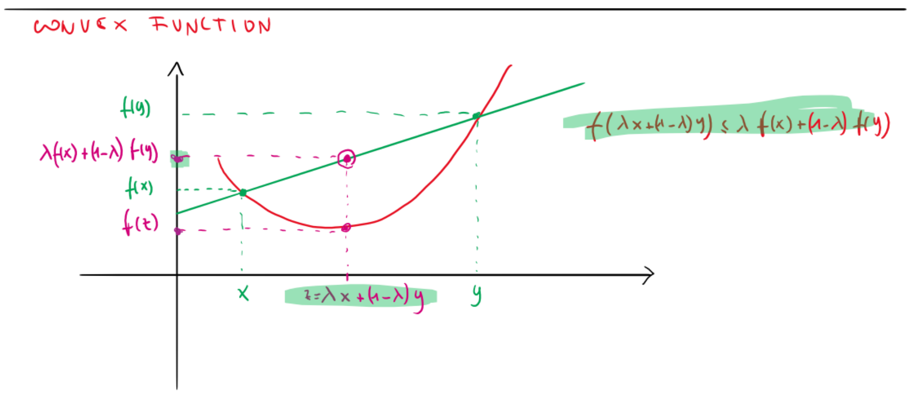
convex function = any function that has a shape like a bowl

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y) \quad \text{for all } x, y \in \mathcal{X}, t \in [0,1]. \quad -(1)$$

Notice the equal in the “<=” sign!

From Rinaldi’s whiteboard:



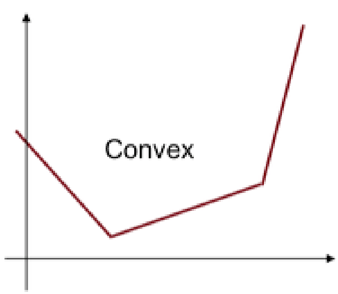
Interpretation: For any two points, we draw a line between them, and for any point on that line, the value of the function in this point is below the point on the line.

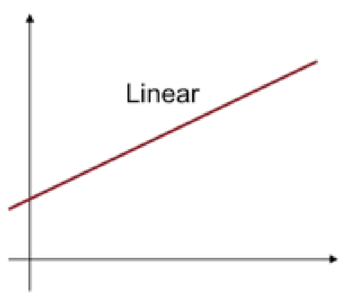
strictly convex function = has only one minimum (local = global)

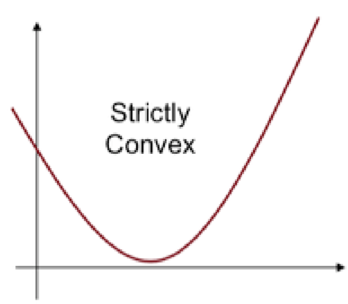
$$f(tx + (1-t)y) < t f(x) + (1-t) f(y) \quad \text{for all } x \neq y \in \mathcal{X}, t \in (0,1).$$

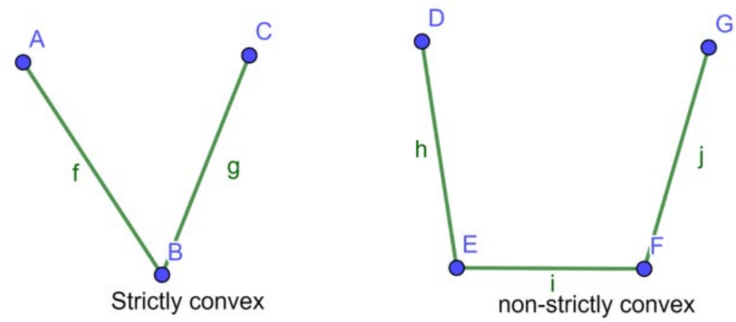
Here we don’t have the equal in the sign!

Examples for strictly convex:









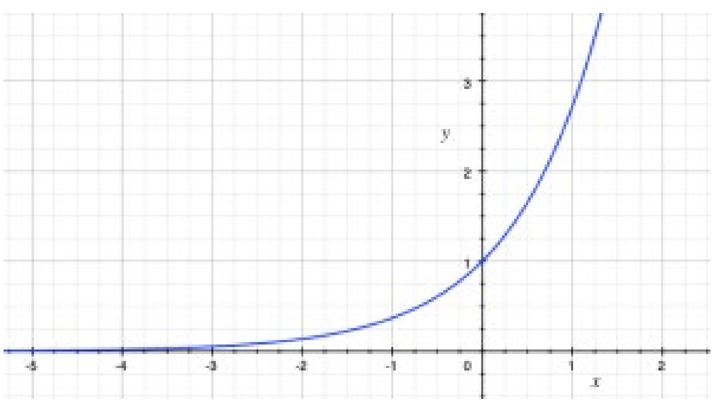
Every strictly convex function is convex, but the reverse is not true. (see pictures above)

Strongly convex function = how "convex" or "curved" a convex function is.

