# Course 8: 26.11.2020

## 3.3 The matrix of a list of vectors

In the previous section, we have seen a matrix as a list of row-vectors. Now we discuss a converse, namely we define the matrix associated to a list of vectors, with respect to a basis.

**Definition 3.3.1** Let $V$ be a vector space over $K$, $B = (v_1, \dots, v_n)$ a basis of $V$ and $X = (u_1, \dots, u_m)$ a list of vectors in $V$. Let

$$\begin{cases} u_1 = a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ u_2 = a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \dots\dots\dots\dots\dots\dots \\ u_m = a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{cases}$$

be the unique writings of the vectors in $X$ as linear combinations of vectors of the basis $B$, for some $a_{ij} \in K$. The *matrix of the list of vectors* $X$ in the basis $B$ is the matrix having as its rows the coordinates the vectors in $X$ in the basis $B$, that is,

$$[X]_B = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

**Example 3.3.2** Consider the canonical basis $B = (e_1, e_2, e_3, e_4)$ and the list $X = (u_1, u_2, u_3)$ in the canonical real vector space $\mathbb{R}^4$, where $u_1 = (1, 2, 3, 4)$, $u_2 = (5, 6, 7, 8)$ and $u_3 = (9, 10, 11, 12)$. Since the coordinates of a vector in the canonical basis are just its components, we get

$$[X]_B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}.$$

Now we give a theorem, whose proof will be omitted and which allows one to determine the dimension of the subspace generated by a list of vectors.

**Theorem 3.3.3** *Let $V$ be a vector space over $K$, $B = (v_1, \dots, v_n)$ a basis of $V$ and $X = (u_1, \dots, u_m)$ a list of vectors in $V$ having the matrix $A$ in the basis $B$. Then:*

*(i)* $\dim\langle X\rangle = \operatorname{rank}(A)$.

*(ii) A basis of $\langle X\rangle$ is the list of non-zero row-vectors $(c_1, \dots, c_r)$ of an echelon form $C$ equivalent to $A$.*

**Example 3.3.4** Let us determine the dimensions of the subspaces $S$, $T$, $S + T$ and $S \cap T$ of the real vector space $\mathbb{R}^4$, where

$$S = \langle(-3, 5, -1, 1), (-1, 1, 0, 1), (1, 1, -1, -3)\rangle,$$

$$T = \langle(1, 0, 2, 0), (2, 1, -1, 2)\rangle.$$

One can easily show that the ranks of the matrices in the canonical basis corresponding to the vectors from $S$ and from $T$ respectively are both 2. Hence $\dim S = \dim T = 2$.

Furthermore, we have $S + T = \langle S \cup T\rangle$. We write the matrix of $S \cup T$ in the canonical basis and we have

$$\begin{pmatrix} -3 & 5 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -3 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & -3 \\ -1 & 1 & 0 & 1 \\ -3 & 5 & -1 & 1 \\ 1 & 0 & 2 & 0 \\ 2 & 1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & -3 \\ 0 & 2 & -1 & -2 \\ 0 & 8 & -4 & -8 \\ 0 & -1 & 3 & 3 \\ 0 & -1 & 1 & 8 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & -3 \\ 0 & -1 & 3 & 3 \\ 0 & 2 & -1 & -2 \\ 0 & 2 & -1 & -2 \\ 0 & -1 & 1 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & -3 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & -2 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & -3 \\ 0 & -1 & 3 & 3 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & \frac{33}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then by Theorem 3.3.3, $\dim(S+T) = 4$ and a basis of $S+T$ consists of the non-zero row-vectors from the echelon form, that is, $((1,1,-1,-3),(0,-1,3,3),(0,0,5,4),(0,0,0,\frac{33}{5}))$. Now by the second dimension formula, it follows that $\dim(S\cap T) = \dim S + \dim T - \dim(S+T) = 2+2-4 = 0$.

Now we are going to define the matrix of a vector in a basis of a vector space. Even if one might expect to define it as a row-matrix, by considering a single vector list, it is more convenient to define it as a column-matrix for our purposes concerning linear maps in order to avoid formulas involving transposes.

**Definition 3.3.5** Let $V$ be a vector space over $K$, $v \in V$ and $B = (v_1,\dots,v_n)$ a basis of $V$. If $v = k_1v_1 + \dots + k_nv_n$ $(k_1,\dots,k_n \in K)$ is the unique writing of $v$ as a linear combination of the vectors of the basis $B$, then the *matrix of the vector* $v$ in the basis $B$ is $[v]_B = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}$.

