

Topic10: Test for Means (Z and T Tests)



Outline

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Example1: Comm Bank's online service

CommBank claims that an online personal loan application takes between 15-20 minutes to complete online. There has been complaints that the applications take longer. A random sample of 26 customers results in the following data.

```
x=c(29.3,23.1,18.5,23.8,24.8,23.8,22.5,26.3,20.8,21.1,21.4,24.0,22.0,28.2,27.3,19.4,20.1,26.4,24.4,24.0,21.0,22.8,29.4,22.9,26.7,24.0)
```

Would this evidence contradict the company's claim?



Example2: ADHD in children in Taiwan

A study from 2013 looked at the long-term effects of stimulants on neurocognitive performance of Taiwanese children with attention deficit hyperactivity disorder (ADHD) using the Wechsler Intelligence Scale (WISC-III).

"In Taiwan, a high prevalence rate of ADHD was noticed about ten years ago, but there is still little research comparing neurocognitive function between children with ADHD and healthy children."

"Due to the nature of the populations sampled, diagnostic criteria used, cultural differences, and methodological limitations, the prevalence of ADHD in various cultures varies. The prevalence is estimated to be about 8.4–11.7% in Taiwan; 2.4% in Australia; and 4% in Japan."

It was found that the 47 children in the control group (without ADHD) had a BMI (body mass index) of 18.8 with standard deviation 3.3, and the group of 171 children with ADHD had a BMI of 18.5 with standard deviation 3.7.

Is there evidence that children with ADHD have a different BMI than the general population?

[▶ Abstract](#)[▶ TableT1](#)[▶ TableT2](#)[▶ BMI](#)

Example3: Weight Loss

The amount of weight loss by a person taking a weight loss program depends on many factors, including gender, age, height, genetic background and fitness. To compare 2 programs, 20 people were 'matched' for gender/age/height. Each person in the pair was expected to act similarly to the other, but possibly quite differently to those in other pairs. This is called a 'Case-Control' study.

The weight losses (in kg) under the 2 programs A and B were as follows.

Pair	1	2	3	4	5	6	7	8	9	10
Diet <i>A</i>	3.6	1.1	6.2	11.3	0.5	8.5	12.4	2.5	4.1	6.2
Diet <i>B</i>	3.2	0.7	7.1	9.8	1.1	7.1	11.5	2.5	2.9	5.0

Is there a difference between the 2 diets?

Steps for 1 Sample Z Test

Context Consider a population with unknown mean μ and known variance σ^2 . We want to test a hypothesis about μ .

H $H_0 : \mu = \mu_0$ vs $H_1 : \mu < \mu_0$.

A The population variance σ^2 is known. The population is Normal or we have a large enough sample size n to ensure Normality of \bar{X} by the CLT.

T

- ▶ $\tau = Z = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$ under H_0 .
- ▶ Small values of z will argue against H_0 for H_1 .
- ▶ The observed value is z .

P $P\text{-value} = P(Z \leq z).$

C Weigh up the size of P -value.

Notes on the Test Statistic and P -value.

- ▶ If the alternate hypotheses is $H_1 : \mu > \mu_0$, then large values of z will argue against H_0 for H_1 and the associated P -value is $P(Z \geq z)$.
- ▶ If the alternate hypotheses is 2 sided $H_1 : \mu \neq \mu_0$, then both small and large values of z will argue against H_0 for H_1 and the associated P -value is $2P(Z \geq z)$ due to symmetry.
- ▶ We can also use the unstandardised random variable \bar{X} as the test statistic, where $\bar{X} \sim N(\mu_0, \frac{\sigma^2}{n})$ under H_0 .

Example

CommBank

Do CommBank online personal loan applications take longer than advertised?

Let μ = Mean time for CommBank online personal loan application. Note this is a 1 sided test as we are testing whether the time is 'longer'.

[H] We will use the upper bound (most conservative) of the claim, so $H_0 : \mu = 20$ vs $H_1 = \mu > 20$.

[A] The $n = 26$ people in the survey are sampled randomly. Here we do not have a given value of σ , but to illustrate the Z test, we will choose $\sigma = 5$.

T

- ▶ $\tau = Z = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$ under H_0 .
- ▶ Large values of z will argue against H_0 for H_1 .
- ▶ As $\bar{x} = 23$ and we have assumed $\sigma = 5$, the observed value is $z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{23 - 20}{\frac{5}{\sqrt{26}}} \approx 3.06$.

P $P\text{-value} = P(Z \geq 3.06) \approx 0.001.$

```
1-pnorm(3.06)
```

```
## [1] 0.001106685
```

```
1-pnorm(23,20,5/sqrt(26))
```

```
## [1] 0.001108861
```

C As the P -value is so small, we would question whether the claim is true.

