Smooth Manifolds

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Contents

Chapter 1

Manifolds and Smoothness

1.1 Topological Manifolds

1.1.1 Elements of a Manifold

We start with the most basic type of manifolds: topological manifolds, and then equip them smooth structures.

Definition 1.1.1. Suppose M is a topological space. We say that M is a **topological manifold of dimension** n if it has the following properties:

- M is a Hausdorff space.
- M is second-countable.
- M is **locally Euclidean of dimension** n: each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

"Locally Euclidean of dimension n" means that for each point $p \in M$ we can find an open neighborhood U of p and an open set $\widehat{U} \subset \mathbb{R}^n$, and a homeomorphism $\varphi : U \to \widehat{U}$. There are equivalent definitions of "locally Euclidean".

EXERCISE 1 . Show that equivalent definitions of manifolds are obtained if instead of allowing U to be homeomorphic to any open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

Definition 1.1.2 (chart)**.** Let M be a topological n-manifold. A **coordinate chart** (or chart) on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to \varphi(U)$ is a homeomorphism.

Given a chart (U, φ), we call U a **coordinate domain** of each of its points. If in adition $\varphi(U)$ is an open ball in $\mathbb{R}^n$. then U is called a **coordinate ball**. The map φ is called a **(local) coordinate map**, and the component functions $x^1, \cdots, x^n$ in $\varphi(p) = (x^1(p), \cdots, x^n(p))$ are called **local coordinates** on U.

Example 1 . Consider 1-sphere $\mathbb{S}^1$, a topological subspace of $\mathbb{R}^2$, we show it is locally Euclidean. Denote

$$U_i^+ = \{(x^1, x^2) \in \mathbb{R}^2 : x^i > 0\}, \quad U_i^- = \{(x^1, x^2) \in \mathbb{R}^2 : x^i < 0\}.$$

Let $f : \mathbb{B}^1 \to \mathbb{R}$ be the continuous function

$$f(u) = \sqrt{1 - |u|^2}.$$

Then $U_i^+ \cap S^1$ is the graph of the function

$$x^1 = f(0, x^2) = \sqrt{1 - |x_2|^2}, \quad x^2 = f(x_1, 0) = \sqrt{1 - |x_1|^2}.$$

(the unit circle is given by the equation $(x^1)^2 + (x^2)^2 = 1$) Similarly, $U_i^- \cap S^1$ is the graph of

$$x^1 = -f(0, x^2) = -\sqrt{1 - |x_2|^2}, \quad x^2 = -f(x_1, 0) = -\sqrt{1 - |x_1|^2}.$$

Thus, each $U_i^\pm \cap S^1$ is locally Euclidean of dimension 2 since each point of $\mathbb{S}^1$ is in the domain of at least one of these charts.

1.1.2 Topological Properties of Manifolds

Compactness

Lemma 1.1.1. *Every topological manifold has a countable basis of precompact coordinate balls.*

Proof. Let M be a topological n-manifold, then M admits a trivial covering $M = \bigcup\{U : (U, \varphi) \text{ is a chart}\}$. Since M is second countable, there is a countable subcover, say, $M = \bigcup_{i=1}^\infty U_i$, where (U_i, φ_i) is a chart. For each i let $\mathcal{B}_i$ be the set of all rational balls $B_r(x)$ such that $B_{r'}(x) \subset \varphi_i(U_i)$ for some $r' > r$, that is,

$$\mathcal{B}_i = \{B_r(x) \subset \mathbb{R}^n : x \in \mathbb{Q}^n, r \in \mathbb{Q}, \exists r' > r : B_{r'}(x) \subset \varphi_i(U_i)\}.$$

Because φ is a homeomorphism and $\mathcal{B}_i$ is a basis for $\phi_i(U_i)$, $\varphi^{-1}(\mathcal{B}_i)$ is a basis for U_i, hence $\mathcal{B} = \bigcup_{i=1}^\infty \varphi_i^{-1}(\mathcal{B}_i)$ is a countable basis for M.

Now we show each basis element is precompact. Let $B_r(x) \in \mathcal{B}_i$, then $\overline{B_r(x)} \subset \varphi_i(U_i)$. As a continuous image of a compact set, $\varphi_i^{-1}(\overline{B_r(x)})$ is compact in M. Since M is Hausdorff, $\varphi_i^{-1}(\overline{B_r(x)})$ is closed, so

$$\overline{\varphi_i^{-1}(B_r(x))} \subset \varphi_i^{-1}(\overline{B_r(x)}).$$

Thus $\varphi_i^{-1}(\overline{B_r(x)})$ is precompact. $\square$

Local compactness and paracompactness

Proposition 1.1.1. *Every topolocial manifold is locally compact.*

Proof. This is because every topological manifold has a countable basis of precompact sets. □

Definition 1.1.3. Let M be a topological space.

- A collection $\mathcal{X}$ of subsets of M is said to be **locally finite** if each point of M has a neighborhood that intersects at most finitely many of the sets in $\mathcal{X}$.
- Given a cover $\mathcal{U}$ of M, another cover $\mathcal{V}$ is called a **refinement** of $\mathcal{U}$ if for each $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ such that $V \subset U$.
- We say that M is **paracompact** if every open cover of M admits an open, locally finite refinement.

Lemma 1.1.2. *Every topological manifold M can be exhausted by compact sets: there is a sequence of compact sets $\{K_j\}$ such that $K_j \subset \text{Int } K_{j+1}$ for all j and $M = \bigcup_{j=1}^{\infty} K_j$.*

Proof. M has a countable basis $\mathcal{B} = \{U_i\}$, where each U_i is precompact. Define $K_1, \cdots, K_n$ as follows:

1. $K_1 = \overline{U_1}$.
2. Assume $K_1, \cdots, K_n$ have been defined, then $K_n \subset \bigcup_{i=1}^{\infty} U_i$, hence there is a finite subcover $\bigcup_{i=1}^{N} U_i \supset K_n$. Let $K_{n+1} = \bigcup_{i=1}^{N} \overline{U_i} \cup \overline{U_{n+1}}$, then clearly $\bigcup_{j=1}^{\infty} \supset \bigcup_{i=1}^{\infty} U_n = M$.

□

Theorem 1.1.1. *Given a topological manifold M, an open cover $\mathcal{X}$, and a basis $\mathcal{B}$, there is a countable locally finite refinement of $\mathcal{X}$, consisting of elements of $\mathcal{B}$.*

Proof. Let $\{K_j\}$ be a compact exhaustion of M, define $\widehat{K_j} = K_{j+1} \setminus \text{Int } K_j, O_j = \text{Int } K_{j+2} \setminus K_{j-1}$. Then

- $\widehat{K_j}$ is compact,
- $\widehat{K_j} \subset O_j$,
- $O_j \cap O_l \neq \varnothing \iff |j - l| \leq 2$.

For $x \in \widehat{K_j}$ there exists $U_x \in \mathcal{X}$ such that $U_x \ni x$, then there is a basis element $B_x \in \mathcal{B}$ with $x \in B_x \subset U_x \cap O_j$, then $\widehat{K_j} \subset \bigcup_{x \in \widehat{K_i}} B_x$, so there is a finite subcover $\mathcal{Y}_j$ of $\widehat{K_j}$. Clearly $\bigcup_{j=1}^{\infty} \widehat{K_j} = M$, so $\mathcal{Y} := \bigcup_{k=1}^{\infty} \mathcal{Y}_j$ is a countable refinement of $\mathcal{X}$. Hence $\mathcal{Y}$ is locally finite. □

Connectedness

In a topological manifold, connectedness is equivalent to path-connectedness. Recall that a topological space X is

- **connected** if there do not exist two disjoint, nonempty, open subsets U, V of X such that $U \cup V = X$,
- **path-connected** if every pair of points in X can be joined by a path (continuous image of an interval) in X,
- **locally path-connected** if for every $x \in X$ and open set $U \ni x$ there is a path-connected open set V such that $x \in V \subset U$.

A maximal connected subset of X is called a **component** (or **connected component**) of X.

Proposition 1.1.2 (properties of connected spaces)**.** *Let X, Y be topological spaces.*

1. *If $F : X \to Y$ is continuous and X is connected, then $F(X)$ is connected.*
2. *A union of connected subspaces of X with a point in common is connected.*
3. *The components of X are disjoint nonempty closed subsets whose union is X.*
4. *If S is a subset of X that is both open and closed, then S is a union of components of X.*

Proposition 1.1.3 (properties of locally path-connected spaces)**.** *Let X be a locally path-connected topological space.*

1. *The components of X are open in X.*
2. *The path components of X are equal to its components.*
3. *X is connected if and only if it is path-connected.*
4. *Every open subset of X is locally path-connected.*

Proof.

1. Let C be a component of X, and let $x \in C$, then x has a path-connected neighborhood basis, thus it is a connected neighborhood basis. Any open set in this basis must be contained in C, as C is a maximal connected subsets. This shows that C is open.
2. We show that C is a path component of X iff C is a component of X. Suppose C is a path component, then C itself is connected

$\square$

Proposition 1.1.4. *Let M be a topological manifold.*

1. *M is locally path-connected.*
2. *M is connected $\iff$ M is path-connected.*
3. *The connected components of M are the same as its path components.*
4. *M has countably many components, each of which is open and a connected topological manifold.*

Proof. Since M is locally Euclidean and $\mathbb{R}^n$ is locally path-connected, M is locally path-connected. (2) and (3) comes from Proposition 1.1.3. To prove (4), note that each component is open in M, so the collection of components is an open cover of M. Since M is second countable, this cover has a countable subcover. Since the components are disjoint, this cover must be countable.

□

1.1.3 Quotient Topology and Projective Spaces

Let $\sim$ be an equivalence relation on the set X. We denote the set of equivalence classes by $X/\sim$ and call this set the *quotient* of X by the equivalence relation $\sim$. There is a natural *projection map*

$$\begin{aligned} \pi : X &\to X/\sim, \\ x &\mapsto [x]. \end{aligned}$$

We call a set U in $X/\sim$ *open* if and only if $\pi^{-1}(U)$ is open in X. Clearly $\varnothing$ and $X/\sim$ are open. Since pre-image commutes with unions and intersections, the collection of open sets in $X/\sim$ is closed under arbitrary union and finite intersection, hence is a topology.

Definition 1.1.4. The collection $\{U \subset X/\sim : \pi^{-1}(U) \text{ is open in } X\}$ is called the **quotient topology** on $X/\sim$. With this topology, $X/\sim$ is called the **quotient space** of X by the equivalence relation $\sim$.

Exercise 2 . With the quotient topology, the projection map $\pi : X \to X/\sim$ is continuous.

Proof. Let U be an open set in $X/\sim$, then by definition $\pi^{-1}(U)$ is open in X, so π is continuous. □ □

Let Y be another topological space, and let $f : X \to Y$ be constant on each equivalence class. It induces a map $\overline{f} : X/\sim \to Y$ by

$$\overline{f}([x]) = f(x), \quad x \in X.$$

We can draw a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\pi"'] & Y \\ & X/\sim \arrow[u, "\overline{f}"'] \end{tikzcd}$$

Proposition 1.1.5 (characteristic property)**.** *The induced map $\overline{f} : X/\sim \to Y$ is continuous if and only if the map $f : X \to Y$ is continuous.*

Proof. Suppose $\overline{f}$ is continuous, then since π is continuous, so is $f = \overline{f} \circ \pi$. On the other hand, suppose f is continuous. Let V be an open set in Y, then $f^{-1}(V) = \pi^{-1}(\overline{f}^{-1}(V))$ is open in X. By the definition of quotient topology, $\overline{f}^{-1}(V)$ is open in $X/\sim$, hence $\overline{f}$ is continuous. □ □

If A is a subspace of a topological space X, we define a relation $\sim$ on X by declaring $x \sim x$ for all $x \in X$ and $x \sim y$ for all $x, y \in A$. We say that the quotient space $X/\sim$ is obtained from S by identifying A to a point.

EXAMPLE 2 . Let $I = [0,1]$ and $I/\sim$ be the quotient space obtained from I by identifying the two points $\{0,1\}$ to a point. The function $f : I \to S^1, f(x) = e^{2\pi ix}$ assumes the same value at 0 and 1, and so induces a function $\overline{f} : I/\sim \to S^1$.

Proposition 1.1.6. *The function $\overline{f} : I/\sim \to S^1$ is a homeomorphism.*

Proof. Since f is continuous, $\overline{f}$ is also continuous. $\overline{f}$ is a bijection because $\overline{f}(0) = \overline{f}(1) = e^{i0}$ (we identify 0 and 1 in I), and $\overline{f}$ is clearly a bijection on $[0,1] \setminus \{0,1\}$. The quotient $I/\sim$ is compact as the continuous image of I under the projection map. Thus, $\overline{f}$ is a continuous bijection from the compact space $I/\sim$ to the Hausdorff space S^1, hence $\overline{f}$ is a homeomorphism. □ □

The Hausdorff property is of vital importance in the theory of manifolds.

Proposition 1.1.7. *If the quotient space $X/\sim$ is Hausdorff, then the equivalence class $[p]$ of any point p in X is closed in X.*

Proof. Let $\pi : X \to X/\sim$ be the projection map and let $X/\sim$ be Hausdorff, then for any $p \in X$, $\{\pi(p)\}$ is closed in $X/\sim$. Since π is contiunous, $\pi^{-1}(\{\pi(p)\}) = [p]$ is closed in X. □ □

Open equivalence relations

Now we derive conditions under which a quotient space is Hausdorff or second countable.

Definition 1.1.5. An equivalence relation $\sim$ on a topological space X is said to be **open** if the projection $\pi : X \to X/\sim$ is open.

Proposition 1.1.8. *Let $\sim$ be an equivalence relation on X. Then $\sim$ is open if and only if for every open set $U \subset X$, the set*

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$$

is open.

Proof. Suppose $\sim$ is open, then $\pi(U)$ is open. Since π is continuous, $\pi^{-1}(\pi(U))$ is open. Conversely, let U be open in X. Then

$$\pi(U) = \pi\left(\bigcup_{x \in U} [x]\right)$$

□

EXAMPLE 3 . The projection map to a quotient space is in general not open. Let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points $1, -1$, and let $\pi : \mathbb{R} \to \mathbb{R}/\sim$ be the projection map. Let $V = (-2, 0)$, then

$$\pi^{-1}(\pi(V)) = (-2, 0) \cup \{1\},$$

which is not open in $\mathbb{R}$. Thus, $\pi(V)$ is not open in the quotient space, so $\pi : \mathbb{R} \to \mathbb{R}/\sim$ is not an open map.

Definition 1.1.6. Given an equivalence relation $\sim$ on X, the set

$$R = \{(x, y) \in X \times X : x \sim y\}$$

is called the **graph** of the equivalence relation $\sim$.

Theorem 1.1.2. *Suppose $\sim$ is an open equivalence relation on X. Then the quotient space $X/\sim$ is Hausdorff if and only if the graph R of the equivalence relation is closed in $X \times X$.*

Proof. ($\Longrightarrow$) Suppose $X/\sim$ is Hausdorff, we will show that $X \times X \setminus R$ is open. Let $(x, y) \in X \times X \setminus R$, then x is not equivalent to y, hence $[x] \neq [y]$ in $X/\sim$. Since $X/\sim$ is Hausdorff, there exist disjoint open sets $\tilde{U}, \tilde{V} \subset X/\sim$ with $[x] \in \tilde{U}$ and $[y] \in \tilde{V}$. Since $\tilde{U} \cap \tilde{V} = \varnothing$, no element in $U := \pi^{-1}(\tilde{U})$ is equivalent to an element of $V := \pi^{-1}(\tilde{V})$. Thus $U \times V$ is open and $U \times V \cap R = \varnothing$, so $(x, y) \in U \times V \subset X \times X \setminus R$.
($\Longleftarrow$) Suppose R is closed in $X \times X$ and $[x] \neq [y]$ in $X/\sim$. Then $x \nsim y$. Thus $(x, y) \in X \times X \setminus R$. Since $X \times X \setminus R$ is open, there exists an open set $U \times V$ such that $(x, y) \in U \times V \subset X \times X \setminus R$. Thus no element of U is equivalent to an element of V, so $\pi(U) \cap \pi(V) = \varnothing$. Since π is an open map, $\pi(U)$ and $\pi(V)$ are open in $X/\sim$. Clearly $[x] \in \pi(U)$ and $[y] \in \pi(V)$, hence $X/\sim$ is Hausdorff. □

Theorem 1.1.3. *Let $\sim$ be an open equivalence relation on a space X with projection $\pi : X \to X/\sim$. If $\mathcal{B} = \{B_\alpha\}$ is a basis for X, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $X/\sim$.*

Proof. Let W be an open set in $X/\sim$, we want to find an element $\pi(B_\alpha) \subset W$. Let $[x] \in W$, then $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open, there is a basis element B_α such that $x \in B_\alpha \subset \pi^{-1}(W)$. Then $[x] = \pi(x) \in \pi(B_\alpha) \subset W$. $\square$

Corollary 1.1.1. *If $\sim$ is an open equivalence relation on a second countable space X, then the quotient space $X/\sim$ is second countable.*

Real projective spaces

Define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ by

$$x \sim y \text{ iff } y = tx \text{ for some } t \in \mathbb{R} \setminus \{0\}.$$

Definition 1.1.7 (real projective spaces)**.** The **real projective space** $\mathbb{RP}^n$ is the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by the above equivalence relation. We denote the equivalence class of a point $(a_0, \cdots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ by $[a_0, \cdots, a_n]$ and let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ be the projection. We call $[a_0, \cdots, a_n]$ the **homogeneous coordinates** on $\mathbb{RP}^n$.

We define an equivalence relation $\sim$ on S^n by identifying the antipodal points:

$$x \sim y \text{ iff } x = \pm y, \quad x, y \in S^n.$$

We then have a bijection $\mathbb{RP}^n \leftrightarrow S^n/\sim$.

EXERCISE 3 . Prove that the map

$$\begin{aligned} f : \mathbb{R}^{n+1} \setminus \{0\} &\to S^n \\ f(x) &= \frac{x}{|x|} \end{aligned}$$

induces a homeomorphism $\overline{f} : \mathbb{RP}^n \to S^n/\sim$.
(Hint: Find an inverse map $\overline{g} : S^n/\sim \to \mathbb{RP}^n$ and show that $\overline{f}$ and $\overline{g}$ are continuous.)

Proof. Consider the diagram

$$\begin{tikzcd}
\mathbb{R}^{n+1} \setminus \{0\} \arrow[d, "\pi_1"'] & S^n \arrow[l, "f^{-1}"'] \arrow[d, "\pi_2"] \arrow[dl, "\pi_1 \circ f^{-1}"] \\
\mathbb{RP}^n & S^n/\sim \arrow[l, "\overline{g}"]
\end{tikzcd}$$

Clearly $\pi_1 \circ f^{-1}$ is continuous, hence the induced map $\overline{\pi_1 \circ f^{-1}} : S^n/\sim \to \mathbb{RP}^n$ is continuous, and we denote it $\overline{g}$. Moreover, $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$, and by another diagram

$$\begin{tikzcd}
\mathbb{R}^{n+1} \setminus \{0\} \arrow[r, "f"] \arrow[d, "\pi_1"] \arrow[dr, "\pi_2 \circ f"] & S^n \arrow[d, "\pi_2"] \\
\mathbb{RP}^n \arrow[r, "\overline{f}"] & S^n/\sim
\end{tikzcd}$$

we have obtain the continuous induced map $\overline{f} := \overline{\pi_2 \circ f}$. From the diagrams we also have

$$\overline{f} = \pi_2 \circ f \circ \pi_1^{-1},$$
$$\overline{g} = \pi_1 \circ f^{-1} \circ \pi_2^{-1},$$

hence

$$\overline{f} \circ \overline{g} = \pi_2 \circ f \circ \pi_1^{-1} \circ \pi_1 \circ f^{-1} \circ \pi_2^{-1} = \mathrm{id}_{S^n/\sim},$$
$$\overline{g} \circ \overline{f} = \pi_1 \circ f^{-1} \circ \pi_2^{-1} \circ \pi_2 \circ f \circ \pi_1^{-1} = \mathrm{id}_{\mathbb{RP}^n}.$$

Therefore $\overline{f}$ is a continuous bijection, with its inverse $\overline{g}$ also being continuous. □

Proposition 1.1.9. *The equivalence relation $\sim$ on $\mathbb{R}^{n+1} \setminus \{0\}$ in the definition of $\mathbb{RP}^n$ is an open equivalence relation.*

Proof. Let $U \subset \mathbb{R}^{n+1} \setminus \{0\}$ be open, then $\pi(U)$ is open in $\mathbb{RP}^n$ if and only if $\pi^{-1}(\pi(U))$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$. □

Proposition 1.1.10. *$\mathbb{RP}^n$ is second countable and Hausdorff.*

Proof. Since $\sim$ is an open equivalence relation on the second countable space $\mathbb{R}^{n+1}$, $\mathbb{RP}^n = X/\sim$ is second countable.

Let $S = \mathbb{R}^{n+1} \setminus \{0\}$ and let

$$R = \{(x, y) \in S \times S : y = tx \text{ for some } t \in \mathbb{R} \setminus \{0\}\}.$$

Viewed as column vectors, $[x\ y]$ is an $(n+1) \times 2$ matrix, and R can be identified as the set of matrices $[x\ y]$ in $S \times S$ of rank ≤ 1. Then the matrix $\begin{pmatrix} x_1 & tx_1 \\ x_2 & tx_2 \\ \vdots & \vdots \\ x_{n+1} & tx_{n+1} \end{pmatrix}$ has all 2×2 minors equal to 0.

□

1.2 Smooth Structures

1.2.1 Smooth Functions Between Euclidean Spaces

If U and V are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, a function $F : U \to V$ is called **smooth** (or C^∞, or infinitely differentiable) if each of its component functions has continuous partial derivatives of all orders. If in addition F is bijective and has a smooth inverse map, it is called a **diffeomorphism**.

Theorem 1.2.1 (inverse function theorem)**.** *Suppose U, V are open subsets of $\mathbb{R}^n$, and $F : U \to V$ is a smooth function. If $DF(a)$ is invertible at some point $a \in U$, then there exist connected neighborhoods $U_0 \subset U$ of a and $V_0 \subset V$ of $F(a)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

1.2.2 Smooth Structures on Manifolds

Let M be a topological n-manifold. If $(U, \varphi), (V, \psi)$ are two charts such that $U \cap V \neq \varnothing$, the composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the **transition map** from φ to ψ. Two charts $(U, \varphi), (V, \psi)$ are said to be **smoothly compatible** if either $U \cap V = \varnothing$ or $\psi \circ \varphi^{-1}$ is a diffeomorphism.

Temporarily denote the smooth compatibility of $(U, \varphi), (V, \psi)$ by $(U, \varphi) \sim (V, \psi)$. Then

- clearly $(U, \varphi) \sim (V, \psi)$.
- If $(U, \varphi) \sim (V, \psi)$, then $(\psi \circ \varphi^{-1})^{-1} = \varphi \circ \psi^{-1}$ is still a diffeomorphism, hence $(V, \psi) \sim (U, \varphi)$.
- Let $(U, \varphi) \sim (V, \psi)$ and $(V, \psi) \sim (W, \theta)$, then $\varphi \circ \psi^{-1}$ and $\psi \circ \theta^{-1}$ are diffeomorphisms, hence $(\varphi \circ \psi^{-1}) \circ (\psi \circ \theta^{-1}) = \varphi \circ \theta^{-1}$ is a diffeomorphism.

Definition 1.2.1 (atlas)**.** We define an **atlas** for M to be a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a **smooth atlas** if any two charts in $\mathcal{A}$ are smoothly compatible with each other.

Definition 1.2.2 (smooth structure)**.** Let M be a topological manifold. A smooth atlas $\mathcal{A}$ on M is **maximal** if it is not properly contained in any larger smooth atlas.
A **smooth structure** on M is a maximal smooth atlas.

Definition 1.2.3 (smooth manifold)**.** A **smooth manifold** is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.

Proposition 1.2.1 (smooth compatibility)**.** *Let M be a topological manifold.*

1. *Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the* ***smooth structure determined by*** *$\mathcal{A}$.*
2. *Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.*

Proof. 1. Let $\mathcal{A}$ be a smooth atlas for M, and let $\overline{\mathcal{A}}$ be the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. Then clearly $\mathcal{A} \subset \overline{\mathcal{A}}$. We need to show that for any $(U, \varphi), (V, \psi) \in \overline{\mathcal{A}}$, the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is smooth.
Let $x = \varphi(p) \in \varphi(U \cap V)$ be arbitrary. Since the domains of the charts in $\mathcal{A}$ cover M, there exists a chart $(W, \theta) \in \mathcal{A}$ such that $p \in W$. Because every

chart in $\overline{\mathcal{A}}$ is smoothly compatible with (W, θ), the maps $\theta \circ \varphi^{-1}$ and $\psi \circ \theta^{-1}$ are smooth. Since $p \in U \cap V \cap W$, it follows that $\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\psi \circ \theta^{-1})$ is smooth on a neighborhood of x. Thus $\psi \circ \varphi^{-1}$ is smooth in a neighborhood of each point in $\varphi(U \cap V)$. Therefore $\overline{\mathcal{A}}$ is a smooth atlas. If $\overline{\mathcal{A}}$ is properly contained in a larger smooth atlas $\mathcal{E}$, then there is a smooth chart $(U, \varphi) \in \mathcal{E} \setminus \overline{\mathcal{A}}$ which is not compatible with a chart $(V, \psi) \in \mathcal{A}$. But $\mathcal{A} \subset \overline{\mathcal{A}} \subset \mathcal{E}$ implies (U, φ) is compatible with (V, ψ), a contradiction. This shows the maximality of $\overline{\mathcal{A}}$. If $\mathcal{B}$ is any other maximal smooth atlas containing $\mathcal{A}$, each of its charts is smoothly compatible with each chart in $\mathcal{A}$, so $\mathcal{B} \subset \overline{\mathcal{A}}$. Since $\mathcal{B}$ is maximal, $\mathcal{B} = \overline{\mathcal{A}}$.

2. Let $\mathcal{A}, \mathcal{B}$ be smooth atlases for M.
   ($\Longrightarrow$): Suppose they determine the same smooth structure, then $\mathcal{A}$ and $\mathcal{B}$ are contained in a unique maximal smooth atlas $\overline{\mathcal{M}}$. By the construction in (1), every chart in $\overline{\mathcal{M}}$ is compatible with every chart in $\mathcal{A} \cup \mathcal{B}$.

   To show $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas, we need to show that for any $(U, \varphi), (V, \psi) \in \mathcal{A} \cup \mathcal{B}$, the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is smooth. If $(U, \varphi), (V, \psi) \in \mathcal{A}$ or $(U, \varphi), (V, \psi) \in \mathcal{B}$, then this is the same as part (1). We may assume $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$. If $\mathcal{A} \cap \mathcal{B} = \varnothing$, then $U \cap V = \varnothing$, so by definition they are smoothly compatible, implying that $\mathcal{A} \cup \mathcal{B}$ is a smooth atlas. Now suppose $\mathcal{A} \cap \mathcal{B} \neq \varnothing$ and $U \cap V \neq \varnothing$. Let $x = \varphi(p) \in \varphi(U \cap V)$ be arbitrary, then there exists a chart $(W, \theta) \in \mathcal{A}$ with $p \in W$. Because every chart in $\overline{\mathcal{M}} \supset \mathcal{A} \cup \mathcal{B}$ is smoothly compatible with (W, θ), the maps $\theta \circ \varphi^{-1}$ and $\psi \circ \theta^{-1}$ are smooth. The following is the same as part (1).
   ($\Longleftarrow$): Suppose $\mathcal{A}$ and $\mathcal{B}$ determine different smooth structures $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$, then there is a smooth chart $(U, \varphi) \in \overline{\mathcal{A}}$ such that $(U, \varphi) \notin \overline{\mathcal{B}}$. Then (U, φ) is smoothly compatible with each $(A, \alpha) \in \mathcal{A}$, and there exists $(V, \psi) \in \mathcal{B}$ with which (U, φ) is not compatible. Since smooth compatibility is an equivalence relation, (A, α) is not smoothly compatible with (V, ψ), thus $\mathcal{A} \cup \mathcal{B}$ is not a smooth atlas.

$\square$

1.2.3 Local Coordinate Representations

If M is a smooth manifold, any chart (U, φ) contained in the given maximal smooth atlas is called a **smooth chart**, and the coordinate map φ is called a **smooth coordinate map**. The domain of a smooth coordinate chart is called a **smooth coordinate domain** or **smooth coordinate neighborhood**.

If the image of a smooth coordinate domain under a smooth coordinate map is a ball in Euclidean space, the domain is called a **smooth coordinate ball**. A **smooth coordinate cube** is defined similarly.

Definition 1.2.4. We say a set $B \subset M$ is a **regular coordinate ball** if there is a smooth coordinate ball $B' \supset \overline{B}$ and a smooth coordinate map $\varphi : B' \to \mathbb{R}^n$

such that for some positive real numbers $r > r'$:

$$\varphi(B) = B_r(0), \quad \varphi(\overline{B}) = \overline{B}_r(0), \quad \varphi(B') = B_{r'}(0).$$

Proposition 1.2.2. *Every smooth manifold has a countable basis of regular coordinate balls.*

1.3 Examples of Smooth Manifolds

Example 4 . A topological manifold M of dimension 0 is just a countable discrete space. For each $p \in M$ the only neighborhood of p that is homeomorphic to an open subset of $\mathbb{R}^0$ is $\{p\}$ itself, and there is exactly one coordinate map $\varphi : \{p\} \to \mathbb{R}^0$.

Example 5 (Euclidean spaces). $\mathbb{R}^n$ is a smooth n-manifold with the smooth structure determined by the atlas consisting of the single chart $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$. We call this the **standard smooth structure** on $\mathbb{R}^n$ and the resulting coordinate map **standard coordinates**.

Example 6 (another smooth structure on $\mathbb{R}$). Consider the homeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ given by $\psi(x) = x^3$. The atlas consisting of the single chart $(\mathbb{R}, \psi)$ defines a smooth structure on $\mathbb{R}$.

Example 7 (finite-dimensional vector space). Let V be a finite-dimensional real vector space. Any norm on V determines a topology. With this topology, V is a topological n-manifold, and has a natural smooth structure defined as follows. Each basis $e_1, \cdots, e_n$ of V defines a basis isomorphism $E : \mathbb{R}^n \to V$ by

$$E(x) = \sum_{i=1}^{n} x^i e_i.$$

This map is clearly a homeomorphism, so (V, E^{-1}) is a chart. If $\tilde{e}_1, \cdots, \tilde{e}_n$ is any other basis and $\tilde{E}(x) = \sum_{j=1}^{n} x^j \tilde{e}_j$ is the corresponding isomorphism, then there is an invertible matrix (A_i^j) such that $e_i = \sum_{j=1}^{n} A_i^j \tilde{e}_j$ for each i.

The transition map between the two charts is given by $\tilde{E}^{-1} \circ E(x) = \tilde{x}$, where $\tilde{x} = (\tilde{x}^1, \cdots, \tilde{x}^n)$ is determined by

$$\sum_{j=1}^{n} \tilde{x}^j \tilde{e}_j = \sum_{i=1}^{n} x^i E_i = \sum_{i=1}^{n} x^i \sum_{j=1}^{n} A_i^j \tilde{e}_j.$$

It follows that $\tilde{x}^j = \sum_{j=1}^{n} A_i^j \tilde{e}_j$, thus $\tilde{E}^{-1} \circ E$ is an invertible linear map and hence a diffeomorphism, so any two such charts are smoothly compatible. The collection of all such charts thus defines a smooth structure.

Example 8 (graphs of continuous functions). Let $U \subset \mathbb{R}^n$ be an open set, let $f : U \to \mathbb{R}^k$ be a continuous function. The **graph** of f is the subset of $\mathbb{R}^n \times \mathbb{R}^k$ is defined by

$$\Gamma(f) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : y = f(x)\}$$

with the subspace topology. Let $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be the projection onto the first factor, and let $\varphi = \pi_1|_{\Gamma(f)}$:

$$\varphi(x, y) = x, \quad (x, y) \in \Gamma(f).$$

Then

- φ is continuous,
- φ is a homeomorphism since $\varphi^{-1}(x) = (x, f(x))$.

Thus $\Gamma(f)$ is homeomorphic to $\varphi(\Gamma(f)) = U \subset \mathbb{R}^n$, so $\Gamma(f)$ is a topological manifold of dimension n.
$(\Gamma(f), \varphi)$ is a global coordinate chart, called **graph coordinates**.

In summary, the graph of a continuous function is a topological manifold.

Spheres

Example 9 (spheres). For each $n \in \mathbb{N}$, the unit n-sphere $\mathbb{S}^n$ is Hausdorff and second countable since it is a topological subspace of $\mathbb{R}^{n+1}$. We want to show $\mathbb{S}^n$ is locally Euclidean. For each $i = 1, \cdots, n+1$, let

$$U_i^+ = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i > 0\},$$
$$U_i^- = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_i < 0\}.$$

Let $f : \mathbb{B}^n \to \mathbb{R}$ be the continuous function (Notice: $\mathbb{B}^n \subset \mathbb{R}^n$)

$$f(u) = \sqrt{1 - |u|^2}.$$

The equation of the sphere is given by (temporarily using subscripts)

$$x_1^2 + \cdots + x_{n+1}^2 = 1,$$

hence

$$x_i^2 = 1 - x_1^2 - \cdots - x_{i-1}^2 - x_{i+1}^2 - \cdots - x_{n+1}^2,$$

thus

$$x_1^2 + \cdots + x_{i-1}^2 + x_{i+1}^2 + x_{n+1}^2 \leq 1 \quad (\text{so that } (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1}) \in \mathbb{B}^n)$$

and

$$x^i = \pm\sqrt{1 - x_1^2 - \cdots - x_{i-1}^2 - x_{i+1}^2 - \cdots - x_{n+1}^2} = \pm f((x^1, \cdots, x^{i-1}, x^{i+1}, \ldots, x^{n+1})).$$

Therefore $U_i^+ \cap \mathbb{S}^n$ is the graph of the function[1]

$$x^i = f(x^1, \cdots, \widehat{x^i}, \cdots, x^{n+1}),$$

where $(x^1, \cdots, \widehat{x^i}, \cdots, x^{n+1}) := (x^1, \cdots, x^{i-1}, x^{i+1}, \ldots, x^{n+1})$. Similarly, $U_i^- \cap \mathbb{S}^n$ is the graph of

$$x^i = -f(x^1, \cdots, \widehat{x^i}, \cdots, x^{n+1}).$$

Thus, each $U_i^\pm \cap \mathbb{S}^n$ is locally Euclidean of dimension n. The maps $\varphi_i^\pm : U_i^\pm \cap \mathbb{S}^n \to \mathbb{B}^n$ given by

$$\varphi_i^\pm(x^1, \cdots, x^{n+1}) = (x^1, \cdots, \widehat{x^i}, \cdots, x^{n+1})$$

are graph coordinates for $\mathbb{S}^n$. Since each point of $\mathbb{S}^n$ is in the domain of at least one of these $2n+2$ charts, $\mathbb{S}^n$ is a topological n-manifold (See Example 1 for the case $n = 1$).

Real Projective Spaces

$\mathbb{RP}^n$ is the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation

$$x \sim y \text{ iff } y = tx \text{ for some } t \in \mathbb{R} \setminus \{0\}.$$

The equivalence class of a point $(a_0, \cdots, a_n) \in \mathbb{R}^{n+1} \setminus \{0\}$ is denoted by $[a_0, \cdots, a_n]$, called the homogeneous coordinates on $\mathbb{RP}^n$.

In $\mathbb{R}^3$, if $a_0 = 0$, then $[a_0, a_1, a_2]$ is the x-axis. In fact, the condition $a_0 \neq 0$ is independent of the choice of a representative for $[a_0, \cdots, a_n]$.

