LECTURE 18

CLASSIFYING THE BEHAVIOR OF VOLATILITY CHANGES…

THE SKEW STICKINESS RATIO

STOCHASTIC VOLATILITY

9 CLASSES LEFT:

(Weighted variance swaps)
(Programming local volatility trees)

18.1 Benefits and Problems of Local Volatility Models

18.1.1 Local Vol May Provide Better Hedge Ratios For Index Volatility

Because it reflects the correlation between implied volatility and index level and results in a lower hedge ratio.

18.1.2 Local Vol Models Provide a Way to Value Exotics Consistent with Vanillas.

The let you determine how to hedge exotics with vanillas consistently.

18.1.3 Inadequacy of the Forward Short-Term Skew in the Model

For equity indexes, because the skew declines with increasing expiration, future local volatilities in the implied tree have less skew than current short-term implied volatilities.Therefore the short-term forward skew in a local volatility model is too flat. One needs the threat of jumps or stochastic volatility in the near future to produce a short-term skew.

A good model would look more or less time-invariant and not require recalibration.

On the other hand, all financial models need recalibration; even in Black-Scholes, the implied volatility changes from day to day. Local volatility models, like Black-Scholes, must be recalibrated regularly

The question is: to what extent do local vol models mirror the behavior of realized volatility?

18.1.4 Similarly, local vol models are not good for options on volatility.

Volatility is stochastic in real life. But in local volatility model the local volatility is not independent of the skew. It is too small, too determined by the current skew, to represent stochastic volatility realistically. This is related to the short-term forward skew being relatively flat.

$$\frac{dS}{S} = \mu dt + \sigma(S, t)dZ$$

$$d\sigma = \frac{\partial\sigma}{\partial t}dt + \frac{\partial\sigma}{\partial S}dS + \frac{1}{2}\frac{\partial^2\sigma}{\partial S^2}\sigma^2 S^2 dt$$

The *dS* terms contains *dZ* and thus indicates the lognormal volatility of the volatility ξ.

$$d\sigma = \frac{\partial\sigma}{\partial S}\sigma S dZ + \dots = \xi\sigma dZ + \dots$$

$$\xi = S\frac{\partial\sigma}{\partial S} = \frac{\partial}{\partial \ln S}\sigma(S, t)$$

If the skew flattens out as time increases, if it becomes more Black-Scholes-like, then the local volatility becomes less variable as a function of *S* as time increases, and therefore the volatility of volatility ξ decreases when the skew flattens.

This is not good. You don't want volatility to *have to* decline when the skew declines with expiration. You want to be able to specify an independent volatility of volatility. Local volatility models therefore often significantly undervalue structured products that depend on volatility of volatility -- e.g. forward start options, cliquets.

An improvement would be to add an independent stochastic volatility to the local vol model.

18.2 Dynamics: Different Patterns of Volatility Change

Local Volatility Predictions, Which We've Seen Before

- Local volatility models relate the slope of the current skew, $\frac{\partial\Sigma}{\partial K}$, to the rate of change of volatility, $\frac{\partial\Sigma}{\partial S}$.
- There are various possible heuristic relationships between $\frac{\partial\Sigma}{\partial K}$ and $\frac{\partial\Sigma}{\partial S}$.
- The sticky strike rule, the sticky delta rule, and the sticky local volatility model are examples.
- Index option markets do not perfectly satisfy any one of these models or rules.

In the local volatility model we had, near at-the-money, from the averaging, the symmetric relation

$$\Sigma(S, K) \approx \sigma_0 + 2\beta S_0 - \beta(S+K)$$

so that

$$\frac{\partial\Sigma}{\partial S} = \frac{\partial\Sigma}{\partial K} = -\beta$$

This relates statics to dynamics, and forecasts how volatility will change.

Note that in this model, if we know the skew as a function of K for fixed S, then we know the implied volatility for a fixed K at all values of S, by symmetry.

Heuristic Rules & Models for Variation of Implied Volatility Σ: The Relation of the (Instantaneous) Skew to Dynamics

We are assuming a skew independent of expiration below, to keep it simple.

Traders like heuristic rules for how the B-S *quoting parameter* Σ varies.

Specify what **doesn't change** rather than what changes.

The Sticky Strike Rule.
The Sticky Delta Rule.
The Sticky Local Volatility Rule.

The Sticky Strike Rule (A rule, not a consistent model)

Each option of a definite strike **maintains its initial implied volatility** – hence the "sticky strike" appellation. This is the simplest "model" of implied volatility:

$$\Sigma(S, K) = f(K)$$

In the linear approximation,

$$\Sigma(S, K, t) = \Sigma_0 - \beta(K - S_0) \qquad \text{Sticky Strike Rule, independent of } S \text{ for all } t$$

(We have assumed $\beta(t) \equiv \beta$, independent of t. β can change, especially during crisis periods.)

