STAT/MATH 395 A - PROBABILITY II – UW
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Continuous Random Variables

Notation.

- The indicator function of a set S is a real-valued function defined by :

$$\mathbb{1}_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

- Suppose that $f : D \to \mathbb{R}$ is a real-valued function whose domain is an arbitrary set D. The support of f, written $\text{supp}(f)$, is the set of points in D where f is nonzero

$$\text{supp}(f) = \{x \in D \,|\, f(x) \neq 0\}.$$

1 Probability Density Function and Cumulative Distribution Function

Definition 1.1 (Probability density function)**.** *A rrv is said to be (absolutely)* ***continuous*** *if there exists a real-valued function f_X such that, for any subset $B \subset \mathbb{R}$:*

$$\mathbb{P}(X \in B) = \int_B f_X(x)\,dx \tag{1}$$

Then f_X is called the ***probability density function*** *(pdf) of the random variable X.*

In particular, for any real numbers a and b, with $a < b$, letting $B = [a, b]$, we obtain from Equation (1) that :

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)\,dx \tag{2}$$

Property 1.1. *If X is a continuous rrv, then*

- *For all $a \in \mathbb{R}$,*

$$\mathbb{P}(X = a) = 0 \tag{3}$$

In other words, the probability that a continuous random variable takes on any fixed value is zero.

- *For any real numbers a and b, with $a < b$*

$$\mathbb{P}(a \le X \le b) = \mathbb{P}(a \le X < b) = \mathbb{P}(a < X \le b) = \mathbb{P}(a < X < b) \tag{4}$$

  The above equation states that including or not the bounds of an interval does not modify the probability of a continuous rrv.

Proof.

- Let us first prove Equation (3) :

$$\mathbb{P}(X = a) = \mathbb{P}(X \in [a, a]) = \int_a^a f_X(x)\,dx = 0$$

- To prove Equation (4), we simply notice that

$$\mathbb{P}(a \le X \le b) = \mathbb{P}(a \le X < b) = \mathbb{P}(a < X \le b) = \mathbb{P}(a < X < b) = \int_a^b f_X(x)\,dx$$

□

Theorem 1.1. *A probability density function completely determines the distribution of a continuous real-valued random variable.*

Remark : This theorem means that two continuous real-valued random variables X and Y that have exactly the same probability density functions follow the same distribution. We say that they are **identically distributed**.

Definition 1.2 (Cumulative distribution function)**.** *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. The **(cumulative) distribution function** (cdf) of a real-valued random variable X is the function F_X given by*

$$F_X(x) = \mathbb{P}(X \le x), \quad \textit{for all } x \in \mathbb{R} \tag{5}$$

Property 1.2. *Let F_X be the cdf of a random variable X. Following are some properties of F_X:*

- *F_X is increasing : $x \le y \Rightarrow F_X(x) \le F_X(y)$*
- $\lim_{x\to\infty} F_X(x) = 1$ *and* $\lim_{x\to-\infty} F_X(x) = 0$
- *F_X is càdlàg :*
  - *F_X is right continuous :* $\lim_{x\downarrow x_0} F_X(x) = F_X(x_0)$ *, for $x_0 \in \mathbb{R}$*
  - *F_X has left limits :* $\lim_{x\uparrow x_0} F_X(x)$ *exists, for $x_0 \in \mathbb{R}$*

Property 1.3 (Cumulative distribution function of a continuous rrv)**.** *Let X be a continuous rrv with pdf f_X. Then the cumulative distribution function F_X of X is given by :*

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \tag{6}$$

Proof. We have the following :

$$\begin{aligned} F_X(x) &= \mathbb{P}(X \leq x) \\ &= \mathbb{P}(X \in (-\infty, x)) \\ &= \int_{-\infty}^{x} f_X(t)\, dt \end{aligned}$$

□

Theorem 1.2. *A cumulative distribution function completely determines the distribution of a continuous real-valued random variable.*

Lemma 1.3. *Let X be a continuous rrv with pdf f_X and cumulative distribution function F_X. Then*

- *$F'_X(x) = f_X(x)$, if f_X is continuous at x;*
- *For any real numbers a and b with $a < b$,*

$$\mathbb{P}(a \leq X \leq b) = F_X(b) - F_X(a) \tag{7}$$

Proof. This is a direct application of the Fundamental Theorem of Calculus. □

Proposition 1.1. *Let X be a continuous rrv on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with pdf f_X. Then, we have :*

- *f_X is nonnegative on $\mathbb{R}$:*

$$f_X(x) \geq 0, \quad \textit{for all } x \in \mathbb{R}; \tag{8}$$

- *f_X is integrable on $\mathbb{R}$ and*

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \tag{9}$$

Proof.