We may define

$$U_0 = \{[a_0, \cdots, a_n] \in \mathbb{RP}^n : a_0 \neq 0\}.$$

Similarly, for each $i = 1, \cdots, n$ let

$$U_i = \{[a_0, \cdots, a_n] \in \mathbb{RP}^n : a_i \neq 0\}.$$

Define

$$\begin{aligned} \phi_0 : U_0 &\to \mathbb{R}^n \\ [a_0, \cdots, a_n] &\mapsto \left(\frac{a_1}{a_0}, \cdots, \frac{a_n}{a_0} \right). \end{aligned}$$

This map has a continuous inverse

$$(b_1, \cdots, b_n) \mapsto [1, b_1, \cdots, b_n]$$

and is therefore a homeomorphism. Similarly, there are homeomorphisms for each $i = 1, \cdots, n$:

$$\begin{aligned} \phi_0 : U_i &\to \mathbb{R}^n \\ [a_0, \cdots, a_n] &\mapsto \left(\frac{a_1}{a_i}, \cdots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \cdots, \frac{a_n}{a_i} \right) = \left(\frac{a_1}{a_i}, \widehat{\frac{a_i}{a_i}}, \cdots, \frac{a_n}{a_i} \right). \end{aligned}$$

This shows $\mathbb{RP}^n$ is locally Euclidean with the (U_i, ϕ_i) as charts.

[1] think $U_i^\pm \cap \mathbb{S}^n$ of north and south hemisphere

Grassman manifolds

EXAMPLE 10 (GRASSMAN MANIFOLDS). Let V be a real vector space and $\dim V = d < \infty$, the **Grassmannian** $\mathrm{Gr}_k(V)$ is the set of k-dimensional vector subspaces of V.

The Einstein Summation Convention

We often abbreviate expressions such as $\sum_i x^i e_i$ to $x^i e_i$. This is a rule called the **Einstein summation convention**.

1.4 Smooth Functions

1.4.1 Smooth Functions on Manifolds

Suppose M is a smooth n-manifold, $k \in \mathbb{N}^+$, $f : M \to \mathbb{R}^k$ is any function.

Definition 1.4.1 (smooth function)**.** We say that f is a **smooth function** if for every $p \in M$ there exists a smooth chart (U, φ) for M whose domain contains p and such that

$$f \circ \varphi^{-1} \text{ is smooth on } \varphi(U) \subset \mathbb{R}^n.$$

EXERCISE 4 . Let M be a smooth manifold with or without boundary. Show that pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative algebra over $\mathbb{R}$. (See Appendix B, p. 624, for the definition of an algebra.)

Proof. Let $f, g \in C^\infty(M)$, then clearly $(C^\infty(M), +)$ is an abelian group. Multiplicative associativity is obvious, so it remains to show $fg \in C^\infty(M)$. By definition, there are smooth charts near a point $(U, \varphi), (V, \psi)$ near a point $p \in M$ such that $f \circ \varphi^{-1}$ is smooth on $\varphi(U)$ and $g \circ \psi^{-1}$ is smooth on $\psi(V)$. $\square$

EXERCISE 5 . Let M be a smooth manifold with or without boundary, and suppose $f : M \to \mathbb{R}^k$ is a smooth function. Show that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth for every smooth chart (U, φ) for M.

Proof. Recall that any two smooth charts are smoothly compatible. Let (U, φ) be an arbitrary smooth chart and let $p \in U$. Then for this p there exists a smooth chart (V, ψ) such that $V \ni p$ and $f \circ \psi^{-1}$ is smooth on $\psi(U)$. Since (U, φ) is smoothly compatible with (V, ψ), $\psi \circ \varphi^{-1}$ is a diffeomorphism. Now

$$f \circ \varphi^{-1} = (f \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})$$

is smooth on $\varphi(U)$, so f is a smooth function. $\square$

Definition 1.4.2 (coordinate representation)**.** Given a function $f : M \to \mathbb{R}^k$ and a chart (U, φ) for M, the function $\widehat{f} : \varphi(U) \to \mathbb{R}^k$ defined by $\widehat{f}(x) = f \circ \varphi^{-1}(x)$ is called the **coordinate representation** of f.

EXAMPLE 11 . Consider $f(x, y) = x^2 + y^2$ defined on $\mathbb{R}^2$. In polar coordinates on the set $U = \{(x, y) : x > 0\}$, it has the coordinate representation $\widehat{f}(r, \theta) = r^2$.

1.4.2 Smooth Maps Between Manifolds

Definition 1.4.3 (smooth map)**.** Let M, N be smooth manifolds, $F : M \to N$ be any map. We say that F is a **smooth map** if for every $p \in M$, there exists smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that

- $F(U) \subset V$, and
- $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$.

Proposition 1.4.1. *Every smooth map is continuous.*

Proof. Let M, N be smooth manifolds and $F : M \to N$ be smooth. Let $p \in M$, then there are smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is a smooth function, hence is continuous. Then

$$F|_U = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi : U \to V$$

is continuous in a neighborhood of p. Since p is arbitrary, F is continuous on M. □

Proposition 1.4.2. *Let M and N be smooth manifolds and let $F : M \to N$ be a map.*

1. *If every $p \in M$ has a neighborhood U such that $F|_U$ is smooth, then F is smooth.*
2. *If F is smooth, then its restriction to every open subset is smooth.*

Proof.

1. Let $p \in M$, then there are smooth charts (U, φ) containing p and (V, ψ) containing $F|_U(p)$ such that $F|_U(U) \subset V$ and $\psi \circ F|_U \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is a smooth function. It is obvious that $F|_U(U) = F(U)$, so all $F|_U$'s appeared in the above expressions can be substituted by F, thus F is smooth.
2. Let W be an open set in M. By the smoothness of F, for every $p \in W$ there exists smooth charts (U_p, φ_p) and (V_p, ψ_p) such that $F(U_p) \subset V_p$ and $\psi_p \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$. By considering the set $W \cap U_p$ it is easy to see that $F|_W$ is smooth.

□

Theorem 1.4.1 (gluing lemma)**.** *Let M, N be smooth manifolds and $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Suppose for each α we are given a smooth map $F_\alpha : U_\alpha \to N$ such that*

$$F_\alpha|_{U_\alpha\cap U_\beta} = F_\beta|_{U_\alpha\cap U_\beta}\ \forall \alpha, \beta.$$

Then there exists a unique smooth map $F : M \to N$ such that

$$F|_{U_\alpha} = F_\alpha\ \forall \alpha \in A.$$

A smooth map $F : M \to N$ induces a smooth structure on N.

1.4.3 Diffeomorphisms

Definition 1.4.4. If M, N are smooth manifolds, a **diffeomorphism** from M to N is a smooth bijective map $F : M \to N$ that has a smooth inverse. We say that M and N are **diffeomorphic** if there exists a diffeomorphism between them.

Example 12 . Let $F : \mathbb{B}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{B}^n$ be

$$F(x) = \frac{x}{\sqrt{1-|x|^2}}, \quad G(y) = \frac{y}{\sqrt{1+|y|^2}}.$$

Then F, G are diffeomorphisms.

Proof. First we compute $F \circ G$ and $G \circ F$.

$$(F\circ G)(y) = F(G(y)) = \frac{G(y)}{\sqrt{1-|G(y)|^2}},$$

where

$$1-|G(y)|^2 = 1 - \frac{|y|^2}{1+|y|^2} = \frac{1}{1+|y|^2},$$

hence

$$(F\circ G)(y) = \frac{y}{\sqrt{1+|y|^2}} \Big/ \frac{1}{\sqrt{1+|y|^2}} = y.$$

Similarly,

$$(G\circ F)(x) = x.$$

Now consider each component $F_i(x) = \frac{x_i}{\sqrt{1-|x|^2}}$, then F_i is smooth by elementary calculus. Hence $\mathbb{B}^n$ is diffeomorphic to $\mathbb{R}^n$. □

Example 13 . If M is a smooth manifold and (U, φ) is a smooth coordinate chart on M, then $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a diffeomorphism.

Proof. Since $\varphi \circ \varphi^{-1} = \mathrm{id}$ is clearly smooth, φ is a smooth map. To see φ^{-1} is smooth, we view φ^{-1} as a map between manifolds $\mathbb{R}^n$ and M. Choose a smooth chart $(\varphi(U), \varphi^{-1})$ on $\mathbb{R}$, then $\varphi^{-1} \circ (\varphi^{-1})^{-1} = \mathrm{id}$ is clearly smooth, completing the proof. □

Proposition 1.4.3 (properties of diffeomorphisms)**.** *Let M, N, P be manifolds.*

1. *If $F : M \to N, G : N \to P$ are diffeomorphisms, then $G \circ F : M \to P$ is a diffeomorphism.*

2. *If F, G are diffeomorphisms from M to N, then the Cartesian product $F \times G$ is a diffeomorphism.*

3. *"Diffeomorphic" is an equivalence relation on the class of all smooth manifolds.*

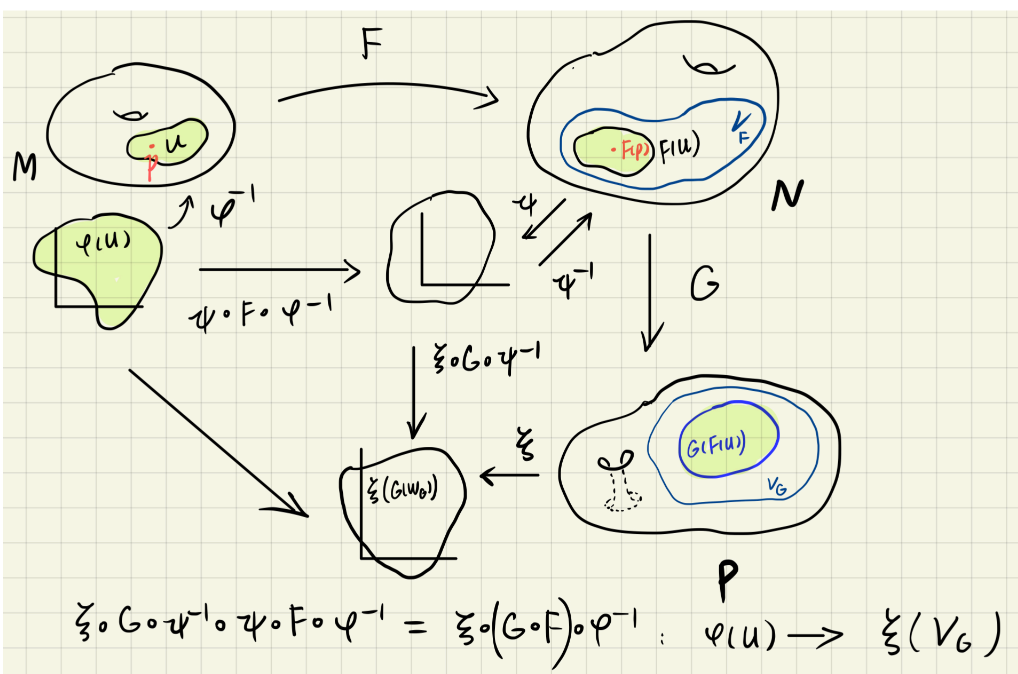
Proof. 1. Let $p \in M$. Since F is smooth, there are smooth charts (U, φ) on M and (V_F, ψ) on N such that $p \in U, F(U) \subset V_F$ and $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V_F)$ is a smooth function. Since G is smooth, $G|_{F(U)}$ is smooth, so there are smooth charts $(F(U), \psi)$ on N and (W, ξ) on P such that

- $F(p) \in F(U), (G \circ F)(p) \in G(F(U)) \subset W$,
- $\xi \circ G \circ \psi^{-1}$ is a smooth function.

Then

$$\xi \circ G \circ \psi^{-1} \circ \psi \circ F \circ \varphi^{-1} = \xi \circ (G \circ F) \circ \varphi^{-1} : \varphi(U) \to \xi(W)$$

is a smooth function, hence $G \circ F$ is a diffeomorphism.



2. Let $p \in M$, then there are smooth charts $(U_F, \varphi_F), (U_G, \varphi_G)$ and $(V_F, \psi_F), (V_G, \psi_G)$ such that

- $p \in U_f, p \in U_G$ and $F(p) \in V_F, G(p) \in V_G$,
- $\psi_F \circ F \circ \varphi_F^{-1}$ and $\psi_G \circ G \circ \varphi_G^{-1}$ are smooth functions.

3. Let M, N, P be smooth manifolds.
   - M is clearly diffeomorphic with M via the identity map.
   - If $F : M \to N$ is a diffeomorphism, so is $F^{-1} : N \to M$.
   - Suppose $M \simeq N$ and $N \simeq P$. Since composition of diffeomorphisms is a diffeomorphism, $M \simeq P$.

$\square$

1.5 Partitions of Unity

Lemma 1.5.1. *The function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

is smooth.

Proof. This is a calculus exercise. $\square$

Lemma 1.5.2. *For any $0 < r_1 < r_2$ there is a smooth functions $h : \mathbb{R}^n \to \mathbb{R}$, where*

1. *$H = 1$ on $\overline{B_{r_1}(0)}$;*
2. *$0 < H < 1$ on $B_{r_2}(0) \setminus \overline{B_{r_1}(0)}$;*
3. *$H = 0$ on $\mathbb{R}^n \setminus B_{r_2}(0)$.*

Using the notation in Urysohn's lemma, we can write

$$\overline{B_{r_1}(0)} \prec h \prec B_{r_2}(0).$$

Proof. Let

$$f(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and define

$$H(x) = \frac{f(r_2 - |x|)}{f(r_2 - |x|) + f(|x| - r_1)}.$$

Then

1. If $x \in \overline{B_{r_1}(0)}$, then $0 \leq |x| \leq r_1 < r_2$, so $f(|x| - r_1) = 0$, hence $H(x) = f(r_2 - |x|)/f(r_2 - |x|) = 1$.

2. Let $x \in B_{r_2}(0) \backslash \overline{B_{r_1}(0)}$, then $r_1 < |x| < r_2$, so $f(r_2 - |x|) > 0$ and $f(|x| - r_1) > 0$, hence $0 < H(x) < 1$.

3. If $x \in \mathbb{R}^n \setminus B_{r_2}(0)$, then $|x| \geq r_2$, so $f(r_2 - |x|) = 0$.

Therefore H is as desired. □

Definition 1.5.1. Let M be a topological space, $\mathcal{X} = \{X_\alpha\}_{\alpha \in A}$ be an open cover of M. A *partition of unity subordinate to* $\mathcal{X}$ is a family $\{\psi_\alpha\}_{\alpha \in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ with the following properties:

1. $0 \leq \psi_\alpha(x) \leq 1 \ \forall \alpha \in A$ and $\forall x \in M$.

2. supp $(\psi_\alpha) \subset X_\alpha$ for each $\alpha \in A$.

3. $\{\text{supp } (\psi_\alpha)\}_{\alpha \in A}$ is locally finite.

4. $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in M$.

If M is a smooth manifold and each ψ_α is smooth, then $\{\psi_\alpha\}_{\alpha \in A}$ is called a smooth partition of unity.

Theorem 1.5.1. *Suppose M is a smooth manifold and $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ is an open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.*

If M is a topological space, $A \subset M$ is a closed set, $U \supset A$ is open, a continuous function $\psi : M \to \mathbb{R}$ is called a **bump function** for A supported in U if $\psi = 1$ on A, supp $\psi \subset U$, and $0 \leq \psi \leq 1$. Analysts use the notation $A \prec f \prec U$ to describe the above properties.

Proposition 1.5.1. *Let M be a smooth manifold. For any closed $A \subset M$ and any open $U \supset A$, there exists a smooth bump function for A supported in U.*

Proof. Let $U_0 = U$ and $U_1 = M \setminus A$, and let $\{\psi_0, \psi_1\}$ be a smooth partition of unity subordinate to the open cover $\{U_0, U_1\}$. Since $\psi_0 = 0$ on A, it follows that $\psi_0 = \sum_i \psi_i = 1$ there, the function ψ_0 has the desired properties. □ □

Lemma 1.5.3 (extension lemma)**.** *Suppose M is a smooth manifold, $A \subset M$ is closed, $f : A \to \mathbb{R}^k$ is a smooth function. Then for any open $U \supset A$, there is a smooth function $\tilde{f} : M \to \mathbb{R}^k$ such that $\tilde{f}|_A = f$ and* supp $\tilde{f} \subset U$*.*

1.6 Manifolds with Boundary

We define the **closed n-dimensional upper half-space** $\mathbb{H}^n = \{(x^1, \cdots, x^n) \in \mathbb{R}^n : x^n \geq 0\}$. When $n > 0$, the interior and boundary of $\mathbb{H}^n$ are given by

$$\begin{aligned} \text{Int } \mathbb{H}^n &= \{(x^1, \cdots, x^n) \in \mathbb{R}^n : x^n > 0\}, \\ \partial \mathbb{H}^n &= \{(x^1, \cdots, x^n) \in \mathbb{R}^n : x^n = 0\}. \end{aligned}$$

In the $n = 0$ case, $\mathbb{H}^0 = \mathbb{R}^0 = \{0\}$, so Int $H^0 = \mathbb{R}^0$ and $\partial \mathbb{H}^0 = \varnothing$.

Chapter 2

Tangent Spaces

2.1 Tangent Vectors

2.1.1 Geometric Tangent Vectors

In $\mathbb{R}^n$, we always identify a **point** with a **vector**, expressed by the coordinates $(x^1, \cdots, x^n)$. However, when come to tangent vectors, it is convenient to think of a point as a location, and think of a vector as have magnitude and direction.

Let us begin with a prototype definition of tangent vectors in Euclidean space. Given a point $a \in \mathbb{R}^n$, define the **geometric tangent space** to $\mathbb{R}^n$ at a, denoted by $\mathbb{R}^n_a$, to be the set

$$\{(a, v) : v \in \mathbb{R}^n\} = \{a\} \times \mathbb{R}^n.$$

A **geometric tangent vector** in $\mathbb{R}^n$ is an element of $\mathbb{R}^n_a$ for some $a \in \mathbb{R}^n$. We abbreviate (a, v) as v_a and think of v_a as the vector v with its initial point at a. These definitions will serve as prototypes of tangent spaces on a manifold. So far they are of no practical uses because there is nothing to “tangent" $\mathbb{R}^n$!

Remark. We can regard a geometric tangent vector as a special type of vector which is not unique up to translation. Two *vectors* are identical if they have the same direction and magnitude, but two *geometric tangent vectors* are distinct even if

- their initial points are different, and
- they have the same direction and magnitude.

Example 1 (directional derivatives). Any geometric tangent vector $v_a \in \mathbb{R}^n_a$ yields a map

$$\begin{aligned} D_v|_a &: C^\infty(\mathbb{R}^n) \to \mathbb{R} \\ D_v|_a f &= D_v f(a) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(a + tv). \end{aligned}$$

This operation is linear over $\mathbb{R}$ and satisfies the Leibniz's rule:

$$D_v|_a(fg) = f(a)D_v|_a g + g(a)D_v|_a f.$$

If $v_a = v^i e_i|_a$ in terms of the standard basis, then by the chain rule we have

$$D_v|_a f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(a_1 + tv^1, \cdots, a_n + tv^n) = v^i \frac{\partial f}{\partial x^i}(a),$$

where we are using the summation convention.

Definition 2.1.1. If a is a point of $\mathbb{R}^n$, a map $w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a **derivation** at a if is linear over $\mathbb{R}$ and satsifies the **Leibniz's rule**:

$$w(fg) = f(a)w(g) + g(a)w(f).$$

Denote the set of all derivations of $C^\infty(\mathbb{R}^n)$ at a by $T_a\mathbb{R}^n$.

Lemma 2.1.1. *Let $a \in \mathbb{R}^n, w \in T_a\mathbb{R}^n$, and $f, g \in C^\infty(\mathbb{R}^n)$.*

1. *If f is a constant function, then $wf = 0$.*
2. *If $f(a) = g(a) = 0$, then $w(fg) = 0$.*

Proof. If $f_1 = 1$, then $wf_1 = w(f_1 f_1) = f_1(a)wf_1 + (wf_1)f_1(a) = 2wf_1$, hence $wf_1 = 0$. If $f = c$, then by linearity $wf = w(cf_1) = cwf_1 = 0$. If $f(a) = g(a) = 0$, then by the Leibniz's rule $w(fg) = 0$. □ □

Proposition 2.1.1 ($\mathbb{R}^n_a \simeq T_a\mathbb{R}^n$). *Let $a \in \mathbb{R}^n$.*

1. *For each gemoetric tangent vector $v_a \in \mathbb{R}^n_a$, the map*

$$\begin{aligned} &D_v|_a : C^\infty(\mathbb{R}^n) \to \mathbb{R} \\ &D_v|_a f = D_v f(a) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(a + tv). \end{aligned}$$

   is a derivation at a.

2. *The map $v_a \mapsto D_v|_a$ is an isomorphism from $\mathbb{R}^n_a$ to $T_a\mathbb{R}^n$.*
3. *The n partial derivative operators*

$$\left.\frac{\partial}{\partial x^1}\right|_a, \cdots, \left.\frac{\partial}{\partial x^n}\right|_a$$

   form a basis for $T_a\mathbb{R}^n$.

Proof. The directional derivative operator is the same as the differentiation in calculus, so $\mathbb{D}_v|_a$ is clearly a derivation. Now we show that $\mathbb{R}^n_a \simeq T_a\mathbb{R}^n$. Write $u = u^i e_i|_a, v = v^i e_i|_a$ in terms of the standard basis, then

$$D_{u+v}|_a(f) = (u^i + v^i)\frac{\partial f}{\partial x^i}(a) = u^i \frac{\partial f}{\partial x^i}(a) + v^i \frac{\partial f}{\partial x^i}(a) = D_u|_a(f) + D_v|_a(f),$$

and $D_u|_a(cf) = cD_u|_a(f)$ is easy to see. This shows the linearity.

Denote the map by $T : v_a \mapsto Tv_a = D_v|_a$. To see that T is injective, suppose $Tv_a = D_v|_a = 0$, then take f to be the jth coordinate function: $f(x^1, \cdots, x^j, \cdots, x^n) = x^j$, we obtain

$$0 = D_v|_a(f) = v^i \frac{\partial f}{\partial x^i}(f)\bigg| = v^j.$$

Since this is true for each j, it follows that v_a is the zero vector.

Now we show the surjectivity. Let $w \in T_a\mathbb{R}^n$ be arbitrary, and define $v = v^i e_i$, where the coefficients $v^1, \cdots, v^n$ are given by $v^i = w(x^i)$. Here x^i is the ith coordinate function: $(x^1, \cdots, x^i, \cdots, x^n) \mapsto x^i$. We will show that $w = D_v|_a$. Let f be any smooth real-valued function on $\mathbb{R}^n$. By Taylor's theorem, we can write

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + \sum_{i,j=1}^{n} (x^i - a^i)(x^j - a^j) \int_0^1 (1-t)\frac{\partial^2 f}{\partial x^i \partial x^j}(a + t(x-a))\, dt.$$

Each term in the last sum is a product of two smooth functions of x that vanish at $x = a$. Thus

$$\begin{aligned} wf &= w(f(a)) + \sum_{i=1}^{n} w\left(\frac{\partial f}{\partial x^i}(a)(x^i - a^i)\right) \\ &= 0 + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a)(w(x^i) - w(a^i)) \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(a) v^i = D_v|_a f. \end{aligned}$$

□ □

2.1.2 Tangent Vectors on Manifolds

Definition 2.1.2. Let M be a smooth manifold, and let $p \in M$. A linear map $v : C^\infty(M) \to \mathbb{R}$ is called a **derivation** at p if it satisfies

$$v(fg) = f(p)vg + g(p)vf \quad \forall f, g \in C^\infty(M).$$

Denote the set of all derivations of $C^\infty(M)$ at p by T_pM, which is a vector space called the **tangent space** to M at p. An element of T_pM is called a tangent vector at p.

Lemma 2.1.2. *Suppose M is a smooth manifold, $p \in M, v \in T_pM$, and $f, g \in C^\infty(M)$.*

1. *If f is a constant function, then $vf = 0$.*
2. *If $f(p) = g(p) = 0$, then $v(fg) = 0$.*

Proof. First let $f_1 = 1$, then $vf_1 = v(f_1 f_1) = f_1(p)vf_1 + (vf_1)f_1(p) = 2vf_1$, hence $vf_1 = 0$. If $f = c$, then by linearity $vf = v(cf_1) = c(vf_1) = 0$. The second assertion is obvious by the Leibniz's rule. □ □

2.2 The Differential of a Smooth Map

Definition 2.2.1. If M, N are smooth manifolds and $F : M \to N$ is a smooth map, for each $p \in M$ we define a map

$$\begin{aligned} dF_p : T_pM &\to T_{F(p)}N, \\ v &\mapsto dF_p(v), \end{aligned}$$

called the **differential** of F at p. Given $v \in T_pM$, we let $dF_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by the rule

$$dF_p(v)(f) = v(f \circ F). \tag{2.1}$$

If $f \in C^\infty(N)$, then $f \circ F \in C^\infty(M)$, so $v(f \circ F)$ makes sense. The operator $dF_p(v) : C^\infty(N) \to \mathbb{R}$ is linear since

- $dF_p(v)(f + g) = v((f + g) \circ F) = v(f \circ F + g \circ F) = v(f \circ F) + v(g \circ F)$.
- $dF_p(v)(cf) = v((cf) \circ F) = v(c(f \circ F)) = cv(f \circ F)$.

And $dF_p(v)$ is a derivation at $F(p)$ because

$$\begin{aligned} dF_p(v)(fg) &= v((fg) \circ F) = v((f \circ F)(g \circ F)) \\ &= (f \circ F)(p)v(g \circ F) + (g \circ F)(p)v(f \circ F) \\ &= f(F(p))dF_p(v)(g) + g(F(p))dF_p(v)(f). \end{aligned}$$

Proposition 2.2.1 (properties of differentials)**.** *Let M, N, P be smooth manifolds, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.*

1. *$dF_p : T_pM \to T_{F(p)}N$ is linear.*
2. *$d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$.*
3. *$d(\mathrm{id}_M)_p = \mathrm{id}_{T_pM} : T_pM \to T_pM$.*
4. *If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.*

Proof. 1. Let $u, v \in T_pM$ and $f \in C^\infty(N)$ be arbitrary, then

$$dF_p(u+v)(f) = (u+v)(f \circ F) = u(f \circ F) + v(f \circ F) = dF_p(u)(f) + dF_p(v)(f).$$

Because f is arbitrary, we have $dF_p(u+v) = dF_p(u) + dF_p(v)$. Let c be a scalar, then $dF_p(cu)(f) = (cu)(f \circ F) = c dF_p(u)(f)$. Therefore dF_p is linear.

2. Let $v \in T_pM$ and $f \in C^\infty(P)$. Let $w = dF_p(v)$, and recall that $dF_p : T_pM \to T_{F(p)}N$, hence $w \in T_{F(p)}N$, and $dG_{F(p)}N \to T_{G(F(p))}P$. Start with

$$(dG_{F(p)} \circ dF_p)(v) = dG_{F(p)}(dF_p(v)) = dG_{F(p)}(w),$$

and then plug f in the right side,

$$dG_{F(p)}(w)(f) = w(f \circ G) = v(f \circ G \circ F) = d(G \circ F)_p(v)(f).$$

3. Let $v \in T_pM$ and $f \in C^\infty(M)$, then

$$d(\mathrm{id}_M)_p(v)(f) = v(f \circ \mathrm{id}_M) = v(f),$$

hence $d(\mathrm{id}_M)_p(v) = v$, implying that $d(\mathrm{id}_M)_p = \mathrm{id}_{T_pM} : T_pM \to T_pM$.

4. First,

$$\begin{aligned}(dF^{-1}_{F(p)} \circ dF_p)(v)(f) &= dF^{-1}_{F(p)}(dF_p(v))(f)\\ &= dF_p(v)(f \circ F^{-1})\\ &= v(f \circ F^{-1} \circ F) = v(f).\end{aligned}$$

On the other hand,

$$\begin{aligned}(dF_p \circ dF^{-1}_{F(p)})(w)(g) &= dF_p(dF^{-1}_{F(p)}(w))(g)\\ &= (dF^{-1}_{F(p)}(w))(g \circ F)\\ &= w(g \circ F \circ F^{-1}) = w(g).\end{aligned}$$

Therefore, dF_p is invertible, and $(dF_p)^{-1} = dF^{-1}_{F(p)}$.

□ □

Proposition 2.2.2. *Let M be a smooth manifold, $p \in M$, and $v \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood of p, then $vf = vg$.*

Proposition 2.2.3 (the tangent space to an open submanifold)**.** *Let M be a smooth manifold, let $U \subset M$ be open, and let $\iota : U \hookrightarrow M$ be the inclusion map. For every $p \in U$, the differential $d\iota_p : T_pU \to T_pM$ is an isomorphism.*

Proof. To show injectivity, let $v \in T_pU$ and $d\iota_p(v) = 0 \in T_pM$. Let B be a neighborhood of p such that $\overline{B} \subset U$. Let $f \in C^\infty(U)$ be arbitrary, then the extension lemma for smooth functions implies that there exists $\tilde{f} \in C^\infty(M)$ such that $\tilde{f} = f$ on $\overline{B}$. Then since $f = \tilde{f}|_U$ in a neighborhood of p, Proposition (2.2.2) implies

$$vf = v(\tilde{f}|_U) = v(\tilde{f} \circ \iota) = d\iota(v)_p(\tilde{f}) = 0.$$

Since this holds for every $f \in C^\infty(U)$, it follows that $v = 0$, so $d\iota_p$ is injective.

On the other hand, suppose $w \in T_pM$ is arbitrary. Define an operator $v : C^\infty(U) \to \mathbb{R}$ by $vf = w\tilde{f}$, where $\tilde{f}$ is any smooth function on M that agrees with f on $\overline{B}$. By Proposition (2.2.2), vf is independent of the choice of $\tilde{f}$, so v is well defined, and it is a derivation of $C^\infty(U)$ at p because w is. For any $g \in C^\infty(M)$,

$$d\iota_p(v)(g) = v(g \circ \iota) = w(\widetilde{g \circ \iota}) = wg,$$

where the last two equalities follow from the facts that $g \circ \iota, \widetilde{g \circ \iota}, g$ all agree on B. Therefore, $d\iota_p$ is surjective. □ □

Proposition 2.2.4 (dimension of the tangent space)**.** *If M is an n-dimensional smooth manifold, then for each $p \in M$, the tangent space T_pM is an n-dimensional vector space.*

Proof. Given $p \in M$, let (U, φ) be a smooth coordinate chart containing p. Since φ is a differomorphism from U to $\varphi(U)$, $d\varphi_p$ is an isomorphism from T_pU to $T_{\varphi(p)}(\varphi(U))$. By Proposition (2.4), $T_pM \simeq T_pU$ and $T_{\varphi(p)}\varphi(U) \simeq T_{\varphi(p)}\mathbb{R}^n$, it follows that

$$\dim T_pM = \dim T_{\varphi(p)}\mathbb{R}^n = n.$$

□ □

2.3 Computations in Coordinates

Suppose M is a smooth manifold, and let (U, φ) be a smooth coordinate chart on M. Then $d\varphi_p : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism. Since the derivations $\partial/\partial x^1 |_{\varphi(p)}, \cdots, \partial/\partial x^n |_{\varphi(p)}$ form a basis for $T_{\varphi(p)}\mathbb{R}^n$, the preimages

$$(d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^1}\right|_{\varphi(p)}\right), \cdots, (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^n}\right|_{\varphi(p)}\right)$$

form a basis for T_pM. We use the notation We use another notation $\partial/\partial x^i |_p$ for these vectors:

$$\left.\frac{\partial}{\partial x^i}\right|_p = (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = d(\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right). \tag{2.2}$$

Notice that $\partial/\partial x^i\mid_{\varphi(p)}$ is a derivation in $T_{\varphi(p)}\mathbb{R}^n$, hence $\partial/\partial x^i\mid_p$ acts on a function $f \in C^\infty(U)$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = d(\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} f\right) = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}(f\circ\varphi^{-1}) = \frac{\partial \widehat{f}}{\partial x^i}(\widehat{p}), \tag{2.3}$$

where $\widehat{f} = f\circ\varphi^{-1}$ is the coordinate representation of f, and $\widehat{p} = \varphi(p)$ is the coordinate representation of p. We summarize these in the following prosposition.

Definition 2.3.1. The vectors $\partial/\partial x^i\mid_p$ are called the **coordinate vectors** at p associated with the given coordinate system.

Proposition 2.3.1. *Let M be a smooth n-manifold, $p\in M$. Then T_pM is an n-dimensional vector space, and for any smooth chart $(U,(x^i))$ containing p, the coordinate vectors $\partial/\partial x^1\mid_p,\cdots,\partial/\partial x^n\mid_p$ form a basis for T_pM.*

Thus, a tangent vector $v\in T_pM$ can be written uniquely as a linear combination

$$v = v^i\left.\frac{\partial}{\partial x^i}\right|_p.$$

The ordered basis $(\partial/\partial x^i\mid_p)$ is called a **coordinate basis** for T_pM, and the numbers $v^1,\cdots,v^n$ are called the **components** of v with respect to the coordinate basis. For each j, the components of v are given by $v^j = v(x^j)$, where x^j is the jth coordinate function.

2.3.1 The Differential in Coordinates

Consider a smooth map $F: U\to V$, $U\subset\mathbb{R}^n, V\subset\mathbb{R}^m$ are open subsets. Let $(x^1,\cdots,x^n)$ denote the coordinates in U and $(y^1,\cdots,y^m)$ denote those in V. $dF_p: T_p\mathbb{R}^n\to T_{F(p)}\mathbb{R}^m$ acts on a basis vector as follows:

$$\begin{aligned} dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f &= \left.\frac{\partial}{\partial x^i}\right|_p (f\circ F) = \sum_{j=1}^m \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) \\ &= \left(\sum_{j=1}^m \frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y_j}\right|_{F(p)}\right) f \\ &= \left(\frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y_j}\right|_{F(p)}\right) f. \quad \text{(Einstein summation)} \end{aligned}$$

Since $f\in C^\infty(V)$ is arbitrary, we have

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y_j}\right|_{F(p)}. \tag{2.4}$$

In terms of the coordinate bases, the matrix of the linear map $dF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$:

$$\begin{pmatrix} \dfrac{\partial F^1}{\partial x^1}(p) & \cdots & \dfrac{\partial F^1}{\partial x^1}(p) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F^1}{\partial x^1}(p) & \cdots & \dfrac{\partial F^1}{\partial x^1}(p) \end{pmatrix}.$$

This is non other than the Jacobian matrix of F at p. In this case, $dF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ corresponds to the *total derivative* $DF(p) : \mathbb{R}^n \to \mathbb{R}^m$.

Now we consider a smooth map $F : M \to N$ between smooth manifolds. Choosing smooth coordinate charts (U, φ) for M containing p and (V, ψ) for N containing $F(p)$, we obtain the coordinate representation $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ from a subset of $\mathbb{R}^n$ to $\mathbb{R}^m$. Next, we will find the domain of $\widehat{F}$. Let's draw a diagram:

$$\begin{tikzcd} M \arrow[r, "F"] & N \arrow[d, "\psi"] \\ \mathbb{R}^n \arrow[u, "\varphi^{-1}"] \arrow[r, "\psi\circ F\circ\varphi^{-1}"] & \mathbb{R}^m \end{tikzcd}$$

But the domain of φ is U and the domain of ψ is V, so we can be a bit more precise step by step:

$$\begin{tikzcd} U \arrow[r, "F"] & V \cap F(U) \arrow[d, "\psi"] \\ \mathbb{R}^n \arrow[u, "\varphi^{-1}"] \arrow[r, "\psi\circ F\circ\varphi^{-1}"] & \mathbb{R}^m \end{tikzcd}$$

ψ should take values on $V \cap F(U)$, but this also affects the domain of F. Instead of starting from U, F will map from $F^{-1}(V \cap F(U)) = F^{-1}(V) \cap U$. Thus the domain of the diffeomorphism φ^{-1} is $\varphi(U \cap F^{-1}(V))$, and our ultimate diagram goes:

$$\begin{tikzcd} F^{-1}(V \cap F(U)) \arrow[r, "F"] & V \cap F(U) \arrow[d, "\psi"] \\ \varphi(U \cap F^{-1}(V)) \arrow[u, "\varphi^{-1}"] \arrow[r, "\psi\circ F\circ\varphi^{-1}"] & \psi(V \cap F(U)) \end{tikzcd}$$

Using the formula 2.2 and chain rule, we compute

$$\begin{aligned}
dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) &= dF_p\left(d(\varphi^{-1})_{\widehat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right)\right)\\
&= dF_{\varphi^{-1}(\widehat{p})}d(\varphi^{-1})_{\widehat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right)\\
&= d(F\circ\varphi^{-1})_{\widehat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right) = d(\psi^{-1}\circ\widehat{F})_{\widehat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right)\\
&= d(\psi^{-1})_{\widehat{F}(\widehat{p})}\left(d\widehat{F}_{\widehat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\widehat{p}}\right)\right) \quad \text{(chain rule)}\\
&= d(\psi^{-1})_{\widehat{F}(\widehat{p})}\left(\frac{\partial\widehat{F}^j}{\partial x^i}(\widehat{p})\left.\frac{\partial}{\partial y^j}\right|_{\widehat{F}(\widehat{p})}\right) \quad (\widehat{F}(\widehat{p}) = F(p))\\
&= (d\psi_{F(p)})^{-1}\left(\frac{\partial}{\partial\widehat{F}^j}x^i(\widehat{p})\left.\frac{\partial}{\partial y^j}\right|_{\widehat{F}(\widehat{p})}\right) \quad (2.2)\\
&= \frac{\partial\widehat{F}^j}{\partial x^i}(\widehat{p})\left.\frac{\partial}{\partial y^j}\right|_{F(p)}.
\end{aligned}$$

2.3.2 Change of Coordinates

Let $(U,\varphi),(V,\psi)$ be two smooth charts on M, and $p\in U\cap V$. Denote the coordinate functions of φ by (x^i) and those of ψ by $(\tilde{x}^i)$. Any tangent vector at p can be represented by either basis $\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)$ or $\left(\left.\frac{\partial}{\partial \tilde{x}^i}\right|_p\right)$.