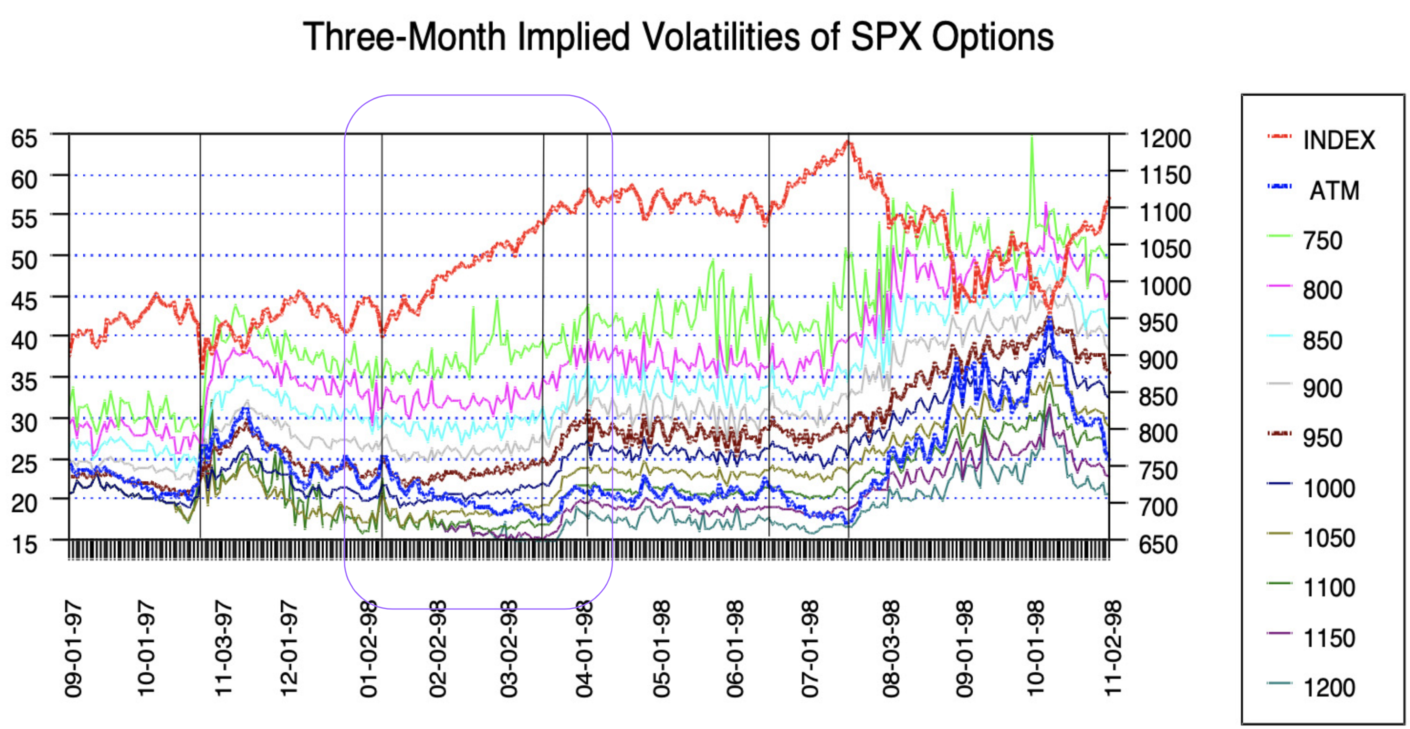
Intuitively, "sticky strike" is a poor man's inconsistent attempt to preserve the BS model.
As S moves, each option keeps the exactly the same constant future instantaneous volatility in its evolution, inconsistently different for different options.

Implied volatility for an option of strike K is independent of S, and therefore $\Delta = \Delta_{BS}$.

TABLE 1. Volatility behavior using the sticky-strike rule.l

Quantity	Behavior
Fixed-strike volatility:	is independent of index level
At-the-money volatility $\Sigma_{atm}(S)$:	$\Sigma_{atm}(S, t) = \Sigma_0 - \beta(S - S_0)$ which decreases as index level increases because of the negative skew.
Exposure Δ:	$= \Delta_{BS}$

You can think of this model as representing Irrational Exuberance (Greenspan 1996). Everything is getting better and safer as the market goes up. Σ_{atm} decreases as S increases.



The Sticky Delta Rule

It's easier to start by explaining the related concept of sticky moneyness.

Sticky moneyness means that an option's volatility depends only on its moneyness K/S or, approximately, $K-S$ for K close to S in the linear approximation.

$$\Sigma(S, K, t) = \Sigma_0 - \beta(K-S) \qquad \text{Sticky Moneyness Rule}$$

Intuition: the volatility of the most liquid option, should stay constant as the index moves. Similarly, a 10% out-of-the-money should always have same volatility.

It's a scale-invariant model of common sense and moderation.

For a roughly linear skew $\Sigma \approx \Sigma(S-K)$

Therefore implied volatility must rise when S rises

Sticky delta means that the implied volatility must be purely a function of Δ_{BS}, i.e. it must be a

function of $\dfrac{\ln(K/S)}{\Sigma(S,K)\sqrt{\tau}}$, so that $\Sigma(S,K) = f\left(\dfrac{\ln\left(\frac{K}{S}\right)}{\Sigma(S,K)\sqrt{\tau}}\right)$ This is a nonlinear equation for $\Sigma(S, K)$.

A linear approximation for the sticky delta rule is

$$\Sigma(S,K) = \Sigma_0 - \beta\frac{\ln\left(\frac{K}{S}\right)}{\Sigma(S,K)\sqrt{\tau}}$$

For a weak skew, low β we can use the approximation

$$\Sigma(S,K) = \Sigma_0 - \beta\frac{\ln\left(\frac{K}{S}\right)}{\Sigma_{\mathrm{ATM}}(S)\sqrt{\tau}}$$

If we set Σ_0 to be the volatility for a 50-delta call option, we can write

$$\Sigma(S,K,\tau) = \Sigma_0 - \beta'(0.5 - \Delta(S,K,\tau,\Sigma_{\mathrm{ATM}}))$$

