Handout 1
BOND MATH

Definition of a Bond

A bond is a security where the issuer of the bond borrows money, called the principal, and agrees to:

1) Pay the lender (the bond holder) periodic interest payments based on the outstanding amount of principal.

2) Return the principal through a “lump sum” payment, periodic payments over time, or in the case of a “perpetual” bond, not at all.

(materials after R. Klotz)

Goals of Bond Math

- Determine the price of a bond.
- Determine if the price of Bond “A” is expensive or cheap when compared with the price of Bond “B”.
- Determine the risks of the bond. How will the price change when the market changes?

The Time Value of Money

- Future value at time t of $1 today

- Present value of $1 received t years from today

The Future Value of $1 in One Year

At a 5% rate, we have:

- Annual compounding--

  $\$1\,(1 + .05) = \$1.05$

- Semi-annual compounding--

  $$\$1\left(1+\frac{.05}{2}\right)\left(1+\frac{.05}{2}\right)=\$1.050625$$

- Quarterly compounding--

  $$\$1\left(1+\frac{.05}{4}\right)\left(1+\frac{.05}{4}\right)\left(1+\frac{.05}{4}\right)\left(1+\frac{.05}{4}\right)=\$1.050945$$

- Continuous compounding--

  $\$1e^{0.05}=\$1.051271$

Future Value Formula

The future value of P dollars t years from now at r% is:

$$FV = \left(1 + \frac{r}{s}\right)^{s \cdot t} \cdot P$$

Where P = number of dollars today

s = number of compounding periods per year

t = number of years

r = interest rate expressed as a decimal

FV = future value

A Future Value Example

Find the future value of 100 dollars at 15% for 30 years:

A) compounded annually

B) compounded semi-annually

C) compounded monthly

$$\text{A) } FV = (1+0.15)^{30} \cdot 100 = 6{,}621.18 \text{ Dollars}$$

$$\text{B) } FV = \left(1+\frac{0.15}{2}\right)^{2\cdot 30} \cdot 100 = 7{,}664.92 \text{ Dollars}$$

$$\text{C) } FV = \left(1+\frac{0.15}{12}\right)^{12\cdot 30} \cdot 100 = 8{,}754.00 \text{ Dollars}$$

Present Value

Example: Find the present value of 100 dollars received one year from today at 5% interest compounded:

A) annually

B) semi-annually

C) monthly

A) $(1+0.05)\cdot \mathrm{PV} = 100$

$$\mathrm{PV} = \frac{100}{1+0.05} = 95.24 \text{ dollars}$$

B) $\left(1+\frac{0.05}{2}\right)^2 \cdot \mathrm{PV} = 100$

$$\mathrm{PV} = \frac{100}{\left(1+\frac{0.05}{2}\right)^2} = 95.18 \text{ dollars}$$

Present Value (cont'd)

C) $\left(1+\frac{0.05}{12}\right)^{12} \cdot PV = 100$

$$PV = \frac{100}{(1+\frac{0.05}{12})^{12}} = 95.13 \,\text{dollars}$$

Present Value Formula

$$PV = \frac{F}{(1+\frac{r}{s})^{s \cdot t}}$$

F = number of dollars being present valued

s = number of compounding periods per year

t = number of years

r = interest rate expressed as a decimal

PV = present value

Pricing a Semi-Annual Pay Bond

Problem: Find the price today of a 3 yr. maturity bond which pays a 10% semi-annual coupon (assume that the principal is 100)

Interval	6 mos.	1 yr.	18 mos.	2 yr.	30 mos.	3 yr.
Amount	5	5	5	5	5	100 + 5

Price = present value (cash flows)

$$\text{Price} = \frac{5}{(1+\frac{r}{2})} + \;.\;\;.\;\;.\; + \frac{5}{(1+\frac{r}{2})^6} + \frac{100}{(1+\frac{r}{2})^6}$$

Bond Pricing Formula (Semi-Annual Pay Bond)

On a coupon date:

$$\text{Price} = \sum_{i=1}^{n} \frac{\frac{c}{2}}{\left(1+\frac{r}{2}\right)^{i}} + \frac{100}{\left(1+\frac{r}{2}\right)^{n}}$$

c = annual coupon rate in percent (10% coupon, c = 10)

r = discount rate (in decimals)

n = number of 6 month periods until the maturity

Bond Price Conventions

- Price represents a percent of the principal

  Example: A price of 102 means for \$50mm principal you pay
  (102%)(\$50mm) = \$51mm

- Prices are generally expressed in 32nds

  102.50 would be written 102-16

- A “+” means a 64th

  102-16+ means 102 and $\frac{33}{64}$

- The smallest unit is generally $\frac{1}{256}$

  102-16 $\frac{1}{8}$ means 102 and $\frac{129}{256}$

Bond Price Conventions (cont'd)

Bond Price	Name
Price = 100	Par Bond
Price < 100	Discount Bond
Price > 100	Premium Bond

- 1 Basis Point (B.P.) = .01%
  100 Basis Points = 1.0%

Example of Pricing a Bond

Example: Find the price of a 3 yr. 10% bond if the discounting rate is 8%.

Time Till Cash Flow	Cash Flow	Present Value at 8%
6 mos.	5	4.8077
1 yr.	5	4.6228
18 mos.	5	4.4450
2 yr.	5	4.2740
30 mos.	5	4.1096
3 yr.	100 + 5	82.9830
		105.2421

Some Bond Facts

- The maximum bond price is when the discount rate is 0%.

$$\text{Maximum price} = \sum_{i=1}^{n} \frac{\frac{c}{2}}{(1+0)^i} + \frac{100}{(1+0)^n}$$

$$= n \cdot \frac{c}{2} + 100$$

- The minimum bond price is zero.