Write the transition map $\psi\circ\varphi^{-1}:\varphi(U\cap V)\to\psi(U\cap V)$ in the following notation:

$$\psi\circ\varphi^{-1}(x) = (y^1(x),\cdots,y^n(x)) \quad (x\in\varphi(U\cap V)).$$

By (2.4), we have

$$d(\psi\circ\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = \frac{\partial\tilde{x}^j}{\partial x^i}(\varphi(p))\left.\frac{\partial}{\partial y^i}\right|_{\psi(p)}.$$

Using the definition of coordinate vectors, we obtain

$$\begin{aligned}
\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) &= d(\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) \\
&= d(\psi^{-1}\circ\psi\circ\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) \\
&= d(\psi^{-1})_{(\psi\circ\varphi^{-1})(\varphi(p))}\circ d(\psi\circ\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) \\
&= d(\psi^{-1})_{\psi(p)}\left(\frac{\partial y^j}{\partial x^i}(\varphi(p))\left.\frac{\partial}{\partial y^i}\right|_{\psi(p)}\right) \\
&= \frac{\partial y^j}{\partial x^i}(\widehat{p})\left.\frac{\partial}{\partial \tilde{x}^j}\right|_p,
\end{aligned}$$

where $\widehat{p} = \varphi(p)$. Applying this to the components of a derivation $v = v^i \left.\frac{\partial}{\partial x^i}\right|_p = \tilde{v}^j \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p$, by the above rule

$$\tilde{v}^j = \frac{\partial y^j}{\partial x^i}(\widehat{p})v^i,$$

where y^j's are the components of the transition map $\psi\circ\varphi^{-1}$.

Example 2 . The transition map between polar coordinates and standard coordinates in suitable open subsets of $\mathbb{R}^2$ is given by $(x, y) = (r\cos\theta, r\sin\theta)$. Let $p \in \mathbb{R}^2$ with polar coordinate represent being $(r, \theta) = (2, \pi/2)$, and $v \in T_p\mathbb{R}^2$ with polar coordinate represent being

$$v = 3\left.\frac{\partial}{\partial r}\right|_p - \left.\frac{\partial}{\partial \theta}\right|_p.$$

We find

$$\begin{aligned}
\left.\frac{\partial}{\partial r}\right|_p &= \left.\frac{\partial(r\cos\theta)}{\partial r}\right|_{(2,\pi/2)}\left.\frac{\partial}{\partial x}\right|_p + \left.\frac{\partial(r\sin\theta)}{\partial r}\right|_{(2,\pi/2)}\left.\frac{\partial}{\partial y}\right|_p = \cos(\pi/2)\left.\frac{\partial}{\partial x}\right|_p + \sin(\pi/2)\left.\frac{\partial}{\partial y}\right|_p = \left.\frac{\partial}{\partial y}\right|_p, \\
\left.\frac{\partial}{\partial \theta}\right|_p &= \left.\frac{\partial(r\cos\theta)}{\partial \theta}\right|_{(2,\pi/2)}\left.\frac{\partial}{\partial x}\right|_p + \left.\frac{\partial(r\sin\theta)}{\partial \theta}\right|_{(2,\pi/2)}\left.\frac{\partial}{\partial y}\right|_p = -2\sin(\pi/2)\left.\frac{\partial}{\partial x}\right|_p + 2\cos(\pi/2) - 2\left.\frac{\partial}{\partial x}\right|_p = \left.\frac{\partial}{\partial y}\right|_p,
\end{aligned}$$

thus v has the coordinate representation in standard coodinates

$$v = 3\left.\frac{\partial}{\partial y}\right|_p + 2\left.\frac{\partial}{\partial x}\right|_p.$$

Exercise 1 . Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\tilde{x}, \tilde{y})$ are global smooth coordinates on $\mathbb{R}^2$, where

$$\tilde{x} = x, \quad \tilde{y} = y + x^3.$$

Let $p = (0, 1) \in \mathbb{R}^2$ (in standarad coordinates), and show that

$$\left.\frac{\partial}{\partial x}\right|_p \neq \left.\frac{\partial}{\partial \tilde{x}}\right|_p.$$

2.4 The Tangent Bundle

Definition 2.4.1. Given a smooth manifold M, we define the **tangent bundle** of M, denoted by TM, to be the disjoint union of the tangent spaces at all points of M:

$$TM = \bigsqcup_{p \in M} T_pM.$$

We often write an element of TM as (p, v), where $p \in M$ and $v \in T_pM$. The next proposition gives a smooth structure on TM, with the idea of divide-and-conquer: utilizing the smooth structure on M and identifying T_pM with $\mathbb{R}^n$.

Proposition 2.4.1. *For any smooth n-manifold M, the tangent bundle TM has a natural topology and smooth structure that make it into a $2n$-dimensional smooth manifold. With respect to this structure, the projection $\pi : TM \to M$ is smooth.*

Proof. Let (U, φ) be a smooth chart for M, then $\pi^{-1}(U)$ is an open subset of TM consisting of all tangent vectors at each point of U. We construct a smooth structure on TM. Let $\varphi = (x^1, \cdots, x^n)$ be the coordinate representation, and define $\widetilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\widetilde{\varphi}\left(v^i \left.\frac{\partial}{\partial x^i}\right|_p\right) = (x^1(p), \cdots, x^n(p), v^1, \cdots, v^n) = (\varphi(p), v),$$

then $\{(\pi^{-1}(U), \widetilde{\varphi}) : (U, \varphi) \text{ is a smooth chart for } M\}$ is a smooth structure on TM. □ □

2.5 Velocity Vectors of Curves

Every element in the tangent space is the velocity of some curve. This insight will be frequently used in Chapter **??**: Integral Curves and Flows.

Definition 2.5.1. A *curve* in M is a continuous map $\gamma : J \to M$, where J is an interval.

Given a smooth curve $\gamma : J \to M$ and $t_0 \in J$, we define the *velocity* of γ at t_0 to be the vector

$$\gamma'(t_0) = d\gamma\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right) \in T_{\gamma(t_0)M}.$$

Here $d\gamma : T_{t_0}J \to T_{\gamma(t_0)}M$, and for any $f \in C^\infty(M)$,

$$d\gamma\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\right)(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}(f\circ\gamma)(t) = (f\circ\gamma)'(t_0).$$

Other common notations are

$$\dot\gamma(t_0) = \frac{\mathrm{d}\gamma}{\mathrm{d}t} = \left.\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|_{t=t_0}.$$

$\gamma'(t_0)$ is the derivation at $\gamma(t_0)$ obtained by taking the derivative of a function along γ.

Let (U,φ) be a smooth chart with coordinate (x^i). If $\gamma(t_0)\in U$, we can write the coordinate representation of γ is

$$\widehat{\gamma}(t) = (\gamma^1(t),\cdots,\gamma^n(t)) \in U \subset \mathbb{R}^n,$$

for t sufficiently close to t_0. Then the coordinate formula for the differential yields

$$\gamma'(t_0) = \frac{\mathrm{d}\widehat{\gamma}^i}{\mathrm{d}t}(t_0)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t_0)}.$$

Every tangent vector on a manifold is the velocity of some curve.

Proposition 2.5.1. *Suppose M is a smooth manifold and $p\in M$. Every $v\in T_pM$ is the velocity of some smooth curve in M.*

Proof. Let (U,φ) be a smooth chart centered at p, and write $v = v^i\,\partial/\partial x^i\,|_p$. For small $\varepsilon>0$ let $\gamma:(-\varepsilon,\varepsilon)\to U$ be the curve whose coordinate representation is $\widehat{\gamma}(t) = (tv^1,\cdots,tv^n)\in\mathbb{R}^n$. Hence $\gamma(t) = \varphi^{-1}\circ\widehat{\gamma}(t) = \varphi^{-1}(tv^1,\cdots,tv^n)$. Then γ is smooth and

- $\gamma(0) = \varphi^{-1}(0) = p$,
- $\gamma'(0) = \dfrac{\mathrm{d}\widehat{\gamma}^i}{\mathrm{d}t}(0)\left.\dfrac{\partial}{\partial x^i}\right|_{\gamma_0} = v^i\left.\dfrac{\partial}{\partial x^i}\right|_p = v.$

□

Proposition 2.5.2 (the velocity of a composite curve)**.** *Let $F: M\to N$ be a smooth map, and let $\gamma: J\to M$ be a smooth curve. For any $t_0\in J$, the velocity at $t=t_0$ of the curve $F\circ\gamma: J\to N$ is given by*

$$(F\circ\gamma)'(t_0) = dF(\gamma'(t_0)).$$

Since every derivation $v\in T_pM$ is the velocity of some curve, we can compute $dF_p(v)$ by choosing a smooth curve γ whose initial tangent vector is v.

Proposition 2.5.3 (computing differential using velocity)**.** *Suppose $F: M\to N$ is a smooth map, $p\in M$, $v\in T_pM$. Then*

$$dF_p(v) = (F\circ\gamma)'(0),$$

for any smooth curve $\gamma: J\to M$ such that $0\in J, \gamma(0)=p, \gamma'(0)=v$.

2.6 Categories and Functors

a **Category** C consists of the following things:

- a class $\mathrm{Ob}(\mathsf{C})$, whose elements are called **objects** of C,
- a class $\mathrm{Hom}(\mathsf{C})$, whose elements are called **morphisms** of C,
- for each morphism $f \in \mathrm{Hom}(\mathsf{C})$, two objects $X, Y \in \mathrm{Ob}(\mathsf{C})$ called the **source** and **target** of f, respectively,
- for each triple $X, Y, Z \in \mathrm{Ob}(\mathsf{C})$, a mapping called **composition**:

$$\mathrm{Hom}_{\mathsf{C}}(X,Y) \times \mathrm{Hom}_{\mathsf{C}}(Y,Z) \to \mathrm{Hom}_{\mathsf{C}}(X,Z),$$

  written $(f,g) \mapsto g \circ f$. $\mathrm{Hom}_{\mathsf{C}}(X,Y)$ denotes the class of all morphisms with source X and target Y.

The morphisms are required to satisfy the following axioms:

1. *Associativity*: $(f \circ g) \circ h = f \circ (g \circ h)$.
2. *Existence of identities*: For each object $X \in \mathrm{Ob}(\mathsf{C})$, there exists an **identity morphism** $\mathrm{id}_X \in \mathrm{Hom}_{\mathsf{C}}(X,X)$ such that $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$ for all $f \in \mathrm{Hom}_{\mathsf{C}}(X,Y)$.

Chapter 3

Submersions, Immersions, Embeddings

3.1 Maps of Constant Rank

Suppose M, N are smooth manifolds. Given a smooth amp $F : M \to N$ and a point $p \in M$, we define the **rank** of F at p to be the rank of the linear map $dF_p : T_pM \to T_{F(p)}N$. If F has the same rank r at every point, we say that it has **constant rank**, and write $\operatorname{rank} F = r$. By the rank-nullity theorem, $\operatorname{rank} F \leq \min(\dim M, \dim N)$. If the rank of dF_p is equal to this upper bound, we say that F has **full rank** at p, and if F has full rank everywhere, we say F has full rank.

Definition 3.1.1 (submersion, immersion)**.** A smooth map $F : M \to N$ is called a **smooth submersion** if its differential is surjective at each point (or equivalently, if $\operatorname{rank} F = \dim N$). It is called a **smooth immersion** if its differential is injective at each point (equivalently, $\operatorname{rank} F = \dim M$).

Example 1 (open submanifolds). Let U be an open set in $\mathbb{R}^n$, then U is a topological n-manifold, and the single chart (U, id_U) defines a smooth structure on U. More generally, let M be a smooth n-manifold and let $U \subset M$ be any open subset. Define an atlas on U by

$$\mathcal{A}_U = \{\text{smooth charts } (V, \varphi) \text{ for } M \text{ such that } V \subset U\}.$$

Example 2 (general linear group). The general linear group $\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\}$. Show that it is an open subset of $\mathbb{R}^{n\times n}$, hence it is a smooth n^2-dimensional manifold.

Proof. Notice that $(\det)^{-1}A(\mathbb{R} \setminus \{0\})$ is open since det is a continuous function of entries of a matrix. $\square$

Example 3 (matrices of full rank). Suppose $m < n$, and let $\mathcal{M}_m(m\times n, \mathbb{R})$ be the subset of $\mathcal{M}(m \times n, \mathbb{R})$ consisting of matrices of rank m. Prove that $\mathcal{M}_m(m \times n, \mathbb{R})$ is an open subset of $\mathcal{M}(m \times n, \mathbb{R})$, and therefore is a smooth mn-manifold.

Proposition 3.1.1. *Suppose $F : M \to N$ is a smooth map and $p \in M$. If dF_p is surjective, then p has a neighborhood U such that $F|_U$ is a submersion. if dF_p is injective, then p has a neighborhood such that $F|_U$ is an immersion.*

3.1.1 Local Diffeomorphisms

A smooth map $F : M \to N$ is called a **local diffeomorphism** if every $p \in M$ has an open neighborhood U where $F(U)$ is open and $F|_U : U \to F(U)$ is a diffeomorphism.

Example 4 . The map $f : \mathbb{R} \to \mathbb{S}^1$ given by $f(t) = (\cos t, \sin t)$ is a local diffeomorphism.

Theorem 3.1.1 (inverse function theorem). *Suppose $F : M \to N$ is a smooth map. If $p \in M$ and dF_p is invertible, then there are connected open neighborhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

Proof. Since dF_p is invertible, $n = \dim M = \dim N$. Fix charts $(U, \varphi), (V, \psi)$ centered at p and $F(p)$ with $F(U) \subset V$. Let

$$\widehat{F} : \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V),$$

then

$$d\widehat{F}_0 = d\psi_{F(p)} \circ dF_p \circ d(\varphi^{-1})_0$$

is invertible. By the classical inverse function theorem there exist connected open neighborhoods $\widehat{U}_0 \subset \varphi(U)$ and $\widehat{V}_0 \subset \psi(V)$ such that $\widehat{F}|_{\widehat{U}_0} : \widehat{U}_0 \to \widehat{V}_0$ is a diffeomorphism. Then letting $U_0 = \varphi^{-1}(\widehat{U}_0)$ and $V_0 = \psi^{-1}(\widehat{U}_0)$ completes the proof. □ □

3.1.2 The Rank Theorem

Theorem 3.1.2. *Suppose $F : M \to N$ is a smooth map,* $\dim M = m, \dim N = n$. *If F has constant rank r (i.e.,* $\operatorname{rank} dF_p = r$ *for all p). then for every $p \in M$ there exist smooth charts $(U, \varphi), (V, \psi)$ of M, N centered at $p, F(p)$ such that $F(U) \subset V$ and*

$$\psi \circ F \circ \varphi^{-1}(x^1, \cdots, x^m) = (x^1, \cdots, x^r, 0, \cdots, 0).$$

In particular,

- *if F is a smooth submersion, then $m \geq n$ and*

$$\psi \circ F \circ \varphi^{-1}(x^1, \cdots, x^m) = (x^1, \cdots, x^n).$$

- *If F is a smooth immersion, then $m \leq n$ and*

$$\psi \circ F \circ \varphi^{-1}(x^1, \cdots, x^m) = (x^1, \cdots, x^m, 0, \cdots, 0).$$

Proof. Linear algebra and inverse function theorem. □

Theorem 3.1.3 (global rank theorem)**.** *Let $F : M \to N$ be a smooth map of constant rank.*

1. *If F is surjective, then F is a smooth submersion.*
2. *If F is injective, then F is a smooth immersion.*
3. *If F is bijective, then F is a diffeomorphism.*

3.2 Topology Review

Theorem 3.2.1 (closed map lemma)**.**

3.3 Embeddings

3.3.1 Topological Embeddings

An injective continuous map that is a homeomorphism onto its image (in the subspace topology) is called a **topological embedding**. If $f : A \to X$ is such a map, we can think of $f(A)$ as a homeomorphic copy of A inside X.

Exercise 1 . Let X be a topological space and let S be a subspace of X. Show that the inclusion $\iota : S \to X$ is a topological embedding.

Proposition 3.3.1. *A continuous injective map that is either open or closed is a topological embedding.*

Proposition 3.3.2. *A surjective topological embeddings is a homeomorphism.*

3.3.2 Smooth Embeddings

A smooth map $F : M \to N$ is called a **smooth embedding** if

1. F is an immersion,
2. F is a topological embedding: M is homeomorphic to $F(M) \subset N$ in the subspace topology.

Example 5 .

Chapter 4

Submanifolds and Sard's Theorem

4.1 Embedded Submanifolds

An embedded submanifold of M is a subset $S \subset M$ which is a topological manifold (w.r.t. the subspace topology) endowed with a smooth structure that makes the inclusion $\iota : S \to M$ a smooth embedding.

EXAMPLE 1 .

- If $U \subset M$ is open, then U is an embedded submanifold.
- $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is an embedded submanifold.

In fact, every embedded submanifold is the image of some manifold under a smooth embedding.

Proposition 4.1.1. *Suppose $F : N \to M$ is a smooth embedding, then $S = F(N)$ is a topological manifold (w.r.t. the subspace topology) and has a unique smooth structure such that*

1. *S is an embedded submanifold.*
2. *$F : N \to S$ is a diffeomorphism.*

EXAMPLE 2 (SLICES OF PRODUCT MANIFOLDS). Suppose M and N are smooth manifolds. For each $p \in N$, the subset $M \times \{p\}$ is an embedded submanifold of $M \times N$ diffeomorphic to M.

Proof. $M \times \{p\}$ is the image of the smooth embedding $x \mapsto (x, p)$. $\square$

Example 3 (Graph of a map). Suppose M, N are smooth m, n-manifolds, $U \subset M$ is open, $f : U \to N$ is smooth. Then

$$\Gamma(f) = \{(x, y) \in M \times N : x \in U, y = f(x)\}$$

is an embedded m-dimensional submanifold of $M \times N$.

Proof. □

Slice Charts

If U is an open subset of $\mathbb{R}^n$ and $k \leq n$, a k-dimensional slice of U (or a k-slice) is a subset of the form

$$S = \{(x^1, \cdots, x^k, x^{k+1}, \cdots, x^n) \in U : x^{k+1} = c^{k+1}, \cdots, x^n = c^n\}$$

for some constants $c^{k+1}, \cdots, c^n$. In practice, those constants are often set to be zero: $c^{k+1} = \cdots = c^n = 0$. Now we define a slice on a manifold.

Definition 4.1.1. Let M be a smooth n-manifold, and (U, φ) be a smooth chart on M. If $S \subset U$ such that $\varphi(S)$ is a k-slice of $\varphi(U)$, then we say that S is a k-slice of U.

Definition 4.1.2 (local k-slice condition)**.** Let $S \subset M$ and $k \in \mathbb{N}$. We say that S satisfies the **local k-slice condition** if each point of S is contained in a smooth chart (U, φ) for M such that $S \cap U$ is a k-slice in U. Any such chart is called a **slice chart** for S in M.

Theorem 4.1.1 (local slice criterion)**.** *Let M be a smooth n-manifold. If $S \subset M$ is an embedded k-dimensional submanifold, then S satisfies the local k-slice condition. Conversely, if $S \subset M$ is a subset that satisfies the local k-slice condition, then S is a k-dimensional topological manifold, and it has a smooth structure making it into an embedded submanifold of M.*

4.2 Level Sets

If $\Phi : M \to N$ is a map, the preimages $\Phi^{-1}(c)$ are called level sets.

Theorem 4.2.1. *Let $\Phi : M \to N$ be a smooth map with constant rank r. Each level set of Φ is embedded submanifold of codimension r in M, i.e.,* $\dim \Phi^{-1}(c) = \dim M - r$

Let $\Phi : M \to N$ be a smooth map.

- $p \in M$ is called a **regular point** if $d\Phi_p$ is surjective, otherwise it is called a **critical point**.

- $c \in N$ is called a **regular value** if every point in $\Phi^{-1}(c)$ is a regular point (in this case $\Phi^{-1}(c)$ is called a regular level set), otherwise c is called a critical value.

Corollary 4.2.1. *Every regular level set of a smooth map $\Phi : M \to N$ is an embedded submanifold with dimension equal to* $\dim M - \dim N$.

Every embedded submanifold is locally a level set of a smooth submersion.

Proposition 4.2.1. *Let S be a subset of a smooth m-manifold M, then S has the structure of an embedded k-dimensional submanifold if and only if every point in S has an open neighborhood U such that $U \cap S$ is a level set of a smooth submersion $\Phi : U \to \mathbb{R}^{m-k}$.*

Proof. First suppose S is an embedded k-submanifold, and let $(x^1, \cdots, x^m)$ be slice coordinates for S on an open subset $U \subset M$. Let $\Phi : U \to \mathbb{R}^{m-k}$ given in coordinates by

$$\Phi(x) = (x^{k+1}, \cdots, x^m).$$

Then Φ is a smooth submersion. By the local slice condition, $S \cap U$ is a k-slice in U, so

$$S \cap U = \{(x^1, \cdots, x^k, x^{k+1}, \cdots, x^m) \in U : x^{k+1} = c^{k+1}, \cdots, x^m = c^m\},$$

hence $S \cap U = \Phi^{-1}(c^{k+1}, \cdots, c^m)$ is a level set of Φ.

Conversely, suppose that every $p \in S$ has a neighborhood U and a smooth submersion $\Phi : U \to \mathbb{R}^{m-k}$ such that $S \cap U$ is a level set of Φ. Then $S \cap U$ is an embedded submanifold of U, so it satisfies the local slice condition. It follows that S itself is an embedded submanifold of M. □

If $S \subset M$ is an embedded submanifold, a smooth map $\Phi : M \to N$ such that S is a regular level set of Φ is called a *defining map* for S. More generally, if U is an open subset of M and $\Phi : U \to N$ is a smooth map such that $S \cap U$ is a regular level set of Φ, then Φ is called a *local defining map* for S. The above proposition says that every embedded submanifold admits a local defining function in a neighborhood of each of its points.

Theorem 4.2.2 (regular level set theorem)**.** *Every regular level set of a smooth map between smooth manifolds is a properly embedded submanifold whose codimension is equal to the dimension of the codomain.*

4.3 Immersed Submanifolds

4.4 The Tangent Space to a Submanifold

Proposition 4.4.1 (characerization of T_pS)**.** *Suppose M is a smooth manifold, $S \subset M$ is an embedded submanifold, and $p \in S$. As a subspace of T_pM, the tangent space T_pS is characterized by*

$$T_pS = \{v \in T_pM : vf = 0 \textit{ whenever } f \in C^\infty \textit{ and } f|_S = 0\}.$$

Proposition 4.4.2. *Suppose M is a smooth manifold and $S \subset M$ is an embedded submanifold. If $\Phi : U \to N$ is any local defining map for S, then*

$$T_pS = \ker d\Phi_p : T_pM \to T_{\Phi(p)}N$$

for each $p \in S \cap U$.

4.5 Sard's Theorem

Theorem 4.5.1 (Sard's theorem)**.** *Suppose M, N are smooth manifolds, $F : M \to N$ is a smooth map. Then the set of critical values of F has measure zero in N.*

Corollary 4.5.1. *Suppose M, N are smooth manifolds, and $F : M \to N$ is a smooth map. if $\dim M < \dim N$, then $F(M)$ has measure zero in N.*

4.6 The Whitney Embedding Theorem

Theorem 4.6.1 (Whitney embedding theorem)**.** *Every smooth n-manifold admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.*

4.7 The Whitney Approximation Theorem

First we use smooth functions to approximate a smooth function.

Definition 4.7.1. If $\delta : M \to \mathbb{R}$ is a positive continuous function, we say that two functions $F, \widetilde{F} : M \to \mathbb{R}^k$ are *δ-close* if

$$|F(x) - \widetilde{F}(x)| < \delta(x) \quad \forall x \in M.$$

Theorem 4.7.1 (Whitney approximation theorem, functions)**.** *Let M be a smooth manifold, $F : M \to \mathbb{R}^k$ be a continuous function. For any positive continuous function $\delta : M \to \mathbb{R}$, there exists a smotoh function $\widetilde{F} : M \to \mathbb{R}^k$ that is δ-close to F. If F is smooth on a closed $A \subset M$, then $\widetilde{F}$ can be chosen to be equal to F on A.*

Proof. Suppose F is smooth on the closed $A \subset M$, then we can extend F to a smooth function $F_0 : M \to \mathbb{R}^k$, and $F_0|_A = F$. Let

$$U_0 = \{y \in M : |F_0(y) - F(y)| > \delta(y)\}.$$

Then U_0 is open and $A \subset U_0$.

For any $x \in M \setminus A$, let U_x be a neighborhood of x contained in $M \setminus A$ and small enough that

$$\delta(y) > \frac{1}{2}\delta(x), \quad |F(y) - F(x)| < \frac{1}{2}\delta(x)$$

for all $y \in U_x$. Note that such a U_x exists by continuity of δ and F. If $y \in U_x$, then

$$|F(y) - F(x)| < \frac{1}{2}\delta(x) < \delta(y).$$

Now $\{U_x : x \in M \setminus A\}$ is an open cover of $M \setminus A$, and we can choose a countable subcover $\{U_{x_i}\}_{i=1}^{\infty} := \{U_i\}_{i=1}^{\infty}$. Then

$$|F(y) - F(x_i)| < \delta(y) \quad \forall y \in U_i.$$

Let $\{\varphi_0, \varphi_i\}_{i=1}^{\infty}$ be a smooth partition of unity subordinate to $\{U_0, U_i\}_{i=1}^{\infty}$, define $\widetilde{F} : M \to \mathbb{R}^k$ by

$$\widetilde{F}(y) = \varphi_0(y)F_0(y) + \sum_{i=1}^{\infty} \varphi_i(y)F(x_i).$$

The sum is finite because the partition of unity is locally finite. Then $\widetilde{F}$ is smooth and $\widetilde{F}|_A = F$. Let $y \in M$,

$$\begin{aligned}
\left|\widetilde{F}(y) - F(y)\right| &= \left|\varphi_0(y)F_0(y) + \sum_{i=1}^{\infty} \varphi_i(y)F(x_i) - \left(\varphi_0(y) + \sum_{i=1}^{\infty} \varphi_i(y)\right) F(y)\right| \\
&\leq \varphi_0(y)|F_0(y) - F(y)| + \sum_{i=1}^{\infty} \varphi_i(y)|F(x_i) - F(y)| \\
&< \varphi_0(y)\delta(y) + \sum_{i=1}^{\infty} \varphi_i(y)\delta(y) = \delta(y).
\end{aligned}$$

□

4.7.1 Normal Bundles and Tubular Neighborhoods

For each $x \in \mathbb{R}^n$, $T_x\mathbb{R}^n$ is identified with $\mathbb{R}^n$, so $T_x\mathbb{R}^n$ inherits a Euclidean inner product.

Definition 4.7.2. For each $x \in M$, define the *normal space* to M at x to be the $(n-m)$-dimensional subspace $N_xM \subset T_x\mathbb{R}^n$ consisting of all vectors that are orthogonal to T_xM. That is, $N_xM = (T_xM)^{\perp}$. Adjoining $x \in M$ together, we get the *normal bundle*

$$NM = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n : x \in M, v \in N_xM\}.$$

Theorem 4.7.2. *If $M \subset \mathbb{R}^n$ is an embedded m-dimensional submanifold, then NM is an embedded n-dimensional submanifold of $T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$.*

Proof. Idea: $n = n - m + m$. □

Define $E : NM \to \mathbb{R}^n$ by

$$E(x, v) = x + v.$$

A *tubular neighborhood* of M is a neighborhood U of M in $\mathbb{R}^n$ that is the diffeomorphic image under E of an open $V \subset NM$ of the form

$$V = \{(x, v) \in NM : |v| < \delta(x)\}$$

for some positive continuous $\delta : M \to \mathbb{R}$.

Theorem 4.7.3 (tubular neighborhood theorem)**.** *Every embedded submanifold of $\mathbb{R}^n$ has a tubular neighborhood.*

Definition 4.7.3. A *retraction* of a topological space X onto a subspace $M \subset X$ is a continuous map $r : X \to M$ such that $X|_M = \mathrm{id}_M$.

Proposition 4.7.1. *Let $M \subset \mathbb{R}^n$ be an embedded submanifold. If U is a tubular neighborhood of M, there exists a smooth map $r : U \to M$ that is both a retraction and a smooth submersion.*

4.7.2 Smooth Approximation of Maps Between Manifolds

Theorem 4.7.4 (Whitney approximation theorem)**.** *Suppose N, M are a smooth manifolds and $F : N \to M$ is a continuous map. Then F is homotopic to a smooth map. If F is smooth on a closed subset $A \subset N$, then the homotopy can be taken to be relative to A.*

Proof. We embed M into $\mathbb{R}^n$. Let U be a tubular neighborhood of M, and $r : U \to M$ be the smooth retraction. For $x \in M$, let

$$\delta(x) = \sup\{\varepsilon \leq 1 : B_\varepsilon(x) \subset U\},$$

then $\delta : M \to \mathbb{R}^+$ is continuous. Let $\widetilde{\delta} = \delta \circ F : N \to \mathbb{R}^+$. □

4.8 Transversality

Definition 4.8.1. Suppose M is a smooth manifold. Two embedded submanifolds $S, S' \subset M$ are *transverse* if

$$T_pS + T_pS' = T_pM \quad \text{for all } p \in S \cap S'$$

Definition 4.8.2. If $F : N \to M$ is smooth and $S \subset M$ is an embedded submanifold, then F is transverse to S if

$$T_{F(x)}S + dF_x(T_xN) = T_{F(x)}M$$

or all $x \in F^{-1}(S)$.

Theorem 4.8.1. *Suppose N, M are smooth manifolds and $S \subset M$ is an embedded submanifold.*

1. *If $F : N \to M$ is transverse to S, then $F^{-1}(S) \subset N$ is an embedded submanifold with codimension of $F^{-1}(S)$ in N equals codimension of S in M.*

2. *If $S' \subset M$ is an embedded submanifold which is transverse to S, then $S' \cap S \subset M$ is an embedded submanifold with*

$$\operatorname{codim} S \cap S' = \operatorname{codim} S + \operatorname{codim} S'.$$

Proof. We use the fact that a subset is a submanifold if and only if it is locally the level set of a submersion (see **Proposition 4.2.1**). Fix $x \in F^{-1}(S)$, then there is an open neighborhood $U \subset M$ of $F(x)$ and a submersion $\varphi : U \to \mathbb{R}^k$ ($k = \dim M - \dim S$), where $\varphi^{-1}(0) = S \cap U$. Let $\widetilde{U} = F^{-1}(U)$ and $\widetilde{\varphi} = \varphi \circ F$, then $\widetilde{\varphi}^{-1}(0) = \widetilde{U} \cap F^{-1}(S)$.

We claim that $\widetilde{\varphi}$ is a submersion in a neighborhood of x. It suffices to show that $d\widetilde{\varphi}_x : T_xN \to T_{\widetilde{\varphi}(x)}\mathbb{R}^k$ is surjective. Fix $v \in T_{\widetilde{\varphi}(x)}\mathbb{R}^k$, since φ is a submersion, $v = d\varphi_{F(x)}w$ for some $w \in T_{F(x)}M$. Since F is transverse to S and $F(x) \in S$,

$$w = w_1 + dF_x w_2 \quad \text{for some } w_1 \in T_{F(x)}S, w_2 \in T_xN.$$

Since $\varphi = 0$ on S, $d\varphi_{F(x)}w_1 = 0$, so

$$v = d\varphi_{F(x)}w = d\varphi_{F(x)}(w_1 + dF_x w_2) = d\varphi_{F(x)}dF_x w_2 = d\widetilde{\varphi}_x w_2.$$

Apply (1) to the inclusion map $S' \hookrightarrow M$, we get (2). $\square$

Deforming To Obtain Transversality

Goal: Given $F : N \to M$ smooth and $X \subset M$ embedded, "deform" F to be transverse.

Theorem 4.8.2. *Suppose N, M are smooth manifolds and $X \subset M$ is an embedded submanifold. If $F : N \times S \to M$ is smooth and transverse to X, then for almost every $s \in S$ the map $F_s = F(\cdot, s) : N \to M$ is transverse to X.*

Proof. Since $F : N \times S \to M$ is transverse to X, $W := F^{-1}(X) \subset N \times S$ is an embedded submanifold. Let $\pi : N \times S \to S$ be the projection. By Sard's theorem it suffices to show that if $s \in S$ is a regular value of $\pi|_W$, then F_s is transverse to X.

Fix $p \in F_s^{-1}(X)$ and $v \in T_{F_s(p)}M$, let $q = F_s(p) = F(p, s)$. By transversality,

$$v = u + dF_{(p,s)}(v_1, v_2)$$

for some $u \in T_qX$ and $(v_1, v_2) \in T_{(p,s)}(N \times S) = T_pN \times T_sS$. Since s is a regular value of $\pi|_W$ and $(p, s) \in W$, $v_2 = d\pi_{(p,s)}(w_1, w_2)$ for some $(w_1, w_2) \in T_{(p,s)}W$. Then $v_2 = w_2$. So $v = u + dF_{(p,s)}(v_1, v_2)$ $\square$

Theorem 4.8.3. *Suppose M, N are smooth manifolds and $X \subset M$ is an embedded submanifold. Every smooth map $f : N \to M$ is homotopic to a smooth map $g : N \to M$ transverse to X.*

Chapter 5

Lie Groups

5.1 Basic Definitions

Definition 5.1.1. A **Lie group** is a smooth manifold G (without boundary) that is also a group, with the property that the multiplication and inversion are both smooth.

Proposition 5.1.1. *If G is a smooth manifold with a group structure such that that map $(g, h) \mapsto gh^{-1}$ is smooth, then G is a Lie group.*

Proof. Let $a, b \in G$, then $(a, b^{-1}) \mapsto ab$ is smooth, so the multiplication is smooth. Let $g = e$, then the inverse map $(e, h) \mapsto h^{-1}$ is smooth. Thus G is a Lie group. $\square$

Let G be a Lie group, then any $g \in G$ defines maps $L_g, R_g : G \to G$, called **left translation** and **right translation** respectively, by

$$L_g(h) = gh, \quad R_g(h) = hg.$$

5.2 Lie Group Homomorphisms

Definition 5.2.1. If G and H are Lie groups, a **Lie group homomorphism** from G to H is a smooth map $F : G \to H$ that is also a group homomorphism. It is called a Lie group **isomorphism** if it is also a diffeomorphism, which implies tht it has an inverse which is also a Lie group homomorphism. In this case we say that G and H are isomorphic Lie groups.

Theorem 5.2.1. *Every Lie group homomorphism has constant rank.*

5.3 Lie Subgroups

Definition 5.3.1. Suppose G is a Lie group. A **Lie subgroup** of G is a subgroup of G endowed with a topology and smooth structure making it into a Lie group and an immersed submanifold of G.

