

# Introduction to Galois Theory



# Requirements

## 0.1 Groups

**Definition 0.1** (Monoid). *The set of elements  $M$  with defined binary operation  $\circ$  we will call as a monoid if the following conditions are satisfied.*

1. *Closure:  $\forall a, b \in M: a \circ b \in G$*
2. *Associativity:  $\forall a, b, c \in M: a \circ (b \circ c) = (a \circ b) \circ c$*
3. *Identity element:  $\exists e \in M$  such that  $\forall a \in G: e \circ a = a \circ e = a$*

**Definition 0.2** (Group). *Let we have a set of elements  $G$  with a defined binary operation  $\circ$  that satisfied the following properties.*

1. *Closure:  $\forall a, b \in G: a \circ b \in G$*
2. *Associativity:  $\forall a, b, c \in G: a \circ (b \circ c) = (a \circ b) \circ c$*
3. *Identity element:  $\exists e \in G$  such that  $\forall a \in G: e \circ a = a \circ e = a$*
4. *Inverse element:  $\forall a \in G \exists a^{-1} \in G$  such that  $a \circ a^{-1} = e$*

*In this case  $(G, \circ)$  is called as group.*

Therefore the group is a Monoid with inverse element property.

**Example 0.1.1** (Group  $\mathbb{Z}/2\mathbb{Z}$ ). *Consider a set of 2 elements:  $G = \{0, 1\}$  with the operation  $\circ$  defined by the table 1.*

*The identity element is 0 i.e.  $e = 0$ . Inverse element is the element itself because  $\forall a \in G: a \circ a = 0 = e$ .*

**Definition 0.3** (Subgroup). *Let we have a Group  $(G, \circ)$ . The subset  $S \subset G$  is called as subgroup if  $(S, \circ)$  is a Group.*

**Definition 0.4** (Abelian group). *Let we have a Group  $(G, \circ)$ . The group is called an Abelian or commutative if  $\forall a, b \in G$  it holds  $a \circ b = b \circ a$ .*

Table 1: Cayley table for  $\mathbb{Z}/2\mathbb{Z}$ 

$\circ$	0	1
0	0	1
1	1	0

Table 2: Cayley table for  $S_2$ 

$\circ$	$e$	$\tau$
$e$	$e$	$\tau$
$\tau$	$\tau$	$e$

### 0.1.1 Permutations

**Example 0.1.2** ( $S_n$  group). *If we have a permutation of  $n$  elements then it's possible to do by means of  $n!$  ways.*

$S_1$  permutation of 1 element consists of only one element  $e$  - the simplest possible group

$S_2$  permutation consists of 2 elements:

1. identity  $e$ :

$$\begin{aligned} 1 &\rightarrow 1 \\ 2 &\rightarrow 2 \end{aligned}$$

2. transposition  $\tau$ :

$$\begin{aligned} 1 &\rightarrow 2 \\ 2 &\rightarrow 1 \end{aligned}$$

*It's easy to see that the Cayley table has the form 2*

$S_3$  permutation consists of 6 elements:  $e, \tau, \tau_1, \tau_2, \sigma, \sigma_1$ . The most important are  $e, \tau$  and  $\sigma$  and all others are represented via them.

1. identity  $e$ :

$$\begin{aligned} 1 &\rightarrow 1 \\ 2 &\rightarrow 2 \\ 3 &\rightarrow 3 \end{aligned}$$

2. transposition  $\tau$ :

$$\begin{aligned} 1 &\rightarrow 2 \\ 2 &\rightarrow 1 \\ 3 &\rightarrow 3 \end{aligned}$$

3. circle  $\sigma$ :

$$\begin{aligned} 1 &\rightarrow 2 \\ 2 &\rightarrow 3 \\ 3 &\rightarrow 1 \end{aligned}$$

## 0.2 Rings, Ideals and Fields

**Definition 0.5** (Ring). Consider a set  $R$  with 2 binary operations defined. The first one  $\oplus$  (addition) and elements of  $R$  forms an Abelian group under this operation. The second one is  $\odot$  (multiplication) and the elements of  $R$  forms a Monoid under the operation. The two binary operations are connected each other via the following distributive law

- Left distributivity:  $\forall a, b, c \in R: a \odot (b \oplus c) = a \odot b \oplus a \odot c$
- Right distributivity:  $\forall a, b, c \in R: (a \oplus b) \odot c = a \odot c \oplus b \odot c$

The identity element for  $(R, \oplus)$  is denoted as 0 (additive identity). The identity element for  $(R, \odot)$  is denoted as 1 (multiplicative identity).

The inverse element to  $a$  in  $(R, \oplus)$  is denoted as  $-a$

In this case  $(R, \oplus, \odot)$  is called as ring.

The Ring is a generalization of integer numbers conception.

**Example 0.2.1** (Ring of integers  $\mathbb{Z}$ ). The set of integer numbers  $\mathbb{Z}$  forms a Ring under  $+$  and  $\cdot$  operations i.e. addition  $\oplus$  is  $+$  and multiplication  $\odot$  is  $\cdot$ . Thus for integer numbers we have the following Ring:  $(\mathbb{Z}, +, \cdot)$

**Definition 0.6** (Ideal). Lets we have the Ring  $(R, \oplus, \odot)$ . Subset  $I \subset R$  will be an ideal if it satisfied the following conditions

1.  $(I, \oplus)$  is Subgroup of  $(R, \oplus)$
2.  $\forall i \in I$  and  $\forall r \in R: i \odot r \in I$  and  $r \odot i \in I$

**Example 0.2.2** (Ideal  $2\mathbb{Z}$ ). Consider even numbers. They forms an Ideal in  $\mathbb{Z}$ . Because multiplication of any even number to any integer is an even. The ideal's symbolic name is  $2\mathbb{Z}$ .

**Example 0.2.3** (Ring of integers modulo  $n$ :  $\mathbb{Z}/n\mathbb{Z}$ ). Let  $n \in \mathbb{Z}$  and  $n > 1$ . Then  $n\mathbb{Z}$  is an Ideal.

Two integer  $a, b \in \mathbb{Z}$  are said to be congruent modulo  $n$ , written

$$a \equiv b \pmod{n}$$

if their difference  $a - b$  is an integer multiple of  $n$ .

Thus we have a separation of set  $\mathbb{Z}$  into subsets of numbers that are congruent. Each subset has the following form

$$\{r\}_n = r + n\mathbb{Z} = \{r + nk \mid k \in \mathbb{Z}\}$$

, thus

$$\mathbb{Z} = \{0\}_n \cup \{1\}_n \cup \dots \cup \{n-1\}_n.$$

Very often use the following notation

$$\bar{r} = \{r\}_n.$$

We can define the following operations

$$\begin{aligned}\bar{k} \oplus \bar{l} &= \overline{k + l} \\ \bar{k} \odot \bar{l} &= \overline{k \cdot l}\end{aligned}$$

The Ring where the objects are defined is called as  $\mathbb{Z}/n\mathbb{Z}$ .