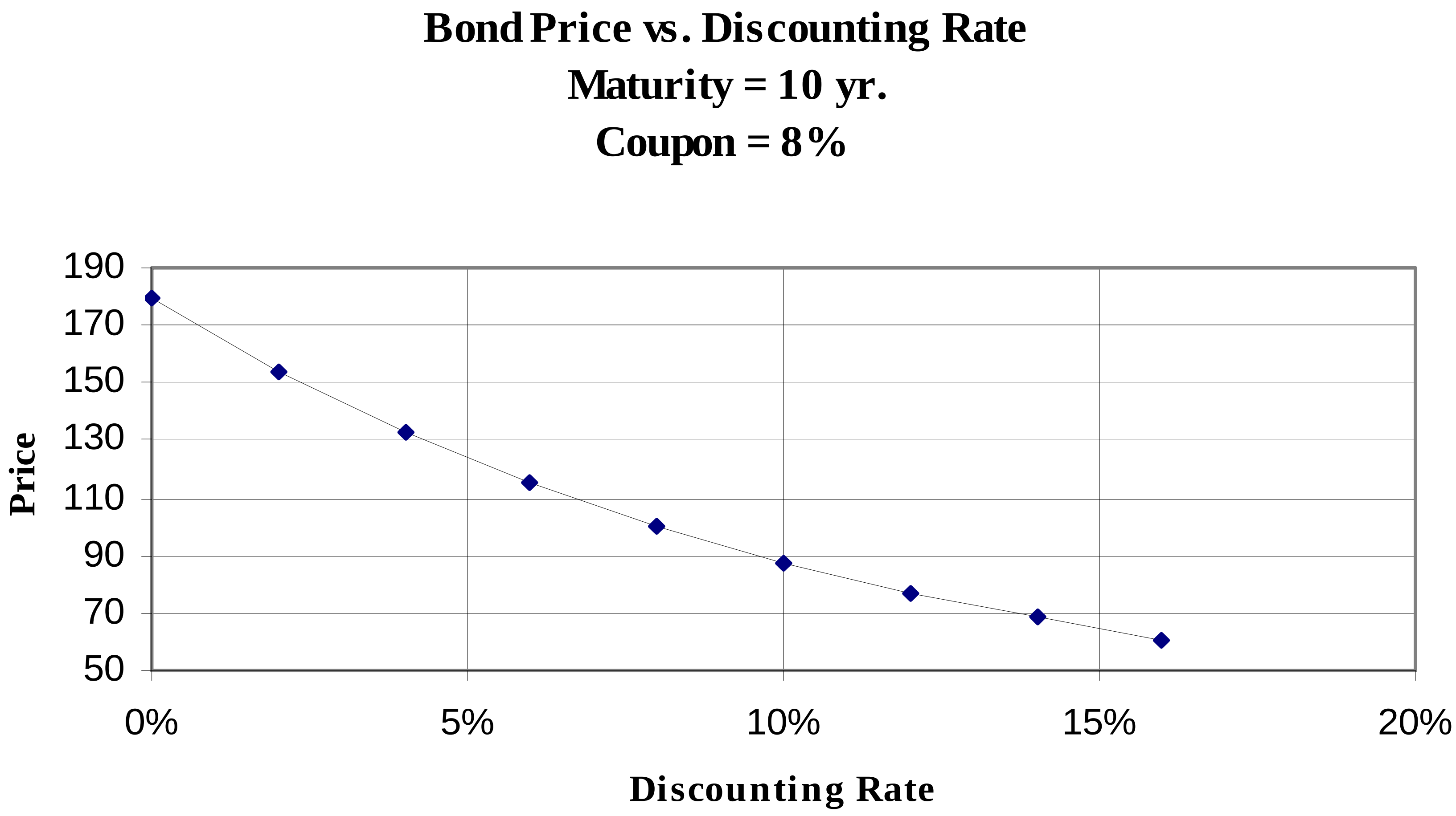
- Zero coupon bonds have the formula:

$$\text{Price} = \frac{100}{(1+\frac{r}{2})^n}$$

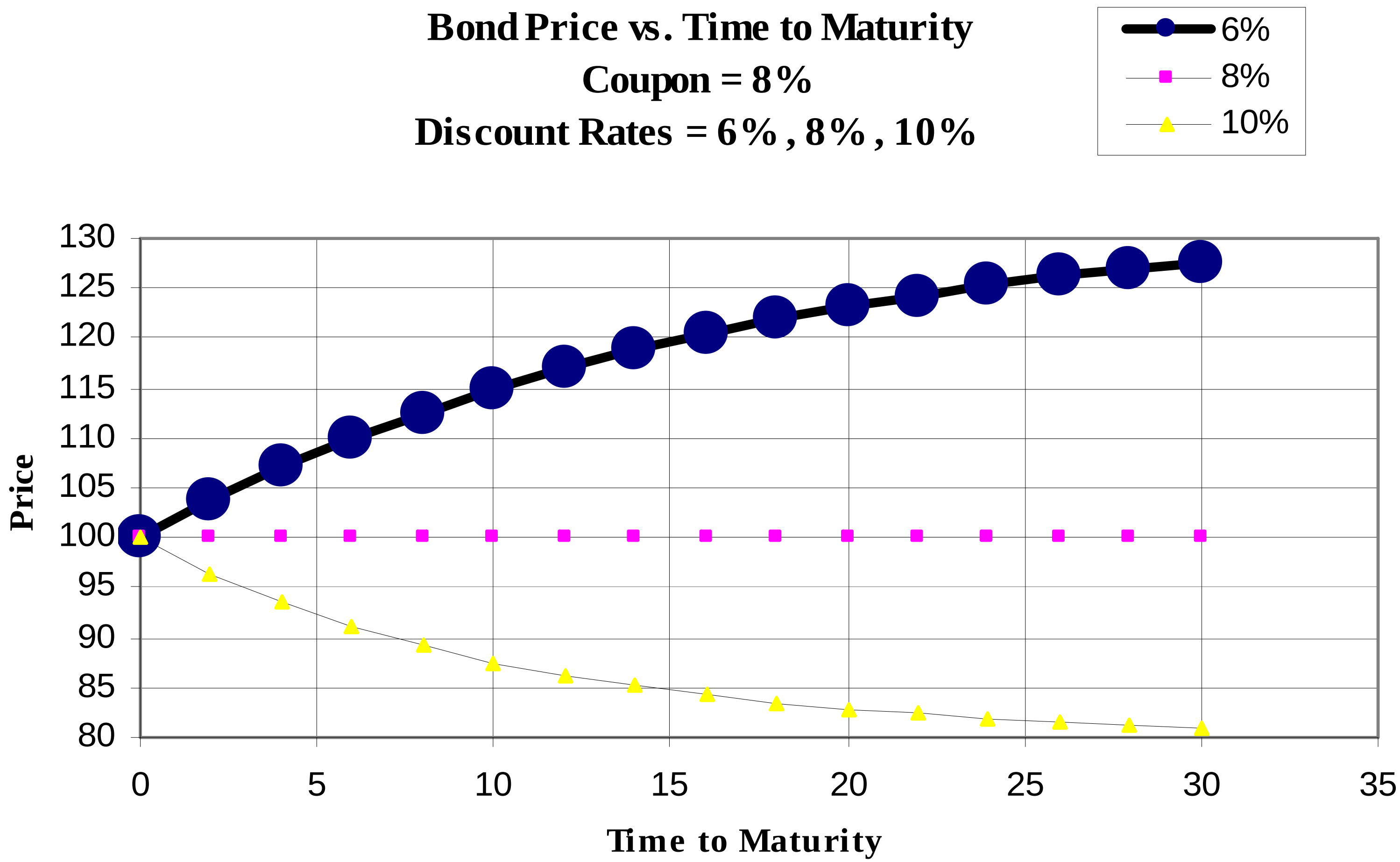
Example: 30 yr. zero coupon bond using a 10% discount rate

