Solving Minimum Enclosing Ball with Outliers: Algorithm, Implementation, and Application

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───── **Abstract** ─────

Motivated by the arising realistic issues in big data, the problem of Minimum Enclosing Ball (MEB) with outliers has attracted a great deal of attention in recent years. Though several methods have been proposed in both theory and practice, most of them still suffer from the issues such as high time complexities or unstable performances for different datasets. For example, Kumar et al. [38] proposed the open problem "Are there practical methods for computing an MEB with outliers?" To answer this question, we extend the idea of core-set and present a randomized algorithm for MEB with outliers in high dimension. In particular, we provide a more "robust" analysis for the core-set construction of MEB, and propose a couple of novel improvements to further reduce the time complexity. The ideas behind are interesting in their own right and we expect to apply them to solve more high-dimensional problems. To show the efficiency and practicality, we test our algorithm on random datasets and also apply it to solve outlier recognition on benchmark image datasets. To our best knowledge, this is the first algorithm yielding both solid theoretical quality guarantee and promising practical performance for MEB with outliers.

1   Introduction

In this big data era, we are confronted with extremely large amount of data and it is important to develop efficient algorithmic techniques to handle the arising realistic issues. For example, the quality of dataset often plays a key role and seriously affects the final result in many machine learning problems. Since manually removing outliers will be very costly, we need to develop efficient algorithms for automatically recognizing outliers. *Outlier recognition* is a typical unsupervised learning problem, and we often model it as an optimization problem based on some reasonable assumption. For instance, it is very natural to assume that the inliers (i.e., normal data) locate in some dense region of the feature space while the outliers are scattered [12, 25, 28, 50]. However, most of the density-based methods are only for low-dimensional space rather than the much more common large-scale high-dimensional data (some existing high-dimensional approaches often are of heuristic natures or suffer from high complexities [4, 35, 37]). We refer the reader to several excellent surveys [13, 30, 36] for outlier recognition.

In this paper, we study the problem of *Minimum Enclosing Ball (MEB) with outliers* in high dimension and apply it to outlier recognition. The MEB based model seems to be very simple but involves a couple of computational challenges. For example, the existence of outliers makes the problem to be not only non-convex but also highly combinatorial; the high dimensionality makes the problem even harder. MEB with outliers is a fundamental problem in computational geometry and has attracted a great deal of attention. For example, Kumar et al. [38] proposed the open problem "Are there practical methods for computing an MEB with outliers?" More prior works are introduced in Section 1.1.

Our main contributions.   To answer Kumar et al.'s question, we adopt the ideas from [8, 21] and present a randomized algorithm using the well known geometric concept "core-set". In theory, our contributions are twofold. To make our method more efficient for MEB with outliers, we provide a more "robust" analysis and new bound for the core-set size in [8]; further, we propose the improved algorithm based on some novel observations in geometry. We provide a thorough analysis on the complexity and quality guarantee for the new algorithm, and show that the running time can be significantly reduced comparing with the basic algorithm. To show the efficiency and practicality, we test our algorithm on both random and benchmark image datasets. In particular, the experimental results on the benchmark datasets reveal the advantages of our approach over a couple of existing methods for outlier recognition. In our experiment on the second benchmark dataset, we apply the emerging popular technique "deep learning" [53] to represent each image as a vector in high dimension; we expect that this work could shed some light on the direction of combining computational geometry and deep learning to solve more real-world problems.

1.1   Preliminaries and Prior Work

Here we introduce several definitions that are used throughout the paper.

▶ **Definition 1** (Minimum Enclosing Ball (MEB)). Given a set P of points in $\mathbb{R}^d$, MEB is the ball covering all the points with the smallest radius. The MEB is denoted by $MEB(P)$.

▶ **Definition 2** (MEB with Outliers). Given a set P of n points in $\mathbb{R}^d$ and a small parameter $\gamma \in (0, 1)$, MEB with outliers is to find the smallest ball that covers at least $(1 - \gamma)n$ points. Namely, the task is to find a subset of P having at least $(1 - \gamma)n$ points such that the resulting MEB is the smallest among all the possible choices; the obtained ball is denoted by $MEB(P, \gamma)$.

The hardness and computational complexity.   We always assume the dimensionality d is large in this paper. We are able to show that the problem of MEB with outliers is NP-complete by the reduction from the problem *covering by two balls* [44]. Due to space limit, we place the details in Appendix. In fact, it is straightforward to obtain a 2-approximation (i.e., the radius is at most two times the optimal). For example, we can enumerate all the points of P and select the one with the smallest induced radius (excluding the farthest γn points) as the ball center; using triangle inequality we know that it is a 2-approximation. Or we can apply random sampling to obtain a linear time 2-approximation with certain probability. Moreover, since an ϵ-core-set of size $\lceil 1/\epsilon \rceil$ for MEB exists [9], we can simply obtain a $n^{O(1/\epsilon)}d$-time $(1 + \epsilon)$-approximation for MEB with outliers by brute force. In this paper, we aim to **design an algorithm being implementable with low complexity, and achieve the approximation ratio significantly lower than** 2.

From Definition 2, we can see that the main issue is to determine the subset of P which makes the problem to be a challenging combinatorial optimization problem. Actually, solving

such combinatorial problems involving outliers are often extremely hard. For example, Mount et al. [46] showed that any approximation for linear regression with n points and γn outliers requires $\Omega\big((\gamma n)^d\big)$ time under the assumption of the hardness of affine degeneracy [24]; then they turned to find an efficient algorithm yielding bi-criteria approximation instead. Similarly, we also relax our goal for MEB with outliers to its bi-criteria approximation in this paper. We leave the lower bound of the complexity for finding the single criterion approximation (i.e., $\delta = 0$ in Definition 3) as an open problem in future work.

For convenience, we always use P_{opt} to denote the optimal subset of P, that is,

$$P_{\mathrm{opt}} = \arg_{P'} \min \Big\{ \text{the radius of } MEB(P') \mid P' \subset P, |P'| \geq (1 - \gamma)n \Big\}, \tag{1}$$

and r_{opt} to denote the radius of $MEB(P_{\mathrm{opt}})$.

▶ **Definition 3** (Bi-criteria Approximation). Given an instance (P, γ) for MEB with outliers and two small parameters $0 < \epsilon, \delta < 1$, an (ϵ, δ)-approximation is a ball that covers at least $(1 - (1 + \delta)\gamma)n$ points and has the radius at most $(1 + \epsilon)r_{\mathrm{opt}}$.

When both ϵ and δ are small, the bi-criteria approximation is very close to the optimal solution with only slight violations on the number of covering points and radius.

