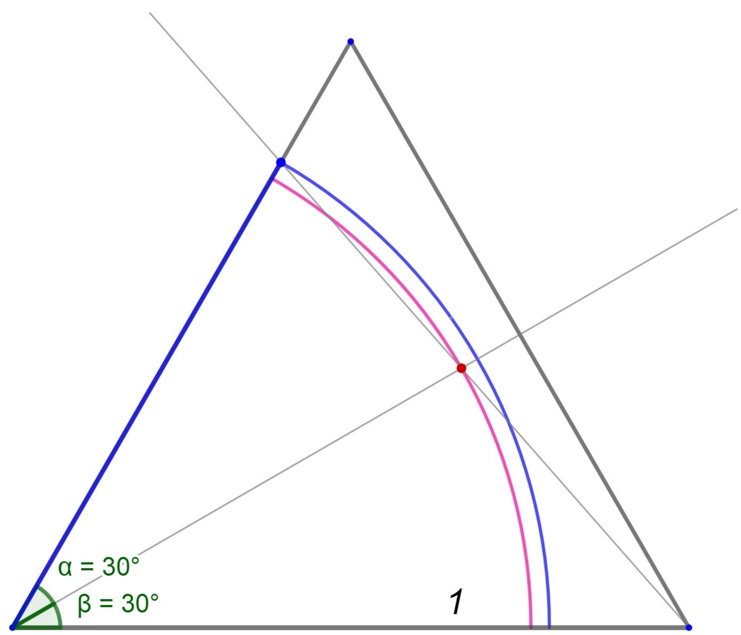
Diamond Section

Doubling the cube. Solution.

Same problem, different angle.



Askarbekov Almas Serikovich

almas@diamondsection.com

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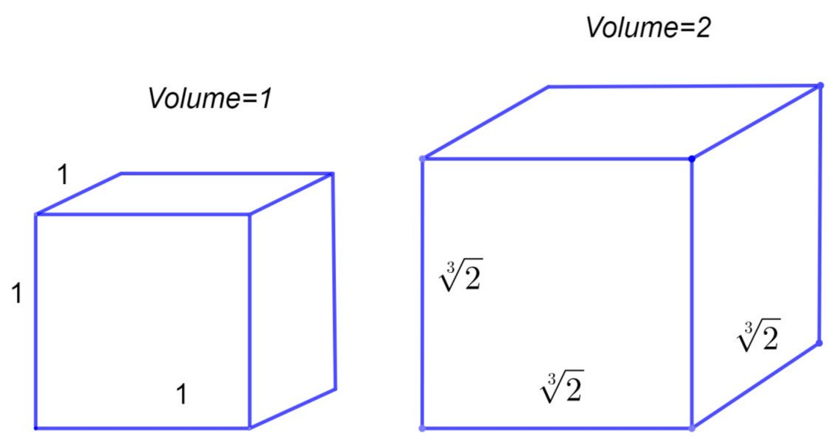
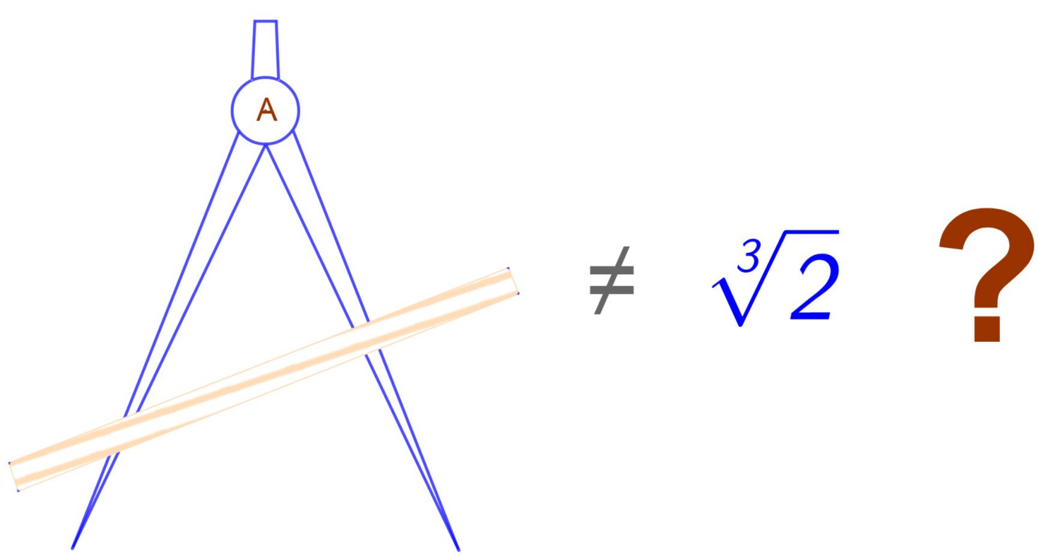
Abstract

Doubling the cube also known as the Delian problem is one of the three famous geometric problems of antiquity, unsolvable by compass and straightedge construction. Given the edge of a cube, the problem requires the construction of the edge of a second cube, whose volume is double that of the first.