$$\text{Price} = \frac{100}{(1+\frac{0.10}{2})^{60}} = 5.35$$

Bond Price vs. Discounting Rate



Bond Price vs. Time to Maturity



Accrued Interest

- If the bond is sold between coupon dates, the seller is entitled to a portion of the next coupon.

- Corporate and municipal bonds - 30/360 day count

$$A.I. = \frac{180 - (30m + d)}{180} \text{ x } \frac{c}{2}$$

Where m = integer number of months till next coupon

d = number of days in excess of the months

- Treasury bonds - actual/actual day count

$$A.I. = \frac{\text{Days in S.A. Period - Days Till Next Coupon}}{\text{Days in S.A. Period}} \text{ x } \frac{c}{2}$$

Example of Accrued Interest

Calculate the accrued interest for an 8% bond with 2 months and 5 days until the next coupon is paid.

1) Corporate or municipal bonds:

$$A.I. = \frac{180 - (30x2 + 5)}{180} x \frac{8}{2} = 2.556$$

2) Treasury Bonds (assume 182 days in S.A. period and 66 days in 2 months and 5 days):

$$A.I. = \frac{182 - 66}{182} x \frac{8}{2} = 2.549$$

Bond Pricing Convention

- The price of a bond is quoted without accrued interest. This is called the flat price, the clean price, or the market price of the bond.

- The bond's price with accrued interest is called the full price or the dirty price of the bond.

Yield to Maturity

The discount rate which gives rise to a given bond price is called the yield to maturity of the bond at that price.

Problem: Given a price P, find y so that

$$P = \sum_{i=1}^{n} \frac{\frac{c}{2}}{\left(1 + \frac{y}{2}\right)^{i}} + \frac{100}{\left(1 + \frac{y}{2}\right)^{n}}$$

Relationships of Coupon, Yield and Price

Coupon, Yield Relationship	Price
Coupon = Yield	Price = 100
Coupon > Yield	Price > 100
Coupon < Yield	Price < 100

Yield as a Measure of Value

Example: Which of these bonds is “cheaper”?

Bond A	Bond B
Maturity = 20 years	Maturity = 20 years
Coupon = 6%	Coupon = 9%
Price = 90	Price = 100

YTM of Bond A = 6.93%
YTM of Bond B = 9.00%

Flaws of YTM as a Measure of Value/Performance

- All cash flows are discounted by the same rate.
- Yield can be a poor approximation of the return when the general level of rates change. Reinvestment rates can also cause problems.
- The notion of a yield to maturity for bonds with uncertain cash flows (callable/putable bonds, mortgages, etc.) may not be very useful.

The Yield Curve and a Yield Curve Spread

- **The Treasury Yield Curve reflects the yields of current issues of various maturities**

- **<u>Example</u>: a *positive* curve, the most common type**

<u>Maturity</u>	<u>1 Yr.</u>	<u>2 Yr.</u>	<u>3 Yr.</u>	<u>5 Yr.</u>	<u>10 Yr.</u>	<u>30 Yr.</u>
Yield (%)	1.5%	2.5%	3.2%	3.9%	4.7%	5.4%

- **<u>*Yield Curve Spread* (or just *Curve Spread*)</u>: is the spread between 2 different points on the curve, e.g, the "*2 vs. 5 yr. treasury spread*" is 140 bps (see above)**

- **<u>Shape of the Curve</u>: positive, flat, or "inverted"**

- **The shape of the curve, the spreads, and their potential changes are greatly studied**
  - **by investors and wall street traders for investment/trading strategies**
  - **by issuers for most efficient funding**

Absolute Price Risk of a Bond: DV01

The dollar value of an 01 (DV01) is change in a bond's price for a 1 B.P. increase in its yield.

Example: Find the DV01 of a 10 yr., 8% bond yielding 7.50%.