Prior work. In computational geometry, a *core-set* [1] is a small set of points that approximates the shape of a much larger point set, and thus can be used to significantly reduce the time complexities for many optimization problems (please refer to a recent survey [48]). In particular, a core-set can be applied to efficiently compute an approximate MEB for a set of points in high-dimensional space [10, 38]. Moreover, [8] showed that it is possible to find a core-set of size $\lceil 2/\epsilon \rceil$ yielding a $(1 + \epsilon)$-approximate MEB, where the advantage is that the size is independent of the data size and dimensionality; [9] further proved that only $\lceil 1/\epsilon \rceil$ is enough but the core-set construction is more complicated. In fact, the algorithm for computing the core-set of MEB is a *Frank-Wolfe* style algorithm [27], which has been systematically studied by Clarkson [16]. Beyond using core-set, several other exact and approximate algorithms for MEB have also been studied [7, 26, 47, 49].

MEB with outliers is a generalization of MEB but much more challenging. Bǎdoiu et al. [10] extended their core-set idea to the problem of MEB with outliers and achieved a bi-criteria approximation; though their running time is linear, the hidden constant is exponential on $O(\frac{1}{\epsilon\delta\gamma})$ which makes it far from being efficient and practical. Several algorithms for low-dimensional instances have been developed, such as [3, 23, 31, 42]. The problem of MEB with outliers also falls under the umbrella of the topic *robust shape fitting* [2, 33], but most of the approaches cannot be applied to high-dimensional data. Dunagan and Vempala [22] studied high dimensional outlier removal based on the theory in statistics but their algorithm yields at least cubic time complexity.

There are also several prior works on k-center clustering with outliers [14, 15, 43] and streaming MEB with outliers [56]. However, all their resulting approximation ratios are at least 2 which makes them out of the scope of this paper (see our comment below Definition 2). Also, some of them assume that the number of outliers is small, but our paper considers the more common case that the outliers take a fraction (e.g., 2%) of the input size which can be large in practice.

The rest is organized as follows. We first present a new analysis for core-set construction and the basic algorithm for MEB with outliers in Section 2, and further introduce our ideas for reducing the time complexity in Section 3. Finally, we show the experimental results including the application on outlier recognition in Section 4.

2 The Basic Algorithm Using Core-sets

Actually, the main idea of our algorithm comes from the work by Bǎdoiu and Clarkson [8] about using core-set to compute MEB in high dimension. Following this work and [29], Ding and Xu [21] developed an efficient algorithm, which is called *Random Gradient Descent (RGD)-Tree*, to solve the problem of SVM with outliers. However, extending the algorithm from SVM to MEB needs some significant new ideas. For example, SVM is equivalent to the problem of polytope distance [29] and we just need to compute a projection to update the solution in each iteration; for MEB, we need a more careful analysis especially when considering outliers, where we discuss the details in Section 2.1.

2.1 A More Robust Analysis for Core-set Construction in [8]

Bǎdoiu and Clarkson [8] provided an elegant algorithm of core-set construction for MEB, where the core-set size is $2/\epsilon$ (for convenience, we always assume that $2/\epsilon$ is integer). However, there is a small issue in their paper. The analysis assumes that the exact MEB (of the current core-set) is computed in each iteration, but in reality we often can only compute an approximate one instead. Therefore, an open technical question is whether the quality is still guaranteed with such a change. Note that Kumar et al. [38] fixed this issue, and showed that computing a $(1 + O(\epsilon^2))$-approximation of the MEB for the current core-set in each iteration still guarantees a core-set with the size of $O(1/\epsilon)$ (the hidden constant > 80). Increasing the core-set size from $2/\epsilon$ to $80/\epsilon$ is neglectable in asymptotic analysis, if only considering MEB; but in Section 2.2 and 2.3, we will show that it could be a serious issue for computing MEB with outliers. Thus, we still want to keep the size to be $2/\epsilon$ and provide a new analysis for this purpose below.

For the sake of completeness, we first briefly introduce the idea in [8]. The algorithm is a simple iterative procedure: initially, it selects an arbitrary point and places it into a set S which is empty at the beginning; in each of the following $2/\epsilon$ steps, the algorithm updates the center of $MEB(S)$ and adds the farthest point from the center to S; finally, the center of $MEB(S)$ induces a $(1 + \epsilon)$-approximation for MEB of the whole input point set. The selected set of $2/\epsilon$ points (i.e., S) is also called the core-set for MEB. To ensure the extent of improvement in each iteration, [8] showed that the following two inequalities hold if the algorithm always selects the farthest point to the temporary center of $MEB(S)$:

$$r_{i+1} \geq (1 + \epsilon)r_{\mathrm{opt}} - L_i; \qquad r_{i+1} \geq \sqrt{r_i^2 + L_i^2}, \tag{2}$$

where r_i and r_{i+1} are the radii of $MEB(S)$ in the i-th and $(i + 1)$-th iterations respectively, r_{opt} is the optimal radius of the MEB, and L_i is the shifting distance of the center of $MEB(S)$ from the i-th to $(i + 1)$-th iteration.

As mentioned before, we often just compute an approximation of $MEB(S)$ in each iteration. In the i-th iteration, we denote by c_i and c_i' the center of $MEB(S)$ and its approximation respectively. Suppose $||c_i - c_i'|| \leq \xi r_i$ where $\xi < \frac{\epsilon}{1+\epsilon}$ (we will see why we need this bound later). Note we only obtain c_i' rather than c_i in reality. As a consequence, we can only select the farthest point (say q) to c_i'. If $||q - c_i'|| \leq (1 + \epsilon)r_{\mathrm{opt}}$, then we are done and the $(1 + \epsilon)$-approximation for MEB is obtained. Otherwise, we have

$$(1 + \epsilon)r_{\mathrm{opt}} < ||q - c_i'|| \leq ||q - c_{i+1}|| + ||c_{i+1} - c_i|| + ||c_i - c_i'|| \leq r_{i+1} + L_i + \xi r_i \tag{3}$$

via applying triangle inequality. In other words, we should replace the first inequality of (2) by $r_{i+1} > (1 + \epsilon)r_{\mathrm{opt}} - L_i - \xi r_i$. Also, the second inequality of (2) still holds since it only depends on the property of exact MEB (please refer to Lemma 2.1 in [8]). So we have

$$r_{i+1} \geq \max\left\{ \sqrt{r_i^2 + L_i^2}, (1 + \epsilon)r_{\mathrm{opt}} - L_i - \xi r_i \right\}. \tag{4}$$

