The University of Sydney

MATH1005 Statistics

Linear Function of a Random Variable

Given a random variable X with $E(X)$ and $Var(X)$.

For constants a and b,

$E(a + bX) = a + bE(X)$ and $Var(a + bX) = b^2 Var(X)$.

Sums of Random Variables

Given a sequence of random variables X_i with $E(X_i)$ and $Var(X_i)$ for $i = 1, \dots, n$.

$$E\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} E(X_i).$$

If X_i are independent, $Var\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} Var(X_i)$.

Sums of Normal Random Variables

Given a sequence of independent random variables $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, \dots, n$.

$$\sum_{i=1}^{n} a_i X_i \sim N\Big(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\Big)$$

Given a sequence of iid random variables $X_i \sim N(\mu, \sigma^2)$ and constants a_i for $i = 1, \dots, n$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\Big(\mu, \frac{\sigma^2}{n}\Big)$$

$$\sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2)$$

Central Limit Theorem (CLT)

Given a sequence of iid random variables $X_i \sim (\mu, \sigma^2)$ for $i = 1, \dots, n$.

where $\sigma^2 < \infty$ and n is large,

then the distribution function of $\dfrac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n\sigma^2}}$ tends to the standard Normal.

Less formally, $\sum_{i=1}^{n} X_i \to N(n\mu, n\sigma^2)$ and $\bar{X} \to N(\mu, \dfrac{\sigma^2}{n})$.

Normal Approximation to the Binomial (CLT special case)

For large n, $X \sim Bin(n, p) \to N(np, np(1-p))$.

Guide: Use when $n > 25$, $np > 5$, $n(1-p) > 5$.

Continuity correction

Given a discrete integer valued RV $X \sim (\mu, \sigma^2)$ and the approximating Normal $Y \sim N(\mu, \sigma^2)$

we adjust by 1/2 to (usually) improve the approximation.

$P(X \geq x) \to P(Y \geq x - 1/2)$

$P(X \leq x) \to P(Y \leq x + 1/2)$

1. Linear Function of a Discrete Random Variable

(a) Consider the following probability distribution function of X.

x	1	2	3	4	Total
$P(X=x)$	0.1	0.2	0.3	0.4	1

(*i*) Show $E(X) = \sum_{i=1}^{4} xP(X=x) = 3$, $E(X^2) = \sum_{i=1}^{4} x^2 P(X=x) = 10$ and $Var(X) = 1$.

(*ii*) Check in R

```
x=c(1:4)
p=x/10
sum(x*p)
sum(x^2*p)-sum(x*p)^2
```

(b) Now consider a linear function of X, namely $Y = 1 + 2X$.

(*i*) Complete the following probability distribution function of Y.

y	3				Total
$P(Y=y)$	0.1				1

(*ii*) Find $E(Y)$ and $Var(Y)$.

(*iii*) Check in R.

```
y=1+2*x
sum(y*p)
sum(y^2*p)-sum(y*p)^2
```

(c) Confirm your answers in (b), by using the formulae for the linear function of a random variable: $E(a+bX) = a + bE(X)$ and $Var(a+bX) = b^2 Var(X)$.

Solution

(a) $E(X) = 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.3 + 4 \times 0.4 = 3$
$E(X^2) = 1^2 \times 0.1 + 2^2 \times 0.2 + 3^2 \times 0.3 + 4^2 \times 0.4 = 10$
$Var(X) = 10 - 3^2 = 1$.

(b)

y	3	5	7	9	Total
$P(Y=y)$	0.1	0.2	0.3	0.4	1

$E(Y) = 3 \times 0.1 + 5 \times 0.2 + 7 \times 0.3 + 9 \times 0.4 = 7$
$E(Y^2) = 3^2 \times 0.1 + 5^2 \times 0.2 + 7^2 \times 0.3 + 9^2 \times 0.4 = 53$
$Var(Y) = 53 - 7^2 = 4$.

(c) $E(Y) = E(2X+1) = 2E(X) + 1 = 2 \times 3 + 1 = 7$.
$Var(Y) = Var(2X+1) = 4Var(X) = 4 \times 1 = 4$.