The only tools allowed for the construction is the unmarked straightedge and compass.

In algebraic terms, doubling a unit cube requires the construction of a line segment of length x, where $x^3 = 2$; in other words $x = \sqrt[3]{2}$.
Pierre Wantzel proved in 1837 that the $\sqrt[3]{2}$ can not be constructed using a compass and straightedge.[1]





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Chapter 1

The solution

1.1 Construction

1. Let's construct a sequence of circles as shown in Figure 1.1, with radius $r = 1$.

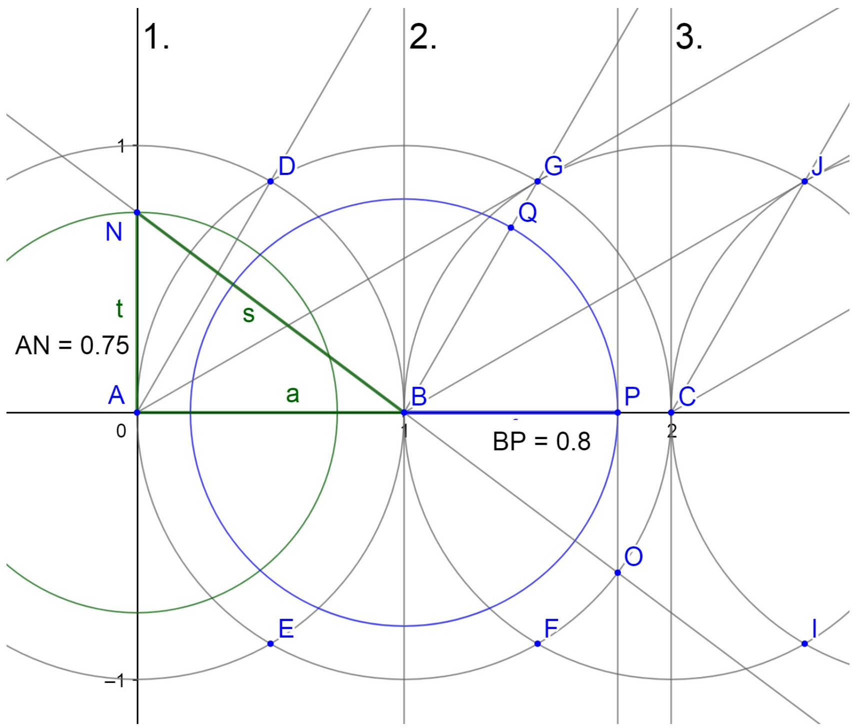

Figure 1.1: Basic circles and triangle

2. Then construct the right triangle $\triangle ANB$ (Figure 1.1) with shortest leg $t = 0.75$. where:

$$s = \overline{NB} = \sqrt{a^2 + t^2} = \sqrt{1^2 + 0.75^2} = 1.25 \tag{1.1}$$

by the Pythagorean theorem.

3. Calculate cosine of $\angle ABN$:

$$\frac{a}{s} = \frac{1}{1.25} = 0.8. \tag{1.2}$$

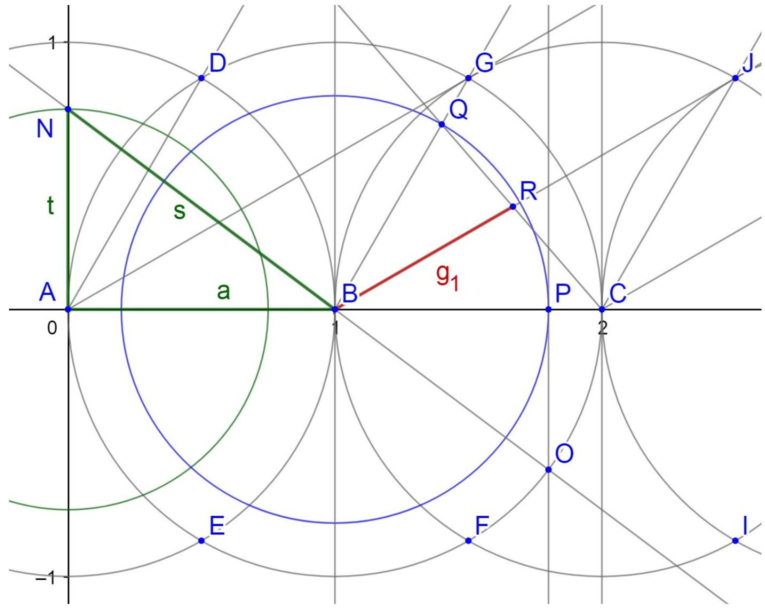

Figure 1.2: Intersection

To construct the ratio we obtain, we need to draw a line through the points N, B. Then we get the point O as the intersection of the line and the circle with center at the point $B(1, 0)$. So we get the line segment $BP = 0.8$, as the projection of the point O on the x-axis. (Figure 1.2)

4. Then construct a ray from point $C(2, 0)$ through the intersection point Q of the circle with radius $BP = 0.8$ and the line segment $\overline{BG}$.

