Monotone Convergence Thm (decreasing)

Let $\{f_n\}$ be nonnegative, $f_n \downarrow f$. $\int f_1 < \infty$, then $\lim\limits_{n\to\infty} \int f_n = \int f$.

Pf. $-f_n \uparrow -f$ not nonnegative!

Let $g_n = f_1 - f_n$, then $g_n \uparrow f_1 - f$

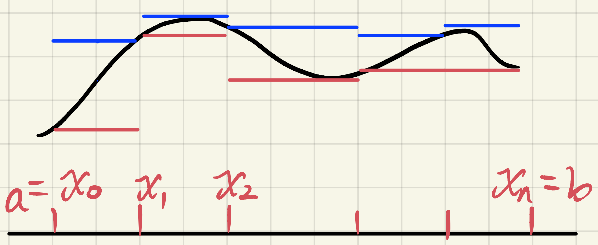
MCT: $\lim\limits_{n\to\infty} \int (f_1 - f_n) = \int f_1 - f$

$\lim\limits_{n\to\infty} \int f_n = \int f$. $\square$

<u>Thm.</u> Suppose f is Riemann integrable, then

$$\int^{R}_{[a,b]} f(x)\,dx = \int^{L}_{[a,b]} f(x)\,dx$$

integrable $\Leftrightarrow$ $\forall \varepsilon > 0\ \exists$ partition P: $|U(f,P) - L(f,P)| < \varepsilon$



$m_i = \inf\limits_{x \in [x_{i-1}, x_i]} f(x)$, $M_i = \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$

a step function

upper sum $U(f,P) = \sum\limits_{i=1}^{n} M_i \Delta x_i = \int_a^b \varphi_P(x)\,dx$

lower sum $L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i = \int_a^b \psi_P(x)\, dx$

$\forall k > 0 \ \exists P_k : U(f, P_k) - L(f, P_k) < \frac{1}{k}$

$$\varphi_1(x) \le \varphi_2(x) \le \cdots \le f(x) \le \cdots \le \psi_2(x) \le \psi_1(x)$$

$$\lim_{k \to \infty} \int_a^b \varphi_k(x)\, dx = \lim_{k \to \infty} \int_a^b \psi_k(x)\, dx = \int_a^b f(x)\, dx$$

note that $\overset{\mathcal{L}}{\int_{[a,b]}} \varphi_k = \int_a^b \varphi_k$, (R) $\int_{[a,b]} \psi_k = \int_a^b \psi_k$

Let $\tilde{\varphi}(x) := \lim_{k \to \infty} \varphi_k(x)$, $\tilde{\psi}(x) := \lim_{k \to \infty} \psi_k(x)$, $\varphi_k \nearrow \tilde{\varphi}$, $\psi_k \searrow \tilde{\psi}$.

$$\Rightarrow \int_{[a,b]} \varphi_k = \int_a^b \varphi_k \nearrow \int_{[a,b]} \tilde{\varphi}$$

$$\int_{[a,b]} \psi_k = \int_a^b \psi_k \searrow \int_{[a,b]} \tilde{\psi}$$

$$\Rightarrow \int_{[a,b]} \tilde{\varphi} = \int_{[a,b]} \tilde{\psi} = \int_{[a,b]} f$$

$\Rightarrow f = \tilde{\varphi} = \tilde{\psi}$ a.e.

$\Rightarrow f$ is measurable, and $\int_{[a,b]} f = \int_a^b f$. $\square$