## 3.4 The matrix of a linear map

**Definition 3.4.1** Let $f : V \to V'$ be a $K$-linear map, $B = (v_1,\dots,v_n)$ a basis of $V$ and $B' = (v'_1,\dots,v'_m)$ a basis of $V'$. Then we can uniquely write the vectors in $f(B)$ as linear combinations of the vectors of the basis $B'$, say

$$\begin{cases} f(v_1) = a_{11}v'_1 + a_{21}v'_2 + \dots + a_{m1}v'_m \\ f(v_2) = a_{12}v'_1 + a_{22}v'_2 + \dots + a_{m2}v'_m \\ \dots\dots\dots\dots\dots\dots \\ f(v_n) = a_{1n}v'_1 + a_{2n}v'_2 + \dots + a_{mn}v'_m \end{cases}$$

for some $a_{ij} \in K$. Then the *matrix of the $K$-linear map* $f$ in the bases $B$ and $B'$ is the matrix having as its columns the coordinates of the vectors of $f(B)$ in the basis $B'$, that is,

$$[f]_{BB'} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

If $V = V'$ and $B = B'$, then we simply denote $[f]_B = [f]_{BB'}$.

**Remark 3.4.2** We have to emphasize that we put the coordinates on the columns of the matrix of a linear map and not on the rows as we did for the matrix of a list of vectors.

**Example 3.4.3** Consider the $\mathbb{R}$-linear map $f : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$f(x,y,z,t) = (x+y+z, y+z+t, z+t+x), \ \forall (x,y,z,t) \in \mathbb{R}^4.$$

Let $E = (e_1,e_2,e_3,e_4)$ and $E' = (e'_1,e'_2,e'_3)$ be the canonical bases in $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively. Since

$$\begin{cases} f(e_1) = f(1,0,0,0) = (1,0,1) = e'_1 + e'_3 \\ f(e_2) = f(0,1,0,0) = (1,1,0) = e'_1 + e'_2 \\ f(e_3) = f(0,0,1,0) = (1,1,1) = e'_1 + e'_2 + e'_3 \\ f(e_4) = f(0,0,0,1) = (0,1,1) = e'_2 + e'_3 \end{cases}$$

it follows that the matrix of $f$ in the bases $E$ and $E'$ is

$$[f]_{EE'} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

**Theorem 3.4.4** *Let $f : V \to V'$ be a $K$-linear map, $B = (v_1, \dots, v_n)$ a basis of $V$, $B' = (v'_1, \dots, v'_m)$ a basis of $V'$ and $v \in V$. Then*

$$[f(v)]_{B'} = [f]_{BB'} \cdot [v]_B \,.$$

*Proof.* Let $[f]_{BB'} = (a_{ij}) \in M_{mn}(K)$. Let $v = \sum_{j=1}^n k_j v_j$ and $f(v) = \sum_{i=1}^m k'_i v'_i$ for some $k_i, k'_i \in K$. On the other hand, using the definition of the matrix of $f$ in the bases $B$ and $B'$, we have

$$f(v) = f\Big(\sum_{j=1}^n k_j v_j\Big) = \sum_{j=1}^n k_j f(v_j) = \sum_{j=1}^n k_j \Big(\sum_{i=1}^m a_{ij} v'_i\Big) = \sum_{i=1}^m \Big(\sum_{j=1}^n a_{ij} k_j\Big) v'_i \,.$$

But the writing of $f(v)$ as a linear combination of the vectors of the basis $B'$ is unique, hence we must have $k'_i = \sum_{j=1}^n a_{ij} k_j$ for every $i \in \{1, \dots, m\}$. Therefore, $[f(v)]_{B'} = [f]_{BB'} \cdot [v]_B$. □

**Definition 3.4.5** Let $f : V \to V'$ be a $K$-linear map. Then the *rank* of $f$ is defined as

$$\operatorname{rank}(f) = \dim(\operatorname{Im} f).$$

Now we give a connection between the ranks of a linear map and of its matrix in a pair of bases.

**Theorem 3.4.6** *Let $f : V \to V'$ be a $K$-linear map. Then*

$$\operatorname{rank}(f) = \operatorname{rank}([f]_{BB'}) \,,$$

*where $B$ and $B'$ are any bases of $V$ and $V'$ respectively.*

*Proof.* Let $B = (v_1, \dots, v_n)$ and $[f]_{BB'} = A$. Using our results relating ranks and dimensions, we have

$$\begin{aligned}\operatorname{rank}(f) &= \dim(\operatorname{Im} f) = \dim f(V) = \dim f(\langle v_1, \dots, v_n \rangle) \\ &= \dim\langle f(v_1), \dots, f(v_n)\rangle = \operatorname{rank}({}^t A) = \operatorname{rank}(A) = \operatorname{rank}([f]_{BB'}) \,.\end{aligned}$$