When $L_i = \frac{\left((1+\epsilon)r_{\text{opt}} - \xi r_i\right)^2 - r_i^2}{2\left((1+\epsilon)r_{\text{opt}} - \xi r_i\right)}$, the lower bound of r_{i+1} achieves the minimum value. Similar to the analysis in [8], we let $\lambda_i = \frac{r_i}{(1+\epsilon)r_{\text{opt}}}$ which is at most $1/(1+\epsilon)$. Therefore,

$$\lambda_{i+1}^2 \geq \lambda_i^2 + \frac{\left((1-\xi\lambda_i)^2 - \lambda_i^2\right)^2}{4(1-\xi\lambda_i)^2} \tag{5}$$

by (4). Because λ_i is always no larger than $1/(1+\epsilon)$ and we assume $\xi < \frac{\epsilon}{1+\epsilon}$, it is easy to see $(1-\xi\lambda_i)^2 - \lambda_i^2 \geq 0$. To simplify the inequality (5), we consider the function $g(x) = \frac{(1-x)^2 - \lambda_i^2}{1-x}$ where $0 < x < \xi$. Its derivative $g'(x) = -1 - \frac{\lambda_i^2}{(1-x)^2}$ is always negative, thus we have

$$g(x) \geq g(\xi) = \frac{(1-\xi)^2 - \lambda_i^2}{1-\xi}. \tag{6}$$

Again, because $\xi < \frac{\epsilon}{1+\epsilon}$ and $\lambda_i \leq 1/(1+\epsilon)$, we know the right-hand side of (6) is always non-negative. Using (6), the inequality (5) can be simplified to be

$$\lambda_{i+1}^2 \geq \lambda_i^2 + \frac{1}{4}\left(g(\xi)\right)^2 = \lambda_i^2 + \frac{\left((1-\xi)^2 - \lambda_i^2\right)^2}{4(1-\xi)^2}. \tag{7}$$

(7) can be further rewritten as

$$\left(\frac{\lambda_{i+1}}{1-\xi}\right)^2 \geq \frac{1}{4}\left(1 + (\frac{\lambda_i}{1-\xi})^2\right)^2 \implies \frac{\lambda_{i+1}}{1-\xi} \geq \frac{1}{2}\left(1 + (\frac{\lambda_i}{1-\xi})^2\right). \tag{8}$$

Now, we can apply a similar transformation of λ_i which was used in [8]. Let $\gamma_i = \frac{1}{1 - \frac{\lambda_i}{1-\xi}}$, and (8) implies that

$$\gamma_{i+1} \geq \frac{\gamma_i}{1 - \frac{1}{2\gamma_i}} = \gamma_i\left(1 + \frac{1}{2\gamma_i} + (\frac{1}{2\gamma_i})^2 + \cdots\right) > \gamma_i + \frac{1}{2} \tag{9}$$

where the equation comes from the fact $\frac{1}{2\gamma_i} \in (0, \frac{1}{2})$ (because $\lambda_i \leq \frac{1}{1+\epsilon}$ and $\xi < \frac{\epsilon}{1+\epsilon}$). Note $\lambda_0 = 0$ and thus $\gamma_0 = 1$. As a consequence, we have $\gamma_i > 1 + \frac{i}{2}$. In addition, since $\lambda_i \leq \frac{1}{1+\epsilon}$, that is, $\gamma_i \leq \frac{1}{1 - \frac{1}{(1+\epsilon)(1-\xi)}}$, we have

$$i < \frac{2}{\epsilon - \xi - \epsilon\xi} = \frac{2}{(1 - \frac{1+\epsilon}{\epsilon}\xi)\epsilon}. \tag{10}$$

According to (10), we directly have the following theorem.
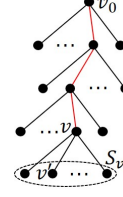
▶ **Theorem 4.** *In the core-set construction algorithm of [8], if one computes an approximate MEB for S in each iteration and the resulting center c_i' has the distance to c_i less than $\xi r_i = \beta\frac{\epsilon}{1+\epsilon}r_i$ with some $\beta \in (0,1)$, the final core-set size is bounded by $\frac{2}{(1-\beta)\epsilon}$. Also, the bound could be arbitrarily close to $2/\epsilon$ when β is small enough.*

2.2 Algorithm for MEB with Outliers

Now, we present the (ϵ, δ)-approximation algorithm for MEB with outliers. To better understand the algorithm, we first illustrate the high-level idea. In the aforementioned core-set construction algorithm [8], Ding and Xu [19] observed that it is not necessary to select the farthest point to the center of $MEB(S)$ in each step. Instead, as long as the selected point has a distance to the center of $MEB(S)$ larger than $(1 + \epsilon)r_{\text{opt}}$, the required extent of improvement will always be guaranteed (because the two inequalities (2) still hold). Following this observation, we investigate the approach below.

We denote by $Ball(c, r)$ the ball centered at point c with radius $r > 0$. Recall that P_{opt} is the subset of P yielding the optimal MEB with outliers, and r_{opt} is the radius of $MEB(P_{\text{opt}})$ (see Section 1.1). In the i-th step, we add an arbitrary point from $P_{\text{opt}} \setminus Ball(c_i, (1+\epsilon)r_{\text{opt}})$ to S where c_i is the current center of S. Based on the observation by Ding and Xu, we know that a $(1+\epsilon)$-approximation is obtained after at most $2/\epsilon$ steps, that is, $|P \cap Ball(c_i, (1+\epsilon)r_{\text{opt}})| \geq (1-\gamma)n$ when $i \geq 2/\epsilon$. According to Theorem 4, c_i can be replaced by an approximate center c_i' and the number of iterations is bounded by $\frac{2}{(1-\beta)\epsilon}$.

However, we need to solve two key issues for carrying out the above approach: **(i)** how to determine the value of r_{opt} and **(ii)**

Figure 1 $\mathcal{P}_v^{v_0}$ contains the four points along the red path from root v_0 to node v. The point set S_v corresponds to the children of v.

how to correctly select a point from $P_{\text{opt}} \setminus Ball(c_i', (1+\epsilon)r_{\text{opt}})$. Actually, we can implicitly avoid the first issue via replacing the radius $(1+\epsilon)r_{\text{opt}}$ by the k-th largest distance from the points of P to c_i', where k is some appropriate number to be determined in our following analysis. For the latter issue, we can take a small random sample, instead of a single point from $P_{\text{opt}} \setminus Ball(c_i', (1+\epsilon)r_{\text{opt}})$, and try each of the sampled points. We present the algorithm in Algorithm 1 and illustrate Step 2(2)(a-c) in Figure 1. We place the detailed discussion in Sections 2.3, and defer the formal analysis on the time complexity (together with the improvement) in Section 3.3.

Algorithm 1 (ϵ, δ)-approximation Algorithm for MEB with Outliers

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Input: A point set P with n points in $\mathbb{R}^d$, the fraction of outliers $\gamma \in (0, 1)$, and the parameters $0 < \epsilon, \delta, \mu < 1$, $h, k \in \mathbb{Z}^+$.

Output: A tree, and each node of the tree has an "attached point" which is a candidate ball center for the (ϵ, δ)-approximation solution.

1: Each node v in the tree is represented by a point (with a slight abuse of notation, we also use v to denote the point). Note that v also has an "attached point" defined below. Initially, pick one point randomly from P as the root node v_0.

