№1

$\xi \sim R(0\,\theta)$ $\quad \theta > 0$ $\quad [0\,\theta]$

выборка $\overline{x_n}$

1) $\widetilde{\theta}_1 = 2\overline{x} = \frac{2}{n}\sum_{i=1}^{n} x_i$

2) $\widetilde{\theta}_2 = x_{min}$

3) $\widetilde{\theta}_3 = x_{max}$

4) $\widetilde{\theta}_4 = x_1 + \frac{1}{n-1}\sum_{k=2}^{n} x_k$

Числ хар

$\cdot\ M\xi = \int_{-\infty}^{\infty} x\, dF(x\theta) = \int_0^{\theta} x \frac{1}{\theta} dx = \frac{\theta}{2}$

$p(x\theta) = \frac{1}{\theta}\{(0\,\theta)\}$

$\cdot\ M\xi^2 = \int_0^{\theta} x^2 \frac{1}{\theta} dx = \frac{\theta^2}{3}$

$\cdot\ D\xi = M\xi^2 - (M\xi)^2 = \frac{\theta^2}{12}$

Проверка

① $\cdot\ \widetilde{\theta}_1 \quad \forall \theta > 0 \quad M\widetilde{\theta}_1 = M\left(\frac{2}{n}\sum x_i\right) = \frac{2}{n}\sum Mx_i =$

$= \frac{2}{n} n M\xi = \theta \Rightarrow \widetilde{\theta}_1$ - несмещ по опр.

$\cdot\ D\widetilde{\theta}_1 = D\left(2\frac{1}{n}\sum x_i\right) = \frac{4}{n^2}\sum Dx_i = \frac{4}{n^2} n D\xi = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0,$

$\forall \theta > 0 \Rightarrow$ сход по доп усл.

② $\widetilde{\theta}_2$

$\cdot\ \forall \theta > 0 \quad M\widetilde{\theta}_2 = M x_{min} = \int_0^{\theta} x\, n \left(1 - \frac{x}{\theta}\right)^{n-1} \frac{1}{\theta} dx =$

/* $\xi_{min} \sim 1 - (1 - F(x))^n$

$\varphi(x) = n(1 - F(x))^{n-1} F'(x)$

$\varphi(x) = n\left(1 - \frac{x}{\theta}\right)^{n-1} \frac{1}{\theta}\{(0\,\theta)\}$ */

$= -\int_1^0 n(1-t)t^{n-1}\theta\, dt =$

$= n\theta \int_0^1 (t^{n-1} - t^n) dt = \frac{\theta}{n+1}$

т.е смещена

Поправим $\widetilde{\theta}_2' = (n+1)x_{min}$ $\Rightarrow$

$\Rightarrow M\widetilde{\theta}_2' = (n+1)\,M x_{min} = \theta$

- $D\widetilde{\theta}_2' = D\left((n+1)x_{min}\right) = (n+1)^2 D x_{min}$

$$M x_{min}^2 = \int_0^\theta x^2 n \underbrace{\left(1-\frac{x}{\theta}\right)}_{t}^{n-1} \frac{1}{\theta}\,dx = \int_0^1 n\theta^2(1-t)^2 t^{n-1}\,dt =$$

$$= n\theta^2 \int_0^1 \left(t^{n+1} - 2t^n + t^{n+1}\right)dt = n\theta^2\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) =$$

$$= \frac{\theta^2 x}{(n+1)(n+2)}$$

$$D x_{min} = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \theta^2 \frac{n}{(n+1)^2(n+2)}$$

$$D\widetilde{\theta}_2' = \frac{n\theta^2}{n+2} \not\to_{n\to\infty} 0$$

$\Rightarrow$ Дост. усл. не подойд $\Rightarrow$

$\Rightarrow$ идем по опред

- $$P\left(|\widetilde{\theta}_2' - \theta| \geq \varepsilon\right) \geqslant P\left(\widetilde{\theta}_2' \geqslant \theta + \varepsilon\right) = P\left((n+1)x_{min} \geqslant \theta + \varepsilon\right) =$$

$$= P\left(x_{min} \geqslant \frac{\theta+\varepsilon}{n+1}\right) = 1 - \Phi\left(\frac{\theta+\varepsilon}{n+1}\right) =$$

$$= 1 - \left[1 - \left(1 - \frac{\theta+\varepsilon}{\theta(n+1)}\right)^n\right] = \left(1 - \frac{\theta+\varepsilon}{\theta(n+1)}\right)^n \underset{n\to\infty}{\to} e^{-\frac{\theta+\varepsilon}{\theta}} \searrow 0$$

$\Rightarrow$ не яв-ся состоятельной

- $$P\left(|x_{min} - \theta| \geq \varepsilon\right) = P\left(x_{min} \leq \theta - \varepsilon\right) = \Phi\left(\theta - \varepsilon\right) =$$

$$= 1 - \left(1 - F(\theta - \varepsilon)\right)^n = 1 - \left(1 - \frac{\theta-\varepsilon}{\theta}\right)^n =$$

$$= 1 - \left(\frac{\varepsilon}{\theta}\right)^n \underset{n\to\infty}{\to} 1 \qquad \text{т.к } \exists \varepsilon > 0\ \exists \theta > 0 : \frac{\varepsilon}{\theta} < 1$$

$\Rightarrow$ не яв-ся состоятельной

③ $\tilde{\theta}_3 = x_{max}$

- $M\tilde{\theta}_3 = Mx_{max} =$ /* $x_{max} \sim \underbrace{(F(x))^n}_{\Psi}$ */

$$= \int_0^\theta x\, n \left(\frac{x}{\theta}\right)^{n-1} \frac{1}{\theta}\, dx = \text{/* } \psi(x) = n\left(\frac{x}{\theta}\right)^{n-1}\frac{1}{\theta}\ \{(0\ \theta)\} \text{ */}$$

$$= \frac{n}{n+1}\theta$$

Подправим $\tilde{\theta}_3' = \frac{n+1}{n} x_{max} \Rightarrow M\tilde{\theta}_3' = \frac{n+1}{n} Mx_{max} = \theta$

– несмещен.