Price at 7.50% = 103.4741

Price at 7.51% = <u>103.4030</u>

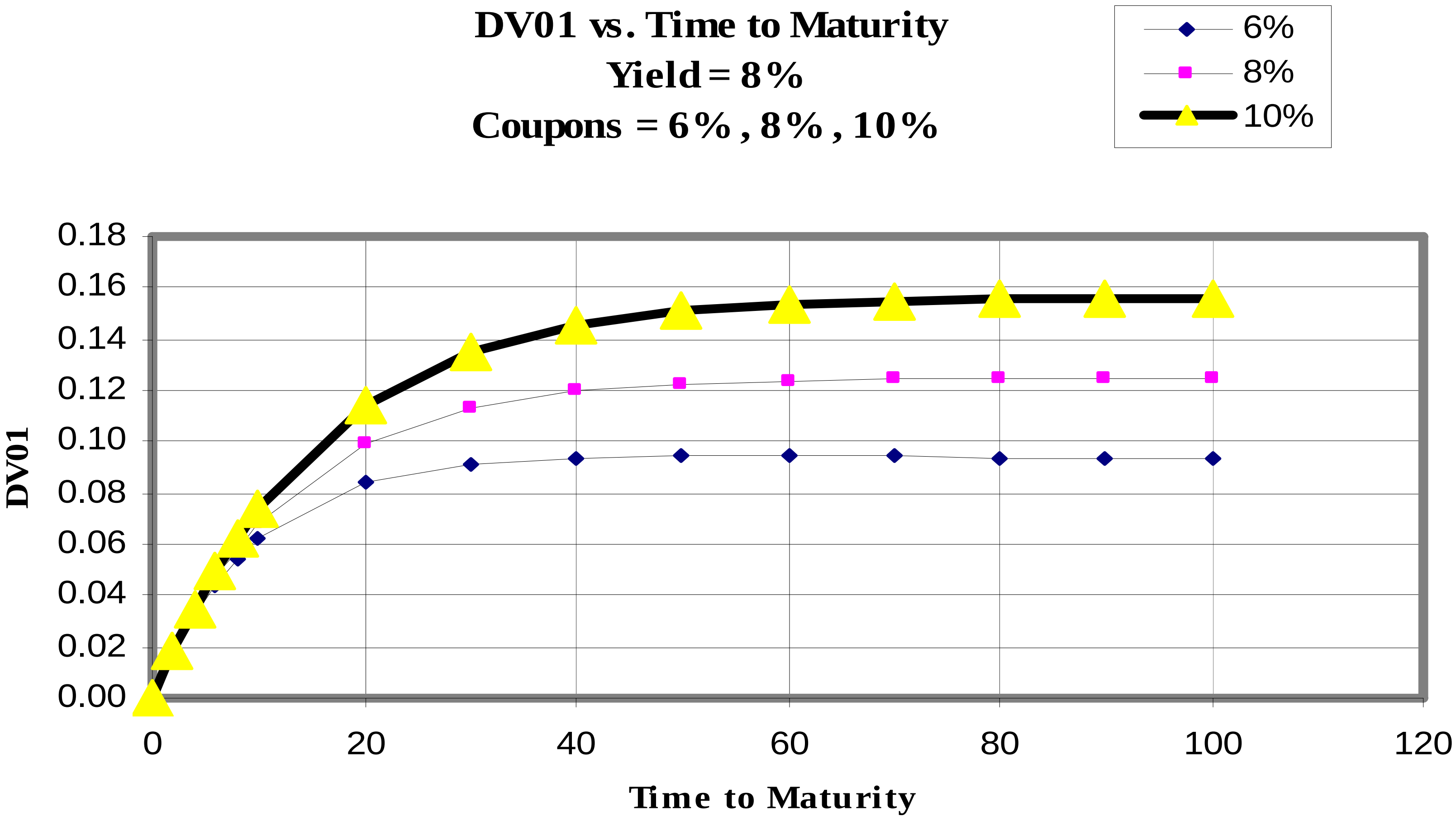
0.0710 = DV01

Notice: DV01 = $\frac{-1}{100}\frac{dP}{dY}$

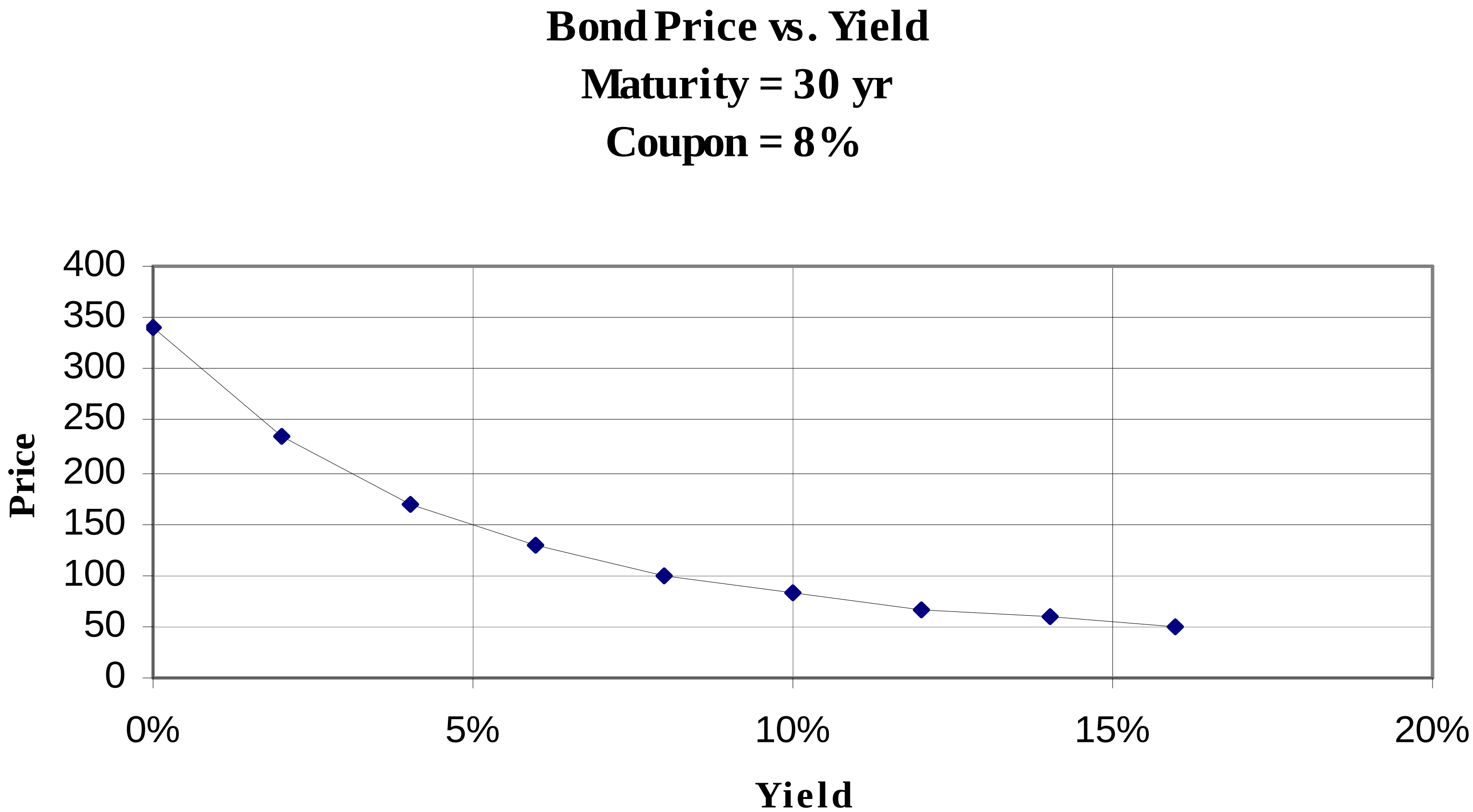
P = Price of bond

Y = Yield in percent

DV01 vs. Maturity



Bond Price vs. Yield



Modified Duration

$$D = \frac{DV01}{\text{Full Price}} x10,000$$

or

$$D = -\frac{100}{P}\frac{dP}{dy}$$

P = full price

y = yield in percent

The modified duration approximates the percentage price change of a bond for 100 B.P. change in yield. This is a portfolio manager's standard measure of risk for a portfolio.