2. Light Bulb Lifetimes

In the light bulb industry, the Average Rated Life of any light bulb is defined by how long it takes for a percentage of the light bulbs in a test batch to fail. For example, if 100,000 bulbs are tested and after 1000 hours 70,000 (70%) of the bulbs have expired, the product is given an Average Rated Life of 1000

hours at L70.
Assume the distribution of light bulb lifetimes is symmetric.
http://www.thelightbulb.co.uk/resources/light_bulb_average_rated_life_time_hours

(a) If a certain light bulb has an Average Rated Life of 700 hours at L50, what is the median of the distribution of the lifetimes?

(b) Assuming the lifetime of the light bulb $X \sim N(700, 100^2)$, what is the chance that a randomly sampled light bulb lasts more than 750 hours.

```
1-pnorm(750,700,100)

## [1] 0.3085375
```

(c) Given 100 randomly sampled light bulbs modelled by $X_i \sim N(700, 100^2)$ $i = 1, 2, \ldots 100$, what is the chance that mean lifetime of the light bulbs is more than 750 hours.

```
1-pnorm(750,700,100/sqrt(100))

## [1] 2.866516e-07
```

(d) Given 10 randomly sampled light bulbs modelled by $X_i \sim N(700, 100^2)$ $i = 1, 2, \ldots 10$, what is the chance that total lifetime of the light bulbs is less than 6800 hours.

```
pnorm(6800,700*10,sqrt(100^2*10))

## [1] 0.2635446
```

Solution

(a) Median is 700 hours.

(b) $X \sim N(700, 100^2)$.
$P(X > 750) = P(\frac{X - 700}{100} > \frac{750 - 700}{100}) = P(Z > 0.5) = 0.3085375.$

(c) $\bar{X} \sim N(700, 100^2/100) = N(700, 100)$
$P(\bar{X} > 750) = P(\frac{\bar{X} - 700}{10} > \frac{750 - 700}{10} = P(Z > 5) = 2.866516e - 07.$

(d) $\sum_{i=1}^{n} X_i \sim N(700 * 10, 100^2 * 10) = N(7000, 100000).$
$P(\sum_{i=1}^{n} X_i < 6800) = P(\frac{\sum_{i=1}^{n} X_i - 7000}{\sqrt(100000)} < \frac{6800 - 7000}{\sqrt(100000)}) = P(Z < -0.6324555) = 0.2635446.$

```
pnorm(6800,7000,sqrt(100000))

## [1] 0.2635446
```

3. MATH1005 Exam multiple choice questions

The MATH1005 exam will have 20 multiple choice questions, each with 5 options. A student randomly guesses each answer independently.

(a) Given X = the number of correct answers, what is the exact distribution of X.

(b) Use R to compute the exact probability that the student gets at least 8 questions correct.

```
1-pbinom(7,20,0.2)

## [1] 0.03214266

x=c(8:20)
sum(dbinom(x,20,0.2))

## [1] 0.03214266
```

(c) Using a normal approximation with continuity correction, approximate the probability that the student gets at least 8 questions correct.

(d) Calculate the relative error of the normal approximation.

Solution

(a) X = number of correct answers $\sim Bin(20, 0.2)$.

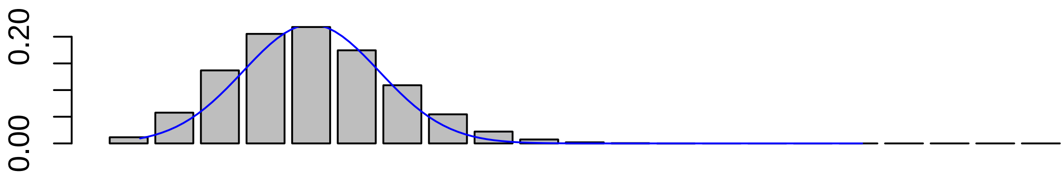
(b) The exact probability is $P(X \geq 8) = 0.03214266$.

```
1-pbinom(7,20,0.2)

## [1] 0.03214266
```

(c) By the CLT, $X \to Y \sim N(20 * 0.2, 20 * 0.2 * 0.8) = N(4, 3.2)$.