- Proof of (8) : Property 1.2 states that the cumulative distribution function F_X is increasing on $\mathbb{R}$. Therefore $F'_X(x) \geq 0$. According to Lemma 1.3, $F'_X(x) = f_X(x)$ if f_X is continuous at x. This completes the proof.

- Proof of (9) :

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \mathbb{P}(X \in (-\infty, \infty)) = \mathbb{P}(X \in \mathbb{R}) = \mathbb{P}(\Omega) = 1$$

  In other words, the event that X takes on some value ($X \in \mathbb{R}$) is the sure event Ω.

□

Definition 1.3. *A real-valued function f is said to be a valid pdf if the following holds :*

- *f is nonnegative on $\mathbb{R}$:*

$$f(x) \geq 0, \quad \textit{for all } x \in \mathbb{R}; \tag{10}$$

- *f is integrable on $\mathbb{R}$ and*

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \tag{11}$$

This means that if f is a valid pdf, then there exists some continuous rrv X that has f as its pdf

Example. A continuous rrv X is said to follow a **uniform distribution** on $[0, 1/2]$ if its pdf is :

$$f_X(x) = \begin{cases} c & \text{if } 0 \leq x \leq 1/2 \\ 0 & \text{otherwise} \end{cases} = c\mathbb{1}_{[0,1/2]}(x)$$

Questions.

(1) Determine c such that f_X satisfies the properties of a pdf.

(2) Give the cdf of X.

Proof. (1) Since f_X is a pdf, $f_X(x)$ should be nonnegative for all $x \in \mathbb{R}$. This is the case for $x \in (-\infty, 0)$ and $x \in (1/2, \infty)$ where $f_X(x)$ equals zero. On the interval $[0, 1/2]$, $f_X(x) = c$. This implies that c should be nonnegative as well.

Let us now focus on the second condition,

$$\begin{aligned} \int_{-\infty}^{\infty} f_X(x)\, dx = 1 &\Leftrightarrow \int_{-\infty}^{0} f_X(x)\, dx + \int_{0}^{1/2} f_X(x)\, dx + \int_{1/2}^{\infty} f_X(x)\, dx = 1 \\ &\Leftrightarrow \int_{-\infty}^{0} 0\, dx + \int_{0}^{1/2} c\, dx + \int_{1/2}^{\infty} 0\, dx = 1 \\ &\Leftrightarrow cx\Big|_{0}^{1/2} = 1 \\ &\Leftrightarrow c \cdot \frac{1}{2} = 1 \\ &\Leftrightarrow c = 2 \end{aligned}$$

And we check that indeed $c = 2$ is nonnegative.

(2) The cumulative distribution function F_X of X is piecewise like its pdf :

- If $x < 0$, then $f_X(t) = 0$ for all $t \in (-\infty, x]$.

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

- If $0 \le x \le 1/2$, then $f_X(t) = 2$ for all $t \in [0, x]$.

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(t)\,dt \\ &= \int_{-\infty}^{0} f_X(t)\,dt + \int_{0}^{x} f_X(t)\,dt \\ &= F_X(0) + \int_{0}^{x} 2\,dt \\ &= 0 + 2\,t\Big|_{0}^{x} \\ &= 2x \end{aligned}$$

- If $x > 1/2$, then $f_X(t) = 0$ for all $t \in [1/2, x]$.

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(t)\,dt \\ &= \int_{-\infty}^{1/2} f_X(t)\,dt + \int_{1/2}^{x} f_X(t)\,dt \\ &= F_X(1/2) + 0 \\ &= 2 \cdot \frac{1}{2} \\ &= 1 \end{aligned}$$

In a nutshell, F_X is given by :

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 2x & \text{if } 0 \le x \le 1/2 \\ 1 & \text{if } x > 1/2 \end{cases}$$

$\square$

Example. Let X be the duration of a telephone call in minutes and suppose X has pdf :

$$f_X(x) = c\,e^{-x/10}\mathbb{1}_{[0,\infty)}(x)$$

Questions.

