Solutions

Problem #1.

$$\boxed{\forall n \in \mathbb{Z}^+ : \sum_{i=1}^{n} 2i-1 = n^2}$$

Claim: $\sum_{i=1}^{n} 2i-1 = n^2$

Proof: By Induction

Base: $n=1$ $\quad \underbrace{2\cdot 1-1}_{1} = \underbrace{1^2}_{1}$ is True

Induction: show that it holds for $n+1$.

$$\sum_{i=1}^{n+1} 2i-1 = (n+1)^2$$

$$n^2+2(n+1)-1 = n^2+2n+1 = n^2+2n+1$$

proven ✓

Problem #2.

$$\boxed{\sum_{i=1}^{n} \frac{1}{i(i+1)}}$$

Claim: $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

Proof: By Induction

Base: $n=1$ $\quad \underbrace{\frac{1}{i(i+1)}}_{1/2} = \underbrace{\frac{n}{n+1}}_{1/2}$ Base is True

Page #2

Induction: We assume that P(1) is true for n=1. Show that P(n+1) is also true.

$$\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \left(\sum_{i=1}^{n} \frac{1}{i(i+1)}\right) + n+1 = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)}$$

$$\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

proven ✓

Problem #3

$\forall n \in \mathbb{Z}: n^2$ is even $\Rightarrow n$ is even

claim: n^2 is even $\Rightarrow n$ is even

proof: For purposes of contradiction, assume that n is odd $\Rightarrow n^2$ is odd

Observation: 1) Assume n is odd

2) Then $n = 2*k+1$ for some integer k

3) Then $n^2 = 4 \cdot k^2 + 4 \cdot k + 1 = 2 \cdot (2k^2 + 2k) + 1$

which is odd

even + even + even + odd = Odd

This completes the proof by contradiction.

Problem #4

Step #4 is incorrect.

We can't divide both sides by $(a-b)$ because division only makes sense when the number we are dividing by is non-zero.

In this proof, $a-b=0$, because $a=b$.

Thus, step #4 is incorrect.

Problem #5

The functions $f(n)$ and $g(n)$ are asymtotically non negative, there exists n_0 such that $f(n) \geq 0$ and $g(n) \geq 0$ $\forall n \geq n_0$. Thus, we have that ~~for~~ $\forall n \geq n_0$,
$f(n) + g(n) \geq f(n) \geq 0$ and $f(n) + g(n) \geq g(n) \geq 0$.
Adding both inequalities we get ~~$f(n)$~~
$f(n) + g(n) \geq \max(f(n), g(n))$ for all $n \geq n_0$.
This proves that $\boxed{\max(f(n), g(n)) \leq c\,(f(n) + g(n))}$
$\forall n \geq n_0$ with $c = 1$. $\Rightarrow$ $\boxed{\max(f(n), g(n)) = O(f(n) + g(n))}$

Similarly, $\max(f(n), g(n)) \geq f(n)$ and
$\max(f(n), g(n)) \geq g(n)$ $\forall n \geq n_0$

Adding these two inequalities:
$$2\max(f(n), g(n)) \geq (g(n) + f(n)) \quad \forall n \geq n_0$$

Problem #6

$$\Theta(\log_{b_1} n) = \Theta(\log_{b_2} n) \quad \text{for any } b_1, b_2 > 1.$$

Whenever the base of the logarithm is a constant, it doesn't matter what base we use in asymptotic notation. Because there is a mathematical formula that says $\log_a n = \frac{\log_b n}{\log_b a}$ for all positive numbers a, b, and n.
Therefore, if a and b are constants, then $\log_a n$ and $\log_b n$ differ only by a factor of $\log_b a$, and that's a constant factor which we can ignore in asymptotic notation.

Page #4

Problem #7

$$\boxed{f(n) = n(-1)^n \ , \ f(n) \in \Theta(n)}$$

This bound doesn't hold because the definition of $\Theta(g(n))$ requires that every member $f(n) \in \Theta(g(n))$ be assymptotically nonnegative, that is, that $f(n)$ be nonnegative whenever n is sufficiently large. Which doesn't hold for $f(n) = n(-1)^n$ if n is odd.