

of a mathematical probability. With this background the student will be better equipped to study the many texts which deal with 'pure' theory based on a system of axioms.

It is hoped that students of these pages will never have to reject any of the ideas given here, no matter how much they may refine them as their knowledge of the subject grows.

The book was written in 1941, while the writer was interned at Hald, in Denmark. The camp was under Danish control. It contained various British subjects caught in Denmark when the Germans overran the country. The Danish authorities saved them from being interned in Germany, on condition that they were interned by the Danes. The writer has great pleasure in thanking the Danish authorities for the measure of protection they were able to afford him, and in congratulating them on the truly admirable manner in which they cared for their internees for so many years.

My thanks are due to my fellow internee Mr. E. Christensen who helped me with the experiments, to Dr. Phil G. Rasch for kindly encouragement and comments, and to my long suffering friend Dr. Paul Johansen who saw the book through the press.

Copenhagen. June 1945

This little book, which was designed as a first chapter to any text book on mathematical statistics, has been kindly received. The time has come to reprint it and the opportunity has been taken to correct a few minor misprints.

Johannesburg. August 1950.

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## AN EXPERIMENTAL INTRODUCTION TO THE THEORY OF PROBABILITY

### § 1.

Scientific knowledge is based on observations of natural phenomena. When studying the elements of a science, a sound method is to consider a set of observations of the type which first attracted attention to that particular subject.

The Theory of Probability began from a study of games of chance, so we shall begin with simple material of this nature, namely a succession of spins of a coin. Everybody knows something about observations of this type; but such knowledge is usually hazy. Our aim is to clarify our ideas on the subject by introducing into them order and number. In so doing we shall be laying the foundations of a mathematical calculus: but a non-mathematician need not be unduly perturbed by this. All we shall require is patience, common sense and graph paper. The complex manipulation of symbols which is so unintelligible to the layman comes later. Perhaps by the time these difficulties are reached the non-mathematician will be sufficiently confident to grapple with them.

It will simplify matters to ignore for the moment all previous knowledge that we possess on the subject of spinning a coin. Somewhere someone must have spun a coin for the first time. Let us go back to that time and start afresh. Having analysed our particular set of observations and built up our fundamental theory then by all means compare these results with our previous ideas and observations. But let us keep this earlier knowledge in the background for the time being.

The data given here were obtained as follows. A small coin, balanced on the writer's forefinger was given a little flip with the thumb so that it spun through the air for about a foot finally landing on a cloth spread out flat over a table. A note was taken of which side of the coin fell uppermost and the experiment was repeated 10,000 times in all. It was convenient to call one side of the coin "head" and the

other "tail" and to note down the number of heads that appeared when the coin was spun. Thus if tail appeared 0 was recorded. If head appeared 1 was recorded. Some of these records are given in Table 1.

A minor point of which mention will be made again later is that if the coin fell heads in one spin it was convenient to balance it head uppermost on the operator's forefinger when preparing for the next, and vice versa. This technique was rigidly adhered to throughout the sequence of experiments.

Table 1.

2000 successive spins of an ordinary coin.

0 denotes tail.

1 denotes head.

000111010011110100011010111000100111001000001110
0010101010010000100110001000011101010100010000101101
0111010000110100101000011110111100110110010101110
0101000001100011100111101010110100110110110110110
011111000011101100010100100001010011111011101011011
10001100011000110011001100100001000011101111000
111111100000000001101011010011111011110010010101100
1110110110010000010001100101100111110100111100010
00001001010111010101111001111001000001101011111
11010001111100101111100111110011111101000010000110000
000011110010101111000011101100100011101000011111
110001010011111101101101110110110010110110011
010100110111111100101110001111011111000001001001
0100111011011010111111000001010101010101001001
111011011100111100000010011010011001000100001100
10111100010011010110110011100100101010000010000
0000101100110101101111100010110010100001110011001
111000101101010000110001001100010010001100100001001
0100001110000011010111100111001101101011010010111
0100000111010100100011100100111000010100000000010
100100010110000100010100011111010110111010101010000
01100010100001000000000010000001100100011011101010
11011000110111010110010010111000101101101010110110
0000101101101010100001100111000110100111011101101
1000110110000010011110001110100001010000111110100
001111111111101010100110001011110010101000111110100
110001101010011010010111110000111011110110011001
11111010000011101010111101011100001000101101001
1001101000010111110101100011010111100001011001001
0011011011111010110010100110100100011000011111000
010100110001101001100001100110001110101011100010001
1101011101110101101110011011100011000011101101000
0101111010011101011001001111011011000011110011111
011010111100110001111000011000111101011101010010010
10100011011110100011111000011000000100111010101001
100010111001000101111101110000011111101101000000010
1011111101110001000011000011000111110100111011011000
0000111101110001110101000101100011011000110111010001111
1000001000011010000010100001010100010110001011110001110
00101110010111010010110010110001100000111000011100001111

Two points should be noted. The experiments made the coin fall in many features in a chaotic manner. The coin was employed throughout the experiment, but the position of the coin at the time of the impulse given to the coin was uppermost. The results we are soon convinced that the coin will be uppermost in any particular case, without exception. It may be remarkable that in many branches of science the theory of probability to a large extent factorially.

