilde{S}_n) \equiv P(S_n)n^{1/\alpha}. \tag{4.28}$$

The normalization

$$\int_{-\infty}^{+\infty} \tilde{P}(\tilde{S}_n)d\tilde{S}_n = 1, \tag{4.29}$$

is assured if

$$\tilde{S}_n \equiv \frac{S_n}{n^{1/\alpha}}. \tag{4.30}$$

When $\alpha = 2$, the scaling relations coincide with what we used for a Gaussian process in Chapter 3, namely Eqs. (3.13) and (3.14).

4.3 Limit theorem for stable distributions

In the previous chapter, we discussed the central limit theorem and we noted that the Gaussian distribution is an attractor in the functional space of pdfs. The Gaussian distribution is a peculiar stable distribution; it is the only stable distribution having all its moments finite. It is then natural to ask if non-Gaussian stable distributions are also attractors in the functional space of pdfs. The answer is affirmative. There exists a limit theorem [65, 66] stating that the pdf of a sum of n i.i.d. random variables x_i converges, in probability, to a stable distribution under certain conditions on the pdf of the random variable x_i. Consider the stochastic process $S_n = \sum_{i=1}^{n} x_i$, with x_i being i.i.d. random variables. Suppose

$$P(x_i) \sim \begin{cases} C_-|x_i|^{-(1+\alpha)} & \text{as } x \to -\infty \\ C_+|x_i|^{-(1+\alpha)} & \text{as } x \to +\infty \end{cases}, \tag{4.31}$$

and

$$\beta \equiv \frac{C_+ - C_-}{C_+ + C_-}. \tag{4.32}$$

Then $\tilde{P}(\tilde{S}_n)$ approaches a stable non-Gaussian distribution $P_{\mathrm{L}}(x)$ of index α and asymmetry parameter β, and $P(S_n)$ belongs to the attraction basin of $P_{\mathrm{L}}(x)$.

Since α is a continuous parameter over the range $0 < \alpha \leq 2$, an infinite number of attractors is present in the functional space of pdfs. They comprise the set of all the stable distributions. Figure 4.1 shows schematically several such attractors, and also the convergence of a certain number of stochastic processes to the asymptotic attracting pdf. An important difference is observed between the Gaussian attractor and stable non-Gaussian attractors: finite variance random variables are present in the Gaussian basin of attraction, whereas random variables with infinite variance are present in the basins of attraction of stable non-Gaussian distributions. We have seen that stochastic processes with infinite variance are characterized by distributions with power-law tails. Hence such distributions with power-law tails are present in the stable non-Gaussian basins of attraction.

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On some expansions of stable distribution functions

By HARALD BERGSTRÖM

1. Introduction

The function e^{-t^α} for any fixed value α in the interval $0 < \alpha < 1$ admits a unique representation.

$$e^{-t^\alpha} = \int_0^\infty e^{-xt} G'_\alpha(t)\, dx, \qquad 0 \leqq t \leqq \infty \tag{1}$$

where $G_\alpha(x)$ is a stable d. f. (distribution function) with $G_\alpha(0) = 0$.[1] P. HUMBERT[2] has formally given the expansion

$$G'_\alpha(x) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} (\sin \pi \alpha k) \frac{\Gamma(\alpha k + 1)}{x^{\alpha k + 1}} \tag{2}$$

for $0 < \alpha < 1$, $x < 0$, which has later been rigorously proved by H. POLLARD.[3]
From (1) follows that $G_\alpha(x)$ has the characteristic function

$$\gamma_\alpha(t) = e^{-|t|^\alpha} \left(\cos \frac{\pi \alpha}{2} - i \sin \frac{\pi \alpha}{2} \operatorname{sgn} t\right)$$

(sgn: read signum). Now owing to P. LÉVY[4] the characteristic function of a stable d.f., when suitably normalized can be written in the form.

$$\gamma_{\alpha\beta}(t) = e^{-|t|^\alpha (\cos \beta - i \sin \beta \operatorname{sgn} t)}, \tag{3}$$

where

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[1] S. BOCHNER, Completely monotone functions of the Laplace operator for torus and sphere, Duke Math. J. vol. 3, 1937.
P. LÉVY, Théorie de l'addition des variables aléatoires, Gauthier-Willars (1937) pp. 94—97, 198—204.
Compare also H. POLLARD, The representation of e^{-x^λ} as a Laplace integral, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 908—910.
[2] P. HUMBERT, Nouvelles correspondances symboliques, Bull. Soc. Math. France vol. 69 (1945) pp. 121—129.
[3] H. POLLARD loc. cit.
[4] P. LÉVY loc. cit.