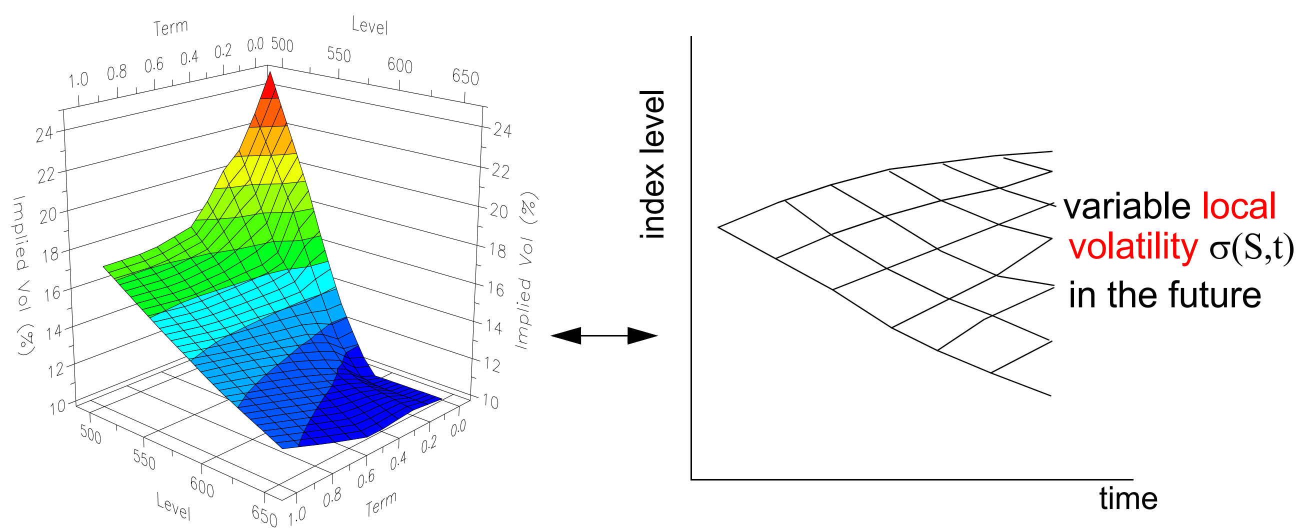
TABLE 2. Index Volatility behavior using the sticky-delta/moneyness rule.

Quantity	Behavior
Fixed-strike volatility:	increases as index level S increases
At-the-money volatility:	is independent of index level
Exposure Δ:	$> \Delta_{BS}$

The (Sticky) Local Volatility Model

All current index options prices determine a single **consistent** unique set of local volatilities.

The implied tree corresponding to a given implied volatility surface.



The implied tree/local volatility model attributes the implied volatility skew to the market's expectation of higher realized volatilities and higher implied volatilities if the index moves down.

As the index level within the tree rises, you can see that the local volatilities decline, monotonically and (roughly) linearly, in order to match the linear strike dependence of the negative skew.

$$\Sigma(S, K, t) = \Sigma_0 - \beta(K + S - 2S_0) \qquad \text{Local Volatility Model, symmetric in K,S}$$

At-the-money volatility is given by

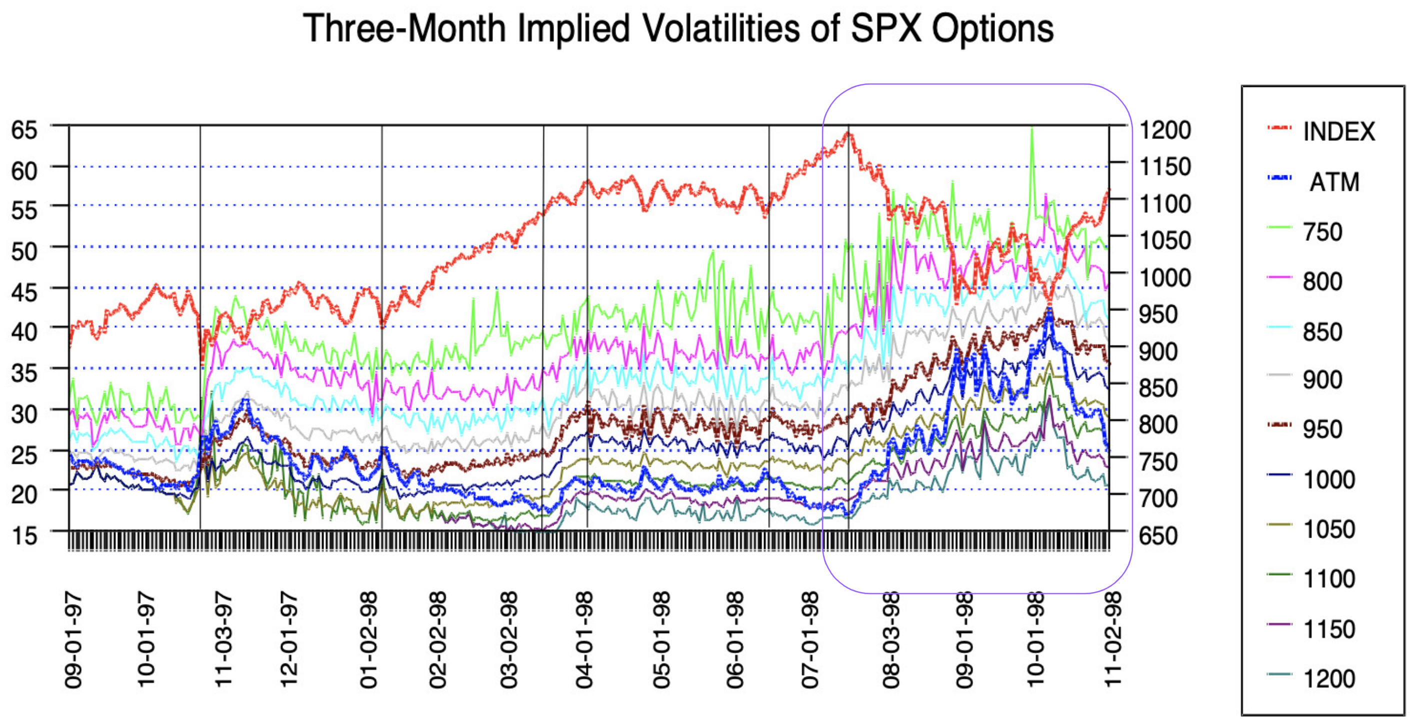
$$\Sigma_{atm}(S, t) = \Sigma_0 - 2\beta(S - S_0)$$

