# Data Structures and Algorithms I

## Hashing

# Objectives

1

- To understand how **hashing** is used to accelerate table lookup

2

- To study the issue of **collision** and techniques to resolve it

# What is Hashing?

- Hashing is an algorithm (via a hash function) that maps large data sets of variable length, called *keys*, to smaller data sets of a fixed length.
- A hash table (or hash map) is a data structure that uses a hash function to efficiently map keys to values, for efficient search and retrieval.
- Widely used in many kinds of computer software, particularly for associative arrays, database indexing, caches, and sets.

# ADT Table Operations

| | **Sorted Array** | **Balanced BST** | **Hashing** |
|---|---|---|---|
| **Insertion** | O(*n*) | O(log *n*) | O(1) avg |
| **Deletion** | O(*n*) | O(log *n*) | O(1) avg |
| **Retrieval** | O(log *n*) | O(log *n*) | O(1) avg |

Note: Balanced Binary Search Tree (BST) will be covered in 502043 Data Structures and Algorithms II.

- Hence, hash table supports the table ADT in constant time on average for the above operations. It has many applications.

# 1 Direct Addressing Table

A simplified version of hash table

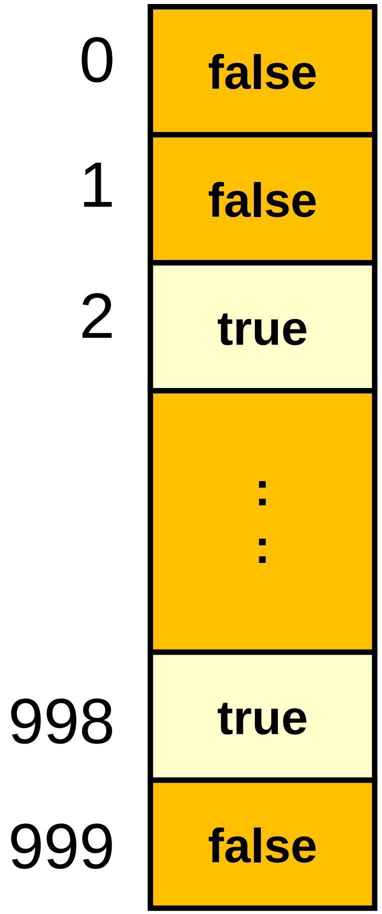
# 1 SBS Transit Problem

- Retrieval: find(*num*)
  - Find the bus route of bus service number *num*
- Insertion: insert(*num*)
  - Introduce a new bus service number *num*
- Deletion: delete(*num*)
  - Remove bus service number *num*

# 1 SBS Transit Problem

Assume that bus numbers are integers between 0 and 999, we can create an array with 1000 Boolean values.

If bus service *num* exists, just set position *num* to true.

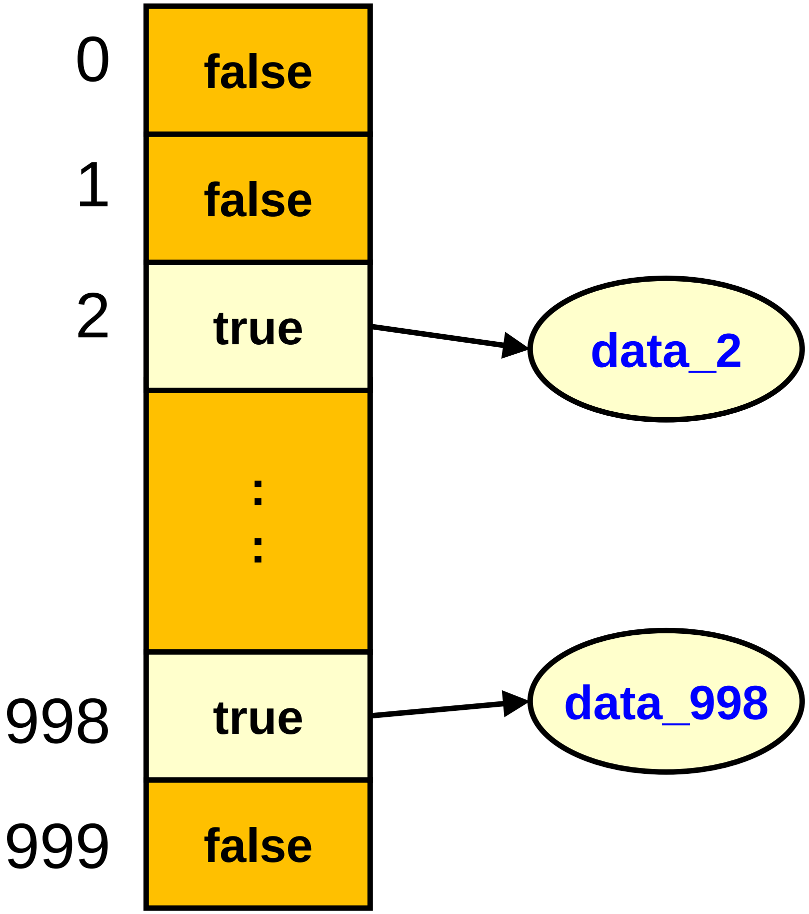


| | |
|---|---|
| 0 | false |
| 1 | false |
| 2 | true |
| | :<br>: |
| 998 | true |
| 999 | false |

# 1 Direct Addressing Table (1/2)

If we want to maintain additional data about a bus, use an array of 1000 slots, each can **reference** to an object which contains the details of the bus route.

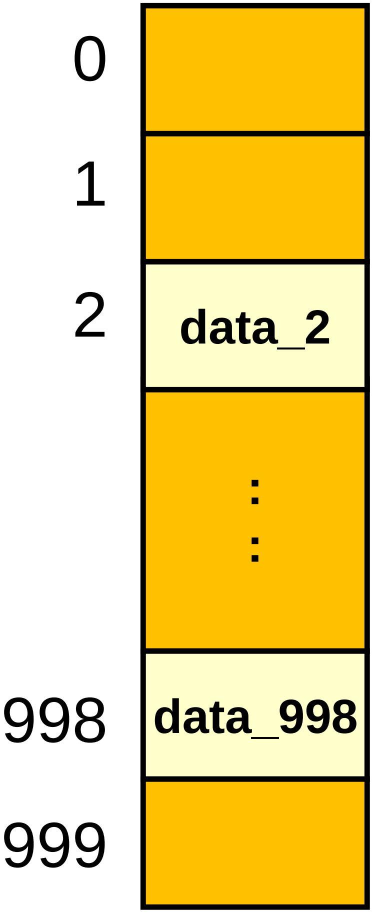
Note: You may want to store the key values, i.e. bus numbers, also.



# 1 Direct Addressing Table (2/2)

Alternatively, we can store the data **directly in the table slots** also.

**Q**: What are the advantages and disadvantages of these 2 approaches?



# 1 Direct Addressing Table: Operations

```
insert (key, data)
    a[key] = data        // where a[] is an array – the table
delete (key)
    a[key] = null
find (key)
    return a[key]
```

# 1 Direct Addressing Table: Restrictions

- Keys must be non-negative integer values
  - What happens for key values 151A and NR10?
- Range of keys must be small
- Keys must be dense, i.e. not many gaps in the key values.
- How to overcome these restrictions?