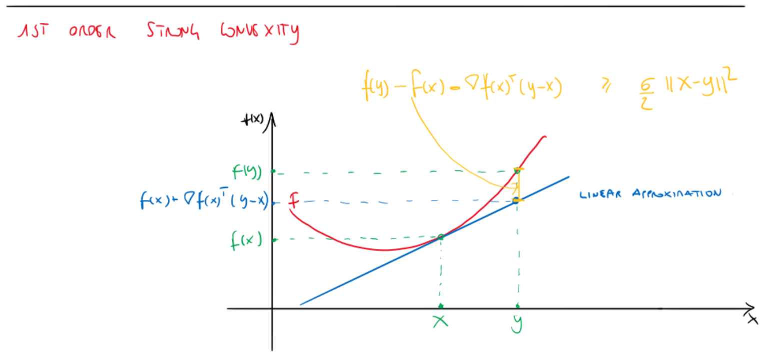
It can be seen as a kind of "parameterized strict convexity".

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - \frac{1}{2}mt(1-t)\|x-y\|_2^2$$
$$\text{for all } x, y \in \mathcal{X}, t \in [0,1]. \quad -(4)$$

You can think of the parameter m as measuring how "curved" the function is: the larger m is the more curved f is. That is why f(x) = exp(x) is not strongly convex: as x goes to infinity, the curve becomes flatter and flatter (m -> 0).



From Rinaldi’s whiteboard: (sigma is the same as m above)



Between two strongly convex functions, the function with the larger m is more strongly convex.
=> this function's curvature grows more rapidly

Extras for strong convexity:

From Rinaldi’s slides (here lambda refers to t in above’s equation):

Finally, by taking limit $\lambda \to 0$, we can write

$$f(x) \geq \underbrace{f(x^*)}_{\text{const}} + \underbrace{\frac{\sigma}{2}}_{\text{const}}\|x - x^*\|^2,$$

Strong Convexity

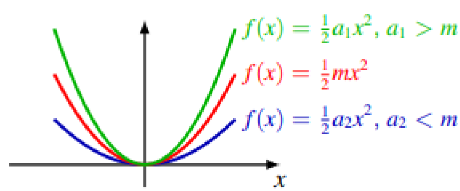
A function f is strongly convex with parameter $m > 0$, or simply m-strongly convex, if

$$\tilde{f}(\boldsymbol{x}) = f(\boldsymbol{x}) - \frac{m}{2}\|\boldsymbol{x}\|^2$$

is convex.

Note. $f(\boldsymbol{x}) = \frac{m}{2}\|\boldsymbol{x}\|^2 + \tilde{f}(\boldsymbol{x})$, i.e. f is $\frac{m}{2}\|\boldsymbol{x}\|^2$ plus an extra convex term. Informally, “m-strongly convex” means at least as “convex” as $\frac{m}{2}\|\boldsymbol{x}\|^2$.

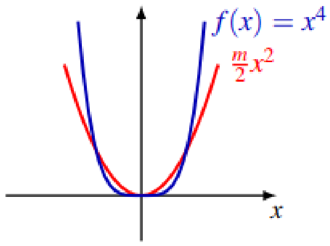
Example. $f(\boldsymbol{x}) = \frac{a}{2}\|\boldsymbol{x}\|^2$ is m-strongly convex iff $a \geq m$

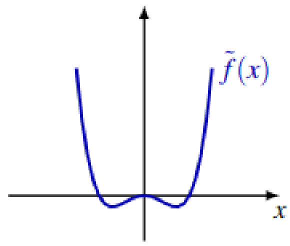


Example. $f(x) = x^4$ is not m-strongly convex for any $m > 0$, as $\tilde{f}(x) = x^4 - \frac{m}{2}x^2$ is not convex,

$$\tilde{f}''(x) = 12x^2 - m < 0$$

for $|x| < \sqrt{m/12}$.





Useful link:

[(Strictly/strongly) convex functions | Statistical Odds & Ends (wordpress.com)]