   Now we get the point R and the line segment $\overline{BR}$. The segment $\overline{BR}$ is the bisector for $\triangle\ CBQ$. (Figure 1.2)

5. The $\angle\alpha$ of $\triangle\ CBQ = \angle CBG = 60°$, because $\triangle CBG$ is an equilateral triangle with sides equal to 1. (Figure 1.3)

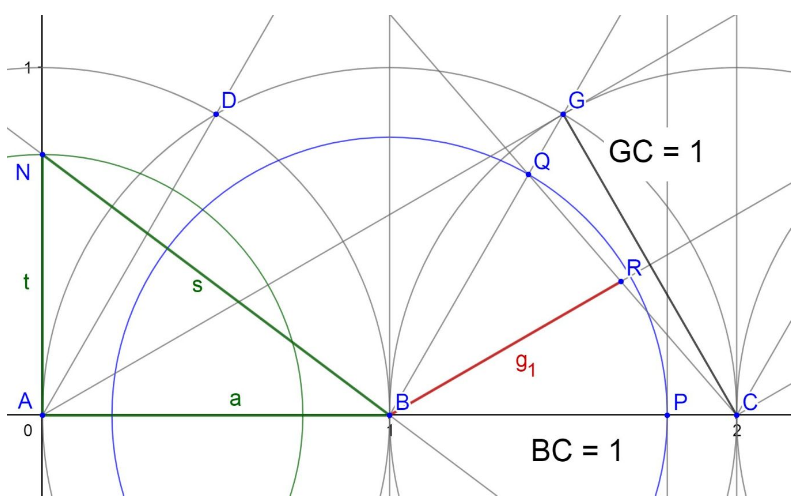

Figure 1.3: Equilateral triangle CBG

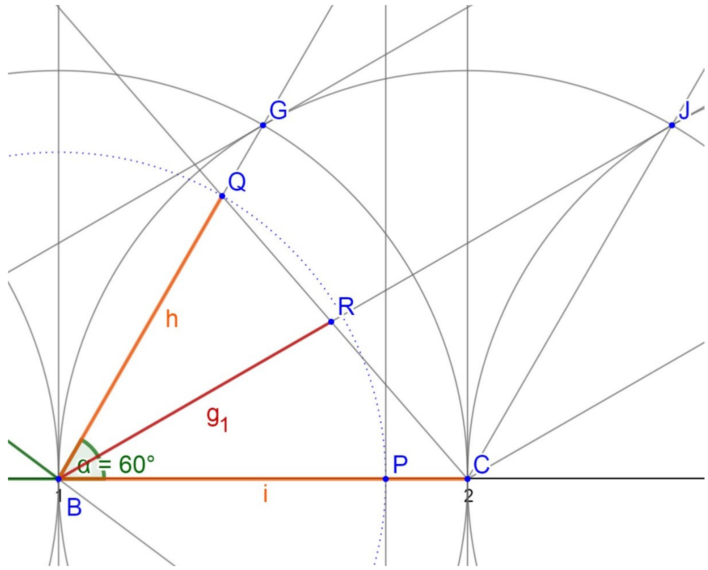

Figure 1.4: Triangle bisector

6. Let's calculate the length of the segment $\overline{BR}$ (Figure 1.4) by the formula for the 60 degree angle bisector[1]:

$$g_1 = \frac{2ih}{i+h} \times \cos 60^\circ = \frac{2ih}{i+h} \times \frac{\sqrt{3}}{2} \tag{1.3}$$

. Where $h = \overline{BQ} = 0.8$; $i = B, C = 1$; therefore:

$$g_1 = \frac{2 \times 0.8}{1+0.8} \times \frac{\sqrt{3}}{2} \approx 0.769800358919501 \tag{1.4}$$

.

7. Now, let's draw a circle with radius g_1 with center at the point $B(1,0)$. Then we get the point S, as the intersection of the circle and the line perpendicular to x-axis through the point B. (Figure 1.5)

