Economics 361
Problem Set #3

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Fall 2022

Question 1: “Moments” of Truth

You may use any properties listed in the “Expectation” Handout except those explicitly asked to be proven.

Let X and Y be two random variables. (a, b, c) are three known real-valued constants. σ_X^2 is the variance of X, σ_Y^2 the variance of Y, and σ_{XY} the covariance of X and Y.

(a) Let $Z \equiv aX + bY + c$. Show that the variance of Z is equal to $a^2\ \sigma_X^2 + b^2\ \sigma_Y^2 + 2ab\ \sigma_{XY}$

(b) A *proportional* predictor of Y given X is a predictor of the form $\hat{Y}(X) = cX$. Show that best proportional predictor under mean squared error (MSE), $BPP_{MSE}(Y|X)$, is c^*X where $c^* = \frac{E[XY]}{E[X^2]}$

For **(c)-(e)**, suppose that X and Y are jointly distributed **bivariate Normal**:

$$f_{XY}(x, y) \;=\; \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]\right\}$$

where ρ is another parameter of the joint distribution, along with $\mu_X, \mu_Y, \sigma_X, \sigma_Y$.

HINT: You may want to read Chapter 5.3 in the Amemiya text or Chapter 7 in the Goldberger text.

(c) Show that the marginal distribution of X is Normal with mean μ_X and variance σ_X^2

(d) Show that $\text{Cov}(X, Y) = \rho\ \sigma_X\ \sigma_Y$. (Alternatively, that the $\text{Correlation}(X, Y) = \rho$)

(e) Many empirical economics studies assume that the relevant conditional expectations function, $E[Y|X]$, is linear in X. Critics have commented that this practice makes sense when the joint distribution of (X, Y) is believed to be bivariate Normal but less so for other distributions. Explain.

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For **(f)-(h)**, suppose X and W are independent random variables with

$$E[X] = 0 \ , \ E[X^2] = 1 \ , \ E[X^3] = 0 \ , \ E[W] = 1 \ , \ E[W^2] = 2$$

Let $Y \equiv W + WX^2$. (This problem borrows from Goldberger Exercise 6.7)

(f) Find the $BP_{MSE}(Y|X)$ and $BLP_{MSE}(Y|X)$

(g) Change the assumption $E[X^3] = 0$ to $E[X^3] = 1$. Now find the $BP_{MSE}(Y|X)$ and $BLP_{MSE}(Y|X)$

(h) Which relation remained the same in going from **(f)** to **(g)**? Which changes? Why?
(Do not simply state "because $E[X^3]$ changed ...")

Question 2: Curved Roof

These questions are modified versions of Exercises 4.1 and 5.1 in Goldberger's *A Course in Econometrics* textbook.

Consider the following joint pdf for continuous random variables X and Y

$$f(x,y) \quad = \quad \begin{cases} \frac{3}{11}\left(x^2 + y\right) & \text{for } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 1 \\ \\ \quad 0 & \text{for all other values of } x \text{ and } y \end{cases}$$

(a) Show that the above joint pdf does not violate $P(S) = 1$. i.e. probability over possible joint realizations "sum up" to 1.

(b) Derive the marginal pdf of X and of Y

(c) Derive the conditional pdf of Y given X for $0 \leq x \leq 2$.

(d) Calculate the following moments: $E[X], E[Y], E[X^2], E[Y^2], E[XY]$.

(e) Find the best predictor of Y given X under MSE: $BP_{MSE}(Y|X)$

(f) Find the best linear predictor of Y given X under MSE: $BLP_{MSE}(Y|X)$

(g) Figure 5.2 (p.55) of the Goldberger text shows $BLP_{MSE}(Y|X)$ closer to $BP_{MSE}(Y|X)$ at high, rather than low, values of x. Explain why.

> **HINT:** Goldberger suggests that you may "see" the answer in Figure 4.5 (p.42), which graphs the marginal pdf of X, $f(x)$

Question 3: Patent Race, Part II

You will want to refer to Problem Set #2 Question 5 for the basic set-up of this problem.

A major dilemma faced by some incumbent R&D firms is the decision whether to "cannibalize" their current patent.[1] An incumbent may succeed in innovating the next generation drug while the patent for the current generation drug is still valid. In such situations, the incumbent has an incentive to delay filing the patent on the new drug. If the incumbent files now, the incumbent effectively terminates the patent for the current drug and starts the patent for the new drug. By filing the patent for the new drug near the end of the patent life for the current drug, the incumbent may earn the full profit stream from both the current and new drug. The main disincentive discouraging such delay is the fear that some entrant will innovate successfully and file the patent for the new drug during the delay.