Implied volatilities decrease as K or S increases.
At-the-money implied volatility decreases twice as fast.

TABLE 3. Equity index volatility behavior in the sticky implied tree model.

Quantity	Behavior
Fixed-strike volatility:	increases as index level decreases
At-the-money volatility:	increases twice as rapidly as index level decreases
Exposure Δ:	$< \Delta_{BS}$

In this regime the options market experiences fear. The implied tree model implicitly assumes the skew arises from a fear of higher market volatility in the event of a fall, and assumes that after the fall, atm market volatility will rise twice as fast.



Summary of the Rules

Assume the current skew linear $\Sigma(S_0, K) = \Sigma_0 - \beta(K - S_0)$.

Sticky	General functional form for future implied volatility	Linear approximation: Future skew when stock price is S	Model with this property
Strike	$\Sigma(S, K) = f(K)$	$\Sigma(S, K) = \Sigma_0 - \beta(K - S_0)$	Black-Scholes[a]
Moneyness	$\Sigma(S, K) = f(K/S)$	$\Sigma(S, K) \approx \Sigma_0 - \beta(K - S)$	Stochastic volatility[b], jump diffusion
Implied tree/local volatility	$\Sigma(S, K) = f(K, S)$	$\Sigma(S, K) \approx \Sigma_0 - \beta(K + S - 2S_0)$ because Σ is approximately the average of the local volatilities between spot and strike.	Local volatility[c]
Delta	$\Sigma(S, K) = f(\Delta)$	$\Sigma(S, K) \approx \Sigma_0 - \beta[0.5 - \Delta_{call}(S, K, t, T)]$ or $\Sigma(S, K) \approx \Sigma_0 - \beta'\left[\frac{\ln K/S}{\Sigma\sqrt{\tau}}\right]$ Note that Δ is itself a function of Σ!	?

a. The Black-Scholes model corresponds roughly to the sticky strike rule of thumb, but it cannot honestly accommodate a skew, because all implied volatilities are the same irrespective of strike in the Black-Scholes model. So, although people use it, it's not really consistent from a theoretical point of view

b. In stochastic volatility models, there is another stochastic variable, the volatility itself, and so $\Sigma(S, K) = f(K/S)$ only if the other stochastic variable doesn't change.

c. **Crepey, Quantitative Finance 4 (Oct. 2004) 559-579,** argues that the local volatility hedging is the best for equities markets, in that it gets things right when the market moves a lot and isn't very wrong otherwise. Because markets crash down and drift up, don't crash up and drift down.

Categorizing Models: The Skew Stickiness Ratio R

A more general linearized formula with a negative slope β:

$$\Sigma(S, K) = \Sigma_0 - \beta(K - S) - B(S - S_0) \qquad \Sigma_{ATM} \equiv \Sigma(S, S) = \Sigma_0 - B(S - S_0)$$

where we assume β and B are constant over the period of interest.

The rules are:

1. Sticky strike: $B = \beta$
2. Sticky moneyness: $B = 0$
3. Sticky local volatility: $B = 2\beta$

Now look at the **evolution of the smile** compared to **the current skew**:

$$\frac{\partial \Sigma_{ATM}}{\partial S} = -B$$

$$\frac{\partial \Sigma(S, K)}{\partial K} = -\beta$$

$$R = \frac{\frac{\partial \Sigma_{ATM}}{\partial S}}{\frac{\partial \Sigma(S, K)}{\partial K}} = \frac{B}{\beta}$$

***R*, the skew stickiness ratio**, is the rate at which atm volatility changes as S changes, divided by the current skew, and is easy to observe in liquid markets.

The three rules correspond to the following predicted values of *C*:

1. Sticky strike: $R = 1$
2. Sticky moneyness: $R = 0$
3. Sticky local volatility: $R = 2$

Kamal and Gatheral in the paper I posted find that *R* for index options is approximately 1.5, somewhere between sticky strike and sticky local volatility.

Which Model to Use?

Use local vol if the asset has a volatility that seems to depend on market level.

Use local vol as a way of hedging exotics from vanillas, heuristically.

But must use stochastic vol if the option has a value strongly dependent on the random level of volatility, e.g. forward start options.

Stochastic Local Volatility (later) can capture both fitting the smile and having enough volatility for the stock at all times.

18.3 Programming Local Volatility Trees

local binomial.pdf in Matlab

18.4 Replicating Weighted Variance Swaps ($r = q = 0$ for simplicity)

An ordinary variance swaps pays $\frac{1}{T}\int_0^T \sigma^2(t)dt \equiv \frac{1}{T}\int_0^T \frac{(dS_t)^2}{S_t^2}$. Eq 18.1

Think of it as a $\frac{1}{S^2}$ weighting of quadratic variation $(dS)^2$ because $(dS)^2 = \sigma(t)^2 S^2(t)dt$.