(1) Which value(s) of c make(s) f_X a valid pdf?

Answer. $c = 1/10$.

(2) Find the probability that the call lasts less than 5 minutes.

Answer. $\mathbb{P}(X < 5) = 1 - e^{-1/2} \approx 0.393$.

2 Expectation and Variance

Definition 2.1 (Expected Value)**.** *Let X be a continuous rrv with pdf f_X. If $\int_{-\infty}^{\infty} x f_X(x)\,dx$ is absolutely convergent, i.e. $\int_{-\infty}^{\infty} |x| f_X(x)\,dx < \infty$[1], then, the **mathematical expectation** (or **expected value** or **mean**) of X exists, is denoted by $\mathbb{E}[X]$ and is defined as follows :*

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx \tag{12}$$

Definition 2.2 (Expected Value of a Function of a Random Variable)**.** *Let X be a continuous rrv with pdf f_X. Let $g : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous function. If random variable $g(X)$ is integrable. Then, the mathematical expectation of $g(X)$ exists, is denoted by $\mathbb{E}[g(X)]$ and is defined as follows :*

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx \tag{13}$$

Example. Compute the expectation of a continuous rrv X following a uniform distribution on $[0, 1/2]$. As seen earlier, its pdf is given by:

$$f_X(x) = \begin{cases} 2 & \text{if } 0 \le x \le 1/2 \\ 0 & \text{otherwise} \end{cases} = 2\ \mathbb{1}_{[0,1/2]}(x)$$

Proof. The expectation of X can be computed as follows :

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\,dx \\
&= \int_{-\infty}^{0} x f_X(x)\,dx + \int_{0}^{1/2} x f_X(x)\,dx + \int_{1/2}^{\infty} x f_X(x)\,dx \\
&= 0 + \int_{0}^{1/2} x\ 2\,dx + 0 \\
&= x^2 \Big|_{0}^{1/2} \\
&= \frac{1}{4}
\end{aligned}$$

□

Property 2.1. *Let X be a continuous rrv with pdf f_X.*

- *for all $c \in \mathbb{R}$,*

$$\mathbb{E}[c] = c \tag{14}$$

[1] We then say that X is **integrable**.

- *If $g : \mathbb{R} \to \mathbb{R}$ is a nonnegative piecewise continuous function and $g(X)$ is integrable. Then, we have :*

$$\mathbb{E}[g(X)] \geq 0 \tag{15}$$

- *If $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ are piecewise continuous functions and $g_1(X)$ and $g_2(X)$ are integrable such that $g_1 \leq g_2$. Then, we have :*

$$\mathbb{E}[g_1(X)] \leq \mathbb{E}[g_2(X)] \tag{16}$$

Proof.

- Proof of Equation (14) : Here we consider the function of the random variable X defined by : $g(X) = c$. We then get :

$$\begin{aligned}\mathbb{E}[c] &= \int_{-\infty}^{\infty} c f_X(x)\,dx \\ &= c \underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx}_{=1 \text{ since } f_X \text{ is a pdf}} \\ &= c\end{aligned}$$

- Proof of Equation (15) : This comes from the nonnegativity of the integral for nonnegative functions.

- Proof of Equation (16) : This is a direct application of Equation (15) applied to function $g_2 - g_1$.

□

Property 2.2 (Linearity of Expectation)**.** *Let X be a continuous rrv with pdf f_X. If $c_1, c_2 \in \mathbb{R}$ and $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ are piecewise continuous functions and $g_1(X)$ and $g_2(X)$ are integrable. Then, we have :*

$$\mathbb{E}[c_1 g_1(X) + c_2 g_2(X)] = c_1 \mathbb{E}[g_1(X)] + c_2 \mathbb{E}[g_2(X)] \tag{17}$$

Note that Equation (17) can be extended to an arbitrary number of piecewise continuous functions.