5.4 Group Actions and Equivariant Maps

5.5 Questions

Example 1.26 Let U be an open set in $\mathbb{R}^n$, then U is a topological n-manifold, and the single chart (U, id_U) defines a smooth structure on U. More generally, let M be a smooth n-manifold and let $U \subset M$ be any open subset. Define an atlas on U by

$$\mathcal{A}_U = \{\text{smooth charts } (V, \varphi) \text{ for } M \text{ such that } V \subset U\}.$$

Chapter 6

Vector Fields, Integral Curves and Flows

6.1 Vector Fields on Manifolds

A **vector field** is a map $X : M \to TM$ such that $X(p) \in T_pM$ for all $p \in M$. We often write X_p for $X(p)$. More generally, a vector field on M is a section of the map $\pi : TM \to M$. Intuitively, the section gives a well-defined rule of assigning each $p \in M$ to an element of $T_pM \in TM$.

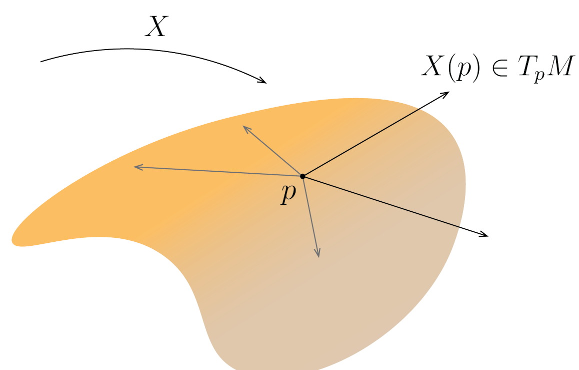


Figure 6.1: There are many tangent vectors in T_pM, a vector field picks a unique one so that the map is well-defined.

It would be wise to review the smooth structure for TM. Recall that $TM = \bigsqcup_{p\in M} T_pM$ and $\pi : TM \to M$ is given by $\pi(p, v) = p$, where $v \in T_pM$. TM is a $2n$-dimensional smooth manifold with the smooth structure

$$\{(\pi^{-1}(U), \widetilde{\varphi} : (U, \varphi = (x^i)) \text{ a smooth chart for } M\},$$

where

$$\widetilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\bigg|_p \right) = \left(x^1(p), \cdots, x^n(p), v^1, \cdots, v^n\right).$$

The coordinates (x^i, v^i) are called the *natural coordinates* on TM.

Given a vector field X and a chart (U, φ), we can write

$$X_p = X^i(p) \frac{\partial}{\partial x^i}\bigg|_p$$

for some functions $X^i : U \to \mathbb{R}$ called the component functions of X.

Proposition 6.1.1. *A vector field $X : M \to TM$ is smooth if and only if for every chart the component functions are smooth.*

Proof. We only need to prove the smoothness of the coordinate representation $\widehat{X}$. Let $(U, \varphi = (x^i))$ be a smooth chart on M and $(\pi^{-1}(U), \widetilde{\varphi})$ be the corresponding smooth chart on TM. Then

$$\begin{aligned}
\widehat{X}(\varphi(p)) &= \widetilde{\varphi} \circ X \circ \varphi^{-1}(\varphi(p)) \\
&= \widetilde{\varphi} \circ X(p) \\
&= \widetilde{\varphi}\left(X^i(p) \frac{\partial}{\partial x^i}\bigg|_p \right) \\
&= \left(x^1(p), \cdots, x^n(p), X^1(p), \cdots, X^n(p) \right).
\end{aligned}$$

It follows that X is smooth if and only if each component X^i is smooth. $\square$

Example 1 . If $(U, (x^i))$ is a smooth chart on M, then

$$p \mapsto \frac{\partial}{\partial x^i}\bigg|_p$$

determines a vector field on U.

Example 2 (Euler's Homogeneous Function Theorem). Let $c \in \mathbb{R}$, let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be a smooth function such that $f(\lambda x) = \lambda^c f(x)$[1] for all $\lambda > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. Let V be the *Euler vector field* on $\mathbb{R}^n$ given by

$$V_x = x^1 \frac{\partial}{\partial x^1}\bigg|_x + \cdots + x^n \frac{\partial}{\partial x^n}\bigg|_x .$$

Prove that $Vf = cf$.

Let $\mathfrak{X}(M)$ denote the set of all smooth vector fields. This is naturally a $\mathbb{R}$-vector space where

- $(X + Y)_p = X_p + Y_p$ for all $X, Y \in \mathfrak{X}(M)$,
- $(aX)_p = aX_p$ for all $a \in \mathbb{R}, X \in \mathfrak{X}(M)$.

[1] Such a function is called *positively homogeneous* of degree c.

Definition 6.1.1. Given $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we can also define $fX \in \mathfrak{X}(M)$ by $(fX)_p = f(p)X(p)$.

Proposition 6.1.2. *Let M be a smooth manifold.*

1. *If $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$, then $fX + gY \in \mathfrak{X}(M)$.*
2. *$\mathfrak{X}(M)$ is a module over $C^\infty(M)$.*

Proof. 1. Let $p \in M$, then

$$\begin{aligned}(fX + gY)(p) &= f(p)X(p) + g(p)Y(p) \\ &= f(p)X^i(p)\frac{\partial}{\partial x^i}\bigg|_p + g(p)Y^i(p)\frac{\partial}{\partial x^i}\bigg|_p\end{aligned}$$

is clearly smooth vector field.

2. Recall that a left $C^\infty(M)$-module $\mathfrak{X}(M)$ consists of an abelian group $\mathfrak{X}(M)$ and a left action $\cdot : C^\infty(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. It is easily seen that $C^\infty(M)$ is a ring and $\mathfrak{X}(M)$ is an abelian group, thus we check that $\cdot$ is an action.

   - Let 1 denote the identity function in $C^\infty(M)$, then
   $$(1 \cdot X)_p = 1(p)X(p) = X(p)$$
   for all $p \in M$.
   - Let $f, g \in C^\infty(M)$, then
   $$[(fg) \cdot X]_p = f(p)g(p)X(p) = f(p)[g(p)X(p)] = [f \cdot (g \cdot X)]_p.$$

$\square$

Definition 6.1.2. If M is a smooth manifold and $A \subset M$, a *vector field along A* is a continuous map $X : A \to TM$ satisfying $\pi \circ X = \mathrm{id}_A$. We call it a *smooth vector field along A* if for each $p \in A$, there is a neighborhood V of p in M and a smooth vector field $\widetilde{X}$ on V that agrees with X on $V \cap A$.

Lemma 6.1.1 (extension lemma for vector fields)**.** *Let M be a smooth manifold with or without boundary, and let $A \subseteq M$ be a closed subset. Suppose X is a smooth vector field along A. Given any open subset U containing A, there exists a smooth global vector field $\tilde{X}$ on M such that $\widetilde{X}\Big|_A = X$ and $\operatorname{supp}\widetilde{X} \subseteq U$.*

Proof. The proof uses the standard smooth structure of TM and essentially follows the proof of the extension lemma of a smooth function. The idea is to "pull back" from $\mathbb{R}^{2n}$ to TM using the diffeomorphism $\widetilde{\varphi}$.

We begin by recalling some notations in the construction of the smooth structure on TM. If (U, φ) is a smooth chart on M with coordinates $(x^1, \cdots, x^n)$ and

$\pi : TM \to M$ given by $\pi(p, v) = p$, then $(\pi^{-1}(U), \widetilde{\varphi})$ is a smooth chart on TM, where

$$\widetilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = (x^1(p), \cdots, x^n(p), v^1, \cdots, v^n).$$

Let $p \in A$, then there is an open neighborhood W_p of p such that there exists a smooth vector field $Y : W_p \to TM$ with $Y|_{W_p \cap A} = X|_{W_p \cap A}$. Without loss of generality we may assume that W_p is the domain of a smooth chart (W_p, φ) for M, Thus $(\pi^{-1}(W_p), \widetilde{\varphi})$ is a smooth chart for TM.

Because $\{W_p\}_{p \in A}$ is an open cover of A, by the gluing lemma, there exists a unique smooth map $Z : A \to TM$ such that

$$Z_{W_p \cap A} = Y|_{W_p \cap A} = X|_{W_p \cap A} \quad \text{for every } p \in A,$$

thus $X : A \to TM$ is a smooth map. Since $\widetilde{\varphi} : \pi^{-1}(W_p) \to \widetilde{\varphi}(\pi^{-1}(W_p)) \subset \mathbb{R}^{2n}$ is a diffeomorphism, the composition $\widetilde{\varphi} \circ X := f : A \to \mathbb{R}^{2n}$ is smooth.

Since $\pi^{-1}(W_p)$ consists of all the tangent vectors at each point of W_p and Y is a vector field, we have $Y(W_p) \subset \pi^{-1}(W_p)$. Thus $\widetilde{f}_p := \widetilde{\varphi} \circ Y : W_p \to TM$ is a smooth map. By replacing W_p by $W_p \cap U$, we may assume that $W_p \subset U$. Now the family $\{W_p\}_{p \in A} \cup \{M \setminus A\}$ is an open cover of M, so there exists a smooth partition of unity $\{\psi_p\}_{p \in A} \cup \{\psi_0\}$ subordinate to this cover such that

- supp $\psi_p \subset W_p$,
- supp $\psi_0 \subset M \setminus A$,
- $\{\text{supp } \psi_p\}$ is locally finite.
- $\bigcup_{p \in A} \text{supp } \psi_p \subset U$.

The product $\psi_p \widetilde{f}_p$ is smooth on W_p and admit a smooth extension to all of M since supp $\psi_p \subset W_p$. Define $\widetilde{f} : M \to \mathbb{R}^{2n}$ by

$$\widetilde{f}(x) = \sum_{p \in A} \psi_p(x) \widetilde{f}_p(x),$$

which is a finite sum due to the local finiteness of $\{\text{supp } \psi_p\}$, thus $\widetilde{f}$ is a smooth map.

Define $\widetilde{X} : M \to TM$ by

$$\widetilde{X}(x) = \widetilde{\varphi}^{-1} \circ \widetilde{f}(x),$$

then

- $\widetilde{X}$ is smooth since it is a composition of smooth maps, and

  $$\widetilde{X}(x) = Y(x) \in T_x M,$$

  so that $\widetilde{X}(x)$ is a smooth vector field.

- If $x \in A$, then

$$\begin{aligned}
\widetilde{\varphi}^{-1} \circ \widetilde{f}(x) &= \widetilde{\varphi}^{-1}\left(\sum_{p\in A} \psi_p(x)\widetilde{f}_p(x)\right) \\
&= \widetilde{\varphi}^{-1}\left(\sum_{p\in A} \psi_p(x) f(x)\right) \\
&= \widetilde{\varphi}^{-1}\left(f(x)\sum_{p\in A} \psi_p(x)\right) \\
&= \widetilde{\varphi}^{-1}(f(x)) \\
&= \widetilde{\varphi}^{-1}(\widetilde{\varphi} \circ X(x)) \\
&= X(x),
\end{aligned}$$

  hence $\widetilde{X}|_A = X|_A$.

- By the coordinate representation of $\widetilde{\varphi}$, we have $\widetilde{\varphi}^{-1}(0) = 0$, thus

$$\operatorname{supp} \widetilde{X} = \operatorname{supp} \widetilde{\varphi}^{-1} \circ \widetilde{f} \subset \operatorname{supp} \widetilde{f} = \bigcup_{p\in A} \operatorname{supp} \psi_p \subset U,$$

completing the proof. □

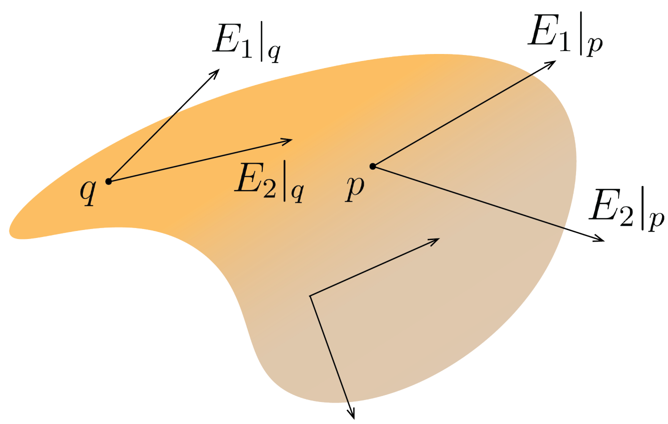
6.1.1 Local and Global Frames

Fix a point p in a smooth chart (U, φ), then $X_p \in T_pM \simeq \mathbb{R}^n$. Now we consider the case when there are finitely many vector fields. Let $X_1, \cdots, X_k$ be a list of vector fields defined on some $A \subset M$. This list

- is called *linearly independent* if the list $X_1|_p, \cdots, X_k|_p$ is linearly independent in T_pM for each $p \in A$.
- said to *span* the tangent bundle if $\operatorname{span}\{X_1|_p, \cdots, X_k|_p\} = T_pM$ for each $p \in A$.

Definition 6.1.3 (frame)**.** Let M be a smooth n-manifold. A *local frame* for M is a list of vector fields $(E_1, \cdots, E_n)$ defined on an open $U \subset M$ that is linearly independent and spans the tangent bundle, equivalently, $E_1|_p, \cdots, E_n|_p$ is a basis for T_pM at each $p \in U$. It is called a *global frame* if $U = M$.

- $(E_1, \cdots, E_n)$ is called a *smooth frame* if each E_i is smooth.
- For $k \in \mathbb{N}$, $(E_1, \cdots, E_k)$ defined on $A \subset \mathbb{R}^n$ is called *orthonormal* if for each $p \in A$, $E_1|_p, \cdots, E_k|_p$ are orthonormal w.r.t. the Euclidean inner product.



We denote a frame $(E_1, \cdots, E_n)$ by (E_i).

Example 3 . The standard coordinate vector fields

$$\left(\left.\frac{\partial}{\partial x^1}\right|_p, \cdots, \left.\frac{\partial}{\partial x^n}\right|_p \right)$$

form a smooth global frame for $\mathbb{R}^n$.

6.1.2 Vector Fields as Derivations of $C^\infty(M)$

A derivation of $C^\infty(M)$ is a map $D : C^\infty(M) \to C^\infty(M)$ such that

$$D(fg) = fD(g) + gD(f) \quad \text{for all } f, g \in C^\infty(M).$$

In the chapter of tangent space, we study the derivation of $f \in C^\infty(M)$ at a point p, satisfying $v(fg) = f(p)vg + g(p)vf$. Here the derivation is global and is a map from $C^\infty(M)$ to $C^\infty(M)$.

If $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we define

$$\begin{aligned} Xf &: M \to \mathbb{R} \\ (Xf)(p) &= X_p(f). \end{aligned}$$

This makes sense since $X_p \in T_pM$, so it acts on any $f \in C^\infty(M)$. It turns out that the smoothness of X and the smoothness of Xf are related.

Proposition 6.1.3. *Let $X : M \to TM$ be a vector field. The following are equivalent:*

1. *X is smooth.*
2. *Xf is a smooth function on M for every $f \in C^\infty(M)$.*
3. *Xf is smooth on U for every open $U \subset M$ and every $f \in C^\infty(U)$.*

Proof. (1) $\implies$ (2): Let X be smooth, $f \in C^\infty(M), p \in M$. Choose smooth coordinates (x^i) on a neighborhood U of p, then for all $x \in U$, we can write

$$Xf(x) = \left(X^i(x) \frac{\partial}{\partial x^i}\bigg|_x \right) f = X^i(x) \frac{\partial f}{\partial x^i}(x).$$

Since $f \in C^\infty(M)$, $\partial f / \partial x^i$ is smooth. Since components X^i are smooth, it follows that Xf is smooth on U. Because U is arbitrary, Xf is smooth on M.
(2) $\implies$ (3): Let open $U \subset M$ and $f \in C^\infty(U)$. For any $p \in U$, let ψ be a smooth bump function such that $\psi = 1$ on a neighborhood of p and supp $\psi \subset U$. Define $\widetilde{f} = \psi f$, extended to be 0 on $M \setminus$ supp ψ, then $X\tilde{f}$ is smooth, and $X\tilde{f} = Xf$ in a neighborhood of p.
(3) $\implies$ (1): Suppose (3) holds, let (x^i) be a smooth local coordinates on $U \subset M$, we can think of each x^i as a smooth function on U. Then

$$Xx^i = X^j \frac{\partial}{\partial x^j}(x^i) = X^i,$$

so each X^i is smooth, hence X is smooth. □

Proposition 6.1.4. *A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if $Df = Xf$ for some $X \in \mathfrak{X}(M)$.*

Proof. First we show that X induces a derivation.

$$\begin{aligned} X(fg)(p) &= X_p(fg) = fX_pg + gX_pf \\ &= f(Xg)(p) + g(Xf)(p), \end{aligned}$$

and X is clearly linear, thus $X : C^\infty(M) \to C^\infty(M)$ is a derivation.
Conversely, we need to construct a vector field X such that $Df = Xf$ for all f. Fix an arbitrary $p \in M$, and define X_p by

$$X_pf = (Df)(p),$$

then $X_p : C^\infty(M) \to \mathbb{R}$ is a tangent vector. Since p is arbitrary, this defines a vector field X. by **Proposition** 6.1.3, X is smooth. □

6.2 Vector Fields and Smooth Maps

We can map a vector field on M to a vector field on N. Let $F : M \to N$ be smooth and X be a vector field on X, then the differential of F at p is $dF_p : T_pM \to T_pN$, thus $dF_p(X_p) \in T_pN$. However, this does not necessarily define a vector field on N.

Example 4 . Let $M = \mathbb{R}^2, N = \mathbb{R}^3, F(x^1, x^2) = (x^1, x^2, 0)$, then dF is not surjective, so there is no vector to assign at $q \in N \setminus F(M)$.

Definition 6.2.1. Suppose $F : M \to N$ is smooth and X is vector field on M, and suppose there is a vector field Y on N such that for each $p \in M$, $dF_p(X_p) = Y_{F(p)}$, then we say X and Y are *F-related.*

Proposition 6.2.1. *Suppose $F : M \to N$ is a smooth map between manifolds, $X \in \mathfrak{X}(M), Y \in \mathfrak{X}(N)$. Then X and Y are F-related if and only if for every $f \in C^\infty(V)$, where V is open in N,*

$$X(f \circ F) = (Yf) \circ F.$$

Proof. Let $p \in M$ and f defined in a neighborhood of $F(p)$ be smooth, then

$$X(f \circ F)(p) = X_p(f \circ F) = dF_p(X_p)f.$$

We also have

$$(Yf) \circ F(p) = (Yf)(F(p)) = Y_{F(p)}f.$$

□

EXAMPLE 5 . Let $F(t) = (\cos t, \sin t)$, then $d/dt \in \mathfrak{X}(\mathbb{R})$ is F-related to the vector field $Y \in \mathfrak{X}(\mathbb{R}^2)$ defined by

$$Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Proof. $\boxed{\text{From definition}}$ To compute $dF_p(X_p)$, we need to plug in an arbitrary $f \in C^\infty(\mathbb{R}^2)$. Thus

$$\begin{aligned} dF_{t_0}\left(\left.\frac{d}{dt}\right|_{t_0}\right) f &= \left.\frac{d}{dt}\right|_{t_0}(f \circ F) \\ &= -\sin t_0 \frac{\partial f}{\partial x}(\cos t_0, \sin t_0) + \cos t_0 \frac{\partial f}{\partial y}(\cos t_0, \sin t_0) \\ &= -\cos t_0 \left.\frac{\partial f}{\partial y}\right|_{F(t_0)} - \sin t_0 \left.\frac{\partial f}{\partial x}\right|_{F(t_0)}. \end{aligned}$$

and

$$Y_{F(t_0)}f = \cos t_0 \left.\frac{\partial f}{\partial y}\right|_{F(t_0)} - \sin t_0 \left.\frac{\partial f}{\partial x}\right|_{F(t_0)}.$$

Since f is arbitrary, we find that $Y_{F(t_0)} = dF_{t_0}\left(\left.\frac{d}{dt}\right|_{t_0}\right)$.

$\boxed{\text{By the above proposition}}$ Let $t_0 \in \mathbb{R}$, then

$$\begin{aligned} X(f \circ F)(t_0) &= X_{t_0}(f \circ F) \\ &= \left.\frac{d}{dt}\right|_{t_0}(f \circ F) \\ &= Y_{F(t_0)}f = (Yf) \circ F(t_0). \end{aligned}$$

□

Remark. A comment on the notation: $\frac{d}{dt}\big|_{t_0} f = f'(t_0)$, and in the computation of $\frac{d}{dt}\big|_{t_0} f \circ F$, it is convenient to formally plug in t so that

$$\begin{aligned} \frac{d}{dt}\bigg|_{t_0} f \circ F(t) &= \frac{d}{dt}\bigg|_{t_0} f(\cos t, \sin t) \\ &= -\sin t_0 \frac{\partial f}{\partial x}(\cos t_0, \sin t_0) + \cos t_0 \frac{\partial f}{\partial x}(\cos t_0, \sin t_0). \end{aligned}$$

Proposition 6.2.2. *Suppose $F : M \to N$ is a diffeomorphism. For every $X \in \mathfrak{X}(M)$, there is a unique smooth vector field on N that is F-related to X.*

Proof. Let $Y \in \mathfrak{X}(N)$ be F-related to X, then $dF_p(X_p) = Y_{F(p)}$ for all $p \in M$. Since F is a diffeomorphism, $p = F^{-1}(q)$ for a unique $q \in N$, then

$$Y_q = dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right).$$

This gives a unique $Y \in \mathfrak{X}(N)$ because q runs through all of N. Since Y is the composition of smooth maps F^{-1}, X, dF, it follows that Y is smooth.

$$\begin{tikzcd} N \arrow[r, "Y"] \arrow[d, "F^{-1}"'] & TN \\ M \arrow[r, "X"] & TM \arrow[u, "dF"'] \end{tikzcd}$$

$\square$

Pushforwards

Given $X \in \mathfrak{X}(M)$ and a diffeomorphism $F : M \to N$, the *pushforward* of X by F is the vector field $F_*X \in \mathfrak{X}(N)$ defined by

$$(F_*X)_q = dF_{F^{-1}(q)}\left(X_{F^{-1}(q)}\right),$$

or equivalently

$$(F_*X)_{F(p)} = dF_p\left(X_p\right).$$

This is the unique smooth vector field that is F-related to X.

Example 6 . Let

$$\begin{aligned} M &= \{(x, y) : y > 0, x + y > 0\}, \\ N &= \{(u, v) : u > 0, v > 0\}, \end{aligned}$$

Define $F(x, y) = (x + y, x/y + 1)$, compute the pushforward F_*X.

Lemma 6.2.1. *If $F : M \to N$ is a diffeomorphism, $X \in \mathfrak{X}(M)$, and $f \in C^\infty(N)$, then*

$$(F_*X)(f) = X(f \circ F) \circ F^{-1}.$$

6.3 Lie Brackets

Definition 6.3.1. The *Lie bracket* of $X, Y \in \mathfrak{X}(M)$ is the map

$$[X,Y] : C^\infty(M) \longrightarrow C^\infty(M)$$
$$[X,Y](f) = XYf - YXf.$$

Lemma 6.3.1. *$[X,Y] \in \mathfrak{X}(M)$.*

Proof. Just compute

$$[X,Y](fg) = f[X,Y]g + g[X,Y]f,$$

where $f \in C^\infty(M)$. □

The calue of the vector field $[X,Y]$ at $p \in M$ is given by

$$[X,Y]_p f = X_p(Yf) - Y_p(Xf).$$

EXAMPLE 7 . In general, $XY \notin \mathfrak{X}(M)$. If $M = \mathbb{R}^2, X = \partial/\partial x \,, Y = x\,\partial/\partial y \,, f(x,y) = x, g(x,y) = y$, then

$$XY(fg) = \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}(xy)\right) = 2x,$$
$$fXY(g) + gXY(f) = x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}(xy)\right) + y\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}(x)\right) = 2x^2$$

Proposition 6.3.1 (coordinate formula)**.** *If $X, Y \in \mathfrak{X}(M)$ and (U,φ) is a smooth chart, and $X = X^i\,\partial/\partial x^i \,, Y = Y^j\,\partial/\partial x^j$, then locally*

$$[X,Y] = \left(X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}.$$

Proof.

$$\begin{aligned}
[X,Y]f &= XYf - YXf \\
&= X\left(Y^j\frac{\partial f}{\partial x^j}\right) - Y\left(X^i\frac{\partial f}{\partial x^i}\right) \\
&= X^i\frac{\partial}{\partial x^i}\left(Y^j\frac{\partial f}{\partial x^j}\right) - Y^j\frac{\partial}{\partial x^j}\left(X^i\frac{\partial f}{\partial x^i}\right) \\
&= X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + X^iY^j\frac{\partial^2 f}{\partial x^i\partial x^j} - Y^j\frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i} - Y^jX^i\frac{\partial^2 f}{\partial x^j\partial x^i} \\
&= X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} - Y^j\frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i} \\
&= X^i\frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} - Y^i\frac{\partial X^j}{\partial x^i}\frac{\partial f}{\partial x^j} \\
&= \left(X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j}.
\end{aligned}$$

□

Proposition 6.3.2 (properties of Lie brackets). 1. *Bilinearity: For* $a, b \in \mathbb{R}$,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z],$$
$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

2. *Antisymmetry:* $[X, Y] = -[Y, X]$.

3. *Jacobi Identity:*

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

4. *For* $f, g \in C^\infty(M)$,

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X.$$

Proof. We prove (4). Just plug in as many variables as you can. Let $p \in M$ and $h \in C^\infty(M)$, then

$$\begin{aligned}
[fX, gY]_p h &= (fX)_p(gYh) - (gY)_p(fXh) \\
&= f(p)X_p(gYh) - g(p)Y_p(fXh) \\
&= f(p)[g(p)X_p(Yh) + Y_phX_p(g)] - g(p)[f(p)Y_p(Xh) + X_phY_p(f)] \quad \text{(Leibniz Rule)} \\
&= f(p)g(p)X_p(Yh) + f(p)Y_phX_p(g) - f(p)g(p)Y_p(Xh) - g(p)X_phY_p(f) \\
&= f(p)g(p)[X, Y]_ph + f(p)(Xg)_pY_ph - g(p)(Yf)_pX_ph \\
&= \{fg[X, Y]\}_ph + (fXg)_pY_ph - (gYf)_pX_ph.
\end{aligned}$$

□

Proposition 6.3.3 (Naturality). *Let* $F : M \to N$ *be smooth,* $X_1, X_2 \in \mathfrak{X}(M), Y_1, Y_2 \in \mathfrak{X}(N)$ *such that* X_i *is* F*-related to* Y_i*. Then* $[X_1, X_2]$ *is* F*-related to* $[Y_1, Y_2]$.

Proof. By **Proposition** 6.2.1, since X_2 and Y_2 is F-related

$$X_2(f \circ F) = (Y_2 f) \circ F,$$

hence

$$X_1X_2(f \circ F) = X_1((Y_2f) \circ F) = (Y_1Y_2f) \circ F. \quad (X_1, Y_1 \text{ are } F\text{-related})$$

Then

$$\begin{aligned}
[X_1, X_2](f \circ F) &= X_1X_2(f \circ F) - X_2X_1(f \circ F) \\
&= (Y_1Y_2f) \circ F - (Y_2Y_1f) \circ F \\
&= ([Y_1, Y_2]f) \circ F.
\end{aligned}$$

□

Corollary 6.3.1. *If $F : M \to N$ is a diffeo and $X_1, X_2 \in \mathfrak{X}(M)$, then*

$$F_*[X_1, X_2] = [F_*X_1, F_*X_2].$$

Proof. Take $Y_i = F_*X_i$, then X_i is F-related to Y_i. By naturality we have

$$[Y_1, Y_2]_{F(p)} = dF_p([X_1, X_2]_p) = (F_*[X_1, X_2])_{F(p)},$$

which is to be shown. □

6.4 Integral Curves

Fix M a smooth manifold.

Definition 6.4.1. If $V \in \mathfrak{X}(M)$, then an *integral curve* of V is a smooth curve $\gamma : I \to M$ with $\gamma'(t) = V_{\gamma(t)}$ for all $t \in I$.

Locally, in a chart (U, φ), we have $\gamma^i = x^i \circ \gamma$ and $V = V^i \frac{\partial}{\partial x^i}$, and

$$\gamma'(t) = \dot{\gamma}^i(t) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)}.$$

Then $\gamma'(t) = V_{\gamma(t)}$ gives a system of ODE

$$\begin{aligned}\dot{\gamma}^1(t) &= V^1(\gamma^1(t), \cdots, \gamma^n(t)),\\ &\vdots\\ \dot{\gamma}^n(t) &= V^n(\gamma^1(t), \cdots, \gamma^n(t)).\end{aligned}$$

The fundamental fact about such systems is the existence, uniqueness, and smoothness theorem, from **Theorem D.1**. of Lee's book.

Proposition 6.4.1. *Let $V \in \mathfrak{X}(M)$, for any $p \in M$ there exists $\varepsilon > 0$ and a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ that is an integral curve of V with $\gamma(0) = p$.*

Proof. Apply the existence statement to the coordinate representation of V. □

Example 8 . Let (x, y) be standard coordinates on $\mathbb{R}^2$, and let $V = \partial/\partial x$ be the first coordinate vector field. Then the integral curves of V are precisely the straight lines parallel to the x-axis.

Example 9 . Let $W = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$ on $\mathbb{R}^2$. Let $\gamma : \mathbb{R} \to \mathbb{R}^2, \gamma(t) = (x(t), y(t))$ be a smooth curve, solve the system of ODE $\gamma'(t) = W_{\gamma(t)}$.

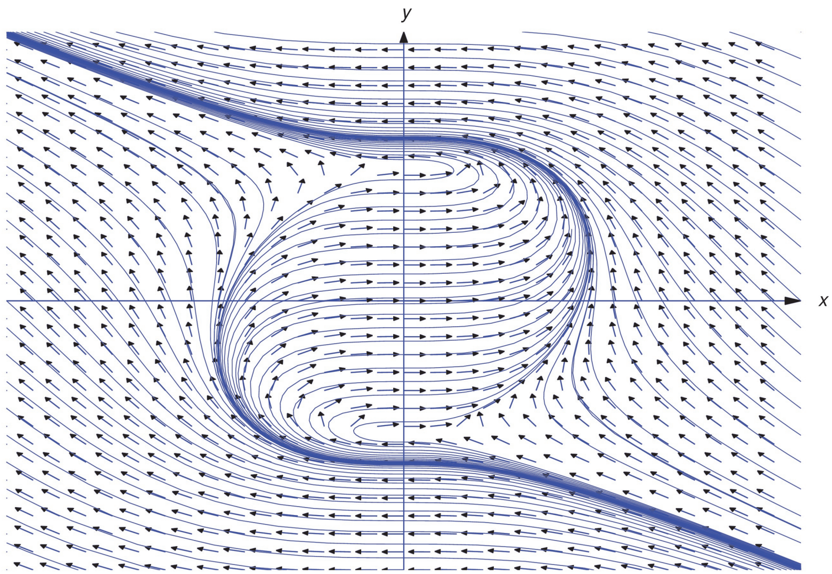


Figure 6.2: flow

6.5 Flows

Intuitively, flow is the set of all integral curves, which can be viewed as the set of all solutions of an ODE. To specify a point in an integral curve, we need two parameters: initial point p and traveling time t (here we assume curves are defind on $[0, \infty)$ for convenience). Fix a smooth manifold M and $V \in \mathfrak{X}(M)$. For each $p \in M$, V has a unique integral curve starting at p and defined for all $t \in [0, \infty)$. Now let p runs through all points of M, we get a map

$$\begin{aligned} \theta : \mathbb{R} \times M &\to M \\ (t, p) &\mapsto \theta(t, p). \end{aligned}$$

Now we can take sections on $\mathbb{R}$ or M.

- Fix p, we get a map $\theta^{(p)} : \mathbb{R} \to M$ given by $\theta^{(p)}(t) = \theta(t, p)$, which is the point on the curve at time t starting at p. This is essentially same as a parametrized curve.
- Fix t, we get a map $\theta_t : M \to M$ given by $\theta_t(p) = \theta(t, p)$. This is to see how far can each integral curve travel in time t.

Suppose a curve $\theta^{(p)}$ is chosen and let q be another point in this curve: $q = \theta^{(p)}(s)$ for some $s \in (0, \infty)$. By concatenating the time interval $[0, s]$, we can

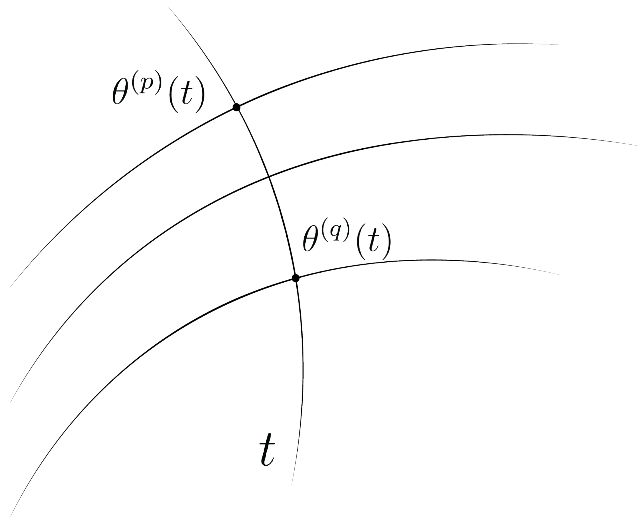


Figure 6.3: Fix t, the position $\theta^{(p)}(t)$ and $\theta^{(q)}(t)$

reparametrize q w.r.t. p:

$$\theta^{(q)}(t) = \theta^{(p)}(t+s), \quad t \in [0, \infty).$$

Now $\theta^{(q)}(0) = \theta^{(p)}(s) = q$ is the initial point of $\theta^{(q)}$. In fact, θ is an action of the group $(\mathbb{R}, +)$ on M. To see this, notice that

- $\theta_t \circ \theta_s(p) = \theta_t(\theta_s(p)) = \theta_t(q) = \theta^{(q)}(t) = \theta^{(p)}(t+s) = \theta_{t+s}(p)$.
- $\theta_0(p) = \theta^{(p)}(0) = p$.

If we denote the the action by $t \cdot p = \theta^{(p)}(t)$, then $0 \cdot p = \theta^{(p)}(0) = p$, and

$$t \cdot (s \cdot p) = t \cdot (\theta^{(p)}(s)) = t \cdot q = \theta^{(q)}(t) = \theta^{(p)}(t+s) = (t+s) \cdot p.$$

Definition 6.5.1 (global flow)**.** A *global flow* is a smooth map $\theta : \mathbb{R} \times M \to M$ such that

- $\theta(0, \cdot) = \mathrm{id}_M$,
- $\theta(t+s, p) = \theta(t, \theta(s, p))$ for all $s, t \in \mathbb{R}$ and $p \in M$.

Note that if $\theta_t = \theta(t, \cdot) : M \to M$, then the map

$$t \in (\mathbb{R}, +) \mapsto \theta_t \in \mathrm{Diff}(M)$$

is a homomorphism, since the second property implies that

$$\theta_{t+s} = \theta_t \circ \theta_s,$$

and $\theta_t^{-1} = \theta_{-t}$ because $\theta_0 = \mathrm{id}_M$.

Flow → Vector Field

It is intuitive that if we take derivative at each point of each integral curve, we will obtain a vector field.

Definition 6.5.2. If $\theta : \mathbb{R} \times M \to M$ is a smooth global flow, for each $p \in M$ we define a tangent vector $V_p \in T_pM$ by

$$V_p = {\theta^{(p)}}'(0).$$

The assignment $p \mapsto V_p$ is a vector field on M, which is called the *infinitesimal generator* of θ.

Proposition 6.5.1. *Let $\theta : \mathbb{R} \times M \to M$ be a global flow. Then*

$$V_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \theta(t, p) \in T_pM$$

defines a smooth vector field on M. Moreover, each curve $\theta^{(p)}$ is an integral curve of V.

