

# Introduction to Galois Theory

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# Requirements

## 0.1 Groups

**Definition 0.1** (Monoid). *The set of elements  $M$  with defined binary operation  $\circ$  we will call as a monoid if the following conditions are satisfied.*

1. *Closure:  $\forall a, b \in M: a \circ b \in G$*
2. *Associativity:  $\forall a, b, c \in M: a \circ (b \circ c) = (a \circ b) \circ c$*
3. *Identity element:  $\exists e \in M$  such that  $\forall a \in G: e \circ a = a \circ e = a$*

**Definition 0.2** (Group). *Let we have a set of elements  $G$  with a defined binary operation  $\circ$  that satisfied the following properties.*

1. *Closure:  $\forall a, b \in G: a \circ b \in G$*
2. *Associativity:  $\forall a, b, c \in G: a \circ (b \circ c) = (a \circ b) \circ c$*
3. *Identity element:  $\exists e \in G$  such that  $\forall a \in G: e \circ a = a \circ e = a$*
4. *Inverse element:  $\forall a \in G \exists a^{-1} \in G$  such that  $a \circ a^{-1} = e$*

*In this case  $(G, \circ)$  is called as group.*

Therefore the group is a Monoid with inverse element property.

**Example 0.1.1** (Group  $\mathbb{Z}/2\mathbb{Z}$ ). *Consider a set of 2 elements:  $G = \{0, 1\}$  with the operation  $\circ$  defined by the table 1.*

*The identity element is 0 i.e.  $e = 0$ . Inverse element is the element itself because  $\forall a \in G: a \circ a = 0 = e$ .*

**Definition 0.3** (Cyclic group). *A cyclic group or monogenous group is a group that is generated by a single element. Note that Group  $\mathbb{Z}/2\mathbb{Z}$  is a cyclic group.*

Table 1: Cayley table for  $\mathbb{Z}/2\mathbb{Z}$ 

$\circ$	0	1
0	0	1
1	1	0

**Definition 0.4** (Order of element in group). *Order, sometimes period, of an element  $a$  of a group is the smallest positive integer  $m$  such that  $a^m = e$  (where  $e$  denotes the identity element of the group, and  $a^m$  denotes the product of  $m$  copies of  $a$ ). If no such  $m$  exists,  $a$  is said to have infinite order.*

**Definition 0.5** (Subgroup). *Let we have a Group  $(G, \circ)$ . The subset  $S \subset G$  is called as subgroup if  $(S, \circ)$  is a Group.*

**Definition 0.6** (Abelian group). *Let we have a Group  $(G, \circ)$ . The group is called an Abelian or commutative if  $\forall a, b \in G$  it holds  $a \circ b = b \circ a$ .*

**Definition 0.7** (Coset). *If  $G$  is a group, and  $H$  is a subgroup of  $G$ , and  $g$  is an element of  $G$ , then*

$$gH = \{gh | h \in H\}$$

*is the left coset of  $H$  in  $G$  with respect to  $g$ , and*

$$Hg = \{hg | h \in H\}$$

*is the right coset of  $H$  in  $G$  with respect to  $g$ .*

### 0.1.1 Permutations

**Example 0.1.2** ( $S_n$  group). *If we have a permutation of  $n$  elements then it's possible to do by means of  $n!$  ways.*

$S_1$  permutation of 1 element consists of only one element  $e$  - the simplest possible group

$S_2$  permutation consists of 2 elements:

1. identity  $e$ :

$$\begin{aligned} 1 &\rightarrow 1 \\ 2 &\rightarrow 2 \end{aligned}$$

2. transposition  $\tau$ :

$$\begin{aligned} 1 &\rightarrow 2 \\ 2 &\rightarrow 1 \end{aligned}$$

Table 2: Cayley table for  $S_2$ 

$\circ$	$e$	$\tau$
$e$	$e$	$\tau$
$\tau$	$\tau$	$e$

It's easy to see that the Cayley table has the form 2

$S_3$  permutation consists of 6 elements:  $e, \tau, \tau_1, \tau_2, \sigma, \sigma_1$ . The most important are  $e, \tau$  and  $\sigma$  and all others are represented via them.

1. identity  $e$ :

$$\begin{aligned} 1 &\rightarrow 1 \\ 2 &\rightarrow 2 \\ 3 &\rightarrow 3 \end{aligned}$$

2. transposition  $\tau$ :

$$\begin{aligned} 1 &\rightarrow 2 \\ 2 &\rightarrow 1 \\ 3 &\rightarrow 3 \end{aligned}$$

3. circle  $\sigma$ :

$$\begin{aligned} 1 &\rightarrow 2 \\ 2 &\rightarrow 3 \\ 3 &\rightarrow 1 \end{aligned}$$

## 0.2 Rings, Ideals and Fields

**Definition 0.8** (Ring). Consider a set  $R$  with 2 binary operations defined. The first one  $\oplus$  (addition) and elements of  $R$  forms an Abelian group under this operation. The second one is  $\odot$  (multiplication) and the elements of  $R$  forms a Monoid under the operation. The two binary operations are connected each other via the following distributive law

- Left distributivity:  $\forall a, b, c \in R: a \odot (b \oplus c) = a \odot b \oplus a \odot c$
- Right distributivity:  $\forall a, b, c \in R: (a \oplus b) \odot c = a \odot c \oplus b \odot c$

The identity element for  $(R, \oplus)$  is denoted as 0 (additive identity). The identity element for  $(R, \odot)$  is denoted as 1 (multiplicative identity).

