Gödel's Incompleteness theorems

This paper is taken from

- *Kurt Gödel Collected Works Volume 1: Publications 1919-1936*,
  Edited by Solomon Feferman et al.
  Oxford University Press, 2001.
  Pages 144-195 (odd pages only).

The translation is by Jean van Heijenoort.

On formally undecidable propositions of *Principia mathematica* and related systems I[1] (*1931*)

1

The development of mathematics toward greater precision has led, as is well known, to the formalization of large tracts of it, so that one can prove any theorem using nothing but a few mechanical rules. The most comprehensive formal systems that have been set up hitherto are the system of *Principia mathematica* (*PM*)[2] on the one hand and the Zermelo–Fraenkel axiom system of set theory (further developed by J. von Neumann)[3] on the other. These two systems are so comprehensive that in them all methods of proof today used in mathematics are formalized, that is, reduced to a few axioms and rules of inference. One might therefore conjecture that these axioms and rules of inference are sufficient to decide *any* mathematical question that can at all be formally expressed in these systems. It will be shown below that this is not the case, that on the contrary there are in the two systems mentioned relatively simple problems in the theory of integers[4] that cannot be decided on the basis of the axioms. This situation is not in any way due to the special nature of the systems that have been set up, but holds for a wide class of formal systems; among these, in particular, are all systems that result from the two just mentioned through the addition of a finite number of axioms,[5] provided no false propositions

[1] See a summary of the results of the present paper in *Gödel 1930b*.

[2] *Whitehead and Russell 1925*. Among the axioms of the system *PM* we include also the axiom of infinity (in this version: there are exactly denumerably many individuals), the axiom of reducibility, and the axiom of choice (for all types).

[3] See *Fraenkel 1927* and *von Neumann 1925*, *1928a*, and *1929*. We note that in order to complete the formalization we must add the axioms and rules of inference of the calculus of logic to the set-theoretic axioms given in the literature cited. The considerations that follow apply also to the formal systems (so far as they are available at present) constructed in recent years by Hilbert and his collaborators. See *Hilbert 1922, 1923, 1928*, *Bernays 1923*, *von Neumann 1927*, and *Ackermann 1924*.

[4] That is, more precisely, there are undecidable propositions in which, besides the logical constants $\overline{}$ (not), $\vee$ (or), (x) (for all), and $=$ (identical with), no other notions occur but $+$ (addition) and $\cdot$ (multiplication), both for natural numbers, and in which the quantifiers (x), too, apply to natural numbers only.

[5] In *PM* only axioms that do not result from one another by mere change of type are

of the kind specified in footnote 4 become provable owing to the added axioms.

Before going into details, we shall first sketch the main idea of the proof, of course without any claim to complete precision. The formulas of a formal system (we restrict ourselves here to the system *PM*) in outward appearance are finite sequences of primitive signs (variables, logical constants, and parentheses or punctuation dots), and it is easy to state with complete precision *which* sequences of primitive signs are meaningful formulas and which are not.[6] Similarly, proofs, from a formal point of view, are nothing but finite sequences of formulas (with certain specifiable properties). Of course, for metamathematical considerations it does not matter what objects are chosen as primitive signs, and we shall assign natural numbers to this use.[7] Consequently, a formula will be a finite sequence of natural numbers,[8] and a proof array a finite sequence of finite sequences of natural numbers. The metamathematical notions (propositions) thus become notions (propositions) about natural numbers or sequences of them;[9] therefore they can (at least in part) be expressed by the symbols of the system *PM* itself. In particular, it can be shown that the notions "formula", "proof array", and "provable formula" can be defined in the system *PM*; that is, we can, for example, find a formula $F(v)$ of *PM* with one free variable v (of the type of a number sequence)[10] such that $F(v)$, interpreted according to the meaning of the terms of *PM*, says: v is a provable formula. We now construct an undecidable proposition of the system *PM*, that is, a proposition A for which neither A nor *not-A* is provable, in the following manner.

A formula of *PM* with exactly one free variable, that variable being of the type of the natural numbers (class of classes), will be called a *class sign*. We assume that the class signs have been arranged in a sequence

[6] Here and in what follows we always understand by "formula of *PM*" a formula written without abbreviations (that is, without the use of definitions). It is well known that [in *PM*] definitions serve only to abbreviate notations and therefore are dispensable in principle.

[7] That is, we map the primitive signs one-to-one onto some natural numbers. (See how this is done on page 157.)

[8] That is, a number-theoretic function defined on an initial segment of the natural numbers. (Numbers, of course, cannot be arranged in a spatial order.)

[9] In other words, the procedure described above yields an isomorphic image of the system *PM* in the domain of arithmetic, and all metamathematical arguments can just as well be carried out in this isomorphic image. This is what we do below when we sketch the proof; that is, by "formula", "proposition", "variable", and so on, *we must always understand the corresponding objects of the isomorphic image.*

[10] It would be very easy (although somewhat cumbersome) to actually write down this formula.

in some way,[11] we denote the nth one by $R(n)$, and we observe that the notion "class sign", as well as the ordering relation R, can be defined in the system *PM*. Let α be any class sign; by $[\alpha; n]$ we denote the formula that results from the class sign α when the free variable is replaced by the sign denoting the natural number n. The ternary relation $x = [y; z]$, too, is seen to be definable in *PM*. We now define a class K of natural numbers in the following way:

$$n \in K \equiv \overline{Bew}[R(n); n] \tag{1}$$

(where $Bew\ x$ means: x is a provable formula).[11a] Since the notions that occur in the definiens can all be defined in *PM*, so can the notion K formed from them; that is, there is a class sign S such that the formula $[S; n]$, interpreted according to the meaning of the terms of *PM*, states that the natural number n belongs to K.[12] Since S is a class sign, it is identical with some $R(q)$; that is, we have

$$S = R(q)$$

for a certain natural number q. We now show that the proposition $[R(q); q]$ is undecidable in *PM*.[13] For let us suppose that the proposition $[R(q); q]$ were provable; then it would also be true. But in that case, according to the definitions given above, q would belong to K, that is, by (1), $\overline{Bew}[R(q); q]$ would hold, which contradicts the assumption. If, on the other hand, the negation of $[R(q); q]$ were provable, then $\overline{q \in K}$, that is, $Bew[R(q); q]$, would hold. But then $[R(q); q]$, as well as its negation, would be provable, which again is impossible.

The analogy of this argument with the Richard antinomy leaps to the eye. It is closely related to the "Liar" too;[14] for the undecidable proposition $[R(q); q]$ states that q belongs to K, that is, by (1), that $[R(q); q]$ is not provable. We therefore have before us a proposition that says about itself

[11] For example, by increasing sum of the finite sequence of integers that is the "class sign", and lexicographically for equal sums.

[11a] The bar denotes negation.

[12] Again, there is not the slightest difficulty in actually writing down the formula S.

[13] Note that "$[R(q); q]$" (or, which means the same, "$[S; q]$") is merely a *metamathematical description* of the undecidable proposition. But, as soon as the formula S has been obtained, we can, of course, also determine the number q and, therewith, actually write down the undecidable proposition itself. [This makes no difficulty in principle. However, in order not to run into formulas of entirely unmanageable lengths and to avoid practical difficulties in the computation of the number q, the construction of the undecidable proposition would have to be slightly modified, unless the technique of abbreviation by definition used throughout in *PM* is adopted.]

[14] Any epistemological antinomy could be used for a similar proof of the existence of undecidable propositions.

that it is not provable [in *PM*].[15] The method of proof just explained can clearly be applied to any formal system that, first, when interpreted as representing a system of notions and propositions, has at its disposal sufficient means of expression to define the notions occurring in the argument above (in particular, the notion "provable formula") and in which, second, every provable formula is true in the interpretation considered. The purpose of carrying out the above proof with full precision in what follows is, among other things, to replace the second of the assumptions just mentioned by a purely formal and much weaker one.

From the remark that $[R(q);q]$ says about itself that it is not provable, it follows at once that $[R(q);q]$ is true, for $[R(q);q]$ *is* indeed unprovable (being undecidable). Thus, the proposition that is undecidable *in the system PM* still was decided by metamathematical considerations. The precise analysis of this curious situation leads to surprising results concerning consistency proofs for formal systems, results that will be discussed in more detail in Section 4 (Theorem XI).

2

We now proceed to carry out with full precision the proof sketched above. First we give a precise description of the formal system P for which we intend to prove the existence of undecidable propositions. P is essentially the system obtained when the logic of *PM* is superposed upon the Peano axioms[16] (with the numbers as individuals and the successor relation as primitive notion).

The primitive signs of the system P are the following:

I. Constants: "$\sim$" (not), "$\vee$" (or), "Π" (for all), "0" (zero), "f" (the successor of), "(", ")" (parentheses).

II. Variables of type 1 (for individuals, that is, natural numbers including 0): "x_1", "y_1", "z_1", ...

Variables of type 2 (for classes of individuals): "x_2", "y_2", "z_2", ...

Variables of type 3 (for classes of classes of individuals): "x_3", "y_3", "z_3", ...

[15] Contrary to appearances, such a proposition involves no faulty circularity, for initially it [only] asserts that a certain well-defined formula (namely, the one obtained from the qth formula in the lexicographic order by a certain substitution) is unprovable. Only subsequently (and so to speak by chance) does it turn out that this formula is precisely the one by which the proposition itself was expressed.