Proof. Rewrite the expression

$$V_p = d\theta_{(0,p)} \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \right),$$

so V is smooth (in local cooridnates $d\theta_{(0,p)}$ is a matrix whose entries depend smoothly on p), hence $V \in \mathfrak{X}(M)$. For the moreover part, fix $p \in M$ and let $\gamma(t) = \theta(t, p)$. Then

$$\gamma'(t) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \gamma(t+s) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \theta(t+s, p) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \theta(s, \theta(t, p)) = V_{\theta(t,p)} = V_{\gamma(t)}.$$

□

Vector Field → Flow

Conversely, we would like to be able to say that every smooth vector field is the infinitesimal generator of a smooth global flow, which is not always the case.

EXAMPLE 10 . Let $M = \mathbb{R}^2 \setminus \{0\}$ and V be the vector field $\partial/\partial x$ on M. The unique integral curve of V starting at $(-1, 0) \in M$ is $\gamma(t) = (t-1, 0)$. In this case, γ cannot be extended continuously past $t = 1$. Suppose this were true, let $\widetilde{\gamma}$ be any continuous extension of γ past $t = 1$, and notice that $\widetilde{\gamma}$ is still a curve in $M = \mathbb{R}^2 \setminus \{0\}$. Then

$$\lim_{t \to 1^-} \gamma(t) = \widetilde{\gamma}(1) \in \mathbb{R}^2 \setminus \{0\},$$

but if we consider $\gamma : M \to \mathbb{R}^2$, then

$$\lim_{t \to 1^-} \gamma(t) = \lim_{t \to 1^-} (t-1, 0) = (0, 0) \neq \widetilde{\gamma}(1),$$

a contradiction. □

It seems that the domain is too large, so we have to restrict integral curves into a *flow domain*.

Definition 6.5.3 (flow domain)**.** A *flow domain* $\mathcal{D} \subset \mathbb{R} \times M$ is an open set such that for each $p \in M$, the section $\mathcal{D}^{(p)} := \{t \in \mathbb{R} : (t, p) \in \mathcal{D}\}$ is an open interval containing 0. A *(partial) flow* is a smooth map $\theta : \mathcal{D} \to M$ defined on a flow domain such that $\theta(0, \cdot) = \mathrm{id}_M$ and $\theta(t + s, p) = \theta(t, \theta(s, p))$ whenever both sides are defined.

EXAMPLE 11 . Let $\mathbb{D}$ be the open disc in $\mathbb{R}^2$, let $V = \partial/\partial x$.

integral curves don't exist for all time. Get a flow $\theta : D \to \mathbb{D}$ where $D \neq \mathbb{R} \times \mathbb{D}$.

Definition 6.5.4. A *maximal integral curve* is one that cannot be extended to an integral curve on any larger open interval, and a *maximal flow* is a flow that admits no extension to a flow on a larger flow domain (i.e., a flow defined on a maximal flow domain)

Theorem 6.5.1 (fundamental theorem on Flows)**.** *Let $V \in \mathfrak{X}(M)$, there is a unique maximal flow $\theta : \mathcal{D} \to M$ where $V_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \theta(t, p)$ for all $p \in M$. Moreover,*

1. *for each $p \in M$, $t \in \mathcal{D}^{(p)} \mapsto \theta(t, p) \in M$ is the unique maximal integral curve starting at p.*
2. *If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\theta(s,p))} = \mathcal{D}^{(p)} \setminus \{s\}$.*

Proof. Use the existence theorem of an ODE to construct the slice $\mathcal{D}^{(p)}$, and define a flow domain $\mathcal{D}$. □

Definition 6.5.5. The flow in **Theorem 6.5.1** is called the *flow generated by V*, or just the *flow of V*.

6.6 Complete Vector Fields

A vector field is *complete* if its flow is a global flow (i.e., defined on $\mathcal{D} = \mathbb{R} \times M$, so that $\mathcal{D}^{(p)} = \mathbb{R}$).

Lemma 6.6.1 (uniform time lemma)**.** *Let $V \in \mathfrak{X}(M)$ and $\theta : \mathcal{D} \to M$ be the flow of V. If there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset \mathcal{D}^{(p)}$ for all $p \in M$, then V is complete.*

Proof. Suppose to the contrary that for some $p \in M$, the domain $\mathcal{D}^{(p)}$ is bounded above. Let $b = \sup \mathcal{D}^{(p)}$, let $t_0 > 0$ be such that $b - \varepsilon < t_0 < b$, and let $q = \theta^p(t_0)$. The hypothesis implies that $\theta^{(q)}(t)$ is defined at least for $t \in (-\varepsilon, \varepsilon)$. Concatenate $\theta^{(p)}$ and $\theta^{(q)}$ by a curve $\gamma : (-\varepsilon, \varepsilon + t_0)$ defined as

$$\gamma(t) = \begin{cases} \theta^{(p)}(t), & \varepsilon < t < b, \\ \theta^{(q)}(t - t_0), & t_0 - \varepsilon < t < t_0 + \varepsilon. \end{cases}$$

Since by the group law we have

$$\theta^{(q)}(t - t_0) = \theta_{t-t_0}(q) = \theta_{t-t_0} \circ \theta_{t_0}(p) = \theta_t(p) = \theta^{(p)}(t),$$

these two definitions agree where they overlap. By the transition lemma, γ is an integral curve starting at p. Since $t_0 + \varepsilon \notin \mathcal{D}^{(p)}$, this contradicts the maximality of the flow domain.

□

Corollary 6.6.1. *On a compact manifold, every vector field is complete.*

Proof. See HW 8, Problem 1(a). □

Lemma 6.6.2. *Suppose M is a smooth manifold, $V \in \mathfrak{X}(M)$, and let $\theta : \mathcal{D} \to M$ be the flow of V. Then for any compact set $K \subset M$, there exists $\epsilon > 0$ such that $(-\epsilon, \epsilon) \times K \subset \mathcal{D}$.*

Proof. Since θ is a flow, $\mathcal{D}$ is a flow domain. Thus for any $p \in M$,

$$\mathcal{D}^{(p)} = \{t \in \mathbb{R} : (t, p) \in \mathcal{D}\} = (a_p, b_p) \ni 0.$$

We also have $(a_p, b_p) \times \{p\} \subset \mathcal{D}$, but $\mathcal{D}$ is open in $\mathbb{R} \times M$, thus there exists an open set $U_p \ni p$ such that $(a_p, b_p) \times U_p \subset \mathcal{D}$. For the cover $\{U_p\}_{p \in K}$ we can extract a finite subcover $\bigcup_{j=1}^n U_{p_j} \supset K$, and each U_{p_j} corresponds to a

$$\mathcal{D}^{(p_j)} = \{t \in \mathbb{R} : (t, p_j) \in \mathcal{D}\} = (a_{p_j}, b_{p_j}) \ni 0$$

and $(a_{p_j}, b_{p_j}) \times U_{p_j} \subset \mathcal{D}$. Choose $0 < \varepsilon < \min_{1 \le j \le n}(|a_{p_j}|, |b_{p_j}|)$, we have $(-\varepsilon, \varepsilon) \subset (a_{p_j}, b_{p_j})$ for $j = 1, \cdots, n$. Hence

$$(-\varepsilon, \varepsilon) \times U_{p_j} \subset \mathcal{D} \quad \text{for each } j = 1, \cdots, n.$$

Therefore,

$$(-\varepsilon, \varepsilon) \times K \subset (-\varepsilon, \varepsilon) \times \bigcup_{j=1}^{n} U_{p_j} \subset \mathcal{D}.$$

□

Lemma 6.6.3 (escape lemma)**.** *If $\gamma : J \to M$ is a maximal integral curve of V whose domain J has a finite least upper bound b, then for any $t_0 \in J$, $\gamma([t_0, b))$ is not contained in any compact subset of M.*

Proof. We may assume that $J = (a, b) \ni 0$. Suppose that there exists $t_0 \in J$ such that $\gamma([t_0, b))$ is contained in a compact set K of M. Set $p = \gamma(a)$, then by the fundamental theorem of flow, there is a maxiaml flow $\theta : \mathcal{D} \to M$. By uniquenss, $\theta^{(p)} : \mathcal{D}^{(p)} \to M$ is exactly γ and $\mathcal{D}^{(p)} = J$. Let $\{t_n\}_{n=1}^{\infty} \subset [t_0, b)$ converge to b, then $\{\gamma(t_n)\}_{n=1}^{\infty} \subset K$. Then there is a subsequence $\{\gamma(t_{n_k})\}_{k=1}^{\infty}$ with $\gamma(t_{n_k})$ converging to some $q \in K$ as $k \to \infty$. By the last lemma, there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset [t_0, b)$ and $(-\varepsilon, \varepsilon) \times K \subset \mathcal{D}$, so θ is defined on $(-\varepsilon, \varepsilon) \times K$. Choose n large so that $t_n > b - \varepsilon$, and define $\beta : (a, t_n + \varepsilon) \to M$ by

$$\beta(t) = \begin{cases} \gamma(t), & a < t < b. \\ \theta_{t-t_n} \circ \theta_{t_n}(p), & t_n - \varepsilon < t < t_n + \varepsilon. \end{cases}$$

These two definitions agree where they overlap since

$$\theta_{t-t_n} \circ \theta_{t_n}(p) = \theta_t(p) = \gamma(t),$$

hence β extends γ past b, a contradiction. □

6.7 Regular Points and Singular Points

If $V \in \mathfrak{X}(M)$, then $p \in M$ is a *regular point* of V if $V_p \neq 0$. Otherwise, if $V_p = 0$, then p is a *singular point* of V.

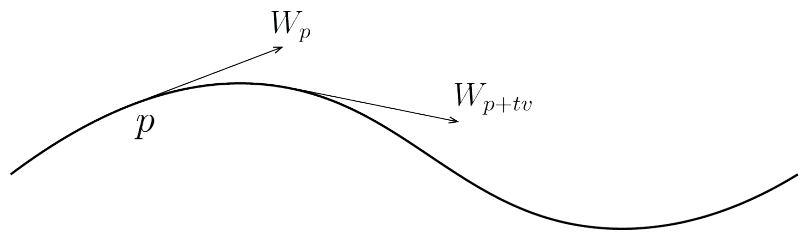
Theorem 6.7.1 (canonical form near a regular point)**.** *If $V \in \mathfrak{X}(M)$, $p \in M$, and $V_p \neq 0$, then there exists a chart $(U, \varphi = (x^1, \cdots, x^n))$ centered at p where $V = \partial / \partial x^i$ on U.*

Proof. □

Remark. No canonical form near singular points: see Lee Figure 9.8.

6.8 Lie Derivatives

How to define the directional derivative of a vector field? In Euclidean space, we can measure the rate of change of a smooth vector field W in the direction of $v \in T_p\mathbb{R}^n$. It is the vector



$$D_vW(p) = \frac{d}{dt}\bigg|_{t=0} W_{p+tv} = \lim_{t\to 0}\frac{W_{p+tv} - W_p}{t}.$$

In attempt to generalizing this definition to manifolds, we replace $p + tv$ by a curve $\gamma(t)$ with $\gamma'(0) = v$. But $W_{\gamma(t)} \in T_{\gamma(t)}M, W_{\gamma(0)} \in T_{\gamma(0)}M$ are in different vector spaces.

If we replace the vector $v \in T_pM$ with a vector field V, then we can use the flow of V to push values of W back to p.

Definition 6.8.1. Suppose $V \in \mathfrak{X}(M)$ has flow θ. Then the *Lie derivative* of $W \in \mathfrak{X}(M)$ w.r.t. V is the vector field where

$$\mathcal{L}_V(W) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = \lim_{t\to 0}\frac{d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) - W_p}{t}$$

Remark. The vector field V is represented in the picture by its flow, and $d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) \in T_{\theta_{-t}(\theta_t(p))}M$. Recall that $\theta_{-t} = \theta_t^{-1}$, see **Definition** 6.5.1.

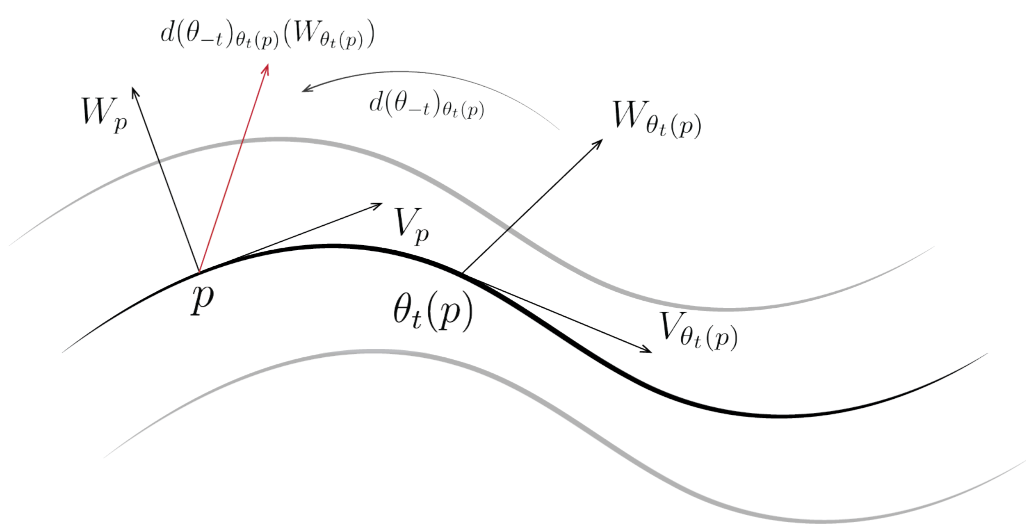


Figure 6.4: Using flow to pull back

There is a simple formula for computing the Lie derivative.

Theorem 6.8.1. *If $V, W \in \mathfrak{X}(M)$, then $\mathcal{L}_V(W) = [V, W]$.*

Proof. Fix a chart (U, φ). In these coordinates

$$\theta(t,p) = (\theta^1(t,p), \cdots, \theta^n(t,p)), \quad W = W^j \frac{\partial}{\partial x^j}.$$

$$\begin{aligned}
\mathcal{L}_V(W) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \frac{\partial \theta^i}{\partial x^j}(-t, \theta(t,x)) W^j(\theta(t,x)) \frac{\partial}{\partial x^i} \\
&= -\frac{\partial^2 \theta^i}{\partial t \partial x^j}(-t, \theta(t,x)) W^j(\theta(t,x)) \frac{\partial}{\partial x^i} + \frac{\partial^2 \theta^i}{\partial x^k \partial x^j} \frac{\partial \theta^k}{\partial t}(t,x) W^j(\theta(t,x)) \frac{\partial}{\partial x^i} \\
&\quad + \left.\frac{\partial \theta^u}{\partial x^j}(-t, \theta(t,x)) \frac{\partial W^j}{\partial x^k}(\theta(t,x)) \frac{\partial \theta^k}{\partial t}(t,x) \frac{\partial}{\partial x^i}\right|_{t=0} \\
&= \frac{\partial^2 \theta}{\partial t \partial x^j}(0,x) W^j(x) \frac{\partial}{\partial x^i} + \frac{\partial^2 \theta^i}{\partial x^k \partial x^j}(0,x) \frac{\partial \theta^k}{\partial t}(0,x) W^j(x) \frac{\partial}{\partial x^i} \\
&\quad + \frac{\partial \theta^i}{\partial x^j}(0,x) \frac{\partial W^j}{\partial x^k}(x) \frac{\partial \theta^k}{\partial t}(0,x).
\end{aligned}$$

Note

$$\frac{\partial \theta^k}{\partial t}(0,x) = V^k, \quad \frac{\partial^2 \theta^i}{\partial t \partial x^j}(0,x) = \frac{\partial}{\partial x^j}\left(\frac{\partial \theta^i}{\partial t}(0,x)\right) = \frac{\partial V^i}{\partial x^j},$$

and $\theta(0, \cdot) = \mathrm{id}_M$, so

$$\begin{aligned}
\frac{\partial \theta^i}{\partial x^j}(0,x) &= \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \\
\frac{\partial^2 \theta^i}{\partial x^k \partial x^j}(0,x) &= \frac{\partial}{\partial x^k}\left(\frac{\partial \theta^i}{\partial x^j}(0,x)\right) = 0,
\end{aligned}$$

then

$$\mathcal{L}_V(W) = -\frac{\partial V^i}{\partial x^j} W^j \frac{\partial}{\partial x^i} + 0 + \frac{\partial W^j}{\partial x^k} V^k \frac{\partial}{\partial x^j} = [V, W].$$

$\square$

Corollary 6.8.1. *If $V, W, X, Y \in \mathfrak{X}(M)$, then*

1. $\mathcal{L}_V(W) = -\mathcal{L}_W(V)$;
2. $\mathcal{L}_V([X,Y]) = [\mathcal{L}_V X, Y] + [X, \mathcal{L}_V Y]$;
3. $\mathcal{L}_{[V,W]} X = \mathcal{L}_V \mathcal{L}_W X - \mathcal{L}_W \mathcal{L}_V X$.
4. *If $g \in C^\infty(M)$, then*

$$\mathcal{L}_V(gW) = (Vg)W + g\mathcal{L}_V(W).$$

5. *If $F : M \to N$ is a diffeomorphism,*

$$F_*\mathcal{L}_V(W) = \mathcal{L}_{F_*V}(F_*W).$$

Proof. 1. $\mathcal{L}_W(V) = [W,V] = -[V,W]$.

2. $\mathcal{L}_V([X,Y]) = [V,[X,Y]]$. Let $f \in C^\infty(M)$, then

$$\begin{aligned}[V,[X,Y]]f &= V[X,Y]f - [X,Y]Vf \\ &= V(XYf - YXf) - (XYVf - YXVf) \\ &= VXYf - VYXf - XYVf + YXVf,\end{aligned}$$

and

$$\begin{aligned}[\mathcal{L}_V X, Y]f &= [[V,X],Y]f \\ &= [V,X]Yf - Y[V,X]f \\ &= VXYf - XVYf - YVXf + YXVf, \\ [X, \mathcal{L}_V Y]f &= [X,[V,Y]]f \\ &= X[V,Y]f - [V,Y]Xf \\ &= XVYf - XYVf - VYXf + YVXf.\end{aligned}$$

3. Similar as (2).

4. Let $f \in C^\infty(M)$, then

$$\begin{aligned}[V,gW]f &= V(gW)f - (gW)Vf \\ &= VgWf - g\mathcal{L}_V(W)f.\end{aligned}$$

∎

6.9 Commuting Vector Fields

Definition 6.9.1. Let M be a smooth manifold and $V, W \in \mathfrak{X}(M)$.

- We say that $V, W \in \mathfrak{X}(M)$ are *commute* if $[V,W] = 0$, i.e., $VWf = WVf$ for every smooth function f.
- If $\theta : \mathcal{D} \to M$ is a flow and $W \in \mathfrak{X}(M)$, then W is *invariant* under θ if

$$d(\theta_t)_p W_p = W_{\theta_t(p)} \quad \text{for all } (t,p) \in \mathcal{D} \tag{6.1}$$

- Two flows θ, ψ *commute* if whenever $p \in M$ and $I, J \subset \mathbb{R}$ are open intervals containing 0 such that one of $\psi_s \circ \theta_t(p)$ or $\theta_t \circ \psi_s(p)$ is defined for all $(s,t) \in I \times J$, then

$$\psi_s \circ \theta_t(p) = \theta_t \circ \psi_s(p)$$

  for all $(s,t) \in I \times J$ (in particular both defined).

Theorem 6.9.1. *If $V, W \in \mathfrak{X}(M)$, then TFAE:*

1. *V, W commute.*
2. *W is invariant under the flow of V.*
3. *V is invariant under the flow of W.*
4. *The flows of V, W commute.*

Proof. (a) $\iff$ (b).

(a) $\iff$ (c): Since $[V,W] = -[W,V]$, this follows from the equivalence of (a) and (b).

(b), (c) $\implies$ (d): By symmetry, it suffices to fix $p \in M$ and open intervals $I, J \subset \mathbb{R}$ containing 0, where $\psi_s \circ \theta_t(p)$ exists for all $(s,t) \in I \times J$, then show that $\psi_s \circ \theta_t(p) = \theta_t \circ \psi_s(p)$ for all $(s,t) \in I \times J$. Fix $s \in I$. Define $\gamma : J \to M$ by $\gamma(t) = \psi_s \circ \theta_t(p)$. Then

$$\gamma'(t) = d(\psi_s)_{\theta_t(p)} V_{\theta_t(p)} = V_{\psi_s(\theta_t(p))} = V_{\gamma(t)},$$

so γ is an integral curve of V. Then $\gamma(t) = \theta_t(\psi_s(p))$ for all $t \in J$ since $\gamma(0) = \psi_s(p)$. Hence $\theta_t(\psi_s(p)) = \psi_s(\theta_t(p))$ for all $t \in J$. Since $s \in I$ was arbitrary, we are done.

(d) $\implies$ (b): Fix $p \in M$, then $\psi_s \circ \theta(p) = \theta_t \circ \psi_s(p)$ for s, t small enough. Then

$$W_{\theta_t(p)} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \psi_s(\theta_t(p)) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \theta_t(\psi_s(p)) = d(\theta_t)_p W_p \quad (*)$$

for t small. To show this for any $t \in D^{(p)}$, use $(*)$ and the fact that

$$d(\theta_{t_1+t_2})_p = d(\theta_t)_{\theta_{t_2}(p)} d(\theta_{t_2})_p.$$

$\square$

Chapter 7

Vector Bundles

7.1 Vector Bundles

Goal: introduce the language of vector bundles.

Definition 7.1.1. Let M be a topological space. A (real) vector bundle of rank k over M is a topological space E and a surjective continuous map $\pi : E \to M$ such that

- for each $p \in M$, the fiber $E_p := \pi^{-1}(p)$ is endowed with a k-dimensional real vector space structure.
- For each $p \in M$, there is a open neighborhood U of p and a homeomorphism

  $$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$$

  (called the *local trivialization* of E over U) such that

  - $\pi_U \circ \Phi = \pi$ where $\pi_U : U \times \mathbb{R}^k \to U$ is the projection.
  - For each $q \in U$, the map $\Phi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^k \simeq \mathbb{R}^k$ is a linear isomorphism.

If E, M are smooth manifolds, π is smooth, and each Φ is a diffeomorphism, then $\pi : E \to M$ is a *smooth vector bundle.*

$E =$ total space of the vector bundle
$\pi =$ projection space of the vector bundle
$M =$ base space of the vector bundle

Example 1 . $M \times \mathbb{R}^k \to M$ is the *trivial bundle.*

Proof. Let $p \in M$, then $\pi^{-1}(p) = \{p\} \times \mathbb{R}^k \simeq \mathbb{R}^k$ is a k-dimensional vector space. Let U be an open neighborhood of p, and define

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$$
$$\Phi(p, x) = (p, x).$$

Then Φ is clearly a homeomorphism. $\square$

EXAMPLE 2 . Given a smooth manifold M, $TM \to M$ is a smooth vector bundle.

Proof. Suppose $\dim M = n$. Let $p \in M$, then $E_p := \pi^{-1}(p) = T_pM = \mathbb{R}^n$. Let U be a neighborhood of p, then $\pi^{-1}(U)$ is the set of all tangent vectors at each point of U. Define

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$$
$$\Phi\left(v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = (p, v^1, \cdots, v^n),$$

then Φ is a homeomorphism. Next,

$$\pi_U \circ \Phi\left(v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = \pi_U(p) = p,$$

and

$$\Phi|_{E_q}\left(v^i \frac{\partial}{\partial x^i}\bigg|_q\right) = (q, v^1, \cdots, v^n)$$

is clearly a linear isomorphism. $\square$

EXAMPLE 3 . If $M \subset \mathbb{R}^n$ is an embedded submanifold, $NM \to M$ is a smooth vector bundle.

EXAMPLE 4 . Let $E = [0,1] \times \mathbb{R}/(0,t) \sim (1,-t)$, $\mathbb{S}^1 = [0,1]/0 \sim -1$. Then $\pi : E \to \mathbb{S}^1$ defined by $\pi([x,t]) = [x]$ is a vector bundle.

7.2 Transition Functions

Lemma 7.2.1. *Let $\pi : E \to M$ be a smooth vector bundle of rank k. Suppose*

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k, \quad \Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$$

are smooth local trivializations with $U \cap V \neq \varnothing$. Then

$$\Phi \circ \Psi^{-1}(p, w) = (p, \tau(p)w)$$

on $(U \cap V) \times \mathbb{R}^k$, where $\tau : U \cap V \to \mathrm{GL}(k, \mathbb{R})$ is smooth. τ is called the transition function between the trivializations.

Proof. By definition,

$$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v)$$

for some map $\tau : U \cap V \to \mathrm{GL}(k, \mathbb{R})$. To show that τ is smooth, it suffices to show that the (i, j)-entry $\tau(p)^i_j$ is smooth. Let $E_1, \cdots, E_k$ be the standard basis of $\mathbb{R}^k$. Let $\pi^i : \mathbb{R}^k \to \mathbb{R}$ be the projection onto the ith entry. Let $\widehat{\pi} : (U \cap V) \times \mathbb{R}^k \to \mathbb{R}^k$ be the projection. Then

$$\tau(p)^i_j = \pi^i(\tau(p)E_j) = \pi^i(\widehat{\pi}(\Phi \circ \Psi^{-1}(p, E_j))$$

is smooth. $\square$

Lemma 7.2.2 (vector bundle chart lemma). *Let M be a smooth manifold. For each $p \in M$, suppose E_p is a real vector space of dimension k. Let $E = \bigsqcup_{p\in M} E_p$ and let $\pi : E \to M$ be the map with $\pi_{E_p} = p$ for all $p \in M$. Suppose we are given:*

1. *an open cover $\{U_\alpha\}_{\alpha\in A}$ of M.*
2. *For each $\alpha \in A$, a bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$, where $\Phi_\alpha|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k \simeq \mathbb{R}^k$ is a linear isomorphism for all $p \in U_\alpha$.*
3. *For each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta$ nonempty, there is a smooth map $\tau_{\alpha\beta}$: $U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ such that*

$$\Phi_a \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v)$$

on $(U_\alpha \cap U_\beta) \times \mathbb{R}^k$.

Then E has a unique topology and smooth structure making $\pi : E \to M$ a smooth vector bundle of rank k where each Φ_α is a local trivialization.

Proof. Topology: See Lee.
Smooth Structure: Let

$$\mathcal{A} = \bigcup_{\alpha\in A} \{(\widetilde{U}, \widetilde{\varphi}) : (U, \varphi) \text{ a smooth chart of } M \text{ where } U \subset U_\alpha\},$$

where

$$\widetilde{U} = \pi^{-1}(U) = \bigcup_{p\in U} E_p, \quad \widetilde{\varphi} = (\varphi \times \mathrm{id}_{\mathbb{R}^k}) \circ \Phi_\alpha.$$

We claim that $\mathcal{A}$ is a smooth atlas covers M, since $M = \bigcup_{\alpha\in A} U_\alpha$. If $(\widetilde{U}, \widetilde{\varphi}), (\widetilde{W}, \widetilde{\psi}) \in \mathcal{A}$, then

$$\begin{aligned}
\widetilde{\varphi} \circ \widetilde{\psi}^{-1}(x, v) &= (\varphi \times \mathrm{id}_{\mathbb{R}^k}) \circ \Phi_\alpha \circ \Phi_\beta^{-1} \circ (\psi \times \mathrm{id}_{\mathbb{R}^k})^{-1}(x, v) \\
&= (\varphi \times \mathrm{id}_{\mathbb{R}^k}) \circ \Phi_\alpha \circ \Phi_\beta^{-1}(\psi^{-1}(x), v) \\
&= (\varphi \times \mathrm{id}_{\mathbb{R}^k})(\psi^{-1}(x), \tau_{\alpha\beta}(\psi^{-1}(x), v)) \\
&= (\varphi \circ \psi^{-1}(x), \tau_{\alpha\beta}(\psi^{-1}(x), v)).
\end{aligned}$$

is smooth, so $\mathcal{A}$ is a smooth atlas.
Vector Bundle: Check that Φ_α are smooth local trivializations. $\square$

Example 5 (Whitney Sums). Suppose $\pi_1 : E_1 \to M, \pi_2 : E_2 \to M$ are smooth vector bundles. Let $E_p = E_{1p} \oplus E_{2p}$ and $E = \bigsqcup_{p \in M} E_p$. Then the projection $\pi : E \to M$ is a smooth vector bundle (called the Whitney sum of E_1 and E_2).

Proof. Fix an open cover $M = \bigcup_\alpha U_\alpha$ such that there are smooth local trivializations $\Phi_{i\alpha} : \pi_i^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{k_i}$. Let $\tau_{i\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k_i, \mathbb{R})$ be the transition functions. Define $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{k_1+k_2}$ by

$$\Phi_\alpha((v_1, v_2)) = (\pi_1(v_1), (\pi_{\mathbb{R}^{k_1}}(\Phi_{i\alpha}(v_1)), \pi_{\mathbb{R}^{k_2}}(\Phi_{i\alpha}(v_2)))).$$

Note that $\pi_1(v_1) = \pi_2(v_2)$ since $(v_1, v_2) \in E_p = E_{1p} \oplus E_{2p}$. Define $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k_1 + k_2, \mathbb{R})$ by

$$\tau_{\alpha\beta}(p) = \begin{pmatrix} \tau_{1\alpha\beta}(p) & 0 \\ 0 & \tau_{2\alpha\beta}(p) \end{pmatrix}.$$

Check that this satisfies the lemma. □

Example 6 (Dual). Suppose $\pi : E \to M$ is a smooth vector bundle. Let $E_p^* = (E_p)^*$ be the dual of E_p. Let $E^* = \bigsqcup_{p \in M} E_p^*$, then the projection $\pi : E^* \to M$ is a smooth vector bundle (called the *dual* to E).

Proof. Check on HW. □

7.3 Sections of Vector Bundles

Let $\pi : E \to M$ be a vector bundle. A *section* of E is a continuous map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}_M$ (so that σ) is injective.

Example 7 . Sections of TM are vector fields on M.

Definition 7.3.1. If $f \in C^\infty(M)$ and σ is a section of E, then define a new section $f\sigma$ by

$$(f\sigma)(p) = f(p)\sigma(p).$$

7.4 Maps Between Bundles

Definition 7.4.1. Suppose $\pi_1 : E_1 \to M_1, \quad \pi_2 : E_2 \to M_2$ are two vector bundles. A continuous map $F : E_1 \to E_2$ is a *vector bundle homomorphism* if

1. there is a continuous map $f : M_1 \to M_2$ such that

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

2. For each p, $F|_{E_{1p}} : E_{1p} \to E_{2f(p)}$ is linear.

Moreover,

- if E, f are homeomorphisms, F is a *bundle isomorphism*.
- If everything is smooth, we add the word smooth to F.

Remark. If F is a bundle homomorphism, then F is a bundle isomorphism if and only if F is bijective and F^{-1} is a bundle homomorphism.

Example 8 . If $F : M \to N$ is smooth, then $dF : TM \to TN$ is a smooth bundle homomorphism.

7.5 Cotangent Bundles

7.5.1 Covector Fields

For the rest of the chapter, fix a smooth manifold M. Let V be a finite-dimensional vector space, a *covector* on V is defined to be a real-valued linear functional on V.

Definition 7.5.1. For each $p \in M$, we define the *cotangent space* at p to be the dual space to T_pM :

$$T_p^*M := (T_pM)^*.$$

Elements of T_p^*M are called *tangent covectors* at p, or just covectors at p.

Definition 7.5.2. Let $T^*M = \bigsqcup_{p\in M} T_p^*M$, then T^*M is called the *cotangent bundle* of TM.

There is a natural projection $\pi : T^*M \to M$ given by $\omega \in T_p^*M \mapsto p \in M$.

Fix a chart (U, φ) of M. If $p \in U$, let $dx^1|_p, \cdots, dx^n|_p \in T_p^*M$ be the dual basis to $\partial/\partial x^1 \mid_p, \cdots, \partial/\partial x^n \mid_p \in T_pM$ so that

$$dx^i|_p \left(\left.\frac{\partial}{\partial x^j}\right|_p \right) = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Next, let $\widetilde{U} = \pi^{-1}(U) \subset T^*M$ and define $\widetilde{\varphi} : \widetilde{U} \to \mathbb{R}^n \times \mathbb{R}^n$ by

$$\widetilde{\varphi}\left(\xi_i dx^i|_p\right) = (\varphi(p), \xi_1, \cdots, \xi_n).$$

Proposition 7.5.1. *$(\widetilde{U}, \widetilde{\varphi})$ is a smooth chart of T^*M.*

Proof. Check. □

A section $w : M \to T^*M$ is called a *covector field*. Given a chart, we can write $w = w_i dx^i$ on U where $w_1, \cdots, w_n : U \to \mathbb{R}$ are the component functions of w.

Proposition 7.5.2. *A covector field is smooth if and only if for every chart its component functions are smooth.*

Proof. □

We let $\mathfrak{X}^*(M)$ be the vector space of covector fields.

7.5.2 Differential of a Function

Given $f \in C^\infty(M)$, if we identify $T_x\mathbb{R} = \mathbb{R}$ for all $x \in \mathbb{R}$, then $df_p \in T_p^*M$.

Definition 7.5.3. Define a covector field df called the *differential* of f, by

$$df_p(v) = vf, \quad v \in T_pM.$$

equivalently,

$$df_p(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ \gamma)(t),$$

where $\gamma : I \to M$ is a smooth curve with $\gamma(0) = p, \gamma'(0) = v$.

Remark. On a chart (U, φ),

- $df = \frac{\partial f}{\partial x^i} dx^i$, so f is smooth.
- $dx^i \in \mathfrak{X}(U)$ is the differential of $x^i : U \to \mathbb{R}$, the ith component of φ.

Proposition 7.5.3. *The differential of a smooth function is a smooth covector field.*

Proof. By the linearity of v as a derivation, df_p is linear, so df_p is a covector at p. Let (x^i) be smooth coordinates on an open $U \subset M$, and let (λ^i) be the corresponding coordinate coframe on U. Write $df_p = A_i(p)\lambda^i|_p$ for some $A_i : U \to \mathbb{R}$, then

$$A_i(p) = df_p \left(\left.\frac{\partial}{\partial x^i}\right|_p \right) = \left.\frac{\partial}{\partial x^i}\right|_p f = \frac{\partial f}{\partial x^i}(p).$$

Then

$$df_p = \frac{\partial f}{\partial x^i}(p)\lambda^i|_p.$$

Apply this to coordinate functions, we obtain

$$dx^j|_p = \frac{\partial x^j}{\partial x^i}(p)\lambda^i|_p = \lambda^j|_p.$$

Hence the coordinate covector field λ^j is just the differential dx^j. Now we obtain the formula

$$df_p = \frac{\partial f}{\partial x^i}(p)dx^i|_p,$$

and

$$df = \frac{\partial f}{\partial x^i}dx^i.$$

□

Remark. The 1-dimensional case reduces to

$$df = \frac{df}{dx}dx.$$

Chapter 8

Tensors

8.1 Multilinear Algebra

Definition 8.1.1. Let $V_1, \cdots, V_k$ and W be vector spaces. A map $F : V_1 \times \cdots \times V_k \to W$ is called **multilinear** if

$$F(u_1, \cdots, au_i + bv_i, \cdots, u_k) = aF(u_1, \cdots, u_i, \cdots, u_k) + bF(u_1, \cdots, v_i, \cdots, u_k) \quad \text{for each } i.$$

Denote $\mathcal{L}(V_1, \cdots, V_k; W)$ for the set of all multilinear maps from $V_1 \times \cdots \times V_k$ to W.

Remark. We all know that $V_1 \times \cdots V_k$ is also a vector space, however, a multilinear map F on this vector space is different from a linear map on this vector space. Take $k = 2$, then

$$F(av_1, av_2) = aF(v_1, av_2) = a^2 F(v_1, v_2),$$

since the scalar multiplication on a product space is defined as $a(v_1, v_2) = (av_1, av_2)$.

Example 1 .

1. The dot product in $\mathbb{R}^n$ is a bilinear function of two vectors.

$$\langle v, w \rangle \in \mathbb{R}^n \times \mathbb{R}^n \mapsto v \cdot w \in \mathbb{R}$$

   is in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n; \mathbb{R})$.

2. The cross product

$$(v, w) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto v \times w \in \mathbb{R}^3$$

   is in $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^3; \mathbb{R}^3)$.