Consider Table 1 in detail. The sequence is chaotic. Sometimes heads, of several heads, follow, and then some sort of regularity appears in the maze of discordant details.

(a) that in a large number of runs there are tails as tails.

(b) long runs of either heads or tails.

It may be noted that a run of thirteen tails, which occurs once in seven is a "long" run.

We now proceed to a detailed analysis of the data.

Column 1 gives the successive numbers of spins made.

Column 2 gives the corresponding number of heads.

Column 3 gives the ratio of heads to tails.

It is clear that  $m/n$  must fluctuate with reference to these values. We find that  $m/n$  fluctuates over a large range. For example, in the first 1000 spins  $m/n = 4/10 = 0.400$ . Then it drops to 0.350, 0.300, 0.250, 0.200, 0.150, 0.100, 0.050, 0.000, 0.050, 0.100, 0.150, 0.200, 0.250, 0.300, 0.350, 0.400, 0.450, 0.500, 0.550, 0.600, 0.650, 0.700, 0.750, 0.800, 0.850, 0.900, 0.950, 1.000, 1.050, 1.100, 1.150, 1.200, 1.250, 1.300, 1.350, 1.400, 1.450, 1.500, 1.550, 1.600, 1.650, 1.700, 1.750, 1.800, 1.850, 1.900, 1.950, 2.000, 2.050, 2.100, 2.150, 2.200, 2.250, 2.300, 2.350, 2.400, 2.450, 2.500, 2.550, 2.600, 2.650, 2.700, 2.750, 2.800, 2.850, 2.900, 2.950, 3.000, 3.050, 3.100, 3.150, 3.200, 3.250, 3.300, 3.350, 3.400, 3.450, 3.500, 3.550, 3.600, 3.650, 3.700, 3.750, 3.800, 3.850, 3.900, 3.950, 4.000, 4.050, 4.100, 4.150, 4.200, 4.250, 4.300, 4.350, 4.400, 4.450, 4.500, 4.550, 4.600, 4.650, 4.700, 4.750, 4.800, 4.850, 4.900, 4.950, 5.000, 5.050, 5.100, 5.150, 5.200, 5.250, 5.300, 5.350, 5.400, 5.450, 5.500, 5.550, 5.600, 5.650, 5.700, 5.750, 5.800, 5.850, 5.900, 5.950, 6.000, 6.050, 6.100, 6.150, 6.200, 6.250, 6.300, 6.350, 6.400, 6.450, 6.500, 6.550, 6.600, 6.650, 6.700, 6.750, 6.800, 6.850, 6.900, 6.950, 7.000, 7.050, 7.100, 7.150, 7.200, 7.250, 7.300, 7.350, 7.400, 7.450, 7.500, 7.550, 7.600, 7.650, 7.700, 7.750, 7.800, 7.850, 7.900, 7.950, 8.000, 8.050, 8.100, 8.150, 8.200, 8.250, 8.300, 8.350, 8.400, 8.450, 8.500, 8.550, 8.600, 8.650, 8.700, 8.750, 8.800, 8.850, 8.900, 8.950, 9.000, 9.050, 9.100, 9.150, 9.200, 9.250, 9.300, 9.350, 9.400, 9.450, 9.500, 9.550, 9.600, 9.650, 9.700, 9.750, 9.800, 9.850, 9.900, 9.950, 10.000, 10.050, 10.100, 10.150, 10.200, 10.250, 10.300, 10.350, 10.400, 10.450, 10.500, 10.550, 10.600, 10.650, 10.700, 10.750, 10.800, 10.850, 10.900, 10.950, 11.000, 11.050, 11.100, 11.150, 11.200, 11.250, 11.300, 11.350, 11.400, 11.450, 11.500, 11.550, 11.600, 11.650, 11.700, 11.750, 11.800, 11.850, 11.900, 11.950, 12.000, 12.050, 12.100, 12.150, 12.200, 12.250, 12.300, 12.350, 12.400, 12.450, 12.500, 12.550, 12.600, 12.650, 12.700, 12.750, 12.800, 12.850, 12.900, 12.950, 13.000, 13.050, 13.100, 13.150, 13.200, 13.250, 13.300, 13.350, 13.400, 13.450, 13.500, 13.550, 13.600, 13.650, 13.700, 13.750, 13.800, 13.850, 13.900, 