Problem - 1:

Is set of odd numbers with binary operations (+), i.e., $\langle O, + \rangle$ an abelian group? If not explain the reasons with necessary notations.

Solution

Step 1: A set G with a binary operation $*$ is a group if it satisfies:

① Closure: For all $a, b \in G$, $a * b \in G$.

② Associativity: $(a * b) * c = a * (b * c)$

③ Identity element: There exists $e \in G$ such that $a * e = e * a = a$ for all $a \in G$

④ Inverse element: For each $a \in G$, there exists $a^{-1} \in G$ such that $a * a^{-1} = e$

Additionally, if $a * b = b * a$ for all $a, b \in G$ then the group is abelian.

Step: 2 Take the set of odd integers

Let $O = \{\ldots, -3, -1, 1, 3, 5 \ldots\}$

with binary operation + (usual addition)

Step 3: Verify group axioms

① Closure:

Odd + odd = Even

Example : $3+5=8$ (not odd)

Thus closure fails

② Associatively:

Addition of integer is associative $(a+b)+c = a+(b+c)$

③ Identity element:

The additive identity in $(\mathbb{Z},+)$ is 0. But 0 is not odd.

So O has no identity element. ✗ Fails

④ Inverse element:

For an odd integer a, its inverse under addition is

Example: If $3 \in O$ inverse is -3, which is also odd

This works.

Step 4: conclusion

Since closure and identity fail, the set of odd integers with + is not a group. Therefore, it can not be an abelian group either.

Final answer:

The set of odd integers under addition is not an abelian group because: (i) it is not closed (odd + odd = even ≠ 0)

(ii) It does not contain the identity element (Since 0 is not odd)

Problem 2

6. Statement: Let G be finite and let p be the smallest prime dividing |G|. Any subgroup of index p in G is normal.

Answer: True. Let H have index p. The action of G on cosets gives

$\varphi: G \to S_p$. By minimality of p the image $\varphi(G)$ must have order either 1 or p. transitivity forces order

D. P fixes a coset, so the kernel of the action equals H. Hence H is normal.

7 Answer: False

explanation: If a and b commute then $(ab)^6 = a^6 b^6$ From $b^2 = a^4$ we get $b^6 = (b^2)^3 = a^{12}$, so, $(ab)^6 = a^{18}$ ∴ There is no reason $a^{18} = e$ in general Then $a^4 = g^4 = b^2$: They commute, but $(ab)^6 = g^{18} \neq e$

The claim needs extra hypotheses (e.g. finite orders forcing $a^{18} = e$) to hold

⑧ Answer: False

correct version: The general true statement is $u^{n!} \in H$ for all $u \in G$. Reason: The permutation action of G on the n cosets gives $\phi : G \to S_n$. The order of $\phi(u)$ divides $n!$, So $u^{n!} \in \ker \phi = \bigcap_{g \in G} gHg^{-1} \subseteq H$. The exponent n is not sufficient in general

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⑨ Answer: True

Why: Let P be the (unique) subgroup of order p^k for each $k \le n$ (and $p^n \mid |G|$). Then G has a normal Sylow p-subgroup. Thus P is normal and is the Sylow p-subgroup.

⑩ Answer: true

Why: A subgroup of order p^n is a Sylow p-subgroup. By Sylow theorems, the number n_p of such subgroups divides m and satisfies $n_p \equiv 1$. Since $p \nmid m$, the only divisor of m congruent to 1 (mod p) is 1. So, $n_p = 1$ uniqueness implies normality.