The inverse element to  $a$  in  $(R, \oplus)$  is denoted as  $-a$

In this case  $(R, \oplus, \odot)$  is called as ring.

The Ring is a generalization of integer numbers conception.

**Example 0.2.1** (Ring of integers  $\mathbb{Z}$ ). *The set of integer numbers  $\mathbb{Z}$  forms a Ring under  $+$  and  $\cdot$  operations i.e. addition  $\oplus$  is  $+$  and multiplication  $\odot$  is  $\cdot$ . Thus for integer numbers we have the following Ring:  $(\mathbb{Z}, +, \cdot)$*

**Definition 0.9** (Ideal). *Lets we have the Ring  $(R, \oplus, \odot)$ . Subset  $I \subset R$  will be an ideal if it satisfied the following conditions*

1.  $(I, \oplus)$  is Subgroup of  $(R, \oplus)$
2.  $\forall i \in I$  and  $\forall r \in R$ :  $i \odot r \in I$  and  $r \odot i \in I$

**Example 0.2.2** (Ideal  $2\mathbb{Z}$ ). *Consider even numbers. They forms an Ideal in  $\mathbb{Z}$ . Because multiplication of any even number to any integer is an even. The ideal's symbolic name is  $2\mathbb{Z}$ .*

**Example 0.2.3** (Ring of integers modulo  $n$ :  $\mathbb{Z}/n\mathbb{Z}$ ). *Let  $n \in \mathbb{Z}$  and  $n > 1$ . Then  $n\mathbb{Z}$  is an Ideal.*

*Two integer  $a, b \in \mathbb{Z}$  are said to be congruent modulo  $n$ , written*

$$a \equiv b \pmod{n}$$

*if their difference  $a - b$  is an integer multiple of  $n$ .*

*Thus we have a separation of set  $\mathbb{Z}$  into subsets of numbers that are congruent. Each subset has the following form*

$$\{r\}_n = r + n\mathbb{Z} = \{r + nk \mid k \in \mathbb{Z}\}$$

*, thus*

$$\mathbb{Z} = \{0\}_n \cup \{1\}_n \cup \dots \cup \{n-1\}_n.$$

*Very often use the following notation*

$$\bar{r} = \{r\}_n.$$

*We can define the following operations*

$$\begin{aligned}\bar{k} \oplus \bar{l} &= \overline{k + l} \\ \bar{k} \odot \bar{l} &= \overline{k \cdot l}\end{aligned}$$

*The Ring where the objects are defined is called as  $\mathbb{Z}/n\mathbb{Z}$ .*

**Definition 0.10** (Principal ideal). *The ideal that is generated by one element  $a$  is called as principal ideal and is denoted as  $(a)$  i.e. left principal ideal:  $(a) = \{ra \mid \forall r \in R\}$  and right principal ideal:  $(a) = \{ar \mid \forall r \in R\}$*



**Definition 0.11** (Integral domain). *In mathematics, and specifically in abstract algebra, an integral domain is a nonzero commutative Ring in which the product of any two nonzero elements is nonzero.*

**Definition 0.12** (Principal ideal domain). *In abstract algebra, a principal ideal domain, or PID, is an Integral domain in which every ideal is principal, i.e., can be generated by a single element.*

**Definition 0.13** (Maximal ideal).  *$I$  is a maximal ideal of a ring  $R$  if there are no other ideals contained between  $I$  and  $R$ .*

**Definition 0.14** (Proper ideal).  *$I$  is a proper ideal of a ring  $R$  if  $I \subsetneq R$ .*

**Definition 0.15** (Quotient ring). *Quotient ring is a construction where one starts with a ring  $R$  and a two-sided ideal  $I$  in  $R$ , and constructs a new ring, the quotient ring  $R/I$ , whose elements are the Cosets of  $I$  in  $R$  subject to special  $+$  and  $\cdot$  operations.*

*Given a ring  $R$  and a two-sided ideal  $I \subset R$ , we may define an equivalence relation  $\sim$  on  $R$  as follows:  $a \sim b$  if and only if  $a - b \in I$ . The equivalence class of the element  $a$  in  $R$  is given by*

$$[a] = a + I := \{a + r : r \in I\}.$$

*This equivalence class is also sometimes written as  $a \bmod I$  and called the "residue class of  $a$  modulo  $I$ ".*

**Definition 0.16** (Field). *The ring  $(R, \oplus, \odot)$  is called as a field if  $(R \setminus \{0\}, \odot)$  is an Abelian group.*

*The inverse element to  $a$  in  $(R \setminus \{0\}, \odot)$  is denoted as  $a^{-1}$*

**Example 0.2.4** (Field  $\mathbb{Q}$ ). *Note that  $\mathbb{Z}$  is not a field because not for every integer number an inverse exists. But if we consider a set of fractions  $\mathbb{Q} = \{a/b \mid a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\}\}$  when it will be a field.*

*The inverse element to  $a/b$  in  $(\mathbb{Q} \setminus \{0\}, \cdot)$  will be  $b/a$ .*

**Definition 0.17** (Unique factorization domain). *Unique factorization domain (UFD) is a commutative ring, which is an Integral domain, and in which every non-zero non-unit element can be written as a product of prime elements (or irreducible elements), uniquely up to order and units, analogous to the fundamental theorem of arithmetic for the integers.*