**Definition 0.7** (Principal ideal). The ideal that is generated by one element  $a$  is called as principal ideal and is denoted as  $(a)$  i.e. left principal ideal:  $(a) = \{ra \mid \forall r \in R\}$  and right principal ideal:  $(a) = \{ar \mid \forall r \in R\}$

**Definition 0.8** (Integral domain). In mathematics, and specifically in abstract algebra, an integral domain is a nonzero commutative Ring in which the product of any two nonzero elements is nonzero.

**Definition 0.9** (Principal ideal domain). In abstract algebra, a principal ideal domain, or PID, is an Integral domain in which every ideal is principal, i.e., can be generated by a single element.

**Definition 0.10** (Maximal ideal).  $I$  is a maximal ideal of a ring  $R$  if there are no other ideals contained between  $I$  and  $R$ .

**Definition 0.11** (Proper ideal).  $I$  is a proper ideal of a ring  $R$  if  $I \subsetneq R$ .

**Definition 0.12** (Field). The ring  $(R, \oplus, \odot)$  is called as a field if  $(R \setminus \{0\}, \odot)$  is an Abelian group.

The inverse element to  $a$  in  $(R \setminus \{0\}, \odot)$  is denoted as  $a^{-1}$

**Example 0.2.4** (Field  $\mathbb{Q}$ ). Note that  $\mathbb{Z}$  is not a field because not for every integer number an inverse exists. But if we consider a set of fractions  $\mathbb{Q} = \{a/b \mid a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\}\}$  when it will be a field.

The inverse element to  $a/b$  in  $(\mathbb{Q} \setminus \{0\}, \cdot)$  will be  $b/a$ .

**Definition 0.13** (Unique factorization domain). Unique factorization domain (UFD) is a commutative ring, which is an Integral domain, and in which every non-zero non-unit element can be written as a product of prime elements (or irreducible elements), uniquely up to order and units, analogous to the fundamental theorem of arithmetic for the integers.

## 0.3 Linear algebra

**Definition 0.14** (Vector space). Let  $F$  is a Field. The set  $V$  is called as vector space under  $F$  if the following conditions are satisfied

1. We have a binary operation  $V \times V \rightarrow V$  (addition):  $(x, y) \rightarrow x + y$  with the following properties:

$$(a) \ x + y = y + x$$

$$(b) \ (x + y) + z = x + (y + z)$$

$$(c) \ \exists 0 \in V \text{ such that } \forall x \in V : x + 0 = x$$

$$(d) \ \forall x \in V \exists -x \in V \text{ such that } x + (-x) = x - x = 0$$

2. We have a binary operation  $F \times V \rightarrow V$  (scalar multiplication) with the following properties

$$(a) \ 1_F \cdot x = x$$

$$(b) \ \forall a, b \in F, x \in V : a \cdot (b \cdot x) = (ab) \cdot x.$$

$$(c) \ \forall a, b \in F, x \in V : (a + b) \cdot x = a \cdot x + b \cdot x$$

$$(d) \ \forall a \in F, x, y \in V : a \cdot (x + y) = a \cdot x + a \cdot y$$

**Lemma 0.15** (About vector space isomorphism). 2 vector spaces  $L$  and  $M$  with same dimension  $\dim L = \dim M$  then there exists an Isomorphism between them

## 0.4 Functions

**Definition 0.16** (Surjection). *The function  $f : X \rightarrow Y$  is surjective (or onto) if  $\forall y \in Y, \exists x \in X$  such that  $f(x) = y$ .*

**Definition 0.17** (Injection). *The function  $f : X \rightarrow Y$  is injective (or one-to-one function) if  $\forall x_1, x_2 \in X$ , such that  $x_1 \neq x_2$  then  $f(x_1) \neq f(x_2)$ .*

**Definition 0.18** (Bijection). *The function  $f : X \rightarrow Y$  is bijective (or one-to-one correspondence) if it is an Injection and a Surjection.*

**Definition 0.19** (Homomorphism). *The homomorphism is a function (map) between two sets that preserves its algebraic structure. For the case of groups  $(X, \circ)$  and  $(Y, \odot)$  the function  $f : X \rightarrow Y$  is called homomorphism if  $\forall x_1, x_2 \in X$  it holds  $f(x_1 \circ x_2) = f(x_1) \odot f(x_2)$ .*

**Definition 0.20** (Isomorphism). *If a map is Bijection as well as Homomorphism when it is called as isomorphism.*

*We use the following symbolic notation for isomorphism between  $X$  and  $Y$ :  $X \cong Y$ .*

**Definition 0.21** (Automorphism). *Automorphism is an isomorphism from a mathematical object to itself.*

## 0.5 Polynomial ring $K[X]$

Let we have a commutative Ring  $K$ . Lets create a new Ring  $B$  with the following infinite sets as elements:

$$f = (f_0, f_1, \dots), f_i \in K, \quad (1)$$

such that only finite number of elements of the sets are non zero.

We can define addition and multiplication on  $B$  as follows

$$\begin{aligned} f + g &= (f_0 + g_0, f_1 + g_1, \dots), \\ f \cdot g &= h = (h_0, h_1, \dots), \end{aligned} \quad (2)$$

where

$$h_k = \sum_{i+j=k} f_i g_j.$$

The sequences (1) forms a Ring with the following identities:



- Additive identity:  $(0, 0, \dots)$
- Multiplicative identity:  $(1, 0, \dots)$

The sequences  $k = (k, 0, \dots)$  added and multiplied as elements of  $K$  this allows say that such elements are elements of original Ring  $K$ . Thus  $K$  is sub-ring of the new ring  $B$ .

Let

$$\begin{aligned} X &= (0, 1, 0, \dots), \\ X^2 &= (0, 0, 1, \dots) \end{aligned}$$

thus if we have

$$f = (f_0, f_1, f_2, \dots, f_n, 0, \dots),$$

where  $f_n$  is the last non-zero element of (1), when one can get

$$f = f_0 + f_1X + f_2X^2 + \dots + f_nX^n.$$

**Definition 0.22** (Polynomial ring). *The Ring of sequences (1) with operations defined by (2) is called as polynomial ring  $K[X]$ .*

**Lemma 0.23** (Bézout's lemma). *Let  $a$  and  $b$  be nonzero integers and let  $d$  be their greatest common divisor. Then there exist integers  $x$  and  $y$  such that*

$$ax + by = d.$$

**Definition 0.24** (Monic polynomial). *Monic polynomial is a univariate polynomial in which the leading coefficient (the nonzero coefficient of highest degree) is equal to 1. Therefore, a monic polynomial has the form*

$$x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$



# Chapter 1

## Generalities on algebraic extensions

We introduce the basic notions such as a field extension, algebraic element, minimal polynomial, finite extension, and study their very basic properties such as the multiplicativity of degree in towers.