As we are approximating a discrete distribution (Binomial) by a continuous distribution (Normal), we use a continuity correction and change 8 to 7.5.



$$
\begin{aligned}
P(X \geq 8) &\approx P(Y \geq 7.5) \\
&= P(\frac{Y-4}{\sqrt{(3.2)}} \geq \frac{7.5-4}{\sqrt{(3.2)}}) \\
&= P(Z \geq 1.956559) \\
&= 1 - \Phi(1.956559) \\
&= 0.02519967
\end{aligned}
$$

```
1-pnorm(7.5,4,sqrt(3.2))

## [1] 0.02519964
```

```r
1-pnorm(1.956559)

## [1] 0.02519967
```

(d) Hence the absolute error is: $|0.03214266 - 0.02519967| = 0.00694299$.

The relative error of the approximation is $\dfrac{|0.03214266 - 0.02519967|}{0.03214266} = 0.2160055$.

A relative error of approximately 22% indicates a poor approximation, but notice here the probabilities are small.

Extra Questions

4. Sums of Random Variables

Given two random variables X: $E(X) = 5$ and $Var(X) = 9$ and Y: $E(Y) = 3$ and $Var(Y) = 4$.

(a) Find $E(X+Y)$ and $Var(X+Y)$. What condition is necessary to calculate the variance?

(b) Find $E(2X+Y+3)$ and $Var(2X+Y+3)$.

(c) Find $E(-5X-6Y)$ and $Var(-5X-6Y)$.

Solution

(a) $E(X+Y) = E(X) + E(Y) = 5 + 3 = 8$
$Var(X+Y) = Var(X) + Var(Y) = 9 + 4 = 13$ (assuming independence)

Condition: Independence of X and Y.

(b) $E(2X+Y+3) = 2E(X) + E(Y) + 3 = 2 \times 5 + 3 + 3 = 16$
$Var(2X+Y+3) = 4Var(X) + Var(Y) = 4 \times 9 + 4 = 40$

(c) $E(-5X-6Y) = -5E(X) - 6E(Y) = -5 \times 5 - 6 \times 3 = -43$
$Var(-5X-6Y) = (-5)^2 Var(X) + (-6)^2 Var(Y) = 25 \times 9 + 36 \times 4 = 369$

5. Mean and Total of iid RVs

Suppose $X_i \sim N(100, 125)$ for $i = 1, \ldots, 5$.

(a) Find the distribution of $\bar{X}$ and $T = \sum_{i=1}^{5} X_i$.

(b) Calculate $P(95 \leq \bar{X} \leq 105)$. Is this what you expected?

(c) $P(T \geq 600)$. Is this what you expected?

Solution

Given: $X_i \sim N(\mu = 100, \sigma^2 = 125)$ for $i = 1, \ldots, n$, where $n = 5$.

(a) $\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) = N(100, \frac{125}{5}) = N(100, 25) = N(100, 5^2)$.

$T = \sum_{i=1}^{5} X_i \sim N(n\mu, n\sigma^2) = N(5 \times 100, 5 \times 125) = N(500, 625) = N(500, 25^2)$.

(b) Standardising we get

$$P(95 \leq \bar{X} \leq 105) = P(\frac{95-100}{5} \leq \frac{\bar{X}-100}{5} \leq \frac{105-100}{5}) = P(-1 \leq Z \leq 1)$$

which is

$$P(Z \leq 1) - P(Z \leq -1) = \Phi(1) - (1 - \Phi(1)) = 2\Phi(1) - 1 \approx 0.68$$

This is what we expect: as 68% of probability lies within 1 standard deviation of the mean.

(c) Standardising we get

$$P(T \geq 600) = P(\frac{T-500}{25} \geq \frac{600-500}{25}) = P(Z \geq 4) \approx 0$$

This is what we expect: as it would be highly unlikely to observe a value more than 4 sds away from the mean.

6. Central Limit Theorem

(a) In your own words, explain the Central Limit Theorem.

(b) Why does the Central Limit Theorem apply to the Binomial Distribution?

