Summer/Winter/Semester2 **Tutorial Solutions 12** 2015

This tutorial explores t tests.

1 Sample t Test

Context	n observations from a population with unknown mean μ and unknown variance σ^2
Hypothesis	$H_0 : \mu = \mu_0$
Test Statistic	$\tau = \dfrac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}} \overset{H_0}{\sim} t_{n-1}$

2 Sample t Test

Context	2 populations with unknown means μ_X and μ_Y and unknown variances σ_X^2 and σ_Y^2 2 samples $\{X_i\}$ and $\{Y_j\}$ of size n_x and n_y
Hypothesis	$H_0 : \mu_X - \mu_Y = 0$
Test Statistic	$\tau = \dfrac{\bar{X} - \bar{Y} - 0}{s_p\sqrt{\frac{1}{n_x} + \frac{1}{n_y}}} \overset{H_0}{\sim} t_{n_x+n_y-2}$ where $s_p^2 = \dfrac{(n_x-1)s_x^2 + (n_y-1)s_y^2}{n_x+n_y-2}$

1. Using t Tables

(a) Find $P(t_5 \geq 3.365)$.

(b) Find $P(t_4 \leq 0.741)$.

(c) Find $P(t_7 \geq 1)$.

(d) Find a, if $P(t_{10} \geq a) = 0.005$.

(e) Find b, if $P(t_{10} \geq b) = 0.9$.

Solution

Note: For all these questions, draw a sketch of the relevant t distribution. This is particularly important for (e), so that you get the right sign.

(a) $P(t_5 \geq 3.365) = 0.01$ (from t table).

Check in R:
```
> 1-pt(3.365,5)
[1] 0.009999236
```

(b) $P(t_4 \leq 0.741) = 0.75$.
Note that the t table is upper tail, so we calculate 1-0.25.

Check in R:
```
> pt(0.741,4)
[1] 0.7500824
```

(c) $P(t_7 \geq 1) \in (0.1, 0.25)$.
Note that 1 is between 0.711 and 1.415, hence from the table, we can only say that the probability is in the interval $(0.1, 0.25)$. Using R, we can get the exact value.

Check in R:

```
> 1-pt(1,7)
[1] 0.1753083
```

(d) From the t table, we can see that $P(t_{10} \geq 3.169) = 0.005$, hence $b = 3.169$.

Check in R:

```
> qt(.995,10)
[1] 3.169273
```

(e) Find b, if $P(t_{10} \geq b) = 0.9$.
From the t table, we can see that $P(t_{10} \geq 1.372) = 0.1$.
Hence by symmetry, $P(t_{10} \leq -1.372) = 0.1$.
Hence by symmetry, $P(t_{10} \geq -1.372) = 0.9$.
So $b = -1.372$.

Check in R:

```
> qt(0.1,10)
[1] -1.372184
```

2. 1 Sample t Test

The breaking strengths of ropes produced by a manufacturer are normally distributed with mean 1800N and standard deviation 100N. By a new technique in the manufacturing process, it is claimed that the breaking strength can be increased, To test this claim a sample of 50 ropes is tested and it is found that the mean breaking strength is 1850N with standard deviation 110N. Can you support the manufacturer's claim? Justify your decision using a statistical argument.

Let X_i $i = 1, \ldots, 50$ be the breaking strengths and let μ be the population mean of the ropes produced by the new technique.

(a) Hypothesis: State the hypotheses to test the manufacturer's claim.

(b) Assumptions: Are the assumptions for a t test valid here?

(c) Test statistic: What is the test statistic and its distribution under H_0?

(d) P-value: Calculate the p-value using the t tables. Confirm using R.

```
> t=(1850-1800)/(110/sqrt(50))
> 1-pt(t,49)
```

(e) Conclusion: Draw your conclusion based on the p-value.