### 0.3 Linear algebra

**Definition 0.18** (Vector space). *Let  $F$  is a Field. The set  $V$  is called as vector space under  $F$  if the following conditions are satisfied*

1. *We have a binary operation  $V \times V \rightarrow V$  (addition):  $(x, y) \rightarrow x + y$  with the following properties:*

$$(a) \ x + y = y + x$$

$$(b) \ (x + y) + z = x + (y + z)$$

$$(c) \ \exists 0 \in V \text{ such that } \forall x \in V : x + 0 = x$$

$$(d) \ \forall x \in V \exists -x \in V \text{ such that } x + (-x) = x - x = 0$$

2. *We have a binary operation  $F \times V \rightarrow V$  (scalar multiplication) with the following properties*

$$(a) \ 1_F \cdot x = x$$

$$(b) \ \forall a, b \in F, x \in V : a \cdot (b \cdot x) = (ab) \cdot x.$$

$$(c) \ \forall a, b \in F, x \in V : (a + b) \cdot x = a \cdot x + b \cdot x$$

$$(d) \ \forall a \in F, x, y \in V : a \cdot (x + y) = a \cdot x + a \cdot y$$

**Lemma 0.19** (About vector space isomorphism). *2 vector spaces  $L$  and  $M$  with same dimension  $\dim L = \dim M$  then there exists an Isomorphism between them*

### 0.4 Functions

**Definition 0.20** (Surjection). *The function  $f : X \rightarrow Y$  is surjective (or onto) if  $\forall y \in Y, \exists x \in X$  such that  $f(x) = y$ .*

**Definition 0.21** (Injection). *The function  $f : X \rightarrow Y$  is injective (or one-to-one function) if  $\forall x_1, x_2 \in X$ , such that  $x_1 \neq x_2$  then  $f(x_1) \neq f(x_2)$ .*

**Definition 0.22** (Bijection). *The function  $f : X \rightarrow Y$  is bijective (or one-to-one correspondence) if it is an Injection and a Surjection.*

**Definition 0.23** (Homomorphism). *The homomorphism is a function (map) between two sets that preserves its algebraic structure. For the case of groups  $(X, \circ)$  and  $(Y, \odot)$  the function  $f : X \rightarrow Y$  is called homomorphism if  $\forall x_1, x_2 \in X$  it holds  $f(x_1 \circ x_2) = f(x_1) \odot f(x_2)$ .*

**Definition 0.24** (Isomorphism). *If a map is Bijection as well as Homomorphism when it is called as isomorphism.*

*We use the following symbolic notation for isomorphism between  $X$  and  $Y$ :  $X \cong Y$ .*

**Definition 0.25** (Automorphism). *Automorphism is an isomorphism from a mathematical object to itself.*

**Definition 0.26** (Embedding). *When some object  $X$  is said to be embedded in another object  $Y$ , the embedding is given by some injective and structure-preserving map  $f : X \rightarrow Y$ . The precise meaning of "structure-preserving" depends on the kind of mathematical structure of which  $X$  and  $Y$  are instances.*

*The fact that a map  $f : X \rightarrow Y$  is an embedding is often indicated by the use of a "hooked arrow", thus:  $f : X \hookrightarrow Y$ . On the other hand, this notation is sometimes reserved for inclusion maps.*

## 0.5 Polynomial ring $K[X]$

Let we have a commutative Ring  $K$ . Lets create a new Ring  $B$  with the following infinite sets as elements:

$$f = (f_0, f_1, \dots), f_i \in K, \quad (1)$$

such that only finite number of elements of the sets are non zero.

We can define addition and multiplication on  $B$  as follows

$$\begin{aligned} f + g &= (f_0 + g_0, f_1 + g_1, \dots), \\ f \cdot g &= h = (h_0, h_1, \dots), \end{aligned} \quad (2)$$

where

$$h_k = \sum_{i+j=k} f_i g_j.$$

The sequences (1) forms a Ring with the following identities:

- Additive identity:  $(0, 0, \dots)$
- Multiplicative identity:  $(1, 0, \dots)$

The sequences  $k = (k, 0, \dots)$  added and multiplied as elements of  $K$  this allows say that such elements are elements of original Ring  $K$ . Thus  $K$  is sub-ring of the new ring  $B$ .

Let

$$\begin{aligned} X &= (0, 1, 0, \dots), \\ X^2 &= (0, 0, 1, \dots) \end{aligned}$$

thus if we have

$$f = (f_0, f_1, f_2, \dots, f_n, 0, \dots),$$

where  $f_n$  is the last non-zero element of (1), when one can get

$$f = f_0 + f_1X + f_2X^2 + \dots + f_nX^n.$$

**Definition 0.27** (Polynomial ring). *The Ring of sequences (1) with operations defined by (2) is called as polynomial ring  $K[X]$ .*

**Lemma 0.28** (Bézout's lemma). *Let  $a$  and  $b$  be nonzero integers and let  $d$  be their greatest common divisor. Then there exist integers  $x$  and  $y$  such that*

$$ax + by = d.$$

**Definition 0.29** (Monic polynomial). *Monic polynomial is a univariate polynomial in which the leading coefficient (the nonzero coefficient of highest degree) is equal to 1. Therefore, a monic polynomial has the form*

$$x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$$

**Theorem 0.30** (About irreducible polynomials). *Let  $\pi(X)$  is an irreducible polynomial in  $K[X]$  and let  $\alpha$  be a root of  $\pi(X)$  in a some larger field.  $\forall h(x) \in K(X)$  if have the following statement:  $h(\alpha) = 0$  if and only if  $\pi(X) \mid h(X)$  in  $K[X]$ .*