Let's re-derive the standard variance swap replication using the static replication formula:

Choose some payoff $W(S)$ to replicate and apply Ito's Lemma to it to generate the weighted variance and a total derivative dW:

$$\frac{dS}{S} = \mu dt + \sigma dZ \qquad\qquad dW = \frac{\partial W}{\partial S}dS + \frac{1}{2}\frac{\partial^2 W}{\partial S^2}(dS)^2 = \frac{\partial W}{\partial S}dS + \frac{1}{2}\frac{\partial^2 W}{\partial S^2}S^2\sigma^2 dt$$

Then integrating

$$W(S_T) - W(S_0) = \int_0^T \frac{1}{2}\frac{\partial^2 W}{\partial S^2}S^2\sigma^2 dt + \int_0^T \frac{\partial W}{\partial S}dS$$

$$\underbrace{\int_0^T \frac{1}{2}\frac{\partial^2 W}{\partial S^2}S^2\sigma^2 dt}_{\text{swap}} = \underbrace{W(S_T) - \underbrace{W(S_0)}_{0}}_{\text{static replicate}} - \underbrace{\int_0^T \frac{\partial W}{\partial S}dS}_{\text{dynamic rebalance}}$$

You can replicate a European payoff $W(S)$ at expiration time T as a function of the terminal stock price S_T by means of the decomposition

$$W(S_T) = W(A) + W'(A)[S_T - A] + \int_0^A P(S_T, K) W''(K) dK + \int_A^\infty C(S_T, K) W''(K) dK$$

nice if this is zero at A and choose A = S_0 for simplicity

where A is an arbitrary positive number, $P(S_T, K)$ is the terminal payoff of a standard put and $C(S_T, K)$ is the terminal payoff of a standard cal.

There are two ways to look at $W(S)$:
(i) replication of this payoff at expiration produces calls and puts with weight W'';
(ii) Ito on $W(S)$ produces the weighted variance as $W(\ )$ evolves.

For a regular variance swap choose $W(S) = \left(\frac{S - S_0}{S_0} - \ln\frac{S}{S_0}\right)$ and note

$$W(S_0) = 0 \qquad W'(S_0) = \left.\frac{1}{S_0} - \frac{1}{S}\right|_{S = S_0} = 0 \qquad W''(S) = S^{-2}$$

$$\int_0^T \frac{1}{2}\sigma^2 dt = W(S_T) + \int_0^T \left(\frac{1}{S} - \frac{1}{S_0}\right) dS = \frac{1}{K^2}[\text{calls} + \text{puts}] + \int_0^T \left(\frac{1}{S} - \frac{1}{S_0}\right) dS$$

from the static replication of the payoff with calls and puts

We can replicate $W(S_T)$ with calls and puts with weights K^{-2} because $W''(K) = K^{-2}$ from the replication general formula, and since its value and derivative vanishes at A, there is no need for zero coupon bonds or forwards.

$$\frac{1}{2}\int_0^T \sigma^2 dt = \underbrace{\frac{1}{K^2}[\text{calls} + \text{puts}]}_{\text{replicate}} + \underbrace{\int_0^T \left(\frac{1}{S} - \frac{1}{S_0}\right) dS}_{\text{dynamic rebalance}}$$

The cost of replicating the variance is only the cost of the weighted puts and calls, and the rebalancing generates the variance.

Note the serendipity that the density of calls and puts is an inverse square, and the weighting of $(dS)^2$ in the definition of the variance swap in Equations 18.1 $\frac{1}{T}\int_0^T \sigma^2(t)dt \equiv \frac{1}{T}\int_0^T \frac{(dS)^2}{S^2}$ is an inverse square in S. **It's because replication of a function involves the second derivative density and Ito term relating the function to variance involves second derivative too.**

Now let's use this to look at weighted variance swaps:

To replicate $\int_0^T f(S_t)\sigma^2(t)dt \equiv \frac{1}{T}\int_0^T f(S_t)\frac{(dS_t)}{S_t^2}$ as the **weighted variance** for some desired weighting function $f(S_u)$ choose the function $W(S_u)$ to replicate such that when you apply Ito's Lemma to it you get $S^2 W''(S) =$ desired weighting $f(S_u)$.

$$dW = W'dS + \frac{1}{2}W''\sigma^2 S^2 dt$$

$$W''\sigma^2 S^2 dt = 2[dW - W'dS]$$

$$\int_0^T W''\sigma^2 S^2 dt = 2\left[W(S_T) - W(S_0) - \int_0^T \left(\frac{\partial W}{\partial S}\right) dS\right]$$

weighted variance
$W''S^2 = f$

replicate
with puts and calls

dynamic rebalancing

1. $W'' = S^{-2}$ gives standard variance capture and $W''(K) = K^{-2}$ is also weight of calls and puts.

2. $W'' = S^{-1}$ and $\int_0^T \sigma^2 S dt$ gives less variance for low S, sometimes called a gamma swap, and note that the weight of calls and puts needed is K^{-1}.

$$W(S) = \frac{S}{S_0}\ln\frac{S}{S_0} - \frac{(S-S_0)}{S_0} \qquad W(S_0) = 0$$

$$W'(S) = \frac{1}{S_0}\ln\frac{S}{S_0} \qquad W'(S_0) = 0 \qquad .$$

$$W''(S) = \frac{1}{S_0}\frac{1}{S}$$

Replicated by 1/K weighted sum of puts and calls.

