A study of standard, implicit, and optimal particle filters applied to a shallow water model (DRAFT, INCOMPLET)

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Abstract

Typical data assimilation techniques including 3/4DVar and ensemble Kalman/particle filters (EnKF/PF) can and do fail on hard nonlinear models due to errors in the linearization in 3 and 4DVar or due to particle significance collapse as in EnKF/PF. In high dimensional models, the space designated by the observations is relatively small. This is hard for a Bayesian filter to produce likely samples. Implicit sampling extends particle filters, in much the same way as in the optimal filter but to a lesser extent, by making use of some of the data during sampling. In this paper, we code and compare standard, implicit, and optimal proposal particle filters to a shallow water model.

0.1 Introduction

In the geosciences, one is concerned with the state of a system. Even in the presence of a perfect model, assimilating sparse observations can be challenging. Sequential data assimilation schemes are used to combine observations of a physical system state into an estimate of it's current state. This state estimate, or analysis, is then used as input to forecasting steps, resulting in probability densities.

TODO: recall the various discrete time filtering algos.

In this paper, we implement the spectral shallow water equations in a spherical geometry model with Rossby-Haurwitz Wave number four initial conditions.

First we will discuss the model as applied in a spherical setting, the Rossby-Haurwitz waves used to test the numerical method, and the spectral . Next we will discuss the data assimilation techniques as applied to the model

0.2 Model

The shallow water equations are a hyperbolic system of partial differential equations. They describe the behavior of a fluid flow assuming pressure invariance, derived from depth-integrating the Navier-Stokes equations, assuming the horizontal length is much larger than the vertical length. The equations are given by:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{u}{a\ cos\ \theta}\frac{\partial u}{\partial \lambda} + \frac{u}{a}\frac{\partial u}{\partial \theta} - \frac{tan\ \theta}{a} vu - fv &= -\frac{g}{a\ cos\ \theta}\frac{\partial h}{\partial \lambda} \\ \frac{\partial v}{\partial t} + \frac{v}{a\ cos\ \theta}\frac{\partial v}{\partial \lambda} + \frac{v}{a}\frac{\partial v}{\partial \theta} + \frac{tan\ \theta}{a} u^2 + fu &= -\frac{g}{a}\frac{\partial h}{\partial \theta} \\ \frac{\partial h}{\partial t} + \frac{h}{a\ cos\ \theta}\frac{\partial h}{\partial \lambda} + \frac{h}{a}\frac{\partial h}{\partial \theta} + \frac{h}{a\ cos\ \theta}\left[\frac{\partial u}{\partial \lambda} + \frac{\partial (cos\ \theta)}{\partial \theta}\right] &= 0 \end{aligned} \tag{1}$$

where g is the acceleration due to gravity, gh is the free surface geopotential, h is the free surface height, $f = 2\Omega sin\,\theta$ is the Coriolis parameter, Ω is Earth's angular velocity, θ is the angle of latitude, λ is the longitude, a is Earth's radius, and $V = u_x + v_y$ is the horizontal velocity vector on the surface of the sphere.

Rossby waves are a β-plane approximation of the barotropic vorticity equation to the deformation of Earth from a perfect sphere, and are related to large scale waves in the atmosphere. Haurwitz later introduced equivalent solutions for the sphere, now called the Rossby-Haurwitz waves, resulting in propagating solutions of the non-divergent barotropic vorticity equation on a sphere. Rossby-Haurwitz waves are widely used as test cases for numerical methods for solving the shallow water equations in spherical geometry [1]. It was shown in [2] that the Rossby-Haurwitz wave number four breaks down into turbulent flows after long term numerical integration.

The initial velocity field is defined as:

$$\begin{aligned} u &= a\omega\, cos\,\phi + aK\, cos^{r-1}\phi\big(r\, sin^2\phi - cos^2\phi\big)\, cos\big(r\lambda\big) \\ v &= -aKr\, cos^{r-1}\phi\, sin\,\phi\, sin\big(r\lambda\big) \end{aligned} \tag{2}$$

and the initial height field is defined as:

$$h = h_0 + \frac{a^2}{g}\big[A(\phi) + B(\phi)\, cos(r\lambda) + C(\phi)\, cos(2r\lambda)\big] \tag{3}$$

with:

$$\begin{aligned} A(\phi) &= \frac{\omega}{2}(2\Omega + \omega)\, cos^2\phi + \frac{1}{4}k^2\, cos^{2r}\phi\big[(r+1)\, cos^2\phi + (2r^2 - 2r - 2) - 2r^2\, cos^2\phi\big] \\ B(\phi) &= \frac{2(\Omega + \omega)}{(r+1)(r+2)}\, cos^r\phi\big[(r^2 + 2r + 2) - (r+1)^2\, cos^2\phi\big] \\ C(\phi) &= \frac{1}{4}k^2\, cos^{2r}\phi\big[(r+1)\, cos^2\phi - (r+2)\big] \end{aligned} \tag{4}$$

where h_0 is the height at the poles, r is the wave number, ω is the west-east zonal wind phase velocity, and k is the wave amplitude.

0.2.1 Discretization by spectral decomposition

The spectral decomposition, as given in [3], is

$$\phi(\lambda, \mu) = \sum_{m=-M}^{M} \sum_{n=|m|}^{\Re(m)} \phi_{m,n} P_{m,n}(\mu) \exp(im\lambda) \tag{5}$$

where $P_{m,n}(\mu)$ is the Legendre polynomial, $P_{m,n}(\mu)\exp(im\lambda)$ are the spherical harmonic functions, M is the highest Fourier wavenumber in the east-west representation, and $\Re$(m) is the highest degree of the corresponding Legendre polynomials for longitudinal wavenumber m.

The coefficients of 5 are determined by:

$$\phi_{m,n} = \int_{-1}^{1} \frac{1}{2\pi} \int_{0}^{2\pi} \phi(\lambda)\exp(-im\lambda)P_{m,n}(\mu)d\lambda d\mu \tag{6}$$

The inner integral is computed with a fast Fourier Transform algorithm. The outer integral is evaluated using Gaussian quadrature on the transform grid

$$\phi_{m,n} = \sum_{j=1}^{J} \phi_m(\mu_j)P_{m,n}(\mu_j)w_j \tag{7}$$

where the μ_js are at the Gaussian latitudes θ_j, which correspond to the J roots of the Legendre polynomial $P_j(sin\,\theta_j) = 0$.

The size of the grid is determined to prevent the aliased representation of quadratic terms:

$$TODO : causes\ terrible\ pixelation \tag{8}$$

where N is the highest wavenumber from the latitudinal Legendre representation.

0.3 Assimilation

The dynamical system of the form:

$$d\boldsymbol{x} = f(\boldsymbol{x}, t) + g(\boldsymbol{x}, t)dw \tag{9}$$

where x is an m-dimensional vector and f is an m-dimensional vector function.

0.4 Experiment

For this paper's experiment, we generate the truth from the model with initial conditions propagated forward in time by the model without noise (ie with deterministic dynamics).

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