[16] The addition of the Peano axioms, as well as all other modifications introduced in the system *PM*, merely serves to simplify the proof and is dispensable in principle.

And so on, for every natural number as a type.[17]

Remark: Variables for functions of two or more argument places (relations) need not be included among the primitive signs, since we can define relations to be classes of ordered pairs, and ordered pairs to be classes of classes; for example, the ordered pair a, b can be defined to be $((a), (a, b))$, where (x, y) denotes the class whose sole elements are x and y, and (x) the class whose sole element is x.[18]

By a *sign of type* 1 we understand a combination of signs that has [any one of] the forms

$$a,\ fa,\ ffa,\ fffa, \ldots, \quad \text{and so on,}$$

where a is either 0 or a variable of type 1. In the first case, we call such a sign a *numeral*. For $n > 1$ we understand by a *sign of type* n the same thing as by a *variable of type* n. A combination of signs that has the form $a(b)$, where b is a sign of type n and a is a sign of type $n+1$, will be called an *elementary formula*. We define the class of *formulas* to be the smallest class[19] containing all elementary formulas and containing $\sim(a)$, $(a) \vee (b)$, $x\Pi(a)$ (where x may be any variable)[19a] whenever it contains a and b. We call $(a) \vee (b)$ the *disjunction* of a and b, $\sim(a)$ the *negation* and $x\Pi(a)$ a *generalization* of a. A formula in which no free variable occurs (*free variable* being defined in the well-known manner) is called a *sentential formula*. A formula with exactly n free individual variables (and no other free variables) will be called an *n-place relation sign*; for $n = 1$ it will also be called a *class sign*.

By Subst $a\binom{v}{b}$ (where a stands for a formula, v for a variable, and b for a sign of the same type as v) we understand the formula that results from a if in a we replace v, wherever it is free, by b.[20] We say that a formula a is a *type elevation* of another formula b if a results from b when the type of each variable occurring in b is increased by the same number.

[17]It is assumed that we have denumerably many signs at our disposal for each type of variables.

[18]Nonhomogeneous relations, too, can be defined in this manner; for example, a relation between individuals and classes can be defined to be a class of elements of the form $((x_2), ((x_1), x_2))$. Every proposition about relations that is provable in *PM* is provable also when treated in this manner, as is readily seen.

[19]Concerning this definition (and similar definitions occurring below) see *Łukasiewicz and Tarski 1930*.

[19a]Hence $x\Pi(a)$ is a formula even if x does not occur in a or is not free in a. In this case, of course, $x\Pi(a)$ means the same thing as a.

[20]In case v does not occur in a as a free variable we put Subst $a\binom{v}{b} = a$. Note that "Subst" is a metamathematical sign.

The following formulas (I–V) are called *axioms* (we write them using the abbreviations ., $\supset$, $\equiv$, (Ex), $=$,[21] defined in the well-known manner, and observing the usual conventions about omitting parentheses):[22]

I. 1. $\sim(fx_1 = 0)$,

2. $fx_1 = fy_1 \supset x_1 = y_1$,

3. $x_2(0) \,.\, x_1\Pi(x_2(x_1) \supset x_2(fx_1)) \supset x_1\Pi(x_2(x_1))$.

II. All formulas that result from the following schemata by substitution of any formulas whatsoever for p, q, r:

1. $p \vee p \supset p$,
2. $p \supset p \vee q$,
3. $p \vee q \supset q \vee p$,
4. $(p \supset q) \supset (r \vee p \supset r \vee q)$.

III. Any formula that results from either one of the two schemata

1. $v\Pi(a) \supset \text{Subst } a\binom{v}{c}$,

2. $v\Pi(b \vee a) \supset b \vee v\Pi(a)$

when the following substitutions are made for a, v, b, and c (and the operation indicated by "Subst" is performed in 1):

For a any formula, for v any variable, for b any formula in which v does not occur free, and for c any sign of the same type as v, provided c does not contain any variable that is bound in a at a place where v is free.[23]

IV. Every formula that results from the schema

1. $(Eu)(v\Pi(u(v) \equiv a))$

when for v we substitute any variable of type n, for u one of type $n + 1$, and for a any formula that does not contain u free. This axiom plays the role of the axiom of reducibility (the comprehension axiom of set theory).

V. Every formula that results from

1. $x_1\Pi(x_2(x_1) \equiv y_2(x_1)) \supset x_2 = y_2$

by type elevation (as well as this formula itself). This axiom states that a class is completely determined by its elements.

A formula c is called an *immediate consequence* of a and b if a is the formula $(\sim(b)) \vee (c)$, and it is called an *immediate consequence* of a if it is the formula $v\Pi(a)$, where v denotes any variable. The class of *provable formulas* is defined to be the smallest class of formulas that contains the

[21] $x_1 = y_1$ is to be regarded as defined by $x_2\Pi(x_2(x_1) \supset x_2(y_1))$, as in *PM*, I, *13 (similarly for higher types).

[22] In order to obtain the axioms from the schemata listed we must therefore
(1) eliminate the abbreviations and
(2) add the omitted parentheses
(in II, III, and IV after carrying out the substitutions allowed).
Note that all expressions thus obtained are "formulas" in the sense specified above. (See also the exact definitions of the metamathematical notions on pp. 163ff.)

[23] Therefore c is a variable or 0 or a sign of the form $f \ldots fu$, where u is either 0 or a variable of type 1. Concerning the notion "free (bound) at a place in a", see I, A5 in *von Neumann 1927*.

axioms and is closed under the relation "immediate consequence".[24]

We now assign natural numbers to the primitive signs of the system P by the following one-to-one correspondence:

$$\begin{array}{lll} \text{"}0\text{"} \ldots 1 & \text{"}f\text{"} \ldots 3 & \text{"}\sim\text{"} \ldots 5 \\ \text{"}\vee\text{"} \ldots 7 & \text{"}\Pi\text{"} \ldots 9 & \text{"(\,"} \ldots 11 \\ & & \text{"\,)"} \ldots 13, \end{array}$$

further to the variables of type n the numbers of the form p^n (where p is a prime number > 13). Thus we have a one-to-one correspondence by which a finite sequence of natural numbers is associated with every finite sequence of primitive signs (hence also with every formula). We now map the finite sequences of natural numbers on natural numbers (again by a one-to-one correspondence), associating the number $2^{n_1} \cdot 3^{n_2} \cdot \ldots \cdot p_k^{n_k}$, where p_k denotes the kth prime number (in order of increasing magnitude), with the sequence $n_1, n_2, \ldots, n_k$. A natural number ⟦out of a certain subset⟧ is thus assigned one-to-one not only to every primitive sign but also to every finite sequence of such signs. We denote by $\Phi(a)$ the number assigned to the primitive sign (or to the sequence of primitive signs) a. Now let some relation (or class) $R(a_1, a_2, \ldots, a_n)$ between [or of] primitive signs or sequences of primitive signs be given. With it we associate the relation (or class) $R'(x_1, x_2, \ldots, x_n)$ between [or of] natural numbers that obtains between $x_1, x_2, \ldots, x_n$ if and only if there are some $a_1, a_2, \ldots, a_n$ such that $x_i = \Phi(a_i)$ $(i = 1, 2, \ldots, n)$ and $R(a_1, a_2, \ldots, a_n)$ holds. The relations between (or classes of) natural numbers that in this manner are associated with the metamathematical notions defined so far, for example, "variable", "formula", "sentential formula", "axiom", "provable formula", and so on, will be denoted by the same words in SMALL CAPITALS. The proposition that there are undecidable problems in the system P, for example, reads thus: There are SENTENTIAL FORMULAS a such that neither a nor the NEGATION of a is a PROVABLE FORMULA.

We now insert a parenthetic consideration that for the present has nothing to do with the formal system P. First we give the following definition: A number-theoretic function[25] $\phi(x_1, x_2, \ldots, x_n)$ is said to be *recursively defined in terms of* the number-theoretic functions $\psi(x_1, x_2, \ldots, x_{n-1})$ and

[24] The rule of substitution is rendered superfluous by the fact that all possible substitutions have already been carried out in the axioms themselves. (This procedure was used also in *von Neumann 1927*.)

[25] That is, its domain of definition is the class of nonnegative integers (or of n-tuples of non-negative integers) and its values are nonnegative integers.

$\mu(x_1, x_2, \ldots, x_{n+1})$ if

$$\begin{aligned}\phi(0, x_2, \ldots, x_n) &= \psi(x_2, \ldots, x_n),\\ \phi(k+1, x_2, \ldots, x_n) &= \mu(k, \phi(k, x_2, \ldots, x_n), x_2, \ldots, x_n)\end{aligned} \qquad (2)$$

hold for all $x_2, \ldots, x_n, k$.[26]

A number-theoretic function ϕ is said to be *recursive* if there is a finite sequence of number-theoretic functions $\phi_1, \phi_2, \ldots, \phi_n$ that ends with ϕ and has the property that every function ϕ_k of the sequence is recursively defined in terms of two of the preceding functions, or results from any of the preceding functions by substitution,[27] or, finally, is a constant or the successor function $x+1$. The length of the shortest sequence of ϕ_i corresponding to a recursive function ϕ is called its *degree*. A relation $R(x_1, \ldots, x_n)$ between natural numbers is said to be *recursive*[28] if there is a recursive function $\phi(x_1, \ldots, x_n)$ such that, for all $x_1, x_2, \ldots, x_n$,

$$R(x_1, \ldots, x_n) \sim [\phi(x_1, \ldots, x_n) = 0].^{29}$$

The following theorems hold:

I. *Every function (relation) obtained from recursive functions (relations) by substitution of recursive functions for the variables is recursive; so is every function obtained from recursive functions by recursive definition according to Schema (2).*

II. *If R and S are recursive relations, so are $\overline{R}$ and $R \vee S$ (hence also $R \,\&\, S$).*

III. *If the functions $\phi(\mathfrak{x})$ and $\psi(\mathfrak{y})$ are recursive, so is the relation $\phi(\mathfrak{x}) = \psi(\mathfrak{y})$.*[30]

IV. *If the function $\phi(\mathfrak{x})$ and the relation $R(x, \mathfrak{y})$ are recursive, so are the relations S and T defined by*

$$S(\mathfrak{x}, \mathfrak{y}) \sim (Ex)[x \leq \phi(\mathfrak{x}) \,\&\, R(x, \mathfrak{y})]$$

[26] In what follows, lower-case italic letters (with or without subscripts) are always variables for nonnegative integers (unless the contrary is expressly noted).