*Proof.* If  $h(X) = \pi(X)g(X)$  then  $h(\alpha) = 0$

From other side let  $\pi \nmid h$  in  $K[X]$  this means that they are relatively prime in  $K[X]$  and by Bézout's lemma we can get  $Q, R \in K[X]$  such that

$$\pi(X)R(X) + h(X)Q(X) = 1,$$

and especially for  $X = \alpha$  we will get that  $0 = 1$  that is impossible.  $\square$

# Chapter 1

## Generalities on algebraic extensions

We introduce the basic notions such as a field extension, algebraic element, minimal polynomial, finite extension, and study their very basic properties such as the multiplicativity of degree in towers.

### 1.1 Field extensions: examples

#### 1.1.1 K-algebra

**Definition 1.1** (K-algebra). *Let  $K$  be a field and  $A$  be a Vector space over  $K$  equipped with an additional binary operation  $A \times A \rightarrow A$  that we denote as  $\cdot$  here. The  $A$  is an algebra over  $K$  if the following identities hold  $\forall x, y, z \in A$  and for every elements (often called as scalar)  $a, b \in K$*

- *Right distributivity:*  $(x + y) \cdot z = x \cdot z + y \cdot z$
- *Left distributivity:*  $z \cdot (x + y) = z \cdot x + z \cdot y$
- *Compatibility with scalars:*  $(ax) \cdot (by) = (ab)(x \cdot y)$

**Example 1.1.1** (Field of complex numbers  $\mathbb{C}$ ). *The field of complex numbers  $\mathbb{C}$  can be considered as a  $K$ -algebra over field of real numbers  $\mathbb{R}$ .*

#### 1.1.2 Field extension

Let  $K$  and  $L$  are fields.

**Definition 1.2** (Field extension).  *$L$  is an extension of  $K$  if  $L \supset K$*

and another definition

**Definition 1.3** (Field extension). *L is an extension of K if L is a K-algebra*

Why the 2 definitions are equivalent?

**Lemma 1.4** (K-algebra and Homomorphism). *Given a K-algebra is the same as having Homomorphism  $f : K \rightarrow A$  of rings.*

*Proof.* Really if I have a K-algebra I can define the Homomorphism  $f(k) = k \cdot 1_A$ , where  $1_A$  is an identity element of  $A$ . Thus  $k \cdot 1_A \in A$ .

And conversely if I have the Homomorphism  $f : K \rightarrow A$  I can define the K-algebra structure by setting  $ka = f(k)a$  because  $f(k), a \in A$  and there is a multiplication defined on  $A$ . As result I have a rule for multiplication a scalar ( $k \in K$ ) on a vector ( $a \in A$ ).  $\square$

**Lemma 1.5** (About Homomorphism of fields). *Any Homomorphism of fields is Injection.*

*Proof.* Lets proof by contradiction. Really if  $f(x) = f(y)$  and  $x \neq y$  then

$$\begin{aligned} f(x) - f(y) &= 0_A, \\ f(x - y) &= 0_A, \\ f(x - y)f((x - y)^{-1}) &= f\left(\frac{x - y}{x - y}\right) = f(1_K) = 1_A = 0_A \end{aligned}$$

that is impossible.  $\square$

There are some comments on the results. We have got that a Homomorphism can be set between field  $K$  and its K-algebra. This means that K-algebra is a field. The Homomorphism is Injection therefore we can allocate a sub-field  $A' \subset A$  for that we will have the Homomorphism is a Surjection and therefore we have an Isomorphism between original field  $K$  and a sub-field  $A'$ . This means that we can say that the original field  $K$  is a sub-field for the K-algebra.

**Example 1.1.2** (Field extensions).  *$\mathbb{C}$  is a field extension for  $\mathbb{R}$ .  $\mathbb{R}$  is a field extension for  $\mathbb{Q}$*

### 1.1.3 Field characteristic

If  $L$  is a field there are 2 possibilities

1.  $1 + 1 + \cdots \neq 0$ . In this case  $\mathbb{Z} \subset L$  but  $\mathbb{Z}$  is not a field therefore  $L$  is an extension of  $\mathbb{Q}$ . In the case  $\text{char} L = 0$
2.  $1 + 1 + \cdots + 1 = \sum_{i=1}^m 1 = 0$  for some  $m \in \mathbb{Z}$ . The first time when it happens is for a prime number i.e. minimal  $m$  with the property is prime. In this case  $\text{char} L = p$ , where  $p = \min m$  - the minimal  $m$  (prime) with the property. In this case  $\mathbb{Z}/p\mathbb{Z} \subset L$ . The  $\mathbb{Z}/p\mathbb{Z}$  is a field denoted by  $\mathbb{F}_p$ . The  $L$  is an extension of  $\mathbb{F}_p$ .

No other possibilities exist. The  $\mathbb{Q}$  and  $\mathbb{F}_p$  are the prime fields. Any field is an extension of one of those.