# 2 Hash Table

Hash Table is a generalization of direct addressing table, to remove the latter’s restrictions.

# 2 Origins of the term Hash

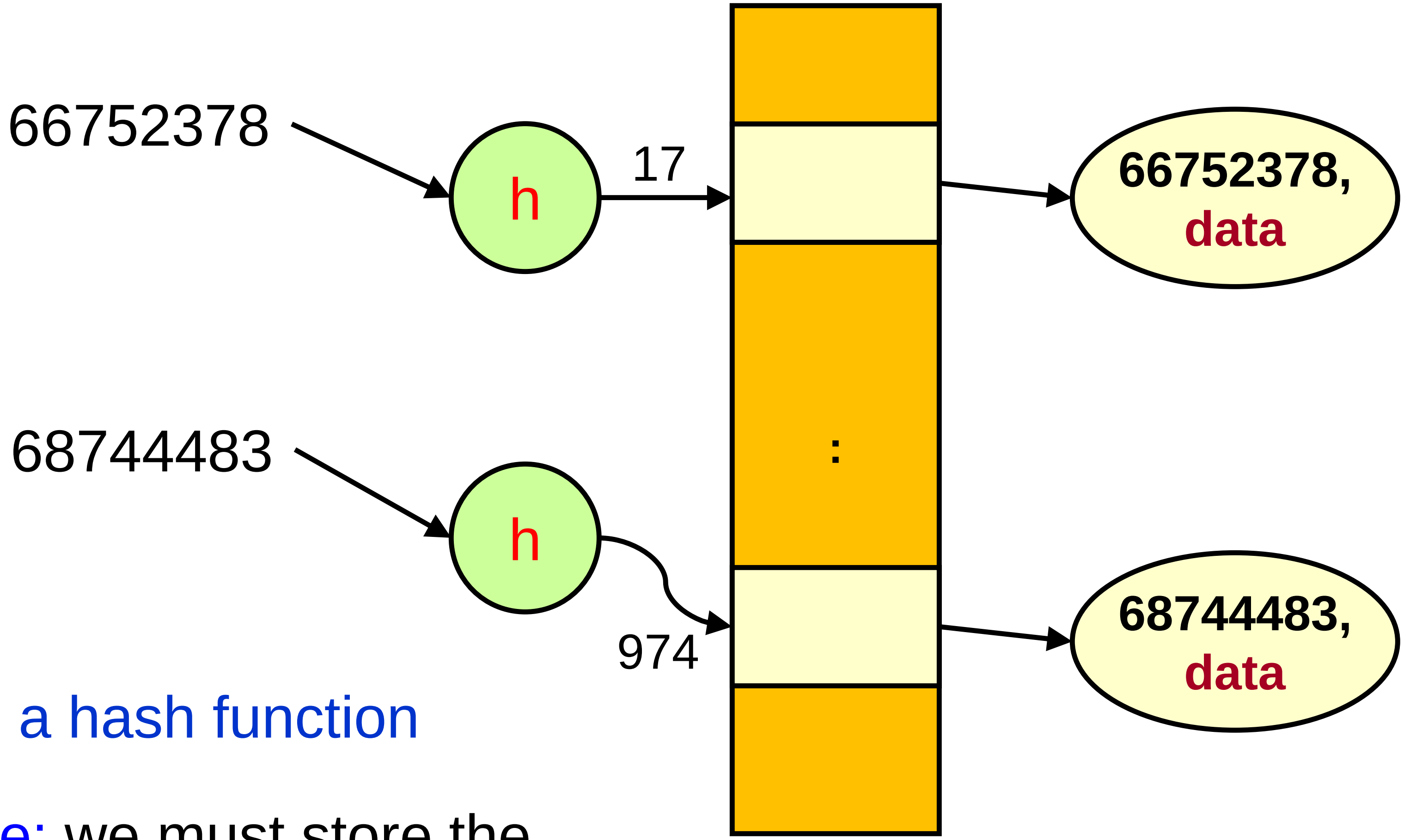
- The term "hash" comes by way of analogy with its standard meaning in the physical world, to "chop and mix".
- Indeed, typical hash functions, like the **mod** operation, “chop” the input domain into many sub-domains that get “mixed” into the output range.
- Donald Knuth notes that Hans Peter Luhn of IBM appears to have been the first to use the concept, in a memo dated January 1953, and that Robert Morris used the term in a survey paper in CACM which elevated the term from technical jargon to formal terminology.

# 2 Ideas

- Map large integers to smaller integers
- Map non-integer keys to integers

HASHING

# 2 Hash Table



**h** is a hash function

Note: we must store the key values. Why?

