

# Cal 2 - Exam Question 34

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## Question 34

Evaluate

$$\int \frac{x^2}{(4-x^2)^{3/2}} dx.$$

a.  $\frac{2x}{\sqrt{4-x^2}} - 2 \sin^{-1}\left(\frac{x}{2}\right) + C$

b.  $\frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$

c.  $\frac{4}{\sqrt{4-x^2}} - 2 \sin^{-1}\left(\frac{x}{2}\right) + C$

d. I do not know

**Solution:** b)  $\frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$

**What this question is asking:** Compute an antiderivative by rewriting the denominator into a form that matches a standard trigonometric substitution pattern, then substitute back to write the final answer in terms of  $x$ .

**Step 1:** Rewrite the denominator  $(4-x^2)^{\frac{3}{2}}$ .

**Step 1.1:**

Rewrite fractional exponents  $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$ ,  $\sqrt{a} = a^{\frac{1}{2}}$ .

**Step 1.2:**

Rewrite the power  $\frac{3}{2}$

$$(4-x^2)^{\frac{3}{2}} = (4-x^2)^{\frac{1}{2} \cdot 3} \quad \text{Rewrite } \frac{3}{2} = \frac{1}{2} \cdot 3$$

$$\begin{aligned} (4-x^2)^{\frac{1}{2} \cdot 3} &= \left((4-x^2)^{\frac{1}{2}}\right)^3 \\ &= (\sqrt{4-x^2})^3 \end{aligned} \quad \begin{array}{l} \text{Use } a^{bc} = (a^b)^c \\ \swarrow \end{array}$$

**Step 2: Set up trigonometric substitution.****Step 2.1:**

Define symbols

 $a$ : constant,  $x$ : unknown variable,  $\theta$ : angle.**Step 2.2:****Choose the substitution**From Step 1, the denominator contains  $\sqrt{4 - x^2}$ , which matches the pattern  $\sqrt{a^2 - x^2}$ . For this specific pattern, use the trig substitution  $x = a \sin \theta$ .**Step 2.3:****Identify  $a$  and  $x$** 

$$4 - x^2 = 2^2 - x^2$$

$$a = 2$$

$$x = a \sin \theta = 2 \sin \theta$$

**Step 3: Rewrite the numerator  $x^2$ .****Step 3.1:****Square both sides**

$$x = 2 \sin \theta$$

$$x^2 = (2 \sin \theta)^2$$

**Step 3.2:****Apply  $(ab)^2 = a^2 b^2$** 

$$a = 2$$

$$b = \sin \theta$$

$$(2 \sin \theta)^2 = 2^2 (\sin \theta)^2$$

**Step 3.3:****Simplify**

$$2^2 = 4$$

$$(\sin \theta)^2 = \sin^2 \theta \quad (\text{notation: shorthand})$$

$$2^2 (\sin \theta)^2 = 4 \sin^2 \theta$$

**Step 4: Rewrite  $4 - x^2$  in terms of  $\theta$ .****Step 4.1:****Substitute  $x^2 = 4 \sin^2 \theta$** 

$$4 - x^2 = 4 - 4 \sin^2 \theta$$

**Step 4.2:****Factor out 4**

$$\begin{aligned} 4 - 4 \sin^2 \theta &= 4(1) - 4(\sin^2 \theta) \\ &= 4(1 - \sin^2 \theta) \end{aligned} \quad \begin{array}{l} \text{Rewrite } 4 = 4 \cdot 1 \text{ and} \\ \text{factor out } 4 \end{array}$$

**Step 4.3:**Use  $1 - \sin^2 \theta = \cos^2 \theta$ 

$$\begin{aligned} 1 - \sin^2 \theta &= \cos^2 \theta \\ 4(1 - \sin^2 \theta) &= 4(\cos^2 \theta) \\ 4 - x^2 &= 4\cos^2 \theta \end{aligned}$$

*Use Pythagorean identity, then multiply both sides by 4*  
*Plug in  $x^2 = 4\sin^2 \theta$*

**Step 5:** Compute  $(4 - x^2)^{\frac{3}{2}}$  after substitution.**Step 5.1:**Substitute into  $(\sqrt{4 - x^2})^3$ 

$$\begin{aligned} (4 - x^2)^{\frac{3}{2}} &= (\sqrt{4 - x^2})^3 \\ (\sqrt{4 - x^2})^3 &= (\sqrt{4\cos^2 \theta})^3 \end{aligned}$$