### 1.1.4 Field $K[X]/(P)$

Let  $K[X]$  Ring of polynomials. The  $P \in K[X]$  is an irreducible.  $(P)$  is an Ideal formed by the polynomial. The set of residues by the polynomial forms a field that denoted by  $K[X]/(P)$ . How we can see it? If  $Q \in K[X]$  is a polynomial that  $Q \notin (P)$  when  $Q$  is prime to  $P$ . Then with Bézout's lemma we can get  $\exists A, B \in K[X]$  such that

$$AP + BQ = 1,$$

or

$$BQ \equiv 1 \pmod{P},$$

thus  $B$  is  $Q^{-1}$  in  $K[X]/(P)$ .

## 1.2 Algebraic elements. Minimal polynomial

### 1.2.1 $K[X]/(P)$ field

Alternative proof that  $K[X]/(P)$  is the Field. The  $(P)$  is a Maximal ideal but a quotient by a Maximal ideal is a Field.

$K[X]/(P)$  is an extension of  $K$  because it's  $K$ -algebra.

**Example 1.2.1** ( $\mathbb{F}_2/(x^2 + x + 1)$ ). Lets consider the following field  $K = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$  in the field polynomial  $x^2 + x + 1$  is irreducible. It's very easy to verify it because  $\mathbb{F}_2$  has only 2 elements that can be (possible) a root:

$$0^2 + 0 + 1 = 1 \neq 0$$

and

$$1^2 + 1 + 1 = 1 \neq 0$$

The polynomial has the following residues:  $\bar{x} = x + (x^2 + x + 1)$  and  $\overline{x+1} = x + 1 + (x^2 + x + 1)$ . Thus the field  $\mathbb{F}_2/(x^2 + x + 1)$  consists of 4 elements:  $\{0, 1, \bar{x}, \overline{x+1}\}$ .

It's easy to see that the third element ( $\bar{x}$ ) is a root of  $P(x) = x^2 + x + 1$ :

$$\bar{x}^2 + \bar{x} + 1 = P(x) + (P(x)) = (P(x)) \equiv 0 \pmod{P}.$$

$$\bar{x}^2 + \bar{x} + 1 = \bar{0},$$

therefore

$$\bar{x}^2 = -\bar{x} - 1 = \bar{x} + 1 = \overline{x+1}.$$

This is because we are in field  $\mathbb{F}_2$  where

$$2(x+1) \pmod{2} = 0$$

and thus

$$-\bar{x} - 1 = \bar{x} + 1$$

Also

$$\overline{x+1}^2 = \bar{x},$$

and they are inverse each other

$$\overline{x+1}\bar{x} = 1,$$

### 1.2.2 Algebraic elements

**Definition 1.6** (Algebraic element). Let  $K \subset L$  and  $\alpha \in L$ .  $\alpha$  is an algebraic element if  $\exists P \in K[X]$  such that  $P(\alpha) = 0$ . Otherwise the  $\alpha$  is called transcendental.

### 1.2.3 Minimal polynomial

**Lemma 1.7** (About minimal polynomial existence). If  $\alpha$  is Algebraic element then  $\exists!$  unitary polynomial  $P$  of minimal degree such that  $P(\alpha) = 0$ . It is irreducible.  $\forall Q$  such that  $Q(\alpha) = 0$  is divisible by  $P$

**Definition 1.8** (Minimal polynomial). Such polynomial is called minimal polynomial and denoted by  $P_{\min}(\alpha, K)$ .



*Proof.* We know that  $K[X]$  is a Principal ideal domain and a polynomial  $Q(\alpha) = 0$  forms an Ideal:  $I = \{Q \in K[X] \mid Q(\alpha) = 0\}$ , so the ideal is generated by one element:  $I = (P)$ . This is an unique (up to constant) polynomial minimal degree in  $I$ . If  $P$  is not irreducible then  $\exists Q, R \in I$  such that  $P = QR$ ,  $Q(\alpha) = 0$  or  $R(\alpha) = 0$  and  $\deg R, Q < \deg P$  that is in contradiction with the definition that  $P$  is a polynomial of minimal degree.  $\square$

### 1.3 Algebraic elements. Algebraic extensions

**Definition 1.9.** Let  $K \subset L$ ,  $\alpha \in L$ . The smallest sub-field contained  $K$  and  $\alpha$  denoted by  $K(\alpha)$ . The smallest sub-ring contained  $K$  and  $\alpha$  denoted by  $K[\alpha]$ .

As soon as  $K[\alpha]$  is a  $K$ -algebra it is a Vector space generated by  $1, \alpha, \alpha^2, \dots, \alpha^n, \dots$ .

**Example 1.3.1** ( $\mathbb{C}$ ).

$$\mathbb{C} = \mathbb{R}(i) = \mathbb{R}[i]$$

$\mathbb{C}$  is also a Vector space generated by  $1$  and  $i$ :  $\forall z \in \mathbb{C}$  it holds  $z = x + iy$  where  $x, y \in \mathbb{R}$ .

**Proposition 1.10.** The following assignment are equivalent

1.  $\alpha$  is algebraic over  $K$
2.  $K[\alpha]$  is a finite dimensional Vector space over  $K$
3.  $K[\alpha] = K(\alpha)$

*Proof.* Lets proof that 1 implies 2. If  $\alpha$  is algebraic over  $K$  then using lemma Minimal polynomial  $\exists P_{min}(\alpha, K)$ :

$$P_{min}(\alpha, K) = \alpha^d + a_{d-1}\alpha^{d-1} + a_1\alpha + a_0 = 0,$$

where  $a_k \in K$ . Then

$$\alpha^d = -a_{d-1}\alpha^{d-1} - a_1\alpha - a_0$$

this means that any  $\alpha^n$  can be represented as a linear combination of finite number of powers of  $\alpha$  i.e.  $K[\alpha]$  generated by  $1, \alpha, \dots, \alpha^{d-1}$  is a finite dimensional Vector space.