### 1.1 Field extensions: examples

#### 1.1.1 K-algebra

**Definition 1.1** (K-algebra). *Let  $K$  be a field and  $A$  be a Vector space over  $K$  equipped with an additional binary operation  $A \times A \rightarrow A$  that we denote as  $\cdot$  here. The  $A$  is an algebra over  $K$  if the following identities hold  $\forall x, y, z \in A$  and for every elements (often called as scalar)  $a, b \in K$*

- *Right distributivity:*  $(x + y) \cdot z = x \cdot z + y \cdot z$
- *Left distributivity:*  $z \cdot (x + y) = z \cdot x + z \cdot y$
- *Compatibility with scalars:*  $(ax) \cdot (by) = (ab)(x \cdot y)$

**Example 1.1.1** (Field of complex numbers  $\mathbb{C}$ ). *The field of complex numbers  $\mathbb{C}$  can be considered as a  $K$ -algebra over field of real numbers  $\mathbb{R}$ .*

#### 1.1.2 Field extension

Let  $K$  and  $L$  are fields.

**Definition 1.2** (Field extension).  *$L$  is an extension of  $K$  if  $L \supset K$*

and another definition

**Definition 1.3** (Field extension). *L is an extension of K if L is a K-algebra*

Why the 2 definitions are equivalent?

**Lemma 1.4** (K-algebra and Homomorphism). *Given a K-algebra is the same as having Homomorphism  $f : K \rightarrow A$  of rings.*

*Proof.* Really if I have a K-algebra I can define the Homomorphism  $f(k) = k \cdot 1_A$ , where  $1_A$  is an identity element of  $A$ . Thus  $k \cdot 1_A \in A$ .

And conversely if I have the Homomorphism  $f : K \rightarrow A$  I can define the K-algebra structure by setting  $ka = f(k)a$  because  $f(k), a \in A$  and there is a multiplication defined on  $A$ . As result I have a rule for multiplication a scalar ( $k \in K$ ) on a vector ( $a \in A$ ).  $\square$

**Lemma 1.5** (About Homomorphism of fields). *Any Homomorphism of fields is Injection.*

*Proof.* Lets proof by contradiction. Really if  $f(x) = f(y)$  and  $x \neq y$  then

$$\begin{aligned} f(x) - f(y) &= 0_A, \\ f(x - y) &= 0_A, \\ f(x - y)f((x - y)^{-1}) &= f\left(\frac{x - y}{x - y}\right) = f(1_K) = 1_A = 0_A \end{aligned}$$

that is impossible.  $\square$

There are some comments on the results. We have got that a Homomorphism can be set between field  $K$  and its K-algebra. This means that K-algebra is a field. The Homomorphism is Injection therefore we can allocate a sub-field  $A' \subset A$  for that we will have the Homomorphism is a Surjection and therefore we have an Isomorphism between original field  $K$  and a sub-field  $A'$ . This means that we can say that the original field  $K$  is a sub-field for the K-algebra.

**Example 1.1.2** (Field extensions).  *$\mathbb{C}$  is a field extension for  $\mathbb{R}$ .  $\mathbb{R}$  is a field extension for  $\mathbb{Q}$*

### 1.1.3 Field characteristic

If  $L$  is a field there are 2 possibilities

1.  $1 + 1 + \cdots \neq 0$ . In this case  $\mathbb{Z} \subset L$  but  $\mathbb{Z}$  is not a field therefore  $L$  is an extension of  $\mathbb{Q}$ . In the case  $\text{char} L = 0$
2.  $1 + 1 + \cdots + 1 = \sum_{i=1}^m 1 = 0$  for some  $m \in \mathbb{Z}$ . The first time when it happens is for a prime number i.e. minimal  $m$  with the property is prime. In this case  $\text{char} L = p$ , where  $p = \min m$  - the minimal  $m$  (prime) with the property. In this case  $\mathbb{Z}/p\mathbb{Z} \subset L$ . The  $\mathbb{Z}/p\mathbb{Z}$  is a field denoted by  $\mathbb{F}_p$ . The  $L$  is an extension of  $\mathbb{F}_p$ .

No other possibilities exist. The  $\mathbb{Q}$  and  $\mathbb{F}_p$  are the prime fields. Any field is an extension of one of those.

### 1.1.4 Field $K[X]/(P)$

Let  $K[X]$  Ring of polynomials. The  $P \in K[X]$  is an irreducible.  $(P)$  is an Ideal formed by the polynomial. The set of residues by the polynomial forms a field that denoted by  $K[X]/(P)$ . How we can see it? If  $Q \in K[X]$  is a polynomial that  $Q \notin (P)$  when  $Q$  is prime to  $P$ . Then with Bézout's lemma we can get  $\exists A, B \in K[X]$  such that

$$AP + BQ = 1,$$

or

$$BQ \equiv 1 \pmod{P},$$

thus  $B$  is  $Q^{-1}$  in  $K[X]/(P)$ .

## 1.2 Algebraic elements. Minimal polynomial

### 1.2.1 $K[X]/(P)$ field

Alternative proof that  $K[X]/(P)$  is the Field. The  $(P)$  is a Maximal ideal but a quotient by a Maximal ideal is a Field.

$K[X]/(P)$  is an extension of  $K$  because it's  $K$ -algebra.

**Example 1.2.1** ( $\mathbb{F}_2/(x^2 + x + 1)$ ). Lets consider the following field  $K = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$  in the field polynomial  $x^2 + x + 1$  is irreducible. It's very easy to verify it because  $\mathbb{F}_2$  has only 2 elements that can be (possible) a root:

$$0^2 + 0 + 1 = 1 \neq 0$$

and

$$1^2 + 1 + 1 = 1 \neq 0$$

The polynomial has the following residues:  $\bar{x} = x + (x^2 + x + 1)$  and  $\overline{x+1} = x + 1 + (x^2 + x + 1)$ . Thus the field  $\mathbb{F}_2/(x^2 + x + 1)$  consists of 4 elements:  $\{0, 1, \bar{x}, \overline{x+1}\}$ .