2: Starting from the root, grow each node as follows:

   (1) Let v be the current node.
   (2) If the height of v is h, v becomes a leaf node. Otherwise, perform the following steps:

      (a) Let $\mathcal{P}_v^{v_0}$ denote the set of points along the path from root v_0 to node v, and c_v denote the approximate center of $MEB(\mathcal{P}_v^{v_0})$. We say that c_v is the "attached point of v".
      (b) Select the top k points having the largest distances to c_v, as the point set O_v.
      (c) Take a random sample S_v of size $(1 + \frac{1}{\delta}) \ln \frac{h}{\mu}$ from O_v, and let each point $v' \in S_v$ be a child of v.

2.3   Parameter Settings and Quality Guarantee

We denote by $\mathbb{H}$ the tree constructed by Algorithm 1. In step 2(2)(a), suppose we compute the approximate center c_v having the distance to the exact one less than $\xi r_v = \beta \frac{\epsilon}{1+\epsilon} r_v$ where r_v is the radius of $MEB(\mathcal{P}_v^{v_0})$ and $\beta \in (0, 1)$. The following theorem shows the success probability of Algorithm 1.

▶ **Theorem 5.** *If we set $h = \frac{2}{(1-\beta)\epsilon} + 1$ and $k = (1+\delta)\gamma n$, then with probability at least $(1-\gamma)(1-\mu)$, there exists at least one node of $\mathbb{H}$ yielding an (ϵ, δ)-approximation for the problem of MEB with outliers.*

Before proving Theorem 5, we need to introduce several important lemmas. Below we always let $k = (1+\delta)\gamma n$ in Algorithm 1. The following Lemma 6 can be obtained by simple calculation, and we put the proof in our Appendix.

▶ **Lemma 6.** *For each node v, the set S_v contains at least one point from P_{opt} with probability $1 - \frac{\mu}{h}$.*

▶ **Lemma 7.** *With probability $(1-\gamma)(1-\mu)$, there exists a leaf node $u \in \mathbb{H}$ such that the corresponding set $\mathcal{P}_u^{v_0} \subset P_{\mathrm{opt}}$.*

Proof. Lemma 6 indicates that each node v has a child corresponding to a point from P_{opt} with probability $1 - \frac{\mu}{h}$. In addition, the probability that the root $v_0 \in P_{\mathrm{opt}}$ is $1 - \gamma$ (recall that γ is the fraction of outliers). Note that the height of $\mathbb{H}$ is h, then with probability at least $(1-\gamma)\left(1 - \frac{\mu}{h}\right)^h > (1-\gamma)(1-\mu)$, there exists one leaf node $u \in \mathbb{H}$ satisfying $\mathcal{P}_u^{v_0} \subset P_{\mathrm{opt}}$. ◀

Now, we always assume that such a root-to-leaf path $\mathcal{P}_u^{v_0}$ described in Lemma 7 exists and only focus on the nodes along this path.

▶ **Lemma 8.** *For each node $v \in \mathcal{P}_u^{v_0}$, at least one of the following two events happens: (1) c_v is the ball center of an (ϵ, δ)-approximation; (2) its child v' on the path $\mathcal{P}_u^{v_0}$ satisfies $r_{v'} > (1+\epsilon)r_{opt} - L_{v \to v'} - \xi r_v$, where $L_{v \to v'}$ denotes the distance between the exact centers of $MEB(\mathcal{P}_v^{v_0})$ and $MEB(\mathcal{P}_{v'}^{v_0})$.*

Proof. Let $\tilde{r}_v$ be the minimum value that $Ball(c_v, \tilde{r}_v)$ covers $P \setminus O_v$. If $\tilde{r}_v \leq (1+\epsilon)r_{opt}$, then we are done; that is, $Ball(c_v, \tilde{r}_v)$ covers $(1 - (1+\delta)\gamma)n$ points and $\tilde{r}_v \leq (1+\epsilon)r_{opt}$. Otherwise, $\tilde{r}_v > (1+\epsilon)r_{opt}$ and we consider the second event. Note that v' (i.e., the point representing the node "v'") lies outside $Ball(c_v, \tilde{r}_v)$. Using the similar idea of (3) we have $r_{v'} \geq \tilde{r}_v - L_{v \to v'} - \xi r_v$. Since we assume $\tilde{r}_v > (1+\epsilon)r_{opt}$, the second event happens and the proof is completed. ◀

Suppose no node in $\mathcal{P}_u^{v_0}$ makes the first event of Lemma 8 occur. As a consequence, we obtain a series of inequalities for each pair of radii $r_{v'}$ and r_v, i.e., $r_{v'} > (1+\epsilon)r_{opt} - L_{v \to v'} - \xi r_v$. Note $\mathcal{P}_u^{v_0} \subset P_{\mathrm{opt}}$. Using almost identical idea in Section 2.1 (just replace the core-set S by $\mathcal{P}_u^{v_0}$), we know that the height of $\mathbb{H}$ is at most $\frac{2}{(1-\beta)\epsilon} + 1$ by Theorem 4. The success probability directly comes from Lemma 7. Overall, we obtain Theorem 5.

3 The Improved Algorithm

In this section, we propose our ideas for reducing the time complexity of Algorithm 1.

3.1 Dimension Reduction via JL Transform

In Section 2.2, we consider the problem in the original input d-dimensional space where d could be very high. To further reduce the complexity, an ideal approach is first reducing the dimensionality and then running our algorithm in the new space. It is well known that the random projection based approach *Johnson-Lindenstrauss (JL) transform* [18], which can be efficiently implemented by random matrix multiplication, can reduce the dimensionality from an arbitrarily large d to only logarithmic on the number of points and approximately preserve the pairwise distances (we refer the reader to the survey by Ailon and Chazelle [6]).

However, we cannot directly apply it to our MEB problem, because the ball center could appear anywhere in the space and hence the resulting radius cannot always be preserved (note JL Transform only preserves the $\binom{n}{2}$ pairwise distances). Thus, the key is to show that a good solution in the reduced space can be efficiently mapped back to the original $\mathbb{R}^d$ while the quality is still preserved.

From Algorithm 1, we know that the approximate MEB center is determined by a core-set with the size of at most $\frac{2}{(1-\beta)\epsilon}$. Namely, the center always locates inside a convex hull generated by a subset of at most $\frac{2}{(1-\beta)\epsilon}$ points from P. Consequently, we just need to guarantee that the pairwise distance of any two points in the union of all these convex hulls is (approximately) preserved. A natural idea is to build a uniform dense enough grid inside each convex hull and apply JL transform to the set of grid points. Through triangle inequality and the fact that JL transform is linear, we can solve the problem in the reduced space and achieve a good approximation in the original $\mathbb{R}^d$. Actually, similar ideas have been well studied in [17, 34] for different purposes, and the resulting reduced dimensionality is roughly $O(\frac{1}{\epsilon^3} \log n)$. In this section, we show that the dimensionality can be further reduced to $O(\frac{1}{\epsilon^2} \log n)$, based on the work from Sheehy [52]. We slightly modify his statement to make it more suitable for our analysis.

