Calculus

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Topics

- Sequences
- Limits
- Differentiation
- Unconstrained optimization
- Plotting functions

Sequences

- A sequence is an enumerated collection of numbers
- The usual notation for the nth element of sequence a is a_n
- Example: The sequence of prime numbers.

$$
\begin{aligned}
a_1 &= 2 \\
a_2 &= 3 \\
a_3 &= 5 \\
&\vdots
\end{aligned}
$$

- We often define sequences by the rule a certain element is calculated. Example:

$$a_n = n/2$$

List the first 5 elements of this sequence!

Types of sequences

Finite and infinite

- A finite sequence has a finite number of elements
- An infinite sequence has infinitely many elements

Increasing or decreasing

- A sequence is monotonically increasing if $a_{n+1} \geq a_n \quad \forall n$
- A sequence is monotonically decreasing if $a_{n+1} \leq a_n \quad \forall n$

Boundedness

- If $\exists\ N$ such that $a_n < N \quad \forall n$ the sequence is bounded from above
- If $\exists\ M$ such that $a_n > M \quad \forall n$ the sequence is bounded from below

Give an example for each type!

Limit of a sequence

The limit of a sequence is a number that the terms of a sequence "tend to". The notation is

$$a_n \to A$$

or

$$\lim_{n\to\infty} a_n = A$$

Examples:

- $a_n = 5 \quad \Longrightarrow \quad a_n \to 5$
- $a_n = \frac{1}{n} \quad \Longrightarrow \quad \lim_{n\to\infty} a_n = 0$

Convergence

If a sequence has a limit, it is called convergent. If it does not, it is divergent.

Formal definition of convergence

A sequence a_n converges to A if $\forall \varepsilon > 0 \quad \exists N$ such that $\forall n > N$ it holds that $|a_n - A| < \varepsilon$.

Examples:

- $a_n = n$ is divergent
- $a_n = \frac{(-1)^n}{n}$ is convergent, $\lim\limits_{n \to \infty} a_n = 0$

Properties of limits

- $\lim\limits_{n\to\infty}(a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$
- $\lim\limits_{n\to\infty} ca_n = c \lim\limits_{n\to\infty} a_n \quad \forall c$
- $\lim\limits_{n\to\infty}(a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right)$
- $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, provided that $\lim_{n\to\infty} b_n \neq 0$
- $\lim\limits_{n\to\infty} a_n^p = \left(\lim\limits_{n\to\infty} a_n\right)^p \quad \forall p > 0$

Let's look at some examples!

$$\lim_{n\to\infty}\frac{n^2-3}{n^3-2}=\lim_{n\to\infty}\frac{\frac{1}{n^3}(n^2-3)}{\frac{1}{n^3}(n^3-2)}$$

$$\lim_{n\to\infty}\frac{\frac{1}{n^3}(n^2-3)}{\frac{1}{n^3}(n^3-2)}=\lim_{n\to\infty}\frac{\left(\frac{1}{n}-\frac{3}{n^3}\right)}{\left(1-\frac{2}{n^3}\right)}$$

$$\lim_{n\to\infty}\frac{\left(\frac{1}{n}-\frac{3}{n^3}\right)}{\left(1-\frac{2}{n^3}\right)}=\frac{\lim\limits_{n\to\infty}\left(\frac{1}{n}-\frac{3}{n^3}\right)}{\lim\limits_{n\to\infty}\left(1-\frac{2}{n^3}\right)}$$

$$\frac{\lim\limits_{n\to\infty}\left(\frac{1}{n}-\frac{3}{n^3}\right)}{\lim\limits_{n\to\infty}\left(1-\frac{2}{n^3}\right)}=\frac{\lim\limits_{n\to\infty}\left(\frac{1}{n}\right)-\lim\limits_{n\to\infty}\left(\frac{3}{n^3}\right)}{\lim\limits_{n\to\infty}(1)-\lim\limits_{n\to\infty}\left(\frac{2}{n^3}\right)}$$