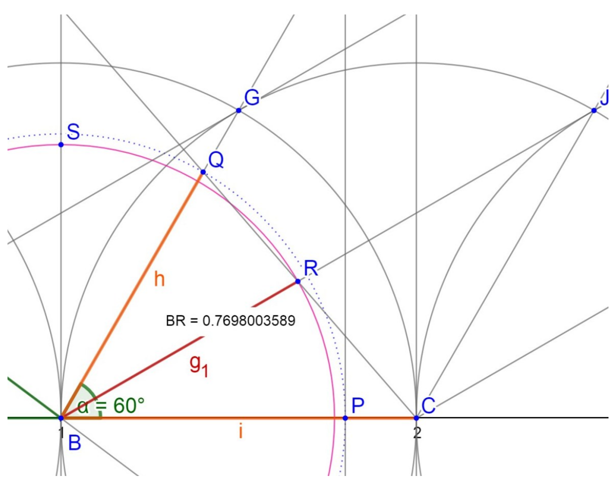

Figure 1.5: Circle with radius = bisector g_1

[1]application of the law of cosines

8. Then construct the right triangle $\triangle CBS$ as shown in the Figure 1.6.

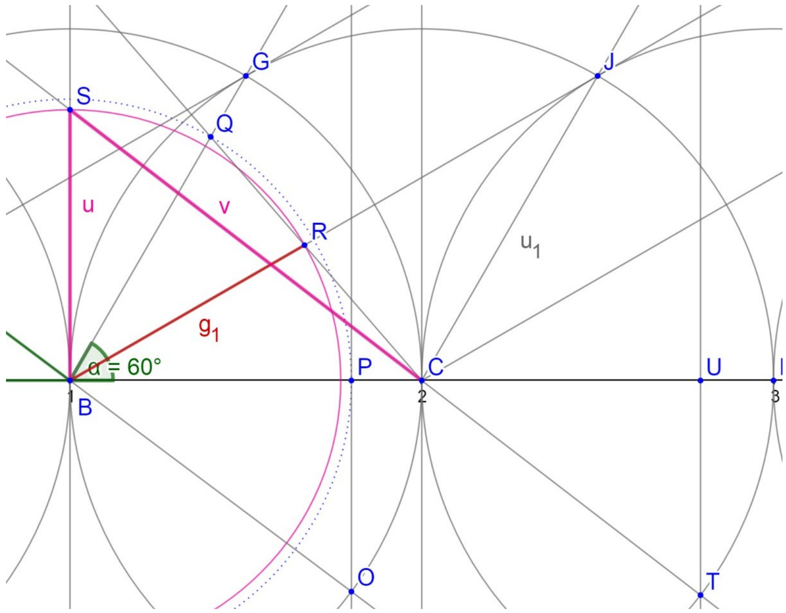

Figure 1.6: Triangle BSC

9. Now we will repeat all the steps from the step 3.
   • We draw a line through the points S, C and get the point T, as the intersection of the line and the circle with center at the point $C(2,0)$.
   Hence:
$$\frac{\overline{SC}}{\overline{BC}} = \overline{CU} = \cos\angle BCS \tag{1.5}$$
   • Then repeat the steps: 4, 5, 6, 7, 8 and repeat again from 3-rd step to the 8 step; and so on...

10. Iterating it over and over we obtain $\sqrt[3]{2}$ as the length of the hypotenuse of the right triangle.

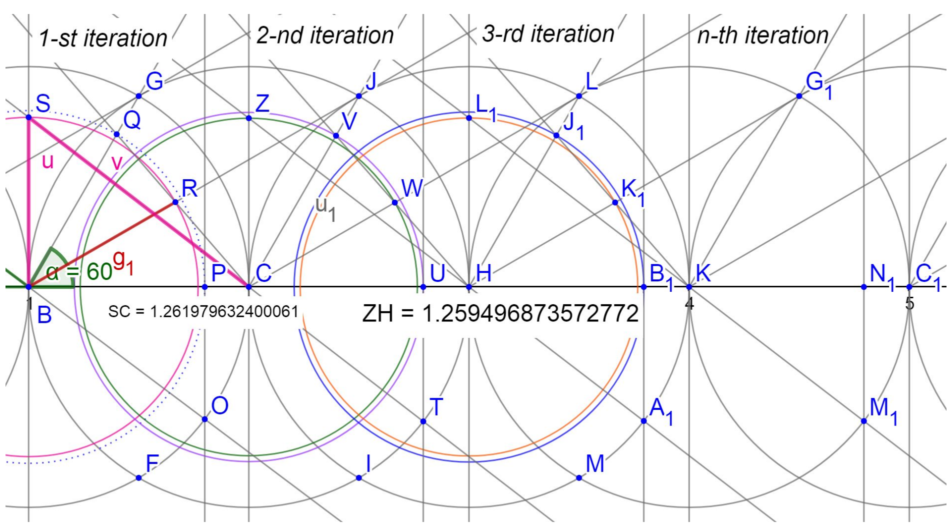

Figure 1.7: Construction

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1.1.1 Proof

The main idea is similar to the recursion concept. The algorithm is looks like this:

1 iteration:

$$\frac{1}{\sqrt{a^2+t_0^2}}=t_1; \quad \frac{2\times t_1}{1+t_1}\times\frac{\sqrt{3}}{2}=t_2 \tag{1.6}$$

2 iteration

$$\frac{1}{\sqrt{a^2+t_2^2}}=t_3; \quad \frac{2\times t_3}{1+t_3}\times\frac{\sqrt{3}}{2}=t_4\ ... \tag{1.7}$$

n-th iteration:

$$\frac{1}{\sqrt{a^2+t_n^2}}=t_{n+1}; \quad \frac{2\times t_{n+1}}{1+t_{n+1}}\times\frac{\sqrt{3}}{2}=t_{n+2}\quad ... \tag{1.8}$$

Where $t_0 = t = 0.75$ (see Figure 1.1). The shortest leg of the first right triangle is 0.75 since this value is easy to construct by a compass and unmarked straightedge, but we will get the cube root of 2 for any arbitrary chosen length. The only rule is that the longest leg should be equal to 1. After n-th iterations we get the length of the hypotenuse $s \approx \sqrt[3]{2}$. The accuracy depends on the number of iterations.