# 2 Hash Table: Operations

**insert (key, data)**

```
    a[h(key)] = data  // h is a hash function and a[] is an array
```

**delete (key)**

```
    a[h(key)] = null
```

**find (key)**

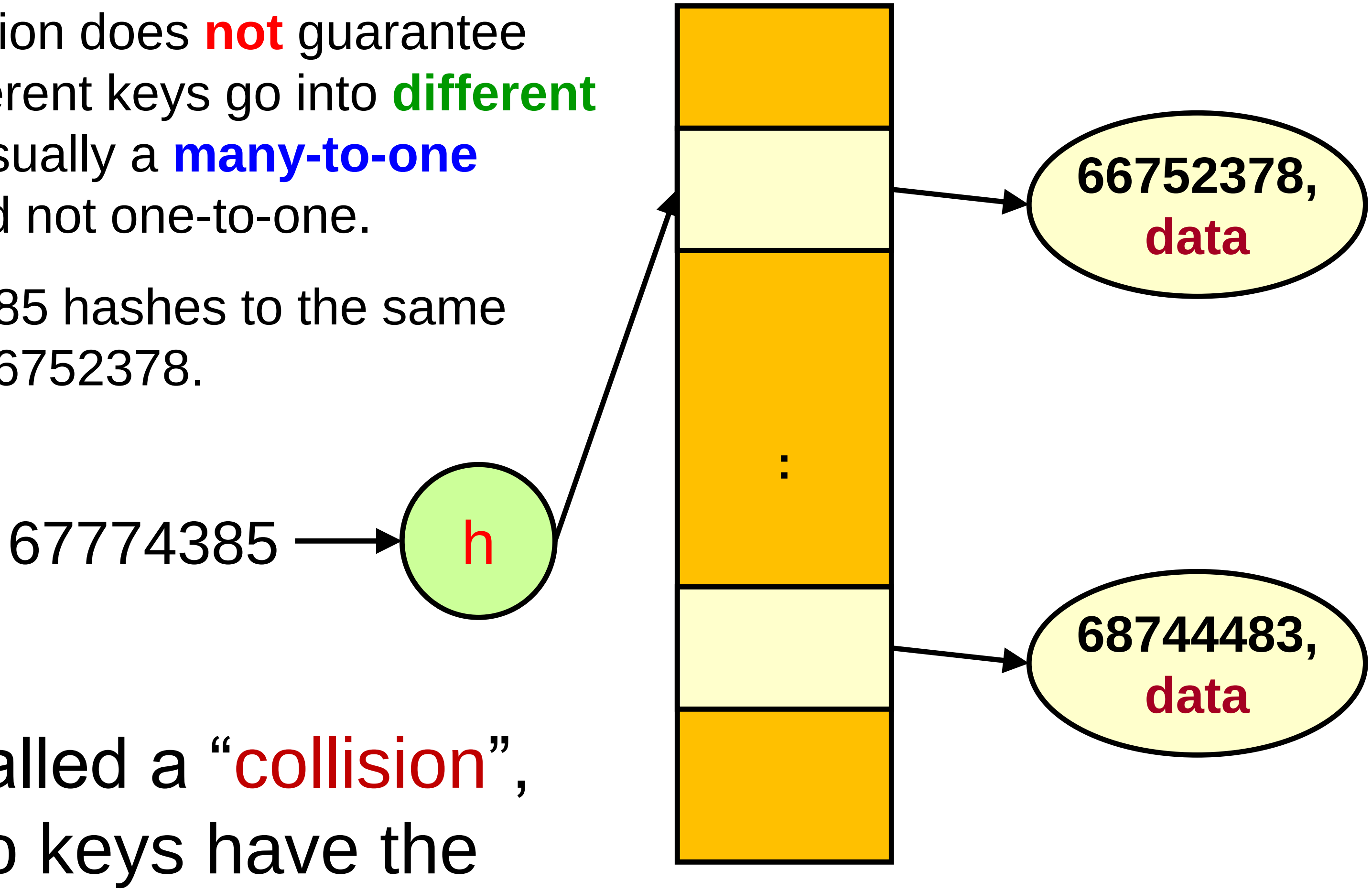
```
    return a[h(key)]
```

However, this does **not** work for **all** cases! (Why?)

# 2 Hash Table: Collision

A hash function does **not** guarantee that two different keys go into **different** slots! It is usually a **many-to-one** mapping and not one-to-one.

E.g. 67774385 hashes to the same location of 66752378.



This is called a "collision", when two keys have the same hash value.

# 2 Two Important Issues

- How to hash?
- How to resolve collisions?
- These are important issues that can affect the efficiency of hashing

# 3 Hash Functions

# 3 Criteria of Good Hash Functions

- Fast to compute
- Scatter keys evenly throughout the hash table
- Less collisions
- Need less slots (space)

# 3 Example of Bad Hash Function

- Select Digits – e.g. choose the 4th and 8th digits of a phone number
  - hash(67754378) = 58
  - hash(63497820) = 90
- What happen when you hash Singapore's house phone numbers by selecting the first three digits?

# 3 Perfect Hash Functions

- Perfect hash function is a **one-to-one** mapping between keys and hash values. So no collision occurs.
- Possible if all keys are known.
- Applications: compiler and interpreter search for reserved words; shell interpreter searches for built-in commands.
- GNU gperf is a freely available perfect hash function generator written in C++ that automatically constructs perfect functions (a C++ program) from a user supplied list of keywords.
- Minimal perfect hash function: The table size is the same as the number of keywords supplied.

# 3 Uniform Hash Functions

- Distributes keys evenly in the hash table
- Example
    - If *k* integers are uniformly distributed among 0 and *X*-1, we can map the values to a hash table of size ***m*** (*m* < *X*) using the hash function below

$$k \in [0, X)$$

$$hash(k) = \left\lfloor \frac{km}{X} \right\rfloor$$

*k* is the key value

[ ]: close interval

( ): open interval

Hence, 0 ≤ *k* < *X*

⌊ ⌋ is the *floor* function

# 3 Division method (mod operator)

- Map into a hash table of ***m*** slots.
- Use the modulo operator (**%** in Java) to map an integer to a value between 0 and *m*-1.
- *n* mod *m* = remainder of *n* divided by *m*, where *n* and *m* are positive integers.

$$hash(k) = k \; \% \; m$$

The most popular method.

# 3 How to pick *m*?

- The choice of *m* (or hash table size) is important. If *m* is power of two, say $2^n$, then key modulo of *m* is the same as extracting the last *n* bits of the key.
- If *m* is $10^n$, then our hash values is the last *n* digit of keys.
- Both are no good.
- Rule of thumb:
  - Pick a prime number close to a power of two to be *m*.

# 3 Multiplication method

1. Multiply by a constant real number A between 0 and 1
2. Extract the fractional part
3. Multiply by *m*, the hash table size

$$hash(k) = \left\lfloor m\left(k\mathbf{A} - \lfloor k\mathbf{A} \rfloor\right)\right\rfloor$$

The reciprocal of the golden ratio = (sqrt(5) - 1)/2 = 0.618033 seems to be a good choice for A (recommended by Knuth).

# 3 Hashing of strings (1/4)

- An example hash function for strings:

```
hash(s)  {    //  s is a string
   sum = 0
   for each character c in s {
        sum += c      //  sum up the ASCII values of all characters
   }
   return sum % m     //  m is the hash table size
}
```

# 3 Hashing of strings: Examples (2/4)

**hash(“Tan Ah Teck”)**

= (“T” + “a” + “n” + “ ” +
“A” + “h” + “ ” +
“T” + “e” + “c” + “k”) % 11 // hash table size is 11

= (84 + 97 + 110 + 32 +
65 + 104 + 32 +
84 + 101 + 99 + 107) % 11

= 825 % 11

= 0

# 3 Hashing of strings: Examples (3/4)

- All 3 strings below have the same hash value! Why?
  - Lee Chin Tan
  - Chen Le Tian
  - Chan Tin Lee
- Problem: This hash function value does not depend on positions of characters! – Bad

# 3 Hashing of strings (4/4)

- A better hash function for strings is to “shift” the sum after each character, so that the positions of the characters affect the hash value.

```
hash(s)
        sum = 0
        for each character c in s {
                sum = sum*31 + c
        }
        return sum % m        // m is the hash table size
```

Java’s String.hashCode() uses *31 as well.

# 4 Collision Resolution

# 4 Probability of Collision (1/2)

- **von Mises Paradox (The Birthday Paradox)**:
  "How many people must be in a room before the probability that some share a birthday, ignoring the year and leap days, becomes at least 50 percent?"

Q($n$) = Probability of unique birthday for $n$ people

$$= \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \cdots \frac{365-n+1}{365}$$

P($n$) = Probability of collisions (same birthday) for $n$ people

= 1 – Q($n$)

P(**23**) = 0.507

Hence, you need only 23 people in the room!