DV01 vs. Duration

- DV01 is a measure of risk for a leveraged portfolio while duration is a measure of risk for an unleveraged portfolio.

- For bonds with prices near 100

  D ≈ DV01 x 100

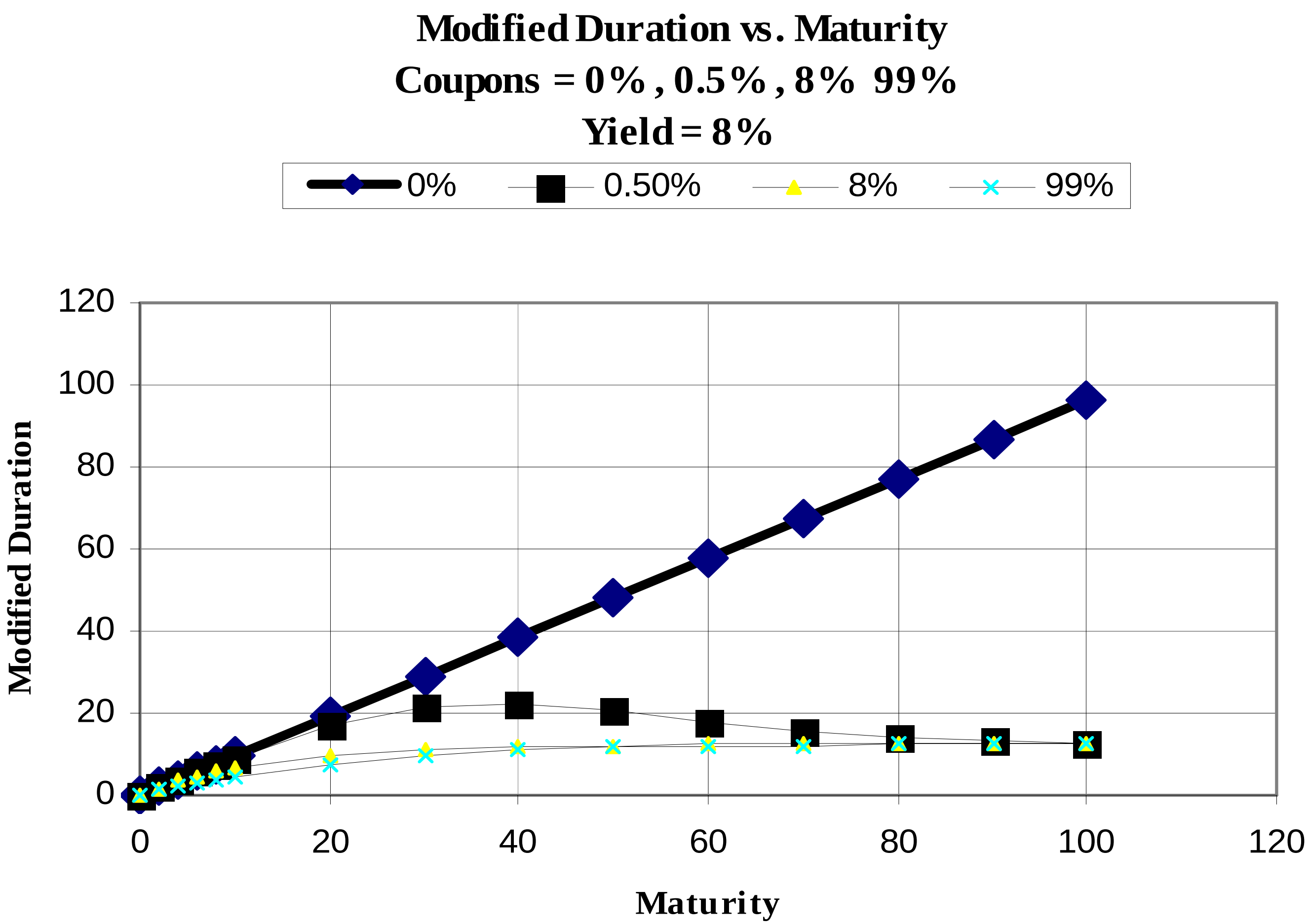
- 30 yr. zeros have very “different” DV01s and durations.

  Example: For a 30 yr. zero yielding 8% (price = 9.51) we have:

  The duration of a 30 yr. zero $= \dfrac{1}{(1+\frac{y}{2})} \text{x} \left(30\right) = 28.85$

  $$\text{DV01} = \frac{1}{(1+\frac{y}{2})} \text{x} \left(30\right) \text{x} \frac{\text{price}}{10{,}000} = 0.0274$$

Modified Duration vs. Maturity



Using the DV01 to Re-Price a Bond

Example: Approximate the price of an 8%, 10 yr. bond yielding 6% assuming that you know the DV01 = .0679 at par.

Approximate price $= 100 - (DV01)\ (\Delta Y)$
$= 100 - (.0679)\ (-200)$
$= 100 + 13.58 = 113.58$

The actual price at 6% is 114.88, an error of 1.30.

Handout 2

DURATION, CONVEXITY, YIELD CURVE

Measuring Risk

The absolute price risk of a portfolio can be measured in terms of the price risk of a portfolio of 10 yr. bonds (for example), DV01 = .0650

Example:	Portfolio	DV01
	10mm 9 yr. bonds	.0584
	15mm 11 yr. bonds	.0714

10mm 9 yr. bonds--> $\frac{.0584}{.0650} x10 = 8.98 mm\ 10\ yr.\ bonds$

15mm 11 yr. bonds--> $\frac{.0714}{.0650} x15 = 16.44 mm\ 10\ yr.\ bonds$

Portfolio Risk = (8.98mm + 16.44mm) 10 yr. bonds

so the portfolio will make (or lose) roughly the same amount of money at 25.46mm 10 yr. bonds if all yields move the same amount.