1 iteration: $\triangle ANB$; (Figure 1.1) where $\overline{AN} = t = 0.75$; therefore hypotenuse and cosine respectively:

$$\overline{NB}=\sqrt{1^2+0.75^2}=1.25 \tag{1.9}$$

$$\cos=\frac{1}{1.25}=0.8 \tag{1.10}$$

the length of the bisector:

$$\frac{2\times 0.8}{1+0.8}\times\frac{\sqrt{3}}{2}\approx 0.769800358919501 \tag{1.11}$$

hypotenuse:

$$\sqrt{1^2+0,769800358919501^2}\approx 1,261979632400061 \tag{1.12}$$

2 iteration:

$$\cos=\frac{1}{1,261979632400061}\approx 0,792405815693061 \tag{1.13}$$

bisector:

$$\frac{2\times 0.792405815693061}{1+0.792405815693061}\times\frac{\sqrt{3}}{2}\approx 0.765723432147395 \tag{1.14}$$

Hypotenuse:

$$\sqrt{1^2+0.765723432147395^2}\approx 1.259496873572772 \tag{1.15}$$

For convenience, I made calculations for 20 triangles, then imported the data into the table and analyzed the values.

The table 1.1 shows the calculated values of the bisector and hypotenuse, except for the cosine.

iteration	**shortest leg=bisector**	**hypotenuse**
1	0.769800358919501	1.261979632400061
2	0.765723432147395	1.259496873572772
3	0.766564817073686	1.260008578849848
4	0.766391253457252	1.259902993637120
5	0.766427060119643	1.259924774930487
6	0.766419673248706	1.259920281423651
7	0.766421197157344	1.259921208430153
8	0.766420882775838	1.259921017189131
9	0.76642094763258	1.259921056642051
10	0.766420934252668	1.259921048502934
11	0.766420937012937	1.259921050182029
12	0.766420936443495	1.259921049835633
13	0.76642093656097	1.259921049907094
14	0.766420936536735	1.259921049892352
15	0.766420936541735	1.259921049895393
16	0.766420936540704	1.259921049894766
17	0.766420936540916	1.259921049894895
18	0.766420936540872	1.259921049894869
19	0.766420936540881	1.259921049894874
20	0.766420936540879	1.259921049894873

Table 1.1: Iterations and values

Cubic equation obtained after plotting the values[2] in column B and applying polynominal regression model to the power of 3:

$$y = 0.375x^3 - 0.5x^2 + 1.25x \tag{1.16}$$

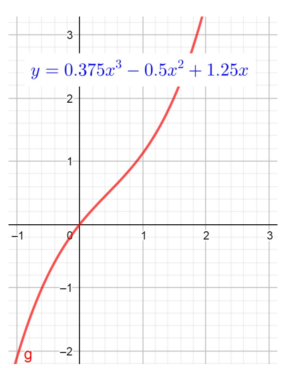


Figure 1.8: Function

[2]I used geogebra.org/classic web app for plotting and analasis

We see that the value in column B on the 20th row (Table 1.1) and the median value (see Figure 1.9) is: **1.259921049894873** $= \sqrt[3]{2}$ with an accuracy of 10^{-15}.

Reference value of the $\sqrt[3]{2} =$ **1.25992104989487316** with an accuracy of 10^{-17} place

During calculations I used rounding to the 10^{-15}, however we can get the final value with any given accuracy. This is because the length of the longest side tend to the cube root of two[3].