3. The determinant is a real-valued multilinear function of n vectors in $\mathbb{R}^n$.

Proof. Let $a \in \mathbb{R}$, then $\langle au + bv, w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$. Let $v_i \in \mathbb{R}^n$, then

$$\det[v_1 \cdots av_i + bv_j \cdots v_n] = a\det[v_1 \cdots v_n] + b\det[v_1 \cdots v_n].$$

$\square$

Definition 8.1.2. Given $F \in \mathcal{L}(V_1, \cdots, V_k; \mathbb{R})$ and $G \in \mathcal{L}(W_1, \cdots, W_l; \mathbb{R})$, define the *tensor (product)* of F and G by $F \otimes G \in \mathcal{L}(V_1, \cdots, V_k, W_1, \cdots, W_l, \mathbb{R})$ by

$$F \otimes G(v_1, \cdots, v_k, w_1, \cdots, w_l) = F(v_1, \cdots, v_k)G(w_1, \cdots, w_l).$$

Example 2 . Let $E_1, \cdots, E_n$ be the standard basis of $\mathbb{R}^n$ and $\varepsilon^1, \cdots, \varepsilon^n$ be the dual basis. Then

$$v \cdot w = \left(\sum_{i=1}^n \varepsilon^i \otimes \varepsilon^i\right)(v, w)$$

for all $v, w \in \mathbb{R}^n$. For each i,

$$\varepsilon^i \otimes \varepsilon^i(v_1, \cdots, v_n)\varepsilon^i(w_1, \cdots, w_n) = v_i w_i.$$

Example 3 . (Need to be justified) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $X_1, \cdots, X_n$ be integrable independent variables, that is, $\mathbb{E}X_j < \infty$ for all j, where $\mathbb{E}$ is the expectation. Note that $\mathbb{E}$ is a linear map on $L^1(\Omega)$, and we write $\mathbb{E} \in \mathcal{L}(L^1, \mathbb{R})$. Then $\mathbb{E}(X_1)\mathbb{E}(X_2) = \mathbb{E} \otimes \mathbb{E}(X_1, X_2) = \mathbb{E}(X_1X_2)$.

Proposition 8.1.1. *Let $V_1, \cdots, V_k$ be a vector space of dimension $n_1, \cdots, n_k$. For each $j = 1, \cdots, k$, let $E_1^{(j)}, \cdots E_{n_j}^{(j)}$ be a basis of V_j and $\varepsilon^1_{(j)}, \cdots, \varepsilon^{n_j}_{(j)}$ be the dual basis, then*

$$\mathcal{B} = \left\{\varepsilon^{i_1}_{(1)} \otimes \cdots \otimes \varepsilon^{i_k}_{(k)} : 1 \le i_j \le n_j \text{ for } j = 1, \cdots, k\right\}$$

is a basis for $\mathcal{L}(V_1, \cdots, V_k; \mathbb{R})$.

vector space	*basis*	*dual basis*
V_1	$E_1^{(1)}, E_2^{(1)}, \cdots, E_{n_1}^{(1)}$	$\varepsilon^1_{(1)}, \cdots, \varepsilon^{n_1}_{(1)}$
V_2	$E_1^{(2)}, E_2^{(2)}, \cdots, E_{n_2}^{(2)}$	$\varepsilon^1_{(2)}, \cdots, \varepsilon^{n_2}_{(2)}$
$\vdots$		
V_k	$E_1^{(k)}, E_2^{(k)}, \cdots, E_{n_k}^{(k)}$	$\varepsilon^1_{(k)}, \cdots, \varepsilon^{n_k}_{(k)}$

Proof. Fix a multi-linear function $F \in \mathcal{L}(V_1, \cdots, V_k; \mathbb{R})$. Define coefficients

$$F_{i_1, \cdots, i_k} = F\left(E^{(1)}_{i_1}, \cdots, E^{(k)}_{i_k}\right),$$

then (using Einstein summation)

$$\begin{aligned}
F(v_1, \cdots, v_k) &= F\left(v_1^{i_1} E_{i_1}^{(1)}, \cdots, v_k^{i_k} E_{i_k}^{(k)}\right) \\
&= v_1^{i_1} \cdots v_k^{i_k} F\left(E_{i_1}^{(1)}, \cdots, E_{i_k}^{(k)}\right) \\
&= v_1^{i_1} \cdots v_k^{i_k} F_{i_1, \cdots, i_k} \\
&= \left(F_{i_1, \cdots, i_k} \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k}\right)(v_1, \cdots, v_k).
\end{aligned}$$

If we do not use Einstein summation, the above sum can be written as

$$\begin{aligned}
&F\left(\sum_{i_1} v_1^{i_1} E_{i_1}^{(1)}, \sum_{i_2} v_2^{i_2} E_{i_2}^{(2)}, \cdots, \sum_{i_k} v_k^{i_k} E_{i_k}^{(k)}\right) \\
&= \sum_{i_1} v_1^{i_1} F\left(E_{i_1}^{(1)}, \sum_{i_2} v_2^{i_2} E_{i_2}^{(2)}, \cdots, \sum_{i_k} v_k^{i_k} E_{i_k}^{(k)}\right) \\
&= \sum_{i_1} v_1^{i_1} \sum_{i_2} v_2^{i_2} F\left(E_{i_1}^{(1)}, E_{i_2}^{(2)}, \cdots, \sum_{i_k} v_k^{i_k} E_{i_k}^{(k)}\right) = \cdots \\
&= \sum_{i_1} v_1^{i_1} \sum_{i_2} v_2^{i_2} \cdots \sum_{i_k} v_k^{i_k} F\left(E_{i_1}^{(1)}, \cdots, E_{i_k}^{(k)}\right) \\
&= \sum_{i_1, \cdots, i_k} v_1^{i_1} \cdots v_k^{i_k} F\left(E_{i_1}^{(1)}, \cdots, E_{i_k}^{(k)}\right) \\
&= \sum_{i_1, \cdots, i_k} v_1^{i_1} \cdots v_k^{i_k} F_{i_1, \cdots, i_k}.
\end{aligned}$$

Recall the definition of a dual basis, we have

$$v_1^{i_1} = \varepsilon_{(1)}^{i_1}\left(\sum_{j=1}^{n_1} v_1^j E_j^{(1)}\right) = \varepsilon_{(1)}^{i_1}(v_1).$$

Then

$$\begin{aligned}
\sum_{i_1, \cdots, i_k} v_1^{i_1} \cdots v_k^{i_k} F_{i_1, \cdots, i_k} &= \sum_{i_1, \cdots, i_k} \varepsilon_{(1)}^{i_1}(v_1) \cdots \varepsilon_{(k)}^{i_k}(v_k) F_{i_1, \cdots, i_k} \\
&= \sum_{i_1, \cdots, i_k} F_{i_1, \cdots, i_k}\left[\varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k}(v_1, \cdots, v_k)\right].
\end{aligned}$$

Now we show $\mathcal{B}$ is linearly independent. Suppose $\alpha_{i_1, \cdots, i_k} \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k} = 0$, then

$$0 = \left(\alpha_{i_1, \cdots, i_k} \varepsilon_{(1)}^{i_1} \otimes \cdots \otimes \varepsilon_{(k)}^{i_k}\right)\left(E_{j_1}^{(1)}, \cdots, E_{j_k}^{(k)}\right) = \alpha_{j_1, \cdots, j_k}$$

for all indices $j_1, \cdots, j_k$. $\square$

8.2 Abstract Tensor Products

8.2.1 Free Vector Spaces

Given a set S,

- A *formal linear combination* of elements in S is a function $f : S \to \mathbb{R}$, where
$$\#\{x \in S : f(x) \neq 0\} < \infty.$$
In this case we write $f = \sum_{i=1}^m c_i x_i$, where $\{x_1, \cdots, x_m\} = \{x \in S : f(x) \neq 0\}$, and $c_i = f(x_i)$.

- The *free vector space* of S denoted by $\mathcal{F}(S)$ is the vector space of formal linear combinations. We view S as a subset of $\mathcal{F}(S)$ by identifying $x \in S$ with $\delta_x \in \mathcal{F}(S)$, where $\delta_x(y) = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$, so $f = \sum_{i=1}^m c_i x_i = \sum_{i=1}^m c_i \delta_{x_i}$.

Rigorously, $\mathcal{F}(S)$ is the vector space of functions $f \in \mathbb{R}^S$, but in practice we think of S as a subset of $\mathcal{F}(S)$ (S is just a set). Typically we identify $x \in S$ with $\delta_x \in \mathbb{R}^S$.

There are many formal linear combinations. Is it possible to construct a sequence of formal linear combinations $\#\{x \in S : f_n(x) \neq 0\} = n$?

Example 4 . Let $S = \{1, 2\}$, then $\#\{x \in S : f_n(x) \neq 0\} \leq 2$, and thus all functions from S to $\mathbb{R}$ is a formal linear combination. Thus $\mathcal{F}(S) = \mathbb{R}^S$.

Example 5 . Let $S = \mathbb{R}$ and f be a formal linear combination, then $\operatorname{supp} f = \{x_1, \cdots, x_m\}$, so we can write $f = \sum_{i=1}^m c_i \delta_{x_i}$. In this case $\operatorname{Im} f = \{c_1, \cdots, c_m\}$. Hence
$$\mathcal{F}(\mathbb{R}) = \left\{ f \in \mathbb{R}^S : f = \sum_{i=1}^m c_i \delta_{x_i}, m \in \mathbb{N}, x_i \in \mathbb{R} \text{ distinct} \right\},$$
which is a vector subspace of $\mathbb{R}^S$ (think of simple functions in real analysis, but assume values on a finite set).

Example 6 . Let $S = \mathbb{R}^d$, then
$$\mathcal{F}(\mathbb{R}^d) = \left\{ f \in (\mathbb{R}^d)^S : f = \sum_{i=1}^m c_i \delta_{x_i}, m \in \mathbb{N}, x_i \in \mathbb{R}^d \text{ distinct} \right\},$$

Example 7 . Let V_1, V_2 be finite-dimensional vector spaces, then
$$\mathcal{F}(V_1 \times V_2) = \left\{ f \in (V_1 \times V_2)^S : f = \sum_{i=1}^m c_i \delta_{(v_1, v_2)}, m \in \mathbb{N}, v_i \in V_i \right\},$$

Proposition 8.2.1. *If W is a vector space, then every map $A : S \to W$ has a unique extension to a linear map $\overline{A} : \mathcal{F}(S) \to W$.*

Proof. Check that

$$\overline{A}\left(\sum_{i=1}^{m} c_i x_i\right) = \sum_{i=1}^{m} c_i A(x_i)$$

defines $\overline{A}$. Let $\sum_{i=1}^{m} c_i x_i, \sum_{i=1}^{m} d_i y_i \in \mathcal{F}(S)$, then

$$\begin{aligned}
\overline{A}\left(\sum_{i=1}^{m} c_i x_i + \sum_{i=1}^{m} d_i y_i\right) &= \overline{A}\left(\sum_{i=1}^{m} (c_i x_i + d_i y_i)\right) \\
&= \sum_{i=1}^{m} \overline{A}(c_i x_i + d_i y_i) \\
&= \sum_{i=1}^{m} c_i A(x_i) + d_i A(y_i) \\
&= \overline{A}\left(\sum_{i=1}^{m} c_i x_i\right) + \overline{A}\left(\sum_{i=1}^{m} d_i y_i\right).
\end{aligned}$$

Let $\lambda \in \mathbb{R}$, then

$$\overline{A}\left(\lambda \sum_{i=1}^{m} c_i x_i\right) = \overline{A}\left(\sum_{i=1}^{m} \lambda c_i x_i\right) = \sum_{i=1}^{m} \lambda c_i A(x_i) = \lambda \overline{A}\left(\sum_{i=1}^{m} c_i x_i\right).$$

□

8.2.2 Tensor Products

Given $V_1, \cdots, V_k$ vector spaces, Let $\mathcal{R} \subset \mathcal{F}(V_1 \times \cdots \times V_k)$ be the linear subspace spanned by elements of the form

$$(w_1, \cdots, w_{i-1}, a w_i, w_{i+1}, \cdots, w_k) - a(w_1, \cdots, w_k)$$

and

$$\begin{aligned}
&(w_1, \cdots, w_{i-1}, w_i + w_i', w_{i+1}, \cdots, w_k) \\
&- (w_1, \cdots, w_i, \cdots, w_k) - (w_1, \cdots, w_{i-1}, w_i', w_{i+1}, \cdots, w_k).
\end{aligned}$$

Then

- The *tensor product* of $V_1, \cdots, V_k$ is the vector space

$$V_1 \otimes \cdots \otimes V_k = \mathcal{F}(V_1 \times \cdots \times V_k)/\mathcal{R}.$$

- Let $\Pi : \mathcal{F}(V_1 \times \cdots \times V_k) \to V_1 \otimes \cdots \otimes V_k$ be the projection and define the *tensor product* of $w_1, \cdots, w_k$ as
$$w_1 \otimes \cdots \otimes w_k = \Pi(w_1, \cdots, w_k)$$
where $w_i \in V_i$. This is the equivalence class of $(v_1, \cdots, v_k)$ in $V_1 \otimes \cdots \otimes V_k$.

Remark. By definition,

$$\begin{aligned} a \cdot (w_1 \otimes \cdots \otimes w_k) &= (aw_1) \otimes w_2 \otimes \cdots \otimes w_k \\ &= w_1 \otimes (aw_2) \otimes w_3 \otimes \cdots \otimes w_k \quad = w_1 \otimes \cdots \otimes w_{k-1} \otimes (aw_k) \end{aligned}$$

and

$$\begin{aligned} &w_1 \otimes \cdots \otimes w_{i-1} \otimes (w_i + w_i') \otimes w_{i+1} \otimes \cdots \otimes w_k \\ &= w_1 \otimes \cdots \otimes w_k + w_1 \otimes \cdots \otimes w_{i-1} \otimes w_i' \otimes w_{i+1} \otimes \cdots \otimes w_k \end{aligned}$$

for all $w_j \in V_j, w_j' \in V_j, a \in \mathbb{R}, i = 1, \cdots, k$.

We review a property of quotient vector space.

Lemma 8.2.1. *Let W be a vector space and V be a subspace of W, let $T : W \to X$ be a linear map. Then T descends*[1] *to $\widetilde{T} : W/V \to X$ iff $V \subset \ker T$, where $\widetilde{T}(w + V) = Tw$.*

Proof. Suppose $V \subset \ker T$. If $u + V = w + V$, then $u - w \in V \subset \ker T$, so $T(u - w) = Tu - Tw = 0$. Hence $\widetilde{T}(u + V) = \widetilde{T}(w + V)$, so $\widetilde{T}$ is well-defined, and it is clearly linear.

Conversely, suppose T descends to a linear map $\widetilde{T} : W/V \to X$. Then $\widetilde{T}(u + V) = Tu = 0$ for all $u \in V$, hence $V \subset \ker T$. □

Proposition 8.2.2 (Characteristic Property)**.** *If $A \in \mathcal{L}(V_1, \cdots, V_k; W)$, then there is a unique linear map $\widetilde{A} : V_1 \otimes \cdots \otimes V_k \to W$ such that the following diagram commutes:*

$$\begin{tikzcd} V_1 \times \cdots \times V_k \arrow[r, "A"] \arrow[d, "\pi"] & W \\ V_1 \otimes \cdots \otimes V_k \arrow[ur, "\widetilde{A}"'] & \end{tikzcd}$$

where $\pi(v_1, \cdots, v_k) = v_1 \otimes \cdots \otimes v_k$.

Proof. First extend A to $\overline{A} : \mathcal{F}(V_1 \times \cdots \times V_k) \to W$, then

$$\overline{A}\left(\sum_{i=1}^{m} c_i x_i\right) = \sum_{i=1}^{m} c_i A(x_i).$$

Since A is multi-linear,

[1] Think of as "T induces a map $\widetilde{T}$".

- $\overline{A}((v_1, \cdots, av_i, \cdots, v_k) - a(v_1, \cdots, v_k)) = 0$,
- $\overline{A}((v_1, \cdots, v_i + v_i', \cdots, v_k) - (v_1, \cdots, v_i, \cdots, v_k) - (v_1, \cdots, v_i', \cdots, v_k)) = 0$,

so $\mathcal{R} \subset \ker \overline{A}$, hence by the above lemma $\overline{A}$ descends to a linear map $\widetilde{A} : \mathcal{F}(V_1 \times \cdots \times V_k)/\mathcal{R} \to W$ satisfying $\widetilde{A}\left(\sum_{i=1}^m c_i x_i + \mathcal{R}\right) = \overline{A}\left(\sum_{i=1}^m c_i x_i\right)$.

$$\begin{tikzcd}
\mathcal{F}(V_1 \times \cdots \times V_k) \arrow[r, "\overline{A}"] \arrow[d, "\Pi"] & W \\
\mathcal{F}(V_1 \times \cdots \times V_k)/\mathcal{R} \arrow[ur, "\widetilde{A}"'] &
\end{tikzcd}$$

Let $\Pi : \mathcal{F}(V_1 \times \cdots \times V_k) \to \mathcal{F}(V_1 \times \cdots \times V_k)/\mathcal{R}$ be the natural projection, then we can write
$$\widetilde{A} \circ \Pi = \overline{A}.$$
The subtle difference between π and Π is that
$$\pi : V_1 \times \cdots \times V_k \to V_1 \otimes \cdots \otimes V_k$$
and
$$\Pi : \mathcal{F}(V_1 \times \cdots \times V_k) \to V_1 \otimes \cdots \otimes V_k.$$
Consider the following diagram,

$$\begin{tikzcd}
V_1 \times \cdots \times V_k \arrow[r, "\iota"] \arrow[d, "\pi"] & \mathcal{F}(V_1 \times \cdots \times V_k) \arrow[dl, "\Pi"] \\
V_1 \otimes \cdots \otimes V_k &
\end{tikzcd}$$

we have $\pi = \Pi \circ \iota$. Then $\widetilde{A} \circ \pi = \widetilde{A} \circ \pi \circ \iota = \overline{A} \circ \iota = A$.

Uniqueness follows from the fact that $\pi(V_1 \times \cdots \times V_k)$ spans $V_1 \otimes \cdots \otimes V_k$. $\square$

Proposition 8.2.3. *There are unique isomorphisms*
$$V_1 \otimes (V_2 \otimes V_3) \simeq V_1 \otimes V_2 \otimes V_3 \simeq (V_1 \otimes V_2) \otimes V_3,$$
where $w_1 \otimes (w_2 \otimes w_3), w_1 \otimes w_2 \otimes w_3, (w_1 \otimes w_2) \otimes w_3$ are identified for all $w_i \in V_i$.

Proposition 8.2.4. *If $V_1, \cdots, V_k$ are finite dimensional vector spaces, then there is a canonical isomorphism*
$$V_1^* \otimes \cdots \otimes V_k^* \simeq \mathcal{L}(V_1, \cdots, V_k; \mathbb{R}).$$

Proof. Fix a basis
$$E_1^{(j)}, \cdots, E_{n_j}^{(j)}$$
of V_j, and let
$$\varepsilon_{(j)}^1, \cdots, \varepsilon_{(j)}^{n_j}$$

denote the dual basis. Define

$$\Psi : \mathcal{L}(V_1, \cdots, V_k; \mathbb{R}) \to V_1^* \otimes \cdots \otimes V_k^*$$

by

$$\Psi(F) = F\left(E_1^{(j)}, \cdots, E_n^{(j)}\right) \varepsilon^1_{(j)} \otimes \cdots \otimes \varepsilon^n_{(j)}.$$

Next, define

$$\Phi : V_1^* \times \cdots \times V_k^* \to \mathcal{L}(V_1, \cdots, V_k; \mathbb{R})$$

by

$$\Phi(w^1, \cdots, w^k)(v_1, \cdots, v_k) = w^1(v_1)w^2(v_2) \cdots w^k(v_k).$$

This is multi-linear, so there is a linear map $\widetilde{\Phi} : V_1^* \otimes \cdots \otimes V_k^* \to \mathcal{L}(V_1, \cdots, V_k; \mathbb{R})$ with

$$\widetilde{\Phi}(w^1 \otimes \cdots \otimes w^k) = \widetilde{\Phi} \circ \pi(w_1, \cdots, w_k) = \Phi(w^1, \cdots, w^k).$$

Check that

$$\widetilde{\Phi} \circ \Psi = \mathrm{id}_{\mathcal{L}(V_1, \cdots, V_k; \mathbb{R})}$$

and

$$\Psi \circ \widetilde{\Phi} = \mathrm{id}_{V_1^* \otimes \cdots \otimes V_k^*}.$$

$\square$

Corollary 8.2.1. *If $V_1, \cdots, V_k$ have finite dimension, then*

1. $V_1 \otimes \cdots \otimes V_k \simeq \mathcal{L}(V_1^*, \cdots, V_k^*; \mathbb{R})$.
2. *If $E_1^{(j)}, \cdots, E_k^{(j)}$ is a basis of V_j, then*

$$\mathcal{B} = \left\{E_{i_1}^{(1)} \otimes \cdots \otimes E_{i_k}^{(k)} : 1 \leq i_j \leq n_j \textit{ for } j = 1, \cdots, k\right\}$$

is a basis for $V_1 \otimes \cdots \otimes V_k$.

Proof. (1) We can identify $V = V^{**}$ by $v \in V \mapsto \psi_V \in V^{**}$, where $\psi_v(f) = f(v)$. Hence

$$V_1 \otimes \cdots \otimes V_k = (V_1^*)^* \otimes \cdots \otimes (V_k^*)^* \simeq \mathcal{L}(V_1^*, \cdots, V_k^*, \mathbb{R}).$$

(2) follows from **Proposition 12.4** of Lee, where we computed a basis of $\mathcal{L}(V_1, \cdots, V_k; \mathbb{R})$. $\square$

EXAMPLE 8 . Show that $M_n(\mathbb{R}) \simeq \mathbb{R}^n \otimes \mathbb{R}^n$.

Proof. $M_n(\mathbb{R}) \simeq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \simeq (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \simeq \mathbb{R}^n \otimes \mathbb{R}^n$. $\square$

8.2.3 Covariant and Contravariant Tensors

Let

$$T^k(V^*) = \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ terms}},$$

which is called the *space of covariant tensors* of rank k.

$$T^k(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ terms}}$$

is called the space of *contravariant* tensors of rank k.

$$T^{(k,l)}(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ terms}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ terms}}$$

is called the space of *mixed* tensors of type (k, l).

Corollary 8.2.2. *Suppose $E_1, \cdots, E_n$ is a basis of V and $\varepsilon^1, \cdots, \varepsilon^n$ is the dual basis. Then*

$$\begin{aligned}
&\{\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} : 1 \leq i_1, \cdots, i_k \leq n\}, \\
&\{E_{i_1} \otimes \cdots \otimes E_{i_k} : 1 \leq i_1, \cdots, i_k \leq n\}, \\
&\{E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l} : 1 \leq i_1, \cdots, i_k \leq n\}
\end{aligned}$$

are bases of $T^k(V^*), T^k(V), T^{(k,l)}(V)$.

8.3 Symmetric and Alternating Tensors

8.3.1 Symmetric Tensors

Let V be a finite-dimensional vector space. A convariant k-densor α on V is said to be *symmetric* if

$$\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = \alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k)$$

whenever $1 \leq i < j \leq k$.

Let S_k be the symmetric group on $\{1, \cdots, k\}$. Then S_k acts on $T^k(V^*) \simeq \mathcal{L}(V, \cdots, V; \mathbb{R})$ by

$$(\sigma \cdot \alpha)(v_1, \cdots, v_k) = \alpha(v_{\sigma^{-1}(1)}, \cdots, v_{\sigma^{-1}(k)}),$$

where $\sigma \in S_k, \alpha \in T^k(V^*), v_1, \cdots, v_k \in V$.

Remark. Lee uses the notation $\sigma \cdot \alpha = \sigma_\alpha$, this is a group action.

Exercise 1 . For a covariant k-tensor α, the following are equivalent:

1. α is symmetric.

2. For any $v_1, \cdots, v_k \in V$, $\alpha(v_1, \cdots, v_k)$ is unchanged when $v_1, \cdots, v_k$ are rearranged in any order.

3. The components $\alpha_{i_1 \cdots i_k}$ of α w.r.t. any basis are unchanged by any permutation of the indices.

Proof. (1) $\implies$ (2): Suppose α is symmetric. Since any permutation is a product of transpositions, we can write $\sigma = \tau_1 \cdots \tau_N$, where τ_n is a transposition: it acts on α by interchanging a pair of arguments, say,

$$\tau_n \cdot \alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = \alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k).$$

But by symmetry,

$$\tau_n \cdot \alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = (v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k).$$

Hence, $\sigma \cdot \alpha = \alpha$.
(2) $\implies$ (1): Since $\sigma \cdot \alpha = \alpha$ for any $\sigma \in \Sigma_k$, taking α to be transpositions shows that α is symmetric. $\square$

Proposition 8.3.1. *The action is by linear transformations*

$$\sigma \cdot (a\alpha + b\beta) = a(\sigma \cdot \alpha) + b(\sigma \cdot \beta)$$

for all $\sigma \in S_k, \alpha, \beta \in T^k(V^*), a, b \in \mathbb{R}$.

Definition 8.3.1. $\alpha \in T^k(V^*)$ is called *symmetric* if $\sigma \cdot \alpha = \alpha$ for all $\sigma \in S_k$. Let $\Sigma^k(V^*) \subset T^k(V^*)$ be the vector space of symmetric tensors. Let $Sym : T^k(V^*) \to \Sigma^k(V^*)$ be the map

$$Sym(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} \sigma \cdot \alpha.$$

Proposition 8.3.2. *If* $\alpha \in T^k(V^*)$, *then*

1. $Sym(\alpha) \in \Sigma^k(V^*)$.

2. $Sym(\alpha) = \alpha$ *if and only if* $\alpha \in \Sigma^k(V^*)$.

Proof. (1) If $\eta \in S_k$, then

$$\eta \cdot Sym(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} (\eta\sigma) \cdot \alpha = \frac{1}{k!} \sum_{\sigma' \in S_k} \sigma' \cdot \alpha = Sym(\alpha).$$

(2) See Lee. $\square$

8.3.2 Symmetric Products

If $\alpha \in \Sigma^k(V^*)$ and $\beta \in \Sigma^l(V^*)$, we define their *symmetric product* by

$$\alpha\beta = Sym(\alpha \otimes \beta).$$

More explicitly,

$$\alpha\beta(v_1, \cdots, v_k, v_{k+1}, \cdots, v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} \alpha\left(v_{\sigma(1)}, \cdots, v_{\sigma(k)}\right) \beta\left(v_{\sigma(k+1)}, \cdots, v_{\sigma(k+l)}\right).$$

8.3.3 Alternating Tensors

A tensor $\alpha \in T^k(V^*)$ is *alternating* if $\sigma \cdot \alpha = (-1)^{\text{sgn } \sigma}\alpha$ for all $\sigma \in S_k$. Equivalently,

$$\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = (-1)\alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k)$$

for all $v_1, \cdots, v_k \in V$ and $1 \leq i < j \leq k$. Sometimes alternating tensors are called *skew* or *anti-symmetric*.

EXAMPLE 9 . Let $\alpha \in \mathcal{L}(\mathbb{R}^n, \cdots, \mathbb{R}^n, \mathbb{R})$ be

$$\alpha(v_1, \cdots, v_k) = \det([v_1 \ \cdots \ v_k]),$$

then α is alternating.

8.4 Tensor and Tensor Fields on Manifolds

Given a smooth manifold M, let

$$T^kT^*M = \bigsqcup_{p \in M} T^k(T_p^*M)$$

be the bundle of covariant k-tensors on M, and

$$T^kTM = \bigsqcup_{p \in M} T^k(T_pM)$$

be the bundle of contravariant k-tensors on M, and

$$T^{(k,l)}TM = \bigsqcup_{p \in M} T^{(k,l)}(T_pM)$$

be the bundle of mixed tensors of type (k, l). These bundles are called *tensor bundles* of M and sections of these bundles are called *tensor fields*. Here is an analogy:

bundle	section
tangent bundle TM	vector field T_pM
tensor bundle $T^{(k,l)}TM$	tensor field $T^{(k,l)}(T_pM)$

Exercise 2 . These are smooth vector bundles over M.

Proof. Check using **Lemma 10.6** of Lee. □

Locally, fix a chart (U,φ) of M. If $p \in U$, then

$$\left\{\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l}|_p : 1 \leq i_1, \cdots, i_k, j_1, \cdots, j_l \leq n = \dim M\right\}$$

is a basis of $T^{(k,l)}T_pM$. Let $\widetilde{U} = \pi^{-1}(U) \subset T^{(k,l)}TM$, define

$$\begin{aligned}
&\widetilde{\varphi} : \widetilde{U} \to \mathbb{R}^n \times T^{(k,l)}(\mathbb{R}^n) \\
&\widetilde{\varphi}\left(\xi^{i_1\cdots i_k}_{j_1\cdots j_l} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l}|_p\right) \\
&= \left(\varphi(p), \xi^{i_1\cdots i_k}_{j_1\cdots j_l} E_{i_1} \otimes \cdots \otimes E_{i_k} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_l}\right),
\end{aligned}$$

where $E_1, \cdots, E_n, \varepsilon^1, \cdots, \varepsilon^n$ are standard basis and dual basis of $\mathbb{R}^n$.

Proposition 8.4.1. *If $L : T^{(k,l)}(\mathbb{R}^n) \to \mathbb{R}^{n^{k+l}}$ is a linear isomorphism, then $(\widetilde{U}, (\mathrm{id}_{\mathbb{R}^n} \times L) \circ \widetilde{\varphi})$ is a smooth chart.*

Proof. □

8.4.1 Basic Properties

Regularity.

Suppose $A : M \to T^{(k,l)}TM$ is a section and (U,φ) is a smooth chart. On U,

$$A = A^{i_1\cdots i_k}_{j_1\cdots j_l} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_l},$$

where $A^{i_1\cdots i_k}_{j_1\cdots j_l} : U \to \mathbb{R}$ are component functions.

Proposition 8.4.2. *Let $A : M \to T^{(k,l)}TM$ be a section. Then TFAE:*

1. *A is smooth.*
2. *For every chart, the component functions are smooth.*
3. *Whenever $X_1, \cdots, X_l \in \mathfrak{X}(M)$ and $Y_1, \cdots, Y_k \in \mathfrak{X}^*(M)$, the function $A(Y_1, \cdots, Y_k, X_1, \cdots, X_l$ defined by*

   $$A(Y_1, \cdots, Y_k, X_1, \cdots, X_l)(p) = A_p\left(Y_1|_p, \cdots, Y_k|_p, X_1|_p, \cdots, X_l|_p\right)$$

   is smooth.

8.4.2 Mixed Tensor Products

The *tensor product* of $\alpha \in T^{(k,l)}T_pM$ and $\beta \in T^{(u,v)}E_pM$ is the element $\alpha \otimes \beta \in T^{(k+u,l+v)}T_pM$ satisfying

$$(\alpha\otimes\beta)(Y_1, \cdots, Y_{k+u}, X_1, \cdots, X_{l+v} = \alpha(Y_1, \cdots, Y_k, X_1, \cdots, X_l)\beta(Y_{k+1}, \cdots, Y_{k+u}, X_{l+1}, \cdots, X_{l+v})$$

for all $Y_1, \cdots, Y_{k+u} \in T_p^*M$ and $X_1, \cdots, X_{l+v} \in T_pM$.

Definition 8.4.1 (tensor product of tensor fields)**.** The *tensor product* of two tensor fields A, B is the tensor field $A \otimes B$ satisfying $(A \otimes B)_p = A_p \otimes B_p$.

Example 10 . If $M = \mathbb{R}^n$, then

$$\left(\frac{\partial}{\partial x^1} \otimes dx^2\right) \otimes \frac{\partial}{\partial x^2} = \frac{\partial}{\partial x^1} \otimes \frac{\partial}{\partial x^2} \otimes dx^2$$

8.5 Pullbacks of Covariant Tensor Fields

Definition 8.5.1. Suppose $F : M \to N$ is smooth.

- Given $p \in M$ and $\alpha \in T^k(T^*_{F(p)}N)$, the *(pointwise) pullback* of α by F is the element $dF_p^*(\alpha) \in T^k(T_p^*M)$ satisfying

  $$dF_p^*(\alpha)(v_1, \cdots, v_k) = \alpha(dF_pv_1, \cdots, dF_pv_k)$$

  for all $v_1, \cdots, v_k \in T_pM$.

- Given a covariant k-tensor field A on N, the *pullback* of α by F is the covariant k-tensor field F^*A where

  $$(F^*A)_p = dF_p^*A_{F(p)}.$$

  This tensor acts on $(v_1, \cdots, v_k) \in T_pM \times \cdots T_pM$ by

  $$(F^*A)_p(v_1, \cdots, v_k) = A_{F(p)}\left(dF_p(v_1), \cdots, dF_p(v_k)\right).$$

Proposition 8.5.1. *Suppose $F : M \to N$ and $G : P \to M$ are smooth, A and B are covariant tensor fields on N, and $f : N \to \mathbb{R}$.*

1. $F^*(fB) = (f \circ F)F^*B$.
2. $F^*(A \otimes B) = F^*A \otimes F^*B$.
3. $F^*(A + B) = F^*A + F^*B$.
4. *If B is smooth, then F^*B is smooth.*

5. $(F \circ G)^* B = G^*(F^* B)$.

Proof. (1) Recall that $fB(p) = f(p)B_p$ (See Section 7.3). Then

$$\begin{aligned}
[F^*(fB)]_p (v_1, \cdots, v_k) &= (fB)_{F(p)} (dF_p(v_1), \cdots, dF_p(v_k)) \\
&= f(F(p))B_{F(p)} (dF_p(v_1), \cdots, dF_p(v_k)) \\
&= (f \circ F)(p)(F^*B)_p(v_1, \cdots, v_k) \\
&= [(f \circ F)F^*B]_p (v_1, \cdots, v_k).
\end{aligned}$$

(2) Let $(v_1, \cdots, v_k), (w_1, \cdots, w_k) \in \underbrace{T_pM \times \cdots \times T_pM}_{k}$, then

$$\begin{aligned}
&(F^*A \otimes F^*B)_p(v_1, \cdots, v_k, w_1, \cdots, w_k) \\
&= (F^*A)_p(v_1, \cdots, v_k)(F^*B)_p(w_1, \cdots, w_k) \\
&= A_{F(p)} (dF_p(v_1), \cdots, dF_p(v_k)) \, B_{F(p)} (dF_p(w_1), \cdots, dF_p(w_k)) \\
&= A_{F(p)} \otimes B_{F(p)} (dF_p(v_1), \cdots, dF_p(v_k), dF_p(w_1), \cdots, dF_p(w_k)) \\
&= (A \otimes B)_{F(p)} (dF_p(v_1), \cdots, dF_p(v_k), dF_p(w_1), \cdots, dF_p(w_k)) \\
&= F^*(A \otimes B)_p(v_1, \cdots, v_k, w_1, \cdots, w_k).
\end{aligned}$$

(5) Let $p \in P$ and $(v_1, \cdots, v_k) \in \underbrace{T_pP \times \cdots \times T_pP}_{k}$, then

$$\begin{aligned}
((F \circ G)^*B)_p (v_1, \cdots, v_k) &= B_{F\circ G(p)} (d(F \circ G)_p(v_1), \cdots, d(F \circ G)_p(v_k)) \\
&= B_{F\circ G(p)} \left(dF_{G(p)} \circ dG_p(v_1), \cdots, dF_{G(p)} \circ dG_p(v_k)\right)
\end{aligned}$$

□

Example 11 . Suppose $F : M \to N$ is smooth and $f : N \to \mathbb{R}$ is smooth. Then $df \in \mathfrak{X}^*(M)$ is a covariant 1-tensor field. For $X \in T_pM$,

$$(F^*df)_p(X) = df_{F(p)}(dF_pX) = d(f \circ F)_p(X),$$

so

$$F^*df = d(f \circ F).$$

Example 12 . Locally a convariant k-tensor field A on N can be written as

$$A = A_{j_1 \cdots j_l} dx^{j_1} \otimes \cdots \otimes dx^{j_l},$$

so locally

$$\begin{aligned}
F^*A &= (A_{1 \cdots j_l} \circ F)F^*dx^{j_1} \otimes \cdots \otimes dx^{j_l} \\
&= (A_{1 \cdots j_l} \circ F)d(x^{j_1} \circ F) \otimes \cdots \otimes d(x^{j_l} \circ F) \\
&= (A_{1 \cdots j_l} \circ F)dF^{j_1} \otimes \cdots \otimes dF^{j_l},
\end{aligned}$$

where F^i is the ith component of F.