▶ **Theorem 9** ([52]). *Let P be a set of n points in $\mathbb{R}^d$, and f be a JL transform from $\mathbb{R}^d$ to $\mathbb{R}^{O(\frac{1}{\epsilon^2} \log n)}$. Then with certain probability, for any subset S of P and any point $p \in S$, if q is inside the convex hull of S, we have $\left| ||p - q||^2 - ||f(p) - f(q)||^2 \right| \leq \epsilon \, rad(S)^2$, where $rad(S)$ indicates the radius of $MEB(S)$. The success probability is equal to that of JL transform which can be boosted to be arbitrarily close to 1.*

A cute part of Theorem 9 is that it relaxes the error bound to be a unified $\epsilon \, rad(S)^2$, rather than the typical JL-lemma style bound $O(\epsilon)||p - q||^2$; this is also the reason that it holds for infinite number of points inside the convex hulls. Using Theorem 9, we have the following theorem.

▶ **Theorem 10.** *Given an instance (P, γ) for the problem of MEB with outliers, one can first apply JL Transform f to reduce the dimensionality to be $O(\frac{1}{\epsilon^2} \log n)$ and run Algorithm 1 on $(f(P), \gamma)$. Let $\bar{c}_v$ be the obtained (ϵ, δ)-approximation center, which is a convex combination of some $f(S) \subset f(P)$, and $f^{-1}(\bar{c}_v)$ be the point having the same combination coefficients of S in the original $\mathbb{R}^d$. Then $f^{-1}(\bar{c}_v)$ is an $(O(\epsilon), \delta)$-approximation center for the original instance (P, γ) in $\mathbb{R}^d$ with high probability.*

Let the set of inliers induced by $\bar{c}_v$ be $f(P')$ which has the size at least $(1-(1+\delta)\gamma)n$. The key for proving Theorem 10 is to show that $\max_{p \in P'} ||p - f^{-1}(\bar{c}_v)||^2$ and $\max_{p \in P'} ||f(p) - \bar{c}_v||^2$ are close. Moreover, $\max_{p \in P'} ||f(p) - \bar{c}_v||^2$ is upper bounded by $(1+\epsilon) \max_{p \in P_{opt}} ||f(p) - f(c_{opt})||^2$, where c_{opt} is the center of $MEB(P_{opt})$, since $\bar{c}_v$ is an (ϵ, δ)-approximation center for the instance $(f(P), \gamma)$. Finally, it is able to show that $f^{-1}(\bar{c}_v)$ is an $(O(\epsilon), \delta)$-approximation center for (P, γ). Due to space limit, we put the full proof in Appendix.

3.2 Random Sampling for Reducing The Data Size

Besides dimension reduction, we can further reduce the data size via random sampling. Consider the range space $\Sigma = (P, \Pi)$ where each range $\pi \in \Pi$ is the complement of a ball in the space. Let $\epsilon \in (0, 1)$, and an "ϵ-sample" S of P is defined as follows: $\forall \pi \in \Pi$, $\left| \frac{|S \cap \pi|}{|S|} - \frac{|P \cap \pi|}{|P|} \right| \leq \epsilon$ [54]; roughly speaking, S is an approximation of P with an additive error inside each range π. The reader can find more details on this topic in Lecture 19 of [45]. Based on the theory of VC dimension, we know that an ϵ-sample can be easily

obtained via uniform sampling, where the success probability is $1 - \lambda$ and the sample size is $O\big(\frac{1}{\epsilon^2}(d \log \frac{d}{\epsilon} + \log \frac{1}{\lambda})\big)$ for any $0 < \lambda < 1$. Thanks to Theorem 10, we can replace the dimension d by $d' = O(\log n/\epsilon^2)$. Also, we need to replace the "ϵ" of the "ϵ-sample" by $\delta\gamma$ in our problem. Therefore, an $(O(\epsilon), O(\delta))$-approximation for MEB with outliers can be obtained by running the algorithm on the $\delta\gamma$-sample, and the sample size is $O\big(\frac{1}{\delta^2\gamma^2}(d' \log \frac{d'}{\delta\gamma} + \log \frac{1}{\lambda})\big)$. The details together with the improvement are shown below.

Actually, the front factor $\frac{1}{\delta^2\gamma^2}$ of the sample size can be further reduced to be $\frac{1}{\delta^2\gamma}$ by a more careful analysis. Our intuition is as follows: we observe that there is no need to guarantee the additive error for each range π (as the definition of ϵ-sample), instead, only a multiplicative error for the ranges covering at least $\gamma|P|$ points should be enough; note that when a range covers more points, the multiplicative error is weaker than the additive error and thus the sample size is reduced. For this purpose, we use the concept of *relative approximation* [32, 41]: if we take a random sample S with the size $O\big(\frac{1}{\delta^2\gamma}(d' \log \frac{1}{\gamma} + \log \frac{1}{\lambda})\big)$ from P, then with probability at least $1 - \lambda$,

$$\forall \pi \in \Pi, \ \left| \frac{|\pi \cap P|}{|P|} - \frac{|\pi \cap S|}{|S|} \right| \le \delta \times \max\left\{ \frac{|\pi \cap P|}{|P|}, \gamma \right\}. \tag{11}$$

We formally state our result below.

▶ **Theorem 11.** *Let (P, γ) be an instance for the problem of MEB with outliers in $\mathbb{R}^{d'}$, and S be a random sample from P with the size $O\big(\frac{1}{\delta^2\gamma}(d' \log \frac{1}{\gamma} + \log \frac{1}{\lambda})\big)$. Let $(S, (1+\delta)\gamma)$ be a new instance for the problem of MEB with outliers, and $Ball(c, r)$ be an (ϵ, δ)-approximation for it. Then, with probability at least $1 - \lambda$, $Ball(c, r)$ is an $(\epsilon, O(\delta))$-approximation for (P, γ).*

Proof. We assume that S is a relative approximation of P and (11) holds (this happens with probability $1 - \lambda$). First, since $MEB(P_{\mathrm{opt}})$ covers $(1 - \gamma)|P|$ points of P, it is a feasible solution for the instance $(S, (1+\delta)\gamma)$ by (11). Since $Ball(c, r)$ is an (ϵ, δ)-approximation of $(S, (1+\delta)\gamma)$, we have

$$r \le (1+\epsilon)r_{\mathrm{opt}}; \quad |S \setminus Ball(c, r)| \le (1+\delta)^2 \gamma |S|. \tag{12}$$

Now, we claim that

$$|P \setminus Ball(c, r)| \le \frac{(1+\delta)^2}{1-\delta} \gamma |P|. \tag{13}$$