13.950, 14.000, 14.050, 14.100, 14.150, 14.200, 14.250, 14.300, 14.350, 14.400, 14.450, 14.500, 14.550, 14.600, 14.650, 14.700, 14.750, 14.800, 14.850, 14.900, 14.950, 15.000, 15.050, 15.100, 15.150, 15.200, 15.250, 15.300, 15.350, 15.400, 15.450, 15.500, 15.550, 15.600, 15.650, 15.700, 15.750, 15.800, 15.850, 15.900, 15.950, 16.000, 16.050, 16.100, 16.150, 16.200, 16.250, 16.300, 16.350, 16.400, 16.450, 16.500, 16.550, 16.600, 16.650, 16.700, 16.750, 16.800, 16.850, 16.900, 16.950, 17.000, 17.050, 17.100, 17.150, 17.200, 17.250, 17.300, 17.350, 17.400, 17.450, 17.500, 17.550, 17.600, 17.650, 17.700, 17.750, 17.800, 17.850, 17.900, 17.950, 18.000, 18.050, 18.100, 18.150, 18.200, 18.250, 18.300, 18.350, 18.400, 18.450, 18.500, 18.550, 18.600, 18.650, 18.700, 18.750, 18.800, 18.850, 18.900, 18.950, 19.000, 19.050, 19.100, 19.150, 19.200, 19.250, 19.300, 19.350, 19.400, 19.450, 19.500, 19.550, 19.600, 19.650, 19.700, 19.750, 19.800, 19.850, 19.900, 19.950, 20.000, 20.050, 20.100, 20.150, 20.200, 20.250, 20.300, 20.350, 20.400, 20.450, 20.500, 20.550, 20.600, 20.650, 20.700, 20.750, 20.800, 20.850, 20.900, 20.950, 21.000, 21.050, 21.100, 21.150, 21.200, 21.250, 21.300, 21.350, 21.400, 21.450, 21.500, 21.550, 21.600, 21.650, 21.700, 21.750, 21.800, 21.850, 21.900, 21.950, 22.000, 22.050, 22.100, 22.150, 22.200, 22.250, 22.300, 22.350, 22.400, 22.450, 22.500, 22.550, 22.600, 22.650, 22.700, 22.750, 22.800, 22.850, 22.900, 22.950, 23.000, 23.050, 23.100, 23.150, 23.200, 23.250, 23.300, 23.350, 23.400, 23.450, 23.500, 23.550, 23.600, 23.650, 23.700, 23.750, 23.800, 23.850, 23.900, 23.950, 24.000, 24.050, 24.100, 24.150, 24.200, 24.250, 24.300, 24.350, 24.400, 24.450, 24.500, 24.550, 24.600, 24.650, 24.700, 24.750, 24.800, 24.850, 24.900, 24.950, 25.000, 25.050, 25.100, 25.150, 25.200, 25.250, 25.300, 25.350, 25.400, 25.450, 25.500, 25.550, 25.600, 25.650, 25.700, 25.750, 25.800, 25.850, 25.900, 25.950, 26.000, 26.050, 26.100, 26.150, 26.200, 26.250, 26.300, 26.350, 26.400, 26.450, 26.500, 26.550, 26.600, 26.650, 26.700, 26.750, 26.800, 26.850, 26.900, 26.950, 27.000, 27.050, 27.100, 27.150, 27.200, 27.250, 27.300, 27.350, 27.400, 27.450, 27.500, 27.550, 27.600, 27.650, 27.700, 27.750, 27.800, 27.850, 27.900, 27.950, 28.000, 28.050, 28.100, 28.150, 28.200, 28.250, 28.300, 28.350, 28.400, 28.450, 28.500, 28.550, 28.600, 28.650, 28.700, 28.750, 28.800, 28.850, 28.900, 28.950, 29.000, 29.050, 29.100, 29.150, 29.200, 29.250, 29.300, 29.350, 29.400, 29.450, 29.500, 29.550, 29.600, 29.650, 29.700, 29.750, 29.800, 29.850, 29.900, 29.950, 30.000, 30.050, 30.100, 30.150, 30.200, 30.250, 30.300, 30.350, 30.400, 30.450, 30.500, 30.550, 30.600, 30.650, 30.700, 30.750, 30.800, 30.850, 30.900, 30.950, 31.000, 31.050, 31.100, 31.150, 31.200, 31.250, 31.300, 31.350, 31.400, 31.450, 31.500, 31.550, 31.600, 31.650, 31.700, 31.750, 