*Substitute  $4 - x^2 = 4\cos^2 \theta$*

**Step 5.2:**Use  $\sqrt{ab} = \sqrt{a}\sqrt{b}$ 

$a = 4$

$b = \cos^2 \theta$

$\sqrt{4\cos^2 \theta} = \sqrt{4}\sqrt{\cos^2 \theta}$  (rule:  $(ab)^{1/2} = a^{1/2}b^{1/2}$ )

**Step 5.3:**Simplify  $\sqrt{4}$ 

$$\begin{aligned} \sqrt{4} &= 2 \\ \sqrt{4} \sqrt{\cos^2 \theta} &= 2 \sqrt{\cos^2 \theta} \end{aligned}$$

*Multiply both sides by  $\sqrt{\cos^2 \theta}$*

**Step 5.4:**Use  $\sqrt{u^2} = |u|$ 

$$\begin{aligned} u &= \cos \theta \\ \sqrt{\cos^2 \theta} &= |\cos \theta| \\ 2\sqrt{\cos^2 \theta} &= 2|\cos \theta| \\ \sqrt{4\cos^2 \theta} &= 2|\cos \theta| \end{aligned}$$

*Let  $u = \cos \theta$*   
*Use  $\sqrt{u^2} = |u|$*   
*Multiply both sides by 2*

**Step 5.5:**

Cube the result

$$\begin{aligned} (\sqrt{4\cos^2 \theta})^3 &= (2|\cos \theta|)^3 \\ &= 2^3 |\cos \theta|^3 \\ &= 8 |\cos \theta|^3 \end{aligned}$$

**Step 5.6:**

Remove the absolute value

$0 \leq \theta \leq \frac{\pi}{2} \Rightarrow \cos \theta \geq 0$

$|\cos \theta| = \cos \theta \text{ since } \cos \theta \geq 0$

$8|\cos \theta|^3 = 8\cos^3 \theta$

$(4 - x^2)^{\frac{3}{2}} = 8\cos^3 \theta$

**Step 6:** Compute  $dx$ .

**Step 6.1:**

Differentiate  $x = 2 \sin \theta$

$$\begin{aligned}x &= 2 \sin \theta \\ \frac{dx}{d\theta} &= \frac{d}{d\theta}(2 \sin \theta) \\ &= 2 \frac{d}{d\theta}(\sin \theta) \\ &= 2 \cos \theta\end{aligned}$$

*Differentiate both sides with respect to  $\theta$*   
*Use constant multiple rule:  $\frac{d}{d\theta}(cf(\theta)) = cf'(\theta)$*   
*Plug in  $\frac{d}{d\theta}(\sin \theta) = \cos \theta$*

**Step 6.2:**

Write  $dx$

$$\begin{aligned}\frac{dx}{d\theta} &= 2 \cos \theta \\ d\theta \left( \frac{dx}{d\theta} \right) &= (2 \cos \theta) d\theta \\ dx &= 2 \cos \theta d\theta\end{aligned}$$

**Step 7:** Substitute into the integral.

**Step 7.1:**

List substitutions

$$\begin{aligned}x^2 &= 4 \sin^2 \theta \\ (4 - x^2)^{\frac{3}{2}} &= 8 \cos^3 \theta \\ dx &= 2 \cos \theta d\theta\end{aligned}$$

**Step 7.2:**

Substitute

$$\int \frac{x^2}{(4 - x^2)^{\frac{3}{2}}} dx = \int \frac{4 \sin^2 \theta}{8 \cos^3 \theta} (2 \cos \theta d\theta)$$

**Step 8:** Simplify the integrand.

**Step 8.1:**

Multiply constants

$$\begin{aligned}\int \frac{4 \sin^2 \theta}{8 \cos^3 \theta} 2 \cos \theta d\theta &= \int \frac{4 \cdot 2 \sin^2 \theta \cos \theta}{8 \cos^3 \theta} d\theta \\ &= \int \frac{8 \sin^2 \theta \cos \theta}{8 \cos^3 \theta} d\theta\end{aligned}$$

**Step 8.2:**

Cancel 8

$$\int \frac{8 \sin^2 \theta \cos \theta}{8 \cos^3 \theta} d\theta = \int \frac{\sin^2 \theta \cos \theta}{\cos^3 \theta} d\theta$$