Lets proof that 2 implies 3. Its enough proof that  $K[\alpha]$  is a field. Let  $x \neq 0 \in K[\alpha]$  then lets look at an operation  $x \cdot K[\alpha] \rightarrow K[\alpha]$ . This is Injection because if  $y, z \in K[\alpha]$  and  $z \neq y$  then  $x \cdot y \neq x \cdot z$ . But the  $K[\alpha]$

is finite dimensional Vector space and a Homomorphism between 2 vector spaces with the same dimension is Surjection thus  $\exists y \in K[\alpha]$  such that  $x \cdot y = 1_{K[\alpha]}$ . Therefore  $x$  is invertable and  $K[\alpha]$  is a Field.

Lets proof that 3 implies 1. Let  $K[\alpha]$  is a Field but  $\alpha$  is not algebraic. Thus  $\forall P \in K[X] P(\alpha) \neq 0$ . Then we have an Injection Homomorphism  $f : K[X] \rightarrow K[\alpha]$  but  $K[X]$  is not a field thus  $K[\alpha]$  should not be a field too that is in contradiction with the initial conditions.  $\square$

**Definition 1.11** (Algebraic extension).  *$L$  an extension of  $K$  is called algebraic if  $\forall \alpha \in L - \alpha$  is algebraic over  $K$ .*

**Proposition 1.12.** *If  $L$  is algebraic over  $K$  then any  $K$ -subalgebra of  $L$  is a Field.*

*Proof.* Let  $L' \subset L$  a subalgebra and let  $\alpha \in L'$ . We want to show that  $\alpha$  is invertable.  $\alpha$  is algebraic therefore  $\alpha \in K[\alpha] \subset L' \subset L$  and it's invertable.  $\square$

**Proposition 1.13.** *Let  $K \subset L \subset M$ .  $\alpha \in M$  - algebraic over  $K$  then  $\alpha$  algebraic over  $L$  and  $P_{min}(\alpha, L)$  divides  $P_{min}(\alpha, K)$ .*

*Proof.* Its clear because  $P_{min}(\alpha, K) \in L[X]$  thus  $\exists P_L \in L[X]$  such that  $P_L(\alpha) = 0$  i.e.  $\alpha$  is algebraic over  $L$ .

As soon as  $P_{min}(\alpha, K) \in L[X]$  then  $\deg P_{min}(\alpha, L) \leq \deg P_{min}(\alpha, K)$  and as soon as  $P_{min}(\alpha, K) \in (P_{min}(\alpha, L))$  then  $P_{min}(\alpha, L)$  divides  $P_{min}(\alpha, K)$ .  $\square$

## 1.4 Finite extensions. Algebraicity and finiteness

**Definition 1.14** (Finite extension).  *$L$  is a finite extension of  $K$  if  $\dim_K L < \infty$ .  $\dim_K L$  is called as degree of  $L$  over  $K$  and is denoted by  $[L : K]$*

**Theorem 1.15** (The multiplicativity formula for degrees). *Let  $K \subset L \subset M$ . Then  $M$  is Finite extension over  $K$  if and only if  $M$  is Finite extension over  $L$  and  $L$  is Finite extension over  $K$ . In this case*

$$[M : K] = [M : L][L : K].$$

*Proof.* Let  $[M : K] < \infty$  but any linear independent set of vectors  $\{m_1, m_2, \dots, m_n\}$  over  $L$  is also linear independent over  $K$  thus

$$[M : K] < \infty \Rightarrow [M : L] < \infty$$

also  $L$  is a vector sub space of  $M$  thus if  $[M : K] < \infty$  then  $[L : K] < \infty$ .

Let  $[M : L] < \infty$  and  $[L : K] < \infty$  then we have the following baseses

- $L$ -basis over  $M$ :  $(e_1, e_2, \dots, e_n)$
- $K$ -basis over  $L$ :  $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d)$

Lets proof that  $e_i \varepsilon_j$  forms a  $K$ -basis over  $M$ .  $\forall x \in M$ :

$$x = \sum_{i=1}^n a_i e_i,$$

where  $a_i \in L$  and can be also written as

$$a_i = \sum_{j=1}^d b_{ij} \varepsilon_j,$$

where  $b_{ij} \in K$ . Thus

$$x = \sum_{i=1}^n \sum_{j=1}^d b_{ij} \varepsilon_j e_i,$$

therefore  $\varepsilon_j e_i = e_i \varepsilon_j$  generates  $M$  over  $K$ . From the other side we should check that  $\varepsilon_j e_i$  linear independent system of vectors. Lets

$$\sum_{i,j} c_{ij} \varepsilon_j e_i = \sum_{i=1}^n \left( \sum_{j=1}^d c_{ij} \varepsilon_j \right) e_i,$$

then  $\forall i$ :

$$\sum_{j=1}^d c_{ij} \varepsilon_j = 0.$$

Thus  $\forall i, j : c_{ik} = 0$  that finishes the proof the linear independence. The number of linear independent vectors is  $n \times d$  i.e.

$$[M : K] = [M : L] [L : K].$$

□

**Definition 1.16** ( $K(\alpha_1, \dots, \alpha_n)$ ).  $K(\alpha_1, \dots, \alpha_n) \subset L$  generated by  $\alpha_1, \dots, \alpha_n$  is the smallest sub field of  $L$  contained  $K$  and  $\alpha_i \in L$ .