[27] More precisely, by substitution of some of the preceding functions at the argument places of one of the preceding functions, for example, $\phi_k(x_1, x_2) = \phi_p[\phi_q(x_1, x_2), \phi_r(x_2)]$ $(p, q, r < k)$. Not all variables on the left side need occur on the right side (the same applies to the recursion schema (2)).

[28] We include classes among relations (as one-place relations). Recursive relations R, of course, have the property that for every given n-tuple of numbers it can be decided whether $R(x_1, \ldots, x_n)$ holds or not.

[29] Whenever formulas are used to express a meaning (in particular, in all formulas expressing metamathematical propositions or notions), Hilbert's symbolism is employed. See *Hilbert and Ackermann 1928.*

[30] We use German letters, $\mathfrak{x}, \mathfrak{y}$, as abbreviations for arbitrary n-tuples of variables, for example, $x_1, x_2, \ldots, x_n$.

and

$$T(\mathfrak{x},\mathfrak{y}) \sim (x)[x \le \phi(\mathfrak{x}) \rightarrow R(x,\mathfrak{y})],$$

as well as the function ψ defined by

$$\psi(\mathfrak{x},\mathfrak{y}) = \varepsilon x[x \le \phi(\mathfrak{x}) \,\&\, R(x,\mathfrak{y})],$$

where $\varepsilon x F(x)$ means the least number x for which $F(x)$ holds and 0 in case there is no such number.

Theorem I follows at once from the definition of "recursive". Theorems II and III are consequences of the fact that the number-theoretic functions

$$\alpha(x), \quad \beta(x,y), \quad \gamma(x,y),$$

corresponding to the logical notions $\overline{}$, $\vee$, and $=$, namely,

$$\alpha(0) = 1, \;\; \alpha(x) = 0 \quad \text{for} \quad x \neq 0,$$
$$\beta(0,x) = \beta(x,0) = 0, \;\; \beta(x,y) = 1 \text{ when } x \text{ and } y \text{ are both} \neq 0,$$
$$\gamma(x,y) = 0 \text{ when } x = y, \;\; \gamma(x,y) = 1 \text{ when } x \neq y,$$

are recursive, as we can readily see. The proof of Theorem IV is briefly as follows. By assumption there is a recursive $\rho(x,\mathfrak{y})$ such that

$$R(x,\mathfrak{y}) \sim [\rho(x,\mathfrak{y}) = 0].$$

We now define a function $\chi(x,\mathfrak{y})$ by the recursion schema (2) in the following way:

$$\chi(0,\mathfrak{y}) = 0,$$

$$\chi(n+1,\mathfrak{y}) = (n+1)\cdot a + \chi(n,\mathfrak{y})\cdot \alpha(a),^{31}$$

where $a = \alpha[\alpha(\rho(0,\mathfrak{y}))]\cdot\alpha[\rho(n+1,\mathfrak{y})]\cdot\alpha[\chi(n,\mathfrak{y})]$.

Therefore $\chi(n+1,\mathfrak{y})$ is equal either to $n+1$ (if $a=1$) or to $\chi(n,\mathfrak{y})$ (if $a=0$).[32] The first case clearly occurs if and only if all factors of a are 1, that is, if

$$\overline{R}(0,\mathfrak{y}) \,\&\, R(n+1,\mathfrak{y}) \,\&\, [\chi(n,\mathfrak{y}) = 0]$$

holds.

[31]We assume familiarity with the fact that the functions $x+y$ (addition) and $x\cdot y$ (multiplication) are recursive.

[32]a cannot take values other than 0 and 1, as can be seen from the definition of α.

From this it follows that the function $\chi(n, \mathfrak{y})$ (considered as a function of n) remains 0 up to ⟦but not including⟧ the least value of n for which $R(n, \mathfrak{y})$ holds and, from there on, is equal to that value. (Hence, in case $R(0, \mathfrak{y})$ holds, $\chi(n, \mathfrak{y})$ is constant and equal to 0.) We have, therefore,

$$\psi(\mathfrak{x}, \mathfrak{y}) = \chi(\phi(\mathfrak{x}), \mathfrak{y}),$$

$$S(\mathfrak{x}, \mathfrak{y}) \sim R[\psi(\mathfrak{x}, \mathfrak{y}), \mathfrak{y}].$$

The relation T can, by negation, be reduced to a case analogous to that of S. Theorem IV is thus proved.

The functions $x + y$, $x \cdot y$, and x^y, as well as the relations $x < y$ and $x = y$, are recursive, as we can readily see. Starting from these notions, we now define a number of functions (relations) 1–45, each of which is defined in terms of preceding ones by the procedures given in Theorems I–IV. In most of these definitions several of the steps allowed by Theorems I–IV are condensed into one. Each of the functions (relations) 1–45, among which occur, for example, the notions "FORMULA", "AXIOM", and "IMMEDIATE CONSEQUENCE", is therefore recursive.

1. $x/y \equiv (Ez)[z \leq x \,\&\, x = y \cdot z]$,[33]
x is divisible by y.[34]

2. $\mathrm{Prim}(x) \equiv \overline{(Ez)}[z \leq x \,\&\, z \neq 1 \,\&\, z \neq x \,\&\, x/z] \,\&\, x > 1$,
x is a prime number.

3. $0 \,Pr\, x \equiv 0$,
$(n+1) \,Pr\, x \equiv \varepsilon y[y \leq x \,\&\, \mathrm{Prim}(y) \,\&\, x/y \,\&\, y > n \,Pr\, x]$,
$n \,Pr\, x$ is the nth prime number (in order of increasing magnitude) contained in x.[34a]

4. $0! \equiv 1$,
$(n+1)! \equiv (n+1) \cdot n!$.

5. $Pr(0) \equiv 0$,
$Pr(n+1) \equiv \varepsilon y[y \leq \{Pr(n)\}! + 1 \,\&\, \mathrm{Prim}(y) \,\&\, y > Pr(n)]$,
$Pr(n)$ is the nth prime number (in order of increasing magnitude).

6. $n \,Gl\, x \equiv \varepsilon y[y \leq x \,\&\, x/(n \,Pr\, x)^y \,\&\, \overline{x/(n \,Pr\, x)^{y+1}}]$,
$n \,Gl\, x$ is the nth term of the number sequence assigned to the number x (for $n > 0$ and n not greater than the length of this sequence).

7. $l(x) \equiv \varepsilon y[y \leq x \,\&\, y \,Pr\, x > 0 \,\&\, (y+1) \,Pr\, x = 0]$,

[33] The sign $\equiv$ is used in the sense of "equality by definition"; hence in definitions it stands for either $=$ or $\sim$ (otherwise, the symbolism is Hilbert's).

[34] Wherever one of the signs (x), (Ex), or εx occurs in the definitions below, it is followed by a bound on x. This bound merely serves to ensure that the notion defined is recursive (see Theorem IV). But in most cases the *extension* of the notion defined would not change if this bound were omitted.

[34a] For $0 < n \leq z$, when z is the number of distinct prime factors of x. Note that $n \,Pr\, x = 0$ for $n = z + 1$.

$l(x)$ is the length of the number sequence assigned to x.

8. $x * y \equiv \varepsilon z\{z \leq [Pr(l(x) + l(y))]^{x+y}$ &
$(n)[n \leq l(x) \rightarrow n\,Gl\,z = n\,Gl\,x]$ &
$(n)[0 < n \leq l(y) \rightarrow (n + l(x))\,Gl\,z = n\,Gl\,y]\}$,

$x * y$ corresponds to the operation of "concatenating" two finite number sequences.

9. $R(x) \equiv 2^x$,

$R(x)$ corresponds to the number sequence consisting of x alone (for $x > 0$).

10. $E(x) \equiv R(11) * x * R(13)$,

$E(x)$ corresponds to the operation of "enclosing within parentheses" (11 and 13 are assigned to the primitive signs "(" and ")", respectively).

11. $n \text{ Var } x \equiv (Ez)[13 < z \leq x \,\&\, \text{Prim}(z) \,\&\, x = z^n] \,\&\, n \neq 0$,

x is a VARIABLE OF TYPE n.

12. $\text{Var}(x) \equiv (En)[n \leq x \,\&\, n \text{ Var } x]$,

x is a VARIABLE.

13. $\text{Neg}(x) \equiv R(5) * E(x)$,

$\text{Neg}(x)$ is the NEGATION of x.