An Application of Modified Duration

- Gives an approximation to the instantaneous return of a portfolio per 100 B.P. move in rates.

$$\text{Instantaneous Percent Return} = \frac{\left(P\left(y+\Delta y\right)-P\left(y\right)\right)}{P\left(y\right)} \text{x} 100 \approx -D\left(\Delta y\right)$$

D = duration

y = yield in percent

P(y) = full price of the bond

Frederick Macaulay (PhD 1924 from Columbia U, research director of the Twentieth Century Fund) introduced the concept of **Macaulay duration** as the weighted average maturity of cash flows:

$$MacD = \sum_{i=1}^{n} \frac{t_i PV_i}{PV} \quad where \quad t_i \quad payment \qquad periods$$

Consider some set of fixed cash flows ti is the time in years until the i-th payment will be received

The present value of these cash flows is PVi

The total present value $$PV = \sum_{i=1}^{n} PV_i$$

For set of positive cash flows the weighted average will fall between the time to the first payment and the time of the final cash flow.

The Macaulay duration will equal the final maturity if and only if there is only a single payment at maturity.

Macaulay duration of zero-coupon bond its time to maturity.

Macaulay Duration

For semiannual coupon bond on coupon date

$$P = \sum_{i=1}^{n} \frac{\frac{c}{2}}{\left(1+\frac{y}{2}\right)^{i}} + \frac{100}{\left(1+\frac{y}{2}\right)^{n}}$$

$$MacD = \left(\sum_{i=1}^{n} \frac{t_i \frac{c}{2}}{\left(1+\frac{y}{2}\right)^{i}} + \frac{t_n 100}{\left(1+\frac{y}{2}\right)^{n}} \right) \Big/ P \qquad where\ t_i = \frac{i}{2}$$

Macaulay Duration vs. Modified Duration

If rates are semiannually compounded Modified duration D is related to Macaulay Duration

$$D = \frac{1}{(1+\frac{y}{2})} MacD$$

If rates are compounded with frequency k times per year Modified duration D is related to Macaulay Duration

$$D = \frac{1}{(1+\frac{y}{k})} MacD$$

Example: For a 30 yr. zero yielding 8% (price = 9.51) we have:

The Modified duration $D = \frac{1}{(1+\frac{y}{2})} \times (30) = 28.85$

The Macaulay duration is *MacD* = 30 (Exactly 30 years maturity)

$$\text{DV01} = \frac{1}{(1+\frac{y}{2})} \text{x} (30) \text{x} \frac{\text{price}}{10{,}000} = 0.0274$$

Modified Duration of a Portfolio

$$D = \frac{(D_1 x P_1) + (D_2 x P_2) + \ldots + (D_n x P_n)}{P_1 + \ldots + P_n}$$

D_1 = Duration of Bond 1

.

.

.

D_n = Duration of Bond n

P_1 = Full price of Bond 1

.

.

.

P_n = Full price of Bond n

Convexity

$$P\left(y+\Delta y\right)=P(y)-D\left(\frac{P(y)}{100}\right)\left(\Delta y\right)+\text{ error term}$$

$$P\left(y+\Delta y\right)=P(y)-D\left(\frac{P(y)}{100}\right)\left(\Delta y\right)+\frac{1}{2}\left(\frac{P(y)}{100}\right)C\left(\Delta y\right)^{2}+\text{ error}$$

$$C=\text{Convexity }=\frac{100}{P(y)}\,x\,\frac{d^{2}P}{dy^{2}}$$

P(y) = full price of the bond

y = yield in percent

Convexity is a measure of how nonlinear a bond's price yield curve is.

Positive and Negative Convexity

- A bond's price-yield curve is <u>positively</u> convex at a given yield if the bond's price rises more when its yield decreases than it falls when its yield increases.

- A bond's price-yield curve is <u>negatively</u> convex at a given yield if the bond's price falls more when its yield increases than it rises when its yield decreases.

- All option-free bonds are positively convex.

Calculating Convexity

$$\text{Convexity} = \frac{\Delta(\text{DV01})}{\text{full price}} \text{x} 1{,}000{,}000$$

$$\text{Where } \Delta(\text{DV01}) = [P(y-.01) - P(y)] - [P(y) - P(y+.01)] = P(y-.01) - 2P(y) + P(y+.01)$$

Convexity of a Portfolio

$$C = \frac{C_1 \text{x} P_1 + C_2 \text{x} P_2 + \ldots + C_n \text{x} P_n}{P_1 + P_2 + \ldots + P_n}$$

C_1 = Convexity of Bond 1

C_2 = Convexity of Bond 2

.

C_n = Convexity of Bond n

P_1 = Full price of Bond 1

P_2 = Full price of Bond 2

.

P_n = Full price of Bond n

Types of Rates

- Treasury rates
- LIBOR rates
- Repo rates

Treasury Rates

- Rates on instruments issued by a government in its own currency

LIBOR and LIBID (London Interbank Offer Rate and Bid Rate)

- LIBOR is the rate of interest at which a bank is prepared to deposit money with another bank. (The second bank must typically have a AA rating)
- LIBOR is compiled once a day by the British Bankers Association on all major currencies for maturities up to 12 months
- LIBID is the rate which a AA bank is prepared to pay on deposits from another bank

Repo Rates

- Repurchase agreement is an agreement where a financial institution that owns securities agrees to sell them today for X and buy them back in the future for a slightly higher price, Y
- The financial institution obtains a loan.
- The rate of interest is calculated from the difference between X and Y and is known as the repo rate

The Risk-Free Rate

- The short-term risk-free rate traditionally used by derivatives practitioners is LIBOR
- The Treasury rate is considered to be artificially low for a number of reasons

Eurodollar futures and swaps are used to extend the LIBOR yield curve beyond one year

The overnight indexed swap rate is increasingly being used instead of LIBOR as the risk-free rate