Now take some other bases $B_1 = (u_1, \dots, u_n)$ of $V$ and $B'_1$ of $V'$ and denote $[f]_{B_1 B'_1} = A_1$. Then

$$\begin{aligned}\operatorname{rank}([f]_{B_1 B'_1}) &= \operatorname{rank}(A_1) = \operatorname{rank}({}^t A_1) = \dim\langle f(u_1), \dots, f(u_n)\rangle \\ &= \dim(\operatorname{Im} f) = \dim\langle f(v_1), \dots, f(v_n)\rangle = \operatorname{rank}([f]_{BB'}) \,.\end{aligned}$$

□

**Remark 3.4.7** Notice that the rank of a linear map does not depend on the pair of bases in which we write its matrix. Also notice that, considering matrices of a linear map in different pairs of bases, their ranks are the same. Some other connection between matrices of a linear map in different pairs of bases will be discussed in the next section.

**Example 3.4.8** Consider the $\mathbb{R}$-linear map $f : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$f(x, y, z, t) = (x + y + z, y + z + t, z + t + x) \,,\ \forall (x, y, z, t) \in \mathbb{R}^4 \,.$$

Let $E = (e_1, e_2, e_3, e_4)$ and $E' = (e'_1, e'_2, e'_3)$ be the canonical bases in $\mathbb{R}^4$ and $\mathbb{R}^3$ respectively. Using Example 3.4.3 it follows that

$$[f]_{EE'} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} .$$

Now by Theorem 3.4.6 it follows that $\operatorname{rank}(f) = \operatorname{rank}([f]_{EE'}) = 3$.

We end this section with a key result in Linear Algebra, connecting linear maps and matrices.

**Theorem 3.4.9** *Let $V$, $V'$ and $V''$ be vector spaces over $K$ with $\dim V = n$, $\dim V' = m$ and $\dim V'' = p$ and let $B$, $B'$ and $B''$ be bases of $V$, $V'$ and $V''$ respectively. Then $\forall f, g \in Hom_K(V, V')$, $\forall h \in Hom_K(V', V'')$ and $\forall k \in K$, we have*

$$[f + g]_{BB'} = [f]_{BB'} + [g]_{BB'} \,,$$

$$[kf]_{BB'} = k \cdot [f]_{BB'} \,,$$

$$[h \circ f]_{BB''} = [h]_{B'B''} \cdot [f]_{BB'} \,.$$

*Proof.* Let $[f]_{BB'} = (a_{ij}) \in M_{mn}(K)$, $[g]_{BB'} = (b_{ij}) \in M_{mn}(K)$ and $[h]_{B'B''} = (c_{ki}) \in M_{pm}(K)$. Then

$$f(v_j) = \sum_{i=1}^{m} a_{ij}v'_i\,, \quad g(v_j) = \sum_{i=1}^{m} b_{ij}v'_i\,, \quad h(v'_i) = \sum_{k=1}^{p} c_{ki}v''_k$$

$\forall j \in \{1,\dots,n\}$ and $\forall i \in \{1,\dots,m\}$.

Then $\forall k \in K$ and $\forall j \in \{1,\dots,n\}$ we have

$$(f+g)(v_j) = f(v_j) + g(v_j) = \sum_{i=1}^{m} a_{ij}v'_i + \sum_{i=1}^{m} b_{ij}v'_i = \sum_{i=1}^{m}(a_{ij}+b_{ij})v'_i\,,$$

$$(kf)(v_j) = kf(v_j) = k\cdot\left(\sum_{i=1}^{m} a_{ij}v'_i\right) = \sum_{i=1}^{m}(ka_{ij})v'_i\,,$$

hence $[f+g]_{BB'} = [f]_{BB'} + [g]_{BB'}$ and $[kf]_{BB'} = k\cdot[f]_{BB'}$.

Finally, $\forall j \in \{1,\dots,n\}$ we have

$$(h\circ f)(v_j) = h(f(v_j)) = h\left(\sum_{i=1}^{m} a_{ij}v'_i\right) = \sum_{i=1}^{m} a_{ij}h(v'_i) = \sum_{i=1}^{m} a_{ij}\left(\sum_{k=1}^{p} c_{ki}v''_k\right) = \sum_{k=1}^{p}\sum_{i=1}^{m}(c_{ki}a_{ij})v''_k\,,$$

hence $[h\circ f]_{BB''} = [h]_{B'B''}\cdot[f]_{BB'}$. □

**Theorem 3.4.10** *Let $V$ and $V'$ be vector spaces over $K$ with $\dim V = n$ and $\dim V' = m$ and let $B$ and $B'$ be bases of $V$ and $V'$ respectively. Then the map*

$$\varphi : Hom_K(V,V') \to M_{mn}(K), \quad \varphi(f) = [f]_{BB'}\,,\ \forall f \in Hom_K(V,V')$$