Assume that (13) does not hold, then (11) implies

$$\left| \frac{|P \setminus Ball(c, r)|}{|P|} - \frac{|S \setminus Ball(c, r)|}{|S|} \right| \le \delta \frac{|P \setminus Ball(c, r)|}{|P|}. \tag{14}$$

So $\frac{|S \setminus Ball(c,r)|}{|S|} \ge (1 - \delta) \frac{|P \setminus Ball(c,r)|}{|P|} > (1+\delta)^2 \gamma$, which is in contradiction with the second inequality of (12), and thus (13) is true. Note $\frac{(1+\delta)^2}{1-\delta} = 1 + O(\delta)$, and consequently (13) and the first inequality of (12) together imply that $Ball(c, r)$ is an $(\epsilon, O(\delta))$-approximation for (P, γ). ◀

3.3 Complexity Analysis

Here, we analyze the time complexity of Algorithm 1. For each node v in $\mathbb{H}$, we need to compute an approximate center c_v of $MEB(\mathcal{P}_v^{v_0})$ having the distance to the exact one less than $\xi r_v = \beta \frac{\epsilon}{1+\epsilon} r_v$, where r_v is the radius of $MEB(\mathcal{P}_v^{v_0})$ and $\beta \in (0, 1)$. Using the proposed algorithm in [8], this can be done in $O(\frac{1}{\xi^2}|\mathcal{P}_v^{v_0}|d) = O(\frac{1}{\xi^2\epsilon}d)$ time. Also, the set O_v can be

obtained by PICK algorithm in linear time [11]. In total, the time complexity of Algorithm 1 is $O\big(C(n + \frac{1}{\xi^2\epsilon})d\big)$ where $C = \left(\left(1 + \frac{1}{\delta}\right)\ln\frac{h}{\mu}\right)^{h-1}$ and $h = \frac{2}{(1-\beta)\epsilon} + 1$.

Using dimension reduction and random sampling. The major drawback in the above time complexity is that the coefficient C could be large and limits the practicality of the algorithm especially when both n and d are large as well. To remedy this issue, we can apply the ideas in Section 3.1 and 3.2 to reduce the complexity. We first randomly project the given data set to a much lower dimensional space and take a small random sample S; then run the algorithm on S to obtain a bi-criteria solution. Finally, Theorem 10 and 11 jointly guarantee that the solution mapped back to $\mathbb{R}^d$ is an $\big(O(\epsilon), O(\delta)\big)$-approximation. The reduced time complexity will be

$$T(n, d, \epsilon) + O\big(C(n' + \frac{1}{\xi^2\epsilon})d'\big), \tag{15}$$

where $T(n, d, \epsilon)$ indicates the time complexity of JL-transform for mapping n points from $\mathbb{R}^d$ to $\mathbb{R}^{O(\log n/\epsilon^2)}$ (a naive implementation of JL-transform by matrix multiplication will result $T(n, d, \epsilon) = O(\frac{1}{\epsilon^2}nd\log n)$; there are also several even faster and practical algorithms [5]), $n' = O(\frac{1}{\delta^2\gamma}(d'\log\frac{1}{\gamma} + \log\frac{1}{\lambda}))$, and $d' = O(\frac{1}{\epsilon^2}\log n)$. To simplify, (15) could be roughly rewritten as $T(n, d, \epsilon) + O\big(C(\log n)^2\big)$ if the values of $\frac{1}{\epsilon}$, $\frac{1}{\delta}$, and $\log\frac{1}{\lambda}$ are not large.

▶ Remark. Some reader may have the natural question that whether we can recursively apply the dimension reduction and random sampling ideas to further reduce the time complexity. However, a more careful analysis reveals that this strategy is not helpful. Let us consider two cases. **i.** The first item in (15) dominates the whole complexity (i.e., C is not very large). Then, no matter how many rounds of dimension reduction and random sampling we apply, the complexity (15) never changes. **ii.** The second item in (15) dominates the whole complexity (i.e., C is very large). In this case, the exponent of C, $O(\frac{1}{\epsilon})$, should be at least $\Omega\big(poly(\log n)\big)$; then, due to Theorem 10, the reduced dimension d' should always be $\Omega\big(poly(\log n)\big)$ too. That is, conducting the JL-transform more times cannot further reduce the dimensionality.

Some heuristic improvements. By Theorem 5, we know the success probability is $(1-\gamma)(1-\mu)$. However, when outlier ratio is high, say $\gamma = 0.5$, the probability $(1-\gamma)(1-\mu)$ will be small. Here, we introduce two boosting methods in practice. **(1) Constructing a forest.** Instead of building a single tree, we randomly initialize multiple root nodes and grow each of them to be a tree. Suppose the number of trees is κ. The probability that there exists an (ϵ, δ)-approximation solution is at least $1 - (1 - (1-\gamma)(1-\mu))^\kappa \approx \kappa(1-\gamma)(1-\mu)$. **(2) Refinement.** Initialize one root node and build a tree; select the node with the smallest resulting radius and set it to be the root node for the next tree. After iteratively performing this procedure several rounds, we can obtain a much more robust solution.

4   Experiments

Our experimental section contains two parts. First, we test our algorithm on random datasets and evaluate its performance with respect to the objective value, i.e., the resulting radius. Second, we apply our algorithm to the problem of outlier recognition; we test it on two popular benchmark datasets **MNIST** [39] and **Caltech** [40]. All of the experimental results are obtained on a Windows workstation with 2.4GHz Intel Xeon E5-2630 v3 CPU and 32GB DDR4 2133MHz Memory; the algorithms are implemented in Matlab R2016b. For each of the random and benchmark datasets, we run 20 trials and report the average results.

From our analysis in Section 2.3, we know that Algorithm 1 results in a tree $\mathbb{H}$ where each node v has a candidate ball center c_v. For each candidate, we identify the nearest

$(1 - (1 + \delta)\gamma)n$ points to c_v as the inliers. To determine the final solution, we select the candidate having the smallest resulting radius. In addition, we ignore the worst case bounds on $|S_v|$ and h given by our (overly conservative) theoretical analysis. We set $|S_v| = 5$-15 and $h = 5$-10 when building the tree $\mathbb{H}$; we also apply the heuristic improvements from Section 3.3 (e.g., let $\kappa = 3$ and run the refinement 2 rounds). For the inner loop about computing the approximate center of $MEB(\mathcal{P}_v^{v_0})$, we set $\xi = 0.02$ (actually in our experiment, we find that varying ξ within a small range, say 0.01-0.05, has very limited influence on the results).

4.1 Random Datasets

We set the dimension $d = 10^4$, and randomly generate 10^5 points by standard normal distribution in each dimension (mean is 0 and variance is 1). Then, we run the algorithm in [8] to compute the MEB as the ground truth. Also, the radius of the obtained MEB is used as the optimal objective value r_{opt}. Finally, for each noise level γ from 0.1 to 0.5, we randomly generate $\frac{\gamma}{1-\gamma} \times 10^5$ points outside the MEB as the outliers.