$$\frac{\lim\limits_{n\to\infty}\left(\frac{1}{n}\right)-\lim\limits_{n\to\infty}\left(\frac{3}{n^3}\right)}{\lim\limits_{n\to\infty}(1)-\lim\limits_{n\to\infty}\left(\frac{2}{n^3}\right)}=\frac{0-0}{1-0}=0$$

$$\lim_{n\to\infty} n^3 - 2 = \lim_{n\to\infty} \frac{\frac{1}{n^2}\left(n^3 - 2\right)}{\frac{1}{n^2}\left(n^2 - 3\right)}$$

$$\lim_{n\to\infty} \frac{\frac{1}{n^2}\left(n^3 - 2\right)}{\frac{1}{n^2}\left(n^2 - 3\right)} = \lim_{n\to\infty} \frac{\left(n - \frac{2}{n^2}\right)}{\left(1 - \frac{3}{n^2}\right)}$$

$$\lim_{n\to\infty} \frac{\left(n - \frac{2}{n^2}\right)}{\left(1 - \frac{3}{n^2}\right)} = \frac{\lim\limits_{n\to\infty} \left(n - \frac{2}{n^2}\right)}{\lim\limits_{n\to\infty} \left(1 - \frac{3}{n^2}\right)}$$

$$\frac{\lim\limits_{n\to\infty} \left(n - \frac{2}{n^2}\right)}{\lim\limits_{n\to\infty} \left(1 - \frac{3}{n^2}\right)} = \frac{\lim\limits_{n\to\infty} (n) - \lim\limits_{n\to\infty} \left(\frac{2}{n^2}\right)}{\lim\limits_{n\to\infty} (1) - \lim\limits_{n\to\infty} \left(\frac{3}{n^2}\right)}$$

$$\frac{\lim\limits_{n\to\infty} (n) - \lim\limits_{n\to\infty} \left(\frac{2}{n^2}\right)}{\lim\limits_{n\to\infty} (1) - \lim\limits_{n\to\infty} \left(\frac{3}{n^2}\right)} = \frac{\infty - 0}{1 - 0} = \infty$$

Thus this sequence is divergent.

$$\lim_{n\to\infty} \frac{1+(-1)^n}{2}$$

Notice that there are two alternating terms: 0 and 1. Thus this sequence doesn't have a limit.

Also good to know

It is not always obvious how to calculate the limit of a sequence. E.g:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

There are some more advanced ways to calculate limits that we don't cover, but they are also good to know:

- Stolz–Cesàro theorem
- L'Hôpital's rule

Solve the following problems

1. $\lim\limits_{n\to\infty} \frac{n^4+5n^3+3n^2-2}{3n^4-6}$
2. $\lim\limits_{n\to\infty} \frac{5}{n+1} + \frac{n}{n+1}$
3. $\lim\limits_{n\to\infty} b^n$ depending on the value of b.
4. $\lim\limits_{n\to\infty} \frac{1}{n(\sqrt{n^2-1}-n)}$
5. $\lim\limits_{n\to\infty} \sqrt[n]{5}$
6. $\lim\limits_{n\to\infty} \ln\left(\frac{1}{n}\right)$
7. $\lim\limits_{n\to\infty} e^{-n}$

Sidenote: Series

Roughly speaking a series is the sum of the elements of a sequence.

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \dots$$

There is one series that you should remember: the geometric series. The sum of a sequence defined by

$$a_n = a \cdot b^n$$

where $0 < b < 1$ is given by

$$\sum_{i=1}^{\infty} a_i = \frac{ab}{1-b}$$

What is the sum of the following sequences?

- $a_n = \frac{3}{5^n}$
- $a_n = 0.5^n$

Limits of functions

Just like for sequences, we can define the limits for functions.

Definition

A function $f(x)$ has a limit L when x approaches to p IF for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all x that satisfies $|x - p| < \delta$ it holds that $|f(x) - L| < \varepsilon$. The notation is

$$\lim_{x \to p} f(x) = L$$

Example: $f(x) = 3x$. Calculate $\lim_{x \to 3} f(x)$. Let's guess this limit first!