31.800, 31.850, 31.900, 31.950, 32.000, 32.050, 32.100, 32.150, 32.200, 32.250, 32.300, 32.350, 32.400, 32.450, 32.500, 32.550, 32.600, 32.650, 32.700, 32.750, 32.800, 32.850, 32.900, 32.950, 33.000, 33.050, 33.100, 33.150, 33.200, 33.250, 33.300, 33.350, 33.400, 33.450, 33.500, 33.550, 33.600, 33.650, 33.700, 33.750, 33.800, 33.850, 33.900, 33.950, 34.000, 34.050, 34.100, 34.150, 34.200, 34.250, 34.300, 34.350, 34.400, 34.450, 34.500, 34.550, 34.600, 34.650, 34.700, 34.750, 34.800, 34.850, 34.900, 34.950, 35.000, 35.050, 35.100, 35.150, 35.200, 35.250, 35.300, 35.350, 35.400, 35.450, 35.500, 35.550, 35.600, 35.650, 35.700, 35.750, 35.800, 35.850, 35.900, 35.950, 36.000, 36.050, 36.100, 36.150, 36.200, 36.250, 36.300, 36.350, 36.400, 36.450, 36.500, 36.550, 36.600, 36.650, 36.700, 36.750, 36.800, 36.850, 36.900, 36.950, 37.000, 37.050, 37.100, 37.150, 37.200, 37.250, 37.300, 37.350, 37.400, 37.450, 37.500, 37.550, 37.600, 37.650, 37.700, 37.750, 37.800, 37.850, 37.900, 37.950, 38.000, 38.050, 38.100, 38.150, 38.200, 38.250, 38.300, 38.350, 38.400, 38.450, 38.500, 38.550, 38.600, 38.650, 38.700, 38.750, 38.800, 38.850, 38.900, 38.950, 39.000, 39.050, 39.100, 39.150, 39.200, 39.250, 39.300, 39.350, 39.400, 39.450, 39.500, 39.550, 39.600, 39.650, 39.700, 39.750, 39.800, 39.850, 39.900, 39.950, 40.000, 40.050, 40.100, 40.150, 40.200, 40.250, 40.300, 40.350, 40.400, 40.450, 40.500, 40.550, 40.600, 40.650, 40.700, 40.750, 40.800, 40.850, 40.900, 40.950, 41.000, 41.050, 41.100, 41.150, 41.200, 41.250, 41.300, 41.350, 41.400, 41.450, 41.500, 41.550, 41.600, 41.650, 41.700, 41.750, 41.800, 41.850, 41.900, 41.950, 42.000, 42.050, 42.100, 42.150, 42.200, 42.250, 42.300, 42.350, 42.400, 42.450, 42.500, 42.550, 42.600, 42.650, 42.700, 42.750, 42.800, 42.850, 42.900, 42.950, 43.000, 43.050, 43.100, 43.150, 43.200, 43.250, 43.300, 43.350, 43.400, 43.450, 43.500, 43.550, 43.600, 43.650, 43.700, 43.750, 43.800, 43.850, 43.900, 43.950, 44.000, 44.050, 44.100, 44.150, 44.200, 44.250, 44.300, 44.350, 44.400, 44.450, 44.500, 44.550, 44.600, 44.650, 44.700, 44.750, 44.800, 44.850, 44.900, 44.950, 45.000, 45.050, 45.100, 45.150, 45.200, 45.250, 45.300, 45.350, 45.400, 45.450, 45.500, 45.550, 45.600, 45.650, 45.700, 45.750, 45.800, 45.850, 45.900, 45.950, 46.000, 46.050, 46.100, 46.150, 46.200, 46.250, 46.300, 46.350, 46.400, 46.450, 46.500, 46.550, 46.600, 46.650, 46.700, 46.750, 46.800, 46.850, 46.900, 46.950, 47.000, 47.050, 47.100, 47.150, 47.200, 47.250, 47.300, 47.350, 47.400, 47.450, 47.500, 47.550, 47.600, 47.650, 47.700, 47.750, 47.800, 47.850, 47.900, 47.950, 48.000, 48.050, 48.100, 48.150, 48.200, 48.250, 48.300, 48.350, 48.400, 48.450, 48.500, 48.550, 48.600, 48.650, 48.700, 48.750, 48.800, 48.850, 48.900, 48.950, 49.000, 49.050, 49.100, 49.150, 49.200, 49.250, 49.300, 49.350, 49.400, 49.450, 49.500, 49.550, 4