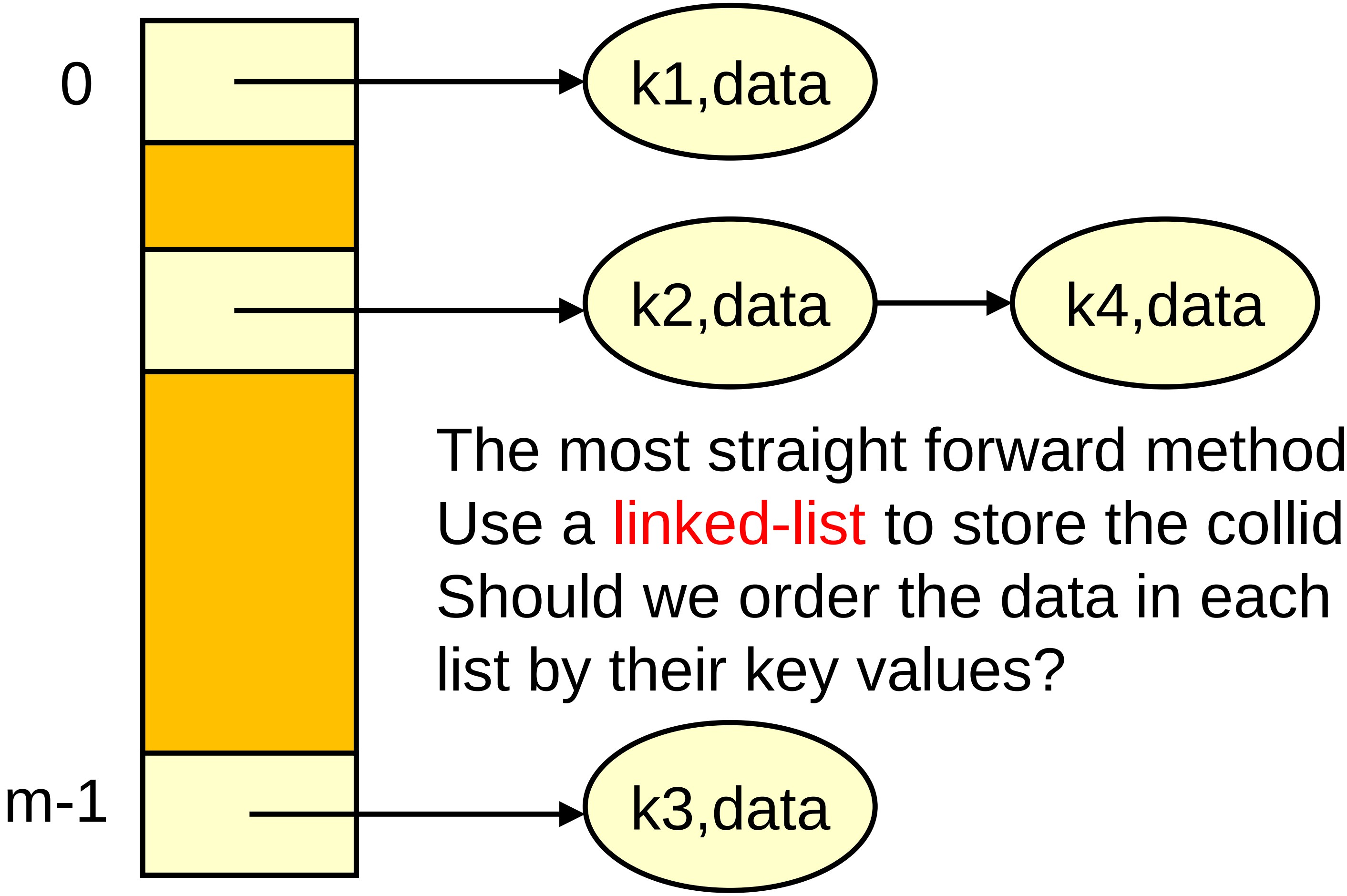
# 4 Probability of Collision (2/2)

- This means that if there are 23 people in a room, the probability that some people share a birthday is 50.7%!
- In the hashing context, if we insert 23 keys into a table with 365 slots, more than half of the time we will get collisions! Such a result is counter-intuitive to many.
- So, collision is very likely!

# 4 Collision Resolution Techniques

- Separate Chaining
- Linear Probing
- Quadratic Probing
- Double Hashing

# 4.1 Separate Chaining



The most straight forward method.
Use a linked-list to store the collided keys.
Should we order the data in each linked list by their key values?

# 4.1 Hash operations

**insert (key, data)**

Insert data into the list a[h(key)]

Takes O(1) time

**find (key)**

Find key from the list a[h(key)]

Takes O(*n*) time, where *n* is length of the chain

**delete (key)**

Delete data from the list a[h(key)]

Takes O(*n*) time, where *n* is length of the chain

# 4.1 Analysis: Performance of Hash Table

- *n*: number of keys in the hash table
- *m*: size of the hash tables – number of slots
- $\alpha$: load factor

$$\alpha = n/m$$

a measure of how full the hash table is. If table size is the number of linked lists, then $\alpha$ is the average length of the linked lists.

# 4.1 Reconstructing Hash Table

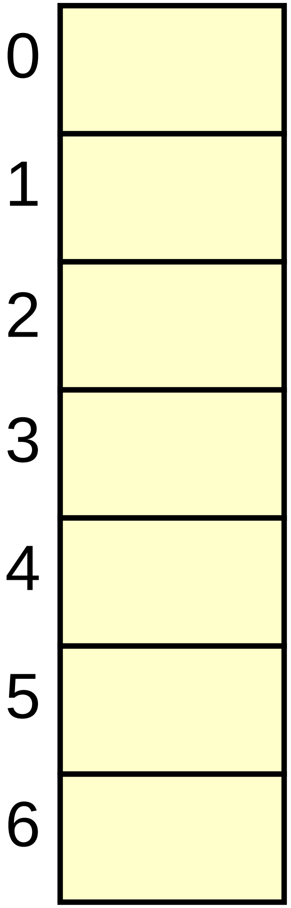
- To keep $\alpha$ bounded, we may need to reconstruct the whole table when the load factor exceeds the bound.
- Whenever the load factor exceeds the bound, we need to rehash all keys into a bigger table (increase *m* to reduce $\alpha$), say double the table size *m*.

# 4.2 Linear Probing

**hash(*k*) = *k* mod 7**

Here the table size m=7

Note: 7 is a prime number.



In linear probing, when we get a collision, we scan through the table looking for the next empty slot (wrapping around when we reach the last slot).

# 4.2 Linear Probing: Insert 18

**hash(*k*) = *k* mod 7**

hash(18) = 18 mod 7 = 4

| | |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | **18** |
| 5 | |
| 6 | |

# 4.2 Linear Probing: Insert 14

**hash(*k*) = *k* mod 7**

hash(18) = 18 mod 7 = 4

hash(14) = 14 mod 7 = 0

| | |
|---|---|
| 0 | 14 |
| 1 | |
| 2 | |
| 3 | |
| 4 | 18 |
| 5 | |
| 6 | |

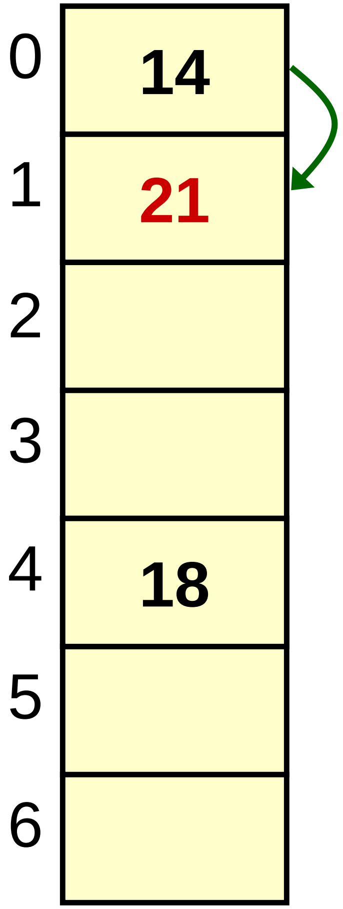
# 4.2 Linear Probing: Insert 21

**hash(*k*) = *k* mod 7**

hash(18) = 18 mod 7 = 4

hash(14) = 14 mod 7 = 0

hash(21) = 21 mod 7 = 0



| | |
|---|---|
| 0 | **14** |
| 1 | **21** |
| 2 | |
| 3 | |
| 4 | **18** |
| 5 | |
| 6 | |

Collision occurs!