Limits of functions

Example: $f(x) = 3x$. Calculate $\lim_{x \to 3} f(x)$. Now let's understand the definition.

- We claim that $\lim_{x \to 3} f(x) = 9$
- Let's have any positive number ε
- There should exist a δ for any ε that if we are in the δ neighborhood of 3, the function value is always closer to 9 than ε
- We can compute this δ depending on ε.

$$|f(x) - 9| < \varepsilon \implies -\varepsilon < f(x) - 9 < \varepsilon \implies -\varepsilon + 9 < f(x) < \varepsilon + 9 \implies$$

$$-\varepsilon + 9 < 3x < \varepsilon + 9 \implies -\frac{\varepsilon}{3} + 3 < x < \frac{\varepsilon}{3} + 3 \implies |x - 3| < \frac{\varepsilon}{3} = \delta$$

- Let's say $\varepsilon = 6$. It implies that $\delta = \frac{6}{3} = 2$, that is, if we are in the $(3-2, 3+2)$ interval, the function value should always be closer to 9 than 6.

Limits of functions

- We don't really want to use the formal definition in most cases to find the limits.
- The graphical approach often helps.
- An important property: For continuous functions the limit is the same as the value of the function.
- We can also use the following properties:

$$\lim_{x\to p}(f(x)+g(x)) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x)$$
$$\lim_{x\to p}(f(x)-g(x)) = \lim_{x\to p} f(x) - \lim_{x\to p} g(x)$$
$$\lim_{x\to p}(f(x)\cdot g(x)) = \lim_{x\to p} f(x) \cdot \lim_{x\to p} g(x)$$
$$\lim_{x\to p}(f(x)/g(x)) = \lim_{x\to p} f(x) / \lim_{x\to p} g(x)$$

Examples

Find $\lim_{x \to 5} e^{x-3}$. Notice that this is a standard exponential function, which is continuous. Thus

$$\lim_{x \to 5} e^{x-3} = e^{5-3} = e^2$$

Find $\lim_{x \to 0} \ln(x)$. Now notice, that $In(0)$ is not defined. However the $\ln(x)$ function is monotonically increasing, thus as we get closer and closer to zero, it's value gets closer and closer to minus infinity. Thus

$$\lim_{x \to 0} \ln(x) = -\infty$$

Examples

Find $\lim\limits_{x\to\infty} \frac{x^4-2x^3+x-3}{x^5-2x}$

$$\lim_{x\to\infty} \frac{x^4-2x^3+x-3}{x^5-2x} = \lim_{x\to\infty} \frac{\frac{1}{x^5}\left(x^4-2x^3+x-3\right)}{\frac{1}{x^5}\left(x^5-2x\right)}$$

$$\lim_{x\to\infty} \frac{\frac{1}{x^5}\left(x^4-2x^3+x-3\right)}{\frac{1}{x^5}\left(x^5-2x\right)} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{2}{x^2}+\frac{1}{x^4}-\frac{3}{x^5}}{1-\frac{2}{x^4}}$$

$$\lim_{x\to\infty} \frac{\frac{1}{x}-\frac{2}{x^2}+\frac{1}{x^4}-\frac{3}{x^5}}{1-\frac{2}{x^4}} = \frac{\lim\limits_{x\to\infty}\frac{1}{x}-\lim\limits_{x\to\infty}\frac{2}{x^2}+\lim\limits_{x\to\infty}\frac{1}{x^4}-\lim\limits_{x\to\infty}\frac{3}{x^5}}{\lim\limits_{x\to\infty}1-\lim\limits_{x\to\infty}\frac{2}{x^4}}$$

$$\frac{\lim\limits_{x\to\infty}\frac{1}{x}-\lim\limits_{x\to\infty}\frac{2}{x^2}+\lim\limits_{x\to\infty}\frac{1}{x^4}-\lim\limits_{x\to\infty}\frac{3}{x^5}}{\lim\limits_{x\to\infty}1-\lim\limits_{x\to\infty}\frac{2}{x^4}} = \frac{0-0+0-0}{1-0} = 0$$