3. $W'' = 1$ and $\int_0^T \sigma^2 S^2 dt$ gives even less variance for low S, sometimes called an **arithmetic variance swap**, and note that the weight of calls and puts to replicate the swap is 1.

$$W(S) = \frac{1}{2}(S-S_0)^2 \qquad W' = (S-S_0) \qquad W'' = 1$$

4. Corridor variance swap -- choose a clever W to capture variance only in a corridor of the stock price by using indicator functions to constrain it. Maybe later …

Stochastic Volatility

18.5 Introduction to Stochastic Volatility

Approaches to Stochastic Volatility Modeling

The local volatility model is a special case of a stochastic volatility in which $\sigma(S, t)$ is 100% correlated with the stochastic stock price. But volatility is independently stochastic too.

Several approaches to introducing a stochastic volatility:

• Allow the instantaneous stock volatility σ itself to be truly stochastic:

(i) **Start with Black-Scholes, no skew**, and let σ be a stochastic variable independent of S, *and then add correlation* to obtain the skew on top of the stochastic volatility;
(ii) **Start with local volatility** $\sigma = \sigma(S)$ so we **begin** with a skew, *and then add volatility* to that skew. (SABR-type models)
(iii) **Stochastic local volatility**. Stochastic vol to get more reality, non-parametric local vol to calibrate to skew.

• BGM-type market models. Let the Black-Scholes implied volatilities $\Sigma(K, t)$ be stochastic. There are then strong constraints on the evolution of the B-S implied volatilities in order to avoid arbitrage. (Analogous to letting zero-coupon yields vary independently.) Not used much.

Comment: Modeling stochastic volatility is much more complex than modeling local volatility. We will develop models and study the character of the solutions and their smile.

Our elliptical path:

1. Heuristic vanna-volga approach to skew by perturbing the Black-Scholes solution with a stochastic volatility.
2. The rigorous SDEs for stochastic volatility and stochastic stock price
3. Mean reversion of volatility
4. Another approach: Stochastic Local Volatility. The parametric SABR model that begins with a local volatility $\sigma = \sigma(S)$ and then adds stochasticity to the evolution of the local volatility
5. Risk-neutral valuation: The riskless hedge and the resultant PDE for the value of the option
6. The Hull-White expected discounted value solution to stochastic volatility when the correlation is zero
7. Monte Carlo solutions to the PDE more generally
8. Semi-analytic solutions and the asymptotic properties of the smile with stochastic volatility.
9. Hedging in a stochastic volatility model

18.6 A HEURISTIC VANNA-VOLGA/TRADER APPROACH FOR INTRODUCING STOCHASTIC VOLATILITY INTO THE BLACK-SCHOLES MODEL

Assume rates and dividends are zero for simplicity. Add stochastic volatility to BS:

$$
\begin{aligned}
dC &= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dSd\sigma \\
&= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dSd\sigma \\
&= \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \cancel{\frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}}d\sigma + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dSd\sigma
\end{aligned}
$$

Now suppose that we constructed a riskless hedge that is long the call C and short just enough stock and enough other options with volatility σ so that the hedged portfolio is instantaneously riskless.

Then the only part of dC that survives is

$$
dC = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2\right)dt + \frac{1}{2}\frac{\partial^2 C}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 C}{\partial S \partial \sigma}dSd\sigma
$$

We don’t know the value of the partial derivatives in the above equation, since we haven’t applied the methods of risk-neutral valuation to determine the partial differential equation for the value of the option with both stochastic volatility and stochastic stock price. In order to proceed further we will replace the unknown partial derivatives by their values in the Black-Scholes model, C_{BSM}, hoping that these capture the approximate contribution to the P&L from the stochastic volatility.

Then for zero rates and dividends

$$\frac{\partial C_{\mathrm{BSM}}}{\partial t} + \frac{1}{2}\frac{\partial^2 C_{\mathrm{BSM}}}{\partial S^2}\sigma^2 S^2 = 0$$

The **expected change** in the value of the hedged portfolio from stochastic volatility is approximately

$$dC \cong \frac{1}{2}\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2} E\left[d\sigma^2\right] + \frac{\partial^2 C_{\mathrm{BSM}}}{\partial S \partial \sigma} E\left[dS d\sigma\right]$$

volga, vol butterfly spread — vanna risk reversal

Approximate by using the BS derivatives of $C_{BS}(S, t, K, T, r, \sigma)$ in the Ito expansion

$$V = \frac{\partial C_{\mathrm{BSM}}}{\partial \sigma} = \frac{\sqrt{\tau}}{\sqrt{2\pi}} S e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right)^2}$$ Vega is always positive

Vega decreases rapidly as ln(S/K) gets more negative or more positive.

$$\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2} = \frac{V}{\sigma}\left[\frac{\ln^2\left(\frac{S}{K}\right)}{\sigma^2 \tau} - \frac{\sigma^2 \tau}{4}\right]$$