Look for next empty slot.

# 4.2 Linear Probing: Insert 1

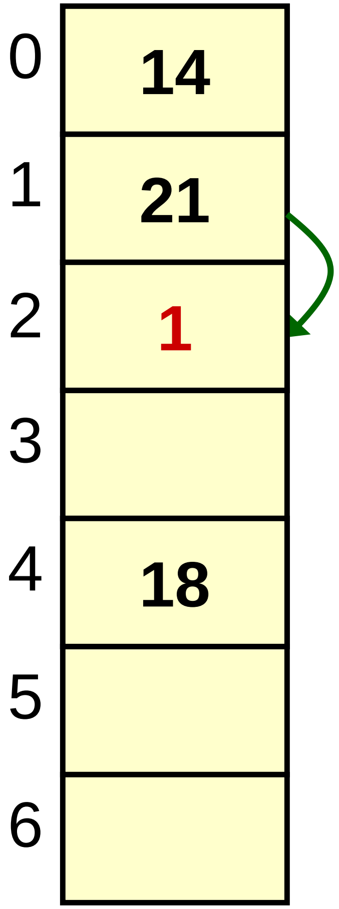
**hash(*k*) = *k* mod 7**

hash(18) = 18 mod 7 = 4

hash(14) = 14 mod 7 = 0

hash(21) = 21 mod 7 = 0

hash(1) = 1 mod 7 = 1



| | |
|---|---|
| 0 | **14** |
| 1 | **21** |
| 2 | **1** |
| 3 | |
| 4 | **18** |
| 5 | |
| 6 | |

Collides with 21 (hash value 0). Look for next empty slot.

# 4.2 Linear Probing: Insert 35

**hash(*k*) = *k* mod 7**

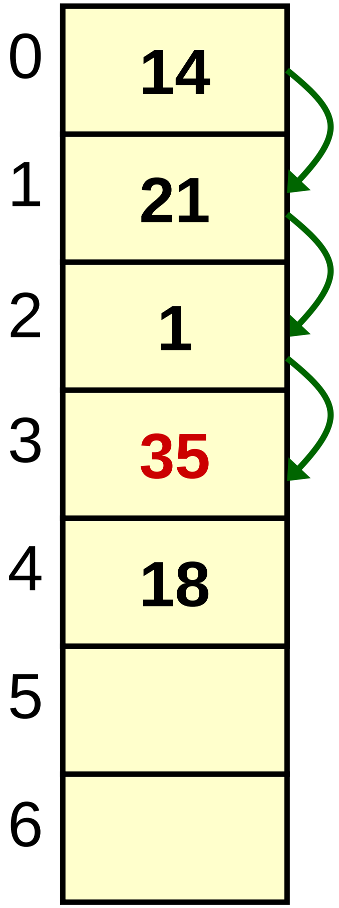
hash(18) = 18 mod 7 = 4

hash(14) = 14 mod 7 = 0

hash(21) = 21 mod 7 = 0

hash(1) = 1 mod 7 = 1

hash(35) = 35 mod 7 = 0



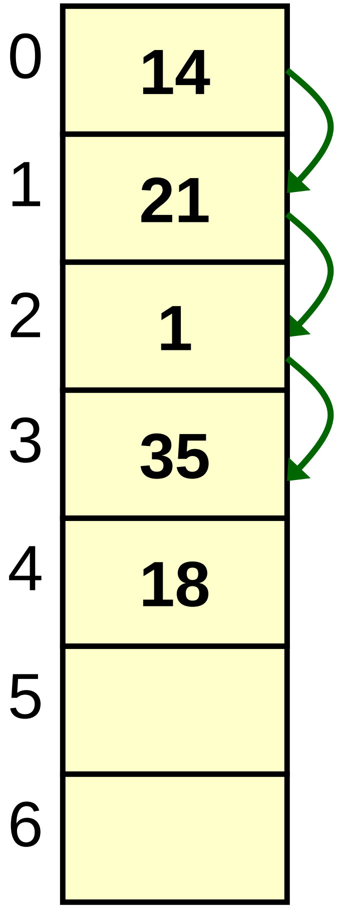
| | |
|---|---|
| 0 | **14** |
| 1 | **21** |
| 2 | **1** |
| 3 | **35** |
| 4 | **18** |
| 5 | |
| 6 | |

Collision, need to check next 3 slots.

# 4.2 Linear Probing: Find 35

**hash(*k*) = *k* mod 7**

hash(35) = 0



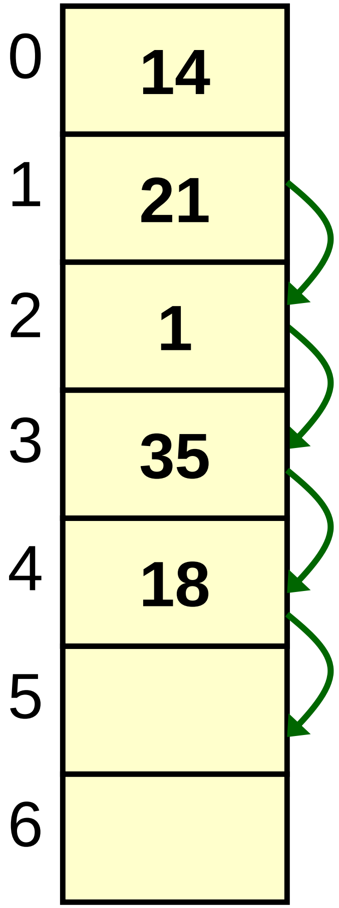
| | |
|---|---|
| 0 | **14** |
| 1 | **21** |
| 2 | **1** |
| 3 | **35** |
| 4 | **18** |
| 5 | |
| 6 | |

Found 35, after 4 probes.