Proof.

$$\begin{aligned}\mathbb{E}[c_1 g_1(X) + c_2 g_2(X)] &= \int_{-\infty}^{\infty} (c_1 g_1(x) + c_2 g_2(x)) f_X(x)\,dx \\ &= c_1 \int_{-\infty}^{\infty} g_1(x) f_X(x)\,dx + c_2 \int_{-\infty}^{\infty} g_2(x) f_X(x)\,dx \\ &= c_1 \mathbb{E}[g_1(X)] + c_2 \mathbb{E}[g_2(X)]\end{aligned}$$

□

Definition 2.3 (Variance–Standard Deviation). *Let X be a real-valued random variable. When $\mathbb{E}[X^2]$ exists*[2]*, the **variance** of X is defined as follows :*

$$Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] \tag{18}$$

*$Var(X)$ is sometimes denoted σ_X^2. The positive square root of the variance is called the **standard deviation** of X, and is denoted σ_X. That is:*

$$\sigma_X = \sqrt{Var(X)} \tag{19}$$

Property 2.3. *The variance of a real-valued random variable X satisfies the following properties :*

- *$Var(X) \geq 0$*
- *If $a, b \in \mathbb{R}$ are two constants, then $Var(aX + b) = a^2\, Var(X)$*

Proof. This property is true for any kind of random variables (discrete or continuous). See proof of Property 4.1 given in the lecture notes of the chapter about discrete rrvs. □

Theorem 2.1 (König-Huygens formula). *Let X be a real-valued random variable. When $\mathbb{E}[X^2]$ exists, the **variance** of X is also given by :*

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \tag{20}$$

Proof. This property is true for any kind of random variables (discrete or continuous). See proof of Theorem 4.1 given in the lecture notes of the chapter about discrete rrvs. □

In the case of continuous random variables, Equation (20) becomes :

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx - \left(\int_{-\infty}^{\infty} x f_X(x)\, dx \right)^2$$

Example. Compute the variance of a continuous rrv X following a uniform distribution on $[0, 1/2]$. As seen earlier, its pdf is given by:

$$f_X(x) = 2\ \mathbb{1}_{[0,1/2]}(x)$$

[2]We then say that X is square integrable.

Proof. We computed previously the expectation of X that is $\mathbb{E}[X] = 1/4$. Computing the variance of X thus boils down to calculating $\mathbb{E}[X^2]$:

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx \\
&= \int_{-\infty}^{0} x^2 f_X(x)\,dx + \int_{0}^{1/2} x^2 f_X(x)\,dx + \int_{1/2}^{\infty} x^2 f_X(x)\,dx \\
&= 0 + \int_{0}^{1/2} x^2 \; 2\,dx + 0 \\
&= 2 \cdot \frac{1}{3} x^3 \Big|_0^{1/2} \\
&= \frac{1}{12}
\end{aligned}$$

Therefore,

$$\mathrm{Var}(X) = \frac{1}{12} - \left(\frac{1}{4}\right)^2 = \frac{1}{48}$$

□

3 Common Continuous Distributions

3.1 Uniform Distribution

Definition 3.1. *A continuous rrv is said to follow a* ***uniform distribution*** $\mathcal{U}(a, b)$ *on a segment* $[a, b]$*, with* $a < b$*, if its pdf is*

$$f_X(x) = \frac{1}{b-a} \mathbb{1}_{[a,b]}(x) \tag{21}$$

Proof. Let us prove that the pdf of a uniform distribution is actually a valid pdf :

- Is $f_X(x) \geq 0$ for $x \in \mathbb{R}$?
  - If $x < a$ or $x > b$, then $f_X(x) = 0 \geq 0$.
  - If $x \in [a, b]$, then $f_X(x) = 1/(b-a) > 0$ since $a < b$.
- Let us show that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$:

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x)\,dx &= \int_{-\infty}^{a} f_X(x)\,dx + \int_{a}^{b} f_X(x)\,dx + \int_{b}^{\infty} f_X(x)\,dx \\
&= 0 + \int_{a}^{b} \frac{1}{b-a}\,dx + 0 \\
&= \frac{1}{b-a}\; x \Big|_a^b \\
&= 1
\end{aligned}$$

$\square$

Property 3.1 (Mean and Variance for a Uniform Distribution)**.** *If X follows a uniform distribution $\mathcal{U}(a,b)$, then*

- *its expected value is given by :*

$$\mathbb{E}[X] = \frac{a+b}{2} \tag{22}$$

- *its variance is given by :*

$$Var(X) = \frac{(b-a)^2}{12} \tag{23}$$

Proof.

