

Cal 2 - Exam Question 34

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Question 34

Evaluate

$$\int \frac{x^2}{(4-x^2)^{3/2}} dx.$$

- a. $\frac{2x}{\sqrt{4-x^2}} - 2 \sin^{-1}\left(\frac{x}{2}\right) + C$
- b. $\frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$
- c. $\frac{4}{\sqrt{4-x^2}} - 2 \sin^{-1}\left(\frac{x}{2}\right) + C$
- d. I do not know

Solution: b) $\frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$

What this question is asking: Compute an antiderivative by rewriting the denominator into a form that matches a standard trigonometric substitution pattern, then substitute back to write the final answer in terms of x .

Step 1: Rewrite the denominator $(4-x^2)^{\frac{3}{2}}$.

Step 1.1:

Rewrite fractional exponents $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$, $\sqrt{a} = a^{\frac{1}{2}}$.

Step 1.2:

Rewrite the power $\frac{3}{2}$

$$(4-x^2)^{\frac{3}{2}} = (4-x^2)^{\frac{1}{2} \cdot 3} \quad \text{Rewrite } \frac{3}{2} = \frac{1}{2} \cdot 3$$

$$\begin{aligned} (4-x^2)^{\frac{1}{2} \cdot 3} &= \left((4-x^2)^{\frac{1}{2}}\right)^3 \\ &= (\sqrt{4-x^2})^3 \end{aligned} \quad \left. \vphantom{\begin{aligned} (4-x^2)^{\frac{1}{2} \cdot 3} &= \left((4-x^2)^{\frac{1}{2}}\right)^3 \\ &= (\sqrt{4-x^2})^3 \end{aligned}} \right) \text{Use } a^{bc} = (a^b)^c$$

Step 2: Set up trigonometric substitution.

Step 2.1:

Define symbols

a : constant, x : unknown variable, θ : angle.

Step 2.2:

Choose the substitution

From Step 1, the denominator contains $\sqrt{4 - x^2}$, which matches the pattern $\sqrt{a^2 - x^2}$. For this specific pattern, use the trig substitution $x = a \sin \theta$.

Step 2.3:

Identify a and x

$$4 - x^2 = 2^2 - x^2$$

$$a = 2$$

$$x = a \sin \theta = 2 \sin \theta$$

Step 3: Rewrite the numerator x^2 .

Step 3.1:

Square both sides

$$x = 2 \sin \theta$$

$$x^2 = (2 \sin \theta)^2$$

Step 3.2:

Apply $(ab)^2 = a^2 b^2$

$$a = 2$$

$$b = \sin \theta$$

$$(2 \sin \theta)^2 = 2^2 (\sin \theta)^2$$

Step 3.3:

Simplify

$$2^2 = 4$$

$$(\sin \theta)^2 = \sin^2 \theta \quad (\text{notation: shorthand})$$

$$2^2 (\sin \theta)^2 = 4 \sin^2 \theta$$

Step 4: Rewrite $4 - x^2$ in terms of θ .

Step 4.1:

Substitute $x^2 = 4 \sin^2 \theta$

$$4 - x^2 = 4 - 4 \sin^2 \theta$$

Step 4.2:

Factor out 4

$$\begin{aligned} 4 - 4 \sin^2 \theta &= 4(1) - 4(\sin^2 \theta) \\ &= 4(1 - \sin^2 \theta) \end{aligned} \quad \left. \begin{array}{l} \text{Rewrite } 4 = 4 \cdot 1 \text{ and} \\ \text{factor out } 4 \end{array} \right\}$$

Step 4.3:Use $1 - \sin^2 \theta = \cos^2 \theta$

$$1 - \sin^2 \theta = \cos^2 \theta$$

$$4(1 - \sin^2 \theta) = 4(\cos^2 \theta)$$

$$4 - x^2 = 4 \cos^2 \theta$$

Use Pythagorean identity, then multiply both sides by 4

Plug in $x^2 = 4 \sin^2 \theta$

Step 5: Compute $(4 - x^2)^{\frac{3}{2}}$ after substitution.**Step 5.1:**Substitute into $(\sqrt{4 - x^2})^3$

$$(4 - x^2)^{\frac{3}{2}} = (\sqrt{4 - x^2})^3$$

$$(\sqrt{4 - x^2})^3 = (\sqrt{4 \cos^2 \theta})^3$$

Substitute $4 - x^2 = 4 \cos^2 \theta$ **Step 5.2:**Use $\sqrt{ab} = \sqrt{a}\sqrt{b}$

$$a = 4$$

$$b = \cos^2 \theta$$

$$\sqrt{4 \cos^2 \theta} = \sqrt{4} \sqrt{\cos^2 \theta} \quad (\text{rule: } (ab)^{1/2} = a^{1/2} b^{1/2})$$