Consider the two firms, incumbent **(I)** and entrant (**E**), from the previous problem set. Suppose that the incumbent has just (time t) succeeded in innovating but the entrant still has not (as of end of t). Furthermore, suppose that the incumbent earns the following profit, depending on its actions and the "state of nature"[2]

- If the incumbent files before the entrant successfully innovates, the incumbent earns $500 million
  - The incumbent is guaranteed the $500 million if it files immediately
  - Each year the incumbent delays, the incumbent risks losing this payoff
- For each year the incumbent delays filing **and** the entrant fails to innovate, the incumbent earns an additional $100 million

Example: if the incumbent delays filing for two year (until $t+2$) and the entrant fails to innovate in $t+1$ and $t+2$, the incumbent earns $700 million. If the incumbent delays filing for two years and the entrant fails to innovate in $t+1$ but succeeds in $t+2$, the incumbent only earns $100 million.

Suppose that the patent life for the incumbent's current drug expires at the end of $t+3$. This means that the longest the incumbent would delay filing the patent is three years.

(a) For each of the four possible actions for the incumbent,(file now, file in $t+1$, file in $t+2$, file in $t+3$), calculate the expected value of the profit the incumbent would earn, in terms of θ_E. **HINT**: the relevant random variable here is the innovation success/failure of the entrant

(b) Suppose that the incumbent must commit to one of the four possible actions now (t). What is the largest value of θ_E for which the incumbent, seeking to maximize its "expected profit," would choose to delay filing the patent (i.e. **not** file now)?

We may re-visit this "patent race" in a later problem set.

[1] The study of this problem can be traced back as far back (at least) as Nobel Laureate Ken Arrow's seminar work on innovation and competition during the 1960s. More recently, the noted microeconomic theorist Jean Tirole has examined the problem using modern game theoretic tools, dubbing this fear the "replacement effect." There has been several attempts to document, empirically, this "replacement effect"

[2] A fancy way of saying "luck of draw" which is itself a colloquial way of saying "realization of random variables"

Question 4: A Past Quiz Problem (Modified)

Consider the discrete random variables X and Y defined as follows

$$X = \begin{cases} +1 & \text{with probability } \frac{1}{3} \\ 0 & \text{with probability } \frac{1}{3} \\ -1 & \text{with probability } \frac{1}{3} \end{cases} \quad \text{and} \quad Y \equiv X^2$$

(a) Derive the (marginal) distribution of Y

(b) Show that (i) $\text{Cov}(X, Y) = 0$ but (ii) X and Y are *not* statistically independent of each other.

(c) Solve for (i) $BP_{MSE}(Y|X)$ and (ii) $BLP_{MSE}(Y|X)$

(d) Often, the $BP_{MSE}(Y|X)$ and $BLP_{MSE}(Y|X)$ share similar intuition as to how X is informative about Y. But for the (X, Y) given for this problem, this does not hold true. For one of the two predictors (under MSE), X is incredibly informative about Y but for the other X is entirely uninformative about Y. Explain which is which and why.

"Food for Thought": Getting Ready to Regress

This question will not be graded. It is meant as "food for thought," to help you start thinking about some key concepts underlying upcoming lectures/discussions.

Let (X, Y) be two random variables with some well defined joint distribution. Consider a third random variable defined as $Z \equiv \gamma_0 + \gamma_1 X + Y$. Also ...

- You are given a **random** sample of N draws of (Z, X): $\{Z_i, X_i\}_{i=1}^N$
- You do **not** know the values of the constants (γ_0, γ_1) or the associated Y draws: $\{Y_i\}_{i=1}^N$

(a) Suppose you were also given the values of the associated Y draws. How would you "estimate" the unknown parameter values (γ_0, γ_1)? How good of an estimate would they be?

(b) Derive the BP and BLP of Z given X under the MSE criterion as a function of $\{\gamma_0, \gamma_1, X\}$ and any relevant moments derived from the joint distribution of $\{X, Y, Z\}$ (and from other distributions that can be derived from that joint distribution).

(c) How does your answer to **(b)** change if you were also told that X and Y were distributed independently of each other and that $E[Y] = 0$?