Solution

Let X_i $i = 1, \ldots, 50$ be the breaking strengths and let μ be the population mean of the ropes produced by the new technique. We have a sample with $n = 50$ and $\bar{x} = 1850$ and $s = 110$.

(a) $\boxed{\text{H}}$ $H_0 : \mu = 1800 \;\; vs \;\; H_1 : \mu > 1800$

(b) $\boxed{\text{A}}$ We are given that the population of ropes is Normally distributed.

(c) $\boxed{\text{T}}$ $\tau = \dfrac{\bar{X} - 1800}{\frac{110}{\sqrt{50}}} \overset{H_0}{\sim} t_{n-1} = t_{49}$.

Observed value: $\tau_0 = \dfrac{1850 - 1800}{\frac{110}{\sqrt{50}}} = 3.214122$

(d) $\boxed{\text{P}}$

Using t Tables:
P-value $= P(t_{49} > 3.214122) \approx P(t_{50} > 3.214122) \in (0.001, 0.0025)$.

Using R:

```
> t=(1850-1800)/(110/sqrt(50))
> t
[1] 3.214122
> 1-pt(t,49)
[1] 0.001158166
```

(e) $\boxed{\text{C}}$ Since p-value $<< 0.05$, we strongly reject H_0.
There appears to be strong evidence that the rope strength has increased.

3. 1 Sample t Test

Suppose that $X_1, X_2, ..., X_{25}$ is a random sample from a normal population with unknown mean μ and unknown variance σ^2. Test the hypothesis $H_0 : \mu = 25$ vs $H_1 : \mu < 25$, given the sum and sum of squares of the sample take the values 587.5 and 14431.25 respectively. Write all your steps.

Solution

Let $X_i \;\; i = 1, \ldots, 25$ be the sample and let μ be the population mean. We have a sample with $n = 25$ and $\sum_{i=1}^{25} x_i = 587.5$ and $\sum_{i=1}^{25} x_i^2 = 14431.25$.

$\boxed{\text{Preparation}}$

$\bar{x} = 587.5/25 = 23.5$

$s = \sqrt{\dfrac{1}{24}(14431.25 - \dfrac{1}{25}587.5^2)} = 5.103104$

$\boxed{\text{H}}$ $H_0 : \mu = 25 \;\; vs \;\; H_1 : \mu < 25$

$\boxed{\text{A}}$ We are given that the population is Normally distributed.

$\boxed{\text{T}}$ $\tau = \dfrac{\bar{X} - 25}{\frac{5.103104}{5}} \overset{H_0}{\sim} t_{n-1} = t_{24}$.

Observed value: $\tau_0 = \dfrac{23.5 - 25}{\frac{5.103104}{5}} = -1.469694$

$\boxed{\text{P}}$

Using t Tables:
P-value $= P(t_{24} < -1.469694) = P(t_{24} > 1.469694) \in (0.05, 0.1)$.

Using R:

```
> t=(23.5-25)/(5.103104/5)
> t
[1] -1.469694
> pt(t,24)
[1] 0.07731647
```

$\boxed{\text{C}}$ Since p-value > 0.05 (just), we would retain H_0 at $\alpha = 0.05$.
Note: This does not mean that H_0 is true. It may be true, or it may be false but we didn't collect enough data to reveal it.

4. Paired t Test

In order to test the difference between two drugs A and B for treatment of high blood pressure, 24 patients are paired according to age. One of each pair is chosen at random to receive drug A and the other receives drug B. The resultant drops in blood pressure are set out below:

Pair	1	2	3	4	5	6	7	8	9	10	11	12
Drug A	5	4	2	6	9	1	1	5	6	3	7	14
Drug B	2	5	1	2	6	3	0	2	5	2	7	12
Difference												

(a) By hand, calculate Difference = Drug A - Drug B.

(b) By hand, calculate the mean and standard deviation of the differences.

(c) Perform a t test for $H_0 : \mu_d = 0$, where μ_d is the population mean of differences.

