Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

Secţia: Informatică engleză, Curs: Dynamical Systems, Primăvara 2021

Laboratory 6: Errors ... where they come from?

1. **Round-off errors.**

Using the command `evalf` in Maple, and, respectively `numerical_approx` in Sage, evaluate $25^{1/8}$. Then, rase the result to the power eight. What do you expect to obtain?

Iterations

Theory. For a function $f : \mathbb{R} \to \mathbb{R}$ we define the sequence $(x_k)_{k\geq 0}$ of iterations of f starting from a given $x_0 \in \mathbb{R}$ as satisfying the recurrence

$$x_{k+1} = f(x_k), \text{ for any } k \geq 0.$$

A fixed point of f is a number $\eta^* \in \mathbb{R}$ such that $f(\eta^*) = \eta^*$. Thus, in order to find the fixed points of f one has to solve the equation

$$f(x) = x.$$

2. **The logistic map: from order to chaos.**

For each $\lambda \in [1, 4]$ we consider $f_\lambda : [0, 1] \to [0, 1], \quad f_\lambda(x) = \lambda x(1 - x)$.

Note that f_λ is a quadratic map, $f_\lambda(0) = f_\lambda(1) = 0$, its maximum value is $f_\lambda'(1/2) = \lambda/4$, and, indeed, $f_\lambda(x) \in [0, 1]$ for all $x \in [0, 1]$.

Thus, if we consider the sequence of iterations of f, we have

$$x_0 \in [0, 1] \quad \Rightarrow \quad x_k \in [0, 1], \quad k \geq 1.$$

For different values of λ and x_0, it is proved that the sequence of iterations has one of the following behaviors:

a) it is constant (this happens only if the starting point is a fixed point of f_λ);

b) it converges monotonically to a certain fixed point of f_λ;

c) it can be split in two subsequences that converge to distinct values;

d) it can be split in four subsequences that converge to distinct values;

e) it has no pattern, is chaotic.

For $\lambda = 1$, consider the function $f1$ having the expression $x(1-x)$. Find the fixed points of $f1$. Then compute the first 200 iterations of $f1$ starting from $x_0 = 0.5$ and represent them using `pointplot` in Maple and, respectively, `Line` in Sage. Interpret what you see and describe in your notebook your observation using expressions like the ones above. Do the same for the initial values $x_0 = 0$ and, respectively, $x_0 = 0.7$. You can try also with other initial values if you want.

Repeat what you have done before for $\lambda = 2$, $\lambda = 3.1$, $\lambda = 3.5$, $\lambda = 3.55$, $\lambda = 3.6$ and, respectively, $\lambda = 3.8$. It is important to consider them all!

3. **Sensitivity with respect to small errors is a feature of chaos.**

(i) Consider the function $f_4 : [0,1] \to [0,1]$, $f_4(x) = 4x(1-x)$. Compute, in the same `for`, the first 40 iterations of f starting with $x_0 = 0.67$ and, respectively, $x_{01} = 0.67001$.

(ii) (only in Maple) Consider now another expression of the same function, $\tilde{f}(x) = 4x - 4x^2$. Compute, in the same `for`, the first 40 iterations of f and, respectively, of $\tilde{f}$ starting with $x_0 = 0.67$. What do you expect and what you obtain?

Numerical methods

Theory. We consider the IVP

$$y' = f(x,y),\ x \in [x_0, x^*], \quad y(x_0) = y_0.$$

The Euler method formula with constant step size $h = (x^* - x_0)/n$ is

$$y_{k+1} = y_k + h\, f(x_k, y_k),\ x_{k+1} = x_k + h, \quad k = \overline{0, n-1}.$$

The improved Euler method formula with constant step size $h = (x^* - x_0)/n$ is

$$y_{k+1} = y_k + \frac{h}{2}\, f(x_k, y_k) + \frac{h}{2}\, f(x_k + h, y_k + h\, f(x_k, y_k)),\ x_{k+1} = x_k + h, \quad k = \overline{0, n-1}.$$

If we denote by φ the exact solution of the given IVP, then the value y_k is an approximation of $\varphi(x_k)$.

4. For the IVP $\quad y' = 2xy, \quad x \in [0,1], \quad y(0) = 1 \quad$ first find its exact solution and then use

(a) Euler's method with step size $h = 0.1$;

(b) the improved Euler's method with step size $h = 0.1$;

to find approximate values of the solution in the interval $[0,1]$.

Compute and write in your notebooks the absolute value of the difference between the correct value and the approximate one at $x = 0.5$ and, respectively, $x = 1$. Formulate a conclusion.

5. For the IVP $\quad y' = y^2 + x^2, \quad y(0) = 0 \quad$, apply the two numerical methods in the interval $[0,2]$ with step size $h = 0.1$.

Use *DEplot* to represent the direction field of the differential equation and the graph of the solution of the IVP. Note that this is the graph of an approximate solution, which is found with a Runge-Kutta type numerical method.

For what reason are these results so different when you approach 2? It seems that on the intervals $[0, 1]$, or even $[0, 1.5]$ the approximations are quite good!

6. **What is a stiff equation?** For the IVP $\quad y' = -250y, \quad y(0) = 1 \quad$, apply the Euler's method and the improved Euler's method in the interval $[0, 1]$ with step size $h = 0.1$. Compare the approximate values of the solution with the exact one. Write the error in each case in your notebook. Have you ever seen such a huge error?