$$\cos\ \beta \geq 0, \quad \left|\sin\beta\cos\frac{\pi\alpha}{2}\right| \leq \cos\beta\sin\frac{\pi\alpha}{2}$$

$0 < \alpha \leq 2$. We omit the uninteresting case $\cos\beta = 0$. Then we can give a generalization of Humbert's expansion corresponding to (3) in the form

$$(4) \qquad G'_{\alpha\beta}(x) = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(\alpha k+1)}{x|x|^{\alpha k}}\sin k\left(\frac{\alpha\pi}{2}+\beta-\alpha\arg x\right)$$

for $0 < \alpha < 1$ ($\arg x = \pi$ for $x < 0$.) The series in the right side of (4) is divergent for $\alpha \geq 1$. However, we can prove that the partial sum of the n first terms in (4) for every n is an asymptotic expansion in the case $1 \leq \alpha < 2$, i.e. the remainder term has smaller order of magnitude (for large $|x|$) then the last term in the partial sum,

$$(5) \quad G'_{\alpha\beta}(x) = -\frac{1}{\pi}\sum_{k=1}^{n}\frac{(-1)^k}{k!}\frac{\Gamma(\alpha k+1)}{x|x|^{\alpha k}}\sin k\left(\frac{\alpha\pi}{2}+\beta-\alpha\arg x\right)+0[|x|^{-\alpha(n+1)-1}]$$

($\arg x = \pi$ for $x < 0$), $|x| \to \infty$.

At the same time we give the convergent series

$$(6) \qquad G'_{\alpha\beta}(x) = \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma\left(\frac{k+1}{\alpha}\right)}{k!\,\alpha}x^k\cos\left[k\left(\frac{\pi}{2}+\frac{\beta}{\alpha}\right)+\frac{\beta}{\alpha}\right]$$

for $\alpha > 1$ and the asymptotic expansion

$$(7) \qquad G'_{\alpha\beta}(x) = \frac{1}{\pi}\sum_{k=0}^{n}(-1)^k\frac{\Gamma\left(\frac{k+1}{\alpha}\right)}{k!\,\alpha}x^k\cos\left[k\left(\frac{\pi}{2}+\frac{\beta}{\alpha}\right)+\frac{\beta}{\alpha}\right]+0(|x|^{n+1})$$

for $0 < \alpha < 2$, $|x| \to 0$.

For the proof we shall use the Fouriertransform.

The asymptotic expansions are of importance in that case when $G_{\alpha\beta}(t)$ is the limiting distribution of a sequence of distributions. We shall return to this question in an other connection.

2. **Proof**

As $|\gamma_{\alpha\beta}(t)|$ is integrable in $(-\infty, \infty)$ for $\alpha > 0$, $\cos\beta > 0$, the inversion

$$(8) \qquad G'_{\alpha\beta}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\,\gamma_{\alpha\beta}(t)\,dt$$

is permitted. We consider real x with $|x| > 0$. Then we can put

$$G'_{\alpha\beta}(x) = \frac{1}{2\pi}[u(x) + \overline{u(x)}] \tag{9}$$

with

$$u(x) = \int_0^\infty e^{-itx}\gamma_{\alpha\beta}(t)\,dt. \tag{10}$$

It is now possible to choose φ_0 in the interval $\frac{-\pi}{2} \le \varphi_0 \le \frac{\pi}{2}$ such that

$$\beta_1 = \pi - \beta + \alpha\varphi_0,$$

and

$$\beta_2 = \frac{3\pi}{2} - \arg x + \varphi_0$$

belong to the interval $\left(\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta\right)$ with some δ, $0 < \delta < \frac{\pi}{2}$. (We choose $\varphi_0 < 0$ for $x > 0$ and $\arg x = \pi$, $\varphi_0 > 0$ for $x < 0$).

We can then transform the integral in the right side of (10) into

$$u(x) = e^{i\varphi_0}\int_0^\infty e^{\tau|x|e^{i\beta_2} + \tau^\alpha e^{i\beta_1}}\,d\tau. \tag{11}$$

For that reason we consider

$$\int e^{-itx - t^\alpha e^{-i\beta}}\,dt$$

taken along a contour C in the $re^{i\varphi}$-plane where C is defined by the following relations:

$\varphi = 0$, $\varphi = \varphi_0$, r between r_0 and r_1, $r = r_0$, $r = r_1$, φ between 0 and φ_0.