Table 3.

Number of Heads and Proportion of Heads, in 10 sequences of 1000 spins of a Coin.

 $n$  = number of spins. $m$  = number of Heads in  $n$  spins. $m/n$  = proportion of Heads.

$n$	(1) $m$	(1) $m/n$	(2) $m$	(2) $m/n$	(3) $m$	(3) $m/n$	(4) $m$	(4) $m/n$	(5) $m$	(5) $m/n$	(6) $m$	(6) $m/n$	(7) $m$	(7) $m/n$	(8) $m$	(8) $m/n$	(9) $m$	(9) $m/n$	(10) $m$	(10) $m/n$
1	0	0	1	1.000	0	0	1	1.000	1	1.000	0	0	1	1.000	0	0	1	1.000	0	0
2	0	0	1	.500	1	.500	2	1.000	2	1.000	1	.500	1	.500	1	.500	2	1.000	1	.500
3	0	0	1	.333	1	.333	2	.667	2	.667	2	.667	1	.333	1	.333	2	.667	1	.333
4	1	.250	2	.500	2	.500	3	.750	3	.750	3	.750	1	.250	1	.250	3	.750	2	.500
5	2	.400	2	.400	3	.600	3	.600	3	.600	4	.800	1	.200	2	.400	4	.800	2	.400
6	3	.500	2	.333	4	.666	4	.667	4	.667	5	.834	2	.333	3	.500	4	.667	2	.333
7	3	.428	2	.286	5	.714	4	.572	5	.714	5	.714	2	.286	4	.572	4	.572	2	.286
8	4	.500	3	.375	5	.625	4	.500	6	.750	5	.625	2	.250	5	.625	5	.625	2	.250
9	4	.444	3	.333	5	.556	5	.556	6	.667	6	.667	3	.333	6	.667	6	.667	2	.222
10	4	.400	4	.400	6	.600	6	.600	7	.700	7	.700	3	.300	6	.600	6	.600	2	.200
11	5	.450	5	.454	7	.637	6	.546	7	.637	8	.726	4	.363	7	.637	7	.637	2	.182
12	6	.500	5	.417	7	.583	7	.584	8	.667	8	.667	5	.417	8	.667	8	.667	2	.167
13	7	.538	5	.385	7	.538	8	.615	8	.615	8	.615	6	.462	9	.692	8	.615	3	.311
14	8	.572	5	.357	7	.500	8	.572	8	.572	9	.642	6	.428	9	.642	8	.572	4	.286
15	9	.600	5	.333	8	.533	8	.534	9	.600	9	.600	6	.400	9	.600	9	.600	4	.267
16	9	.562	6	.375	9	.562	9	.562	10	.625	9	.562	7	.437	10	.625	9	.562	5	.312
17	10	.588	6	.353	9	.528	9	.528	11	.648	10	.588	7	.412	11	.648	9	.528	5	.294
18	10	.556	6	.334	9	.500	9	.500	12	.667	11	.612	7	.399	12	.667	9	.500	6	.333
19	10	.526	7	.369	9	.474	10	.526	12	.632	12	.632	7	.368	13	.684	10	.526	7	.368
20	10	.500	7	.350	10	.500	10	.500	13	.650	13	.650	7	.350	13	.650	11	.550	7	.350
21	11	.524	8	.381	10	.476	10	.476	13	.619	14	.667	7	.333	13	.619	12	.572	8	.381
22	12	.546	8	.364	10	.455	10	.455	13	.591	15	.682	8	.363	13	.591	12	.546	9	.408
23	12	.522	8	.348	11	.478	11	.478	13	.565	15	.652	9	.391	13	.565	12	.522	9	.391
24	13	.542	8	.333	11	.458	11	.458	14	.583	16	.667	10	.417	13	.542	13	.542	10	.417
25	13	.521	9	.360	11	.440	11	.440	15	.600	17	.739	11	.430	13	.520	13	.521	11	.430
26	14	.538	10	.385	12	.452	12	.450	16	.616	18	.692	11	.423	14	.538	13	.500	12	.452
27	15	.556	11	.407	12	.444	13	.482	16	.593	18	.667	12	.444	14	.518	13	.482	12	.444
28	16	.572	12	.428	13	.464	14	.500	17	.608	18	.642	13	.464	14	.500	14	.500	13	.464
29	17	.587	13	.448	14	.483	14	.483	17	.587	18	.621	13	.448	15	.517	14	.483	14	.483
30	17	.567	14	.468	14	.467	14	.467	18	.600	18	.600	14	.467	16	.534	15	.500	15	.500