**Step 8.3:**Cancel  $\cos \theta$ 

$$\cos^3 \theta = \cos^2 \theta \cos \theta$$

$$\begin{aligned} \int \frac{\sin^2 \theta \cos \theta}{\cos^3 \theta} d\theta &= \int \frac{\sin^2 \theta \cos \theta}{\cancel{\cos^2 \theta} \cos \theta} d\theta \\ &= \int \frac{\sin^2 \theta}{\cos^2 \theta} d\theta \end{aligned}$$

**Step 9: Rewrite using tangent.****Step 9.1:**

Use the definition of tangent

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta}$$

$$\int \frac{\sin^2 \theta}{\cos^2 \theta} d\theta = \int \tan^2 \theta d\theta$$

**Step 10: Integrate in  $\theta$ .****Step 10.1:**Use  $\tan^2 \theta = \sec^2 \theta - 1$ 

$$\tan^2 \theta = \sec^2 \theta - 1$$

$$\begin{aligned} \int \tan^2 \theta d\theta &= \int (\sec^2 \theta - 1) d\theta \\ &= \int \sec^2 \theta d\theta - \int 1 d\theta \end{aligned}$$

**Step 10.2:**

Integrate

$$\int \sec^2 \theta d\theta = \tan \theta$$

$$\int 1 d\theta = \theta$$

$$\int \tan^2 \theta d\theta = \tan \theta - \theta + C$$

**Step 11: Convert back to  $x$ .****Step 11.1:**Find  $\theta$ 

$$x = 2 \sin \theta$$

$$\frac{1}{2} x = \frac{1}{2} 2 \sin \theta$$

$$\frac{x}{2} = \sin \theta$$

$$\theta = \sin^{-1}\left(\frac{x}{2}\right)$$

**Step 11.2:****Find**  $\tan \theta$ 

$$\sin \theta = \frac{x}{2}$$

opposite =  $x$ , hypotenuse = 2

$$\text{adjacent}^2 = 2^2 - x^2 = 4 - x^2$$

$$\text{adjacent} = \sqrt{4 - x^2}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{\sqrt{4 - x^2}}$$

**Step 12:** Write the final answer and choose the correct option.**Step 12.1:****Substitute back**

$$\begin{aligned} \int \frac{x^2}{(4 - x^2)^{\frac{3}{2}}} dx &= \tan \theta - \theta + C \\ &= \left[ \frac{x}{\sqrt{4-x^2}} \right] - \left[ \sin^{-1}\left(\frac{x}{2}\right) \right] + C \end{aligned}$$

**Step 12.2:****Final answer**The correct option is **b**.

## Definitions and Notation

**Trigonometric substitution.** A method for integrals involving radicals. Common patterns:

$$\sqrt{a^2 - x^2} : x = a \sin \theta, \quad \sqrt{a^2 + x^2} : x = a \tan \theta, \quad \sqrt{x^2 - a^2} : x = a \sec \theta.$$

**Denominator.** In a fraction  $\frac{p(x)}{q(x)}$ , the denominator is the bottom expression  $q(x)$ . In Q34, it is  $(4 - x^2)^{3/2}$ .

**Fractional exponents.**

$$a^{m/n} = \sqrt[n]{a^m} = \left(a^{1/n}\right)^m.$$

Example:

$$(4 - x^2)^{3/2} = \left((4 - x^2)^{1/2}\right)^3 = \left(\sqrt{4 - x^2}\right)^3.$$

**Notation**  $\sin^2 \theta$ . This is shorthand for  $(\sin \theta)^2$ .

**Pythagorean identity.**

$$\sin^2 \theta + \cos^2 \theta = 1 \iff 1 - \sin^2 \theta = \cos^2 \theta.$$

**Rule**  $\sqrt{ab} = \sqrt{a}\sqrt{b}$ . For nonnegative  $a, b$ :

$$\sqrt{ab} = (ab)^{1/2} = a^{1/2}b^{1/2} = \sqrt{a}\sqrt{b}.$$

**Inverse sine**  $\sin^{-1}(x)$ . Also written  $\arcsin(x)$ , it returns the angle whose sine is  $x$ . If  $\sin \theta = \frac{x}{2}$ , then

$$\theta = \sin^{-1}\left(\frac{x}{2}\right).$$