COMP 149 Homework #3

Due Date: Friday, February 21st, 12:45pm

You may submit at the end of the instructor's office hours on the due date, or at any time sliding under the instructor's office door (Swan 302) before the due.

0. Write your name and student ID Number. Multi-page submissions must be stapled.

1. An integer n is divisible by $k \in \mathbb{Z}$ when there exists another integer m satisfying $n = km$. Integers that divide n are called factors. A proper factor k dividing n satisfies $0 < k < n$. A *perfect number* is an integer $n > 0$ that can be written as the sum of its proper factors. For example, the number 14 is not perfect: the set of proper factors of 14 are $\{1, 2, 7\}$, but $1 + 2 + 7 = 10 \neq 14$.

   Prove the following statements.

   (a) At least one perfect number exists.

   (b) For any prime integer p, the positive integer p^2 is not a perfect prime.

2. Prove the following statements using **weak** induction:

   (a) $\sum_{i=1}^{n} f_i = f_{n+2} - 1$, where f is the Fibonacci Sequence: $f_1 = f_2 = 1, f_n = f_{n-1} + f_{n-2}$.

   (b) $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$, where f is the Fibonacci Sequence.

   (c) $n + n^2 < n^3$, for all integers $n \geq 2$.

3. Suppose you want to write an algorithm to compute a^x, where $x = 2^n$, and a, n are positive integers given as input.

   (a) Using the equality $a^{2^{n+1}} = (a^{2^n})^2$, devise a recursive algorithm which matches the above specifications.

   (b) Prove the correctness of your algorithm using induction.

4. Define $K(0) = 1$, $K(1) = \frac{1}{2}$, and $K(n) = 9(K(\lfloor \frac{n}{3} \rfloor - 1) + n^3 + 2n$. We will attempt to prove the fact that $K(n) \leq \frac{3}{2}n^3$ for all $n \geq 0$, by direction of part (a) and (b).

   (a) First attempt to prove the fact using weak induction. Clearly point out the part where your proof attempt falters.

   (b) Next attempt to prove the same fact using strong induction. You should get further than in part (a), but still fail to finish the proof. Clearly point out the part where your proof attempt falters.

   (c) Now instead prove the stronger claim that $K(n) \leq \frac{3}{2}n^3 - n$ for all $n \geq 0$.

5. Prove that the first player has a winning strategy for the game of Chomp, introduced in class, if the initial board is square. **Hint:** Use strong induction to show that the following strategy works. For the first move, the first player chomps all cookies except those in the left and top edges. On subsequent moves, after the second player has chomped cookies on either the top or left edge, the first player chomps cookies in the same relative positions in the left or top edge, respectively.

Practice Problems on back.

If you need some extra practice, try the following problems. Do not submit them, as they will not be graded. All practice problems and solutions are in the zyBook.

Chapter 8.4: exercises 8.4.1, 8.4.2 (a)-(d), 8.4.3 (a)-(d)
Chapter 8.5: exercises 8.5.1 (d)-(f), 8.5.3 (a)-(b)
Chapter 8.6: exercises 8.6.1, 8.6.2