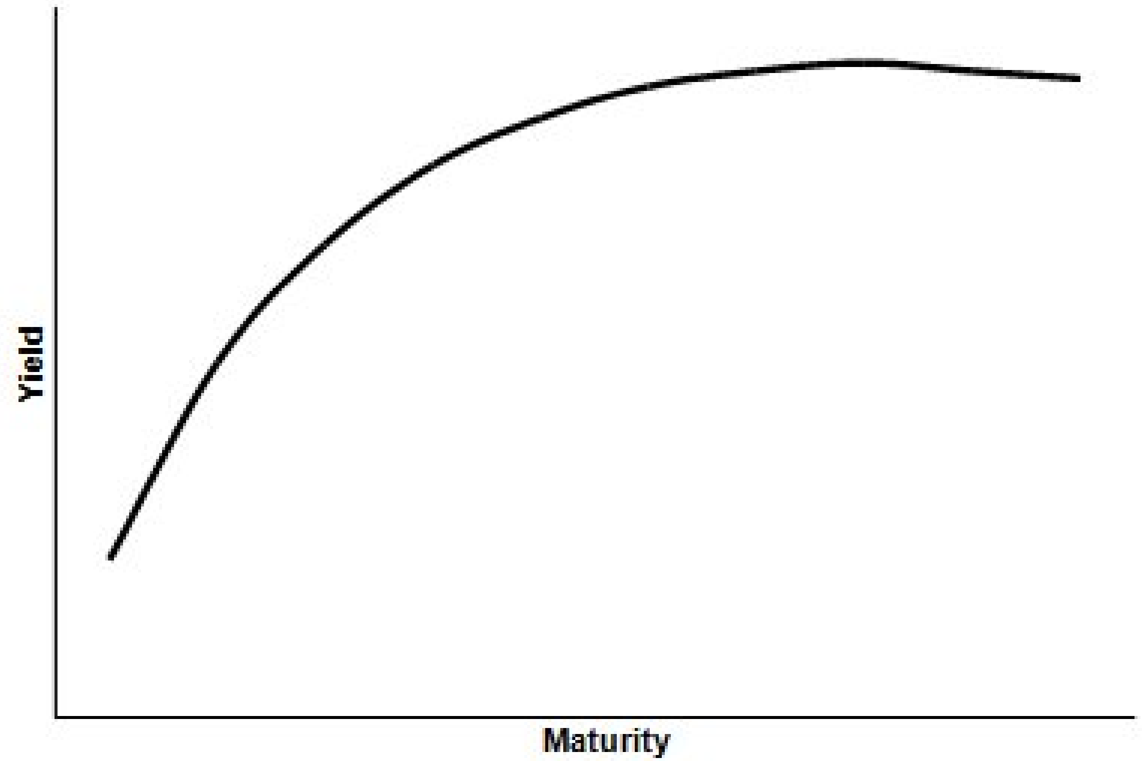
Zero Rates

A zero rate (or spot rate), for maturity T is the rate of interest earned on an investment that provides a payoff only at time T

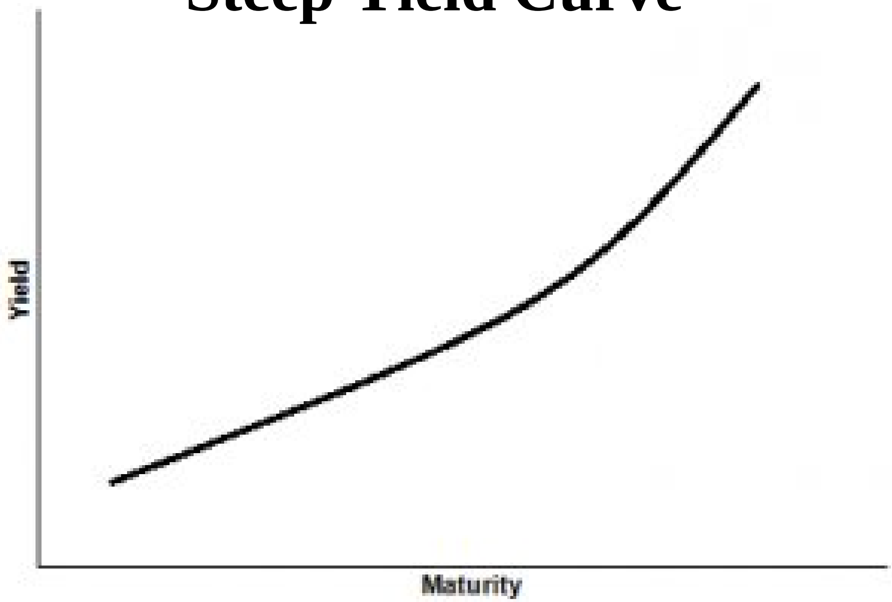
Maturity (years)	Zero rate
0.5	1.0
1.0	1.2
1.5	1.4
2.0	1.8

Yield Curve is a line graph that depicts the relationship between yields to maturity and time to maturity for bonds of the same asset class and credit quality. Different types of yield curves: Treasury, Swap, Corporate. Zero-coupon yield curve.

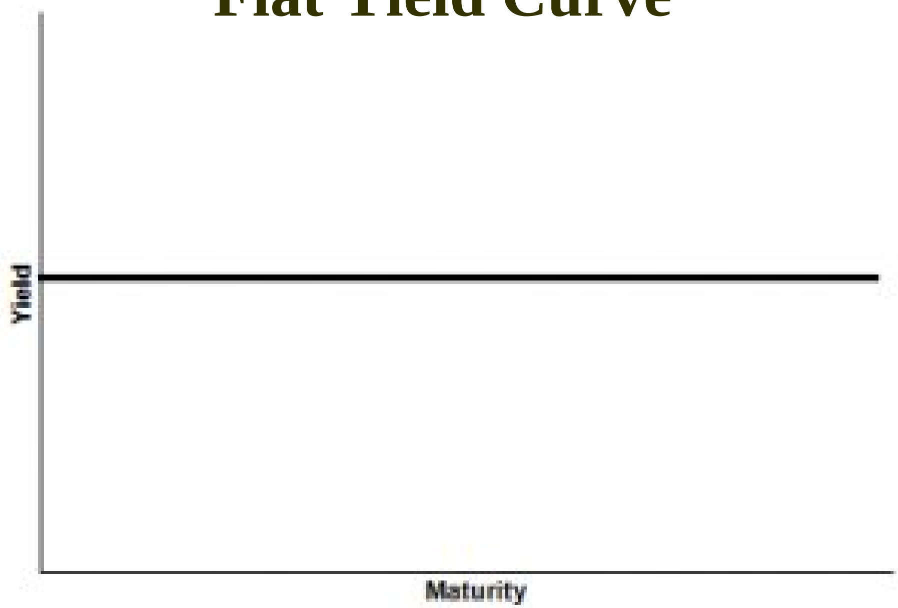
Normal, upper sloping yield curve. Bonds with longer maturities have higher yields relative to shorter-dated bonds. The slope of the curve is not steep, towards the longer maturities, the curve flattens.