(d) Confirm your results using R.

```
> a =c(5,4,2,6,9,1,1,5,6,3,7,14)
> b=c(2,5,1,2,6,3,0,2,5,2,7,12)
> diff=a-b
> mean(diff)
> sd(diff)
> t=mean(diff)/( sd(diff)/sqrt(length(diff)) )
> pt(t,length(diff-1))
> boxplot(diff)
```

(e) Produce a boxplot of the differences. Comment.

```
> boxplot(diff)
```

(f) (Revision of Week 11) Perform a test for $H_0 : \mu_d = 0$, using a sign test.

Solution

(a)

Pair	1	2	3	4	5	6	7	8	9	10	11	12
Drug A	5	4	2	6	9	1	1	5	6	3	7	14
Drug B	2	5	1	2	6	3	0	2	5	2	7	12
Difference = A - B	3	-1	1	4	3	-2	1	3	1	1	0	2

Note: We don't remove the 0 (cf the sign test).

(b) Let $\{x_i\}$ be the differences.
$\sum_{i=1}^{12} x_i = 16$ and $\sum_{i=1}^{12} x_i^2 = 56$.

Hence:
$\bar{x} = 16/12 = \frac{4}{3} \doteq 1.33$.
$s = \sqrt{\frac{1}{11}(56 - \frac{1}{12}16^2)} \doteq 1.78$.

(c) Perform a t test for $H_0 : \mu_d = 0$, where μ_d is the population mean of differences.

$\boxed{\text{H}}$ $H_0 : \mu_d = 0 \;\; vs \;\; H_1 : \mu_d \neq 0$

$\boxed{\text{A}}$ We assume that the population is Normally distributed.

$\boxed{\text{T}}$ $\tau = \dfrac{\bar{X} - 0}{\frac{1.78}{\sqrt{12}}} \overset{H_0}{\sim} t_{n-1} = t_{11}.$

Observed value: $\tau_0 = \dfrac{\frac{4}{3} - 0}{\frac{1.78}{\sqrt{12}}} \doteq 2.59$

$\boxed{\text{P}}$

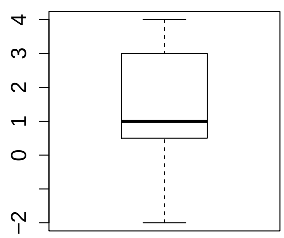
Using t Tables:
P-value $= 2P(t_{11} > 2.59) \in (0.02, 0.05)$.

Using R:
```
> t=4/3/(1.78/sqrt(12))
> t
[1] 2.594833
> 2*(1- pt(t,11))
[1] 0.02492257
```

$\boxed{\text{C}}$ Since p-value < 0.05, we would reject H_0 at $\alpha = 0.05$.
The new technique does seem to be significantly different from the standard technique.
(d) Confirm your results using R.

```
> a=c(5,4,2,6,9,1,1,5,6,3,7,14)
> b=c(2,5,1,2,6,3,0,2,5,2,7,12)
> diff=a-b
> diff
 [1]  3 -1  1  4  3 -2  1  3  1  1  0  2
> mean(diff)
[1] 1.333333
> sd(diff)
[1] 1.775251
> t=mean(diff)/( sd(diff)/sqrt(length(diff)) )
> t
[1] 2.601775
> 2*(1-pt(t,length(diff-1)))
[1] 0.02315006
> boxplot(diff)
```

(e) Comment: Boxplot is a bit left skewed, which may indicate some evidence against the Normal assumption for the t Test. However, given we have a small sample, it may still indicate Normality.



(f) $\boxed{\text{Preparation}}$
The signs of differences are: $\{+ - + + + - + + + + 0 +\}$.

X = Number of + signs $\sim Bin(11, p)$, where $\rho = P$(preference for New Technique).
Note: 1 reading was discarded, so we have $n = 11$ and $x = 9$.