14. $x \text{ Dis } y \equiv E(x) * R(7) * E(y)$,

$x \text{ Dis } y$ is the DISJUNCTION of x and y.

15. $x \text{ Gen } y \equiv R(x) * R(9) * E(y)$,

$x \text{ Gen } y$ is the GENERALIZATION of y with respect to the VARIABLE x (provided x is a VARIABLE).

16. $0\,N\,x \equiv x$,
$(n+1)Nx \equiv R(3) * n\,N\,x$,

$n\,N\,x$ corresponds to the operation of "putting the sign 'f' n times in front of x".

17. $Z(n) \equiv n\,N\,[R(1)]$,

$Z(n)$ is the NUMERAL denoting the number n.

18. $\text{Typ}'_1(x) \equiv (Em, n)\{m, n \leq x \,\&\, [m = 1 \vee 1 \,\text{Var}\, m] \,\&$
$x = n\,N\,[R(m)]\}$,[34b]

x is a SIGN OF TYPE 1.

19. $\text{Typ}_n(x) \equiv [n = 1 \,\&\, \text{Typ}'_1(x)] \vee$
$[n > 1 \,\&\, (Ev)\{v \leq x \,\&\, n \,\text{Var}\, v \,\&\, x = R(v)\}]$,

x is a SIGN OF TYPE n.

20. $\mathit{Elf}(x) \equiv (Ey, z, n)[y, z, n \leq x \,\&\, \text{Typ}_n(y) \,\&\, \text{Typ}_{n+1}(z) \,\&$
$x = z * E(y)]$,

x is an ELEMENTARY FORMULA.

21. $\mathit{Op}(x, y, z) \equiv x = \text{Neg}(y) \vee x = y \text{ Dis } z \vee$
$(Ev)[v \leq x \,\&\, \text{Var}(v) \,\&\, x = v \text{ Gen } y]$.

[34b] $m, n \leq x$ stands for $m \leq x \,\&\, n \leq x$ (similarly for more than two variables).

22. $FR(x) \equiv (n)\{0 < n \leq l(x) \rightarrow Elf(n\,Gl\,x) \vee$
$(Ep, q)[0 < p, q < n \,\&\, Op(n\,Gl\,x, p\,Gl\,x, q\,Gl\,x)]\} \,\&$
$l(x) > 0,$

x is a SEQUENCE OF FORMULAS, each term of which either is an ELEMENTARY FORMULA or results from the preceding FORMULAS through the operations of NEGATION, DISJUNCTION, or GENERALIZATION.

23. $\text{Form}(x) \equiv (En)\{n \leq (Pr([l(x)]^2))^{x \cdot [l(x)]^2} \,\&$
$FR(n) \,\&\, x = [l(n)]\,Gl\,n\},$[35]

x is a FORMULA (that is, the last term of a FORMULA SEQUENCE n).

24. $v\,\text{Geb}\,n, x \equiv \text{Var}(v) \,\&\, \text{Form}(x) \,\&$
$(Ea, b, c)[a, b, c \leq x \,\&\, x = a * (v \text{ Gen } b) * c \,\&$
$\text{Form}(b) \,\&\, l(a) + 1 \leq n \leq l(a) + l(v \text{ Gen } b)],$

the VARIABLE v is BOUND in x at the nth place.

25. $v\,Fr\,n, x \equiv \text{Var}(v) \,\&\, \text{Form}(x) \,\&\, v = n\,Gl\,x \,\&\, n \leq l(x) \,\&\, \overline{v \text{ Geb } n, x},$

the VARIABLE v is FREE in x at the nth place.

26. $v\,Fr\,x \equiv (En)[n \leq l(x) \,\&\, v\,Fr\,n, x],$

v occurs as a FREE VARIABLE in x.

27. $Su\,x\binom{n}{y} \equiv \varepsilon z\{z \leq [Pr(l(x) + l(y))]^{x+y} \,\&\, (Eu, v)[u, v \leq x \,\&$
$x = u * R(n\,Gl\,x) * v \,\&\, z = u * y * v \,\&\, n = l(u) + 1]\},$

$Su\,x\binom{n}{y}$ results from x when we substitute y for the nth term of x (provided that $0 < n \leq l(x)$).

28. $0\,St\,v, x \equiv \varepsilon n\{n \leq l(x) \,\&\, v\,Fr\,n, x \,\&$
$\overline{(Ep)}[n < p \leq l(x) \,\&\, v\,Fr\,p, x]\},$
$(k+1)\,St\,v, x \equiv \varepsilon n\{n < k\,St\,v, x \,\&\, v\,Fr\,n, x \,\&$
$\overline{(Ep)}[n < p < k\,St\,v, x \,\&\, v\,Fr\,p, x]\},$

$k\,St\,v, x$ is the $(k+1)$th place in x (counted from the right end of the FORMULA x) at which v is FREE in x (and 0 in case there is no such place).

29. $A(v, x) \equiv \varepsilon n\{n \leq l(x) \,\&\, n\,St\,v, x = 0\},$

$A(v, x)$ is the number of places at which v is FREE in x.

30. $Sb_0(x^v_y) \equiv x,$
$Sb_{k+1}(x^v_y) \equiv Su[Sb_k(x^v_y)]\binom{k\,St\,v, x}{y}.$

31. $Sb(x^v_y) \equiv Sb_{A(v,x)}(x^v_y),$[36]

$Sb(x^v_y)$ is the notion SUBST $a\binom{v}{b}$ defined above.[37]

[35]That $n \leq (Pr([l(x)]^2))^{x \cdot [l(x)]^2}$ provides a bound can be seen thus: The length of the shortest formula sequence that corresponds to x can at most be equal to the number of subformulas of x. But there are at most $l(x)$ subformulas of length 1, at most $l(x) - 1$ of length 2, and so on, hence altogether at most $l(x)(l(x)+1)/2 \leq [l(x)]^2$. Therefore all prime factors of n can be assumed to be less than $Pr([l(x)]^2)$, their number $\leq [l(x)]^2$, and their exponents (which are subformulas of x) $\leq x$.

[36]In case v is not a VARIABLE or x is not a FORMULA, $Sb(x^v_y) = x$.

[37]Instead of $Sb[Sb(x^v_y)^w_z]$ we write $Sb(x^v_y\,{}^w_z)$ (and similarly for more than two VARIABLES).

32. $x \text{ Imp } y \equiv [\text{Neg}(x)] \text{ Dis } y$,
$x \text{ Con } y \equiv \text{Neg}\{[\text{Neg}(x)] \text{ Dis } [\text{Neg}(y)]\}$,
$x \text{ Aeq } y \equiv (x \text{ Imp } y) \text{ Con } (y \text{ Imp } x)$,
$v \text{ Ex } y \equiv \text{Neg}\{v \text{ Gen } [\text{Neg}(y)]\}$.

33. $$n\, Th x \equiv \varepsilon y\{y \leq x^{(x^n)} \,\&\, (k)[k \leq l(x) \rightarrow (k\, Gl\, x \leq 13 \,\&\, k\, Gl\, y = k\, Gl\, x) \vee (k\, Gl\, x > 13 \,\&\, k\, Gl\, y = k\, Gl\, x \cdot [1\, Pr\, (k\, Gl\, x)]^n)]\},$$

$n\, Th\, x$ is the nTH TYPE ELEVATION of x (in case x and $n\, Th\, x$ are FORMULAS).

Three specific numbers, which we denote by z_1, z_2, and z_3, correspond to the Axioms I, 1–3, and we define

34. $Z\text{-}Ax(x) \equiv (x = z_1 \vee x = z_2 \vee x = z_3)$.

35. $A_1\text{-}Ax(x) \equiv (Ey)[y \leq x \,\&\, \text{Form}(y) \,\&\, x = (y \text{ Dis } y) \text{ Imp } y]$,

x is a FORMULA resulting from Axiom Schema II, 1 by substitution. Analogously, $A_2\text{-}Ax$, $A_3\text{-}Ax$, and $A_4\text{-}Ax$ are defined for Axioms [rather, Axiom Schemata] II, 2–4.

36. $A\text{-}A(x) \equiv A_1\text{-}Ax(x) \vee A_2\text{-}Ax(x) \vee A_3\text{-}Ax(x) \vee A_4\text{-}Ax(x)$,

x is a FORMULA resulting from a propositional axiom by substitution.

37. $$Q(z, y, v) \equiv \overline{(En, m, w)[n \leq l(y) \,\&\, m \leq l(z) \,\&\, w \leq z \,\&\, w = m\, Gl\, z \,\&\, w \text{ Geb } n, y \,\&\, v\, Fr\, n, y]},$$

z does not contain any VARIABLE BOUND in y at a place at which v is FREE.

38. $$L_1\text{-}Ax(x) \equiv (Ev, y, z, n)\{v, y, z, n \leq x \,\&\, n \text{ Var } v \,\&\, \text{Typ}_n(z) \,\&\, \text{Form}(y) \,\&\, Q(z, y, v) \,\&\, x = (v \text{ Gen } y) \text{ Imp } [Sb(y^v_z)]\},$$

x is a FORMULA resulting from Axiom Schema III, 1 by substitution.

39. $$L_2\text{-}Ax(x) \equiv (Ev, q, p)\{v, q, p \leq x \,\&\, \text{Var}(v) \,\&\, \text{Form}(p) \,\&\, \overline{v\, Fr\, p} \,\&\, \text{Form}(q) \,\&\, x = [v \text{ Gen } (p \text{ Dis } q)] \text{ Imp } [p \text{ Dis } (v \text{ Gen } q)]\},$$

x is a FORMULA resulting from Axiom Schema III, 2 by substitution.