It's easy to see that the third element ( $\bar{x}$ ) is a root of  $P(x) = x^2 + x + 1$ :

$$\bar{x}^2 + \bar{x} + 1 = P(x) + (P(x)) = (P(x)) \equiv 0 \text{ mod } P.$$

$$\bar{x}^2 + \bar{x} + 1 = \bar{0},$$

therefore

$$\bar{x}^2 = -\bar{x} - 1 = \bar{x} + 1 = \overline{x+1}.$$

This is because we are in field  $\mathbb{F}_2$  where

$$2(x+1) \text{ mod } 2 = 0$$

and thus

$$-\bar{x} - 1 = \bar{x} + 1$$

Also

$$\overline{x+1}^2 = \bar{x},$$

and they are inverse each other

$$\overline{x+1}\bar{x} = 1,$$

### 1.2.2 Algebraic elements

**Definition 1.6** (Algebraic element). Let  $K \subset L$  and  $\alpha \in L$ .  $\alpha$  is an algebraic element if  $\exists P \in K[X]$  such that  $P(\alpha) = 0$ . Otherwise the  $\alpha$  is called transcendental.

### 1.2.3 Minimal polynomial

**Lemma 1.7** (About minimal polynomial existence). If  $\alpha$  is Algebraic element then  $\exists!$  unitary polynomial  $P$  of minimal degree such that  $P(\alpha) = 0$ . It is ineducable.  $\forall Q$  such that  $Q(\alpha) = 0$  is divisible by  $P$

**Definition 1.8** (Minimal polynomial). Such polynomial is called minimal polynomial and denoted by  $P_{\min}(\alpha, K)$ .

*Proof.* We know that  $K[X]$  is a Principal ideal domain and a polynomial  $Q(\alpha) = 0$  forms an Ideal:  $I = \{Q \in K[X] \mid Q(\alpha) = 0\}$ , so the ideal is generated by one element:  $I = (P)$ . This is an unique (up to constant) polynomial minimal degree in  $I$ . If  $P$  is not ineducable then  $\exists Q, R \in I$  such that  $P = QR$ ,  $Q(\alpha) = 0$  or  $R(\alpha) = 0$  and  $\deg R, Q < \deg P$  that is in contradiction with the definition that  $P$  is a polynomial of minimal degree.  $\square$

### 1.3 Algebraic elements. Algebraic extensions

**Definition 1.9.** Let  $K \subset L$ ,  $\alpha \in L$ . The smallest sub-field contained  $K$  and  $\alpha$  denoted by  $K(\alpha)$ . The smallest sub-ring contained  $K$  and  $\alpha$  denoted by  $K[\alpha]$ .

As soon as  $K[\alpha]$  is a  $K$ -algebra it is a Vector space generated by  $1, \alpha, \alpha^2, \dots, \alpha^n, \dots$

**Example 1.3.1** ( $\mathbb{C}$ ).

$$\mathbb{C} = \mathbb{R}(i) = \mathbb{R}[i]$$

$\mathbb{C}$  is also a Vector space generated by 1 and  $i$ :  $\forall z \in \mathbb{C}$  it holds  $z = x + iy$  where  $x, y \in \mathbb{R}$ .

**Proposition 1.10.** The following assignment are equivalent

1.  $\alpha$  is algebraic over  $K$
2.  $K[\alpha]$  is a finite dimensional Vector space over  $K$
3.  $K[\alpha] = K(\alpha)$

*Proof.* Lets proof that 1 implies 2. If  $\alpha$  is algebraic over  $K$  then using lemma Minimal polynomial  $\exists P_{min}(\alpha, K)$ :

$$P_{min}(\alpha, K) = \alpha^d + a_{d-1}\alpha^{d-1} + a_1\alpha + a_0 = 0,$$

where  $a_k \in K$ . Then

$$\alpha^d = -a_{d-1}\alpha^{d-1} - a_1\alpha - a_0$$

this means that any  $\alpha^n$  can be represented as a linear combination of finite number of powers of  $\alpha$  i.e.  $K[\alpha]$  generated by  $1, \alpha, \dots, \alpha^{d-1}$  is a finite dimensional Vector space.

Lets proof that 2 implies 3. Its enough proof that  $K[\alpha]$  is a field. Let  $x \neq 0 \in K[\alpha]$  then lets look at an operation  $x \cdot K[\alpha] \rightarrow K[\alpha]$ . This is Injection because if  $y, z \in K[\alpha]$  and  $z \neq y$  then  $x \cdot y \neq x \cdot z$ . But the  $K[\alpha]$

is finite dimensional Vector space and a Homomorphism between 2 vector spaces with the same dimension is Surjection thus  $\exists y \in K[\alpha]$  such that  $x \cdot y = 1_{K[\alpha]}$ . Therefore  $x$  is invertable and  $K[\alpha]$  is a Field.

Lets proof that 3 implies 1. Let  $K[\alpha]$  is a Field but  $\alpha$  is not algebraic. Thus  $\forall P \in K[X] P(\alpha) \neq 0$ . Then we have an Injection Homomorphism  $f : K[X] \rightarrow K[\alpha]$  but  $K[X]$  is not a field thus  $K[\alpha]$  should not be a field too that is in contradiction with the initial conditions.  $\square$

**Definition 1.11** (Algebraic extension).  *$L$  an extension of  $K$  is called algebraic if  $\forall \alpha \in L - \alpha$  is algebraic over  $K$ .*

**Proposition 1.12.** *If  $L$  is algebraic over  $K$  then any  $K$ -subalgebra of  $L$  is a Field.*

*Proof.* Let  $L' \subset L$  a subalgebra and let  $\alpha \in L'$ . We want to show that  $\alpha$  is invertable.  $\alpha$  is algebraic therefore  $\alpha \in K[\alpha] \subset L' \subset L$  and it's invertable.  $\square$

**Proposition 1.13.** *Let  $K \subset L \subset M$ .  $\alpha \in M$  - algebraic over  $K$  then  $\alpha$  algebraic over  $L$  and  $P_{min}(\alpha, L)$  divides  $P_{min}(\alpha, K)$ .*

*Proof.* Its clear because  $P_{min}(\alpha, K) \in L[X]$  thus  $\exists P_L \in L[X]$  such that  $P_L(\alpha) = 0$  i.e.  $\alpha$  is algebraic over  $L$ .

As soon as  $P_{min}(\alpha, K) \in L[X]$  then  $\deg P_{min}(\alpha, L) \leq \deg P_{min}(\alpha, K)$  and as soon as  $P_{min}(\alpha, K) \in (P_{min}(\alpha, L))$  then  $P_{min}(\alpha, L)$  divides  $P_{min}(\alpha, K)$ .  $\square$

## 1.4 Finite extensions. Algebraicity and finiteness

**Definition 1.14** (Finite extension).  *$L$  is a finite extension of  $K$  if  $\dim_K L < \infty$ .  $\dim_K L$  is called as degree of  $L$  over  $K$  and is denoted by  $[L : K]$*

**Theorem 1.15** (The multiplicativity formula for degrees). *Let  $K \subset L \subset M$ . Then  $M$  is Finite extension over  $K$  if and only if  $M$  is Finite extension over  $L$  and  $L$  is Finite extension over  $K$ . In this case*

$$[M : K] = [M : L][L : K].$$

*Proof.* Let  $[M : K] < \infty$  but any linear independent set of vectors  $\{m_1, m_2, \dots, m_n\}$  over  $L$  is also linear independent over  $K$  thus

$$[M : K] < \infty \Rightarrow [M : L] < \infty$$

also  $L$  is a vector sub space of  $M$  thus if  $[M : K] < \infty$  then  $[L : K] < \infty$ .

