Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică
Secţia: Informatică engleză, Curs: Dynamical Systems

Laboratory 3. Solving Linear Nonhomogeneous Differential Equations

1. Find the general solution to $x' + x = \frac{2}{\sqrt{\pi}} e^{-t^2-t}$. Compute the primitive of e^{t^2} and then of $\frac{2}{\sqrt{\pi}} e^{-t^2}$. You obtain the results in terms of the error function, denoted **erf**. How is this function defined?

Decide whether the following statements are true or false.

2. All the solutions of $x'' + 3x' + x = 1$ satisfy $\lim\limits_{t\to\infty} x(t) = 1$.
3. The solution of the IVP $x'' + 4x = 1, \quad x(0) = 5/4, \quad x'(0) = 0$ satisfies $x(\pi) = 5/4$.
4. The equation $x' = 3x + t^3$ admits a polynomial solution.

First introduce in Maple/Sage the following function f, then represent its graph in an interval that contains the point where the function changes its expression. In Maple you can use the option discont=true. Is this function continuous in any point? Is this function differentiable in any point? Decide this looking at the graph. In your notebooks draw the graph and write the explanations.

5. $f(t) = \begin{cases} t, & t \le 2 \\ 3-t, & t > 2 \end{cases}$; 6. $f(t) = \begin{cases} t, & 0 \le t \le \pi \\ \pi e^{\pi - t}, & t > \pi \end{cases}$.

7. We consider the IVP $x'' + x = f(t), \quad x(0) = 0, \quad x'(0) = 1,$ where f is given at 6. Find its solution and represent the corresponding integral curve on the interval $(0, 10)$.

In Maple this can be done directly. In Sage we have to do this in two steps. It is good for all of you to understand these two steps. In the first step we find the solution $\varphi(t)$ of the IVP $x'' + x = t, \quad x(0) = 0, \quad x'(0) = 1$. But this is solution of the IVP 7 only on the interval $(0, \pi)$. In order to find the expression of the IVP 7 on the interval (π, ∞) we need to solve the IVP $x'' + x = \pi e^{\pi - t}, \quad x(0) = \varphi(0), \quad x'(0) = \varphi'(0)$.

8. Let $\omega > 0$ and $\varphi(t, \omega)$ be the solution of the IVP

$$x'' + x = \cos(\omega t), \quad x(0) = x'(0) = 0.$$

Find $\varphi(t, \omega)$ and note that it is not defined in $\omega = 1$. Find separately the solution when $\omega = 1$. Prove that $\lim_{\omega\to 1} \varphi(t, \omega) = \varphi(t, 1)$ for each $t \in \mathbb{R}$.

Notice that $\varphi(t, \omega)$ is bounded for each $\omega \ne 1$ fixed. Find a bound for $|\varphi(t, \omega)|$. Plot $\varphi(t, 1)$ and note that it is unbounded. Write everything in your notebooks.

9. Let $\alpha > 0$ and $\varphi(t,\alpha)$ be the solution of the IVP

$$x'' - 4x = e^{\alpha t}, \quad x(0) = x'(0) = 0.$$

Find $\varphi(t,\alpha)$ and note that it is not defined in $\alpha = 2$. Find separately the solution when $\alpha = 2$. Compute $\lim_{\alpha\to 2}\varphi(t,\alpha)$ for each $t \in \mathbb{R}$. If Maple/Sage does not return the result of this limit, please compute it by hand.

10. The motion of a spring-mass system is given by the IVP

$$x'' + x = f(t), \quad x(0) = 5, \quad x'(0) = 0,$$

where $f(t) = \begin{cases} t, & t \in [0,\pi/2) \\ \pi - t, & t \in [\pi/2,\pi] \\ 0, & t \in (\pi,\infty) \end{cases}$. Find the solution of this IVP and draw its graph.

Find the solution of each of the following IVPs. Plot the planar curve with the obtained parametric equations.

11. $x' = -2x, \quad y' = -3y, \quad x(0) = 1, \quad y(0) = 1.$
12. $x' = -2x, \quad y' = 3y, \quad x(0) = 1, \quad y(0) = 1.$
13. $x' = -y, \quad y' = 4x, \quad x(0) = 1, \quad y(0) = 1.$
14. $x' = -x + 3y, \quad y' = -3x - y, \quad x(0) = 1, \quad y(0) = 1.$
15. $x' = -x + 3y, \quad y' = 3x - y, \quad x(0) = 1, \quad y(0) = 1.$