- Expectation :

$$\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\
&= \frac{1}{b-a} \int_a^b x\, dx \\
&= \frac{1}{b-a} \left. \frac{1}{2} x^2 \right|_a^b \\
&= \frac{1}{2} \frac{b^2-a^2}{b-a} \\
&= \frac{a+b}{2}
\end{aligned}$$

- Variance : Let us first compute $\mathbb{E}[X^2]$

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \\
&= \frac{1}{b-a} \int_a^b x^2\, dx \\
&= \frac{1}{b-a} \left. \frac{1}{3} x^3 \right|_a^b \\
&= \frac{1}{3} \frac{b^3-a^3}{b-a} \\
&= \frac{b^2+ab+a^2}{3}
\end{aligned}$$

Then using the shortcut formula (20), we find :

$$\mathrm{Var}(X) = \frac{b^2+ab+a^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}$$

$\square$

Motivation. Most computer programming languages include functions or library routines that provide random number generators. They are often designed to provide a random byte or word, or a floating point number uniformly distributed between 0 and 1.

The quality i.e. randomness of such library functions varies widely from completely predictable output, to cryptographically secure.

There are a couple of methods to generate a random number based on a probability density function. These methods involve transforming a uniform random number in some way. Because of this, these methods work equally well in generating both pseudo-random and true random numbers.

3.2 Gaussian Distribution

Example. A restaurant wants to advertise a new burger they call *The Quarter-kilogram.* Of course, none of their burgers will exactly weigh exactly 250 grams. However, you may expect that most of the burgers' weights will fall in some small interval centered around 250 grams.

Definition 3.2. *A continuous random variable is said to follow a* ***normal*** *(or* ***Gaussian****) distribution* $\mathcal{N}(\mu, \sigma^2)$ *with parameters, mean* μ *and variance* σ^2 *if its pdf* f_X *is given by:*

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad \text{for } x \in \mathbb{R} \tag{24}$$

We also say that X *is normally distributed or* X *is a normal (or Gaussian) rrv with parameters* μ *and* σ^2*.*

Property 3.2 (Mean and Variance for a Normal Distribution)**.** *If* X *follows a normal distribution* $\mathcal{N}(\mu, \sigma^2)$*, then*

- *its expected value is given by :*

$$\mathbb{E}[X] = \mu \tag{25}$$

- *its variance is given by :*

$$Var(X) = \sigma^2 \tag{26}$$

Remark. The normal distribution is one of the most (even perhaps the most) important distributions in Probability and Statistics. It allows to model many natural, physical and social phenomenons. We will see later in this course how all the distributions are somehow related to the normal distribution.

Property 3.3. *If* X *follows a normal distribution* $\mathcal{N}(\mu, \sigma^2)$*. Then its pdf* f_X *has the following properties:*

(1) f_X is symmetric about the mean μ :

$$f_X(\mu - x) = f_X(\mu + x), \quad \text{for } x \in \mathbb{R} \tag{27}$$

(2) f_X is maximized at $x = \mu$.

(3) The limit of $f_X(x)$, as x approches $-\infty$ or ∞, is 0:

$$\lim_{x\to-\infty} f_X(x) = \lim_{x\to\infty} f_X(x) = 0 \tag{28}$$

Definition 3.3 (Standard Normal Distribution)**.** *We say that a continuous rrv X follows a **standard normal** distribution if X follows a normal distribution $\mathcal{N}(0,1)$ with mean 0 and variance 1. We also say that X is a standard normal random variable. In particular, the cdf of a standard normal random variable is denoted Φ, that is :*

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt, \quad \text{for } x \in \mathbb{R} \tag{29}$$

Property 3.4. *The cdf of a standard normal random variable satisfies the following property:*

$$\Phi(-z) = 1 - \Phi(z), \quad \text{for } z \in \mathbb{R} \tag{30}$$

Property 3.5. *Let $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$. If X follows a normal distribution $\mathcal{N}(\mu, \sigma^2)$, then random variable $aX + b$ follows a normal distribution $\mathcal{N}(a\mu + b, a^2\sigma^2)$.*

Proof. Let us assume that $a > 0$. Consider random variable $Y = aX+b$. We will prove that F_Y, the cdf of Y is the cdf of a normal distribution $\mathcal{N}(a\mu+b, a^2\sigma^2)$.