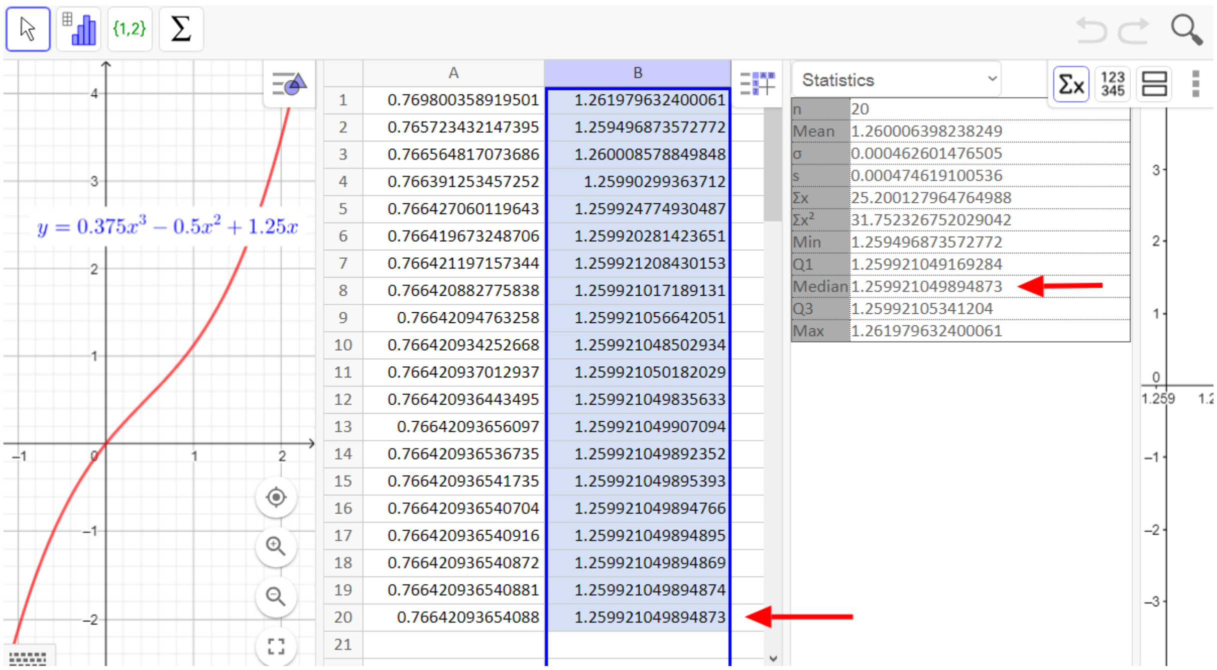


Figure 1.9: Data analysis

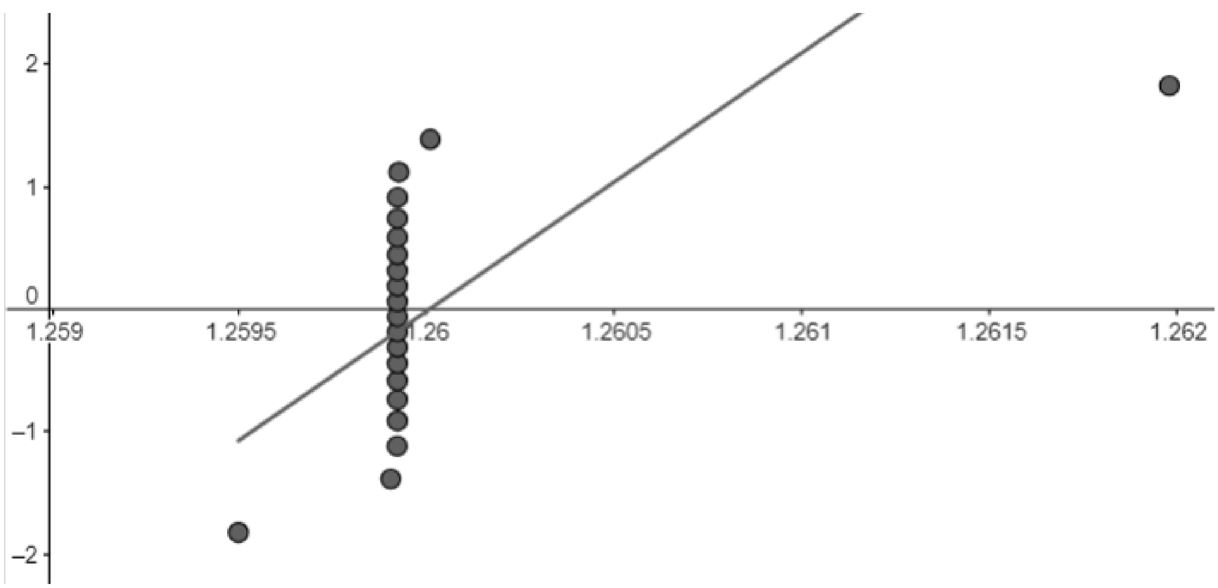


Figure 1.10: Normal Quantile plot

[3]Analysis online and spreadsheet data: https://www.geogebra.org/classic/TpwsEbx2

1.2 Definition

$$\sqrt[3]{2} = \lim_{n\to\infty} \sqrt{1+b_n^2} \tag{1.17}$$

where:

$$b_0 > 0, \qquad b_1 = \frac{1}{\sqrt{1+b_0^2}}, \qquad b_2 = \frac{2\times b_1}{1+b_1} \times \frac{\sqrt{3}}{2} \tag{1.18}$$

$$b_3 = \frac{1}{\sqrt{1+b_2^2}}, \qquad b_4 = \frac{2\times b_3}{1+b_3} \times \frac{\sqrt{3}}{2} \ \ldots \tag{1.19}$$

$$b_{n+1} = \frac{1}{\sqrt{1+b_n^2}}, \qquad b_{n+2} = \frac{2\times b_{n+1}}{1+b_{n+1}} \times \frac{\sqrt{3}}{2} \tag{1.20}$$