$\boxed{\text{H}}$ $H_0 : \rho = 0.5 \ \ vs \ \ H_1 : \rho \neq 0.5$

$\boxed{\text{A}}$ We assume the results were sampled independently with same ρ.

$\boxed{\text{T}}$ $X \overset{H_0}{\sim} Bin(11, 0.5)$.
Observed value: $x = 9$.

$\boxed{\text{P}}$ P-value $= 2P(X \geq 9) = 2(1 - P(X \leq 8) = 2(1 - 0.9673) = 0.0654$.

$\boxed{\text{C}}$ As p-value > 0.05, the evidence is consistent with H_0 (only just).
Hence, there is not significant evidence of difference in techniques.
Note: This is different conclusion to the t Test.

5. Paired t Test

A new measuring technique is being considered to replace the standard technique. When 10 samples are measured by both techniques, the measurements are:

Sample	1	2	3	4	5	6	7	8	9	10
New Technique	2.5	2.2	2.6	2.6	1.9	3.3	3.3	2.8	3.0	2.9
Standard Technique	2.1	2.4	2.1	1.9	2.0	2.8	2.7	2.8	2.8	3.0
Difference										

(a) Show that the mean and the standard deviation of the differences (New -Standard) are 0.25 and 0.331. Assuming that the differences are normally distributed, test the hypothesis that the techniques give the same results.

(b) Produce a boxplot of the differences. Comment.

(c) (Revision of Week 11) Test the hypothesis that there is no long-run systematic difference between the two techniques using a sign test.

Solution

(a)

Sample	1	2	3	4	5	6	7	8	9	10
New Technique	2.5	2.2	2.6	2.6	1.9	3.3	3.3	2.8	3.0	2.9
Standard Technique	2.1	2.4	2.1	1.9	2.0	2.8	2.7	2.8	2.8	3.0
Difference	0.4	-0.2	0.5	0.7	-0.1	0.5	0.6	0	0.2	-0.1

```
> a=c(2.5,2.2,2.6,2.6,1.9,3.3,3.3,2.8,3.0,2.9)
> a
 [1] 2.5 2.2 2.6 2.6 1.9 3.3 3.3 2.8 3.0 2.9
> b=c(2.1,2.4,2.1,1.9,2.0,2.8,2.7,2.8,2.8,3.0)
> diff=a-b
> diff
 [1]  0.4 -0.2  0.5  0.7 -0.1  0.5  0.6  0.0  0.2 -0.1
[1] 0.25
> sd(diff)
[1] 0.3308239
```

$\boxed{\text{H}}$ $H_0 : \mu_d = 0 \ \ vs \ \ H_1 : \mu_d \neq 0$

$\boxed{\text{A}}$ We assume that the population is Normally distributed.

$\boxed{\text{T}}$ $\tau = \dfrac{\bar{X} - 0}{\frac{0.3308239}{\sqrt{12}}} \overset{H_0}{\sim} t_{n-1} = t_{11}.$

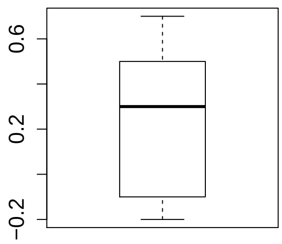
Observed value: $\tau_0 = \dfrac{0.25 - 0}{\frac{0.3308239}{\sqrt{12}}} \doteq 2.617784$

$\boxed{\text{P}}$

Using t Tables:
P-value $= 2P(t_{11} > 2.62) \in (0.02, 0.05)$.

$\boxed{\text{C}}$ Since p-value < 0.05, we would reject H_0 at $\alpha = 0.05$.
The new technique does seem to be significantly different from the standard technique.

(b) Comment: Boxplot is fairly symmetric, so Normality assumption for t Test seems appropriate.



(c) $\boxed{\text{Preparation}}$
The signs of differences are: $\{+ - + + - + + 0 + -\}$.

X = Number of + signs $\sim Bin(9, p)$, where $\rho = P$(preference for New Technique).
Note: 1 reading was discarded, so we have $n = 9$ and $x = 6$.