35	18	.514	17	.486	17	.486	19	.543	19	.543	16	.457	19	.543	18	.514	18	.514	18	.514
40	21	.524	21	.524	20	.500	20	.500	21	.524	19	.476	22	.550	21	.524	20	.500	20	.500
45	22	.488	23	.511	23	.511	21	.467	26	.578	22	.488	24	.533	23	.511	23	.511	23	.511
50	25	.500	24	.481	26	.520	24	.481	23	.460	29	.580	27	.540	26	.520	23	.460	26	.520
55	27	.492	26	.473	29	.527	27	.492	26	.473	32	.582	30	.546	28	.509	29	.527	29	.527
60	29	.483	28	.467	31	.517	29	.483	28	.467	34	.567	31	.517	30	.500	32	.534	33	.550
65	30	.462	29	.446	34	.523	32	.492	31	.477	37	.569	33	.508	33	.508	38	.585	38	.585
70	32	.457	29	.414	37	.528	34	.486	34	.486	38	.543	35	.500	34	.486	40	.571	40	.571
75	34	.467	30	.400	40	.533	35	.467	39	.520	40	.533	36	.480	36	.480	36	.480	43	.573
80	35	.437	30	.375	43	.538	39	.488	42	.525	41	.513	39	.488	40	.500	37	.462	45	.562
85	38	.447	32	.376	46	.542	42	.494	47	.553	43	.506	41	.482	43	.506	38	.477	47	.553
90	40	.444	34	.378	51	.566	46	.511	50	.556	44	.488	42	.467	45	.500	39	.433	49	.544
95	41	.432	38	.400	54	.568	50	.526	51	.537	48	.505	44	.463	47	.494	41	.432	53	.558
100	44	.440	40	.400	56	.560	51	.510	52	.520	52	.520	47	.470	50	.500	45	.450	55	.550
110	48	.436	46	.418	60	.546	57	.518	56	.508	57	.518	51	.463	57	.518	46	.418	62	.564
120	53	.442	52	.433	67	.558	61	.508	63	.525	61	.508	58	.483	62	.516	52	.433	69	.575
130	58	.446	57	.438	75	.577	64	.493	70	.538	66	.507	65	.500	66	.507	60	.462	72	.554
140	65	.464	62	.444	80	.572	70	.500	75	.536	75	.536	72	.514	75	.536	64	.457	75	.536
150	71	.473	68	.454	84	.560	76	.507	79	.527	82	.546	78	.520	79	.527	66	.448	80	.534
160	74	.462	72	.450	91	.568	81	.507	85	.532	87	.543	82	.513	87	.543	73	.457	84	.525
170	79	.465	78	.459	96	.564	85	.500	91	.535	93	.547	89	.524	92	.542	80	.471	90	.529
180	86	.477	82	.456	100	.556	92	.511	94	.522	100	.556	91	.506	95	.528	87	.483	98	.534
190	92	.484	87	.458	105	.553	97	.511	99	.521	106	.558	94	.495	100	.526	91	.478	101	.532
200	98	.490	94	.470	109	.546	99	.495	106	.530	110	.550	99	.495	107	.535	95	.475	106	.530
250	125	.500	117	.468	134	.537	126	.504	128	.512	128	.512	126	.504	131	.524	115	.460	126	.504
300	146	.487	147	.492	162	.540	155	.518	161	.538	155	.517	144	.480	156	.520	148	.494	154	.514
350	173	.494	175	.494	185	.528	183	.508	186	.532	174	.512	168	.480	175	.500	177	.506	183	.522
400	199	.498	202	.498	207	.518	212	.530	213	.532	192	.493	192	.480	202	.505	194	.485	210	.525
450	226	.502	229	.509	235	.522	239	.531	240	.534	221	.492	213	.473	232	.515	220	.488	238	.529
500	255	.510	254	.508	259	.518	262	.524	264	.528	239	.482	236	.472	264	.528	250	.500	265	.530
550	280	.509	277	.503	279	.508	285	.518	292	.526	271	.494	263	.478	287	.522	280	.509	293	.533
600	312	.520	308	.513	303	.506	316	.526	317	.528	291	.485	287	.479	312	.520	208	.513	320	.533
650	341	.525	338	.520	327	.503	341	.525	345	.530	318	.489</								

Example 2. The flowers of a certain kind of bulb have been found to possess either 5 or 6 or .... 9 petals. Here we let  $x$  denote the number of petals on a given flower.

In both these cases we find ourselves talking about the probability that  $x$  has a certain integral value.

