

# ALGEBRA - Course 1: 01.10.2020

## 0.0 Coordinates

- **Structure:**

Chapter 1: Preliminaries

Chapter 2: Vector Spaces

Chapter 3: Matrices and Linear Systems

Chapter 4: Introduction to Coding Theory

- **Bibliography:**

1. N. Both, S. Crivei, *Culegere de probleme de algebră*, Lito UBB Cluj-Napoca, 1996.
2. G. Călugăreanu, *Lecții de algebră liniară*, Lito UBB, Cluj-Napoca, 1995.
3. S. Crivei, *Basic abstract algebra*, Casa Cărții de Știință, Cluj-Napoca, 2002, 2003.
4. J. Gilbert, L. Gilbert, *Elements of Modern Algebra*, PWS-Kent, Boston, 1992.
5. W.J. Gilbert, W.K. Nicholson, *Modern Algebra with Applications*, John Wiley, 2004.
6. R. Lidl, G. Pilz, *Applied Abstract Algebra*, Springer-Verlag, 1998.
7. I. Purdea, C. Pelea, *Probleme de algebră*, Eikon, Cluj-Napoca, 2008.

- **Course:**

Course materials will be uploaded on the Microsoft Teams platform.

Students may get up to 1 bonus point from course projects to the final grade: up to 5 projects, each for 0.2 points [you will receive details in due time...].

- **Seminar:**

Minimum attendance: 75% for seminar classes in order to be allowed to participate in the second partial exam.

Problems for the next week will be uploaded on the Microsoft Teams platform after the course.

Students may get up to 0.5 bonus points from seminar to the final grade: 5 problems solved during the seminar, each for 0.1 points [you will receive details during seminars...].

- **Exam:**

Written partial exams in Week 8 (Chapters 1-2) and Week 14 (Chapters 3-4).

The final grade is computed as follows:

$$G = 1 + P_1 + P_2 + B,$$

where:

$G$  = the final grade

$P_1$  = the grade from the first partial exam (max. 4)

$P_2$  = the grade from the second partial exam (max. 5)

$B$  = bonus points from seminar or course (max. 1.5)

Students may not pass the exam unless they participate in the second partial exam.

# Chapter 1 PRELIMINARIES

## 1.1 Relations

**Definition 1.1.1** A triple  $r = (A, B, R)$ , where  $A, B$  are sets and

$$R \subseteq A \times B = \{(a, b) \mid a \in A, b \in B\},$$

is called a *(binary) relation*.

The set  $A$  is called the *domain*, the set  $B$  is called the *codomain* and the set  $R$  is called the *graph* of the relation  $r$ .

If  $A = B$ , then the relation  $r$  is called *homogeneous*.

If  $(a, b) \in R$ , then we sometimes write  $a r b$  and we say that  $a$  has the relation  $r$  to  $b$  or  $a$  and  $b$  are related with respect to the relation  $r$ .

**Definition 1.1.2** Let  $r = (A, B, R)$  be a relation and let  $X \subseteq A$ . Then the set

$$r(X) = \{b \in B \mid \exists x \in X : x r b\}$$

is called the *relation class of  $X$  with respect to  $r$* . If  $x \in X$ , then we denote

$$r < x > = r(\{x\}) = \{b \in B \mid x r b\}.$$

**Remark 1.1.3** One may represent relations (defined on finite sets) by diagrams. E.g., let  $r = (A, B, R)$ , where  $A = \{1, 2, 3\}$ ,  $B = \{1, 2\}$  and  $R = \{(1, 1), (1, 2), (3, 1)\}$ . As in the case of functions, one may draw the two sets  $A$  and  $B$ , and arrows between the elements related by  $R$ , namely arrows from 1 to 1, from 1 to 2 and from 3 to 1. Also note that  $r < 1 > = \{1, 2\} = r(A)$ .

**Example 1.1.4** (a) Let  $C$  be the set of all children and  $P$  be the set of all parents. Then we may define the relation  $r = (C, P, R)$ , where

$$R = \{(c, p) \in C \times P \mid c \text{ is a child of } p\}.$$

(b) The triple  $r = (\mathbb{R}, \mathbb{R}, R)$ , where

$$R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \leq y\}$$

is a homogeneous relation, called the *inequality relation* on  $\mathbb{R}$ . We have  $r < 1 > = [1, \infty)$  and  $r([1, 2]) = [1, \infty)$ .