Our experimental results are shown in Table 1. The approximation ratio is about the obtained radius to r_{opt}; the last column shows the total running time (including the time for random projection and sampling) on the 5 noise levels. As discussed in Section 1.1, most of existing algorithms for MEB with outliers have high complexities, and some of them are only applicable for the case with small number of outliers (e.g., if the number of outliers takes a constant fraction of input size, the complexity will be at least quadratic). Therefore, we directly use our proposed algorithm without dimension reduction and sampling as the baseline (the first line in Table 1). We also test our algorithm with the reduced dimensions $d/4$, $d/8$, and $d/16$, by random projection. Using the idea in Section 3.1, we map the obtained ball center in the lower dimension back to $\mathbb{R}^d$, and compute the resulting radius. In particular, for the case $d/8$, we take random samples with the sizes $n/4$, $n/8$, and $n/16$, and run our algorithm only on the samples. From Table 1, we can see that our algorithm performs quite stably regarding two aspects. First, when γ increases, the approximation ratios are always lower than 1.3; second, when reducing the dimensionality and running the algorithm on small samples, the ratios keep in the same level generally. More importantly, the running time could be dramatically reduced by dimension reduction and random sampling (e.g., the running time for the case $(d/8, n/16)$ is less than 1/20 of that for the case (d, n)).

■ **Table 1** Random Dataset: the approximation ratio to r_{opt} and running time.

γ / Methods	0.1	0.2	0.3	0.4	0.5	Total time (s)
Ours-d	1.127	1.124	1.240	1.245	1.125	1.3×10^3
Ours-$d/4$	1.127	1.127	1.247	1.246	1.244	2.9×10^2
Ours-$d/8$	1.122	1.130	1.106	1.129	1.126	1.6×10^2
Ours-$d/16$	1.132	1.125	1.131	1.239	1.250	< 100
Ours-$(d/8, n/4)$	1.089	1.113	1.114	1.147	1.135	< 50
Ours-$(d/8, n/8)$	1.086	1.128	1.132	1.154	1.154	< 50
Ours-$(d/8, n/16)$	1.079	1.081	1.127	1.152	1.150	< 50

4.2 The Application for Outlier Recognition

The two datasets. MNIST contains $70,000$ handwritten digits from 0 to 9. For each of the 10 digits, we add the outliers by randomly selecting a certain number of images from

the other 9 digits. Each image in MNIST has a 28×28 grayscale and is represented by a 784-dimensional vector. Caltech-256 dataset includes $30,607$ colored images. We choose 11 categories from Caltech-256 as the inliers for 11 instances respectively: *airplane, binocular, bonsai, cup, face, ketch, laptop, motorbike, sneaker, t-shirt*, and *watch*; for each instance, we also randomly select a certain number of images from other categories as the outliers. Using deep learning techniques, we apply **VGG net** [53] to extract the image features and obtain a 4096-dimensional vector for each image from the second fully-connected layer.

We compare our algorithm with three well known outlier recognition methods: angle-based outlier detection **(ABOD)** [37], one-class SVM **(OCSVM)** [51], and discriminative reconstructions in an autoencoder **(DRAE)** [55]. Specifically, ABOD distinguishes the inliers and outliers by assessing the distribution of the angles determined by each 3-tuple data points; OCSVM models the problem as a soft-margin one-class SVM; DRAE applies autoencoder to separate the inliers and outliers based on their reconstruction errors. For a fair comparison, we have also tuned the parameters in these three methods, such as the parameters for the kernel used by OCSVM, to achieve their best experimental results.

The performances are measured by the commonly used $F1\ score = 2 \times \frac{\text{Precision} \times \text{Recall}}{\text{Precision} + \text{Recall}}$, where *precision* is the fraction of the correctly identified positives over the total number of identified positives, and *recall* is the fraction of the correctly identified positives over the total number of positives in the dataset. For each outlier ratio γ, we compute the average $F1$ score over all the 10 digits (resp., 11 categories) for MNIST (resp., Caltech-256).

The experimental results are shown in Table 2 and 3. Similar to random datasets, the performances of our method remain quite stable with respect to noise levels. For both datasets, our method significantly outperforms the first two baselines, ABOD and OCSVM, especially on higher noise levels . Compared with the third baseline DRAE, our method always achieves slightly lower $F1$ scores (the differences are normally within 2%); however, our running times are much lower. For example, the running times of our algorithm on the case $(d/8, n/4)$ are less than $1/100$ and $1/1000$ of theirs on MNIST and Caltech, respectively.

Table 2 MNIST: $F1$ score and running time.

Methods \ γ	0.05	0.10	0.15	0.20	0.25	0.30	Total time (s)
ABOD	0.950	0.896	0.841	0.781	0.717	0.648	2×10^4
OCSVM	0.965	0.934	0.903	0.868	0.833	0.797	3.4×10^3
DRAE	0.974	0.949	0.925	0.898	0.867	0.838	1×10^5
Ours-d	0.965	0.935	0.907	0.883	0.857	0.831	1.6×10^4
Ours-$d/8$	0.964	0.936	0.909	0.882	0.853	0.826	2×10^3
Ours-$(d/8, n/2)$	0.965	0.937	0.912	0.880	0.864	0.817	1×10^3
Ours-$(d/8, n/3)$	0.965	0.937	0.911	0.880	0.848	0.829	0.8×10^3
Ours-$(d/8, n/4)$	0.965	0.939	0.909	0.886	0.862	0.834	0.7×10^3

Table 3 Caltech: $F1$ score and running time.

γ Methods	0.05	0.10	0.15	0.20	0.25	0.30	Total time (s)
ABOD	0.971	0.940	0.904	0.828	0.760	0.695	3×10^3
OCSVM	0.974	0.943	0.925	0.898	0.872	0.845	0.5×10^3
DRAE	0.992	0.983	0.980	0.974	0.967	0.957	6×10^5
Ours-d	0.986	0.973	0.967	0.951	0.951	0.936	1×10^4
Ours-$d/8$	0.982	0.968	0.963	0.951	0.947	0.938	1×10^3
Ours-$(d/8, n/2)$	0.980	0.969	0.969	0.950	0.941	0.943	0.7×10^3
Ours-$(d/8, n/3)$	0.981	0.967	0.969	0.956	0.943	0.941	0.6×10^3
Ours-$(d/8, n/4)$	0.981	0.968	0.964	0.946	0.945	0.938	0.5×10^3

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5   Appendix

5.1   MEB with Outliers Is NP-complete

Megiddo [44] studied the problem of covering by two balls in high dimension: given a set of points $S \subset \mathbb{R}^d$, find two unit balls B_1 and B_2 so that $S \subset B_1 \cup B_2$, or recognize that no such balls exist. The paper showed that the problem is NP-complete when both $|S|$ and d are large, and the proof is by the reduction from 3-SAT.