A different and very important type of example is

Example 3. Certain screws made by mass production are reputedly all one inch long. Careful measurement shows that they differ slightly in length, and any particular screw will fall within one or other of the following classes

0.999 ins. or over, but less than 1.000 ins.	.....
1.000 ins. - - - - -	1.001 ins.
1.001 ins. - - - - -	1.002 ins.
.....	

where the classification is extended as far as required in either direction. Here the screw is the "experiment". The result of the experiment is the length of the screw, which falls into one or other of certain mutually exclusive classes, which represent our "events". We let  $x$  denote the length of the screw, and have an example of a statistical variable that can take on *any* value within a certain range. We find ourselves talking about the probability that  $x$  lies between certain limits such as 0.990 to 1.010 ins.

Yet another type arises as follows:

Example 4. Suppose we study the question of eyecolour among schoolchildren. Here the child is the "experiment"; the colour of his eyes the "event". We might classify the possible mutually exclusive events as possessing (a) blue eyes, (b) brown eyes, (c) grey eyes or (d) eyes of any other colour, and talk about the probability of a child being in one or other of these classes. In this example we cannot easily, if at all, introduce the idea of a statistical variable, a fact which is found to hamper our analysis considerably.

Similar situations arise in astronomy, biology, chemistry, dermatology, epidemiology, field experiments in agricultural research, or any other source of observations that one cares to name.

## § 2.

The simple experiment of spinning a coin has yielded valuable results, and further results will be obtained later. But to suggest the next steps in the development of the theory it is convenient to have data of a rather more complicated nature. These were obtained as follows: —

Four ping-pong balls were placed in a box. Two balls were of one brand and bore a red trade-mark. They will be referred to as red balls. The other two were of a different brand and bore a green trade-mark. They will be referred to as green balls. Apart from this difference the balls appeared to be practically identical. The box was shaken with a rotary as well as an up and down motion. It was then held level so that all the balls lay on the bottom of the box and a ball drawn from it without the observer looking at it. The colour of the ball was noted. Then without replacing the first ball a second ball was drawn and its colour noted. This process was repeated 5000 times, by the same experimenter (Mr. Eric Christensen), at a rate of about 400 an hour, with — need it be stated — periods of rest between successive hours. Some of the results are given in Table 6.

The experiment was suggested by the standard texts, which for generations past have contained problems concerning the drawing of balls from urns. So we shall refer to it as our "urn" experiment.

Before enumerating the various events connected with this experiment we shall introduce the useful notation that  $P\{E\}$  denotes the mathematical probability associated with a given event  $E$ .

The primary events associated with the urn experiment are that either the first ball drawn is RED and the second is RED

or - - - - - RED - - - - - GREEN

or - - - - - GREEN - - - - - RED

or - - - - - GREEN - - - - - GREEN

They are four mutually exclusive events, one of which must happen. We shall denote them by  $R_1 R_2$ ,  $R_1 G_2$ ,  $G_1 R_2$  and  $G_1 G_2$  respectively.

In  $n$  repetitions of the experiment let

$m_1$  denote the number of times that  $R_1 R_2$  occurs

$m_2$  - - - - -  $R_1 G_2$  -

$m_3$  - - - - -  $G_1 R_2$  -

$m_4$  - - - - -  $G_1 G_2$  -

Table 6.

500 consecutive drawings of two balls from an urn. Arranged in batches of 50.