# 4.2 Linear Probing: Find 8

**hash(*k*) = *k* mod 7**

hash(8) = 1



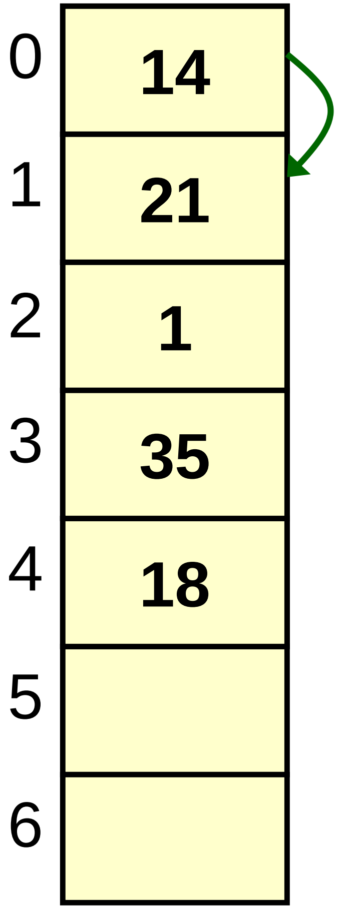
| | |
|---|---|
| 0 | 14 |
| 1 | 21 |
| 2 | 1 |
| 3 | 35 |
| 4 | 18 |
| 5 | |
| 6 | |

8 NOT found.
Need 5 probes!

# 4.2 Linear Probing: Delete 21

**hash(*k*) = *k* mod 7**

hash(21) = 0



| | |
|---|---|
| 0 | **14** |
| 1 | **21** |
| 2 | **1** |
| 3 | **35** |
| 4 | **18** |
| 5 | |
| 6 | |

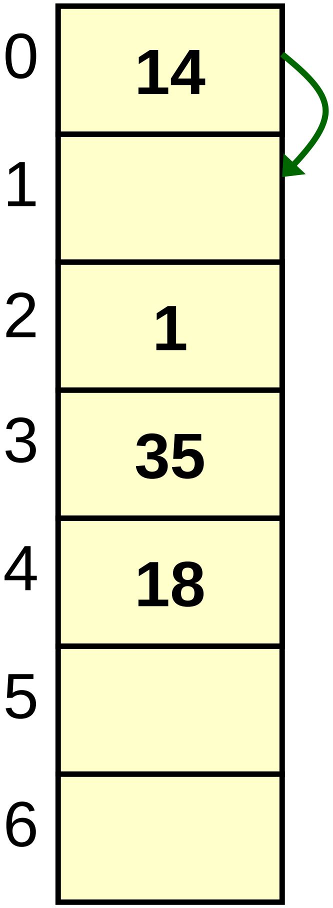
We cannot simply remove a value, because it can affect find()!

# 4.2 Linear Probing: Find 35

**hash(*k*) = *k* mod 7**

hash(35) = 0

Hence for deletion, cannot simply remove the key value!



|  |  |
|---|---|
| 0 | **14** |
| 1 |  |
| 2 | **1** |
| 3 | **35** |
| 4 | **18** |
| 5 |  |
| 6 |  |

We cannot simply remove a value, because it can affect find()!

35 NOT found!

Incorrect!

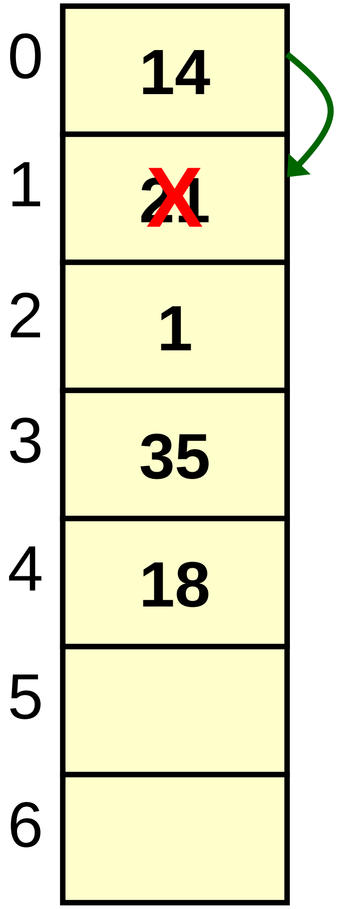
# 4.2 How to delete?

- Lazy Deletion
- Use three different states of a slot
  - Occupied
  - Occupied but mark as deleted
  - Empty
- When a value is removed from linear probed hash table, we just mark the status of the slot as “deleted”, instead of emptying the slot.

# 4.2 Linear Probing: Delete 21

**hash(*k*) = *k* mod 7**

hash(21) = 0



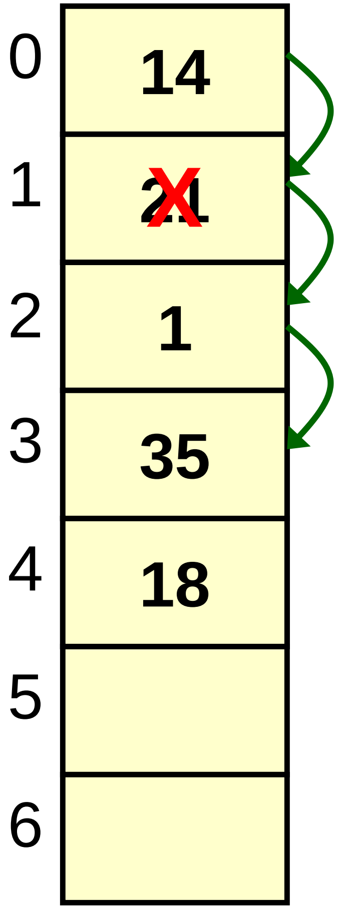
| | |
|---|---|
| 0 | **14** |
| 1 | **~~21~~** |
| 2 | **1** |
| 3 | **35** |
| 4 | **18** |
| 5 | |
| 6 | |

Slot 1 is occupied but now marked as deleted.

# 4.2 Linear Probing: Find 35

**hash(*k*) = *k* mod 7**

hash(35) = 0



| | |
|---|---|
| 0 | 14 |
| 1 | 21 X |
| 2 | 1 |
| 3 | 35 |
| 4 | 18 |
| 5 | |
| 6 | |

Found 35

Now we can find 35

# 4.2 Linear Probing: Insert 15 (1/2)

**hash(*k*) = *k* mod 7**

hash(15) = 1

| | |
|---|---|
| 0 | **14** |
| 1 | ~~**21**~~ X |
| 2 | **1** |
| 3 | **35** |
| 4 | **18** |
| 5 | |
| 6 | |

Slot 1 is marked as deleted.

We continue to search for 15, and found that 15 is not in the hash table (total 5 probes).

So, we insert this new value 15 into the slot that has been marked as deleted (i.e. slot 1).

