

Algorithm Analysis - Math CheatSheet

Logarithm properties

- Definition: $\text{base}^{\log_{\text{base}} x} = x$
- Base change: $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$
- Product rule: $\log_b(x * y) = \log_b(x) + \log_b(y)$
- Quotient rule: $\log_b(x/y) = \log_b(x) - \log_b(y)$
- $a^{\log_B(b)} = b^{\log_B(a)}$

Common derivatives

F(x)	F(x)'
x^n	$n * x^{(n-1)}$
b^x	$b^x * \ln(b)$
$\log_b(x)$	$\frac{1}{x * \ln(b)}$
$f(x) * g(y)$	$f(x)' * g(y) + f(x) * g(y)'$
$\frac{f(x)}{g(y)}$	$\frac{f(x)' * g(y) - f(x) * g(y)'}{g(x)^2}$
$f(g(x))$	$f'(g(x)) * g'(x)$

L'Hospital's Rule

Assuming:

- $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ are either 0 or ∞
- $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ exists

Then: $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$

Useful sums

- $\sum_{i=0}^n i = \frac{n * (n+1)}{2}$
- $\sum_{i=0}^n i^2 = \frac{n * (n+1) * (2n+1)}{6}$
- $\sum_{i=0}^n i^p \in \theta(n^{p+1})$
- $\sum_{i=0}^n 2^i = 2^{(n+1)} - 1$
- $\sum_{i=1}^n a * r^{i-1} = \frac{a * (1 - r^n)}{1 - r}$
- $\sum_{i=0}^n i * x^i = \frac{x - (n+1) * x^{n+1} + n * x^{n+2}}{(x-1)^2}$
- $\sum_{i=1}^n \frac{1}{x} \in \theta(\log(n))$
- $\sum_{i=1}^n \ln(j) = \ln(n!) = n * \ln(n) - n + O(\ln(n))$

Asymptotic Complexities

Assuming: $\lim_{n \rightarrow \infty} \left(\frac{f(n)}{g(n)} \right) = L$

- $0 \leq L < \infty \Rightarrow f(n) \in O(g(n))$
- $0 < L < \infty \Rightarrow f(n) \in \Theta(g(n))$
- $0 < L \leq \infty \Rightarrow f(n) \in \Omega(g(n))$
- $L = 0 \Rightarrow f(n) \in o(g(n))$
- $L = \infty \Rightarrow f(n) \in \omega(g(n))$

Master Theorem

Suppose: $T(n) = a * T(n/b) + f(n), T(1) = \theta(1)$.

Subject to: $a \geq 1, b > 1$, f is an asymptotically positive function.

Then: $T(n) = \sum_{i=0}^{\log_b(n)-1} a^i * f(n/b^i) + \theta(n^{\log_b(a)})$

Proof using recursion trees:

- $\log_b(n)$ - height of the recursion tree
- $a^i * f(n/b^i)$ - cost of all internal nodes at depth i
- $\theta(n^{\log_b(a)})$ - cost associated with the leaves

Master Theorem

1. $f(n) = O(n^{\log_b(a)-\epsilon})$, for some constant $\epsilon > 0$ then $T(n) = \theta(n^{\log_b(a)})$
2. $f(n) = \theta(n^{\log_b(a)})$, then $T(n) = \theta(n^{\log_b(a)} \log(n))$
3. $f(n) = \Omega(n^{\log_b(a)+\epsilon})$, for some constant $\epsilon > 0$ and if $a * f(n/b) \leq c * f(n)$, for some constant $c < 1$ and all sufficiently large n , then $T(n) = \theta(f(n))$

Extension of Master Theorem

Assuming: $f(n) = O(n^{\log_b(a)} * \log_b(n)^\alpha)$. Then:

1. $\alpha < -1 \rightarrow T(n) = \theta(n^{\log_b(a)})$
2. $\alpha = -1 \rightarrow T(n) = \theta(n^{\log_b(a)} \log_b(\log_b(n)))$
3. $\alpha > -1 \rightarrow T(n) = \theta(n^{\log_b(a)} \log_b(n)^{\alpha+1})$

- Sometimes we may obtain a simpler expression by changing variables:

Example: $T(n) = 2 * T(\sqrt{n}) + \log(n)$

1. Let $n = 2^m \rightarrow m = \log(n) \rightarrow \sqrt{n} = 2^{m/2}$
2. Let $S(m) = T(2^m)$

Then: $T(2^m) = 2 * T(2^{m/2}) + m$

$\rightarrow S(m) = 2 * S(m/2) + m$

$\rightarrow S(m) = \theta(m * \log(m))$ - Master Th.

$\rightarrow T(n) = T(2^m) = S(m) = \theta(m * \log(m))$

$\rightarrow T(n) = \theta(\log(n) * \log(\log(n)))$

- It may be useful to search for a lower and an upper bound that is easier to compute.

Example: $T(n) = T(n/2 - \log_2(n)) + 1$

Consider:

$T_1(n) = T_1(n/4) + 1 \rightarrow T_1(n) = \theta(\log(n))$

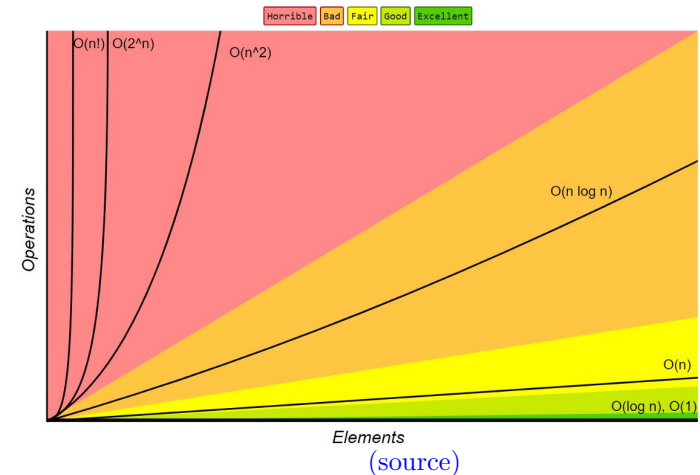
$T_2(n) = T_2(n/2) + 1 \rightarrow T_2(n) = \theta(\log(n))$

Since: $n/4 \leq n/2 - \log_2(n) \leq n/2$

$\rightarrow T_1(n) \leq T(n) \leq T_2(n)$

$\rightarrow T(n) \in \theta(\log(n))$

Big-O Complexity Chart



(source)