40. $$R\text{-}Ax(x) \equiv (Eu, v, y, n)[u, v, y, n \leq x \,\&\, n \text{ Var } v \,\&\, (n+1) \text{ Var } u \,\&\, \overline{u\, Fr\, y} \,\&\, \text{Form}(y) \,\&\, x = u \text{ Ex } \{v \text{ Gen } [[R(u) * E(R(v))] \text{ Aeq } y]\}],$$

x is a FORMULA resulting from Axiom Schema IV, 1 by substitution.

A specific number z_4 corresponds to Axiom V, 1, and we define

41. $M\text{-}Ax(x) \equiv (En)[n \leq x \,\&\, n\, Th\, z_4]$.

42. $$Ax(x) \equiv Z\text{-}Ax(x) \vee A\text{-}A(x) \vee L_1\text{-}Ax(x) \vee L_2\text{-}Ax(x) \vee R\text{-}Ax(x) \vee M\text{-}Ax(x),$$

x is an AXIOM.

43. $Fl(x, y, z) \equiv y = z \text{ Imp } x \vee (Ev)[v \leq x \,\&\, \text{Var}(v) \,\&\, x = v \text{ Gen } y]$,

x is an IMMEDIATE CONSEQUENCE of y and z.

44. $Bw(x) \equiv (n)\{0 < n \leq l(x) \rightarrow Ax(n\, Gl\, x) \vee$
$(Ep, q)[0 < p, q < n \,\&\, Fl(n\, Gl\, x, p\, Gl\, x, q\, Gl\, x)]\}$
$\&\, l(x) > 0,$

x is a PROOF ARRAY (a finite sequence of FORMULAS, each of which is either an AXIOM or an IMMEDIATE CONSEQUENCE of two of the preceding FORMULAS).

45. $x\, B\, y \equiv Bw(x) \,\&\, [l(x)]\, Gl\, x = y,$

x is a PROOF of the FORMULA y.

46. $\mathrm{Bew}(x) \equiv (Ey)y\, B\, x,$

x is a PROVABLE FORMULA. ($\mathrm{Bew}(x)$ is the only one of the notions 1–46 of which we cannot assert that it is recursive.)

The fact that can be formulated vaguely by saying that every recursive relation is definable in the system P (if the usual meaning is given to the formulas of this system) is expressed in precise language, *without* reference to any interpretation of the formulas of P, by the folllowing theorem:

Theorem V. *For every recursive relation $R(x_1, \ldots, x_n)$ there exists an n-place* RELATION SIGN *r (with the* FREE VARIABLES[38] *$u_1, u_2, \ldots, u_n$) such that for all n-tuples of numbers $(x_1, \ldots, x_n)$ we have*

$$R(x_1, \ldots, x_n) \rightarrow \mathrm{Bew}\left[Sb(r^{u_1}_{Z(x_1)} \cdots^{u_n}_{Z(x_n)})\right], \tag{3}$$

$$\overline{R}(x_1, \ldots, x_n) \rightarrow \mathrm{Bew}\left[\mathrm{Neg}(Sb(r^{u_1}_{Z(x_1)} \cdots^{u_n}_{Z(x_n)}))\right]. \tag{4}$$

We shall give only an outline of the proof of this theorem because the proof does not present any difficulty in principle and is rather long.[39] We prove the theorem for all relations $R(x_1, \ldots, x_n)$ of the form $x_1 = \phi(x_2, \ldots, x_n)$[40] (where ϕ is a recursive function) and we use induction on the degree of ϕ. For functions of degree 1 (that is, constants and the function $x + 1$) the theorem is trivial. Assume now that ϕ is of degree m. It results from functions of lower degrees, $\phi_1, \ldots, \phi_k$, through the operations of substitution or recursive definition. Since by the inductive hypothesis everything has already been proved for $\phi_1, \ldots, \phi_k$, there are corresponding RELATION SIGNS, $r_1, \ldots, r_k$, such that (3) and (4) hold. The processes of definition by which ϕ results from $\phi_1, \ldots, \phi_k$ (substitution and recursive definition) can both be formally reproduced in the system P. If this is

[38]The VARIABLES $u_1, \ldots, u_n$ can be chosen arbitrarily. For example, there always is an r with the FREE VARIABLES $17, 19, 23, \ldots$, and so on, for which (3) and (4) hold.

[39]Theorem V, of course, is a consequence of the fact that in the case of a recursive relation R it can, for every n-tuple of numbers, be decided *on the basis of the axioms of the system P* whether the relation R obtains or not.

[40]From this it follows at once that the theorem holds for every recursive relation, since any such relation is equivalent to $0 = \phi(x_1, \ldots, x_n)$, where ϕ is recursive.

done, a new RELATION SIGN r is obtained from $r_1, \ldots, r_k$,[41] and, using the inductive hypothesis, we can prove without difficulty that (3) and (4) hold for it. A RELATION SIGN r assigned to a recursive relation[42] by this procedure will be said to be *recursive.*

We now come to the goal of our discussions. Let κ be any class of FORMULAS. We denote by $\mathrm{Flg}(\kappa)$ (the set of consequences of κ) the smallest set of FORMULAS that contains all FORMULAS of κ and all AXIOMS and is closed under the relation "IMMEDIATE CONSEQUENCE". κ is said to be ω-consistent if there is no CLASS SIGN a such that

$$(n)[Sb(a_{Z(n)}^{v}) \in \mathrm{Flg}(\kappa)] \,\&\, [\mathrm{Neg}(v \text{ Gen } a)] \in \mathrm{Flg}(\kappa),$$

where v is the FREE VARIABLE of the CLASS SIGN a.

Every ω-consistent system, of course, is consistent. As will be shown later, however, the converse does not hold.

The general result about the existence of undecidable propositions reads as follows:

Theorem VI. *For every ω-consistent recursive class κ of* FORMULAS *there are recursive* CLASS SIGNS r *such that neither* v Gen r *nor* $\mathrm{Neg}(v \text{ Gen } r)$ *belongs to* $\mathrm{Flg}(\kappa)$ (where v is *the* FREE VARIABLE of r).

Proof: Let κ be any recursive ω-consistent class of FORMULAS. We define

$$\begin{gathered} Bw_\kappa(x) \equiv (n)[n \leq l(x) \rightarrow Ax(n\,Gl\,x) \vee (n\,Gl\,x) \in \kappa \vee \\ (Ep,q)\{0 < p,\ q < n \,\&\, Fl(n\,Gl\,x,\ p\,Gl\,x,\ q\,Gl\,x)\}] \,\&\, l(x) > 0 \end{gathered} \tag{5}$$

(see the analogous notion 44),

$$x\,B_\kappa\,y \equiv Bw_\kappa(x) \,\&\, [l(x)]\,Gl\,x = y \tag{6}$$

$$\mathrm{Bew}_\kappa(x) \equiv (Ey)y\,B_\kappa\,x \tag{6.1}$$

(see the analogous notions 45 and 46).

We obviously have

$$(x)[\mathrm{Bew}_\kappa(x) \sim x \in \mathrm{Flg}(\kappa)] \tag{7}$$

and

$$(x)[\mathrm{Bew}(x) \rightarrow \mathrm{Bew}_\kappa(x)]. \tag{8}$$

[41] When this proof is carried out in detail, r, of course, is not defined indirectly with the help of its meaning but in terms of its purely formal structure.

[42] Which, therefore, in the usual interpretation expresses the fact that this relation holds.

We now define the relation

$$Q(x,y) \equiv \overline{x\,B_\kappa\,[Sb(y^{19}_{Z(y)})]}. \tag{8.1}$$

Since $x\,B_\kappa\,y$ (by (6) and (5)) and $Sb(y^{19}_{Z(y)})$ (by Definitions 17 and 31) are recursive, so is $Q(x,y)$. Therefore, by Theorem V and (8) there is a RELATION SIGN q (with the FREE VARIABLES 17 and 19) such that

$$\overline{x\,B_\kappa\,[Sb(y^{19}_{Z(y)})]} \rightarrow \mathrm{Bew}_\kappa\,[Sb(q^{17}_{Z(x)}\,{}^{19}_{Z(y)})] \tag{9}$$

and

$$x\,B_\kappa\,[Sb(y^{19}_{Z(y)})] \rightarrow \mathrm{Bew}_\kappa[\mathrm{Neg}(Sb(q^{17}_{Z(x)}\,{}^{19}_{Z(y)}))]. \tag{10}$$

We put

$$p = 17\ \mathrm{Gen}\ q \tag{11}$$

(p is a CLASS SIGN with the FREE VARIABLE 19) and

$$r = Sb(q^{19}_{Z(p)}) \tag{12}$$

(r is a recursive CLASS SIGN[43] with the FREE VARIABLE 17).
Then we have

$$Sb(p^{19}_{Z(p)}) = Sb([17\ \mathrm{Gen}\ q]^{19}_{Z(p)}) = 17\ \mathrm{Gen}\ Sb(q^{19}_{Z(p)}) = 17\ \mathrm{Gen}\ r \tag{13}$$

(by (11) and (12));[44] furthermore

$$Sb(q^{17}_{Z(x)}\,{}^{19}_{Z(p)}) = Sb(r^{17}_{Z(x)}) \tag{14}$$

(by (12)). If we now substitute p for y in (9) and (10), and take (13) and (14) into account, we obtain

$$\overline{x\,B_\kappa\,(17\ \mathrm{Gen}\ r)} \rightarrow \mathrm{Bew}_\kappa[Sb(r^{17}_{Z(x)})], \tag{15}$$

$$x\,B_\kappa\,(17\ \mathrm{Gen}\ r) \rightarrow \mathrm{Bew}_\kappa[\mathrm{Neg}(Sb(r^{17}_{Z(x)}))]. \tag{16}$$

This yields:

[43] Since r is obtained from the recursive RELATION SIGN q through the replacement of a VARIABLE by a definite number p. [Precisely stated the final part of this footnote (which refers to a side remark unnecessary for the proof) would read thus: "REPLACEMENT of a VARIABLE by the NUMERAL for p".]