**Theorem 1.17** (About towers).  $L$  is finite over  $K$  if and only if  $L$  is generated by a finite number of algebraic elements over  $K$ .

*Proof.* If  $L$  is finite then  $\alpha_1, \dots, \alpha_d$  is a basis. In this case  $L = K[\alpha_1, \dots, \alpha_d] = K(\alpha_1, \dots, \alpha_d)$ . Moreover each  $K[\alpha_i]$  is finite dimensional thus by proposition 1.10  $\alpha_i$  is algebraic.

From other side if we have a finite set of algebraic elements  $\alpha_1, \dots, \alpha_d$  then  $K[\alpha_1]$  is a finite dimensional Vector space over  $K$ ,  $K[\alpha_1, \alpha_2]$  is a finite dimensional Vector space over  $K[\alpha_1]$  and so on  $K[\alpha_1, \dots, \alpha_d]$  is a finite dimensional Vector space over  $K[\alpha_1, \dots, \alpha_{d-1}]$ . All elements are algebraic thus

$$K[\alpha_1, \dots, \alpha_i] = K(\alpha_1, \dots, \alpha_i)$$

Then using theorem 1.15 we can conclude that  $K(\alpha_1, \dots, \alpha_d)$  has finite dimension.  $\square$

## 1.5 Algebraicity in towers. An example

**Theorem 1.18.**  $K \subset L \subset M$  then  $M$  Algebraic extension over  $K$  if and only if  $M$  algebraic over  $L$  and  $L$  algebraic over  $K$ .

*Proof.* If  $\alpha \in M$  is an Algebraic element over  $K$  then  $\exists P \in K[X]$  such that  $P(\alpha) = 0$  but the polynomial  $P \in K[X] \subset L[X]$  thus  $\alpha$  is algebraic over  $L$ . If  $\alpha \in L \subset M$  then  $\alpha$  is algebraic over  $K$  thus  $L$  is algebraic over  $K$ .

Let  $M$  algebraic over  $L$  and  $L$  algebraic over  $K$  and let  $\alpha \in M$ . We want to prove that  $\alpha$  is algebraic over  $K$ . Lets consider  $P_{min}(\alpha, L)$  the polynomial coefficients are from  $L$  and they (as soon as they count is a finite) generate a finite extension  $E$  over  $K$  thus  $E(\alpha)$  is finite over  $E$  (exists a relation between powers of  $\alpha$ ) and by theorem 1.17 is finite over  $K$  thus  $\alpha$  is algebraic over  $K$ .  $\square$

**Example 1.5.1** ( $\mathbb{Q}$  extension).  $\mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$  algebraic and finite over  $\mathbb{Q}$ :

$$\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2}) \subset \mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$$

*Minimal polynomial*

$$P_{min}(\sqrt[3]{2}, \mathbb{Q}) = x^3 - 2.$$

$\mathbb{Q}(\sqrt[3]{2})$  is generated over  $\mathbb{Q}$  by  $1, \sqrt[3]{2}, \sqrt[3]{4}$  thus  $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ .

But  $\sqrt{3} \notin \mathbb{Q}(\sqrt[3]{2})$  because otherwise  $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$  must divide  $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$  that is impossible.

Therefore  $x^2 - 3$  is irreducible over  $\mathbb{Q}(\sqrt[3]{2})$  and

$$P_{\min}(\sqrt{3}, \mathbb{Q}(\sqrt[3]{2})) = x^2 - 3.$$

$$[\mathbb{Q}(\sqrt[3]{2}, \sqrt{3}) : \mathbb{Q}] = 3 \cdot 2 = 6.$$

**Proposition 1.19** (On dimension of extension).

$$[K(\alpha) : K] = \deg P_{\min}(\alpha, K),$$

if  $\alpha$  is algebraic.

*Proof.* If  $\deg P_{\min}(\alpha, K) = d$  then  $1, \alpha, \dots, \alpha^{d-1}$  -  $d$  independent vectors and dimension  $K(\alpha)$  is  $d$ .  $\square$

**Proposition 1.20** (About algebraic closure). If  $K \subset L$  ( $L$  extension of  $K$ ). Consider

$$L' = \{\alpha \in L \mid \alpha \text{ algebraic over } K\},$$

then  $L'$  sub-field of  $L$  and is called as algebraic closure of  $K$  in  $L$ .

*Proof.* We have to prove that if  $\alpha, \beta$  are algebraic then  $\alpha + \beta$  and  $\alpha \cdot \beta$  are also algebraic. This is trivial because

$$\alpha + \beta, \alpha \cdot \beta \in K[\alpha, \beta] = K(\alpha, \beta)$$

$\square$

## 1.6 A digression: Gauss lemma, Eisenstein criterion

What we have seen so far:

- $K$  is a field,  $\alpha$  is an Algebraic element over  $K$  if it is a root of a polynomial  $P \in K[X]$ .
- $L$  is an Algebraic extension over  $K$  if  $\forall \alpha \in L$ :  $\alpha$  is an algebraic over  $K$
- $L$  is a Finite extension over  $K$  if  $\dim_K L < \infty$ .
- If an extension is finite then it is algebraic

- An extension is finite if and only if it is algebraic and generated by a finite number of algebraic elements (see theorem 1.17)
- $[K[\alpha] : K] = \deg P_{\min}(\alpha, K)$  (see proposition 1.19).

