The University of Sydney

MATH1005 Statistics

maths.usyd.edu.au/MATH1005/

Summer/WinterSemester2 **Tutorial Solutions 4** 2015

This tutorial explores bivariate data.
Most of this week's tutorial class will consist of the Report 1 presentations.
Complete the rest of the tutorial questions at home.

Bivariate Data

For paired observations $\{(x_i, y_i)\}$ for $i = 1, \ldots, n$

summary statistics

$$S_{xy} = \sum_{i=1}^{n} x_i y_i - \frac{1}{n}\Big(\sum_{i=1}^{n} x_i\Big)\Big(\sum_{i=1}^{n} y_i\Big)$$

$$S_{xx} = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\Big(\sum_{i=1}^{n} x_i\Big)^2 = (n-1)s_x^2$$

$$S_{yy} = \sum_{i=1}^{n} y_i^2 - \frac{1}{n}\Big(\sum_{i=1}^{n} y_i\Big)^2$$

least squares regression line $y = a + bx$

where $a = \bar{y} - b\bar{x}$ and $b = \dfrac{S_{xy}}{S_{xx}}$

correlation coefficient $r = \dfrac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} = b\dfrac{s_x}{s_y}$

1. Bivariate Data by hand

Of environmental interest is the relation between carbon monoxide concentration and traffic density. The following table gives the traffic density (vehicles per hour to the nearest 500 vehicles) and carbon monoxide concentration (CO) in ppm for a particular street corner in Newtown.
Note the table reads as $(x_1, y_1) = (1.0, 9), (x_2, y_2) = (1.0, 6.8), \ldots (x_{12}, y_{12}) = (3.0, 20.6)$

x: Traffic density (in thousands)	y: CO concentration (in ppm)
1.0	9.0 6.8 7.7
1.5	9.6 6.8 10.3
2.0	12.3 11.8
3.0	20.7 20.2 21.6 20.6

(a) Produce a scatter plot. Does a linear fit seem reasonable?

(b) Calculate the summary statistics from

$$\sum_{i=1}^{12} x_i y_i = 361.05, \sum_{i=1}^{12} x_i^2 = 53.75, \sum_{i=1}^{12} y_i^2 = 2449, \sum_{i=1}^{12} x_i = 23.5, \sum_{i=1}^{12} y_i = 157.4$$

(c) Calculate the mean and variance of x and y.

(d) Find the least squares regression line and correlation coefficient. What does r suggest?

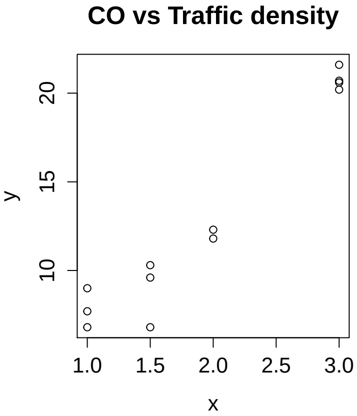
(e) What is the predicted CO concentration on a day with traffic density 2,500?

(f) For the reading (2.0,12.3), what is the residual?

(g) On a day with traffic density of 5,000, would it be appropriate to use the least squares regression line to predict the CO concentration? Explain.

Solution

(a) Draw a scatter plot by hand. Check against this R output.



Comment: The linear fit seems reasonable.

(b) $S_{xx} = 53.75 - \frac{1}{12}(23.5)^2 \approx 7.73$
$S_{yy} = 2449 - \frac{1}{12}(157.4)^2 \approx 384.44$
$S_{xy} = 361.05 - \frac{1}{12}(23.5)(157.4) \approx 52.81$

(c) $\bar{x} = \frac{23.5}{12} \approx 1.96$
$\bar{y} = \frac{157.4}{12} \approx 13.12$
$s_x^2 = \frac{1}{11}(53.75 - \frac{1}{12}(23.5)^2) \approx 0.70$
$s_y^2 = \frac{1}{11}(2449 - \frac{1}{12}(157.4)^2) \approx 34.95$

(d) $b = \frac{52.81}{7.73} \approx 6.83$
$a = 13.12 - (6.83)(1.96) \approx -0.27$
Hence the LSR line is $\hat{y} = -0.27 + 6.83x$.

$$r = \frac{52.81}{\sqrt{(7.73)(384.44)}} \approx 0.97$$

r suggests a high positive linear correlation between y and x. The points are closely scattered about a line with positive slope. The reduction in variance from y values to the residuals is $r^2 \approx 94\%$.

(e) $\hat{y} = -0.27 + (6.83)(2.5) \approx 16.81$.