Examples

Find $\lim\limits_{x\to 2} \frac{3x^2+3x-18}{x-2}$

$$\lim_{x\to 2} \frac{3x^2+3x-18}{x-2} = \lim_{x\to 2} \frac{3(x^2+x-6)}{x-2}$$
$$\lim_{x\to 2} \frac{3(x^2+x-6)}{x-2} = \lim_{x\to 2} \frac{3(x+3)(x-2)}{x-2}$$
$$\lim_{x\to 2} \frac{3(x+3)(x-2)}{x-2} = \lim_{x\to 2} 3(x+3) = 3\cdot 5 = 15$$

Solve the following problems

1. $\lim_{x \to 0} (3 + 2x^2)$
2. $\lim_{x \to -1} \frac{3+2x}{x-1}$
3. $\lim_{x \to 1} \frac{x^2+7x-8}{x-1}$
4. $\lim_{x \to \infty} \frac{x^3-3x^2+x-5}{3x^3+5x^2-2}$
5. $\lim_{x \to 1} \frac{x^2-1}{x-1}$
6. $\lim_{h \to 0} \frac{\sqrt{h+1}-1}{h}$
7. $\lim_{x \to 5} \frac{3x^2-9x-30}{x-5}$

Differentiation

- We are often interested in the slope of the tangent line of a curve at a given point.
- To get this, we use differentiation.
- It is especially useful in case of optimization problems.
- Why? Consider for example the case when you are looking for the maximum of $f(x) = 3 - x^2$.
- What is the slope of the tangent line at the maximum point?

Differentiation

- The first differential $f'(x_0)$ of a function $f(x)$ at a given point x_0 is given by the limit:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

- Notice that $\frac{f(a)-f(b)}{a-b}$ is the slope of the section connecting the function at a and b.
- What we do here, is we get these two points closer and closer.
- Once they are infinitesimally close, it gives the slope of the tangent line.

Differentiation

Our workhorse function will be $f(x) = x^2$. Let's find $f'(1)$. By definition:

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x - 1}$$

$$\lim_{x \to 1} \frac{(x+1)(x-1)}{x - 1} = \lim_{x \to 1} x + 1 = 2$$

Differentiation

Still working with $f(x) = x^2$, let's find $f'(2)$. By definition:

$$f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x - 2}$$

$$\lim_{x \to 2} \frac{(x+2)(x-2)}{x - 2} = \lim_{x \to 2} x + 2 = 4$$

Solve the following problems

Still working with $f(x) = x^2$ find

1. $f'(5)$
2. For any general x_0 find $f'(x_0)$

A bit more difficult problem: Consider now $g(x) = x^3$.

1. First find $g'(2)$
2. Now find $g'(-2)$
3. For any general x_0 try to find $g'(x_0)$

The derivative function

- We have shown that $f'(x_0) = 2x_0$ if $f(x) = x^2$
- We have also shown that $g'(x_0) = 3x_0^2$ if $g(x) = x^3$
- These are the first derivative functions, that give the derivative of a function at any point.
- The usual notation is either $f'(x)$ or

$$\frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x}$$

- Let's find it for the general power function $f(x) = x^n$

Properties of derivative functions

- $c' = 0 \quad \forall c \in \mathbb{R}$
- $(af)' = af'$
- $(af + bg)' = af' + bg'$
- $(fg)' = f'g + fg'$
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Solve the following problems

Find the derivative of the following functions:

- $f(x) = x^3 + 2x^2 - x$
- $g(x) = (x^2 + 2)(x - 4)$
- $h(x) = \frac{x^{12} - 15x^2}{x - 5}$

Some additional useful derivatives

- $\frac{d}{dx}e^x = e^x$
- $\frac{d}{dx}a^x = a^x \ln(a)$
- $\frac{d}{dx}\ln(x) = \frac{1}{x}, \quad \forall x > 0$
- $\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$