Solution

(a) The Central Limit Theorem allows a *sum* of non-Normal random variables to be approximated by a Normal random variable.

(b) The Binomial is a *sum* of Bernouli random variables.

7. Central Limit Theorem

A new type of electronics flash for cameras will last an average of 5000 hours with a standard deviation of 400 hours. A company quality control engineer intends to select a random sample of 100 of these flashes and use them until they fail.

(a) What is the approximate probability that the mean life time of 100 flashes will be less than 4940 hours?

(b) What is the approximate probability that the mean life time of 100 flashes will be between 4960 and 5040 hours (ie. within 40 hours of μ, the population mean)

Solution

Sample: Flash Lifetimes $= X_i \sim N(5000, 400^2), i = 1, 2, \ldots, 100$.

(a) By the CLT, $\bar{X} \to N(5000, 400^2/100) = N(5000, 40^2)$.

So

$$\begin{aligned} P(\bar{X} < 4940) &= P(\frac{\bar{X} - 5000}{40} < \frac{4940 - 5000}{40}) \\ &= P(Z < -1.5) \\ &= 1 - \Phi(1.5) \\ &= 0.0668072 \end{aligned}$$

```
pnorm(4940,5000,40)

## [1] 0.0668072

1-pnorm(1.5)

## [1] 0.0668072
```

(b)

$$\begin{aligned} P(4960 < \bar{X} < 5040) &= P(\frac{4960 - 5000}{40} < \frac{\bar{X} - 5000}{40} < \frac{5040 - 5000}{40}) \\ &= P(-1 < Z < 1) \\ &= \Phi(1) - (1 - \Phi(1)) \\ &= 2\Phi(1) - 1 \\ &= 0.6826895 \end{aligned}$$

Note: This is as we expect - approximately 68% chance of being 1 sd away from the mean.

8. Combinations of Light Bulb Lifetimes

An electrical firm manufactures light bulbs. The lifetime of the bulbs is approximately normally distributed, with average lifetime of 800 hours and variance of 1600 hours.

(a) Find the probability that a randomly chosen light bulb lasts less than 790 hours.

(b) Find the probability that the average lifetime of a sample of 25 bulbs is less than 790 hours.

Solution

X = lifetime of bulb $\sim N(800, 1600)$.

(a) $P(X < 790) = P(\frac{X - 800}{40} < \frac{790 - 800}{40}) = P(Z < -1/4) = P(Z > 1/4) = 1 - \Phi(1/4) = 0.4013$

```
pnorm(790,800,40)

## [1] 0.4012937
```

(b) $\bar{X}$ = average lifetime of bulb $\sim N(\mu, \sigma^2/n) = N(800, 1600/25) = N(800, 64)$.

$P(\bar{X} < 790) = P(\frac{\bar{X} - 800}{8} < \frac{790 - 800}{8}) = P(Z < -10/8) = P(Z > 1.25) = 1 - \Phi(1.25) = 0.1056$

```
pnorm(790,800,8)

## [1] 0.1056498
```

9. Poor Normal approximation to Binomial

(a) Find the relative error of the Normal approximation to $P(X \leq 2)$, when $X \sim Bin(25, 0.25)$.

(b) Can you explain why this is a poor appoximation?

Solution

(a) By the CLT, $X \to Y \sim N(6.25, 4.6875)$.

As we are approximating a discrete distribution (Binomial) by a continuous distribution (Normal), we use a continuity correction and change 2 to 2.5.

$$\begin{aligned} P(X \leq 2) &\approx P(Y \leq 2.5) \\ &= P(\frac{Y - 6.25}{\sqrt{4.6875}} \leq \frac{2.5 - 6.25}{\sqrt{4.6875}}) \\ &= P(Z \leq -1.732051) \\ &= 1 - \Phi(1.732051) \\ &= 0.04163224 \end{aligned}$$

(b) Exact probability is 0.03210852.

```
pbinom(2,25,0.25)

## [1] 0.03210852
```

Hence the relative error of approximation is $\frac{0.03210852 - 0.04163224}{0.03210852} = -0.2966104$.