Volga is mostly positive except atm

Thus Volga has two peaks

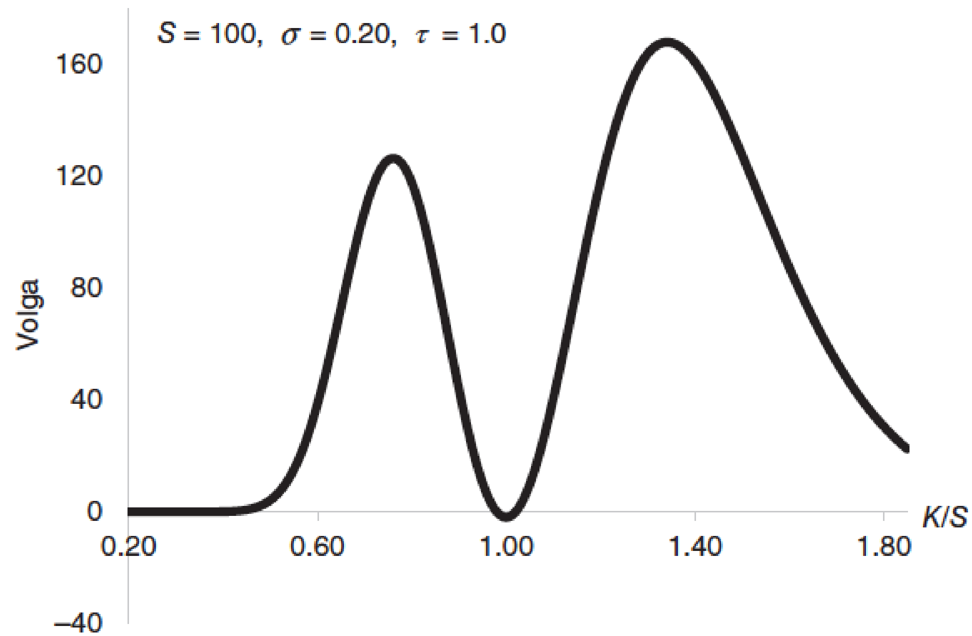


FIGURE 19.1 BSM Volga of a Standard Call Option

Mostly positive convexity, with peaks on either side

$$dC = \frac{1}{2}\frac{\partial^2 C_{\mathrm{BSM}}}{\partial \sigma^2} E\left[d\sigma^2\right] + \frac{\partial^2 C_{\mathrm{BSM}}}{\partial S \partial \sigma} E\left[dS d\sigma\right]$$

A hedged option is long gamma $\frac{\partial^2 C}{\partial S^2}$, long volatility $\frac{\partial C}{\partial \sigma}$ and **long volatility of volatility** $\frac{\partial^2 C}{\partial \sigma^2}$, esp out of money or deep in the money. Check that C_{BSM}(high vol) + C_{BSM}(low vol) > C_{BSM}(av vol) because the second derivative w.r.t volatility is positive -- convexity in volatility.

If volatility is volatile, then the convexity in volatility adds value to the option away from at-the-money, and adds value to out-of-the-money options relative to at-the-money options in a U-shaped smile.

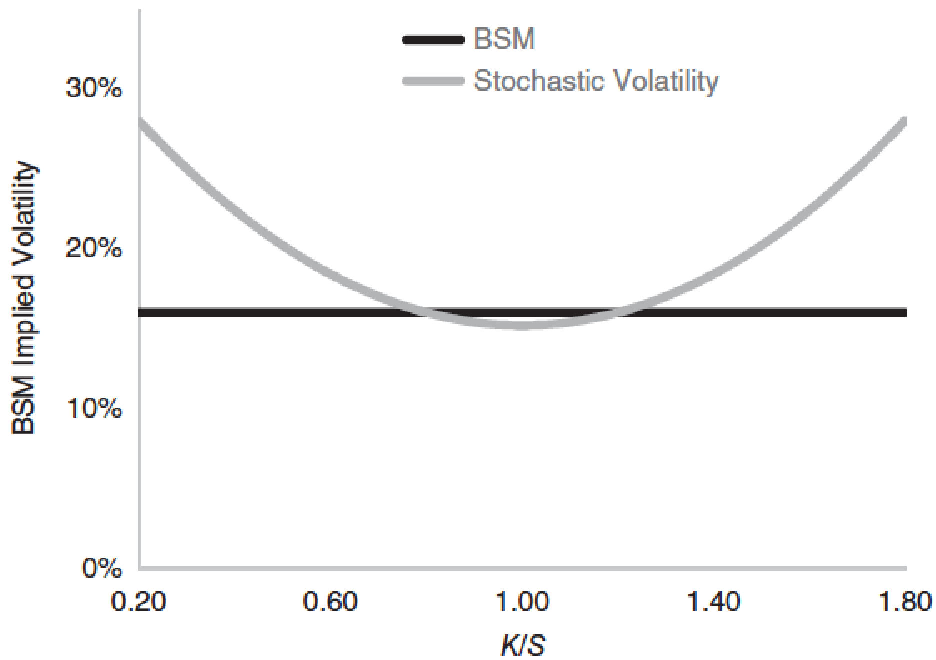


FIGURE 19.2 BSM Implied Volatility versus Moneyness

Similarly, Vanna

$$\frac{\partial^2 C_{\mathrm{BSM}}}{\partial S \partial \sigma} = \frac{V}{S}\left(\frac{1}{2} - \frac{1}{\sigma^2 \tau} \ln\left(\frac{S}{K}\right)\right)$$