Solve the following problems

Find the derivative of the following functions:

- $f(x) = \frac{x^2}{\ln x}$
- $g(x) = e^x(x^3 - x^2)$
- $h(x) = \frac{5^x}{x^2-2}$

The chain rule

You can take the derivative of a function of a function the following way:

$$(f(g(x)))' = f'(g(x))g'(x)$$

Example: e^{-x^2}. Here $f(x) = e^x$ and $g(x) = -x^2$. Thus:

$$(e^{-x^2})' = -2xe^{-x^2}$$

Find the derivative of $\ln(x^2 + 2x)$

Unconstrained optimization

We often want to solve so-called unconstrained optimization problems. Examples:

- What is the optimal quantity to produce in order to maximize your profit?
- What is the optimal length of sleep if you want to be as productive as possible?

If we can characterize these problems with functions, we can optimize them.

- We want to find their minima/maxima
- At these points, the tangent line should be horizontal
- Thus the derivative should be equal to zero

A quick example

Assume that you want to find the minimum of $f(x) = x^2 - 2x - 3$.

- We can either notice that it is equivalent to $f(x) = (x+1)(x-3)$ and infer that it's minimum is at $x = 1$
- Or simply take it's first derivative and find it's root

$$
\begin{aligned}
\frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x} &= 0 \\
2x - 2 &= 0 \\
x &= 1
\end{aligned}
$$

Minimum or maximum? Maybe neither?

- In the previous case we knew that we had a minimum, as it was a simple convex parabola
- But the derivative is 0 at minima and maxima as well
- Also, there is something called an inflection point that we will see by checking $f(x) = x^3$
- If we try to find it's minimum/maximum we get

$$\begin{aligned} \frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x} &= 0 \\ 3x^2 &= 0 \\ x &= 0 \end{aligned}$$

- Thus we should have a minimum/maximum at $x = 0$
- But we don't have one! The derivative can be zero, where the function changes convexity (inflection point)

How to decide?

- Notice that if it is a minimum point, the function has to be convex around the point
- For a maximum point, the function has to be concave around the point
- In case of an inflection point, the function is convex on one side but concave on the other side
- We should look at convexity

How to decide convexity?

- Notice that for convex functions the slope of the tangent line is continuously increasing (or at least not decreasing).
- For concave functions, this is the opposite. The slope of the tangent line is continuously decreasing (or at least not increasing).
- We already know a method to show whether a function is increasing or decreasing: taking it's derivative
- Thus if the derivative shows the slope of the function (how the function values change), the derivative of the derivative shows how the slope of the function changes (convexity).
- Therefore we will need to check the sign of the second derivative denoted by $f''(x)$ or $\frac{\mathrm{d}^2 f(x)}{\mathrm{d} x^2}$

An example

Find the minima/maxima of $f(x) = \frac{1}{3}x^3 - 1.5x^2 - 4x + 10$

- First find the points of minima/maxima:

$$\frac{\mathrm{d}\, f(x)}{\mathrm{d}\, x} = 0$$
$$x^2 - 3x - 4 = 0$$
$$x_1 = 4 \qquad x_2 = -1$$

- Take the second derivative and substitute these values

$$\frac{\mathrm{d}^2\, f(x)}{\mathrm{d}\, x^2} = 2x - 3$$
$$f''(4) = 5 \qquad f''(-1) = -5$$

Thus the function is concave at $x = -1$, and that point should be a local maximum. It is convex at $x = 4$, and it should be a local minimum. Check on WolframAlpha!