[44] The operations Gen and Sb, of course, can always be interchanged in case they refer to different VARIABLES.

1. 17 Gen r is not κ-PROVABLE.[45] For, if it were, there would (by (6.1)) be an n such that $n\,B_\kappa$ (17 Gen r). Hence by (16) we would have

$$\mathrm{Bew}_\kappa[\mathrm{Neg}(Sb(r^{17}_{Z(n)}))],$$

while, on the other hand, from the κ-PROVABILITY of 17 Gen r that of $Sb(r^{17}_{Z(n)})$ follows. Hence, κ would be inconsistent (and a fortiori ω-inconsistent).

2. Neg(17 Gen r) is not κ-PROVABLE. Proof: As has just been proved, 17 Gen r is not κ-PROVABLE; that is (by (6.1)),

$$(n)\overline{n\,B_\kappa\,(17 \text{ Gen } r)}$$

holds. From this,

$$(n)\mathrm{Bew}_\kappa[Sb(r^{17}_{Z(n)})]$$

follows by (15), and that, in conjunction with

$$\mathrm{Bew}_\kappa[\mathrm{Neg}(17 \text{ Gen } r)],$$

is incompatible with the ω-consistency of κ.

17 Gen r is therefore undecidable on the basis of κ, which proves Theorem VI.

We can readily see that the proof just given is constructive;[45a] that is, the following has been proved in an intuitionistically unobjectionable manner: Let an arbitrary recursively defined class κ of FORMULAS be given. Then, if a formal decision (on the basis of κ) of the SENTENTIAL FORMULA 17 Gen r (which [for each κ] can actually be exhibited) is presented to us, we can actually give

1. a PROOF of Neg(17 Gen r),
2. for any given n, a PROOF of $Sb(r^{17}_{Z(n)})$.

That is, a formal decision of 17 Gen r would have the consequence that we could actually exhibit an ω-inconsistency.

We shall say that a relation between (or a class of) natural numbers $R(x_1, \ldots, x_n)$ is *decidable* if there exists an n-place RELATION SIGN r such that (3) and (4) (see Theorem V) hold. In particular, therefore, by Theorem V every recursive relation is decidable. Similarly, a RELATION SIGN will be said to be *decidable* if it corresponds in this way to a decidable relation. Now it suffices for the existence of propositions undecidable on the basis of κ that the class κ be ω-consistent and decidable. For the decidability

[45] By "x is κ-PROVABLE" we mean $x \in \mathrm{Flg}(\kappa)$, which, by (7), means the same thing as $\mathrm{Bew}_\kappa(x)$.

[45a] Since all existential statements occurring in the proof are based upon Theorem V, which, as is easily seen, is unobjectionable from the intuitionistic point of view.

carries over from κ to $x\,B_\kappa\,y$ (see (5) and (6)) and to $Q(x, y)$ (see (8.1)), and only this was used in the proof given above. In this case the undecidable proposition has the form v Gen r, where r is a decidable CLASS SIGN. (Note that it even suffices that κ be decidable in the system enlarged by κ.)

If, instead of assuming that κ is ω-consistent, we assume only that it is consistent, then, although the existence of an undecidable proposition does not follow [by the argument given above], it does follow that there exists a property (r) for which it is possible neither to give a counterexample nor to prove that it holds of all numbers. For in the proof that 17 *Gen* r is not κ-PROVABLE only the consistency of κ was used (above, page 177). Moreover from $\overline{\mathrm{Bew}_\kappa}$(17 Gen r) it follows by (15) that, for every number x, $Sb(r^{17}_{Z(x)})$ is κ-PROVABLE and consequently that $\mathrm{Neg}(Sb(r^{17}_{Z(x)}))$ is not κ-PROVABLE for any number.

If we adjoin Neg(17 Gen r) to κ, we obtain a class of FORMULAS κ' that is consistent but not ω-consistent. κ' is consistent, since otherwise 17 Gen r would be κ-PROVABLE. However, κ' is not ω-consistent, because, by $\overline{\mathrm{Bew}_\kappa}$(17 Gen r) and (15),

$$(x)\mathrm{Bew}_\kappa(Sb(r^{17}_{Z(x)}))$$

and, a fortiori,

$$(x)\mathrm{Bew}_{\kappa'}(Sb(r^{17}_{Z(x)}))$$

hold, while on the other hand, of course,

$$\mathrm{Bew}_{\kappa'}(\mathrm{Neg}(17 \text{ Gen } r))$$

holds.[46]

We have a special case of Theorem VI when the class κ consists of a finite number of FORMULAS (and, if we so desire, of those resulting from them by TYPE ELEVATION). Every finite class κ is, of course, recursive. Let a be the greatest number contained in κ. Then we have for κ

$$x \,\epsilon\, \kappa \sim (Em, n)[m \leq x \,\&\, n \leq a \,\&\, n \,\epsilon\, \kappa \,\&\, x = m \; Th \; n].$$

Hence κ is recursive. This allows us to conclude, for example, that, even with the help of the axiom of choice (for all types) or the generalized continuum hypothesis, not all propositions are decidable, provided these hypotheses are ω-consistent.

[46] Of course, the existence of classes κ that are consistent but not ω-consistent is thus proved only on the assumption that there exists some consistent κ (that is, that P is consistent).

In the proof of Theorem VI no properties of the system P were used besides the following:

1. The class of axioms and the rules of inference (that is, the relation "immediate consequence") are recursively definable (as soon as we replace the primitive signs in some way by natural numbers).

2. Every recursive relation is definable (in the sense of Theorem V) in the system P.

Therefore, in every formal system that satisfies the assumptions 1 and 2 and is ω-consistent, there are undecidable propositions of the form $(x)F(x)$, where F is a recursively defined property of natural numbers, and likewise in every extension of such a system by a recursively definable ω-consistent class of axioms. As can easily be verified, included among the systems satisfying the assumptions 1 and 2 are the Zermelo–Fraenkel and the von Neumann axiom systems of set theory,[47] as well as the axiom system of number theory consisting of the Peano axioms, recursive definition (by Schema (2)), and the rules of logic.[48] Assumption 1 is satisfied by any system that has the usual rules of inference and whose axioms (like those of P) result from a finite number of schemata by substitution.[48a]

3

We shall now deduce some consequences from Theorem VI, and to this end we give the following definition:

A relation (class) is said to be *arithmetical* if it can be defined in terms of the notions $+$ and $\cdot$ (addition and multiplication for natural numbers)[49] and the logical constants $\vee$, $\overline{}$, (x), and $=$, where (x) and $=$ apply to natural numbers only.[50] The notion "arithmetical proposition" is defined

[47]The proof of assumption 1 turns out to be even simpler here than for the system P, since there is just one kind of primitive variables (or two in von Neumann's system).

[48]See Problem III in *Hilbert 1929a*.

[48a]As will be shown in Part II of this paper, the true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite (see *Hilbert 1926*, page 184), while in any formal system at most denumerably many of them are available. For it can be shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added (for example, the type ω to the system P). An analogous situation prevails for the axiom system of set theory.

[49]Here and in what follows, zero is always included among the natural numbers.

[50]The definiens of such a notion, therefore, must consist exclusively of the signs listed, variables for natural numbers, $x, y, \ldots$, and the signs 0 and 1 (variables for functions and sets are not permitted to occur). Instead of x any other number variable, of course, may occur in the quantifiers.

accordingly. The relations "greater than" and "congruent modulo n", for example, are arithmetical because we have

$$x > y \sim \overline{(Ez)}[y = x + z],$$
$$x \equiv y\,(\mathrm{mod}\, n) \sim (Ez)[x = y + z \cdot n \vee y = x + z \cdot n].$$

We now have

Theorem VII. *Every recursive relation is arithmetical.*

We shall prove the following version of this theorem: Every relation of the form $x_0 = \phi(x_1, \ldots, x_n)$, where ϕ is recursive, is arithmetical, and we shall use induction on the degree of ϕ. Let ϕ be of degree s $(s > 1)$. Then we have either

1. $\phi(x_1, \ldots, x_n) = \rho[\chi_1(x_1, \ldots, x_n), \chi_2(x_1, \ldots, x_n), \ldots,$
$\chi_m(x_1, \ldots, x_n)]$

(where ρ and all χ_1 are of degrees less than s)[51] or

2. $\phi(0, x_2, \ldots, x_n) = \psi(x_2, \ldots, x_n),$
$\phi(k + 1, x_2, \ldots, x_n) = \mu[k, \phi(k, x_2, \ldots, x_n), x_2, \ldots, x_n]$

(where ψ and μ are of degrees less than s).