(c) Other examples from Number Theory (divisibility on  $\mathbb{N}$ :  $r < 3 > = \{3k \mid k \in \mathbb{N}\}$ ), Geometry (parallelism of lines, perpendicularity of lines, congruence of triangles, similarity of triangles).

(d) Let  $A$  and  $B$  be two sets. Then the triples

$$o = (A, B, \emptyset), \quad u = (A, B, A \times B)$$

are relations, called the *void relation* and the *universal relation* respectively.

(e) Let  $A$  be a set. Then the triple  $\delta_A = (A, A, \Delta_A)$ , where

$$\Delta_A = \{(a, a) \mid a \in A\}$$

is a relation called the *equality relation* on  $A$ .

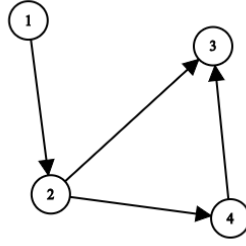
(f) Every function is a relation. Indeed, a function  $f : A \rightarrow B$  is determined by its domain  $A$ , its codomain  $B$  and its graph

$$G_f = \{(x, y) \in A \times B \mid y = f(x)\}.$$

Then the triple  $(A, B, G_f)$  is a relation.

(g) Every directed graph is a relation. Indeed, a directed graph  $(V, E)$  consists of a set  $V$  of vertices and a set  $E$  of directed edges ("arrows") between vertices. We may identify each directed edge with a pair in  $V \times V$ , where the first and the second component are respectively the starting and the ending vertex of that directed edge. Denote by  $P$  the set of those pairs. Then the triple  $(V, V, P)$  is a relation.

For instance, the directed graph



may be seen as a relation  $(A, A, R)$ , where  $A = \{1, 2, 3, 4\}$  and  $R = \{(1, 2), (2, 3), (2, 4), (4, 3)\}$ .

## 1.2 Functions

**Definition 1.2.1** A relation  $r = (A, B, R)$  is called a *function* if  $\forall a \in A, |r < a >| = 1$ .

In other words, a relation  $r$  is a function if and only if every element of the domain has the relation  $r$  to exactly one element of the codomain.

In what follows, if  $f = (A, B, F)$  is a function, we will mainly use the classical notation for a function, namely  $f : A \rightarrow B$  or sometimes  $A \xrightarrow{f} B$ . The unique element of the set  $f < a >$  will be denoted by  $f(a)$ . Then we have  $(a, b) \in F \iff f(a) = b$ .

**Example 1.2.2** Let  $A = \{1, 2, 3\}$ ,  $B = \{1, 2\}$  and let  $r = (A, B, R)$ ,  $s = (A, B, S)$ ,  $t = (A, B, T)$  be the relations having the graphs

$$R = \{(1, 1), (2, 1), (3, 2)\},$$

$$S = \{(1, 2), (3, 1)\},$$

$$T = \{(1, 1), (1, 2), (2, 1), (3, 2)\}.$$

Since  $|r < a >| = 1$  for every  $a \in A$ , the relation  $r$  is a function. But  $s$  and  $t$  are not functions, because, for instance, we have  $|s < 2 >| = 0$  and  $|t < 1 >| = 2$ .

**Homework:** recall from High School the definitions and the basic properties of injective, surjective and bijective functions.

## 1.3 Equivalence relations and partitions

**Definition 1.3.1** A homogeneous relation  $r = (A, A, R)$  on  $A$  is called:

- *reflexive* (r) if:  $\forall x \in A, x r x$ ;
- *transitive* (t) if:  $x, y, z \in A, x r y \text{ and } y r z \implies x r z$ ;
- *symmetric* (s) if:  $x, y \in A, x r y \implies y r x$ .

A homogeneous relation  $r = (A, A, R)$  is called an *equivalence relation* if  $r$  has the properties (r), (t) and (s).

**Example 1.3.2** (a) The equality relation  $\delta_A$  on a set  $A$  has all 3 properties, hence  $\delta_A$  is an equivalence relation on  $A$ .

(b) The similarity of triangles is an equivalence relations on the set of all triangles.

(c) The inequality relation “ $\leq$ ” on  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$  or  $\mathbb{R}$  has (r) and (t), but not (s). Hence it is not an equivalence relation on the corresponding set.