A relative error of approximately 30% indicates a poor approximation.

(c) This is surprising given that the 'Guide' is satisfied, which reminds us that the 'Guide' is merely a rule of thumb.

10. Normal approximation to Discrete RV

Let X_1, X_2, X_3 be independent rolls of a fair 6-sided die and let $S = X_1 + X_2 + X_3$.

(a) Show that $E(X_i) = 3.5$ and $\text{Var}(X_i) = \frac{35}{12}$.

(b) Compute a normal approximation with continuity correction for $P(S \leq 6)$.

(c) Using probability, compute $P(S \leq 6)$ exactly. What is the relative error of the approximation in (b)?

Solution

Let X_1, X_2, X_3 be independent rolls of a fair 6-sided die and let $S = X_1 + X_2 + X_3$.

(a) Let $X_i = i$th toss of a fair die, $i = 1, 2, 3$.

x_i	1	2	3	4	5	6	Total
$P(X_i = x_i)$	1/6	1/6	1/6	1/6	1/6	1/6	1

Then:
$E(X_i) = 1 \times 1/6 + 2 \times 1/6 + \ldots 6 \times 1/6 = 3.5$
$E(X_i^2) = 1^2 \times 1/6 + 2^2 \times 1/6 + \ldots 6^2 \times 1/6 = 91/6$
$Var(X_i) = 91/6 - 3.5^2 = 35/12$

(b) By the CLT, $S = \sum_{i=1}^{3} X_i \to Y \sim N(3 \times 3.5, 3 \times 35/12) = N(10.5, 8.75)$.

As we are approximating a discrete distribution by a continuous distribution (Normal), we use a continuity correction and change 6 to 6.5.

$$\begin{aligned} P(S \leq 6) &\approx P(Y \leq 6.5) \\ &= P(\frac{Y - 10.5}{\sqrt{8.75}} \leq \frac{6.5 - 10.5}{\sqrt{8.75}}) \\ &= P(Z \leq -1.352247) \\ &= 1 - \Phi(1.352247) \\ &= 0.08814816 \end{aligned}$$

(c) Consider the number of ways to get a total of 6: $\{111, 112, 113, 114, 121...222\}$, altogether there are 20 ways, and each has a probability of $\frac{1}{6^3}$.

Hence $P(S \leq 6) = \frac{20}{6^3} \approx 0.093$.

Hence the relative error of approximation in (b) is $\frac{\frac{20}{6^3} - 0.08814816}{\frac{20}{6^3}} = 0.04799987$.

A relative error of approximately 5% is excellent, given we are approximating a discrete sum of only 3 variables.

11. Normal Approximation to Binomial

Suppose that $X \sim B(20, 0.25)$.

(a) Write down $E(X)$ and $Var(X)$.

(b) Compute a normal approximation to $P(X \geq 5)$ using a continuity correction.

(c) Use R to compute the exact probability. What is the relative error of the approximation in (b).

(d) Repeat (a)-(c) for $X \sim B(20, 0.1)$.

(e) Why is the relative error in (d) so poor compared to (c)?

Solution

(a) $E(X) = np = 5$ and $Var(X) = np(1-p) = 3.75$.

(b) By the CLT, $X \sim Bin(20, 0.25) \rightarrow Y \sim N(5, 3.75)$.

As we are approximating a discrete distribution (Binomial) by a continuous distribution (Normal), we use a continuity correction and change 5 to 4.5.

$$\begin{aligned}
P(X \geq 5) &\approx P(Y \geq 4.5) \\
&= P(\frac{Y-5}{\sqrt{3.75}} \geq \frac{4.5-5}{\sqrt{3.75}}) \\
&= P(Z \geq -0.2581989) \\
&= \Phi(0.2581989) \\
&= 0.6018733
\end{aligned}$$

(c)

```
1-pbinom(4,20,0.25)

## [1] 0.5851585
```

Hence the relative error of approximation in (a) is $\dfrac{0.5851585 - 0.6018733}{0.5851585} = -0.02856457$.
A relative error of approximately 3% again reflects an accurate approximation.