In the first case we have

$$x_0 = \phi(x_1, \ldots, x_n) \sim (Ey_1, \ldots, y_m)[R(x_0, y_1, \ldots, y_m)\ \&$$
$$S_1(y_1, x_1, \ldots, x_n)\ \&\ \ldots\ \&\ S_m(y_m, x_1, \ldots, x_n)],$$

where R and S_i are the arithmetical relations, existing by the inductive hypothesis, that are equivalent to $x_0 = \rho(y_1, \ldots, y_m)$ and $y = \chi_i(x_1, \ldots, x_n)$, respectively. Hence in this case $x_0 = \phi(x_1, \ldots, x_n)$ is arithmetical.

In the second case we use the following method. We can express the relation $x_0 = \phi(x_1, \ldots, x_n)$ with the help of the notion "sequence of numbers" (f)[52] in the following way:

$$x_0 = \phi(x_1, \ldots, x_n) \sim (Ef)\{f_0 = \psi(x_2, \ldots, x_n)\ \&\ (k)[k < x_1 \rightarrow$$
$$f_{k+1} = \mu(k, f_k, x_2, \ldots, x_n)]\ \&\ x_0 = f_{x_1}\}.$$

If $S(y, x_2, \ldots, x_n)$ and $T(z, x_1, \ldots, x_{n+1})$ are the arithmetical relations, existing by the inductive hypothesis, that are equivalent to $y = \psi(x_2, \ldots, x_n)$ and $z = \mu(x_1, \ldots, x_{n+1})$, respectively, then

$$x_0 = \phi(x_1, \ldots, x_n) \sim (Ef)\{S(f_0, x_2, \ldots, x_n)\ \&\ (k)[k < x_1 \rightarrow$$
$$T(f_{k+1}, k, f_k, x_2, \ldots, x_n)]\ \&\ x_0 = f_{x_1}\}. \qquad (17)$$

[51] Of course, not all $x_1, \ldots, x_n$ need occur in the χ_i (see the example in footnote 27).

[52] f here is a variable with the [infinite] sequences of natural numbers as its range of values. f_k denotes the $(k + 1)$th term of a sequence f (f_0 denoting the first).

We now replace the notion "sequence of numbers" by "pair of numbers", assigning to the number pair n, d the number sequence $f^{(n,d)}$ ($f_k^{(n,d)} = [n]_{1+(k+1)d}$, where $[n]_p$ denotes the least nonnegative remainder of n modulo p).

We then have

Lemma 1. If f is any sequence of natural numbers and k any natural number, there exists a pair of natural numbers, n, d, such that $f^{(n,d)}$ and f agree in the first k terms.

Proof: Let l be the maximum of the numbers $k, f_0, f_1, \ldots, f_{k-1}$. Let us determine an n such that

$$n \equiv f_i \left[\operatorname{mod}(1 + (i+1)l!)\right] \quad \text{for} \quad i = 0, 1, \ldots, k-1,$$

which is possible, since any two of the numbers $1 + (i+1)l!$ ($i = 0, 1, \ldots, k-1$) are relatively prime. For a prime number contained in two of these numbers would also be contained in the difference $(i_1 - i_2)l!$ and therefore, since $|i_1 - i_2| < l$, in $l!$; but this is impossible. The number pair $n, l!$ then has the desired property.

Since the relation $x = [n]_p$ is defined by

$$x \equiv n \pmod{p} \,\&\, x < p$$

and is therefore arithmetical, the relation $P(x_0, x_1, \ldots, x_n)$, defined by

$$\begin{aligned} P(x_0, \ldots, x_n) \equiv (En, d)\{&S([n]_{d+1}, x_2, \ldots, x_n) \,\&\, (k)[k < x_1 \rightarrow \\ &T([n]_{1+d(k+2)}, k, [n]_{1+d(k+1)}, x_2, \ldots, x_n)] \,\&\, x_0 = [n]_{1+d(x_1+1)}\}, \end{aligned}$$

is also arithmetical. But by (17) and Lemma 1 it is equivalent to $x_0 = \phi(x_1, \ldots, x_n)$ (the sequence f enters in (17) only through its first $x_1 + 1$ terms). Theorem VII is thus proved.

By Theorem VII, for every problem of the form $(x)F(x)$ (with recursive F) there is an equivalent arithmetical problem. Moreover, since the entire proof of Theorem VII (for every particular F) can be formalized in the system P, this equivalence is provable in P. Hence we have

Theorem VIII. *In any of the formal systems mentioned in Theorem VI,*[53] *there are undecidable arithmetical propositions.*

By the remark on page 181 above, the same holds for the axiom system of set theory and its extensions by ω-consistent recursive classes of axioms.

Finally, we derive the following result:

[53] These are the ω-consistent systems that result from P when recursively definable classes of axioms are added.

Theorem IX. *In any of the formal systems mentioned in Theorem VI,*[53] *there are undecidable problems of the restricted functional calculus*[54] (that is, formulas of the restricted functional calculus for which neither validity nor the existence of a counterexample is provable).[55]

This is a consequence of

Theorem X. *Every problem of the form $(x)F(x)$ (with recursive F) can be reduced to the question whether a certain formula of the restricted functional calculus is satisfiable* (that is, for every recursive F, we can find a formula of the restricted functional calculus that is satisfiable if and only if $(x)F(x)$ is true).

By formulas of the restricted functional calculus (r. f. c.) we understand expressions formed from the primitive signs $\overline{}$, $\vee$, (x), $=$, x, y, ... (individual variables), $F(x), G(x,y), H(x,y,z), \ldots$ (predicate and relation variables), where (x) and $=$ apply to individuals only.[56] To these signs we add a third kind of variables, $\phi(x)$, $\psi(x,y)$, $\chi(x,y,z)$, and so on, which stand for functions of individuals (that is, $\phi(x)$, $\psi(x,y)$, and so on denote single-valued functions whose arguments and values are individuals).[57] A formula that contains variables of the third kind in addition to the signs of the r. f. c. first mentioned will be called a formula in the extended sense (i. e. s.).[58] The notions "satisfiable" and "valid" carry over immediately to formulas i. e. s., and we have the theorem that, for any formula A i. e. s., we can find a formula B of the r. f. c. proper such that A is satisfiable if and only if B is. We obtain B from A by replacing the variables of the third kind, $\phi(x), \psi(x,y), \ldots$, that occur in A with expressions of the form $(\iota z)F(z,x)$, $(\iota z)G(z,x,y), \ldots$, by eliminating the "descriptive" functions by the method used in *PM* (I, *14), and by logically multiplying[59] the formula thus obtained by an expression stating about each $F, G, \ldots$ put in place of

[54]See *Hilbert and Ackermann 1928*. In the system P we must understand by formulas of the restricted functional calculus those that result from the formulas of the restricted functional calculus of *PM* when relations are replaced by classes of higher types as indicated on page 153 above.

[55]In *1930* I showed that every formula of the restricted functional calculus either can be proved to be valid or has a counterexample. However, by Theorem IX the existence of this counterexample is *not* always provable (in the formal systems we have been considering).

[56]Hilbert and Ackermann (*1928*) do not include the sign $=$ in the restricted functional calculus. But for every formula in which the sign $=$ occurs there exists a formula that does not contain this sign and is satisfiable if and only if the original formula is (see *Gödel 1930*).

[57]Moreover, the domain of definition is always supposed to be the *entire* domain of individuals.

[58]Variables of the third kind may occur at all argument places occupied by individual variables, for example, $y = \phi(x)$, $F(x, \phi(y))$, $G(\psi(x, \phi(y)), x)$, and the like.

[59]That is, by forming the conjunction.

some $\phi, \psi, \ldots$ that it holds for a unique value of the first argument [for any choice of values for the other arguments].

We now show that, for every problem of the form $(x)F(x)$ (with recursive F), there is an equivalent problem concerning the satisfiability of a formula i. e. s., so that, on account of the remark just made, Theorem X follows.

Since F is recursive, there is a recursive function $\Phi(x)$ such that

$$F(x) \sim [\Phi(x) = 0],$$

and for Φ there is a sequence of functions, $\Phi_1, \Phi_2, \ldots, \Phi_n$, such that $\Phi_n = \Phi$, $\Phi_1(x) = x + 1$, and for every Φ_k $(1 < k \leq n)$ we have either

$$\begin{aligned} &1. \qquad (x_2, \ldots, x_m)[\Phi_k(0, x_2, \ldots, x_m) = \Phi_p(x_2, \ldots, x_m)], \\ &(x, x_2, \ldots, x_m)\{\Phi_k[\Phi_1(x), x_2, \ldots, x_m] = \\ &\qquad \Phi_q[x, \Phi_k(x, x_2, \ldots, x_m), x_2, \ldots, x_m]\}, \\ &\qquad \text{with } p, q < k,^{59a} \end{aligned} \tag{18}$$

or

$$\begin{aligned} &2. \qquad (x_1, \ldots, x_m)[\Phi_k(x_1, \ldots, x_m) = \Phi_r(\Phi_{i_1}(\mathfrak{x}_1), \ldots, \Phi_{i_s}(\mathfrak{x}_s))], \\ &\qquad \text{with } r < k, \quad i_v < k \quad (\text{for } v = 1, 2, \ldots, s),^{60} \end{aligned} \tag{19}$$

or

$$3. \qquad (x_1, \ldots, x_m)[\Phi_k(x_1, \ldots, x_m) = \Phi_1(\Phi_1(\ldots(\Phi_1(0))\ldots))]. \tag{20}$$

We then form the propositions

$$(x)\overline{\Phi_1(x) = 0} \,\&\, (x, y)[\Phi_1(x) = \Phi_1(y) \to x = y], \tag{21}$$

$$(x)[\Phi_n(x) = 0]. \tag{22}$$

In all of the formulas (18), (19), (20) (for $k = 2, 3, \ldots, n$) and in (21) and (22) we now replace the functions Φ_i by function variables ϕ_i and the number 0 by an individual variable x_0 not used so far, and we form the conjunction C of all the formulas thus obtained.

