Cal 2 - Exam Question 34

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Question 34

Evaluate

$$\int \frac{x^2}{(4-x^2)^{3/2}}\,dx.$$

a. $\dfrac{2x}{\sqrt{4-x^2}} - 2\sin^{-1}\left(\dfrac{x}{2}\right) + C$

b. $\dfrac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\dfrac{x}{2}\right) + C$

c. $\dfrac{4}{\sqrt{4-x^2}} - 2\sin^{-1}\left(\dfrac{x}{2}\right) + C$

d. I do not know

Solution: b) $\dfrac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\dfrac{x}{2}\right) + C$

What this question is asking: Compute an antiderivative by rewriting the denominator into a form that matches a standard trigonometric substitution pattern, then substitute back to write the final answer in terms of x.

Step 1: Rewrite the denominator $(4-x^2)^{\frac{3}{2}}$**.**

Step 1.1:
Rewrite fractional exponents $\quad a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m, \qquad \sqrt{a} = a^{\frac{1}{2}}.$

Step 1.2:
Rewrite the power $\frac{3}{2}$

$$(4-x^2)^{\boxed{\frac{3}{2}}} = (4-x^2)^{\boxed{\frac{1}{2}\cdot 3}} \quad \text{Rewrite } \frac{3}{2} = \frac{1}{2}\cdot 3$$

$$\begin{aligned}(4-x^2)^{\frac{1}{2}\cdot 3} &= \left((4-x^2)^{\frac{1}{2}}\right)^3 \\ &= \left(\sqrt{4-x^2}\right)^3\end{aligned} \quad \textit{Use } a^{bc} = (a^b)^c$$

Step 2: Set up trigonometric substitution.

Step 2.1:

Define symbols a: constant, x: unknown variable, θ: angle.

Step 2.2:

Choose the substitution From Step 1, the denominator contains $\sqrt{4-x^2}$, which matches the pattern $\sqrt{a^2-x^2}$. For this specific pattern, use the trig substitution $x = a\sin\theta$.

Step 2.3:

Identify a and x

$$\boxed{4} - x^2 = \boxed{2^2} - x^2$$
$$a = \boxed{2}$$
$$x = a\sin\theta = \boxed{2\sin\theta}$$

Step 3: Rewrite the numerator x^2.

Step 3.1:

Square both sides

$$x = 2\sin\theta$$
$$x^2 = (2\sin\theta)^2$$

Step 3.2:

Apply $(ab)^2 = a^2b^2$

$$a = \boxed{2}$$
$$b = \boxed{\sin\theta}$$
$$(2\sin\theta)^{\boxed{2}} = 2^{\boxed{2}}(\sin\theta)^{\boxed{2}}$$

Step 3.3:

Simplify

$$2^2 = 4$$
$$(\sin\theta)^2 = \sin^2\theta \quad (\text{notation: shorthand})$$
$$2^2(\sin\theta)^2 = 4\sin^2\theta$$

Step 4: Rewrite $4 - x^2$ in terms of θ.

Step 4.1:

Substitute $x^2 = 4\sin^2\theta$

$$4 - \boxed{x^2} = 4 - \boxed{4\sin^2\theta}$$

Step 4.2:

Factor out 4

$$4 - 4\sin^2\theta = \boxed{4}(1) - \boxed{4}(\sin^2\theta)$$
$$= \boxed{4}(1 - \sin^2\theta)$$

Rewrite $4 = \boxed{4}\cdot 1$ *and factor out* $\boxed{4}$

Step 4.3:

Use $1-\sin^2\theta=\cos^2\theta$

$$\begin{aligned} 1-\sin^2\theta &= \cos^2\theta \\ \boxed{4}(1-\sin^2\theta) &= \boxed{4}(\cos^2\theta) \\ 4-x^2 &= 4\cos^2\theta \end{aligned}$$

Use Pythagorean identity, then multiply both sides by $\boxed{4}$

Plug in $x^2=4\sin^2\theta$

Step 5: Compute $(4-x^2)^{\frac{3}{2}}$ **after substitution.**

Step 5.1:

Substitute into $\left(\sqrt{4-x^2}\right)^3$

$$\begin{aligned} (4-x^2)^{\frac{3}{2}} &= \left(\sqrt{4-x^2}\right)^3 \\ \left(\sqrt{4-x^2}\right)^3 &= \left(\sqrt{4\cos^2\theta}\right)^3 \end{aligned}$$

Substitute $\boxed{4-x^2}=\boxed{4\cos^2\theta}$

Step 5.2:

Use $\sqrt{ab}=\sqrt{a}\sqrt{b}$

$$\begin{aligned} a &= \boxed{4} \\ b &= \boxed{\cos^2\theta} \\ \sqrt{4\cos^2\theta} &= \sqrt{4}\sqrt{\cos^2\theta} \quad (\text{rule: } (ab)^{1/2}=a^{1/2}b^{1/2}) \end{aligned}$$

Step 5.3:

Simplify $\sqrt{4}$

$$\begin{aligned} \sqrt{4} &= 2 \\ \sqrt{4}\,\boxed{\sqrt{\cos^2\theta}} &= 2\,\boxed{\sqrt{\cos^2\theta}} \end{aligned}$$

Multiply both sides by $\boxed{\sqrt{\cos^2\theta}}$

Step 5.4:

Use $\sqrt{u^2}=|u|$

$$\begin{aligned} u &= \boxed{\cos\theta} \\ \sqrt{\cos^2\theta} &= |\cos\theta| \\ \boxed{2}\sqrt{\cos^2\theta} &= \boxed{2}|\cos\theta| \\ \sqrt{4\cos^2\theta} &= 2|\cos\theta| \end{aligned}$$

Let $u=\boxed{\cos\theta}$

Use $\sqrt{u^2}=|u|$

Multiply both sides by $\boxed{2}$

Step 5.5:

Cube the result

$$\begin{aligned} \left(\sqrt{4\cos^2\theta}\right)^{\boxed{3}} &= (2|\cos\theta|)^{\boxed{3}} \\ &= \boxed{2^3}|\cos\theta|^3 \\ &= \boxed{8}|\cos\theta|^3 \end{aligned}$$

Step 5.6:

Remove the absolute value

$$\begin{aligned} 0\le\theta\le\frac{\pi}{2} &\Rightarrow \cos\theta\ge 0 \\ |\cos\theta| &= \cos\theta since \cos\theta\ge 0 \\ 8|\cos\theta|^3 &= 8\cos^3\theta \\ (4-x^2)^{\frac{3}{2}} &= 8\cos^3\theta \end{aligned}$$

Step 6: Compute dx.

Step 6.1:

Differentiate $x = 2\sin\theta$

$$\begin{aligned} x &= 2\sin\theta && \text{\textit{Differentiate both sides with respect to } } \theta \\ \tfrac{dx}{d\theta} &= \tfrac{d}{d\theta}(2\sin\theta) && \text{\textit{Use constant multiple rule: }} \tfrac{d}{d\theta}(cf(\theta)) = cf'(\theta) \\ &= 2\,\tfrac{d}{d\theta}(\sin\theta) && \text{\textit{Plug in }} \tfrac{d}{d\theta}(\sin\theta) = \cos\theta \\ &= 2\cos\theta \end{aligned}$$

Step 6.2:

Write dx

$$\begin{aligned} \frac{dx}{d\theta} &= 2\cos\theta \\ d\theta\left(\frac{dx}{d\theta}\right) &= (2\cos\theta)\,d\theta \\ dx &= 2\cos\theta\,d\theta \end{aligned}$$

Step 7: Substitute into the integral.

Step 7.1:

List substitutions

$$\begin{aligned} x^2 &= 4\sin^2\theta \\ (4-x^2)^{\frac{3}{2}} &= 8\cos^3\theta \\ dx &= 2\cos\theta\,d\theta \end{aligned}$$

Step 7.2:

Substitute

$$\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\,dx = \int \frac{4\sin^2\theta}{8\cos^3\theta}\,(2\cos\theta\,d\theta)$$

Step 8: Simplify the integrand.