Let  $[M : L] < \infty$  and  $[L : K] < \infty$  then we have the following baseses



- $L$ -basis over  $M$ :  $(e_1, e_2, \dots, e_n)$
- $K$ -basis over  $L$ :  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d)$

Lets proof that  $e_i \varepsilon_j$  forms a  $K$ -basis over  $M$ .  $\forall x \in M$ :

$$x = \sum_{i=1}^n a_i e_i,$$

where  $a_i \in L$  and can be also written as

$$a_i = \sum_{j=1}^d b_{ij} \varepsilon_j,$$

where  $b_{ij} \in K$ . Thus

$$x = \sum_{i=1}^n \sum_{j=1}^d b_{ij} \varepsilon_j e_i,$$

therefore  $\varepsilon_j e_i = e_i \varepsilon_j$  generates  $M$  over  $K$ . From the other side we should check that  $\varepsilon_j e_i$  linear independent system of vectors. Lets

$$\sum_{i,j} c_{ij} \varepsilon_j e_i = \sum_{i=1}^n \left( \sum_{j=1}^d c_{ij} \varepsilon_j \right) e_i,$$

then  $\forall i$ :

$$\sum_{j=1}^d c_{ij} \varepsilon_j = 0.$$

Thus  $\forall i, j : c_{ik} = 0$  that finishes the proof the linear independence. The number of linear independent vectors is  $n \times d$  i.e.

$$[M : K] = [M : L] [L : K].$$

□

**Definition 1.16** ( $K(\alpha_1, \dots, \alpha_n)$ ).  $K(\alpha_1, \dots, \alpha_n) \subset L$  generated by  $\alpha_1, \dots, \alpha_n$  is the smallest sub field of  $L$  contained  $K$  and  $\alpha_i \in L$ .

**Theorem 1.17** (About towers).  $L$  is finite over  $K$  if and only if  $L$  is generated by a finite number of algebraic elements over  $K$ .

*Proof.* If  $L$  is finite then  $\alpha_1, \dots, \alpha_d$  is a basis. In this case  $L = K[\alpha_1, \dots, \alpha_d] = K(\alpha_1, \dots, \alpha_d)$ . Moreover each  $K[\alpha_i]$  is finite dimensional thus by proposition 1.10  $\alpha_i$  is algebraic.

From other side if we have a finite set of algebraic elements  $\alpha_1, \dots, \alpha_d$  then  $K[\alpha_1]$  is a finite dimensional Vector space over  $K$ ,  $K[\alpha_1, \alpha_2]$  is a finite dimensional Vector space over  $K[\alpha_1]$  and so on  $K[\alpha_1, \dots, \alpha_d]$  is a finite dimensional Vector space over  $K[\alpha_1, \dots, \alpha_{d-1}]$ . All elements are algebraic thus

$$K[\alpha_1, \dots, \alpha_i] = K(\alpha_1, \dots, \alpha_i)$$

Then using theorem 1.15 we can conclude that  $K(\alpha_1, \dots, \alpha_d)$  has finite dimension.  $\square$

## 1.5 Algebraicity in towers. An example

**Theorem 1.18.**  $K \subset L \subset M$  then  $M$  Algebraic extension over  $K$  if and only if  $M$  algebraic over  $L$  and  $L$  algebraic over  $K$ .

*Proof.* If  $\alpha \in M$  is an Algebraic element over  $K$  then  $\exists P \in K[X]$  such that  $P(\alpha) = 0$  but the polynomial  $P \in K[X] \subset L[X]$  thus  $\alpha$  is algebraic over  $L$ . If  $\alpha \in L \subset M$  then  $\alpha$  is algebraic over  $K$  thus  $L$  is algebraic over  $K$ .

Let  $M$  algebraic over  $L$  and  $L$  algebraic over  $K$  and let  $\alpha \in M$ . We want to prove that  $\alpha$  is algebraic over  $K$ . Lets consider  $P_{min}(\alpha, L)$  the polynomial coefficients are from  $L$  and they (as soon as they count is a finite) generate a finite extension  $E$  over  $K$  thus  $E(\alpha)$  is finite over  $E$  (exists a relation between powers of  $\alpha$ ) and by theorem 1.17 is finite over  $K$  thus  $\alpha$  is algebraic over  $K$ .  $\square$

**Example 1.5.1** ( $\mathbb{Q}$  extension).  $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$  algebraic and finite over  $\mathbb{Q}$ :

$$\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2}) \subset \mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$$

*Minimal polynomial*

$$P_{min}(\sqrt[3]{2}, \mathbb{Q}) = x^3 - 2.$$

$\mathbb{Q}(\sqrt[3]{2})$  is generated over  $\mathbb{Q}$  by  $1, \sqrt[3]{2}, \sqrt[3]{4}$  thus  $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ .

But  $\sqrt{3} \notin \mathbb{Q}(\sqrt[3]{2})$  because otherwise  $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$  must divide  $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$  that is impossible.

Therefore  $x^2 - 3$  is ineducable over  $\mathbb{Q}(\sqrt[3]{2})$  and

$$P_{\min}(\sqrt{3}, \mathbb{Q}(\sqrt[3]{2})) = x^2 - 3.$$

$$[\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) : \mathbb{Q}] = 3 \cdot 2 = 6.$$

**Proposition 1.19** (On dimension of extension).

$$[K(\alpha) : K] = \deg P_{\min}(\alpha, K),$$

if  $\alpha$  is algebraic.

*Proof.* If  $\deg P_{\min}(\alpha, K) = d$  then  $1, \alpha, \dots, \alpha^{d-1}$  -  $d$  independent vectors and dimension  $K(\alpha)$  is  $d$ .  $\square$

**Proposition 1.20** (About algebraic closure). If  $K \subset L$  ( $L$  extension of  $K$ ). Consider

$$L' = \{\alpha \in L \mid \alpha \text{ algebraic over } K\},$$

then  $L'$  sub-field of  $L$  and is called as algebraic closure of  $K$  in  $L$ .