# 4.2 Linear Probing: Insert 15 (2/2)

**hash(*k*) = *k* mod 7**

hash(15) = 1

| | |
|---|---|
| 0 | **14** |
| 1 | **15** |
| 2 | **1** |
| 3 | **35** |
| 4 | **18** |
| 5 | |
| 6 | |

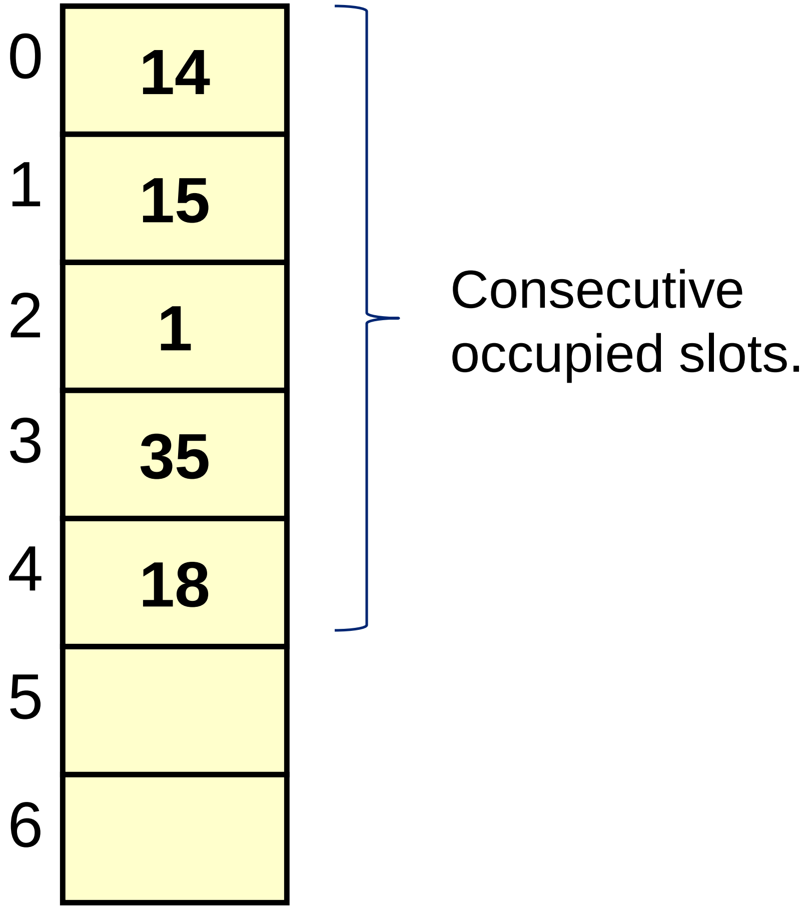
So, 15 is inserted into slot 1, which was marked as deleted.

Note: We should insert a new value in first available slot so that the find operation for this value will be the fastest.

# 4.2 Problem of Linear Probing

It can create many **consecutive occupied slots**, increasing the running time of find/insert/delete.

This is called Primary Clustering



| | |
|---|---|
| 0 | 14 |
| 1 | 15 |
| 2 | 1 |
| 3 | 35 |
| 4 | 18 |
| 5 | |
| 6 | |

Consecutive occupied slots.

# 4.2 Linear Probing

The probe sequence of this linear probing is:

$$
\begin{array}{l}
\text{hash(key)} \\
(\ \text{hash(key)} + \mathbf{1}\ )\ \%\ m \\
(\ \text{hash(key)} + \mathbf{2}\ )\ \%\ m \\
(\ \text{hash(key)} + \mathbf{3}\ )\ \%\ m \\
\vdots
\end{array}
$$

# 4.2 Modified Linear Probing

Q: How to modify linear probing to avoid primary clustering?

We can modify the probe sequence as follows:

hash(key)

( hash(key) + **1** * ***d*** ) % *m*

( hash(key) + **2** * ***d*** ) % *m*

( hash(key) + **3** * ***d*** ) % *m*

:

where *d* is some constant integer >1 and is co-prime to *m*.

Note: Since *d* and *m* are co-primes, the probe sequence covers all the slots in the hash table.

# 4.3 Quadratic Probing

For quadratic probing, the probe sequence is:

$$\text{hash(key)}$$

$$(\ \text{hash(key)} + 1\ )\ \%\ m$$

$$(\ \text{hash(key)} + 4\ )\ \%\ m$$

$$(\ \text{hash(key)} + 9\ )\ \%\ m$$

:

$$(\ \text{hash(key)} + k^2\ )\ \%\ m$$

# 4.3 Quadratic Probing: Insert 3

**hash(*k*) = *k* mod 7**

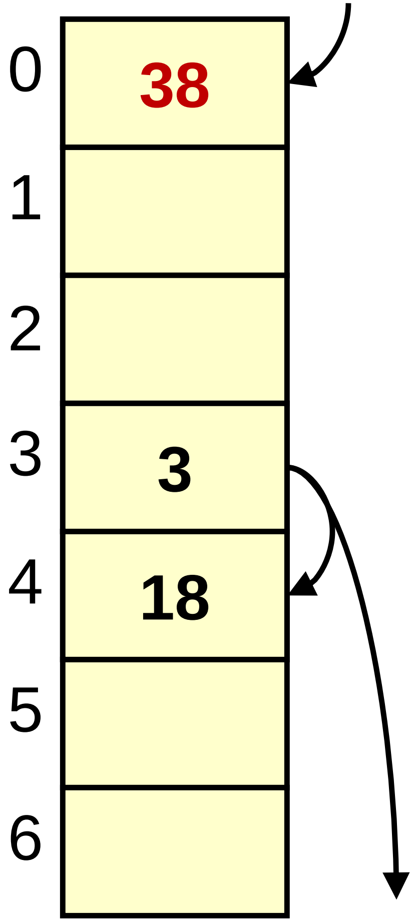
hash(3) = 3

| | |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | **3** |
| 4 | **18** |
| 5 | |
| 6 | |

# 4.3 Quadratic Probing: Insert 38

**hash(*k*) = *k* mod 7**

hash(38) = 3



# 4.3 Theorem of Quadratic Probing

- If $\alpha < 0.5$, and $m$ is prime, then we can always find an empty slot.
  ($m$ is the table size and $\alpha$ is the load factor)
- Note: $\alpha < 0.5$ means the hash table is less than half full.
- Q: How can we be sure that quadratic probing always terminates?
- Insert 12 into the previous example, followed by 10. See what happen?

# 4.3 Problem of Quadratic Probing

- If two keys have the same initial position, their probe sequences are the same.
- This is called secondary clustering.
- But it is not as bad as linear probing.

# 4.4 Double Hashing

Use 2 hash functions:

$$\text{hash(key)}$$
$$(\ \text{hash(key)} + 1 * \text{hash}_2\text{(key)}\ )\ \%\ m$$
$$(\ \text{hash(key)} + 2 * \text{hash}_2\text{(key)}\ )\ \%\ m$$
$$(\ \text{hash(key)} + 3 * \text{hash}_2\text{(key)}\ )\ \%\ m$$
$$\vdots$$

$\text{hash}_2$ is called the secondary hash function, the number of slots to jump each time a collision occurs.