8.6 Lie Derivative of Tensor Fields

Definition 8.6.1. Suppose $V \in \mathfrak{X}(M)$ and $\theta : \mathcal{D} \to M$ is the flow of V. Given a smooth covariant k-tensor field A, the *Lie derivative* of A with respect to V is the smooth covariant k-tensor field where

$$(\mathcal{L}_V A)_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\theta_t^* A)_p = \lim_{t \to 0} \frac{d(\theta_t)_p^* A_{\theta_t(p)} - A_p}{t}.$$

Note that $(\theta_t^* A)_p \in T^k(T_p^* M)$ and $T^k(T_p^* M)$ is just a vector space, so we can use the limit definition of the derivative.

Proposition 8.6.1. *Suppose $V \in \mathfrak{X}(M), f \in C^\infty(M)$ and A, B are smooth covariant tensor fields on M. Then:*

1. $\mathcal{L}_V(f) = Vf$.
2. $\mathcal{L}_V(fA) = \mathcal{L}_V(f)A + f\mathcal{L}_V(A)$.
3. $\mathcal{L}_V(A \otimes B) = \mathcal{L}_V(A) \otimes B + A \otimes \mathcal{L}_V(B)$.
4. *If $X_1, \cdots, X_k \in \mathfrak{X}(M)$ and A is a covariant k-tensor field, then*

$$\begin{aligned}&\mathcal{L}_V(A(X_1, \cdots, X_k)) \\ &= (\mathcal{L}_V A)(X_1, \cdots, X_k) + A(\mathcal{L}_V(X_1), X_2, \cdots, X_k) + \cdots + A(X_1, \cdots, X_{k-1}, \mathcal{L}_V(X_k)).\end{aligned}$$

Note that in (1), we regard elements of $C^\infty(M)$ as smooth covariant 0-tensors. If we do this, $f \otimes A = fA$, then (2) is a consequence of (3)

Corollary 8.6.1. *If $f \in C^\infty(M)$ and $V \in \mathfrak{X}(M)$, then*

$$\mathcal{L}_V(df) = d(V(f)) = d(\mathcal{L}_V(f)).$$

Proof. Use (4): If $X \in \mathfrak{X}(M)$, then

$$\begin{aligned}\mathcal{L}_V(df)(X) &= \mathcal{L}_V(df(X)) - df(\mathcal{L}_V(X)) \\ &= V(X(f)) - df([V, X]) \\ &= V(Xf) - [V, X](f) \\ &= VXf - VXf + XVf \\ &= X(Vf) = d(V(f))X.\end{aligned}$$

Since $X \in \mathfrak{X}(M)$ was arbitrary, $\mathcal{L}_V(df) = d(V(f))$. ∎

EXAMPLE 13 . Suppose A is a smooth covariant k-tensor field and $V \in \mathfrak{X}(M)$. Fix a chart (U, φ), then on U we can write

$$V = V^i \frac{\partial}{\partial x^i},$$

$$A = A_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}.$$

Note

$$\mathcal{L}_V(dx^i) = d(V(x^i)) = dV^i = \frac{\partial V^i}{\partial x^j} dx^j.$$

By proposition (3)

$$\begin{aligned}
\mathcal{L}_V(A) &= V(A_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \\
&+ A_{i_1 \cdots i_k} \mathcal{L}_V(dx^{i_1}) \otimes \cdots \otimes dx^{i_k} \\
&+ \cdots + A_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes \mathcal{L}_V dx^{i_k} \\
&= V(A_{i_1 \cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \\
&+ A_{i_1 \cdots i_k} \frac{\partial V^{i_1}}{\partial x^j} dx^j \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k} \\
&+ \cdots + A_{i_1 \cdots i_k} \frac{\partial V^{i_k}}{\partial x^j} dx^{i_1} \otimes \cdots \otimes dx^{i_{k-1}} \otimes dx^j.
\end{aligned}$$

8.7 Recapitulation

tensor product of	
multi-linear functions	
vectors	
tensor fields	

Chapter 9

Differential Forms and Integration

9.1 Differential Forms

9.1.1 Alternating Tensors: Review

Recall that $\alpha \in T^k(V^*)$ is *alternating* if $\sigma \cdot \alpha = (\text{sgn } \sigma)\alpha$ for all $\sigma \in S_k$. Equivalently,

$$\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = (-1)\alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k)$$

for all $v_1, \cdots, v_k \in V$ and $1 \leq i < j \leq k$. Let $\Lambda^k(V^*) \subset T^k(V^*)$ be the vector space of alternating tensors.

Lemma 9.1.1. *If $\alpha \in T^k(V^*)$, then TFAE*

1. *$\alpha \in \Lambda^k(V^*)$.*
2. *$\alpha(v_1, \cdots, v_k) = 0$ whenever $v_1, \cdots, v_k$ is linearly dependent.*
3. *$\alpha(v_1, \cdots, v_k) = 0$ whenever $v_i = v_j$ for some $i \neq j$.*

Proof. Suppose $v_1, \cdots, v_k$ is linearly dependent. Without loss of generality, let $v_1 = a_2v_2 + \cdots + a_kv_k$ with $a_2, \cdots, a_k$ not all zero, then

$$\begin{aligned}\alpha(v_1, \cdots, v_k) &= \alpha(a_2v_2 + \cdots + a_kv_k, v_2, \cdots, v_k) & (9.1)\\ &= a_2\alpha(v_2, v_2, v_3, \cdots, v_k) + \cdots + a_k\alpha(v_k, v_2, v_3, \cdots, v_k). & (9.2)\end{aligned}$$

(2) $\implies$ (3): If $v_i = v_j$ for some $i \neq j$, the $v_1, \cdots, v_k$ is linearly dependent, hence by (2), $\alpha(v_1, \cdots, v_k) = 0$.
(1) $\implies$ (3): If $v_i = v_j$ for some $i \neq j$, then

$$\alpha(v_1, \cdots, v_i, \cdots, v_i, \cdots, v_k) = (-1)\alpha(v_1, \cdots, v_i, \cdots, v_i, \cdots, v_k)$$

implies $\alpha(v_1, \cdots, v_i, \cdots, v_i, \cdots, v_k) = 0$.
(3) $\implies$ (1) and (2): By the observation (9.1) made in the beginning of the proof we get (2). Next,

$$\begin{aligned}
0 &= \alpha(v_1, \cdots, v_i + v_j, \cdots, v_i + v_j, \cdots, v_k) \\
&= \alpha(v_1, \cdots, v_i, \cdots, v_i + v_j, \cdots, v_k) + \alpha(v_1, \cdots, v_j, \cdots, v_i + v_j, \cdots, v_k) \\
&= \alpha(v_1, \cdots, v_i, \cdots, v_i, \cdots, v_k) + \alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) \\
&\quad + \alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k) + \alpha(v_1, \cdots, v_j, \cdots, v_j, \cdots, v_k) \\
&= \alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) + \alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k),
\end{aligned}$$

hence

$$\alpha(v_1, \cdots, v_i, \cdots, v_j, \cdots, v_k) = (-1)\alpha(v_1, \cdots, v_j, \cdots, v_i, \cdots, v_k).$$

□

Definition 9.1.1. Let $\mathrm{Alt} : T^k(V^*) \to \Lambda^k(V^*)$ be the map given by

$$\mathrm{Alt}(\alpha) = \frac{1}{k!} \sum_{\sigma \in S_k} (\mathrm{sgn}\ \sigma)\sigma \cdot \alpha.$$

This means

$$\mathrm{Alt}(\alpha)(v_1, \cdots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\mathrm{sgn}\ \sigma)\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}).$$

We present a technical result which will be used to show $\mathrm{Alt}(\alpha)$ is an alternating tensor.

Proposition 9.1.1. *Let $\sigma, \tau \in S_k$ and f be a k-linear function on V, then*

$$\tau(\sigma f) = (\tau\sigma) f.$$

Proof.

$$\begin{aligned}
\tau(\sigma f)(v_1, \cdots, v_k) &= (\sigma f)(v_{\tau(1)}, \cdots, v_{\tau(k)}) \quad \text{let } w_\square = v_{\tau(\square)} \\
&= (\sigma f)(w_1, \cdots, w_k) \\
&= f(w_{\sigma(1)}, \cdots, w_{\sigma(k)}) \\
&= f(v_{\tau(\sigma(1))}, \cdots, v_{\tau(\sigma(k))}) \\
&= (\tau\sigma) f(v_1, \cdots, v_k).
\end{aligned}$$

□

Proposition 9.1.2. *If $\alpha \in T^k(V^*)$, then*

1. $\mathrm{Alt}(\alpha) \in \Lambda^k(V^*)$.

2. $\mathrm{Alt}(\alpha) = \alpha \iff \alpha \in \Lambda^k(V^*)$

Proof. Let $\tau \in S_k$. We will use the identity sgn $\tau\sigma = (\text{sgn } \tau)(\text{sgn } \sigma)$.

$$\begin{aligned}
\tau\left(\mathrm{Alt}(\alpha)\right) &= \frac{1}{k!}\sum_{\sigma \in S_k} (\text{sgn } \sigma)\tau(\sigma\alpha) \\
&= \frac{1}{k!}\sum_{\sigma \in S_k} (\text{sgn } \alpha)(\tau\sigma)\alpha \\
&= \frac{1}{k!}(\text{sgn } \tau)\sum_{\sigma \in S_k} (\text{sgn } \tau\sigma)(\tau\sigma)\alpha \\
&= (\text{sgn } \tau)\mathrm{Alt}(\alpha),
\end{aligned}$$

since $\tau\sigma$ runs through all of S_k. For the second part, $\alpha = \mathrm{Alt}(\alpha)$ is obviously alternating. Now suppose α is alternating, then

$$(\text{sgn } \sigma)\alpha(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) = \alpha(v_1, \cdots, v_k),$$

hence

$$\mathrm{Alt}(\alpha)(v_1, \cdots, v_k) = \frac{1}{k!}k!\alpha(v_1, \cdots, v_k) = \alpha(v_1, \cdots, v_k).$$

□

We will use terms "k-linear functions", "multilinear functions", "covariant k-tensors" in a mixed way, thanks to the isomorphism

$$V^* \otimes \cdots \otimes V^* \simeq \mathcal{L}(V_1, \cdots, V_k).$$

In many cases it suffices to consider multilinear functions to express the idea.

Exercise 1 . Let f be an alternating 3-linear function on a vector space V, compute $\mathrm{Alt}(f)(v_1, v_2, v_3)$, where $v_i \in V$.

Proof.

$$\begin{aligned}
\mathrm{Alt}(f)(v_1, v_2, v_3) &= \frac{1}{3!}\sum_{\sigma \in S_3} (\text{sgn } \sigma)f(v_{\sigma(1)}, v_{\sigma(2)}, v_{\sigma(3)}) \\
&= \frac{1}{3!}(f(v_1, v_2, v_3) - f(v_1, v_3, v_2) - f(v_2, v_1, v_3) + f(v_2, v_3, v_1) \\
&\quad - f(v_3, v_2, v_1) + f(v_3, v_1, v_2)).
\end{aligned}$$

□

EXAMPLE 1 . Let $E_1, \cdots, E_n$ and $\varepsilon^1, \cdots, \varepsilon^n$ be the standard basis and dual basis of $\mathbb{R}^n$. Then

$$\begin{aligned} Alt(\varepsilon^1 \otimes \cdots \otimes \varepsilon^n)(v_1, \cdots, v_n) &= \frac{1}{n!} \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) \varepsilon^1 \otimes \cdots \otimes \varepsilon^n (v_{\sigma(1)}, \cdots, v_{\sigma(n)}) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) v^1_{\sigma(1)}, \cdots, v^n_{\sigma(n)} \\ &= \frac{1}{n!} \det[v_1 \cdots v_n]. \end{aligned}$$

9.1.2 Elementary Alternating Tensors

$\boxed{\text{GOAL}}$ Find a nice basis of $\Lambda^k(V^*)$.
Fix V a vector space and a basis $E_1, \cdots, E_n$, and let $\varepsilon^1, \cdots, \varepsilon^n$ be the dual basis.

Definition 9.1.2. Given $I = (i_1, \cdots, i_k) \in \{1, \cdots, n\}^k$ (called the *multi-index* of length k), let $\varepsilon^I \in \Lambda^k(V^*)$ be the element where

$$\varepsilon^I(v_1, \cdots, v_k) = \det \begin{pmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k) \\ \vdots & & \vdots \\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k) \end{pmatrix} = \det \begin{pmatrix} {v_1}^{i_1} & \cdots & {v_k}^{i_1} \\ \vdots & & \vdots \\ {v_1}^{i_k} & \cdots & {v_k}^{i_k} \end{pmatrix}.$$

Note that $\varepsilon^I \in \Lambda^k(V^*)$ because of the column swapping property of the determinant. We call ε^I an *elementary alternating tensor*.

The multi-index can also be permuted: for $\sigma \in S_k$ let

$$I_\sigma = (i_{\sigma(1)}, \cdots, i_{\sigma(k)})$$

and for $J = (j_1, \cdots, j_k)$ let

$$\delta^I_J = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_k} \\ \vdots & & \vdots \\ \delta^{i_k}_{j_1} & \cdots & \delta^{i_k}_{j_k} \end{pmatrix}.$$

EXAMPLE 2 . Let $v, w, x \in \mathbb{R}^3$ and let e^1, e^2, e^3 be the standard dual basis of $\mathbb{R}^3$. Then

$$e^{13}(v, w) = \det \begin{pmatrix} v^1 & w^1 \\ v^3 & w^3 \end{pmatrix} = v^1 w^3 - w^1 v^3.$$

$$e^{123}(v, w, x) = \det \begin{pmatrix} v^1 & w^1 & x^1 \\ v^2 & w^2 & x^2 \\ v^3 & w^3 & x^3 \end{pmatrix} = \det(v, w, x).$$

Lemma 9.1.2. *Let (E_i) be a basis of V and (ε^i) be the dual basis, and let ε^I be as defined above*
(a) If I has repeated entries, then $\varepsilon^I = 0$.
(b) If $J = I_\sigma$ for some $\sigma \in S_k$, then $\varepsilon^I = (\operatorname{sgn}\sigma)\varepsilon^J$.
(c) $\varepsilon^I(E_{j_1}, \cdots, E_{j_k}) = \delta^I_J$.

Proof. (a) In this case, the matrix has a repeated row, hence $\varepsilon^I = 0$.
(b) In this case, the matrices are equal after n row swaps where $(-1)^n = \operatorname{sgn}\sigma$. Hence $\varepsilon^I = (\operatorname{sgn}\sigma)\varepsilon^J$.
(c) By definition $\varepsilon^i(E_j) = \delta^i_j$. □

EXERCISE 2 . Show that

$$\delta^I_J = \begin{cases} 0, & I \text{ or } J \text{ has a repeated entry, or } J \text{ is not a permutation of } I \\ \operatorname{sgn}\sigma, & I \text{ and } J \text{ has no repeated entries and } J = I_\sigma \end{cases}$$

Proof. If I or J have a repeated index, then there is a repeated column or row. Hence $\delta^I_J = 0$. Otherwise:

- If J is not a permutation of I, then there is a zero column (there is some $i \in I - J$). Hence $\delta^I_J = 0$.
- If $J = I_\sigma$ for some σ, then

$$\delta^I_J = \varepsilon^I(E_{j_1}, \cdots, E_{j_k} = (\operatorname{sgn}\sigma)\varepsilon^J(E_{j_1}, \cdots, E_{j_k}) = (\operatorname{sgn}\sigma)\det(\mathrm{id}_k) = \operatorname{sgn}\sigma.$$

□

$\boxed{\text{NOTATION}}$ A multi-index $I = (i_1, \cdots, i_k)$ is said to be *increasing* if $i_1 < \cdots < i_k$. We use a primed summation sign to denote a sum over only increasing multi-indices:

$$\sum_I{}' \alpha_I \varepsilon^I = \sum_{1 \le i_1 < \cdots < i_k \le n} \alpha_I \varepsilon^I.$$

Proposition 9.1.3. *The collection of k-covectors*

$$\mathcal{E} = \{\varepsilon^I : I = (i_1, \cdots, i_k), 1 \le i_1 < i_2 < \cdots < i_k \le n\}$$

is a basis for $\Lambda^k(V^*)$. Hence, $\dim \Lambda^k(V^*) = \binom{n}{k}$.

Proof. If $k > n$, then any collection of k vectors in V are linearly dependent. Hence

$$\Lambda^k(V^*) = \{0\},$$

so $B = \varnothing$ is a basis. Suppose $k \leq n$, fix $\alpha \in \Lambda^k(V^*)$. For each multi-index I let $\alpha_I = \alpha(E_{i_1}, \cdots, E_{i_k})$, then for any $J = (j_1, \cdots, j_k)$ we have

$$\begin{aligned}(\sum_I{}' \alpha_I \varepsilon^I)(E_{j_1}, \cdots, E_{j_k}) &= \sum_I{}' \alpha_I \delta^I_J = \alpha_J \\ &= \alpha(E_{j_1}, \cdots, E_{j_k}),\end{aligned}$$

so $\sum_I' \alpha_I \varepsilon^I = \alpha$. Suppose $\sum_I' \alpha_I \varepsilon^I = 0$. Fix J an increasing multi-index, then

$$0 = (\sum_I{}' \alpha_I \varepsilon^I)(E_{j_1}, \cdots, E_{j_k}) = \alpha_J.$$

□

Example 3 . $\Lambda^n(V^*)$ is 1-dimensional (because $\binom{n}{n} = 1$) and spanned by $\varepsilon^{(1,\cdots,n)}$.

Proposition 9.1.4. *Suppose V is an n-dimensional vector space and $\omega \in \Lambda^n(V^*)$. If $T : V \to V$ is a linear map and $v_1, \cdots, v_n \in V$, then*

$$\omega(Tv_1, \cdots, Tv_n) = (\det T)\omega(v_1, \cdots, v_n).$$

Proof. It suffices to consider $\omega = \varepsilon^{(1,\cdots,n)}$. Since an element of $\Lambda^n(V^*)$ is determined by its value on $E_1, \cdots, E_n$, it suffices to show that

$$\varepsilon^{(1,\cdots,n)}(TE_1, \cdots, TE_n) = \det(T)\varepsilon^{(1,\cdots,n)}(E_1, \cdots, E_n).$$

Note

$$\det(T) = \varepsilon^{(1,\cdots,n)}(E_1, \cdots, E_n) = \det(T)\det(\mathrm{id}_n) = \det(T).$$

Let (T_i^J) be the matrix representative of T relative to $E_1, \cdots, E_n$. That is,

$$Tv = v^i T_i^j E_j$$

when $v = v^i E_i$. Then

$$\varepsilon^j(TE_i) = \varepsilon^j(T_i^j E_j) = T_i^j,$$

so

$$w(TE_1, \cdots, TE_n) = \det\big(\varepsilon^j(TE_i)\big) = \det\Big(T_i^j\Big) = \det(T).$$

□

9.1.3 Wedge Product

The *wedge product* of $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$ is defined by

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!}\mathrm{Alt}(\omega \otimes \eta) \in \Lambda^{k+l}(V^*).$$

In the following lemma we will see where the coefficient comes from.

Lemma 9.1.3. *If $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_l)$, then*

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ},$$

where $IJ = (i_1, \cdots, i_k, j_1, \cdots, j_l)$.

Proof. Long calculation. □

Proposition 9.1.5. *Let $\omega, \omega', \eta, \eta', \xi$ be alternating tensors.*
(a) The map

$$(\omega, \eta) \in \Lambda^k(V^*) \times \Lambda^l(V^*) \to \omega \wedge \eta \in \Lambda^{k+l}(V^*)$$

is bilinear:

$$(a\omega + a'\omega') \wedge (b\eta + b'\eta') = (ab)\omega \wedge \eta + (ab')\omega \wedge \eta' + (a'b)\omega' \wedge \eta + (a'b')\omega' \wedge \eta'.$$

(b) $\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi$.
(c) For all $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, then $\omega \wedge \eta = (-1)^{kl}\eta \wedge \omega$.
(d) If $I = (i_1, \cdots, i_k)$, then

$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} = \varepsilon^I.$$

(e) If $v_1, \cdots, v_k \in V$ and $\omega^1, \cdots, \omega^k \in V^*$, then

$$(\omega^1 \wedge \cdots \wedge \omega^k)(v_1, \cdots, v_k) = \det\big(\omega^j(v_i)\big).$$

Proof. (a) By definition $(\omega, \eta) \to \omega \otimes \eta$ is bilinear and $\xi \to Alt(\xi)$ is linear. So there composition is bilinear.
(b) By multilinearity, we can assume that $\omega = \varepsilon^I, \eta = \varepsilon^J$ and $\xi = \varepsilon^K$. Then

$$\varepsilon^I \wedge (\varepsilon^J \wedge \varepsilon^K) = \varepsilon^I \wedge \varepsilon^{JK} = \varepsilon^{IJK} = \varepsilon^{IJ} \wedge \varepsilon^K = (\varepsilon^I \wedge \varepsilon^J) \wedge \varepsilon^K.$$

(c) By multi-linearity, we can assume $\omega = \varepsilon^I$ and $\eta = \varepsilon^J$. Let τ be the permutation that maps IJ to JI. Note this can be done in kl swaps, so $\mathrm{sgn}\ \tau = (-1)^{kl}$. Then

$$\omega \wedge \eta = \varepsilon^{IJ} = (\mathrm{sgn}\ \tau)\varepsilon^{JI} = (-1)^{kl}\eta \wedge \omega.$$

(d) follows from $\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}$.
(e) By multi-linearity we can assume $\omega^j = \varepsilon^{a_j}$ and $v_i = E_{b_i}$. Then

$$\varepsilon^{a_1} \wedge \cdots \wedge \varepsilon^{a_k}(E_{b_1}, \cdots, E_{b_k}) = \det(\varepsilon^{a_j} E_{b_i})$$

by 14.7(c) of Lee. □

By part (d) we can use the notations ε^I and $\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}$ interchangeably.

Corollary 9.1.1. *If $v_1, \cdots, v_k \in V$ and $\omega^1, \cdots, \omega^k \in V^*$, then*

$$(\omega^1 \wedge \cdots \wedge \omega^k)(v_1, \cdots, v_k) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(v_1)(\omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \omega^k)(v_2, \cdots, v_k)$$

where the hat means that ω^i is omitted.

Proof.

$$\begin{aligned}(w^1 \wedge \cdots \wedge w^k)(v_1, \cdots, v_k) &= \det\big(w^j(v_i)\big) \\ &= \sum_{i=1}^{k} (-1)^{i-1} w^i(v_1) \det\big(V_1^i\big)\end{aligned}$$

where V_1^i is the submatrix of $(w^j(v_i))$ obtained by deleting the ith column and 1st row. Then

$$\det\big(V_1^i\big) = (w_1 \wedge \cdots \wedge \widehat{w}^i \wedge \cdots \wedge w^k)(v_2, \cdots, v_k).$$

□

EXERCISE 3 . The wedge product is the unique associative bilinear and anticommutative map
$$\Lambda^k(V^*) \times \Lambda^l(V^*) \to \Lambda^{k+l}(V^*)$$
satisfying
$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} = \varepsilon^I$$
for any multi-index $I = (i_1, \cdots, i_k)$.

9.1.4 Interior Multiplication

Given $v \in V$, define $i_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$ by

$$(i_v\omega)(v_1, \cdots, v_{k-1}) = \omega(v, v_1, \cdots, v_{k-1}).$$

Another common notation is
$$v \lrcorner\, \omega = i_v\omega.$$

Lemma 9.1.4. *If $v \in V$, then*

(a) $i_v \circ i_v = 0$.

(b) If $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, then

$$i_v(\omega \wedge \eta) = (i_v\omega) \wedge \eta + (-1)^k \omega \wedge (i_v\eta).$$

Proof. (a) $i_v i_v \omega(v_1, \cdots, v_{k-2}) = \omega(v, v, v_1, \cdots, v_{k-2}) = 0$.
(b) By multi-linearity, we can assume

$$\omega = \omega^1 \wedge \cdots \wedge \omega^k$$

and

$$\eta = \omega^{k+1} \wedge \cdots \wedge \omega^{k+l}$$

for some $\omega^1, \cdots, \omega^{k+l} \in V^*$. By the corollary,

$$i_v(\omega \wedge \eta) = \sum_{i=1}^{k+l} (-1)^{i-1} \omega^i(v) \omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \omega^{k+l}.$$

Likewise,

$$i_v(\omega) = \sum_{i=1}^{l} (-1)^{i-1} \omega^i(v) \omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \omega^k$$

and

$$i_v(\eta) = \sum_{i=l+1}^{k+l} (-1)^{i-k-1} \omega^i(v) \omega^{k+1} \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \omega^{k+l},$$

so

$$i_v(\omega \wedge \eta) = i_v(\omega) \wedge \eta + (-1)^k \omega \wedge i_v(\omega).$$

□

9.2 Differential Forms on Manifolds

9.2.1 Differential of a Function

In calculus, given $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$, the gradient of f at a point $x \in \mathbb{R}^n$ is given by

$$\nabla f(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x).$$

For V a finite-dimensional vector space, recall that a *covector* on V is a real-valued linear functional on V. For each $p \in M$, the *cotangent space* at p is the dual space of T_pM:

$$T_p^*M := (T_pM)^*.$$

Elements of T_p^*M are called *tangent covectors/covectors* at p. Taking union over all $p \in M$ gives the *cotangent bundle* $T^*M = \bigsqcup_{p\in M} T_p^*M$. A *covector field* is just a section of T^*M (so that we can specify a component of the union).

Definition 9.2.1 (differential as a covector)**.** Let $f \in C^\infty(M)$, we define a covector field *df* called the *differential* of f by

$$df_p(v) = vf, \quad v \in T_pM.$$

Here df_p is a covector (a linear functional on T_pM).

The coordinate covector field λ^j is precisely the differential dx^j. Now we can write

$$df_p = \frac{\partial f}{\partial x^i}(p)dx^i|_p.$$

In 1-dimensional case, this reduces to the familiar expression

$$df = \frac{df}{dx}dx.$$

EXAMPLE 4 . Let $f(x, y) = x^2 y \cos x$, then

$$df(x, y) = (2xy \cos x - x^2 y \sin x)\ dx + x^2 \cos x\ dy.$$

9.2.2 Local Expressions

Let $\Lambda^k T^* M = \bigsqcup_{p \in M} \Lambda^k(T_p^* M)$. $\Lambda^k T^* M$ is a smooth vector bundle over M by Lemma 10.6 of Lee.

- A section $M \to \Lambda^k T^* M$ is called a *differential k-form* or just a *k-form*.
- $\Omega^k(M)$ denotes the vector space of smooth k-forms.
- The *wedge* $\omega \wedge \eta$ of two forms is defined by

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p.$$

- Let $\Omega^0(M) = C^\infty(M)$. If $f \in \Omega^0(M)$, then $f \wedge \omega = f\omega$.

Remark. $\Omega^1(M) = \mathfrak{X}^*(M)$ is a covector field. If $f \in C^\infty(M) = \Omega^0(M)$, then

$$df \in \Omega^1(M) = \mathfrak{X}^*(M).$$

For simplicity, one can think of $\Omega^k(M)$ as the set of all alternating multi-linear functions on $T_p^* M \times \cdots \times T_p^* M$ for some $p \in M$.

Since a k-form is a section, we can consider it as an element in $\Lambda^k T_p^* M$ for some p. Recall that (dx^i) is a basis for $T_p^* M$ and a basis for $\Lambda^k T_p^* M$ is given by

$$\{dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k} : I \text{ runs through all increasing multi-indices}\}.$$

Locally, given a smooth chart (U, φ) and a k-form ω, then

$$\omega = \sum_I{}' \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_I{}' \omega_I dx^I$$

on U. The standard basis for T_pM is $\left(\frac{\partial}{\partial x^{j_1}}, \cdots, \frac{\partial}{\partial x^{j_k}}\right)$, and hence

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} \left(\frac{\partial}{\partial x^{j_1}}, \cdots, \frac{\partial}{\partial x^{j_k}}\right) = \delta^I_J.$$

Thus the component functions ω_I are given by

$$\omega_I = \omega \left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_k}}\right).$$

The functions $\omega_I : U \to \mathbb{R}$ are called component functions. See Page 281 of Lee to refresh on the notation $dx^{i_\circ}$.

Proposition 9.2.1. *A form is smooth if and only if its component functions are smooth in every chart.*

9.2.3 Pullbacks

If $F : M \to N$ is smooth and $\omega \in \Omega^k(N)$, then $F^*\omega \in \Omega^k(M)$ satisfies

$$(F^*\omega)_p(v_1, \cdots, v_k) = \omega_{F(p)}(dF_p v_1, \cdots, dF_p v_k).$$

Lemma 9.2.1. *Suppose $F : M \to N$ smooth.*

(a) $F^* : \Omega^k(N) \to \Omega^k(M)$ *is linear over* $\mathbb{R}$.

(b) $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$.

(c) In any smooth chart,

$$F^*\left(\sum_I{}' \omega_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F).$$

Proof. (a) Let $\omega, \eta \in \Omega^k(N)$, then

$$\begin{aligned}
(F^*(\omega + \eta))_p(v_1, \cdots, v_k) &= (\omega + \eta)_{F(p)}(dF_p v_1, \cdots, dF_p v_k) \\
&= (\omega_{F(p)} + \eta_{F(p)})(dF_p v_1, \cdots, dF_p v_k) \\
&= \omega_{F(p)}(dF_p v_1, \cdots, dF_p v_k) + \eta_{F(p)}(dF_p v_1, \cdots, dF_p v_k) \\
&= (F^*\omega)_p(v_1, \cdots, v_k) + (F^*\eta)_p(v_1, \cdots, v_k), \\
(F^*(\lambda\omega))_p(v_1, \cdots, v_k) &= (\lambda\omega)_{F(p)}(dF_p v_1, \cdots, dF_p v_k) \\
&= \lambda\omega_{F(p)}(dF_p v_1, \cdots, dF_p v_k) \\
&= \lambda(F^*\omega)_p(v_1, \cdots, v_k).
\end{aligned}$$

(b) This is also a long calculation:

$$
\begin{aligned}
&(F^*\omega \wedge F^*\eta)_p (v_1, \cdots, v_k, v_{k+1}, \cdots, v_{2k}) \\
&= (F^*\omega)_p \wedge (F^*\eta)_p (v_1, \cdots, v_{2k}) \\
&= \frac{(2k)!}{k!k!} \mathrm{Alt}\left((F^*\omega)_p \wedge (F^*\eta)_p\right) (v_1, \cdots, v_k, v_{k+1}, \cdots, v_{2k}) \\
&= \frac{(2k)!}{k!k!k!} \sum_{\sigma \in S_k} (\mathrm{sgn}\ \sigma) \left((F^*\omega)_p \otimes (F^*\eta)_p\right) (v_{\sigma(1)}, \cdots, v_{\sigma(k)}, v_{\sigma(k+1)}, \cdots, v_{\sigma(2k)}) \\
&= \frac{(2k)!}{k!k!k!} \sum_{\sigma \in S_k} (\mathrm{sgn}\ \sigma)(F^*\omega)_p(v_{\sigma(1)}, \cdots, v_{\sigma(k)})(F^*\eta)_p(v_{\sigma(k+1)}, \cdots, v_{\sigma(2k)}) \\
&= \frac{(2k)!}{k!k!k!} \sum_{\sigma \in S_k} (\mathrm{sgn}\ \sigma)\omega_{F(p)} \left(dF_p v_{\sigma(1)}, \cdots, dF_p v_{\sigma(k)}\right) \eta_{F(p)} \left(dF_p v_{\sigma(k+1)}, \cdots, dF_p v_{\sigma(2k)}\right) \\
&= \frac{(2k)!}{k!k!k!} \sum_{\sigma \in S_k} (\mathrm{sgn}\ \sigma)(\omega_{F(p)} \otimes \eta_{F(p)}) \left(dF_p v_{\sigma(1)}, \cdots, dF_p v_{\sigma(k)}, dF_p v_{\sigma(k+1)}, \cdots, dF_p v_{\sigma(2k)}\right) \\
&= \frac{(2k)!}{k!k!} \mathrm{Alt}(\omega_{F(p)} \otimes \eta_{F(p)}) \left(dF_p v_{\sigma(1)}, \cdots, dF_p v_{\sigma(k)}, dF_p v_{\sigma(k+1)}, \cdots, dF_p v_{\sigma(2k)}\right) \\
&= \left(\omega_{F(p)} \wedge \eta_{F(p)}\right) (dF_p v_1, \cdots, dF_p v_{2k}) \\
&= (\omega \wedge \eta)_{F(p)} (dF_p v_1, \cdots, dF_p v_{2k}) \\
&= (F^*(\omega \wedge \eta))_p (dF_p v_1, \cdots, dF_p v_{2k}).
\end{aligned}
$$

(c) Fix an increasing multi-index $I = (i_1, \cdots, i_k)$, then by **Proposition** 8.5.1 (1),

$$
\begin{aligned}
\left(F^*\left(\omega_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right)\right)_p &= (\omega_I \circ F(p)) \left(F^*\left(\omega_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right)\right)_p \\
&= (\omega_I \circ F(p))(F^* dy^{i_1})_p \wedge \cdots \wedge (F^* dy^{i_k})_p \\
&= (\omega_I \circ F(p)) d(y^{i_1} \circ F)_p \wedge \cdots \wedge d(y^{i_1} \circ F)_p.
\end{aligned}
$$

Since F^* is linear, it follows that

$$
F^*\left(\sideset{}{'}\sum_I \omega_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right) = \sideset{}{'}\sum_I (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F).
$$

□

Example 5 . Let $\omega = dx \wedge dy$ on $\mathbb{R}^2$. Consider polar coordinates (r, θ) on $V = \mathbb{R}^2 \setminus \{(x, 0) : x \geq 0\}$ with $0 < r$ and $0 < \theta < 2\pi$, $x = r\cos\theta, y = r\sin\theta$. Let $F : V \to \mathbb{R}^2$ be $F(r, \theta) = (r\cos\theta, r\sin\theta)$.

$$
\begin{aligned}
F^*(dx \wedge dy) &= d(\underbrace{r\cos\theta}_{x \circ F}) \wedge d(\underbrace{r\sin\theta}_{y \circ F}) \\
&= (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \\
&= (r(\cos\theta)^2 + r(\sin\theta)^2) dr \wedge d\theta \\
&= r dr \wedge d\theta.
\end{aligned}
$$

Proposition 9.2.2. *Suppose $F : M \to N$ is smooth and $\dim M = n = \dim N$. If $(U, \varphi = (x^i)), (V, \psi = (y^i))$ are smooth charts with $F(U) \subset V$, then*

$$F^*(u dy^1 \wedge \cdots \wedge dy^n) = (u \circ F)(\det DF) dx^1 \wedge \cdots \wedge dx^n,$$

where DF is the derivative matrix in these coordinates.

Proof. Note that

$$\begin{aligned}
&F^*(u dy^1 \wedge \cdots \wedge dy^n)\left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right) \\
&= (u \circ F) dF^1 \wedge \cdots \wedge dF^n \left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right) \\
&= (u \circ F) \det \left(dF^j (\frac{\partial}{\partial x^i})\right) \\
&= (u \circ F) \det \left(\frac{\partial}{\partial x^i}\right) \\
&= (u \circ F)(\det DF)(dx^1 \wedge \cdots \wedge dx^n)\left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\right).
\end{aligned}$$

□

9.3 Exterior Derivatives

Let M be a smooth manifold, $\Omega^k(M)$ be the vector space of k-forms. Goal: Define a "derivative" $d : \Omega^k(M) \to \Omega^{k+1}(M)$.