Geometric represantation:

$$d = \frac{a}{s} = \frac{1}{\sqrt{1+b^2}} \tag{1.21}$$

$$b = g = \frac{2\times d}{1+d} \times \frac{\sqrt{3}}{2} \tag{1.22}$$

$$s = \frac{1}{d} = \sqrt[3]{2} \tag{1.23}$$

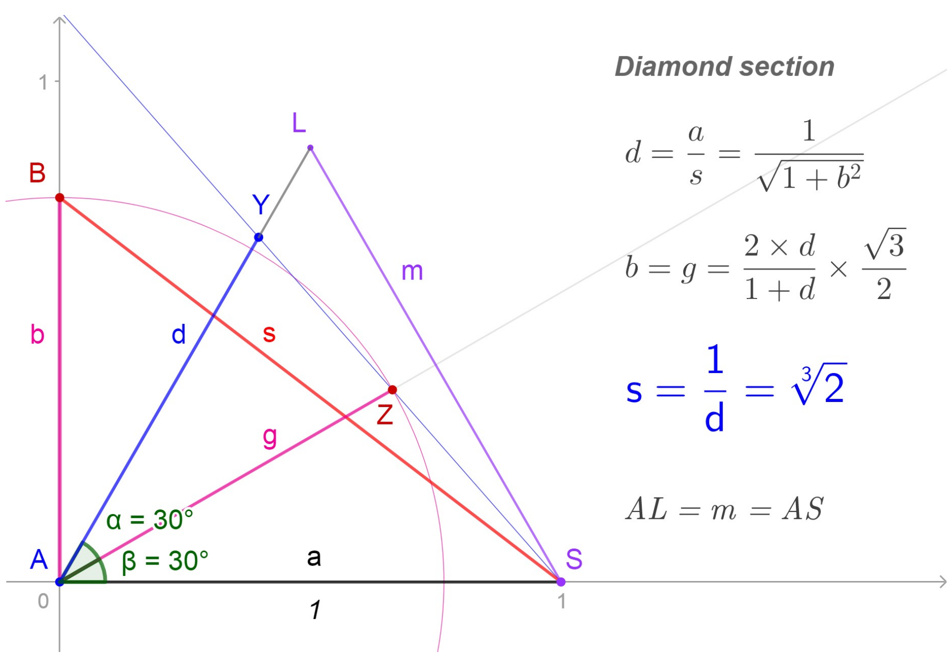


Figure 1.11: Definition

1.3 Why

A diamond is a crystall of tetrahedrally bonded carbon atoms. Each carbon atom in the diamond structure is located at the center of the tetrahedron, whose vertices are the four nearest atoms.[2] The orthogonal vertex projection of a tetrahedron on a 2-D plane is a triangle. (Figure 1.12).

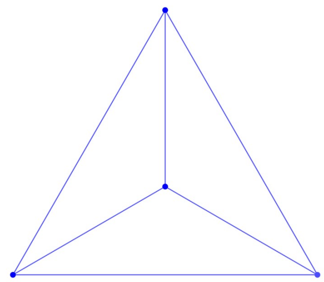
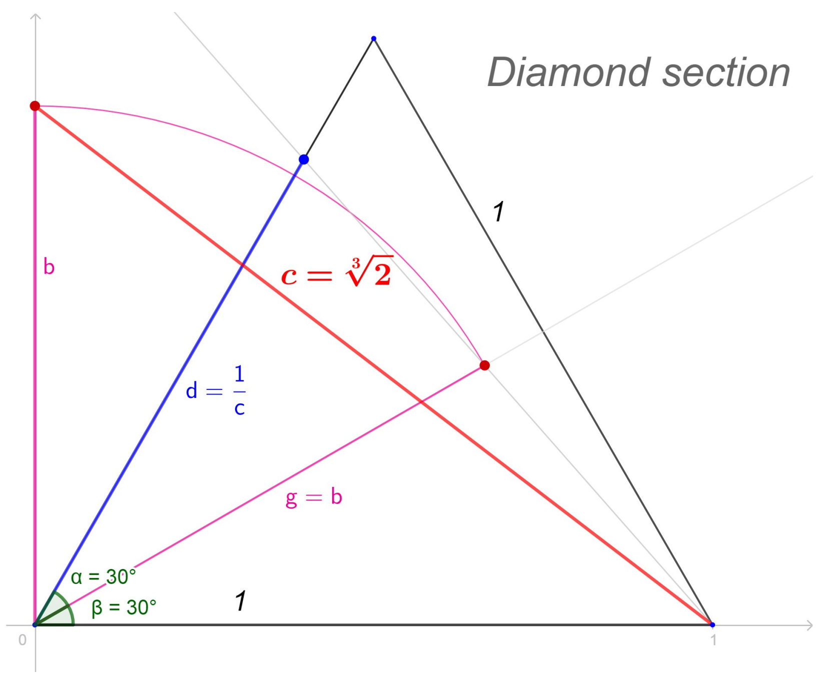


Figure 1.12: Vertex projection of tetrahedron

1.4 Conclusion

We have seen that the **cube root of two can be constructed** with any given accuracy by a compass and unmarked straightedge.