Step 5.3:Simplify $\sqrt{4}$

$$\sqrt{4} = 2$$

$$\sqrt{4} \sqrt{\cos^2 \theta} = 2 \sqrt{\cos^2 \theta}$$

Multiply both sides by $\sqrt{\cos^2 \theta}$ **Step 5.4:**Use $\sqrt{u^2} = |u|$

$$u = \cos \theta$$

$$\sqrt{\cos^2 \theta} = |\cos \theta|$$

$$2 \sqrt{\cos^2 \theta} = 2 |\cos \theta|$$

$$\sqrt{4 \cos^2 \theta} = 2 |\cos \theta|$$

Let $u = \cos \theta$ Use $\sqrt{u^2} = |u|$

Multiply both sides by 2

Step 5.5:

Cube the result

$$(\sqrt{4 \cos^2 \theta})^3 = (2 |\cos \theta|)^3$$

$$= 2^3 |\cos \theta|^3$$

$$= 8 |\cos \theta|^3$$

Step 5.6:

Remove the absolute value

$$0 \leq \theta \leq \frac{\pi}{2} \Rightarrow \cos \theta \geq 0$$

$$|\cos \theta| = \cos \theta \text{ since } \cos \theta \geq 0$$

$$8 |\cos \theta|^3 = 8 \cos^3 \theta$$

$$(4 - x^2)^{\frac{3}{2}} = 8 \cos^3 \theta$$

Step 6: Compute dx .**Step 6.1:****Differentiate** $x = 2 \sin \theta$

$$\begin{aligned}
 x &= 2 \sin \theta \\
 \frac{dx}{d\theta} &= \frac{d}{d\theta}(2 \sin \theta) && \left. \begin{array}{l} \text{Differentiate both sides with respect to } \theta \\ \text{Use constant multiple rule: } \frac{d}{d\theta}(cf(\theta)) = cf'(\theta) \end{array} \right\} \\
 &= 2 \frac{d}{d\theta}(\sin \theta) && \left. \begin{array}{l} \text{Plug in } \frac{d}{d\theta}(\sin \theta) = \cos \theta \end{array} \right\} \\
 &= 2 \cos \theta
 \end{aligned}$$

Step 6.2:**Write** dx

$$\begin{aligned}
 \frac{dx}{d\theta} &= 2 \cos \theta \\
 d\theta \left(\frac{dx}{d\theta} \right) &= (2 \cos \theta) d\theta \\
 dx &= 2 \cos \theta d\theta
 \end{aligned}$$

Step 7: Substitute into the integral.**Step 7.1:****List substitutions**

$$\begin{aligned}
 x^2 &= 4 \sin^2 \theta \\
 (4 - x^2)^{\frac{3}{2}} &= 8 \cos^3 \theta \\
 dx &= 2 \cos \theta d\theta
 \end{aligned}$$

Step 7.2:**Substitute**

$$\int \frac{x^2}{(4 - x^2)^{\frac{3}{2}}} dx = \int \frac{4 \sin^2 \theta}{8 \cos^3 \theta} (2 \cos \theta d\theta)$$

Step 8: Simplify the integrand.**Step 8.1:****Multiply constants**

$$\begin{aligned}
 \int \frac{4 \sin^2 \theta}{8 \cos^3 \theta} 2 \cos \theta d\theta &= \int \frac{4 \cdot 2 \sin^2 \theta \cos \theta}{8 \cos^3 \theta} d\theta \\
 &= \int \frac{8 \sin^2 \theta \cos \theta}{8 \cos^3 \theta} d\theta
 \end{aligned}$$

Step 8.2:**Cancel 8**

$$\int \frac{8 \sin^2 \theta \cos \theta}{8 \cos^3 \theta} d\theta = \int \frac{\sin^2 \theta \cos \theta}{\cos^3 \theta} d\theta$$