- $D\tilde{\theta}_3' = \left(\frac{n+1}{n}\right)^2 D x_{max}$

$$Mx_{max}^2 = \int_0^\theta x^2\, n \left(\frac{x}{\theta}\right)^{n-1} \frac{1}{\theta}\, dx = \frac{n\theta^2}{n+2}$$

$$Dx_{max} = \theta^2\left(\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right) = \theta^2 \cdot \left(\frac{n}{(n+2)(n+1)^2}\right)$$

$$D\tilde{\theta}_3 = \left(\frac{n+1}{n}\right)^2 D x_{max} = \frac{\theta^2}{n}\cdot\frac{1}{n+2} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ состоят по дост усл.

Исслед $\tilde{\theta}_3'$ на состоят по опред

- $P(|\tilde{\theta}_3' - \theta| \geq \varepsilon) = P(\tilde{\theta}_3' \leq \theta - \varepsilon) + P(\tilde{\theta}_3' \geq \theta + \varepsilon) =$

$= P\left(x_{max} \leq \frac{n}{n+1}(\theta - \varepsilon)\right) + P\left(x_{max} \geq \frac{n}{n+1}(\theta + \varepsilon)\right) =$

$= 1 - \underbrace{\left(\frac{n(\theta+\varepsilon)}{\theta(n+1)}\right)^n}_{A} + \underbrace{\left(\frac{n}{\theta}\frac{(\theta-\varepsilon)}{(n+1)}\right)^n}_{B}$

A: Если $\frac{n}{n+1}(\theta+\varepsilon) > \theta \Rightarrow$ т.к $x_{max} \leq \theta \Rightarrow P(-1-) = 1-1 = 0$

Если $\frac{n}{n+1}(\theta+\varepsilon) < \theta \Rightarrow n < \frac{\theta}{\varepsilon} \Rightarrow$

$\Rightarrow \exists N = \frac{\theta}{\varepsilon}$: $\forall n > N \hookrightarrow \frac{n}{n+1}(\theta+\varepsilon) > \theta \Rightarrow$

$\Rightarrow P(-1-) = 1 - 1 = 0$

B: Если $\varepsilon < \theta \Rightarrow \underbrace{\left(1 - \frac{\varepsilon}{\theta}\right)^n}_{e} \xrightarrow[n\to\infty]{} 0$

Если $\varepsilon > \theta \Rightarrow \left.\begin{array}{l} \frac{n}{n+1}(\theta - \varepsilon) < 0 \\ x_{max} \geq 0 \end{array}\right| \Rightarrow P\left(x_{max} \leq \frac{n}{n+1}(\theta-\varepsilon)\right) = 0$

$\Rightarrow$ состоятельна

- Исслед $\tilde{\theta}_3$ на состоят по опред

$P(|\tilde{\theta}_3 - \theta| \geq \varepsilon) = \underbrace{P(x_{max} \geq \theta + \varepsilon)}_{0} + P(x_{max} \leq \theta - \varepsilon) =$

$= \left(\frac{\theta - \varepsilon}{\theta}\right)^n \Rightarrow$ состоятельна

Если $\varepsilon < \theta \Rightarrow \left(\frac{\theta-\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 0$ Если $\varepsilon > \theta \to P(-1-) = 0$

т.к $x_{max} \geq 0$

④ $\tilde{\theta}_4 = X_1 + \frac{1}{n-1}\sum_{i=2}^{n} X_i$

- $M\tilde{\theta}_4 = MX_1 + \frac{1}{n-1}\sum_{i=2}^{n} MX_i = M\xi + \frac{1}{n-1}\sum_{i=2}^{n} M\xi =$

$= \frac{\theta}{2} + \frac{\theta}{2} = \theta$

$\Rightarrow$ несмещен.

- $D\tilde{\theta}_4 = DX_1 + \frac{1}{(n-1)^2}\sum_{i=2}^{n} DX_i = \frac{\theta^2}{12} + \frac{1}{n-1}\frac{\theta^2}{12} \not\to 0$ $(n \to \infty)$

$\Rightarrow$ Дост усл не выполн $\Rightarrow$ смотр на опред

- $X_1 \xrightarrow{P} \xi$; $\frac{1}{n-1}\sum_{i=2}^{n} X_i \xrightarrow{P} \frac{\theta}{2}$

$X_1 + \frac{1}{n-1}\sum_{i=2}^{n} X_i \xrightarrow{P} \xi + \frac{\theta}{2}$ $\Rightarrow$ не явл состоят.

Наиболее эффект

$$\tilde{\theta}_3' = \frac{n+1}{n} x_{max} \qquad \tilde{\theta}_1 = 2\bar{x}$$

$$D\tilde{\theta}_3' = \frac{\theta^2}{(n+2)n}$$

$$D\tilde{\theta}_1 = \frac{\theta^2}{3n}$$

$\Rightarrow n^2 + 2n > 3n$ при $n > 1$ $\Rightarrow$

$\Rightarrow \tilde{\theta}_3'$ более эффект