Given an instance of 3-SAT including m clauses with a set of $d-1$ variables, [44] generated a set of points $S = U \cup P$ in $\mathbb{R}^d$: $U = \{\pm e^i, i = 1, \cdots, d\}$ where each e^i is the unit vector having 1 in the i-th position; P is a set of m points corresponding to the m clauses based on some careful construction. Then, [44] proved that the given 3-SAT instance is satisfiable iff $U \cup P$ is covered by two balls both having the radius $\sqrt{1 - 1/d}$ (we can simply scale all the points by $\frac{1}{\sqrt{1-1/d}}$ to make the two balls to be unit). As a by-product in the proof, it reveals that the given 3-SAT instance is satisfiable iff the whole P is covered by one of the two balls. In addition, it is easy to see that each pair $\pm e^i$ in U are always separated by the two balls. Following this reduction, we can further reduce 3-SAT to our MEB with outliers problem.

Consider the instance $(U \cup P, \gamma = \frac{d}{2d+m})$ of MEB with outliers (e.g., we can set $d = m$ and $\gamma = \frac{d}{2d+m} = 1/3$). First, we know the optimal radius $r_{opt} \geq \sqrt{1 - 1/d}$, because the inliers should contain at least d points from U and any ball covering d points of U should has the radius at least $\sqrt{1 - 1/d}$. Moreover, if $r_{opt} = \sqrt{1 - 1/d}$, then the optimal ball of the instance $(U \cup P, \gamma)$ should always cover the whole P (that is, $U \cup P$ is covered by two balls both having the radius $\sqrt{1 - 1/d}$). Then, according to the above reduction from 3-SAT to the problem of covering by two balls, we know that the given instance of 3-SAT is satisfiable iff $r_{opt} = \sqrt{1 - 1/d}$. Therefore, the problem of MEB with outliers is NP-complete.

5.2   Proof of Lemma 6

First, we need the following lemma in our proof.

▶ **Lemma 12.** *[20] Let Q be a set of elements, and Q' be a subset of Q with size $|Q'| = \beta |Q|$ for some $\beta \in (0,1)$. If one randomly samples $\frac{\ln 1/\eta}{\ln 1/(1-\beta)} \leq \frac{1}{\beta} \ln \frac{1}{\eta}$ elements from Q, then with probability at least $1 - \eta$, the sample contains at least one element of Q' for any $\eta \in (0,1)$.*

Since $|O_v| = (1 + \delta)\gamma n$ and $|P \backslash P_{\text{opt}}| = \gamma n$, we have

$$\frac{|O_v \cap P_{\text{opt}}|}{|O_v|} = 1 - \frac{|O_v \backslash P_{\text{opt}}|}{|O_v|} \geq 1 - \frac{|P \backslash P_{\text{opt}}|}{|O_v|} = \frac{\delta}{1 + \delta}. \tag{16}$$

Note $|S_v| = (1 + \frac{1}{\delta}) \ln \frac{h}{\mu}$. If we apply Lemma 12 via setting $\beta = \frac{\delta}{1+\delta}$ and $\eta = \frac{\mu}{h}$, it is easy to know that S_v contains at least one point from P_{opt} with probability $1 - \frac{\mu}{h}$.

5.3 Proof of Theorem 10

Recall we use P_{opt} and r_{opt} to denote the optimal subset and radius for the instance (P, γ) in Section 1.1. Let c_{opt} be the center of $MEB(P_{opt})$. Obviously c_{opt} is inside the convex hull of P_{opt}. Thus, using Theorem 9 we have

$$\max_{p \in P_{opt}} ||f(p) - f(c_{opt})||^2 \leq (1 + \epsilon) r_{opt}^2. \tag{17}$$

Further, we let the set of inliers induced by $\bar{c}_v$ be $f(P') \subset f(P)$ which has the size at least $(1 - (1 + \delta)\gamma)n$. Also, due to the nature of our algorithm we know that $f(S) \subset f(P')$, that is, $\bar{c}_v$ is inside the convex hull of $f(P')$ as well; in other words, $f^{-1}(\bar{c}_v)$ is inside the convex hull of P'. Applying Theorem 9 again, we have

$$\begin{aligned}
\forall p \in P', \quad ||p - f^{-1}(\bar{c}_v)||^2 &\leq ||f(p) - \bar{c}_v||^2 + \epsilon \ rad(P')^2 \\
\implies \max_{p \in P'} ||p - f^{-1}(\bar{c}_v)||^2 &\leq \max_{p \in P'} ||f(p) - \bar{c}_v||^2 + \epsilon \ rad(P')^2.
\end{aligned} \tag{18}$$

Let the exact center of $MEB(f(P'))$ be $\bar{c}_v'$. Then $\bar{c}_v'$ and $f^{-1}(\bar{c}_v')$ (defined similarly as $f^{-1}(\bar{c}_v)$) locate inside the convex hulls of $f(P')$ and P', respectively. So we have

$$\begin{aligned}
\forall p \in P', \quad ||f(p) - \bar{c}_v'||^2 &\geq ||p - f^{-1}(\bar{c}_v')||^2 - \epsilon \ rad(P')^2 \\
\implies \forall p \in P', \quad rad(f(P'))^2 &\geq ||p - f^{-1}(\bar{c}_v')||^2 - \epsilon \ rad(P')^2 \\
\implies rad(f(P'))^2 &\geq (1 - \epsilon) \ rad(P')^2.
\end{aligned} \tag{19}$$

Plugging (19) into (18), we have

$$\begin{aligned}
\max_{p \in P'} ||p - f^{-1}(\bar{c}_v)||^2 &\leq \max_{p \in P'} ||f(p) - \bar{c}_v||^2 + \frac{\epsilon}{1 - \epsilon} \ rad(f(P'))^2 \\
&\leq \frac{1}{1 - \epsilon} \max_{p \in P'} ||f(p) - \bar{c}_v||^2.
\end{aligned} \tag{20}$$

Recall that $\bar{c}_v$ is an (ϵ, δ)-approximation center for $f(P)$. Together with (17) and (20), we have

$$\begin{aligned}
\max_{p \in P'} ||p - f^{-1}(\bar{c}_v)||^2 &\leq \frac{1}{1 - \epsilon}(1 + \epsilon) \max_{p \in P_{opt}} ||f(p) - f(c_{opt})||^2 \\
&\leq \frac{(1 + \epsilon)^2}{1 - \epsilon} r_{opt}^2 = (1 + O(\epsilon)) r_{opt}^2.
\end{aligned} \tag{21}$$

(21) implies $\max_{p \in P'} ||p - f^{-1}(\bar{c}_v)|| \leq \sqrt{(1 + O(\epsilon))} r_{opt} = (1 + O(\epsilon)) r_{opt}$, that is, $f^{-1}(\bar{c}_v)$ is an $(O(\epsilon), \delta)$-approximation center in $\mathbb{R}^d$.