The formula $(Ex_0)C$ then has the required property, that is,

[59a][The last clause of footnote 27 was not taken into account in the formulas (18). But an explicit formulation of the cases with fewer variables on the right side is actually necessary here for the formal correctness of the proof, unless the identity function, $I(x) = x$, is added to the initial functions.]

[60]The $\mathfrak{x}_i$ $(i = 1, \ldots, s)$ stand for finite sequences of the variables $x_1, x_2, \ldots, x_m$; for example, x_1, x_3, x_2.

1. If $(x)[\Phi(x) = 0]$ holds, $(Ex_0)C$ is satisfiable. For the functions $\Phi_1, \Phi_2, \ldots, \Phi_n$ obviously yield a true proposition when substituted for $\phi_1, \phi_2, \ldots, \phi_n$ in $(Ex_0)C$.

2. If $(Ex_0)C$ is satisfiable, $(x)[\Phi(x) = 0]$ holds.

Proof: Let $\Psi_1, \Psi_2, \ldots, \Psi_n$ be the functions (which exist by assumption) that yield a true proposition when substituted for $\phi_1, \phi_2, \ldots, \phi_n$ in $(Ex_0)C$. Let $\mathfrak{J}$ be their domain of individuals. Since $(Ex_0)C$ holds for the functions Ψ_i, there is an individual a (in $\mathfrak{J}$) such that all of the formulas (18)–(22) go over into true propositions, (18′)–(22′), when the Φ_i are replaced by the Ψ_i and 0 by a. We now form the smallest subclass of $\mathfrak{J}$ that contains a and is closed under the operation $\Psi_1(x)$. This subclass ($\mathfrak{J}'$) has the property that every function Ψ_i, when applied to elements of $\mathfrak{J}'$, again yields elements of $\mathfrak{J}'$. For this holds of Ψ_1 by the definition of $\mathfrak{J}'$, and by (18′), (19′), and (20′) it carries over from Ψ_i with smaller subscripts to Ψ_i with larger ones. The functions that result from the Ψ_i when these are restricted to the domain $\mathfrak{J}'$ of individuals will be denoted by Ψ_i'. All of the formulas (18)–(22) hold for these functions also (when we replace 0 by a and Φ_i by Ψ_i').

Because (21) holds for Ψ_1' and a, we can map the individuals of $\mathfrak{J}'$ one-to-one onto the natural numbers in such a manner that a goes over into 0 and the function Ψ_1' into the successor function Φ_1. But by this mapping the functions Ψ_i' go over into the functions Φ_i; and, since (22) holds for Ψ_n' and a,

$$(x)[\Phi_n(x) = 0],$$

that is, $(x)[\Phi(x) = 0]$, holds, which was to be proved.[61]

Since (for each particular F) the argument leading to Theorem X can be carried out in the system P, it follows that any proposition of the form $(x)F(x)$ (with recursive F) can in P be proved equivalent to the proposition that states about the corresponding formula of the r. f. c. that it is satisfiable. Hence the undecidability of one implies that of the other, which proves Theorem IX.[62]

4

The results of Section 2 have a surprising consequence concerning a consistency proof for the system P (and its extensions), which can be stated as follows:

[61]Theorem X implies, for example, that Fermat's problem and Goldbach's problem could be solved if the decision problem for the r. f. c. were solved.

[62]Theorem IX, of course, also holds for the axiom system of set theory and for its extensions by recursively definable ω-consistent classes of axioms, since there are undecidable propositions of the form $(x)F(x)$ (with recursive F) in these systems too.

Theorem XI. *Let κ be any recursive consistent*[63] *class of* FORMULAS; *then the* SENTENTIAL FORMULA *stating that κ is consistent is not κ-*PROVABLE; in particular, the consistency of P is not provable in P,[64] provided P is consistent (in the opposite case, of course, every proposition is provable [in P]).

The proof (briefly outlined) is as follows: Let κ be some recursive class of FORMULAS chosen once and for all for the following discussion (in the simplest case it is the empty class). As appears from 1, page 177 above, only the consistency of κ was used in proving that 17 Gen r is not κ-PROVABLE;[65] that is, we have

$$\mathrm{Wid}(\kappa) \rightarrow \overline{\mathrm{Bew}_\kappa}(17 \text{ Gen } r), \tag{23}$$

that is, by (6.1),

$$\mathrm{Wid}(\kappa) \rightarrow (x)\overline{x\, B_\kappa\, (17 \text{ Gen } r)}.$$

By (13), we have

$$17 \text{ Gen } r = Sb(p^{19}_{Z(p)}),$$

hence

$$\mathrm{Wid}(\kappa) \rightarrow (x)\overline{x\, B_\kappa\, Sb(p^{19}_{Z(p)})},$$

that is, by (8.1),

$$\mathrm{Wid}(\kappa) \rightarrow (x)Q(x, p). \tag{24}$$

We now observe the following: all notions defined (or statements proved) in Section 2,[66] and in Section 4 up to this point, are also expressible (or provable) in P. For throughout we have used only the methods of definition and proof that are customary in classical mathematics, as they are formalized in the system P. In particular, κ (like every recursive class) is definable in P. Let w be the SENTENTIAL FORMULA by which $\mathrm{Wid}(\kappa)$ is expressed in P. According to (8.1), (9), and (10), the relation $Q(x, y)$ is expressed by the RELATION SIGN q, hence $Q(x, p)$ by r (since, by (12), $r = Sb(q^{19}_{Z(p)})$), and the proposition $(x)Q(x, p)$ by 17 Gen r.

Therefore, by (24), w Imp (17 Gen r) is provable in P (and a fortiori κ-PROVABLE).[67] If now w were κ-PROVABLE, then 17 Gen r would also be κ-PROVABLE, and from this it would follow, by (23), that κ is not consistent.

[63] "κ is consistent" (abbreviated by "$\mathrm{Wid}(\kappa)$") is defined thus: $\mathrm{Wid}(\kappa) \equiv (Ex)(\mathrm{Form}(x) \,\&\, \overline{\mathrm{Bew}_\kappa}(x))$.

[64] This follows if we substitute the empty class of FORMULAS for κ.

[65] Of course, r (like p) depends on κ.

[66] From the definition of "recursive" on page 159 above to the proof of Theorem VI inclusive.

[67] That the truth of w Imp (17 Gen r) can be inferred from (23) is simply due to the fact that the undecidable proposition 17 Gen r asserts its own unprovability, as was noted at the very beginning.

Let us observe that this proof, too, is constructive; that is, it allows us to actually derive a contradiction from κ, once a PROOF of w from κ is given. The entire proof of Theorem XI carries over word for word to the axiom system of set theory, M, and to that of classical mathematics,[68] A, and here, too, it yields the result: There is no consistency proof for M, or for A, that could be formalized in M, or A, respectively, provided M, or A, is consistent. I wish to note expressly that Theorem XI (and the corresponding results for M and A) do not contradict Hilbert's formalistic viewpoint. For this viewpoint presupposes only the existence of a consistency proof in which nothing but finitary means of proof is used, and it is conceivable that there exist finitary proofs that *cannot* be expressed in the formalism of P (or of M or A).

Since, for any consistent class κ, w is not κ-PROVABLE, there always are propositions (namely w) that are undecidable (on the basis of κ) as soon as Neg(w) is not κ-PROVABLE; in other words, we can, in Theorem VI, replace the assumption of ω-consistency by the following: The proposition "κ is inconsistent" is not κ-PROVABLE. (Note that there are consistent κ for which this proposition is κ-PROVABLE.)

In the present paper we have on the whole restricted ourselves to the system P, and we have only indicated the applications to other systems. The results will be stated and proved in full generality in a sequel to be published soon.[68a] In that paper, also, the proof of Theorem XI, only sketched here, will be given in detail.

Note added 28 August 1963. In consequence of later advances, in particular of the fact that due to A. M. Turing's work[69] a precise and unquestionably adequate definition of the general notion of formal system[70] can now be given, a completely general version of Theorems VI and XI is now possible. That is, it can be proved rigorously that in *every* consistent formal system that contains a certain amount of finitary number theory there exist undecidable arithmetic propositions and that, moreover, the consistency of any such system cannot be proved in the system.

[68] See *von Neumann 1927*.

[68a] ⟦This explains the "I" in the title of the paper. The author's intention was to publish this sequel in the next volume of the *Monatshefte*. The prompt acceptance of his results was one of the reasons that made him change his plan.⟧

[69] See *Turing 1937*, page 249.

[70] In my opinion the term "formal system" or "formalism" should never be used for anything but this notion. In a lecture ⟦*1946*⟧ at Princeton (mentioned in *Princeton University 1947*, p. 11) I suggested certain transfinite generalizations of formalisms; but these are something radically different from formal systems in the proper sense of the term, whose characteristic property is that reasoning in them, in principle, can be completely replaced by mechanical devices.