(f) $\hat{y}_{x=2.0} = -0.27 + (6.83)(2) = 13.39$.
So $res = y - \hat{y} = 12.3 - 13.39 = -1.09$.

(g) As 5000 is far away from the range of $x\epsilon(1,3)$ (in thousands), it is not appropriate to use the LSR line, unless it can be argued that that same linear trend continues till $x = 5$.

2. Bivariate Data in R

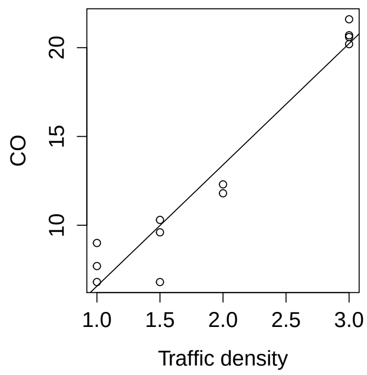
Using the data in Question 1, produce a scatterplot, least squares regression line, correlation coefficient and residual plot. Is the linear fit appropriate?

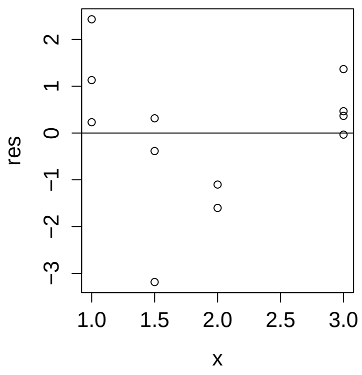
```
> x=c(1,1,1,1.5,1.5,1.5,2,2,3,3,3,3)
> y=c(9.0,6.8,7.7,9.6,6.8,10.3,12.3,11.8,20.7,20.2,21.6,20.6)
> cor(x,y)
> fit <- lm(y~x)
> a= fit$coefficients[[1]]
> b= fit$coefficients[[2]]
> plot(x,y,xlab="Traffic density", ylab="CO")
> abline(fit)

> res = y - (a + b * x)
> plot(x, res)                   OR > plot(x,fit$residuals)
> abline(h = 0)                     > abline(h = 0)
```

Solution

R Commands given.





(1) Scatter plot with LSR line added. On first inspection, linear fit looks appropriate.
(2) Residual plot with res=0 added. Notice the quadratic pattern, indicating that the linear fit is not appropriate. A better model might be based on $y = x^2$.

Hence the predictions in both Q1(e) and (g) are not valid.

3. Bivariate Data

J.B. Haldane is responsible for showing how carbon dioxide levels in the blood influences breathing rates by affecting the acidity of the blood. In one experiment he administered varying doses of sodium bicarbonate with the following results:

Dose (in grams)	x	30	40	50	60	70	80	90	100
Breathing rate	y	16	14	13	13	11	12	9	9

(a) By hand, find the least squares regression line and correlation coefficient. What breathing rate would you predict for a dose of 85g?

(b) Using R, produce a scatter plot and residual plot. Is the linear fit appropriate?

(c) Refit the model, so that you could predict the dose from the breathing rate.

Solution

(a) First calculate the summary statistics:

$$\sum_{i=1}^{8} x_i = 520 \sum_{i=1}^{8} x_i^2 = 38000 \sum_{i=1}^{8} y_i = 97 \sum_{i=1}^{8} y_i^2 = 1217 \sum_{i=1}^{8} x_i y_i = 5910$$

Next calculate the sums:

$$S_{xx} = \sum_{i=1}^{8} x_i^2 - \frac{1}{8}(\sum_{i=1}^{8} x_i)^2 = 3800 - \frac{1}{8}520^2 = 4200$$

$$S_{yy} = \sum_{i=1}^{8} y_i^2 - \frac{1}{8}(\sum_{i=1}^{8} y_i)^2 = 1217 - \frac{1}{8}97^2 = 40.875$$

$$S_{xy} = \sum_{i=1}^{8} x_i y_i - \frac{1}{8}(\sum_{i=1}^{8} x_i)(\sum_{i=1}^{8} y_i) = 5910 - -\frac{1}{8}(520)(97) = -395.$$

Finally calculate the estimates of the regression parameters:
$b = \frac{S_{xy}}{S_{xx}} = \frac{-395}{4200} = -0.094$ (3dp) and $a = \bar{y} - b\bar{x} = \frac{97}{8} - b\frac{520}{8} = 18.2381.$

