Summer/Winter/Semester2 **Tutorial Solutions 10** 2015

This tutorial explores hypothesis testing, the proportion test and the sign test.

Proportion Test	
Context	n independent Binary trials with unknown success probability ρ
Hypothesis	$H_0 : \rho = \rho_0$
Test Statistic	$X = \#$ successes $\overset{H_0}{\sim} Bin(n, \rho_0)$

Sign Test	
Context	n observations $\{x_i\}$ from a continuous distribution with unknown mean μ We want to test $H_0 : \mu = \mu_0$
Working	Work out signs $\{x_i - \mu_0\}$ (eliminating any zeroes)
Hypothesis	$H_0 : \rho_+ = 0.5$
Test Statistic	$X = \# +$ signs $\overset{H_0}{\sim} Bin(n, 0.5)$

1. Hypothesis Testing

In your own words, explain the following concepts:

(a) The purpose of hypothesis testing

(b) The importance of assumptions

(c) Test statistic

(d) P-value

Solution

(a)The purpose of hypothesis testing is to make a robust decision about an unknown population parameter (specified in H_0 vs H_1), based on an observed sample.

(b) A hypothesis test is invalid if the assumptions are not satisfied.

(c) A test statistic is a random variable consisting of a function of the observed values, with a distribution depending on the unknown parameter.

(c) The p-value is the weighting of the evidence for H_1 from the sample, assuming H_0 is true.

2. Proportion test

The proportion of families buying a certain brand of orange juice in a certain city is believed to be 0.6. A consumer group claims that this particular brand is now less popular than it was before. A random sample of 50 families from this city shows that only 22 bought that brand of juice.

Let X be the number of families in the sample buying the brand and let ρ be the probability that a randomly picked family prefers the brand.

(a) Hypothesis: Explain why a one-sided test of $H_0 : \rho = 0.6 \ \ vs \ \ H_1 : \rho < 0.6$ is suitable to test the claim of this consumer group.

(b) Assumptions: To model X as a Binomial random variable, what further assumptions are needed?

(c) Test statistic: What is the test statistic and its distribution under H_0?

(d) P-value: Calculate the p-value using R, and by hand using a normal approximation. What is the relative error of this approximation?

```
> pbinom(22,50,0.6)
```

(e) Conclusion: Draw your conclusion based on the p-value.

Solution

(a) $\boxed{\text{H}}$ $H_0 : \rho = 0.6 \ \ vs \ \ H_1 : \rho < 0.6$
We choose a one-sided alternative because the consumer group believes that the brand is now *less* popular than before.

(b) $\boxed{\text{A}}$ We assume each of the 50 families are sampled independently and have the same preference ρ for the orange juice brand.

(c) $\boxed{\text{T}}$ $X =$ Number of families buying the brand $\overset{H_0}{\sim} Bin(50, 0.6)$.
The observed value is $x = 22$.

(d) $\boxed{\text{P}}$

Using R:
p-value $= P(X \leq 22) = 0.01603476$.

```
> pbinom(22,50,0.6)
[1] 0.01603476
```

Using Normal Approximation:
By the CLT, $X \to Y \sim N(np, np(1-p)) = N(30, 12)$.
As we are approximating a discrete distribution (Binomial) by a continuous distribution (Normal), we use a continuity correction and change 22 to 22.5.

$$
\begin{aligned}
P(X \leq 22) &\doteq P(Y \leq 22.5) \\
&= P(\frac{Y-30}{\sqrt{12}} \leq \frac{22.5-30}{\sqrt{12}}) \\
&= P(Z \leq -2.165064) \\
&= 1 - \Phi(2.165064) \\
&= 0.01519139
\end{aligned}
$$

Hence the relative error of approximation is $\dfrac{0.01603476 - 0.01519139}{0.01603476} = 0.05259636$.
A relative error of approximately 5% reflects an accurate approximation.

(e) $\boxed{\text{C}}$ As p-value < 0.05, we have evidence against H_0 at $\alpha = 0.05$.
Hence there is evidence for the consumer claim that the brand is losing market share.

3. Proportion Test

A certain family has 7 children and they are all girls. Perform a 2 sided test of the hypothesis that in that family each child is either a boy or a girl independently with equal probability. Write all your steps. Calculate the p-value using both Binomial tables and R.

Solution

Let X = the number of girls in family of 7. Then $X \sim Bin(7, \rho)$, where $\rho = P(X = \text{girl})$.

$\boxed{\text{H}}$ $H_0 : \rho = 0.5 \;\; vs \;\; H_1 : \rho \neq 0.5$

$\boxed{\text{A}}$ We assume that the gender of each of the children is independent of the others and that they have the same probability $\rho = P(girl)$.

$\boxed{\text{T}}$ X = Number of girls $\overset{H_0}{\sim} Bin(7, 0.5)$.
The observed value is $x = 7$.

$\boxed{\text{P}}$

Using Formula:
p-value $= 2P(X \geq 7) = 2P(X = 7) = 2\binom{7}{7}(0.5)^7(0.5)^0 = 0.015625$.

Using Binomial Tables:
p-value $= 2P(X \geq 7) = 2(1 - P(X \leq 6)) = 2(1 - 0.992) = 0.016$.

Using R:
p-value $\doteq 0$

```
> 2*(1-pbinom(6,7,0.5))
[1] 0.015625
```

$\boxed{\text{C}}$ As p-value $<< 0.05$, we have evidence against H_0 at $\alpha = 0.05$.
Hence there is evidence against the family having girls with $\rho = 0.5$.