(d) Given $X \sim B(20, 0.1)$, $E(X) = np = 2$ and $Var(X) = np(1-p) = 1.8$.

By the CLT, $X \rightarrow Y \sim N(2, 1.8)$.

As we are approximating a discrete distribution (Binomial) by a continuous distribution (Normal), we use a continuity correction and change 5 to 4.5.

$$\begin{aligned}
P(X \geq 5) &\approx P(Y \geq 4.5) \\
&= P(\frac{Y-2}{\sqrt{1.8}} \geq \frac{4.5-2}{\sqrt{1.8}}) \\
&= P(Z \geq 1.86339) \\
&= 1 - \Phi(1.86339) \\
&= 0.03120371
\end{aligned}$$

Exact probability is 0.0431745.

```
1-pbinom(4,20,0.1)

## [1] 0.0431745
```

Hence the relative error of approximation is $\dfrac{0.0431745 - 0.03120371}{0.0431745} = 0.2772653$.
A relative error of approximately 28% indicates a very poor approximation.

(e) Comparing $Bin(20, 0.1)$ to the 'Guide' for using Normal approximation, we find that $np < 5$ and $n(1-p) > 5$, hence it is not surprising that the approximation is not accurate.

12. Sum of iid Normal Random Variables

Suppose that the weight (in pounds) of a North American adult can be represented by a normal random variable with mean 150 and variance 625. An elevator contains a sign "Maximum 10 people". It can in fact safely carry 2000 lbs.

(a) (Extension) If we want to be at least 99% certain that the elevator will not be overloaded, what is the maximum number of people that should be allowed into the elevator?

(b) If we want an elevator that we are 99% certain can carry 10 people without overloading, how much weight should it be able to safely carry?

Solution

X_i = weight of a North American adult $\sim N(150, 625)$, $i = 1, 2, \ldots, n$

Safety limit for the lift:
Sign says: 'Maximum 10 people'
Actual limit: 2000 pounds

(a) We want to find n such that $P(\text{total weight} \leq \text{safety limit}) = P(\sum_{i=1}^{n} X_i \leq 2000) \geq 0.99$.

Now $\sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2) = N(150n, 625n)$.

So $P(\sum_{i=1}^{n} X_i \leq 2000) = P(\frac{\sum_{i=1}^{n} X_i - 150n}{25\sqrt{n}} \leq \frac{2000 - 150n}{25\sqrt{n}}) = P(Z \leq \frac{2000 - 150n}{25\sqrt{n}})$

Hence we want to find n such that $P(Z \leq \frac{2000 - 150n}{25\sqrt{n}}) \geq 0.99$.

From the Normal table, the 99% quantile of Z is 2.33.

Hence we need to solve $\frac{2000 - 150n}{25\sqrt{n}} \geq 2.33$.

(1) Method1: Using quadratic
Rearranging, we get $150n + 58.25\sqrt{n} - 2000 = 0$, which is a quadratic in $\sqrt{n}$.

Using the quadratic formula,
$\sqrt{n} = \frac{-58.25 \pm \sqrt{58.25^2 - 4 \times 150 \times -2000}}{2 \times 150}$
so $\sqrt{n} = -3.850809$ or 3.462476, giving $n = (3.462476)^2 = 11.98874$.
Hence, we would choose $n = 11$ to be safe.

(2) Method2: Using trial and error

n	$\frac{2000 - 150n}{25\sqrt{n}}$
9	8.7
10	6.3
11	4.2
12	2.3
13	0.6

To get $\frac{2000 - 150n}{25\sqrt{n}} \geq 2.33$, we need a maximum of $n = 11$.

(b) We want to find L such that $P(\sum_{i=1}^{10} X_i \leq L) \geq 0.99$.

Now $\sum_{i=1}^{10} X_i \sim N(1500, 6250)$.

So $P(\sum_{i=1}^{10} X_i \leq L) = P(\frac{\sum_{i=1}^{10} X_i - 1500}{\sqrt{6250}} \leq \frac{L-1500}{\sqrt{6250}}) = P(Z \leq \frac{L-1500}{\sqrt{6250}})$

Hence we want $P(Z \leq \frac{L-1500}{\sqrt{6250}}) \geq 0.99$.