*Proof.* We have to prove that if  $\alpha, \beta$  are algebraic then  $\alpha + \beta$  and  $\alpha \cdot \beta$  are also algebraic. This is trivial because

$$\alpha + \beta, \alpha \cdot \beta \in K[\alpha, \beta] = K(\alpha, \beta)$$

$\square$

## 1.6 A digression: Gauss lemma, Eisenstein criterion

What we have seen so far:

- $K$  is a field,  $\alpha$  is an Algebraic element over  $K$  if it is a root of a polynomial  $P \in K[X]$ .
- $L$  is an Algebraic extension over  $K$  if  $\forall \alpha \in L$ :  $\alpha$  is an algebraic over  $K$
- $L$  is a Finite extension over  $K$  if  $\dim_K L < \infty$ .
- If an extension is finite then it is algebraic

- An extension is finite if and only if it is algebraic and generated by a finite number of algebraic elements (see theorem 1.17)
- $[K[\alpha] : K] = \deg P_{\min}(\alpha, K)$  (see proposition 1.19).

How to decide that a polynomial  $P$  is irreducible over  $K$ ? About polynomial  $x^3 - 2$  it is easy to decide that it's irreducible over  $\mathbb{Q}$ , but what's about  $x^{100} - 2$ ?

**Lemma 1.21** (Gauss). *Let  $P \in \mathbb{Z}[X]$ , i.e. a polynomial with integer coefficients, then if  $P$  decomposes over  $\mathbb{Q}$  ( $P = Q \cdot R, \deg Q, R < \deg P$ ) then it also decomposes over  $\mathbb{Z}$ .*

*Proof.* Let  $P = QR$  over  $\mathbb{Q}$ . Then

$$\begin{aligned} Q &= mQ_1, Q_1 \in \mathbb{Z}[X], \\ R &= nR_1, R_1 \in \mathbb{Z}[X], \end{aligned}$$

thus

$$nmP = Q_1R_1.$$

There exists  $p$  that divides  $mn$ :  $p \mid mn$  thus in modulo  $p$  we have

$$0 = \overline{Q_1R_1}$$

but  $p$  is prime and the equation is in the field  $\mathbb{F}_p$  thus either  $\overline{Q_1} = 0$  or  $\overline{R_1} = 0$ . Let  $\overline{Q_1} = 0$  thus  $p$  divides all coefficients in  $Q_1$  and we can take  $\frac{Q_1}{p} = Q_2 \in \mathbb{Z}[X]$ . Continue for all primes in  $mn$  we can get that

$$P = Q_s R_t,$$

where  $Q_s, R_t \in \mathbb{Z}[X]$ . □

**Example 1.6.1** (Eisenstein criterion). *Lets consider the following polynomial  $x^{100} - 2$ . It's irreducible. Lets prove it. If it reducible then  $\exists Q, R \in \mathbb{Z}[X]$  such that*

$$x^{100} - 2 = QR \tag{1.1}$$

*Lets consider (1.1) modulo 2. In the case we will have*

$$QR \equiv x^{100} \pmod{2},$$

*therefore*

$$\begin{aligned} Q &\equiv x^k \pmod{2}, \\ R &\equiv x^l \pmod{2}, \end{aligned}$$

or

$$Q = x^k + \cdots + 2 \cdot m$$

and

$$R = x^l + \cdots + 2 \cdot n$$

thus

$$QR = x^{100} + 4 \cdot nm$$

that is impossible because  $n, m \in \mathbb{Z}$  and  $nm \neq -\frac{1}{2}$ .

**Lemma 1.22** (Eisenstein criterion). *Lets  $P \in \mathbb{Z}[X]$  and  $P = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$ . If  $\exists p$  - prime such that  $p \nmid a_n$ ,  $p \mid a_i \forall i < n$  and  $p^2 \nmid a_0$ , then  $P \in \mathbb{Z}[X]$  is irreducible.*

*Proof.* the same as for example 1.6.1. □

Note: that both: Gauss and Eisenstein criterion are valid by replacing  $\mathbb{Z}$  with an Unique factorization domain  $R$  and  $\mathbb{Q}$  by its factorization field.



# Chapter 2

## Stem field, splitting field, algebraic closure

We introduce the notion of a stem field and a splitting field (of a polynomial). Using Zorn's lemma, we construct the algebraic closure of a field and deduce its unicity (up to an isomorphism) from the theorem on extension of homomorphisms.

### 2.1 Stem field. Some irreducibility criteria

#### 2.1.1 Stem field

**Definition 2.1** (Stem field). *Let  $P \in K[X]$  is an irreducible Monic polynomial. Field extension  $E$  is a stem field of  $P$  if  $\exists \alpha \in E$  - the root of polynomial  $P$  and  $E = K[\alpha]$ .*

Such things exist, for instance we can take  $K[X]/(P)$ . It is a field because  $P$  is irreducible moreover the root of the  $P$  is in the field (see example 1.2.1).

We also can say that for any stem field  $E$ :

$$K[X]/(P) \cong E.$$

We can use the following Isomorphism:  $f : \forall p \in K[X]/(P) \rightarrow p(\alpha)$ , there  $\alpha$  is a root of polynomial  $P$ . To summarize we have the following

**Proposition 2.2** (About stem field existence). *The stem field exist and if we have 2 stem fields  $E$  and  $E'$  which correspond 2 roots of  $P$ :  $E = K[\alpha]$ ,  $E' = K[\alpha']$  then  $\exists! f : E \cong E'$  (Isomorphism of  $K$ -algebras) such that  $f(\alpha) = \alpha'$ .*

*Proof.* Existence:  $K[X]/(P)$  can be took as the stem field.

Uniqueness of the Isomorphism is easy because it is defined by its value on argument  $\alpha$ :

$$\begin{aligned}\phi : K[X]/(P) &\cong_{x \rightarrow \alpha} E, \\ \psi : K[X]/(P) &\cong_{x \rightarrow \alpha'} E',\end{aligned}$$

thus

$$\phi^{-1} \circ \psi : E \cong_{\alpha \rightarrow \alpha'} E'.$$

□

**Remark 2.3** (About stem field). 1. In particular: If a stem field contains 2 roots of  $P$  then  $\exists!$  Automorphism taking one root into another.

2. If  $E$  stem field then  $[E : K] = \deg P$

3. If  $[E : K] = \deg P$  and  $E$  contains a root of  $P$  then  $E$  is a stem field

4. If  $E$  is not a stem field but contains root of  $P$  then  $[E : K] > \deg P$   
(???)

### 2.1.2 Some irreducibility criteria

**Corollary 2.4.**  $P \in K[X]$  is irreducible over  $K$  if and only if it does not have a root in Field extension  $L$  of  $K$  of such that  $[L : K] \leq \frac{n}{2}$ , where  $n = \deg P$ .