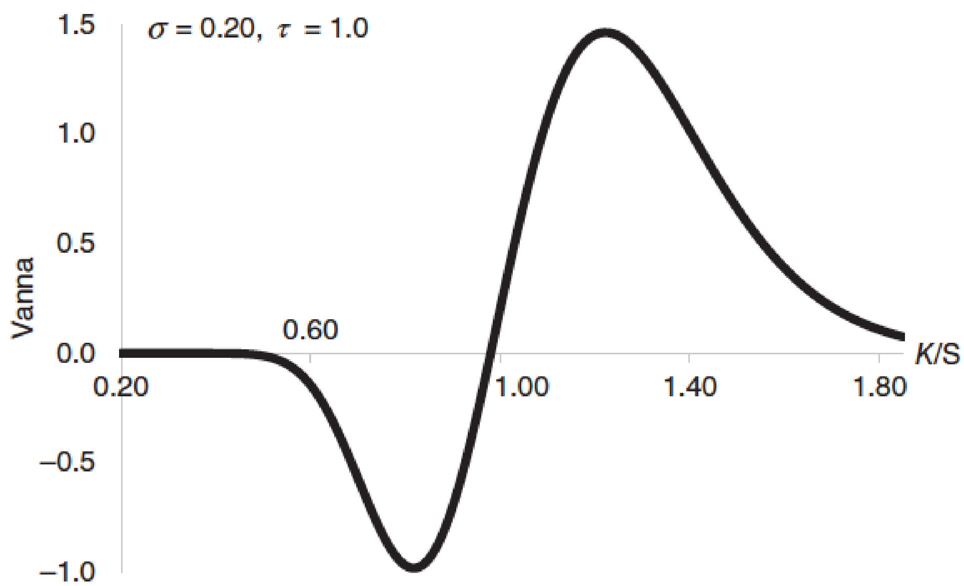


FIGURE 19.3 BSM Vanna, DvegaDspot, or Ddelta/Dsigma

This adds value to one out-of-the-money wing of the implied volatility and subtracts from the opposite one. For typical values of σ and τ, vanna will be positive when the call option is out of the money ($K > S$) and negative when the call option is in the money ($K < S$). If $E[dSd\sigma]$ is positive (if the stock price and its volatility are positively correlated), the vanna term will enhance the P&L and hence value of a Black-Scholes option at high strikes and reduce it at low strikes. The opposite is the case if the correlation is negative. Since the equity index skew is typically negative, with low strikes carrying greater implied volatility than high ones, we can guess that in a stochastic volatility model we will require a negative correlation between the index and its volatility in order to reflect the skew.

Crude usefully intuitive ways to understand the effect of stochastic volatility on the smile.

18.1 Aside: Vanna Volga Models for Exotics in a Skew

One good reference is Castagna and Mercurio, Risk Magazine 2007: *The Vanna Volga Method for Implied Volatilities.* You can find it on the internet.

For risk management with a stochastic volatility model, we will see that it's hard to understand really well the connection between **observed changes in implied volatility** Σ and **changes in the stochastic parameters** $\sigma, \xi \ldots$ **of the model**. In the local volatility model, in contrast, there was a simple (averaging) relation and the Dupire equation.

The Vanna Volga method tries to heuristically cut out the model parameters and deal directly with the implied volatilities. In its simplest form, it uses the Black-Scholes values for three different strikes as "control variates" for a stochastic volatility world to then estimate all other values for different strikes. Given the price of three different strike options of a given expiration, it lets us value any other option in terms of them. Here is a sketch of a proof. P_{BS} is an exotic option we can value in a BS world.

$$dP_{BS} = \frac{\partial P_{BS}}{\partial t} + dS\frac{\partial P_{BS}}{\partial S} + d\sigma\frac{\partial P_{BS}}{\partial \sigma} + \frac{1}{2}\frac{\partial^2 P_{BS}}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 P_{BS}}{\partial \sigma^2}d\sigma^2 + \frac{\partial^2 P_{BS}}{\partial S\partial \sigma}dSd\sigma$$

We replicate the exotic target option $P_{BS}(T;\sigma)$ in a GBM BS world approximately by writing its volatility sensitivities as a combination of vanilla options with the same volatility sensitivities:

$$\frac{\partial}{\partial\sigma}P_{BS}(T;\sigma) = \sum_{i=1}^{3} w_i\left[\frac{\partial}{\partial\sigma}C_{BS}(K_i, T;\sigma)\right]$$

$$\frac{\partial^2}{\partial\sigma^2}P_{BS}(T;\sigma) = \sum_{i=1}^{3} w_i\left[\frac{\partial^2}{\partial\sigma^2}C_{BS}(K_i, T;\sigma)\right] \quad \sigma \text{ is atm volatility of vanilla options w maturity } T$$

$$\frac{\partial^2}{\partial\sigma\partial S}P_{BS}(T;\sigma) = \sum_{i=1}^{3} w_i\left[\frac{\partial^2}{\partial\sigma\partial S}C_{BS}(K_i, T;\sigma)\right]$$

To fix the weights w_i of the replicating portfolio, we require that the vega, volga and vanna of the exotic option in a BS world, which we know, equals the vega, volga and vanna of the three replicating options at the current time and stock price. This gives three equations for three unknowns in a BS world.

Now we turn on the skew and depart from the “flat-smile” BS world. We “hope” that to first order the volatility derivatives tell us how much the price will change as the volatility and the model changes. The new option price is defined by adding to the “flat-smile” BS price the price difference between market price and BS theory price for each vanilla option.

$$P_{VannaVolga}(\ ) = P_{BS}(\ ) + \sum_{i=1}^{3} w_i [C_{MKT}(K_i) - C_{BS}(K_i, \sigma)]$$

When $P(\ )$ is a vanilla option with one of the three strikes K_i, $w_i = 1$ for just that option, and we reproduce the market price of the three vanilla options. This lets us estimate the price and hence the implied vol in a smile of any exotic option, or vanilla option with a different strike, from the implied vols of three strikes, by a rule for extrapolation and interpolation.

This formula lets us estimate how the observed change in one option's implied volatility will affect the value of an option of any strike, and in particular the value of an exotic option such as a barrier option.

We are using the familiar BS for three strikes as a control variate.

We'll cover this again towards the end of the stochastic volatility section.