1 Sample T Test

In some contexts, it is reasonable to assume that σ^2 is known, and the Z Test hinges on this. But what happens when this assumption is not valid?

Can we simply replace σ (population) by s (sample) in the Z test?

Can we change the Test Statistic from $\tau = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$ to

$$\tau = \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}} \sim N(0, 1)?$$

This is not appropriate because S is itself a random variable, not a constant like σ . Hence, we find that $\tau = \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$ which is a t distribution with $n - 1$ degrees of freedom. However, this also requires the extra assumption that the population is Normal.

Steps for the Student's T Test

Context Consider a population with unknown mean μ and unknown variance σ^2 . We want to test a hypothesis about μ .

H $H_0 : \mu = \mu_0$ vs $H_1 : \mu < \mu_0$.

A The population is Normal. Check this by a boxplot or histogram of the sample.

T

▶ $\tau = T = \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$ (under H_0).

▶ Small values of T will argue against H_0 for H_1 .

▶ The observed value is t .

P $P\text{-value} = P(t_{n-1} \leq t)$.

C Weigh up size of P -value.

Example

CommBank

Do CommBank online personal loan applications take longer than advertised?

We now use the T test. This is preferable as the previous usage of Z test required assuming that $\sigma = 5$. However, here we don't know whether the distribution is Normal, so this is a untested assumption. From the sample of 26 application times $\{23.6, 26.7, 22.9, \dots, 24.3\}$, we have $\bar{x} \approx 23.8$ and $s \approx 2.92$.

```
mean(x)

## [1] 23.76923

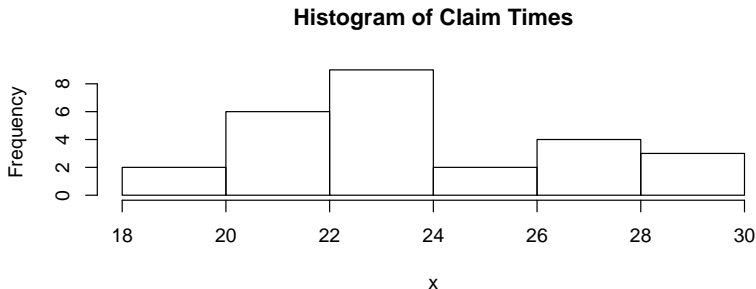
sd(x)

## [1] 2.928176
```

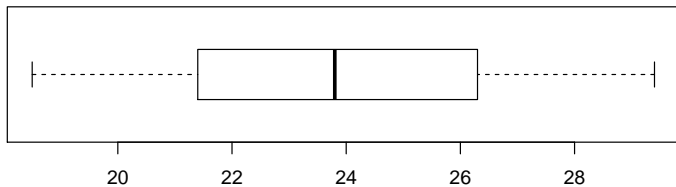
H $H_0 : \mu = 20$ vs $H_1 = \mu > 20$.

A We assume that the population of claim times is Normally distributed.

```
hist(x, main="Histogram of Claim Times")
```



```
boxplot(x, horizontal=T)
```



T

- ▶ $\tau = T = \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}} \sim t_{25}$ under H_0 .
- ▶ Large values of T will argue against H_0 for H_1 .
- ▶ The observed value is $t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{23.8 - 20}{\frac{2.92}{\sqrt{26}}} \approx 6.64$

P

$P\text{-value} = P(t_{25} \geq 6.64) \approx 0.0000007.$

```
1-pt(6.64, 25)
```

```
## [1] 2.93773e-07
```

C

As the P -value is so small, again we question whether the claim is true.

Paired T Test

The T Test can also be applied to paired data. While the Sign Test only requires the population is continuous, the T Test requires a stronger assumption: that the population of differences are Normal.

Sleep Study

Is there a difference between the affect of drugs on sleep?

```
a=c(0.7,-1.6,-0.2,-1.2,-0.1,3.4,3.7,0.8,0.0,2.0)
b=c(1.9,0.8,1.1,0.1,-0.1,4.4,5.5,1.6,4.6,3.4)
diff=b-a
mean(diff)

## [1] 1.58

sd(diff)

## [1] 1.229995

tobs = (mean(diff)-0)/(sd(diff)/sqrt(10))
2*(1-pt(tobs,9))

## [1] 0.00283289
```

H $H_0 : \mu = 0$ vs $H_0 : \mu \neq 0$, where μ is the population mean of the differences $B - A$.

A The set of differences is Normal.