1.5 Appendix

I found an interesting property of the right triangle with sides: $a = 1$; $b = \sqrt{\varphi - 1}$; $c = \sqrt{\varphi}$ where $\varphi \approx 1.6180339887$ (see Figure 1.13)

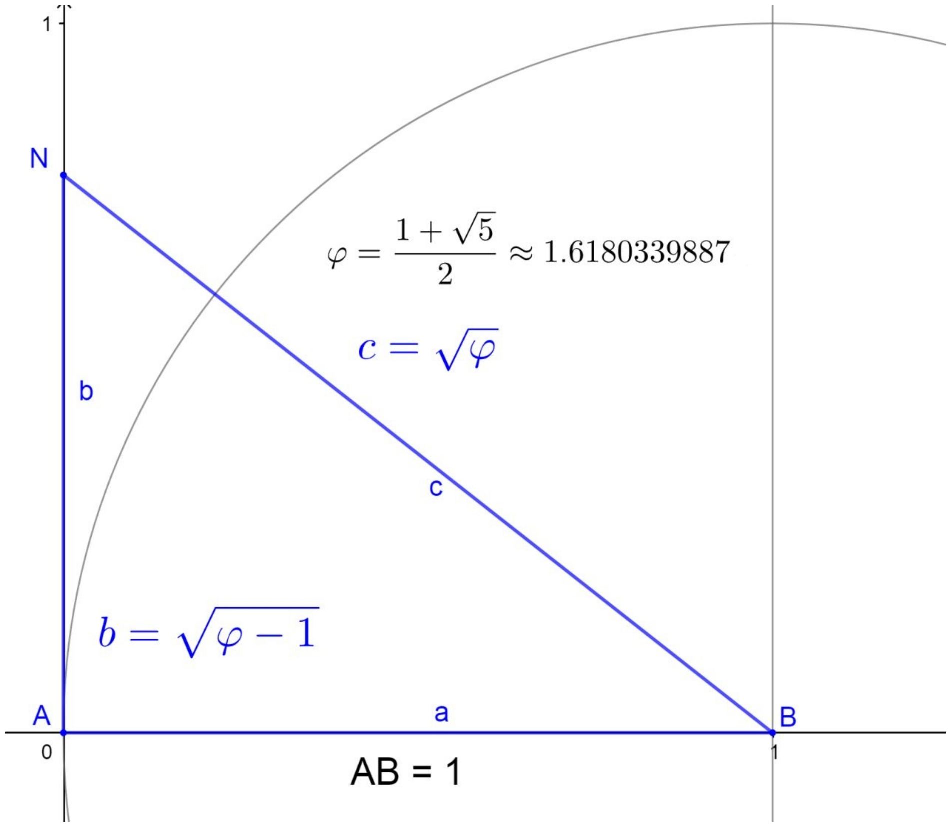


Figure 1.13: Interesting triangle

If we take an **arbitrary** length as the shortest leg b of the right triangle with longest leg $a = 1$, and recursively apply this:

$$\frac{1}{\sqrt{a^2+b^2}} = b_1;\ \frac{1}{\sqrt{a^2+b_1^2}} = b_2;\ \frac{1}{\sqrt{a^2+b_2^2}} = b_3;\ \ldots \frac{1}{\sqrt{a^2+b_{n-1}^2}} = b_n \tag{1.24}$$

Then, after the n-th iteration[4] we get:

$$a = 1; \tag{1.25}$$

$$b = \sqrt{\varphi - 1} \tag{1.26}$$

$$c = \sqrt{\varphi} \tag{1.27}$$

Therefore, our triangle with an arbitrary shortest leg became to the triangle shown in the Figure 1.13.
In other words the shortest leg b tend to $\sqrt{\varphi - 1}$

[4]The accuracy depends on the number of iterations.

1.6 Resources

website: `http://diamondsection.com`
e-mail: almas@diamondsection.com
github: `https://github.com/AlmasAskarbekov`
spreadsheet values: `https://www.geogebra.org/classic/TpwsEbx2`

1.7 Thanks

`https://geogebra.org`

1.8 Notes

Note: I used a standard calculator built into the operating system to get the values from the spreadsheet (Table 1.1 1.6). The geogebra web application uses javascript, which means that if I used a web application to calculate all the values measuring the length between two points, I would get a slightly different value. This is due to how javascript stores floating point numbers[3]. However, in the end we get the cube root of two in any case, because the length of the longest side tends to the cube root of two.

Keywords: Geometric problems of Antiquity, geometry, doubling the cube, the delians problem

Bibliography

[1] https://en.wikipedia.org/wiki/Doubling_the_cube

[2] https://en.wikipedia.org/wiki/Diamond

[3] https://en.wikipedia.org/wiki/IEEE_754

@miscDiamondSection2018, author = Almas Askarbekov, title = Doubling the cube. Solution, year = 2018, howpublished = https://github.com/DiamondSection/Doubling-the-cube-Solution_Latex, note = commit dbgsxxx