Solving $2.33 = \frac{L-1500}{\sqrt{6250}}$, we get $L = 2.33 * \sqrt{6250} + 1500 = 1684.203$.

Hence, choose a safety limit of 1684 pounds.

13. A pharmaceutical firm produces a rare antibiotic. It is believed that daily production (in mg) of the drug may be represented by a normal random variable with mean 500 and variance 1600.

(a) A hospital wants as much of the antibiotic as possible five weeks from now. Assuming independence of the production each day and 5 working days per week, how much of the antibiotic can the firm promise, and be 90% certain of keeping its promise?

(b) (Extension) After delivering the antibiotic to the hospital, the firm receives an urgent order for 10 grams of the drug from a hospital in Iqaluit. How quickly can the hospital promise to deliver the order, and be 95% certain of keeping its promise?

(c) After studying its production more closely, the firm realizes that its production (in mg/day) varies due to absenteeism. The daily means and variances are as follows:

Day	Mean	Variance
Monday	300	900
Tuesday	500	1000
Wednesday	600	1100
Thursday	500	1000
Friday	300	900

If it can be assumed that each day's production is normally distributed and independent of the other days, what is the probability that the total production in a week exceeds 2 grams?

(d) Verify your answers in (a) and (c) by using the R commands `pnorm` and `qnorm`.

Solution

Daily Production $= X_i \sim N(500, 1600), i = 1, 2, 3, \ldots$

(a) Total production in 25 days $= \sum_{i=1}^{25} X_i \sim N(25 \times 500, 25 \times 1600) = N(12500, 200^2)$

We want L such that $P(\sum_{i=1}^{25} X_i \geq L) \geq 0.90$.

Hence $P(\frac{\sum_{i=1}^{25} X_i - 12500}{200} \geq \frac{L-12500}{200}) \geq 0.90$.

Solving $-1.28 = \frac{L-12500}{200}$, we get $L = -1.28 \times 200 + 12500 = 12244$.

Hence, the firm could promise a production of 12244g.

(b) We want n (number of days) such that $P(\sum_{i=1}^{n} X_i \geq 10000) \geq 0.95$.

Total production in n days $= \sum_{i=1}^{n} X_i \sim N(500n, 1600n)$

So we want $P(\sum_{i=1}^{n} X_i \geq 10000) = P(\frac{\sum_{i=1}^{n} X_i - 500n}{\sqrt{1600n}} \geq \frac{10000 - 500n}{\sqrt{1600n}}) = P(Z \geq \frac{10000 - 500n}{\sqrt{1600n}}) \geq$ 0.95

We need to solve $-1.645 = \frac{10000 - 500n}{\sqrt{1600n}}$.

(1) Method1: Using quadratic
Rearranging gives the quadratic in $\sqrt{n}$: $500n - 65.8\sqrt{n} - 10000 = 0$.

Using the quadratic formula, we get $\sqrt{n} = \frac{65.8 \pm \sqrt{-65.8^2 - 4 \times 500 \times -10000}}{2 \times 500}$
which gives $\sqrt{n} = -4.40682$ and 4.53842.
Hence $n = 21$. The hospital expects to deliver within 21 days.

(2) Method2: Using trial and error

n	$\frac{10000 - 500n}{\sqrt{1600n}}$
19	2.867697
20	0
21	-2.727724
22	-5.330018

To get $\frac{10000 - 500n}{\sqrt{1600n}} \geq 1.645$, we need a maximum of $n = 21$.

(c) Total Production in a week $= T \sim N(300 + 500 + 600 + 500 + 300, 900 + 1000 + 1100 + 1100 + 900)$.
So $T \sim N(2200, 5000)$.
Hence $P(T \geq 2000) = P(\frac{T - 2200}{\sqrt{5000}} \geq \frac{2000 = 2200}{\sqrt{5000}}) = P(Z \geq -2.828427) \doteq 0.998$

(d)

```
qnorm(0.1,12500,200)

## [1] 12243.69

1-pnorm(2000,2200,70)

## [1] 0.9978626
```