$\boxed{\text{H}}$ $H_0 : \rho = 0.5 \ \ vs \ \ H_1 : \rho \neq 0.5$

$\boxed{\text{A}}$ We assume the results were sampled independently with same ρ.

$\boxed{\text{T}}$ $X \overset{H_0}{\sim} Bin(9, 0.5)$.
Observed value: $x = 6$.

$\boxed{\text{P}}$ P-value $= 2P(X \geq 6) = 2(1 - P(X \leq 5) = 2(1 - 0.7461) = 0.5078$.

$\boxed{\text{C}}$ As p-value $>> 0.05$, the evidence is consistent with H_0.
Hence, there is not evidence of difference in techniques.

Note: Here we get a different conclusion from the t Test. The reason is that the Sign Test only uses the signs (not magnitudes) of differences. We can see that the positive differences are much bigger than the negative ones, and this is lost in the Sign test. Hence, it would better to use the t Test.

6. 2 Sample t Test

Two samples have been taken from two independent normal populations with equal variances. From these samples ($n_x = 12, n_y = 15$) we calculate $\bar{x} = 119.4$, $\bar{y} = 112.7$, $s_x = 9.2$, $s_y = 11.1$.

(a) Calculate the pooled standard deviation.

(b) Perform a test for $H_0 : \mu_x = \mu_y$ vs $H_1 : \mu_x \neq \mu_y$. Write down all your steps.

Solution

Let $X_i \quad i = 1, \ldots, 12$ and $Y_i \quad i = 1, \ldots, 15$ be the 2 samples from the independent populations with means μ_X and μ_Y and common variance σ^2.

We have $n_x = 12$, $n_y = 15$, $\bar{x} = 119.4$, $\bar{y} = 112.7$, $s_x = 9.2$ and $s_y = 11.1$.

(a) $s_p^2 = \dfrac{(n_x - 1)s_x^2 + (n_y - 1)s_y^2}{n_x + n_y - 2} = \dfrac{11 \times 9.2^2 + 14 \times 11.1^2}{12 + 15 - 2} = 106.2392$

Hence the pooled standard deviation is $s_p = 10.30724$.

(b)

$\boxed{\text{H}}$ $H_0 : \mu_X - \mu_Y = 0 \;\; vs \;\; H_1 : \mu_X - \mu_Y \neq 0$

$\boxed{\text{A}}$ We are given that the 2 populations are independent, Normally distributed, and have equal variances.

$\boxed{\text{T}}$ $\tau = \dfrac{\bar{X} - \bar{Y} - 0}{s_p\sqrt{\frac{1}{12} + \frac{1}{15}}} \sim t_{12+15-2} = t_{25}$

Observed value: $\tau = \dfrac{119.4 - 112.7 - 0}{10.30724\sqrt{\frac{1}{12} + \frac{1}{15}}} = 1.678366$

$\boxed{\text{P}}$

Using t Tables:
P-value $= 2P(t_{25} > 1.68) = \in \; (0.1, 0.2)$.

Using R:

```
> t=t=(119.4-112.7)/(10.30724*sqrt(1/12 + 1/15))
> t
[1] 1.678366
> 2*(1-pt(t,25))
[1] 0.1057345
```

$\boxed{\text{C}}$ Since p-value > 0.05, we retain H_0.
The evidence is consistent with the 2 populations having the same means.

7. 2 Sample t Test

Using the following R output, test the hypothesis $H_0 : \mu_x = \mu_y$ vs $H_1 : \mu_x \neq \mu_y$. Write down all your steps.

```
> length(x)
[1] 12
> mean(x)
[1] 119.4
> sd(x)
[1] 9.2
> length(y)
[1] 15
> mean(y)
[1] 112.7
> sd(y)
[1] 11.1
```

Solution

As the Output includes the summary statistics for Q6, the same test follows.