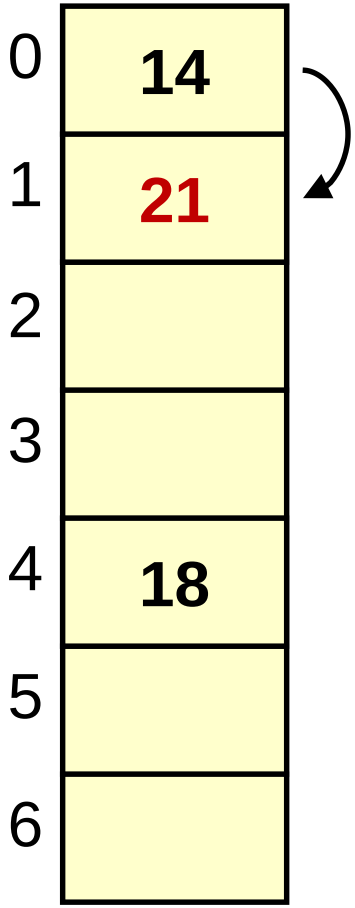
# 4.4 Double Hashing: Insert 21

**hash($k$) = $k$ mod 7**

**hash$_2$($k$) = $k$ mod 5**

hash(21) = 0

hash$_2$(21) = 1

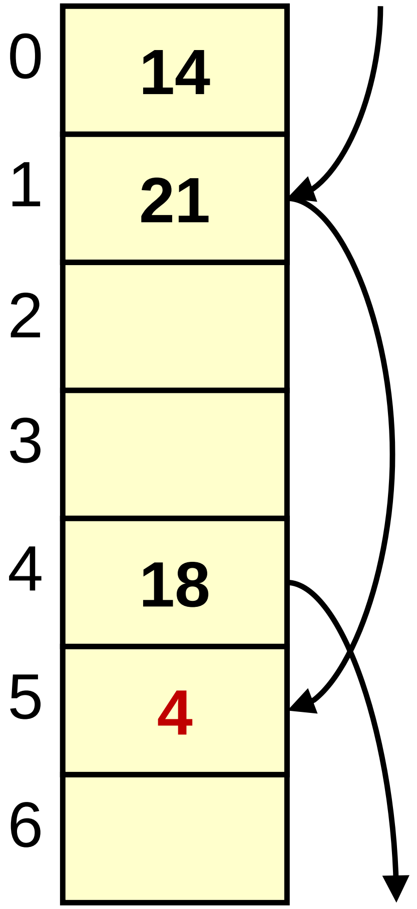


| | |
|---|---|
| 0 | 14 |
| 1 | 21 |
| 2 | |
| 3 | |
| 4 | 18 |
| 5 | |
| 6 | |

# 4.4 Double Hashing: Insert 4

**hash(*k*) = *k* mod 7**
**$\text{hash}_2(k) = k \bmod 5$**

$\text{hash}(4) = 4$
$\text{hash}_2(4) = 4$

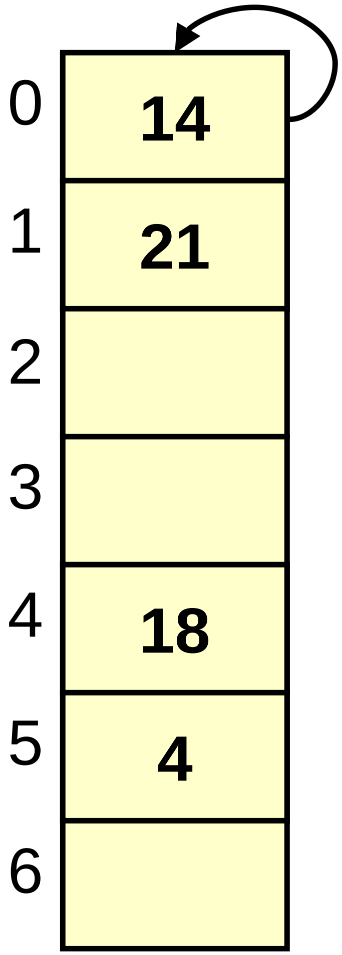


If we insert 4, the probe sequence is 4, 8, 12, …

# 4.4 Double Hashing: Insert 35

**hash(*k*) = *k* mod 7**
**$\text{hash}_2$(*k*) = *k* mod 5**

hash(35) = 0
$\text{hash}_2$(35) = 0



But if we insert 35, the probe sequence is **0, 0, 0,** …

What is wrong?
Since $\text{hash}_2$(35)=**0**.
**Not acceptable!**

# 4.4 Warning

- **Secondary hash function must not evaluate to 0!**
- To solve this problem, simply change $\text{hash}_2(\text{key})$ in the above example to:

$$\text{hash}_2(\text{key}) = 5 – (\text{key} \% 5)$$

**Note**:

- If $\text{hash}_2(\text{k}) = 1$, then it is the same as linear probing.
- If $\text{hash}_2(\text{k}) = d$, where $d$ is a constant integer > 1, then it is the same as modified linear probing.

# 4.5 Criteria of Good Collision Resolution Method

- Minimize clustering
- Always find an empty slot if it exists
- Give different probe sequences when 2 initial probes are the same (i.e. no secondary clustering)
- Fast

# ADT Table Operations

| | Sorted Array | Balanced BST | Hashing |
|---|---|---|---|
| **Insertion** | O(*n*) | O(log *n*) | O(1) avg |
| **Deletion** | O(*n*) | O(log *n*) | O(1) avg |
| **Retrieval** | O(log *n*) | O(log *n*) | O(1) avg |

Note: Balanced Binary Search Tree (BST) will be covered in 502043 Data Structures and Algorithms II.

# 5 Summary

- How to hash? Criteria for good hash functions?
- How to resolve collision?

  Collision resolution techniques:
  - separate chaining
  - linear probing
  - quadratic probing
  - double hashing
- Problem on deletions
- Primary clustering and secondary clustering.

# 6 Java HashMap Class

java.util.HaspMap

# 6 Class HashMap <K, V>

```
public class HashMap<K,V>
  extends AbstractMap<K,V>
  implements Map<K,V>, Cloneable, Serializable
```

- This class implements a hash map, which maps keys to values. Any non-null object can be used as a key or as a value.

  **e.g.** We can create a hash map that maps people names to their ages. It uses the names as keys, and the ages as the values.
- The AbstractMap is an abstract class that provides a skeletal implementation of the Map interface.
- Generally, the default load factor (0.75) offers a good tradeoff between time and space costs.
- The default HashMap capacity is 16.

# 6 Class HashMap <K, V>

- **Constructors** summary
  - `HashMap()`

    Constructs an empty HashMap with a default initial capacity (16) and the default load factor of 0.75.
  - `HashMap(int initialCapacity)`

    Constructs an empty HashMap with the specified initial capacity and the default load factor of 0.75.
  - `HashMap(int initialCapacity, float loadFactor)`

    Constructs an empty HashMap with the specified initial capacity and load factor.
  - `HashMap(Map<? extends K, ? extends V> m)`

    Constructs a new HashMap with the same mappings as the specified Map.

# 6 Class HashMap <K, V>

## Some methods

- `void clear()`

  Removes all of the mappings from this map.
- `boolean containsKey(Object key)`

  Returns true if this map contains a mapping for the specified key.
- `boolean containsValue(Object value)`

  Returns true if this map maps one or more keys to the specified value.
- `V get(Object key)`

  Returns the value to which the specified key is mapped, or null if this map contains no mapping for the key.
- `V put(K key, V value)`

  Associates the specified value with the specified key in this map.
- …

# 6 Example

- Example: Create a hashmap that maps people names to their ages. It uses names as key, and the ages as their values.

```
HashMap<String, Integer> hm = new HashMap<String, Integer>();
// placing items into the hashmap
hm.put("Mike", 52);
hm.put("Janet", 46);
hm.put("Jack", 46);
// retrieving item from the hashmap
System.out.println("Janet => " + hm.get("Janet"));
```

**TestHash.java**

The output of the above code is:

```
Janet => 46
```

End of file