Local versus global

- If you look at the previous example, you can see that the function actually takes higher values than the maximum we found
- It also takes lower values than the minimum we found
- By looking at the derivatives, we find so-called local minima/maxima
- These are the highest/lowers values of the function in it's surrounding
- It is not necessarily the same as the global maximum/minimum
- We should also check the limits of the function at the endpoints of the domain

Plotting functions

- Using the tools we have studied, we can easily plot even quite difficult functions.
- We can decide whether they are increasing or decreasing using the first derivative
- We can also find local minima and maxima using the first derivative
- We can find out their convexity using the second derivative
- Let's try to plot $f(x) = \frac{\ln x}{x}$

Plotting functions

- First find the root

$$\frac{\ln x}{x} = 0$$
$$\ln x = 0$$
$$x = 1$$

- Next take the first derivative and find its root. Notice that $\frac{\ln x}{x} = x^{-1} \ln x$

$$f'(x) = -x^{-2} \ln x + x^{-1} x^{-1} = x^{-2}(1 - \ln x)$$
$$x^{-2}(1 - \ln x) = 0 \quad \implies \ln x = 1$$
$$x = e$$

The function crosses the x axis at 1, and has a minimum/maximum at e.

Plotting functions

- Take a further look at the derivative. Check whether it is negative/positive for $x < e$ and $x > e$. We can for example check it at 1 and at e^2.

$$f'(1) = 1^{-2}(1 - \ln 1) = 1 > 0$$

$$f'(2^2) = (e^2)^{-2}(1 - \ln(e^2)) = -\frac{1}{e^4} < 0$$

- The function is increasing until $x = e$ and decreases after $x = e$. This also means that $x = e$ should be a local maximum.
- Check the second derivative nevertheless.

Plotting functions

The second derivative is

$$f''(x) = \left[x^{-2}(1 - \ln x)\right]' = \left[x^{-2} - x^{-2}\ln x\right]'$$

$$\left[x^{-2} - x^{-2}\ln x\right]' = -2x^{-3} - \left(-2x^{-3}\ln x + x^{-2}x^{-1}\right)$$

$$-2x^{-3} - \left(-2x^{-3}\ln x + x^{-2}x^{-1}\right) = x^{-3}(2\ln x - 3)$$

At e this is

$$f''(e) = e^{-3}(2\ln e - 3) = -\frac{1}{e^3} < 0$$

Thus at e the function is concave, we have a local maximum.

Plotting functions

Let's check whether the function changes convexity anywhere. If it does, the second derivative should change from positive to negative at that point (or the other way around), thus it has to be zero.

$$\begin{aligned} f''(x) = x^{-3}(2\ln x - 3) &= 0 \\ 2\ln x - 3 &= 0 \\ x &= e^{3/2} \end{aligned}$$

We know that if $x < e^{3/2}$ (for example e), the second derivative is negative, and the function is concave. What if it is larger? Let's check e^2.

$$f''(e^2) = (e^2)^{-3}(2\ln e^2 - 3) = \frac{1}{e^6} > 0$$

Thus if $x > e^{3/2}$, the function is convex.

Plotting functions

We should also find the limits at the ends of the domain. Since $\ln x$ requires $x > 0$, the function's domain is $\mathbb{R}^+$. Finding these limits is quite difficult without using L'Hôpital's rule, but we can use a trick and some intuition. Since $x \in \mathbb{R}^+$, we can write any x as $x = e^y$, where $y = \ln x$. Thus we can transform the limit we are looking for a bit:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{y\to\infty} \frac{\ln e^y}{e^y} = \lim_{y\to\infty} \frac{y}{e^y} = 0$$

Notice that in the last step we divide a linear function with an exponential, and the exponential grows much faster. This is why the limit is zero. With the other limit:

$$\lim_{x\to 0} \frac{\ln x}{x} = \lim_{y\to -\infty} \frac{\ln e^y}{e^y} = \lim_{y\to -\infty} \frac{y}{e^y} = -\infty$$

Notice that since we defined $x = e^y$, $x \to 0$ is the same as $y \to -\infty$.