Step 8.1:

Multiply constants

$$\begin{aligned} \int \frac{4\sin^2\theta}{8\cos^3\theta}\,2\cos\theta\,d\theta &= \int \frac{4\cdot 2\sin^2\theta\cos\theta}{8\cos^3\theta}\,d\theta \\ &= \int \frac{8\sin^2\theta\cos\theta}{8\cos^3\theta}\,d\theta \end{aligned}$$

Step 8.2:

Cancel 8

$$\int \frac{8\sin^2\theta\cos\theta}{8\cos^3\theta}\,d\theta = \int \frac{\sin^2\theta\cos\theta}{\cos^3\theta}\,d\theta$$

Step 8.3:

Cancel $\cos\theta$

$$\cos^3\theta = \cos^2\theta\cos\theta$$

$$\int \frac{\sin^2\theta\cos\theta}{\cos^3\theta}\,d\theta = \int \frac{\sin^2\theta\cos\theta}{\cos^2\theta\cos\theta}\,d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$$

Step 9: Rewrite using tangent.

Step 9.1:

Use the definition of tangent

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$$

$$\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$$

Step 10: Integrate in θ.

Step 10.1:

Use $\tan^2\theta = \sec^2\theta - 1$

$$\tan^2\theta = \sec^2\theta - 1$$

$$\int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta$$

$$= \int \sec^2\theta\,d\theta - \int 1\,d\theta$$

Step 10.2:

Integrate

$$\int \sec^2\theta\,d\theta = \tan\theta$$

$$\int 1\,d\theta = \theta$$

$$\int \tan^2\theta\,d\theta = \tan\theta - \theta + C$$

Step 11: Convert back to x.

Step 11.1:

Find θ

$$x = 2\sin\theta$$

$$\tfrac{1}{2}x = \tfrac{1}{2}2\sin\theta$$

$$\frac{x}{2} = \sin\theta$$

$$\theta = \sin^{-1}\left(\frac{x}{2}\right)$$

Step 11.2:

Find $\tan\theta$

$$\sin\theta = \frac{x}{2}$$

$$\text{opposite} = x, \quad \text{hypotenuse} = 2$$

$$\text{adjacent}^2 = 2^2 - x^2 = 4 - x^2$$

$$\text{adjacent} = \sqrt{4 - x^2}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{4 - x^2}}$$

Step 12: Write the final answer and choose the correct option.

Step 12.1:

Substitute back

$$\int \frac{x^2}{(4 - x^2)^{\frac{3}{2}}}\,dx = \tan\theta - \theta + C$$

$$= \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$$

Step 12.2:

Final answer

The correct option is b.

Definitions and Notation

Trigonometric substitution. A method for integrals involving radicals. Common patterns:

$$\sqrt{a^2 - x^2} : x = a\sin\theta, \qquad \sqrt{a^2 + x^2} : x = a\tan\theta, \qquad \sqrt{x^2 - a^2} : x = a\sec\theta.$$

Denominator. In a fraction $\frac{p(x)}{q(x)}$, the denominator is the bottom expression $q(x)$. In Q34, it is $(4 - x^2)^{3/2}$.

Fractional exponents.

$$a^{m/n} = \sqrt[n]{a^m} = \left(a^{1/n}\right)^m.$$

Example:

$$(4 - x^2)^{3/2} = \left((4 - x^2)^{1/2}\right)^3 = \left(\sqrt{4 - x^2}\right)^3.$$

Notation $\sin^2\theta$. This is shorthand for $(\sin\theta)^2$.

Pythagorean identity.

$$\sin^2\theta + \cos^2\theta = 1 \quad \Longleftrightarrow \quad 1 - \sin^2\theta = \cos^2\theta.$$

Rule $\sqrt{ab} = \sqrt{a}\sqrt{b}$. For nonnegative a, b:

$$\sqrt{ab} = (ab)^{1/2} = a^{1/2}b^{1/2} = \sqrt{a}\sqrt{b}.$$

Inverse sine $\sin^{-1}(x)$. Also written $\arcsin(x)$, it returns the angle whose sine is x. If $\sin\theta = \frac{x}{2}$, then

$$\theta = \sin^{-1}\left(\frac{x}{2}\right).$$