For $y \in \mathbb{R}$, we have that :

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(aX + b \leq y) \\ &= \mathbb{P}\left(X \leq \frac{y-b}{a}\right) \\ &= F_X\left(\frac{y-b}{a}\right) \\ &= \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx \end{aligned}$$

Now, consider the change of variable : $t = ax + b \Rightarrow dt = a\, dx$. Hence, we have :

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{y} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\frac{t-b}{a} - \mu}{\sigma}\right)^2\right\} \frac{dt}{a} \\ &= \int_{-\infty}^{y} \frac{1}{a\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{t-(a\mu+b)}{a\sigma}\right)^2\right\} dt \end{aligned}$$

We now recognize the cdf of a normal distribution $\mathcal{N}(a\mu + b, a^2\sigma^2)$, which completes the proof.

The proof in the case where a is negative is left as an exercise. $\square$

Corollary 3.1. *If X follows a normal distribution $\mathcal{N}(\mu, \sigma^2)$, then random variable Z defined by:*

$$Z = \frac{X - \mu}{\sigma}$$

is a standard normal random variable.

Finding Normal probabilities. If $X \sim \mathcal{N}(\mu, \sigma^2)$. The above property leads us to the following strategy for finding probabilities $\mathbb{P}(a \leq X \leq b)$:

(1) Transform X, a, and b, by:

$$Z = \frac{X - \mu}{\sigma}$$

(2) Use the standard normal $\mathcal{N}(0, 1)$ Table to find the desired probability.

Example. Let X be the weight of a so-called 'quarter-kilogram' burger. Assume X follows a normal distribution with mean 250 grams and standard deviation 15 grams.

(a) What is the probability that a randomly selected burger has a weight below 240 grams?

(b) What is the probability that a randomly selected burger has a weight above 270 grams?

(c) What is the probability that a randomly selected burger has a weight between 230 and 265 grams?

3.3 Quantiles

Previously, we learned how to use the standard normal curve $\mathcal{N}(0, 1)$ to find probabilities concerning a normal random variable X. Now, what would we do if we wanted to find some range of values for X in order to reach some probability?

Definition 3.4 (Quantile)**.** *Let X be a rrv with cumulative distribution function F_X and $\alpha \in [0, 1]$. A quantile of order α for the distribution of X, denoted q_α, is defined as follows :*

$$q_\alpha = \inf \{x \in \mathbb{R} \; | F_X(x) \geq \alpha \; \} \tag{31}$$

Remarks :

- The quantile of order 1/2 is called the median of the distribution
- If F_X is a bijective function, then $q_\alpha = F_X^{-1}(\alpha)$

Property 3.6. *For $\alpha \in (0, 1)$, the quantile function of a standard normal random variable is given by :*

$$\Phi^{-1}(\alpha) = -\Phi^{-1}(1 - \alpha) \tag{32}$$

Finding the Quantiles of a Normal Distribution. In order to find the value of a normal random variable $X \sim \mathcal{N}(\mu, \sigma^2)$:

(1) Find in the Table the $z = (x - \mu)/\sigma$ value associated with the desired probability.

(2) Use the transformation $x = \mu + z\sigma$.

Example. Suppose X, the grade on a midterm exam, is normally distributed with mean 70 and standard deviation 10.

(a) The instructor wants to give 15% of the class an A. What cutoff should the instructor use to determine who gets an A?

(b) The instructor now wants to give the next 10% of the class an A-. For which range of grades should the instructor assign an A-?

3.4 Exponential Distribution

Motivating example. Let Y be the discrete rrv equal to the number of people joining the line to visit the Eiffel Tower in an interval of one hour. If λ, the mean number of people arriving in an interval of one hour, is 800, we are interested in the continuous random variable X, the waiting time until the first visitor arrives.

Definition 3.5. *A continuous random variable is said to follow an exponential distribution $\mathcal{E}(\lambda)$ with $\lambda > 0$ if its pdf f_X is given by:*

$$f_X(x) = \lambda e^{-\lambda x} \mathbb{1}_{[0,\infty)}(x) \tag{33}$$

Property 3.7 (Mean and Variance for an Exponential Distribution)**.** *If X follows an exponential distribution $\mathcal{E}(\lambda)$, then*

- *its expected value is given by :*

$$\mathbb{E}[X] = \frac{1}{\lambda} \tag{34}$$

- *its variance is given by :*

$$Var(X) = \frac{1}{\lambda^2} \tag{35}$$

Example. Suppose that people join the line to visit the Eiffel Tower according to an approximate Poisson process at a mean rate of 800 visitors per hour. What is the probability that nobody joins the line in the next 30 seconds?