*is an isomorphism of vector spaces.*

*Proof.* One may show that $Hom_K(V,V')$ is a vector space over $K$ with respect to the following addition and scalar multiplication: $\forall f,g \in Hom_K(V,V')$ and $\forall k \in K$, $f+g, k\cdot f \in Hom_K(V,V')$, where $\forall x \in V$,

$$(f+g)(x) = f(x) + g(x)\,,$$
$$(kf)(x) = kf(x)\,.$$

Also, $M_{mn}(K)$ is a vector space over $K$. By Theorem 3.4.9 it follows that $\varphi$ is a $K$-linear map.

Finally, let us prove that $\varphi$ is bijective. Let $f,g \in Hom_K(V,V')$ be such that $\varphi(f) = \varphi(g)$. Then $[f]_{BB'} = [g]_{BB'} = (a_{ij}) \in M_{mn}(K)$, hence $f(v_j) = a_{1j}v'_1 + a_{2j}v'_2 + \dots + a_{mj}v'_m = g(v_j)$, $\forall j \in \{1,\dots,n\}$. We have seen that two $K$-linear maps are equal if and only if they have the same values at all vectors of a basis. Hence $f = g$, which shows that $\varphi$ is injective. Now let $A = (a_{ij}) \in M_{mn}(K)$, seen as a list of column-vectors $(a^1,\dots,a^n)$, where $a^j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{pmatrix}$. Consider $B = (v_1,\dots,v_n)$ and $B' = (v'_1,\dots,v'_m)$ and define a $K$-linear map $f : V \to V'$ on the basis of the domain by $f(v_j) = a_{1j}v'_1 + \dots + a_{mj}v'_m$, $\forall j \in \{1,\dots,n\}$. Then $\varphi(f) = [f]_{BB'} = (a_{ij}) = A$. Thus, $\varphi$ is surjective. □

**Remark 3.4.11** The extremely important isomorphism given in Theorem 3.4.10 allows us to work with matrices instead of linear maps, which is much simpler from a computational point of view.

**Theorem 3.4.12** *Let $V$ be a vector space over $K$ with $\dim V = n$ and let $B$ be a basis of $V$. Then the map*

$$\varphi : End_K(V) \to M_n(K), \quad \varphi(f) = [f]_B\,,\ \forall f \in End_K(V)$$

*is an isomorphism of vector spaces and of rings.*

*Proof.* Note that $(End_K(V),+,\circ)$ and $(M_n(K),+,\cdot)$ are rings. The required isomorphisms follow by Theorem 3.4.10. □

**Corollary 3.4.13** *Let $V$ be a vector space over $K$ and $f \in End_K(V)$. Then*

$$f \in Aut_K(V) \iff \det[f]_B \neq 0\,,$$

*where $B$ is any basis of $V$.*

*Proof.* Let $B$ a basis of $V$. By Theorem 3.4.12, $f \in Aut_K(V) \iff f$ is invertible in the ring $(End_K(V),+,\circ) \iff [f]_B$ is invertible in the ring $(M_n(K),+,\cdot) \iff \det[f]_B \neq 0$. □

# Extra: Image transformations

Suppose that we have a 2D-image that we want to rotate counterclockwise with $\theta$ degrees around the origin. By such a rotation, the point of coordinates $(1,0)$ becomes the point of coordinates $(\cos\theta, \sin\theta)$, while the point of coordinates $(0,1)$ becomes the point of coordinates $(-\sin\theta, \cos\theta)$.

We look for an $\mathbb{R}$-linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the following conditions:

$$f(1,0) = (\cos\theta, \sin\theta)$$
$$f(0,1) = (-\sin\theta, \cos\theta).$$

Recall that every linear map is determined by its values at the elements of a basis (the canonical basis in our case). Hence the matrix of the linear map $f$ in the canonical basis $E$ of the canonical real vector space $\mathbb{R}^2$ is:

$$[f]_E = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For any point $v = (x,y) \in \mathbb{R}^2$ of a 2D-image, its corresponding point in the rotated image is computed as $f(v) = (x', y') \in \mathbb{R}^2$, where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = [f(v)]_E = [f]_E \cdot [v]_E = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

For instance, for a counterclockwise rotation of $90^0$ around the origin one has the matrix:

$$[f]_E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$