APS1022 PROJECT 1 REPORT

Group 3

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Introduction

Options are a type of financial derivative that represents a contract sold from one party to another. In this project, we will be dealing with the following option types:

- Asian options: payoff is determined by the average underlying price over a specified time period
- American options: the right to exercise can occur any time before or during maturity
- Lookback and floating lookback options: a type of exotic option with path dependency where the payoff depends on the optimal underlying asset's price occurring over the life of the option

We will be using two different approaches to simulate the behaviours of the underlying assets: the Monte Carlo method and the lattice approach.

Monte Carlo Simulations generate a specific number of random paths of the underlying asset through simulation. The payoff of the options can be obtained from these simulations. The number of paths, or sample size, that we will be using in this project is 10000.

Lattice approach, or binomial options pricing model uses a binomial price tree to calculate an option’s payoff by following the paths on the lattice.

The Monte Carlo method and lattice approach will be used to calculate the price of seven different types of options: Asian call option, Asian put option, lookback call option, lookback put option, floating lookback call option, floating lookback float option, and American put option.

Monte Carlo Simulation Method

Parameter	Value used
Risk-free rate (r)	0.02
Current price (S)	100
Volatility (σ)	0.25
Strike price (K)	105
Time to maturity	2 months
Simulation unit time	1 week

The following equations are used to generate the asset price through stochastic process:

Equation 1:

$$S_{t_j} = S_{t_{j-1}} e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t} Z_j}$$

r = risk-free rate

σ = volatility

Z_j = random variable with standard normal distribution N(0,1)

Equation 2:

$$\bar{h}_i = e^{-rT} h_i$$

h_i = payoff of option in path i

$\bar{h}_i$ = present value of payoff

Asian and lookback options algorithm for Monte Carlo method:

For each scenario, the price at time t is calculated using Equation 1. The variables $\bar{S}$, S_{max}, S_{min}, and S_T can be obtained after the asset price is simulated to maturity. The average price can be obtained using the following formula:

$$\bar{S} = \frac{1}{m+1} \sum_{j=0}^{m} S_{t_j} \omega_i$$

For Asian options, the present value of the payoff is determined by K and $\bar{S}$. For lookback options, the value is determined by K, S_{max}, and S_{min}. For floating lookback options, the value is determined by S_T, S_{max}, and S_{min}. Equation 2 is used to obtain the present value of the payoff at maturity. Finally, the premium price can be determined using sample mean:

$$\bar{h} = \frac{1}{n}\sum_{i=1}^{n} h_i$$

American put option algorithm for Monte Carlo method:

For the American put option, the following equations are used:

Equation 3:

$$E[S_T|S_t] = Ke^{-r(T-t)}N(-d_2) - S_tN(-d_1)$$

Equation 4:

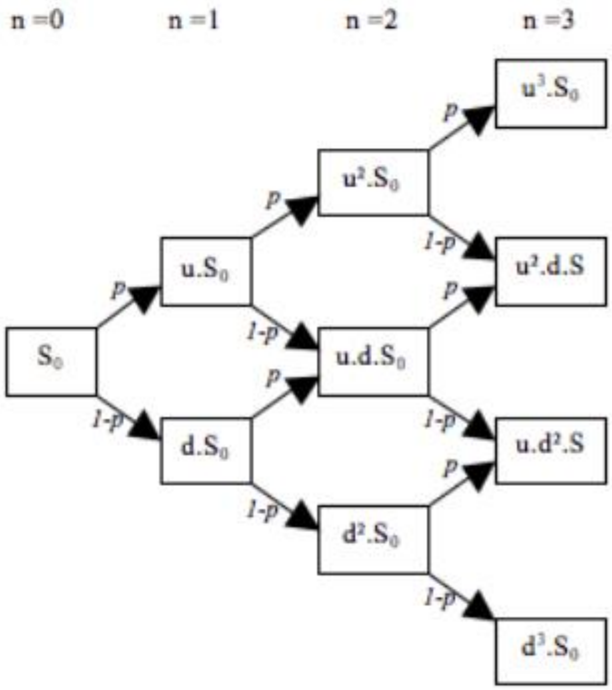
$$d_1 = \frac{\log\left(\frac{S}{k}\right) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}$$

Equation 5:

$$d_2 = d_1 - \sigma\sqrt{T - t}$$

Each scenario i is divided into j different time steps. For each time step, d_1 and d_2 are first calculated using Equation 4 and 5. The expected value $E[S_T|S_t]$ is then calculated using Equation 3. Next, we obtain the payoff at time t, Z_t. This process is repeated until we obtain a Z_t that is greater or equal to the expected value $E[S_T|S_t]$. Once we obtain the optimal payoff, we again use Equation 2 to obtain the present value of the payoff at maturity. The optimal stopping time is then calculated by taking the average values. Finally, the premium price is calculated using the sample mean formula.