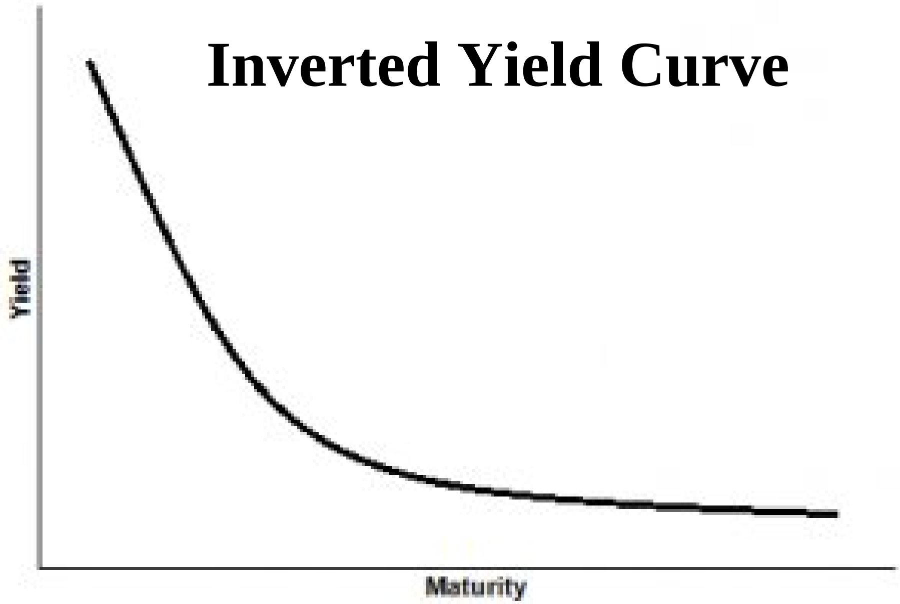
Steep Yield Curve



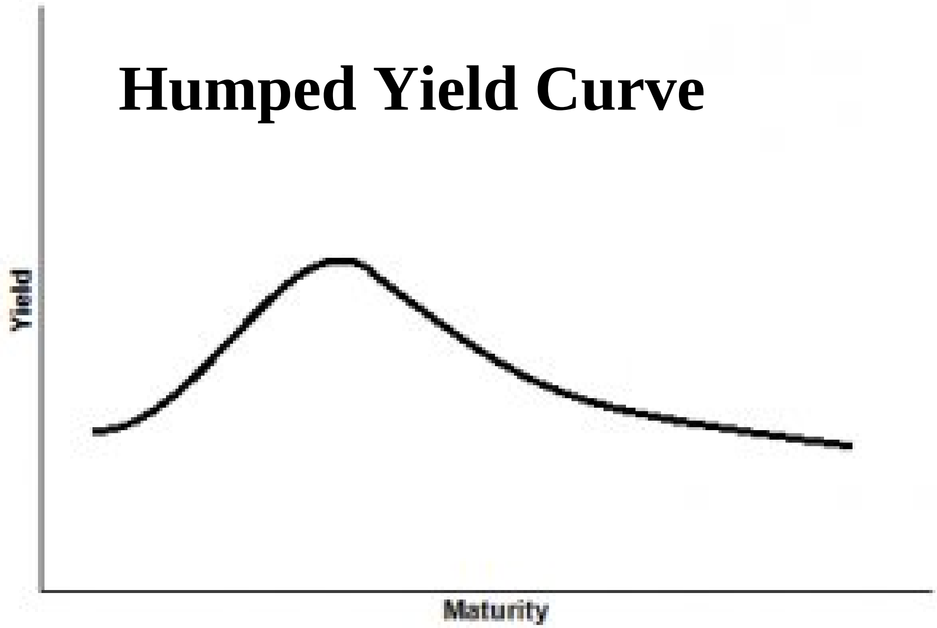
Flat Yield Curve



Inverted Yield Curve



Humped Yield Curve



YIELD CURVE

Dynamics of the yield curve. Changes in yield curve can be explained using Principal Components Analysis

- The first factor is a roughly parallel shift (83.1% of variation explained)
- The second factor is a twist (10% of variation explained)
- The third factor is a bowing (2.8% of variation explained)

Forward Rates

The forward rate is the future zero rate implied by today's term structure of zero coupon interest rates

Example: 2 year continuously compounded rate is 2%

$$\text{FV } 1\$ \text{ in 2 years} = 1 \cdot \exp(0.02 \cdot 2)$$

5 year continuously compounded rate is 3%

$$\text{FV } 1\$ \text{ in 5 years} = 1 \cdot \exp(0.03 \cdot 5)$$

Forward rate between years 2 and 5 implied by these two rates is such rate *r* that

$$\text{FV } 1\$ \text{ in 5 years} = 1 \cdot \exp(0.03 \cdot 5) = 1 \cdot \exp(0.02 \cdot 2) \cdot \exp(r \cdot 3)$$

Forward Rates Continued

Example: T_1 years continuously compounded rate is r_1

$$\text{FV 1\$ in T}_1 \text{ years} = 1 \cdot \exp(r_1 \cdot T_1)$$

T_2 years continuously compounded rate is r_2

$$\text{FV 1\$ in T}_2 \text{ years} = 1 \cdot \exp(r_2 \cdot T_2)$$

Forward rate between years T_1 and T_2 implied by these 2 rates is such rate r_{12}

$$\text{FV 1\$ in T}_2 \text{ years} = 1 \cdot \exp(r_2 \cdot T_2) = 1 \cdot \exp(r_1 \cdot T_1) \cdot \exp(r_{12} \cdot (T_2 - T_1))$$

Taking logarithms $r_2 \cdot T_2 = r_1 \cdot T_1 + r_{12} \cdot (T_2 - T_1)$

$$\frac{r_2 \cdot T_2 - r_1 \cdot T_1}{T_2 - T_1} = r_{12}$$

BOOTSTRAP METHOD
To Construct Zero Curve

Bond Principal	Time in yrs to Maturity	Coupon per year ($) each half paid semiannually	Bond price ($)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

The Bootstrap Method

- An amount 2.5 can be earned on 97.5 during 3 months.

- Because $100=97.5e^{0.10127\times 0.25}$

  the 3-month rate is 10.127% with continuous compounding

- Similarly the 6 month and 1 year rates are 10.469% and 10.536% with continuous compounding

The Bootstrap Method continued

- To calculate the 1.5 year rate we solve

$$4e^{-0.10469\times 0.5} + 4e^{-0.10536\times 1.0} + 104e^{-R\times 1.5} = 96$$

to get R = 0.10681 or 10.681%

- Similarly the two-year rate is 10.808%