*Proof.*  $\Rightarrow$ : If  $P$  is not irreducible then it has a polynomial  $Q$  that divides  $P$  and  $\deg Q \leq \frac{n}{2}$  ( $P = RQ$  and if  $\deg Q > \frac{n}{2}$  then we can take  $R$  as  $Q$ ). The Stem field  $L$  for  $Q$  exists and its degree is  $\deg Q \leq \frac{n}{2}$ .  $L$  should have root of  $Q$  (as soon as root of  $P$ ) by definition.

$\Leftarrow$ : If  $P$  has a root  $\alpha$  in  $L$  then  $\exists P_{\min}(\alpha, K)$  with degree  $\leq \frac{n}{2} < n$  (because  $[L : K] \leq \frac{n}{2}$ ) that divides  $P$  i.e.  $P$  become reducible. □

**Corollary 2.5.**  $P \in K[X]$  irreducible with  $\deg P = n$ . Let  $L$  be an extension of  $K$  such that  $[L : K] = m$ . If  $\gcd(n, m) = 1$  then  $P$  is irreducible over  $L$ .

*Proof.* If it is not a case and  $\exists Q$  such that  $Q \mid P$  in  $L[X]$ . Let  $M$  be a Stem field of  $Q$  over  $L$ .

So we have  $K \subset L \subset M = L(\alpha)$ .  $M$  is a stem field that  $[M : L] = \deg Q = d < n$ . Thus  $[M : K] = md$

Lets  $K(\alpha)$  is a stem field of  $P$  over  $K$  then  $[K(\alpha) : K] = \deg P = n$ .

$K(\alpha) \subseteq M$  and therefore  $n \mid md$  thus using  $\gcd(m, n) = 1$  one can get that  $n \mid d$  but this is impossible because  $d < n$ . □



## 2.2 Splitting field

**Definition 2.6** (Splitting field). *Let  $P \in K[X]$ . The splitting field of  $P$  over  $K$  is an extension  $L$  where  $P$  is split (i.e. is a product of linear factors) and roots of  $P$  generate  $L$*

**Theorem 2.7** (About splitting fields). 1. *Splitting field  $L$  exists and  $[L : K] \leq d!$ , where  $d = \deg P$ .*

2. *If  $L$  and  $M$  are 2 splitting fields then  $\exists \phi : L \cong M$  (an Isomorphism). But the Isomorphism is not necessary to be unique.*

*Proof.* Lets prove by induction on  $d$ . The first case ( $d = 1$ ) is trivial the  $K$  itself is the splitting field. Now assume  $d > 1$  and that the theorem is valid for any polynomial of degree  $< d$  over any field  $K$ . Let  $Q$  be any irreducible factor of  $P$ . We can create a Stem field  $L_1 = K(\alpha)$  for  $Q$  that will be also a Stem field for  $P$ .

Over  $L_1$  we have  $P = (x - \alpha)R$ , where  $R$  is a polynomial with  $\deg R = d - 1$ . We know (see remark 2.3) that there exists a Splitting field  $L$  for  $R$  over  $L_1$  and its degree:  $[L : L_1] \leq (d - 1)!$  We have  $K \subset L_1 \subset L$ . The  $L$  will be a splitting field for original polynomial  $P$ . Its degree (by The multiplicativity formula for degrees) is  $\leq d \cdot (d - 1)! = d!$ .

Uniqueness: Let  $L$  and  $M$  are 2 splitting fields. Let  $\beta$  is a root of  $Q$  (irreducible factor of  $P$ ) in  $M$ . We have 2 stem fields:  $L_1 = K(\alpha)$  and  $M_1 = K(\beta)$ . Proposition 2.2 says as that

$$\exists \phi : L_1 = K(\alpha) \cong K(\beta) = M_1,$$

such that  $\phi(\alpha) = \beta$ .

Over  $M_1$  we have  $P = (x - \beta)S$ , where  $S = \phi(R)$ <sup>1</sup>

$M$  is splitting field for  $S$  over  $K(\beta) = M_1$ .  $M$  is also  $L_1$ -algebra (via the Isomorphism  $\phi$ ) and as such it's a splitting field for  $R$  over  $L_1$ . As soon as  $[L : L_1] = [M : M_1]$  the  $M/L_1 \cong L/L_1$  because the  $L_1$ -algebras with the same dimension are isomorphic (see lemma 0.15). Therefore we have an  $L_1 = K(\alpha)$  Isomorphism  $L \cong M$  and therefore  $K$  Isomorphism  $L \cong M$ .  $\square$

**Remark 2.8.** *The Isomorphism is not unique. A splitting field can have many Automorphism and this is in fact the subject of Galois theory.*

---

<sup>1</sup> We have  $\phi : K(\alpha) \rightarrow K(\beta)$ . The  $\phi : K \rightarrow K$  because  $K \subset K(\alpha)$  as well as  $K \subset K(\beta)$ . Therefore  $\phi(P) = P$  because  $P \in K[X]$ . Thus

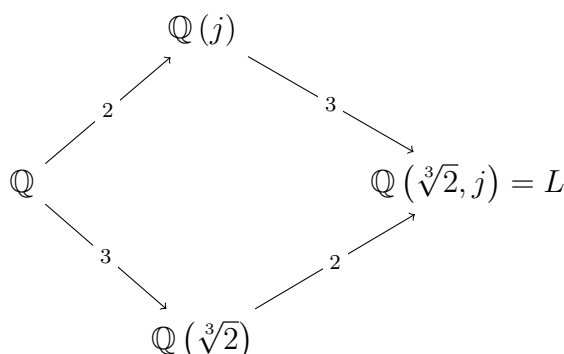
$$P = (x - \beta)S = \phi(P) = \phi((x - \alpha)R) = (x - \beta)\phi(R)$$

and  $S = \phi(R)$ .

## 2.3 An example. Algebraic closure

### 2.3.1 An example of automorphism

**Example 2.3.1** ( $x^3 - 2$  over  $\mathbb{Q}$ ). Let us have the following polynomial  $x^3 - 2$  over  $\mathbb{Q}$ . It has the following roots:  $\sqrt[3]{2}, j\sqrt[3]{2}$  and  $j^2\sqrt[3]{2}$ , where  $j = e^{\frac{2\pi i}{3}}$ . Splitting field is the following  $L = \mathbb{Q}(\sqrt[3]{2}, j)$ . Let's find Automorphisms of the field.