4. Sign Test

The following data are 8 measurements of moisture retention (%) using a new scaling system. This system is expected to be better (ie have greater retention) than the system previously in use, for which the mean retention was 96%.

```
97.5 96.2 97.3 96.0 99.8 93.0 94.2 95.5
```

(a) Preparation: By comparing the data to 96, write down the signs.

(b) Hypothesis: Explain why $H_0 : \rho = 0.5 \;\; vs \;\; H_1 : \rho > 0.5$ is the appropriate hypothesis.

(c) Assumptions: To perform a sign test, what assumptions are needed?

(d) Test statistic: What is the test statistic and its distribution under H_0?

(e) P-value: Calculate the p-value using Binomial Tables.

(f) Conclusion: Draw your conclusion based on the p-value.

Solution

(a) $\boxed{\text{Preparation}}$ $\{\text{Signs}\} = \{+ + + 0 + - - -\}$. (We discard the 0).

(b) $\boxed{\text{H}}$ $H_0 : \mu = 0.96 \;\; vs \;\; H_1 : \mu > 0.96$ is equivalent to $H_0 : \rho = 0.5 \;\; vs \;\; H_1 : \rho > 0.5$ where $\rho = P(X > 0.96)$.
We use a one-sided test as the system is expected to be better.

(c) $\boxed{\text{A}}$ We assume the data is from a continuous distribution.

(d) $\boxed{\text{T}}$ X = Number of + signs $\overset{H_0}{\sim} Bin(7, 0.5)$.
Observed value: $x = 4$.

(e) $\boxed{\text{P}}$ P-value $= P(X \geq 4) = 1 - P(X \leq 3) = 1 - 0.5 = 0.5$.

(f) $\boxed{\text{C}}$ As p-value >> 0.05, the evidence is consistent with H_0.
Hence, there is no evidence that the new system has improved.
Note: This was based on a small sample. Increasing the sample size might change the result.

5. Sign Test

14 students taste-tested two different brands of drink (brand A and brand B), with the brands being hidden from them. The object of the exercise was to see if students preferred one brand over the other, but there was no indication of which this might be before the test. Overall, 8 subjects preferred brand A, 4 preferred brand B and 2 had no preference either way. Use a sign test to test whether there is a difference between the 2 brands. Write all your steps. Calculate the p-value using Binomial tables.

Solution

X = Number of + signs $\sim Bin(12, p)$, where $\rho = P$(preference for brand A) and $x = 8$.
Note: 2 readings were discarded ('no preference'), giving us $n = 12$.

$\boxed{\text{H}}$ $H_0 : \rho = 0.5 \;\; vs \;\; H_1 : \rho \neq 0.5$

$\boxed{\text{A}}$ We assume the students were sampled independently with same ρ.

$\boxed{\text{T}}$ $X \overset{H_0}{\sim} Bin(12, 0.5)$.
Observed value: $x = 8$.

$\boxed{\text{P}}$ P-value $= 2P(X \geq 8) = 2(1 - P(X \leq 7) = 2(1 - 0.8062) = 0.3876$.

$\boxed{\text{C}}$ As p-value >> 0.05, the evidence is consistent with H_0.
Hence, there is no evidence of student preference for Brand A.

6. Sign Test for paired data

A new measuring technique is being considered to replace the standard technique. When 10 samples are measured by both techniques, the measurements are

Sample:	1	2	3	4	5	6	7	8	9	10
New	2.5	2.2	2.6	2.6	1.9	3.3	3.3	2.8	3.0	2.9
Standard	2.1	2.4	2.1	1.9	2.0	2.8	2.7	2.8	2.8	3.0

Test the hypothesis that there is no long-run systematic difference between the two techniques.

Hint: First, calculate Difference = New-Standard. Second, write down the signs of the Differences. Third, perform the sign test.

Solution

$\boxed{\text{Preparation}}$

Sample:	1	2	3	4	5	6	7	8	9	10
New	2.5	2.2	2.6	2.6	1.9	3.3	3.3	2.8	3.0	2.9
Standard	2.1	2.4	2.1	1.9	2.0	2.8	2.7	2.8	2.8	3.0
Differences = New - Standard	0.4	-0.2	0.5	0.7	-0.1	0.5	0.6	0	0.2	-0.1
Signs	+	-	+	+	-	+	+	0	+	-

So we have a sample of signs of size $n = 9$ with x = number of + signs = 6.

Define X = Number of + signs $\sim Bin(9, \rho)$, where $\rho = P(+sign) = P(\text{New different than Standard})$.

$\boxed{\text{H}}$ $H_0 : \rho = 0.5 \;\; vs \;\; H_1 : \rho \neq 0.5$

$\boxed{\text{A}}$ We assume the measurements were sampled independently with same ρ.

$\boxed{\text{T}}$ $X \overset{H_0}{\sim} Bin(9, 0.5)$.
Observed value: $x = 6$.

$\boxed{\text{P}}$ P-value $= 2P(X \geq 6) = 2(1 - P(X \leq 5)) = 2(1 - 0.7461) = 0.5078$.

$\boxed{\text{C}}$ As p-value >> 0.05, the evidence is consistent with H_0.
Hence, there is no evidence that the standard and new techniques are systematically different.