Step 8.3:Cancel $\cos \theta$

$$\begin{aligned}\cos^3 \theta &= \cos^2 \theta \cos \theta \\ \int \frac{\sin^2 \theta \cos \theta}{\cos^3 \theta} d\theta &= \int \frac{\sin^2 \theta \cos \theta}{\cos^2 \theta \cos \theta} d\theta \\ &= \int \frac{\sin^2 \theta}{\cos^2 \theta} d\theta\end{aligned}$$

Step 9: Rewrite using tangent.**Step 9.1:**

Use the definition of tangent

$$\begin{aligned}\tan \theta &= \frac{\sin \theta}{\cos \theta} \\ \tan^2 \theta &= \frac{\sin^2 \theta}{\cos^2 \theta} \\ \int \frac{\sin^2 \theta}{\cos^2 \theta} d\theta &= \int \tan^2 \theta d\theta\end{aligned}$$

Step 10: Integrate in θ .**Step 10.1:**Use $\tan^2 \theta = \sec^2 \theta - 1$

$$\begin{aligned}\tan^2 \theta &= \sec^2 \theta - 1 \\ \int \tan^2 \theta d\theta &= \int (\sec^2 \theta - 1) d\theta \\ &= \int \sec^2 \theta d\theta - \int 1 d\theta\end{aligned}$$

Step 10.2:

Integrate

$$\begin{aligned}\int \sec^2 \theta d\theta &= \tan \theta \\ \int 1 d\theta &= \theta \\ \int \tan^2 \theta d\theta &= \tan \theta - \theta + C\end{aligned}$$

Step 11: Convert back to x .**Step 11.1:**Find θ

$$\begin{aligned}x &= 2 \sin \theta \\ \frac{1}{2}x &= \frac{1}{2}2 \sin \theta \\ \frac{x}{2} &= \sin \theta \\ \theta &= \sin^{-1}\left(\frac{x}{2}\right)\end{aligned}$$

Step 11.2:

Find $\tan \theta$

$$\sin \theta = \frac{x}{2}$$

$$\text{opposite} = x, \quad \text{hypotenuse} = 2$$

$$\text{adjacent}^2 = 2^2 - x^2 = 4 - x^2$$

$$\text{adjacent} = \sqrt{4 - x^2}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{4 - x^2}}$$

Step 12: Write the final answer and choose the correct option.

Step 12.1:

Substitute back

$$\begin{aligned} \int \frac{x^2}{(4 - x^2)^{\frac{3}{2}}} dx &= \tan \theta - \theta + C \\ &= \frac{x}{\sqrt{4 - x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C \end{aligned}$$

Step 12.2:

Final answer

The correct option is **b**.

Definitions and Notation

Trigonometric substitution. A method for integrals involving radicals. Common patterns:

$$\sqrt{a^2 - x^2} : x = a \sin \theta, \quad \sqrt{a^2 + x^2} : x = a \tan \theta, \quad \sqrt{x^2 - a^2} : x = a \sec \theta.$$

Denominator. In a fraction $\frac{p(x)}{q(x)}$, the denominator is the bottom expression $q(x)$. In Q34, it is $(4 - x^2)^{3/2}$.

Fractional exponents.

$$a^{m/n} = \sqrt[n]{a^m} = \left(a^{1/n}\right)^m.$$

Example:

$$(4 - x^2)^{3/2} = \left((4 - x^2)^{1/2}\right)^3 = \left(\sqrt{4 - x^2}\right)^3.$$

Notation $\sin^2 \theta$. This is shorthand for $(\sin \theta)^2$.

Pythagorean identity.

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \Longleftrightarrow \quad 1 - \sin^2 \theta = \cos^2 \theta.$$

Rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$. For nonnegative a, b :

$$\sqrt{ab} = (ab)^{1/2} = a^{1/2}b^{1/2} = \sqrt{a}\sqrt{b}.$$

Inverse sine $\sin^{-1}(x)$. Also written $\arcsin(x)$, it returns the angle whose sine is x . If $\sin \theta = \frac{x}{2}$, then

$$\theta = \sin^{-1}\left(\frac{x}{2}\right).$$