As soon as  $L$  is a stem field for  $\mathbb{Q}(j)$  and for  $\mathbb{Q}(\sqrt[3]{2})$  then 2 types of automorphism exist:

1.  $\mathbb{Q}(\sqrt[3]{2})$  Automorphism. We have  $x^2 + x + 1$  as  $P_{\min}(j, \mathbb{Q}(\sqrt[3]{2}))$ . The polynomial has 2 roots:  $j$  and  $j^2$  and there is an Automorphism that exchanges the root. Let's call it  $\tau$
2.  $\mathbb{Q}(j)$  Automorphism. In this case the automorphism of exchanging  $\sqrt[3]{2}$  and  $j\sqrt[3]{2}$ .<sup>2</sup> Let's call it  $\sigma$

The group of automorphism of  $L$   $\text{Aut}(L/K)$  is embedded into permutation group of 3 elements  $S_3$  (see example 0.1.2):

$$\text{Aut}(L/K) \hookrightarrow S_3.$$

It's embedded because the automorphism exchanges the roots of  $x^3 - 2$ . Moreover

$$\text{Aut}(L/K) = S_3,$$

because  $\sigma$  and  $\tau$  generates  $S_3$  because

- $\sigma: \sqrt[3]{2} \rightarrow j\sqrt[3]{2} \rightarrow j^2\sqrt[3]{2} \rightarrow \sqrt[3]{2}$ . This is a circle.
- $\tau$  - it keeps  $\sqrt[3]{2}$  and exchanges  $j$  and  $j^2$ :  $\sqrt[3]{2}j \leftrightarrow \sqrt[3]{2}j^2$  (???). This is a transposition.

---

<sup>2</sup> ??? The minimal polynomial is  $x^3 - 2$  there and thus we have 3 roots:  $\sqrt[3]{2}, j\sqrt[3]{2}$  and  $j^2\sqrt[3]{2}$

### 2.3.2 Algebraic closure

**Definition 2.9** (Algebraically closed field).  *$K$  is algebraically closed if any non constant polynomial  $P \in K[X]$  has a root in  $K$  or in other words if any  $P \in K[X]$  splits*

**Example 2.3.2** ( $\mathbb{C}$ ).  *$\mathbb{C}$  is an Algebraically closed field. This will be proved later.*

**Definition 2.10** (Algebraic closure). *An algebraic closure of  $K$  is a field  $L$  that is Algebraically closed field and Algebraic extension over  $K$ .*

**Theorem 2.11** (About Algebraic closure). *Any field  $K$  has an Algebraic closure*

*Proof.* Lets discuss the strategy of the prove. First construct  $K_1$  such that  $\forall P \in K[X]$  has a root in  $K_1$ . There is not a victory because  $K_1$  can introduce new coefficients and polynomials that can be irreducible over  $K_1$ . Then construct  $K_2$  such that  $\forall P \in K_1[X]$  has a root in  $K_2$  and so forth. As result we will have

$$K \subset K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$$

Take  $\bar{K} = \cup_i K_i$  and we claim that  $\bar{K}$  is algebraically closed. Really  $\forall P \in \bar{K}[X] \exists j : P \in K_j[X]$  thus it has a root in  $K_{j+1}$  and as result in  $\bar{K}$ .

Now how can we construct  $K_1$ . Let  $S$  be a set of all irreducible  $P \in K[X]$ . Let  $A = K[(X_p)_{p \in S}]$  - multi-variable (one variable  $X_p$  for each  $p \in S$ ) polynomial ring.

Let  $I \subset A$  is an Ideal generated by  $P(X_p) \forall p \in S$ .<sup>3</sup> We claim that  $I$  is a Proper ideal i.e.  $I \neq A$ . If not then we can write

$$1_A = \sum_i^n \lambda_i P_i(X_{p_i}), \quad (2.1)$$

where  $\lambda_i \in A$  and the sum is the finite. As soon as the sum is finite then I can take the product of the polynomials in the sum:  $P = \prod_i^n P_i$  and I can create a Splitting field  $L$  for the polynomial  $P$  over  $K$ .<sup>4</sup>

$A$  is a polynomial ring and it's very easy produce a homomorphism between polynomial algebra and any other algebra. Therefore there is a homomorphism between rings  $A$  and  $L$  such that  $\phi : A \rightarrow L$  where  $X_{p_i} \rightarrow \alpha_i$  if  $P = P_i$  and  $X_{p_i} \rightarrow 0$  otherwise. From (2.1) we have

$$\phi(1_A) = \sum_i^n \lambda_i \phi(P_i(X_{p_i})) = \sum_i^n \lambda_i P_i(\alpha_i) = 0$$

---

<sup>3</sup>  $I = \sum_i \lambda_i P_i(X_{p_i})$ , where  $\lambda_i \in A$

<sup>4</sup>  $\alpha_i$  is a root of  $P_i$

that is impossible.

Fact: Any Proper ideal  $I \subset A$  is contained in the Maximal ideal  $m$  and  $A/m$  is a field.

Thus I can take  $K_1 = A/m$  and continue in the same way to construct  $K_2, K_3, \dots, K_n, \text{dots}$ .  $\square$

### 2.3.3 Ideals in a ring

The ring is commutative, associative with unity. Any Proper ideal is in a Maximal ideal. This is a consequence of what one calls Zorn's lemma

**Definition 2.12** (Chain). *Let  $\mathcal{P}$  is a partially ordered set ( $\leq$  is the order relation).  $\mathcal{C} \subset \mathcal{P}$  is a chain if  $\forall \alpha, \beta \in \mathcal{C}$  exists a relation between  $\alpha$  and  $\beta$  i.e.  $\alpha \leq \beta$  or  $\beta \leq \alpha$ .*

**Lemma 2.13** (Zorn). *If any non-empty Chain  $\mathcal{C}$  in a non-empty set  $\mathcal{P}$  has an upper bound (that is  $M \in \mathcal{P}$  such that  $M \geq x, \forall x \in \mathcal{C}$ ) then  $\mathcal{P}$  has a maximal element.*

We can use Zorn lemma to prove that any proper ideal is in a Maximal ideal.

Let  $\mathcal{P}$  is the set of proper ideals in  $A$  containing  $I$ . The set is not empty because it has at least one element  $I$ . Any Chain  $\mathcal{C} = \{I_\alpha\}$ <sup>5</sup> has an upper bound: it's  $\cup_\alpha I_\alpha$  (exercise that the union is an ideal). So  $\mathcal{P}$  has a maximal element  $m$  and  $I \subset m$ .

If we take a quotient by maximal ideal it's always a field otherwise it will have a proper ideal:  $\exists a \in A/m$  such that  $(a)$  is a proper ideal and its pre-image in  $\pi : A \rightarrow A/m$  should strictly contain  $m$ <sup>6</sup>.

## 2.4 Extension of homomorphisms. Uniqueness of algebraic closure

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<sup>5</sup> The order is the following  $I_\alpha \leq I_\beta$  if  $I_\alpha \subset I_\beta$

<sup>6</sup> ??? i.e.  $m$  is not a maximal ideal in the case.