Plotting functions

We can summarize everything in a table:

x	<1	1	>1 and < e	e	> e and < $e^{3/2}$	$e^{3/2}$	> $e^{3/2}$
f(x)	-	0	+	+	+	+	+
f'(x)	+	+	+	0	-	-	-
Slope	$\nearrow$	$\nearrow$	$\nearrow$	MAX	$\searrow$	$\searrow$	$\searrow$
f''(x)	-	-	-	-	-	0	+
Convexity	$\cap$	$\cap$	$\cap$	$\cap$	$\cap$	INF	$\cup$

Now we know everything to plot it! Let's do the same with $f(x) = (x-1)(x+3)^2$

Partial derivatives

A partial derivative is the derivative of a multivariate function with respect to one of it's variables, while we consider all other variables to be constant. An example for the notation if we have a function of two variables $f(x, y)$:

- The partial derivative w.r.t. x is $f'_x(x, y)$ or $\frac{\partial f(x,y)}{\partial x}$
- The partial derivative w.r.t. y is $f'_y(x, y)$ or $\frac{\partial f(x,y)}{\partial y}$

All the rules of differentiation still hold!

Example

Let's find the partial derivatives of $f(x, y) = x^2 + 2xy + y^2$

$$\frac{\partial f(x,y)}{\partial x} = 2x + 2y$$

$$\frac{\partial f(x,y)}{\partial y} = 2x + 2y$$

Example

Let's find the partial derivatives of $f(x, y) = x \ln y + \frac{e^x}{y^4}$

$$\frac{\partial f(x, y)}{\partial x} = \ln y + \frac{e^x}{y^4}$$

$$\frac{\partial f(x, y)}{\partial y} = \frac{x}{y} - 4\frac{e^x}{y^5}$$

Example

Let's find the partial derivatives of $f(x,y,z) = xyz + e^y \ln(x) z^5 + e^x$

$$\frac{\partial f(x,y,z)}{\partial x} = yz + e^y z^5 \frac{1}{x} + e^x$$

$$\frac{\partial f(x,y,z)}{\partial y} = xz + e^y \ln(x) z^5$$

$$\frac{\partial f(x,y,z)}{\partial z} = xy + e^y \ln(x) 5z^4$$

Solve the following problems

Find all partial derivatives of the following functions

1. $g(x, y) = 42x + 42y$
2. $f(x, y) = x^2 \ln(y) + \frac{e^y}{\ln x}$
3. $h(x, y, z) = \frac{z^5 e^y}{y^2 \ln(z) x}$

Multivariate unconstrained optimization

Just like in the one variable case, the derivatives at the minima/maxima have to be equal to zero. The difference is that now we need all partial derivatives to be equal to zero. If we put all these partial derivatives in a vector, it is called the **gradient**. You don't have to use it right now, but it is good to know, as it will come up later (E.g.: Gradient descent in ML courses).

Example

Let's find the maxima/minima of the following function: $f(x,y) = x^2y^2 - 5x - 5y$ The derivatives are:

$$\frac{\partial f(x,y)}{\partial x} = 2xy^2 - 5$$

$$\frac{\partial f(x,y)}{\partial y} = 2yx^2 - 5$$

Thus we need to solve

$$2xy^2 - 5 = 0 \quad \text{AND} \quad 2yx^2 - 5 = 0$$

By setting them equal, we find that $x = y$, which implies that $x = y = \sqrt[3]{5/2}$

Example

Let's find the maxima/minima of the following function: $f(x,y) = -xye^{-x^2-y^2}$ The derivatives are:

$$\frac{\partial f(x,y)}{\partial x} = -ye^{-y^2}(e^{-x^2} - 2x^2e^{-x^2}) = e^{-x^2-y^2}y(2x^2-1)$$

$$\frac{\partial f(x,y)}{\partial y} = -xe^{-x^2}(e^{-y^2} - 2y^2e^{-y^2}) = e^{-x^2-y^2}x(2y^2-1)$$

Thus we need to solve

$$e^{-x^2-y^2}y(2x^2-1) \quad \text{AND} \quad e^{-x^2-y^2}x(2y^2-1)$$

We find 5 solutions: $(x,y) = (0,0)$, $(x,y) = \left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$, $(x,y) = \left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$, $(x,y) = \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$, $(x,y) = \left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$

Are these minima or maxima?