Lattice Approach



The above figure illustrates a 3-period lattice. In our case, since each period is 1 week we will have a total of 8 periods for the lattice model. Unlike the Monte Carlo method, for the lattice approach we assume that the asset price can either go up or down for each period. In the above figure, u means the price went up, and d means the price went down.

Asian and lookback options algorithm for lattice approach:

The lattice approach algorithm for Asian and lookback options is very similar to the Monte Carlo method. Rather than using Equation 1 to simulate the price for each period, the paths generated in the lattice model are used instead. In our case, since we are dealing with 8 periods, the total number of paths that will be simulated is 2^8 = 256. For Asian call and put options, the 9 prices in each scenario are used to compute the average price. We are then able to compute option payoff for the specified path. This process is then repeated for all the other paths. Next, all the payoff values are converted into present values using the discount factor. Finally, the average of all the present payoff values is computed and that will be the price of the premium. The above procedure is the exact same for lookback options and floating lookback options.

American put option algorithm for lattice approach:

For American put options, the payoff between exercising and holding will be compared at each node. For each node we will choose the higher payoff between these two actions. Next, the value of the option at each of the previous nodes are calculated. For a 3-period lattice, the following equations are used to calculate the option price at maturity:

$$C_{uuu} = \max\left(u^3S - K, 0\right)$$

$$C_{uud} = \max\left(u^2dS - K, 0\right)$$

$$C_{udd} = \max\left(ud^2S - K, 0\right)$$

$$C_{ddd} = \max\left(d^3S - K, 0\right)$$

Once the above prices are obtained, the maturity payoff values are calculated. Then, the payoffs of the nodes from the previous period are obtained (to the left) using the following equations:

$$q = e^{r\Delta t}$$

$$C_{uu} = \max\left(u^2S - K, e^{-r\Delta t}(qC_{uuu} + (1-q)C_{uud})\right)$$

$$C_{ud} = \max\left(udS - K, e^{-r\Delta t}(qC_{uud} + (1-q)C_{udd})\right)$$

$$C_{dd} = \max\left(d^2S - K, e^{-r\Delta t}(qC_{udd} + (1-q)C_{ddd})\right)$$

$$C_{u} = \max\left(uS - K, e^{-r\Delta t}(qC_{uu} + (1-q)C_{ud})\right)$$

$$C_{d} = \max\left(dS - K, e^{-r\Delta t}(qC_{ud} + (1-q)C_{dd})\right)$$

$$C = \max\left(S - K, e^{-r\Delta t}(qC_{u} + (1-q)C_{d})\right)$$

In the above maximum functions, the first term returns the payoff if the option is exercised. Likewise, the second term returns the option value if we hold it until the next period. The greater value of the two is selected to be the option value at the specified time period. This process is repeated until all nodes (including the leftmost root mode) are calculated.

Monte Carlo Simulation Results

Plot of simulated paths with sample size of 100000:

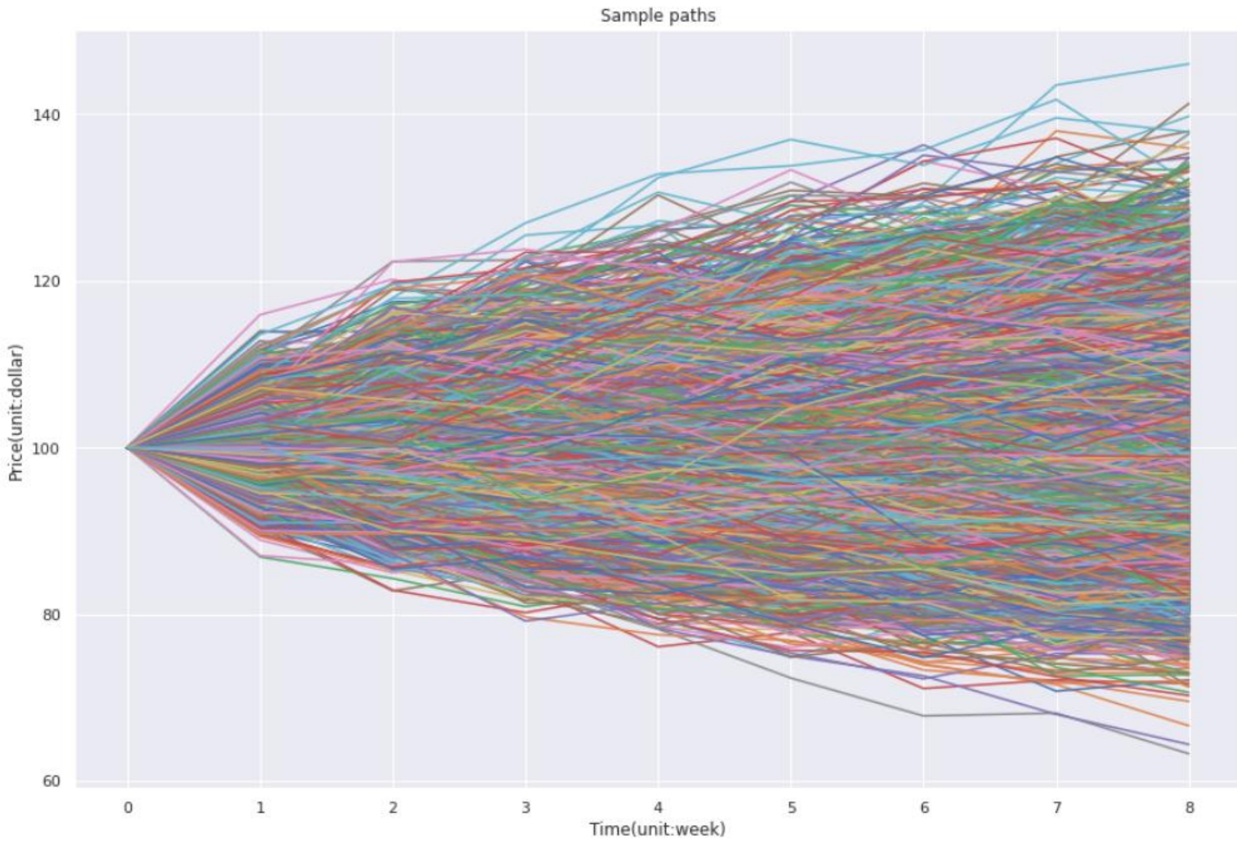


Table of option prices:

Option type	**Average price ($)**	**95% confidence interval ($)**	
Asian call	0.67	0.63	0.71
Asian put	5.59	5.50	5.68
Lookback call	3.26	3.15	3.36
Lookback put	11.08	10.97	11.19
Floating lookback call	6.23	6.10	6.35
Floating lookback put	6.32	6.20	6.43
American put	6.91	6.79	7.03

Lattice Approach Simulation Results

Table of option prices:

Option prices	Average price ($)
Asian call	0.71
Asian put	5.44
Lookback call	3.65
Lookback put	11.16
Floating lookback call	6.70
Floating lookback put	6.46
American put	6.44

First 10 rows of lattice approach:

Time (unit: week)								
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
100	103.6743	107.4837	111.433	115.5274	119.7723	124.1731	128.7356	133.4658
100	103.6743	107.4837	111.433	115.5274	119.7723	124.1731	128.7356	124.1731
100	103.6743	107.4837	111.433	115.5274	119.7723	124.1731	119.7723	124.1731
100	103.6743	107.4837	111.433	115.5274	119.7723	124.1731	119.7723	115.5274
100	103.6743	107.4837	111.433	115.5274	119.7723	115.5274	119.7723	124.1731
100	103.6743	107.4837	111.433	115.5274	119.7723	115.5274	119.7723	115.5274
100	103.6743	107.4837	111.433	115.5274	119.7723	115.5274	111.433	115.5274
100	103.6743	107.4837	111.433	115.5274	119.7723	115.5274	111.433	107.4837
100	103.6743	107.4837	111.433	115.5274	111.433	115.5274	119.7723	124.1731
100	103.6743	107.4837	111.433	115.5274	111.433	115.5274	119.7723	115.5274

Analysis of Results

For both simulation methods, call options for Asian and lookback types seem have a lower price compared to put options. This is mainly due to the fact that the initial price is lower than the strike price. Having an initial price that is lower than the strike price means that it would have been very hard to obtain a price that is higher than the strike price along all the paths. Therefore, it was more likely for a lot of the call options to have 0 payoff. We also obtained similar results for the floating lookback options. The price of the floating lookback options for both simulation methods seems to be stable with very little fluctuations.

We can see that our Monte Carlo method used stochastic processes throughout the simulation process. Through the use of random walks, we were able to reduce uncertainties by taking into account a very large range of possible option prices. One downside of the Monte Carlo method is that the simulations may sometimes undervalue the probabilities of outliers. We think that this may have affected our results for the floating lookback options, since the prices of these options are highly dependant on the highest and lowest values of the paths that were simulated.

Similar to the Monte Carlo method, the lattice approach can also provide a fairly accurate price for options as long as a suitable number of timesteps is used. The lattice approach is a very reliable simulation method in that we are able to change the actual values in each specific node thus giving us a very detailed visual outline of the entire simulation process. A downside of the lattice approach is that the running time of the computations for the lattice scales exponentially. Because of this, the lattice approach is subject to the curse of dimensionality constraint since having a lattice with a very large number of periods will result in a much higher computation time.

All in all, we have determined that both the Monte Carlo method and lattice approach are suitable for the simulation of option prices through our results and analysis.