**Definition 1.3.3** Let  $A$  be a non-empty set. Then a family  $(A_i)_{i \in I}$  of non-empty subsets of  $A$  is called a *partition* of  $A$  if:

- (i) The family  $(A_i)_{i \in I}$  covers  $A$ , that is,

$$\bigcup_{i \in I} A_i = A;$$

(ii) The  $A_i$ 's are pairwise disjoint, that is,

$$i, j \in I, i \neq j \implies A_i \cap A_j = \emptyset.$$

**Example 1.3.4** (a) Let  $A = \{1, 2, 3, 4, 5\}$  and  $A_1 = \{1, 2, 3\}$ ,  $A_2 = \{4\}$ ,  $A_3 = \{5\}$ . Then  $\{A_1, A_2, A_3\}$  is a partition of  $A$ .

(b) Let  $A_1$  be the set of even integers and  $A_2$  the set of odd integers. Then  $\{A_1, A_2\}$  is a partition of  $\mathbb{Z}$ .

(c) Let  $A$  be a set. Then  $\{\{a\} \mid a \in A\}$  and  $\{A\}$  are partitions of  $A$ .

Denote by  $E(A)$  the set of all equivalence relations and by  $P(A)$  the set of all partitions on a set  $A$ .

**Definition 1.3.5** Let  $r \in E(A)$ .

The relation class  $r < x >$  of an element  $x \in A$  with respect to  $r$  is called the *equivalence class of  $x$  with respect to  $r$* , while the element  $x$  is called a *representative* of  $r < x >$ .

The set  $A/r = \{r < x > \mid x \in A\}$ , which is the set of all equivalence classes of elements of  $A$  with respect to  $r$ , is called the *quotient set of  $A$  by  $r$* .

The following theorem establishes the fundamental connection between equivalence relations and partitions.

**Theorem 1.3.6** (i) Let  $r \in E(A)$ . Then

$$A/r = \{r < x > \mid x \in A\} \in P(A).$$

(ii) Let  $\pi = (A_i)_{i \in I} \in P(A)$  and define the relation  $r_\pi$  on  $A$  by

$$x r_\pi y \iff \exists i \in I : x, y \in A_i.$$

Then  $r_\pi \in E(A)$ .

(iii) Let  $F : E(A) \rightarrow P(A)$  be defined by  $F(r) = A/r$ ,  $\forall r \in E(A)$ . Then  $F$  is a bijection, whose inverse is  $G : P(A) \rightarrow E(A)$ , defined by  $G(\pi) = r_\pi$ ,  $\forall \pi \in P(A)$ .

**Example 1.3.7** (a) Let  $A = \{1, 2, 3\}$  and let  $r$  and  $s$  be the homogeneous relations defined on  $A$  with the graphs

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\},$$

$$S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}.$$

Then  $r$  is an equivalence relation, but  $s$  is not. The partition corresponding to  $r$  is

$$A/r = \{\{1, 2\}, \{3\}\}.$$

(b) Let

$$\pi = \{\{1\}, \{2, 3\}, \{4\}\},$$

$$\pi' = \{\{1, 2\}, \{2, 3\}, \{4\}\}.$$

Then  $\pi$  is a partition of  $A = \{1, 2, 3, 4\}$ , but  $\pi'$  is not. The equivalence relation corresponding to  $\pi$  has the graph

$$R_\pi = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4)\}.$$

## Extra: Relational Database

**Definition 1.3.8** A (finite) tuple  $r = (A_1, \dots, A_n, R)$ , where  $A_1, \dots, A_n$  are sets and

$$R \subseteq A_1 \times \dots \times A_n = \{(a_1, \dots, a_n) \mid a_1 \in A_1, \dots, a_n \in A_n\},$$

is called an ( $n$ -ary) *relation*. The sets  $A_1, \dots, A_n$  are called the *domains* of  $r$ , and the set  $R$  is called the *graph* of  $r$ . The number  $n$  is called the *degree (arity)* of  $r$ .

A *relational database* is a (finite) set of relations.

**Example 1.3.9** Consider the relation  $student = (Integer, String, String, Integer, Student)$ , where

$$Student \subseteq Integer \times String \times String \times Integer$$

is given by the following table:

ID (Integer)	Surname (String)	Name (String)	Grade (Integer)
7	Ionescu	Alina	9
11	Ardelean	Cristina	10
23	Ionescu	Dan	7

**Remark 1.3.10** Some known Relational Database management systems are:

- Oracle and RDB Oracle
- SQL Server and Access - Microsoft