- ▶ $\tau = T = \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}} \sim t_9$ under H_0 .
- ▶ Large and small values of T will argue against H_0 for H_1 .
- ▶ The observed value is $t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{1.58 - 0}{\frac{1.229995}{\sqrt{10}}} \approx 4.06$

P $P\text{-value} = 2P(t_9 \geq 4.06) \approx 0.003$.

```
2*(1-pt(4.06,9))
```

```
## [1] 0.002841947
```

C As the P -value is so small, again we would question whether the drugs are equivalent.

2 Sample T Test

The T test (or Z Test) can be generalised to cover 2 populations and samples as follows.

Context Consider 2 populations with unknown means μ_X and μ_Y and unknown common variance σ^2 . We take 2 independent samples. We want to test a hypothesis about $\mu_X - \mu_Y$.

H $H_0 : \mu_X - \mu_Y = c$ vs $H_1 : \mu_X - \mu_Y < c$. (Note: Often $c = 0$.)

A The 2 populations are Normal with common σ^2 . The 2 samples are independent.

T

► $\tau = T = \frac{\bar{X} - \bar{Y} - c}{s_p \sqrt{\frac{1}{n_x} + \frac{1}{n_y}}} \sim t_{n_x + n_y - 2}$ (under H_0), where the

pooled standard deviation is $s_p = \sqrt{\frac{(n_x - 1)s_x^2 + (n_y - 1)s_y^2}{n_x + n_y - 2}}$.

- Small values of T will argue against H_0 for H_1 .
- The observed value is t .

P

$P\text{-value} = P(t_{n_x + n_y - 2} \leq t).$

C

Weigh up size of P -value.

Example

ADHD in children in Taiwan

Is there evidence that children with ADHD have a different BMI to the general population?

Preparation

Let Population X = Control and Population Y = ADHD.

μ_X = Mean BMI of children in general population and μ_Y = Mean BMI of children with ADHD.

$n_x = 47$, $\bar{x} = 18.8$, $s_x = 3.3$, $n_y = 171$, $\bar{y} = 18.5$, $s_y = 3.7$.

So $s_p = \sqrt{\frac{(n_x-1)s_x^2 + (n_y-1)s_y^2}{n_x + n_y - 2}} = 3.618522$.

```
n_x= 47
xbar=18.8
s_x=3.3
n_y=171
ybar=18.5
s_y=3.7
sp = sqrt( ((n_x-1)*s_x^2 + (n_y-1)*s_y^2)/(n_x+n_y-2) )
sp

## [1] 3.618522

t = (xbar-ybar)/(sp*sqrt(1/n_x + 1/n_y))
t

## [1] 0.5033948
```


H $H_0 : \mu_X - \mu_Y = 0$ vs $H_1 : \mu_X - \mu_Y \neq 0$.

A We assume that both population are Normally distributed with common variance, and that the 2 samples are independent. We do not have the raw data to be able to do histograms as a diagnostic.

T

► $\tau = T = \frac{\bar{X} - \bar{Y} - c}{s_p \sqrt{\frac{1}{n_x} + \frac{1}{n_y}}} \sim t_{n_x + n_y - 2} \sim t_{47 + 171 - 2} = t_{216}$ under H_0 .

- Large and small values of T will argue against H_0 for H_1 .
- The observed value is $t = 0.5033948$

P $P\text{-value} = 2P(t_{216} \geq 0.5033948) \approx 0.615$.

```
2*(1-pt(0.5033948,216))
```

```
## [1] 0.6151997
```

C As the P -value is so big, there does not appear to be a difference in the BMIs of children with and without ADHD.

Why do we use the pooled variance?

As with the 1 sample t test, we need an estimate of the unknown variance σ^2 (here by assumption, common for both populations).

Options for the estimate include:

- ▶ The sample variance of Population X : s_x^2 .
- ▶ The sample variance of Population Y : s_y^2 .
- ▶ The average of the sample variances: $\frac{s_x^2 + s_y^2}{2}$.
- ▶ The weighted average of the sample variances: s_p^2 . This takes into account both sample variances and the size of the samples. s_p^2 is always between s_x^2 and s_y^2 .

Summary: What do we do with 2 samples?

We need to distinguish between a 1 sample test on differences (for paired data) and a 2 sample test on independent data. It is easy to make a mistake with wrong type of analysis. So we ask 2 critical questions:

Q1: Are the 2 samples paired? Leads to 1 sample test of differences.

Q2: Are the 2 samples independent? Leads to the 2 sample test.

ANOVA (Not examinable)

We can extend this theory to 3 or more populations and samples. This is very useful in many research contexts, and is called Analysis of Variance.