- That, again, depends on the convexity
- But it is significantly more difficult to check the convexity here
- We would need to check whether the so-called Hessian matrix (see below) is positive or negative definite, which we can not do without a decent knowledge on linear algebra

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n\,\partial x_1} & \dfrac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

Solve the following problems

Find the points where the following functions obtain their minimum/maximum:

1. $f(x, y) = x^2 + 9x + y^2 - 6y + 15$
2. $h(x, y) = e^{-x^2-y^2}$

Constrained optimization

- Sometimes we want to maximize/minimize a function, but we face a constraint
- For example: You have 20m of fence-material and you want to fence the largest possible rectangle area
- In this case the sides of the rectangle are a and b, thus we want to maximize $f(a,b) = ab$. But $2a + 2b = 20$ also has to hold.
- To solve such problems we will use a function called the Lagrangian

The Lagrangian

If we want to find the minima/maxima of a function $f(x, y)$ with a constraint $g(x, y) = 0$, then we can formulate the following function:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

We call this function the Lagrangian, and λ is called a Lagrange multiplier. We can find the solution(s) to out constrained optimization problem by taking the partial derivatives of the Lagrangian w.r.t. x, y and λ. These all should be equal to zero at the minimum/maximum.

How does this method work?

Notice that the partial derivative w.r.t. λ gives back the constraint, thus it makes sure that the constraint holds. Let's just take two other partial derivatives:

$$\frac{\partial \mathcal{L}}{\partial x} = f'_x(x, y) - \lambda g'_x(x, y) = 0$$

$$\frac{\partial \mathcal{L}}{\partial x} = f'_y(x, y) - \lambda g'_y(x, y) = 0$$

We can rearrange them as:

$$\frac{f'_x(x, y)}{g'_x(x, y)} = \lambda$$

$$\frac{f'_y(x, y)}{g'_y(x, y)} = \lambda$$

How does this method work?

It means that at the solution:

$$\frac{f'_x(x,y)}{g'_x(x,y)} = \frac{f'_y(x,y)}{g'_y(x,y)}$$

- $f'_x(x,y)$ shows how much the objective function would increase if we managed to increase x marginally
- $g'_x(x,y)$ shows how much we would violate the constraint by raising x marginally
- Thus $\frac{f'_x(x,y)}{g'_x(x,y)}$ basically shows how much we can increase the objective function by violating the constraint marginally and increasing x.
- The same way $\frac{f'_y(x,y)}{g'_y(x,y)}$ shows how much we can increase the objective function by violating the constraint marginally and increasing y.
- Thus we basically make sure that the effect of a marginal increase in x on the value of the objective function is the same as the effect of a marginal increase in y

How does this method work?

Consider a case when $\frac{f'_x(x,y)}{g'_x(x,y)} > \frac{f'_y(x,y)}{g'_y(x,y)}$

- Then we can decrease y a bit, and make the constraint slack.
- We can use this to increase x, and the constraint would hold again.
- But since the effect of changing x is larger than changing y, the original point could not be an optimum.

The same way consider a case when $\frac{f'_x(x,y)}{g'_x(x,y)} < \frac{f'_y(x,y)}{g'_y(x,y)}$

- Then we can decrease x a bit, and make the constraint slack.
- We can use this to increase y, and the constraint would hold again.
- But since the effect of changing y is larger than changing x, the original point could not be an optimum.

Example

Let's solve $f(x, y) = xy \to \max$ s.t. $x + y = 10$.

- The Lagrangian is $\mathcal{L} = xy - \lambda(x + y - 10)$
- The derivatives have to be zero:

$$\frac{\partial \mathcal{L}}{x} = y - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{y} = x - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\lambda} = x + y - 10 = 0$$

- From the first two $x = y$, which implies $x = y = 5$.

Solve the following problems

Find the constrained maxima/minima of the following functions:

1. $g(x, y) = x^2y^4 \to \max \text{ s.t. } x + y = 9$
2. $f(x, y) = e^{xy} \to \max \text{ s.t. } x + y = 2$