This integral is zero and the part of the integral taken along the curved portions of C vanishes when $r_1 \to \infty$, $r_0 \to 0$. Thus (11) holds. Consider the expansion

$$e^z = \sum_{k=0}^{n}\frac{z^k}{k!} + z^{n+1}M_{n+1}. \tag{12}$$

Here M_{n+1} is smaller than a constant only depending on n, if $R(z) \le 0$.[1]

Putting $z = \tau^\alpha e^{i\beta_1}$ and combining (11) and (12), we obtain

$$u(x) = \sum_{k=0}^{n}\frac{1}{k!}e^{ik\beta_1 + i\varphi_0}\int_0^\infty e^{\tau|x|e^{i\beta_2}}\tau^{\alpha k}\,d\tau + e^{i\varphi_0 + i(n+1)\beta_1}\cdot \int_0^\infty M_{n+1}\,e^{\tau|x|e^{i\beta_2}}\tau^{\alpha(n+1)}\,d\tau. \tag{13}$$

[1] We derive the expansion from Cauchy's integral formula and then we can obtain the remainder term in the form $z^{n+1}M_{n+1}$,

$$M_{n+1} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{a+iy}}{(a+iy)^{n+1}(a+iy-z)}\,dy$$

with any $a > 0$.

The remainder term is here smaller than

$$\operatorname*{Max}_{z} | M_{n+1} | \cdot \frac{\Gamma[\alpha(n+1)+1]}{|x \cos \beta_2|^{\alpha(n+1)+1}} = 0 [|x|^{-\alpha(n+1)-1}].$$

By a transformation analogous to that one used for (9) we get

$$(14) \qquad \int_0^\infty e^{\tau |x| e^{i\beta_2}} \tau^{\alpha k} d\tau = e^{i(\pi-\beta_2)(\alpha k+1)} \int_0^\infty e^{-\varrho |x|} \varrho^{\alpha k} d\varrho = $$
$$= e^{i(\pi-\beta_2)(\alpha k+1)} \frac{\Gamma(\alpha k+1)}{|x|^{\alpha k+1}}.$$

From (9), (13) and (14) we get (5).

In order to obtain (4) in the case $0 < \alpha < 1$ we change e^z in (10) against the infinite Taylor-expansion and observe that term by term integration is permitted.[1]

Considering the expansion (10) and the corresponding Taylor-expansion with $z = \tau |x| e^{i\beta_2}$ we prove (7) and (6) in the same way as we have proved (5) and (4). Then we have to observe the relation[2]

$$\int_0^\infty \varrho^k e^{-\varrho^\alpha} d\varrho = \frac{1}{\alpha} \Gamma\left(\frac{k+1}{\alpha}\right).$$

We consider the following special cases.

a) $$\alpha < 1, \quad \beta = \frac{\alpha\pi}{2}.$$

Then every member in the right side of (4) is equal to 0 for $x < 0$, i.e. we have $G'_{\alpha\beta}(x) = 0$ for $x < 0$.

b) $$\alpha < 1, \quad \beta = -\frac{\alpha\pi}{2}.$$

Then every member in the right side of (4) is equal to 0 for $x > 0$, i.e. we have $G'_{\alpha\beta}(x) = 0$ for $x > 0$

c) $$1 < \alpha < 2, \quad \beta = -\pi + \frac{\alpha\pi}{2}.$$

Then every member except the remainder term in the right side of (5) is equal to 0 for $x < 0$, i.e. we have $G'_{\alpha\beta}(x) = 0(|x|^{-m})$ with arbitrarily large m for $x < 0$.

d) $$1 < \alpha < 2, \quad \beta = \pi - \frac{\alpha\pi}{2}.$$

Then every member except the remainder term is equal to 0 for $x > 0$, i.e. we have $G'_{\alpha\beta}(x) = 0(x^{-m})$ with arbitrarily large m for $x > 0$.

[1] We can apply a test in E. W. Hobson, The theory of functions of a real variable II, Cambridge (1950) p. 306.

[2] Compare W. Gröbner und N. Hofreiter, Integraltafel II, Springer-Verlag (1950) p. 67.

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