9.3.1 Euclidean Case

If $U \subset \mathbb{R}^n$ is open, define $d : \Omega^k(U) \to \Omega^{k+1}(U)$ by

$$d\left(\sum_I{}' \omega_I dx^I\right) = \sum_I{}' d\omega_I \wedge dx^I.$$

Example 6 . Let $U = \mathbb{R}^3, k = 1$. If $\omega = P dx + Q dy + R dz \in \Omega^1(\mathbb{R}^3)$, then

$$d\omega = dP \wedge dx + dQ \wedge dy + dR \wedge dz,$$

$$\begin{aligned}
dP \wedge dx &= \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz\right) \wedge dx \\
&= -\frac{\partial P}{\partial y} dx \wedge dy - \frac{\partial P}{\partial z} dx \wedge dz.
\end{aligned}$$

Expand other terms,

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) dx \wedge dz + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy \wedge dz.$$

If we define isomorphisms

$$\begin{aligned} \flat : \mathfrak{X}(\mathbb{R}^3) &\to \Omega^1(\mathbb{R}^3) \\ \flat(X)(\cdot) &= \langle X, \cdot \rangle \end{aligned}$$

$$\begin{aligned} \beta : \mathfrak{X}(\mathbb{R}^3) &\to \Omega^2(\mathbb{R}^3) \\ \beta(X) &= i_X(dx \wedge dy \wedge dz) \\ \beta(X)(y_1, y_2) &= (dx \wedge dy \wedge dz)(X, y_1, y_2) \end{aligned}$$

One can show

$$\begin{tikzcd} \mathfrak{X}(\mathbb{R}^3) \arrow[r, "curl"] \arrow[d, "\flat"] & \mathfrak{X}(\mathbb{R}^3) \arrow[d, "\beta"] \\ \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \end{tikzcd}$$

Proposition 9.3.1. *(a) d is linear over $\mathbb{R}$.*

(b) If $\omega \in \Omega^k(U)$ and $\eta \in \Omega^l(U)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

(c) $d \circ d = 0$.

(d) If $F : U \to V$ is smooth and $\omega \in \Omega^k(U)$, then $F^*(d\omega) = dF^*(\omega)$.

Proof. (a) By definition.

(b) By (a) we can assume that $\omega = u dx^I$ and $\eta = v dx^J$, then

$$\begin{aligned} d(\omega \wedge \eta) &= d(uv dx^{IJ}) \\ &= d(uv) \wedge dx^{IJ} \\ &= (v du + u dv) \wedge dx^{IJ} \\ &= du \wedge dx^I \wedge (v dx^J) + dv \wedge (u dx^I) \wedge dx^J \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned}$$

(c) By (a) we can assume $\omega = u dx^I$, then

$$\begin{aligned}
d(d\omega) &= d(du \wedge dx^I) \\
&= d\left(\frac{\partial u}{\partial x_j} dx^j \wedge dx^I\right) \\
&= d\left(\frac{\partial u}{\partial x^j}\right) dx^j \wedge dx^I \\
&= \frac{\partial^2 u}{\partial x^k \partial x^j} dx^k \wedge dx^j \wedge dx^I \\
&= \sum_{j<k} \left(\frac{\partial^2 u}{\partial x^k \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^k}\right) dx^k \wedge dx^j \wedge dx^I \\
&= 0.
\end{aligned}$$

(d) By (a) we can assume $\omega = u dx^I$, then

$$\begin{aligned}
F^*(d\omega) &= F^*(du \wedge dx^I) \\
&= d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)
\end{aligned}$$

□

9.3.2 Manifold Case

Theorem 9.3.1. *There are unique operators*

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

for $K = 0, 1, 2, \cdots$ called exterior differentiation such that

1. *d is linear over $\mathbb{R}$.*
2. *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then*
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$
3. *$d \circ d = 0$.*
4. *For $f \in \Omega^0(M) = C^\infty(M)$, df is the previously defined differential.*

Proof. First note that if $\omega \in \Omega^k(M)$ and $(U, \varphi), (V, \psi)$ are smooth charts, then on $U \cap V$

$$\begin{aligned}
\varphi^* d(\underbrace{\varphi^{-1*}\omega}_{\text{in } \Omega^k(\varphi(U))}) &= \psi^* \psi^{-1*} \varphi^* d(\varphi^{-1*}\omega) \\
&= \psi^* (\varphi \circ \psi^{-1})^* d(\varphi^{-1*}\omega) \\
&= \psi^* d\left((\varphi \circ \psi^{-1})^* \varphi^{-1*}\omega\right) \quad \text{by 14.24(d)} \\
&= \psi^* d\left((\psi^{-1})^* \varphi^* \varphi^{-1*}\omega\right) \\
&= \psi^* d\left(\psi^{-1*}\omega\right).
\end{aligned}$$

Then define $d\omega \in \Omega^{k+1}(M)$ to the element where $d\omega = \varphi^* d(\varphi^{-1*}\omega)$ on every chart (U, φ). By 14.23, d satisfies (1) - (4). This proves existence. See Lee for uniqueness. ☐

Proposition 9.3.2. *If $F : M \to N$ is smooth, then the pullback commutes with d:*

$$F^* d\omega = dF^*\omega \quad \text{for all } \omega \in \Omega^k(N).$$

Proof. Fix charts $(U, \varphi), (V, \psi)$ with $F(U) \subset V$. Then on U,

$$\begin{aligned}
F^*(d\omega) &= F^*\psi^* d(\psi^{-1*}\omega) \\
&= \varphi^*(\psi \circ F \circ \varphi^{-1})^* d(\psi^{-1*}\omega) \\
&= \psi^* d((\psi \circ F \circ \varphi^{-1})^* \psi^{-1*}\omega) \\
&= \varphi^* d(\varphi^{-1*} F^*\omega) \\
&= d(F^*\omega).
\end{aligned}$$

☐

9.3.3 a Non-Local (Invariant) Formula for d

Proposition 9.3.3. *Let $\omega \in \Omega^k(M)$ and $X_1, \cdots, X_{k+1} \in \mathfrak{X}(M)$. Then*

$$\begin{aligned}
d\omega(X_1, \cdots, X_{k+1}) = &\sum_{1 \le i \le k+1} (-1)^{i-1} X_i \left(\omega(X_1, \cdots, \widehat{X_i}, \cdots, X_{k+1}\right) \\
&+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega\left([X_i, X_j], X_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_n\right).
\end{aligned}$$

Note: If $k = 1$ (so ω is a contangent vector) then

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof. Let $D_\omega(X_1, \cdots, X_{k+1})$ be the right hand side. Observe that

1. we can work locally.
2. The expressions are
    - linear in ω over $\mathbb{R}$.
    - linear in $X_1, \cdots, X_{k+1}$ over $C^\infty(M)$. (Check this when $k = 1$)

so it suffices to fix a chart and assume $\omega = u dx^I$ and $X_1, \cdots, X_{k+1} = \frac{\partial}{\partial x^{j_1}}, \cdots, \frac{\partial}{\partial x^{j_{k+1}}}$. Then

$$
\begin{aligned}
d\omega(X_1, \cdots, X_{k+1}) &= \left(\sum_m \frac{\partial u}{\partial x^m} dx^m \wedge dx^I \right) (X_1, \cdots, X_{k+1}) \\
&= \left(\sum_m \frac{\partial u}{\partial x^m} \delta_J^{mI} \right),
\end{aligned}
$$

where $J = (j_1, \cdots, j_{k+1})$. Note that

$$
\left[\frac{\partial}{\partial x^{j_p}}, \frac{\partial}{\partial x^{j_q}} \right] = 0,
$$

so

$$
\begin{aligned}
D_\omega(X_1, \cdots, X_{k+1}) &= \sum_p (-1)^{p-1} X_p \left(\omega(X_1, \cdots, \widehat{X_p}, \cdots, X_{k+1} \right) \\
&= \sum_p (-1)^{p-1} \frac{\partial}{\partial x^{j_p}} \left(u dx^I \left(\frac{\partial}{\partial x^{j_1}}, \cdots, \widehat{\frac{\partial}{\partial x^{j_p}}}, \cdots, \frac{\partial}{\partial x^{j_{k+1}}} \right) \right) \\
&= \sum_p (-1)^{p-1} \frac{\partial u}{\partial x^{j_p}} \delta^I_{\widehat{J}_p},
\end{aligned}
$$

where $\widehat{J}_p = (j_1, \cdots, \widehat{j_p}, \cdots, j_{k+1})$. Note at most one term in (**) is nonzero and for this term I is a permutation of $\widehat{J}_p$. Then $j_p I$ is a permutation of J and the $m = j_p$ term is the only non-vanishing term in (*). Finally, by row/column expansion of determinant

$$
(-1)^{p-1} \delta^I_{\widehat{J}_p} = \delta_J^{j_p I}.
$$

□

9.3.4 Lie Derivatives

We previously defined Lie derivatives for tensor fields, hence we can consider Lie derivatives of forms (i.e. alternating tensor fields)

Proposition 9.3.4. *If $V \in \mathfrak{X}(M), \omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then*

$$
\mathcal{L}_V(\omega \wedge \eta) = \mathcal{L}_V(\omega) \wedge \eta + \omega \wedge \mathcal{L}_V(\eta).
$$

Proof. HW. □

Theorem 9.3.2 (Cartan's magic formula)**.** *If $V \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, then*

$$
\mathcal{L}_V(\omega) = i_V(d\omega) + d(i_V(\omega)).
$$

Proof. We induct on k.

$k = 0$. Recall $\omega^0(M) = C^\infty(M)$. Fix $f \in C^\infty(M)$, then $i_V(f) = 0$ (by definition). Also

$$i_V(df) = df(V) = V(f),$$

so $\mathcal{L}_V(f) = V(f) = i_V(df) + d(i_V(f))$.

Suppose $k > 0$. It suffices to work locally. Then since both sides are linear over $\mathbb{R}$ in ω, we can assume that $\omega = f dx^I$. Let $u = x^{i_1}, \beta = f dx^{i_2} \wedge \cdots \wedge dx^{i_k}$, then $\omega = du \wedge \beta$.

Reminders:

$$\begin{aligned}
&i_V(\alpha \wedge \xi) = i_V(\alpha) \wedge \xi + (-1)^k \alpha \wedge i_V(\xi), \quad \alpha \in \Omega^k(M).\\
&d(\alpha \wedge \xi) = d\alpha \wedge \xi + (-1)^k \alpha \wedge d\xi, \quad \alpha \in \Omega^k(M).\\
&d \circ d = 0.\\
&\mathcal{L}_V(f) = V(f), \quad f \in C^\infty(M).\\
&\mathcal{L}_V(df) = d\mathcal{L}_V(f), \quad f \in C^\infty(M).
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{L}_V(\omega) &= \mathcal{L}_V(du \wedge \beta)\\
&= \mathcal{L}_V(du) \wedge \beta + du \wedge \mathcal{L}_V(\beta)\\
&= d\mathcal{L}_V(u) \wedge \beta + du \wedge (i_V(d\beta) + d(i_V\beta)),
\end{aligned}$$

and

$$\begin{aligned}
i_V(d\omega) &= i_V(d(du \wedge \beta))\\
&= i_V(0 - du \wedge d\beta)\\
&= -i_V(du) \wedge d\beta + du \wedge i_V(d\beta)\\
&= -V(u)d\beta + du \wedge i_V(d\beta)
\end{aligned}$$

and

$$\begin{aligned}
d(i_V(\omega)) &= d(i_V(du \wedge \beta))\\
&= d(i_V(du) \wedge \beta - du \wedge i_V(\beta))\\
&= d(V(u)\beta - du \wedge i_V(\beta))\\
&= d(V(u))\beta + V(u)d\beta - 0 + du \wedge d(i_V(\beta))
\end{aligned}$$

so

$$\mathcal{L}_V(\omega) = i_V(d\omega) + d(i_V\omega).$$

Note $d\mathcal{L}_V(u) \wedge \beta = d(V(u) \wedge \beta)$. □

Corollary 9.3.1. *If $V \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, then*

$$\mathcal{L}_V(d\omega) = d\mathcal{L}_V(\omega).$$

Proof. By Cartan,

$$\mathcal{L}_V(d\omega) = i_V(dd\omega) = di_V(d\omega) = d(i_V d\omega)$$

and

$$d\mathcal{L}_V(\omega) = d\left[i_V(d\omega) + d(i_V\omega)\right] = d(i_V(d\omega)).$$

□

9.4 Orientations

Goal: Define orientable manifolds

9.4.1 Orientations of Vector Spaces

Let V be a vector space with $\dim V = n$. Two ordered bases $E_1, \cdots, E_n$ and $E'_1, \cdots, E'_n$ are *consistently oriented* if the transition matrix (the matrix (B_i^j) satisfying $E_i = B_i^j E'_j$) has positive determinant.

Exercise 4 . Show that being consistently oriented is an equivalence relation and there are two equivalence classes.

Definition 9.4.1. An *orientation* for V is a choice of one of this equivalence classes. An ordered basis in this class is called *positively oriented*, otherwise it is called *negatively oriented*.

Example 7 . Let $V = \mathbb{R}^2$ with the orientation making the standard basis positive. If $\dim V = 0$, then an orientation is a choice of $+1$ or -1.

9.4.2 Orientations of Manifolds

Definition 9.4.2. An *orientation* on a smooth n-manifold M is a choice of orientation on each tangent space which is continuous in the following sense: every $p \in M$ has an open neighborhood U where there exist $X_1, \cdots, X_n \in \mathfrak{X}(M)$ such that

$$(X_1|_q, \cdots, X_n|_q)$$

is a positively oriented basis of T_qM for all $q \in U$.

Remark. If M has an orientation, $U \subset M$ is open and connected, and there exist $X_1, \cdots, X_n \in \mathfrak{X}(U)$, where $(X_1|_q, \cdots, X_n|_q)$ is a basis for all $q \in U$, then $(X_1|_q, \cdots, X_n|_q)$ is either always positively oriented or always negatively oriented.

Definition 9.4.3. M is *orientable* if it has an orientation. An *oriented manifold* is an ordered pair $(M, \mathcal{O})$, there $\mathcal{O}$ is a choice of orientation for M. For each $p \in M$, the orientation of T_pM determined by $\mathcal{O}$ is denoted by $\mathcal{O}_p$.

Example 8 . The Möbius band is not orientable.

9.4.3 Oriented Atlases

Definition 9.4.4. A smooth atlas $\mathcal{A}$ of M is *oriented* if for all $(U,\varphi),(V,\psi)\in\mathcal{A}$,

$$\det\big(D(\psi\circ\varphi^{-1})\big)>0 \text{ on } \varphi(U\cap V).$$

A smooth atlas $\mathcal{A}$ of M is *compatible* with an orientation on M if for every $(U,\varphi=(x^i))\in\mathcal{A}$ and $p\in U$ the basis $\left(\left.\frac{\partial}{\partial x^1}\right|_p,\cdots,\left.\frac{\partial}{\partial x^n}\right|_p\right)$ of T_pM is positively oriented.

Proposition 9.4.1. *Suppose M is a smooth manifold.*

1. *If $\mathcal{A}$ is an oriented smooth atlas, then there is an orientation on M compatible with $\mathcal{A}$.*

2. *If M has an orientation, then there exists a compatible oriented smooth atlas.*

Proof. For each $p\in M$, give T_pM the orientation where $\left(\left.\frac{\partial}{\partial x^1}\right|_p,\cdots,\left.\frac{\partial}{\partial x^n}\right|_p\right)$ is positively oriented for every chart containing p. This is well-defined since $\mathcal{A}$ is oriented.

Let

$$\mathcal{A}=\Big\{(U,\varphi):(U,\varphi)\text{ smooth chart}, U\text{ is connected, and }\left(\left.\frac{\partial}{\partial x^1}\right|_p,\cdots,\left.\frac{\partial}{\partial x^n}\right|_p\right)\text{ is positively oriented for all }p\in U\Big\},$$

then $\det\big(D(\psi\circ\varphi^{-1})\big)>0$ for all $(U,\varphi),(V,\psi)\in\mathcal{A}$. We need to show that $M=\bigcup_{(U,\varphi)\in\mathcal{A}}U$. Fix $p\in M$ and fix a smooth chart (V,ψ) with $p\in V$. By shrinking, we can assume that V is connected. Since V is connected, $\left(\left.\frac{\partial}{\partial x^1}\right|_q,\cdots,\left.\frac{\partial}{\partial x^n}\right|_q\right)$ is either always positively oriented or always negatively oriented for $q\in V$. In the first case, $(V,\psi)\in\mathcal{A}$. In the second case, let

$$(U,\varphi)=(V,\varphi=(-x^1,x^2,x^3,\cdots,x^n)).$$

Then $(U,\varphi)\in\mathcal{A}$. $\square$

9.4.4 n-Forms

Recall if $\dim V=n$, $\omega\in T^n(V^*)$ and $L:V\to V$ is linear, then

$$\omega(Lv_1,\cdots,Lv_n)=(\det L)\omega(v_1,\cdots,v_n)$$

for all $v_1,\cdots,v_n\in V$. So any nonzero $\omega\in T^n(V^*)$ determines an orientation on V where $(E_1,\cdots,E_n)$ is positively oriented if and only if $\omega(E_1,\cdots,E_n)>0$.

Definition 9.4.5. An orientation on a smooth n-manifold M is *compatible* with a non-vanishing n-form $\omega \in \Omega^n(M)$ if for every $p \in M$ the orientation on T_pM is determined by ω_p.

Proposition 9.4.2. *1. If $\omega \in \Omega^n(M)$ is non-vanishing, then M has an orientation compatible with ω.*

2. If M has an orientation, then there is a non-vanishing compatible n-form.

Proof. 1.

2. Fix a compatible smooth atlas $\mathcal{A} = \{(U_i, \varphi_i) : i \in I\}$, then fix a partition of unity $\{\chi_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$. Define

$$\omega = \sum_{i\in I} \chi_i \varphi_i^* \left(dx^1 \wedge \cdots \wedge dx^n\right).$$

Note, on $U_i \cap U_j$ we have

$$\begin{aligned} \varphi_i^*(dx^1 \wedge \cdots \wedge dx^n) &= \varphi_j^*(\varphi_j^{-1})^* \varphi_i^*(dx^1 \wedge \cdots \wedge dx^n) \\ &= \varphi_j^*(\varphi_i \circ \varphi_j^{-1})^*(dx^1 \wedge \cdots \wedge dx^n) \\ &= \det\Big(D(\phi_i \circ \varphi_j^{-1})\Big)\varphi_j^*(dx^1 \wedge \cdots \wedge dx^n). \end{aligned}$$

So since $\det\Big(D(\varphi_i \circ \varphi_j^{-1})\Big) > 0$, ω is non-vanishing.

∎

$\boxed{\text{TOPOLOGY}}$

Theorem 9.4.1. *Every connected non-orientable manifold M admits a 2-sheeted covering map $\pi : \widehat{M} \to M$ where $\widehat{M}$ is connected and orientable.*

Corollary 9.4.1. *Any simply connected manifold is orientable.*

9.5 Integration

$\boxed{\text{GOALS}}$

- Given an oriented manifold M and $\omega \in \Omega^n(M)$ where $n = \dim M$. Define $\int_M \omega$.
- Prove Stokes' Theorem.

9.5.1 Integration on $\mathbb{R}^n$

Given an open set $U \subset \mathbb{R}^n$, let $\int_U f\ dV$ denote the Lebesgue integral.

Definition 9.5.1. Given an n-form ω on an open set $U \subset \mathbb{R}^n$, we can write $\omega = f dx^1 \wedge \cdots \wedge dx^n$ where $f : U \to \mathbb{R}$. If f is Lebesgue integrable, then ω is *integrable* and the integral of ω over U is

$$\int_U \omega = \int_U f\ dV.$$

This can be written more suggestively as

$$\int_U f\ dx^1 \wedge \cdots \wedge dx^n = \int_U f\ dx^1 \cdots dx^n.$$

To compute the integral of a form, just "erase the wedges".

EXAMPLE 9 . Let $n = 1, U = (a, b), \omega = f\ dx$, then

$$\int_U \omega = \int_{(a,b)} f\ dx.$$

Proposition 9.5.1. *Suppose $U, W \subset \mathbb{R}^n$ are open and $G : U \to W$ is a diffeomorphism which either preserves or reverses orientation (i.e. $\det DG$ is either always positive or always negative).*

If ω is an integrable n-form on W, then

$$\int_U G^*\omega = \begin{cases} \int_W \omega & \text{if } G \text{ preserves orientation,} \\ -\int_W \omega & \text{if } G \text{ reverses orientation.} \end{cases}$$

Proof. If $\omega = f dx^1 \wedge \cdots \wedge dx^n$, then

$$\begin{aligned} G^*\omega &= f \circ G(\det DG) dx^1 \wedge \cdots \wedge dx^n \\ &= s(f \circ G)|\det DG| dx^1 \wedge \cdots \wedge dx^n, \end{aligned}$$

where $s = 1$ if G preserves orientation and $s = -1$ if G reverses orientation. Then

$$\begin{aligned} \int_W \omega = \int_W f\ dV &= \int_U f \circ G|\det DG|\ dV \\ &= s \int_U G^*\omega. \end{aligned}$$

$\square$

9.5.2 Integration on Manifolds

Fix an oriented smooth n-manifold M. A chart (U, φ) is *positively* (*negatively*, resp.) *oriented* if $\left(\frac{\partial}{\partial x^1}\bigg|_p, \cdots, \frac{\partial}{\partial x^n}\bigg|_p \right)$ is positively (negatively, resp.) oriented for all $p \in U$.

$\boxed{\text{Goal}}$ Define $\int_M \omega$ when $\omega \in \Omega^n(M)$ is compactly suppoerted.

$\boxed{\text{Case 1}}$ Suppose supp $\omega \subset U$ where (U, φ) is a positively or negatively oriented chart. Then define

$$\int_M \omega = \begin{cases} \displaystyle\int_{\varphi(U)} (\varphi^{-1})^* \omega & \text{if positively oriented,} \\ \displaystyle -\int_{\varphi(U)} (\varphi^{-1})^* \omega & \text{if positively oriented.} \end{cases}$$

Note

$$\int_{\varphi(U)} (\varphi^{-1})^* \omega = \int_{\varphi(U)} \omega \left(\frac{\partial}{\partial x^1}\bigg|_{\varphi(p)}, \cdots, \frac{\partial}{\partial x^n}\bigg|_{\varphi(p)} \right) dV(p).$$

Recall

$$\frac{\partial}{\partial x^i}\bigg|_p = d(\varphi^{-1})_{\varphi(p)} \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}.$$

Proposition 9.5.2. *$\int_M \omega$ does not depend on the choice of charts containing* supp ω.

Proof. Suppose $(U, \varphi), (W, \psi)$ are smooth charts, each of which are positively or negatively oriented, and where supp $\omega \subset U \cap W$. Then

$$\begin{aligned} \int_{\psi(W)} (\psi^{-1})^* \omega &= \int_{\psi(U \cap W)} (\psi^{-1})^* \omega \\ &= s \int_{\varphi(U \cap W)} (\psi \circ \psi^{-1})^* (\psi^{-1})^* \omega \\ &= s \int_{\varphi(U \cap W)} (\varphi^{-1})^* \psi^* (\psi^{-1})^* \omega \\ &= s \int_{\varphi(U \cap W)} (\varphi^{-1})^* \omega \\ &= s \int_{\varphi(U)} (\varphi^{-1})^* \omega, \end{aligned}$$

where $s = 1$ if $\psi \circ \varphi^{-1}$ preserves orientation (equivalently, ψ, φ have same orientation), $s = -1$ if $\psi \circ \varphi^{-1}$ reverses orientation (equivalently, ψ, φ have opposite orientation) $\square$

$\boxed{\text{General Case}}$ Fix finitely many charts $\{(U_i, \psi_i)\}_{i\in I}$, where $\operatorname{supp} \omega \subset \bigcup_{i\in I} U_i$ and each chart is either negatively or positively oriented. Fix a partition of unity $\{\chi_i\}_{i\in I}$ subordinate to $\{U_i\}_{i\in I}$. Then define

$$\int_M \omega = \sum_{i\in I} \int_M \chi_i\omega = \sum_{i\in I} (\pm 1) \int_{\varphi_i(U_i)} (\varphi_i)^*(\chi_i\omega).$$

Proposition 9.5.3. *$\int_M \omega$ is well defined.*

Proof. Suppose $\{(\widetilde{U}_j, \widetilde{\varphi}_j\}_{j\in J}$ and $\{\widetilde{\chi}_j\}_{j\in J}$ are other choices. Then

$$\begin{aligned} \int_M \chi_i\omega &= \int_M \left(\sum_j \widetilde{\chi}_j\right) \chi_i\omega \\ &= \sum_j \int_M \widetilde{\chi}_j\chi_i\omega. \end{aligned}$$

Also,

$$\int_M \widetilde{\chi}_j\omega = \sum_i \int_M \widetilde{\chi}_j\chi_i\omega.$$

So

$$\sum_i \int_M \chi_i\omega = \sum_j \int_M \widetilde{\chi}_j\omega.$$

□

$\boxed{\text{Corner Case}}$ If $M = \{p\}$ is a single point, then

- $n = \dim M = 0$,
- $\Omega^n(M) = \Omega^0(M) = C^\infty(M) \simeq \mathbb{R}$,
- an orientation on M is a choice of $+1$ or -1.

So we define

$$\int_M \omega = (\text{orientation of } M)\ \omega(p).$$

Proposition 9.5.4 (Properties)**.** *Suppose $\omega, \eta \in \Omega^n(M)$ are compactly supported.*

1. *If $a, b \in \mathbb{R}$, then*
$$\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta.$$
2. *If $-M$ is M with the opposite orientation, then*
$$\int_{-M} \omega = \int_M \omega.$$

3. *If N is an oriented n-manifold and $F : N \to M$ is a diffeomorphism, then*

$$\int_N F^*\omega = \begin{cases} \int_M \omega, & F \text{ preserves orientation} \\ -\int_M \omega, & F \text{ reserves orientation} \end{cases}$$

Proposition 9.5.5. *Let $\omega \in \Omega^n(M)$ be compactly supported. Suppose $D_1, \cdots, D_k \in \mathbb{R}^n$ are open and $F_i : \overline{D_i} \to M$ is smooth for $i = 1, \cdots, k$.*

1. *D_i is bounded and ∂D_i has Lebesgue measure 0.*
2. *F_i induces an orientation-preserving diffeomorphism of D_i onto an open set $W_i \subset M$.*
3. *$W_i \cap W_j = \varnothing$ if $i \neq j$.*
4. *$\operatorname{supp} \omega \subset \overline{W_1} \cup \cdots \overline{W_k}$.*

Then

$$\int_M \omega = \sum_{i=1}^k \int_{D_i} F_i^*\omega.$$

EXAMPLE 10 . Consider $\mathbb{S}^2$ with the orientation induced by $\mathbb{R}^3$. Let $\omega = x\ dy \wedge dz$ and $\int_{\mathbb{S}^2} \omega$. Let $F(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$. $D = (0, \pi) \times (-\pi, \pi)$, $W = \mathbb{S}^2 \setminus \{(0, 0, -1)\}$. Then $F : D \to W$ is a diffeomorphism which preserves orientation and $\overline{W} = \mathbb{S}^2$. Then

$$\begin{aligned}
\int_{\mathbb{S}^2} \omega &= \int_D F^*\omega \\
&= \int_D (\sin\phi\cos\theta)\ d(\sin\phi\sin\theta) \wedge d(\cos\phi) \\
&= \int_D \sin\phi\cos\theta(\sin\theta\cos\phi\ d\phi + \sin\phi\cos\theta\ d\theta) \wedge (-\sin\phi\ d\phi) \\
&= \int_D (\sin\phi)^3(\cos\theta)^2\ d\phi \wedge d\theta \\
&= \int_0^\pi \int_{-\pi}^\pi (\sin\phi)^3(\cos\theta)^2\ d\phi\ d\theta \\
&= \frac{4\pi}{3}.
\end{aligned}$$

9.6 Stokes Theorem

Theorem 9.6.1 (Stokes Theorem)**.** *Let M be an oriented smooth n-manifold and let $\omega \in \Omega^{n-1}(M)$ be compactly supported. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Remark. If $\partial M = \varnothing$, then we define $\int_{\partial M} \omega = 0$. If $\partial M \neq \varnothing$, then ∂M is given the induced orientation.

Example 11 (Fundamental Theorem of Calculus). Let $M = [a, b] \subset \mathbb{R}$ with standard orientation. Then $\partial M = \{a, b\}$, $\{a\}$ has -1 and $\{b\}$ has $+1$ orientation. If $f \in \Omega^{n-1}(M) = \Omega^0(M) = C^\infty([a, b])$, then $df = f'(x)dx$. Then

$$\int_M df = \int_a^b f'(x)dx$$
$$\int_{\partial M} f = f(b) - f(a).$$

By Stokes, $\int_a^b f'(x)\ dx = f(b) - f(a)$.

Example 12 (Green's Theorem). If $D \subset \mathbb{R}^2$ is a bounded open set, ∂D is an embedded submanifold, and $P, Q : \overline{D} \to \mathbb{R}$ are smooth, then

$$\int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dV = \int_{\partial D} P dx + Q dy.$$

Apply Stokes to $\omega = P dx + Q dy$.

Proof of Stokes

Assume $M = \mathbb{H}^n = \{x \in \mathbb{R}^n : x^n \geq 0\}$. Pick $R > 0$ such that

$$\text{supp } \omega \subset (-R, R)^{n-1} \times [0, R).$$

Then

$$\omega = \sum_{i=1}^n \omega_i\ dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

for some $\omega_i : \mathbb{H}^n \to \mathbb{R}$. So

$$\begin{aligned} d\omega &= \sum_{i,j=1}^n \frac{\partial \omega_i}{\partial x^j}\ dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n, \end{aligned}$$

then

$$\int_M d\omega = \sum_{i=1}^n (-1)^{i-1} \int_{x^1=-R}^R \cdots \int_{x^{i-1}=-R}^R \cdots \int_{x^n=0}^R \frac{\partial \omega_i}{\partial x^i}\ dV$$

where we can integrate in any order. If $1 \leq i \leq n-1$,

$$\int_{-R}^R \frac{\partial \omega_i}{\partial x^i}\ dx^i = \omega_i \Big|_{x^i=-R}^R = 0 \tag{9.3}$$

since supp $\omega \subset (-R,R)^{n-1} \times [0,R)$. If $i=n$,

$$\int_0^R \frac{\partial \omega_n}{\partial x^n}\, dx^n = -\omega_n(x^1,\cdots,x^{n-1},0),$$

so

$$\int_{\mathbb{H}^n} d\omega = (-1)^n \int_{x^1=-R}^{R} \cdots \int_{x^{n-1}=-R}^{R} \omega_n(x^1,\cdots,x^{n-1},0)\; dx^1\cdots dx^n.$$

Next we compute $\int_{\partial M}\omega = \int_{\partial\mathbb{H}^n}\omega$. Note $\frac{\partial}{\partial x^1},\cdots,\frac{\partial}{\partial x^{n-1}}$ are positively oriented on $\partial\mathbb{H}^n$, so by definition $-\frac{\partial}{\partial x^n},\frac{\partial}{\partial x^1},\cdots,\frac{\partial}{\partial x^{n-1}}$ are positively oriented on $\mathbb{R}^n$ $\iff (-1)^n = 1 \iff n$ is even. Also, $\omega|_{\partial\mathbb{H}^n} = \omega_n\; dx^1\wedge\cdots\wedge dx^{n-1}$, so

$$\int_{\partial\mathbb{H}^n}\omega = (-1)^n \int_{\partial\mathbb{H}^n} \omega_n\; dx^1\wedge\cdots\wedge dx^{n-1},$$

so $\int_{\partial M}\omega = \int_M d\omega$.

$\boxed{\text{Special Case 2: } M=\mathbb{R}^n}$ Now $\partial M=\varnothing$, so $\int_{\partial M}=0$ by definition. Further by (9.3), we can show $\int_M\omega=0$.

$\boxed{\text{Special Case 3: supp } \omega \text{ is contained in a positively or negatively oriented chart } (U,\varphi)}$
Then

$$\begin{aligned}
\int_M d\omega &= \pm\int_{\varphi(U)} (\varphi^{-1})^*\; d\omega \\
&= \pm\int_{\varphi(U)} d(\varphi^{-1})^*\omega \quad \text{by (14.26) of Lee} \\
&= \begin{cases} \pm\int_{\mathbb{R}^n} d(\varphi^{-1})^*\omega, & \varphi \text{ is an interior chart} \\ \pm\int_{\mathbb{H}^n} d(\varphi^{-1})^*\omega, & \varphi \text{ is an exterior chart} \end{cases} \\
&= \begin{cases} 0, & U\cap\partial M=\varnothing \\ \pm\int_{\varphi(U)\cap\partial\mathbb{H}^n} (\varphi^{-1})^*\omega, & U\cap\partial M\neq\varnothing \end{cases} \\
&= \int_{\partial M}\omega.
\end{aligned}$$

$\boxed{\text{General Case}}$ Fix finitely many smooth charts $\{(U_i,\varphi_i)\}_{i\in I}$ such that

1. supp $\omega \subset \bigcup_{i\in I} U_i$.
2. Each (U_i,φ_i) is either positively or negatively oriented.

Fix a partition of unity $\{\chi_i\}$ subordinate to $\{U_i\}$, then

$$\begin{aligned}\int_M d\omega &= \int_M d\left(\sum \chi_i \omega\right) \\ &= \sum \int_M d(\chi_i \omega) \\ &= \sum \int_{\partial M} \chi_i \omega \\ &= \int_{\partial M} \sum \chi_i \omega \\ &= \int_{\partial M} \omega.\end{aligned}$$

9.6.1 An Application

Theorem 9.6.2 (Brouwer Fixed Point Theorem)**.** *Let $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$. If $f : B \to B$ is continuous, then f has a fixed point in B.*

Lemma 9.6.1. *There does not exist a smooth retraction of B onto ∂B.*

Proof. Suppose $r : B \to \partial B$ is a smooth retraction (i.e. $r|_{\partial B} = \mathrm{id}|_{\partial B}$). Since $\partial B \simeq \mathbb{S}^{n-1}$ is oriented, there is a non-vanishing $(n-1)$-form ω on ∂B which is compatible with the orientation. Then

$$\begin{aligned}0 < \int_{\partial B} \omega &= \int_{\partial B} r^* \omega \\ &= \int_B dr^* \omega = \int_B r^* d\omega \\ &= 0\end{aligned}$$

because $d\omega \in \Omega^n(\partial B)$, a contradiction. □

Lemma 9.6.2. *Every smooth map $f : B \to B$ has a fixed point.*

Proof. Suppose $f(x) \neq x$ for all $x \in B$. Define

$$\begin{aligned}&\mu : B \to \mathbb{R} \\ &\mu(x) = \frac{-2\langle x, f(x) - x\rangle + \sqrt{4(\langle x, f(x) - x\rangle)^2 + 4|f(x) - x|^2}}{2|f(x) - x|^2},\end{aligned}$$

then

1. μ is smooth.
2. $\mu \geq 0$.
3. $|x + \mu(x)(f(x) - x)|^2 = 1$.

4. If $|x| = 1$, then $\mu(x) = 0$.

Then $r(x) = x + \mu(x)(f(x) - x)$ is a retraction from B to ∂B, a contradiction. □

Lemma 9.6.3. *Every continuous map $f : B \to B$ has a fixed point.*

Proof. Using approximation, we can find a sequence $\{f_n\}$ of smooth functions from B to B where $f_n \to f$ uniformly. Each f_n has a fixed point $x_n \in B$. Passing to a subsequence, we can assume $x_n \to x_\infty \in B$. Then $f(x_\infty) = \lim_{n\to\infty} f_n(x_n) = \lim_{n\to\infty} x_n = x_\infty$. □

Appendix A

Set Theory

A.1 Cartesian Products

Definition A.1.1. Let $\{X_\alpha\}_{\alpha \in A}$ be an indexed family of sets, their **Cartesian product** $\prod_{\alpha \in A} X_\alpha$ is the set of all maps $f : A \to \bigcup_{\alpha \in A} X_\alpha$ such that $f(\alpha) \in X_\alpha$ for all $\alpha \in A$.

Definition A.1.2. If $X = \prod_{\alpha \in A} X_\alpha$ and $\alpha \in A$, we define the αth **projection** or **coordinate map** $\pi_\alpha : X \to X_\alpha$ by $\pi_\alpha(f) = f(\alpha)$.