Hence the LSR line is

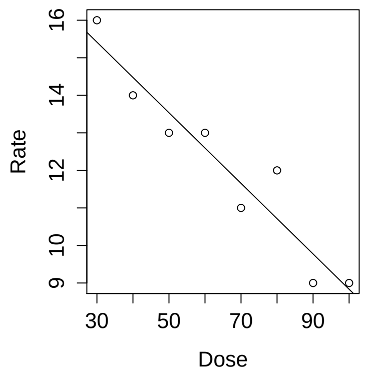
$$\hat{y} = 18.2381 - 0.09404762x$$

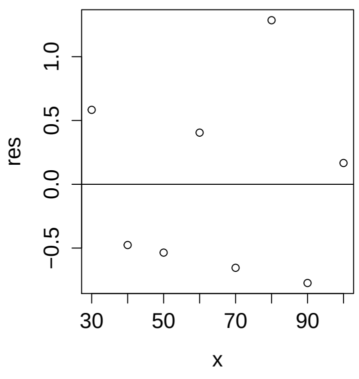
The correlation coefficient is $r = \frac{S_{xy}}{\sqrt{S_{xx}S_{yy}}} = \frac{-395}{\sqrt{4200 \times 40.875}} = -0.953$ (3dp).

For a dose of 85g, we would predict a breathing rate of: $\hat{y}_{x=85} = 18.2381 - (0.09404762)(85) = 10.24.$

(b)

```
> x=c(30,40,50,60,70,80,90,100)
> y=c(16,14,13,13,11,12,9,9)
> cor(x,y)
[1] -0.9533307
> fit <- lm(y˜x)
> a= fit$coefficients[[1]]
> b= fit$coefficients[[2]]
> a
[1] 18.2381
> b
[1] -0.09404762
> plot(x,y,xlab="Dose", ylab="Rate")
> abline(fit)
> res = y - (a + b * x)
> plot(x, res)
> abline(h = 0)
```





Comment: Based on both plots, it seems the linear fit is appropriate: The scatter plot shows a linear trend, and residual plot shows no observable pattern.

(c) Reverse the roles of x and y, and then repeat the R commands.

```
> x=c(16,14,13,13,11,12,9,9)
> y=c(30,40,50,60,70,80,90,100)
> cor(x,y)
[1] -0.9533307
> fit <- lm(y~x)
> a= fit$coefficients[[1]]
> a
[1] 182.1713
> b= fit$coefficients[[2]]
> b
[1] -9.663609
```

To predict dose from breathing rate, the LSR model is $\hat{y} = 182.1713 - 9.663609x$, where y is dose and x is breathing rate.

4. Bivariate Data in R

The following data describes the relationship between obesity measured as the percentage over ideal weight (x) and individual's response to pain measured on a certain scale (y):

x	89%	90%	75%	30%	51%
y	2	3	4	4.5	5.5

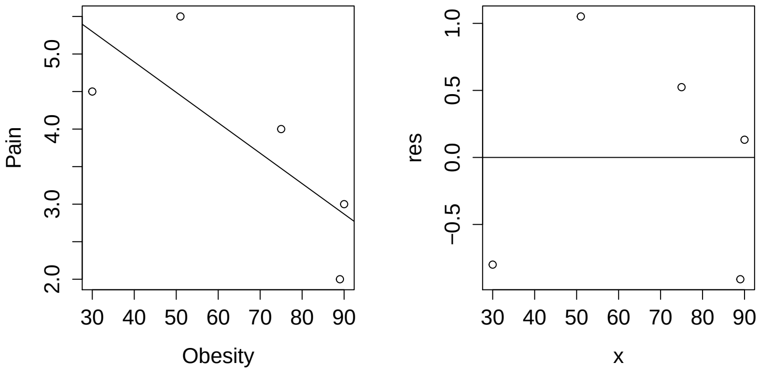
(a) Using R, produce a scatter plot, least squares regression line, correlation coefficient and residual plot.

(b) Do the plots suggest that the linear fit is appropriate? If so, use the least squares line to predict the y-value for an x-value of 60%.

Solution

(a)

```
> x=c(89,90,75,30,51)
> x
[1] 89 90 75 30 51
> y=c(2,3,4,4.5,5.5)
> y
[1] 2.0 3.0 4.0 4.5 5.5
> cor(x,y)
[1] -0.7796686
> fit <- lm(y~x)
> a= fit$coefficients[[1]]
> a
[1] 6.515211
> b= fit$coefficients[[2]]
> b
[1] -0.04052554
> res = y - (a + b * x)
> plot(x, res)
> abline(h = 0)
```



(b) The scatter plot suggests a linear fit. The residual plot may show a quadratic trend (which would make the linear fit invalid), but it is hard to tell with only 5 data points.

IF the fit is valid, then $\hat{y} = 6.515211 - (0.04052554)(60) \approx 4.08$.