n	$\begin{smallmatrix} R \\ R \\ G \end{smallmatrix}$		$\begin{smallmatrix} G \\ R \\ G \end{smallmatrix}$		$\begin{smallmatrix} R \\ R \\ G \end{smallmatrix}$		$\begin{smallmatrix} G \\ R \\ G \end{smallmatrix}$		$\begin{smallmatrix} R \\ R \\ G \end{smallmatrix}$		$\begin{smallmatrix} G \\ R \\ G \end{smallmatrix}$	
	$R$	$G$										
1	1		1		1		1		1		1	
2	1		1		1		1		2		2	
3		1		1	2		1		1		3	
4		2		2	3		2		2		2	
5		3	1		4		2		3		3	
6	2		2		5		3		3		4	
7	1		2		1		3		4		3	
8	3		3		6		4		2		3	
9	2		2		7		1		3		5	
10	2		3		8		2		3		2	
11	3		4		9		4		3		4	
12	4		4		10		5		5		6	
13		4	5		11		7		4		3	
14	3		6		12		3		2		4	
15	4		5		13		6		3		5	
16	5		6		14		7		6		6	
17	6		7		15		5		7		7	
18	7		6		16		2		8		8	
19	4		7		17		8		5		1	
20	5		8		18		4		4		6	
21	6		9		19		3		9		7	
22		5	10		20		10		10		9	
23	8		11		21		5		11		5	
24	5		9		22		6		12		2	
25	6		10		23		7		13		10	
26	7		11		24		11		4		9	
27	9		12		25		8		5		10	
28	10		13		26		12		6		6	
29	7		14		27		13		10		3	
30	11		15		28		6		11		11	
31	8		16		29		7		12		12	
32		3	17		30		14		13		7	
33	12		18		31		5		14		13	
34	6		19		32		9		15		4	
35	13		20		33		4		16		12	
36	9		21		34		16		8		13	
37	9		22		35		5		9		8	
38	10		23		36		15		15		14	
39	11		24		37		16		11		4	
40	14		25		38		17		12		5	
41	15		26		39		6		17		14	
42	16		27		40		18		9		15	
43	17		28		41		13		18		16	
44	12		29		42		4		12		15	
45	7		30		43		7		19		6	
46	18		31		44		8		10		9	
47	19		32		45		14		13		16	
48	13		33		46		8		11		17	
49	20		34		47		5		10		18	
50	10	13	20	7	48		19		12		18	
	5	15	20	10	49		8		15		16	
	8	14	22	6	50		5		20		7	
	5	15	22	8			8		15		15	
	8	20	15	7			8		20		8	
	8	20	14	8			5		22		9	
	8	20	14	8			5		21		18	
	8	20	14	8			10		15		16	
	8	20	14	8			10		15		7	

Table 6. (continued).

	$\begin{smallmatrix} R \\ R \\ G \end{smallmatrix}$		$\begin{smallmatrix} G \\ R \\ G \end{smallmatrix}$		$\begin{smallmatrix} R \\ R \\ G \end{smallmatrix}$		$\begin{smallmatrix} G \\ R \\ G \end{smallmatrix}$		$\begin{smallmatrix} R \\ R \\ G \end{smallmatrix}$		$\begin{smallmatrix} G \\ R \\ G \end{smallmatrix}$		$\begin{smallmatrix} R \\ R \\ G \end{smallmatrix}$		$\begin{smallmatrix} G \\ R \\ G \end{smallmatrix}$		1st Draw	
	$R$	$G$	2nd Draw															
	1		2		3		4		1		2		3		4		1	
	2		3		4		1		2		3		4		5		2	
	3		4		5		2		3		4		5		6		3	
	4		5		6		3		4		5		6		7		4	
	5		6		7		4		5		6		7		8		5	
	6		7		8		5		6		7		8		9		6	
	7		8		9		6		7		8		9		10		7	
	8		9		10		7		8		9		10		11		8	
	9		10		11		8		9		10		11		12		9	
	10		11		12		9		10		11		12		13		10	
	11		12		13		10		11		12		13		14		11	
	12		13		14		11		12		13		14		15		12	
	13		14		15		12		13		14		15		16		13	
	14		15		16		13		14		15		16		17		14	
	15		16		17		14		15		16		17		18		15	
	16		17		18		15		16		17		18		19		16	
	17		18		19		16		17		18		19		20		17	
	18		19		20		17		18		19		20		21		18	
	19		20		21		18		19		20		21		22		19	
	20		21		22		19		20		21		22		23		20	
	21		22		23		20		21		22		23		24		21	
	22		23		24		21		22		23		24		25		22	
	23		24		25		22		23		24		25		26		23	
	24		25		26		23		24		25		26		27		24	
	25		26		27		24		25		26		27		28		25	
	26		27		28		25		26		27		28		29		26	
	27		28		29		26		27		28		29		30		27	
	28		29		30		27		28		29		30		31		28	
	29		30		31		28		29		30		31		32		29	
	30		31		32		29		30		31		32		33		30	
	31		32		33		30		31		32		33		34		31	
	32		33		34		31		32		33		34		35		32	
	33		34		35		32		33		34		35		36		33	
	34		35		36		33		34		35		36		37		34	
	35		36		37		34		35		36		37		38		35	
	36		37		38		35		36		37		38		39		36	
	37		38		39		36		37		38		39		40		37	
	38		39		40		37		38		39		40		41		38	
	39		40		41		38		39		40		41		42		39	
	40		41		42		39		40		41		42		43		40	
	41		42		43		40		41		42		43		44		41	
	42		43		44		41		42		43		44		45		42	
	43		44		45		42		43		44		45		46		43	
	44		45		46		43		44		45		46		47		44	
	45		46		47		44		45		46		47		48		45	
	46		47		48		45		46		47		48		49		46	
	47																	