How to decide that a polynomial  $P$  is irreducible over  $K$ ? About polynomial  $x^3 - 2$  it is easy to decide that it's irreducible over  $\mathbb{Q}$ , but what's about  $x^{100} - 2$ ?

**Lemma 1.21** (Gauss). *Let  $P \in \mathbb{Z}[X]$ , i.e. a polynomial with integer coefficients, then if  $P$  decomposes over  $\mathbb{Q}$  ( $P = Q \cdot R$ ,  $\deg Q, R < \deg P$ ) then it also decomposes over  $\mathbb{Z}$ .*

*Proof.* Let  $P = QR$  over  $\mathbb{Q}$ . Then

$$\begin{aligned} Q &= mQ_1, Q_1 \in \mathbb{Z}[X], \\ R &= nR_1, R_1 \in \mathbb{Z}[X], \end{aligned}$$

thus

$$nmP = Q_1R_1.$$

There exists  $p$  that divides  $mn$ :  $p \mid mn$  thus in modulo  $p$  we have

$$0 = \overline{Q_1R_1}$$

but  $p$  is prime and the equation is in the field  $\mathbb{F}_p$  thus either  $\overline{Q_1} = 0$  or  $\overline{R_1} = 0$ . Let  $\overline{Q_1} = 0$  thus  $p$  divides all coefficients in  $Q_1$  and we can take  $\frac{Q_1}{p} = Q_2 \in \mathbb{Z}[X]$ . Continue for all primes in  $mn$  we can get that

$$P = Q_s R_t,$$

where  $Q_s, R_t \in \mathbb{Z}[X]$ . □

**Example 1.6.1** (Eisenstein criterion). *Lets consider the following polynomial  $x^{100} - 2$ . It's irreducible. Lets prove it. If it reducible then  $\exists Q, R \in \mathbb{Z}[X]$  such that*

$$x^{100} - 2 = QR \tag{1.1}$$

*Lets consider (1.1) modulo 2. In the case we will have*

$$QR \equiv x^{100} \pmod{2},$$

*therefore*

$$\begin{aligned} Q &\equiv x^k \pmod{2}, \\ R &\equiv x^l \pmod{2}, \end{aligned}$$

or

$$Q = x^k + \cdots + 2 \cdot m$$

and

$$R = x^l + \cdots + 2 \cdot n$$

thus

$$QR = x^{100} + 4 \cdot nm$$

that is impossible because  $n, m \in \mathbb{Z}$  and  $nm \neq -\frac{1}{2}$ .

**Lemma 1.22** (Eisenstein criterion). *Lets  $P \in \mathbb{Z}[X]$  and  $P = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$ . If  $\exists p$  - prime such that  $p \nmid a_n$ ,  $p \mid a_i \forall i < n$  and  $p^2 \nmid a_0$ , then  $P \in \mathbb{Z}[X]$  is irreducible.*

*Proof.* the same as for example 1.6.1. □

Note: that both: Gauss and Eisenstein criterion are valid by replacing  $\mathbb{Z}$  with an Unique factorization domain  $R$  and  $\mathbb{Q}$  by its factorization field.





# Chapter 2

## Stem field, splitting field, algebraic closure

We introduce the notion of a stem field and a splitting field (of a polynomial). Using Zorn's lemma, we construct the algebraic closure of a field and deduce its unicity (up to an isomorphism) from the theorem on extension of homomorphisms.

### 2.1 Stem field. Some irreducibility criteria

#### 2.1.1 Stem field

**Definition 2.1** (Stem field). *Let  $P \in K[X]$  is an irreducible Monic polynomial. Field extension  $E$  is a stem field of  $P$  if  $\exists \alpha \in E$  - the root of polynomial  $P$  and  $E = K[\alpha]$ .*

Such things exist, for instance we can take  $K[X]/(P)$ . It is a field because  $P$  is irreducible moreover the root of the  $P$  is in the field (see example 1.2.1).

We also can say that for any stem field  $E$ :

$$K[X]/(P) \cong E.$$

We can use the following Isomorphism:  $f : \forall p \in K[X]/(P) \rightarrow p(\alpha)$ , there  $\alpha$  is a root of polynomial  $P$ . To summarize we have the following

**Proposition 2.2** (About stem field existence). *The stem field exist and if we have 2 stem fields  $E$  and  $E'$  which correspond 2 roots of  $P$ :  $E = K[\alpha]$ ,  $E' = K[\alpha']$  then  $\exists! f : E \cong E'$  (Isomorphism of  $K$ -algebras) such that  $f(\alpha) = \alpha'$ .*

*Proof.* Existence:  $K[X]/(P)$  can be took as the stem field.

Uniqueness of the Isomorphism is easy because it is defined by its value on argument  $\alpha$ :

$$\begin{aligned}\phi : K[X]/(P) &\cong_{x \rightarrow \alpha} E, \\ \psi : K[X]/(P) &\cong_{x \rightarrow \alpha'} E',\end{aligned}$$

thus

$$\phi^{-1} \circ \psi : E \cong_{\alpha \rightarrow \alpha'} E'.$$

□

**Remark 2.3** (About stem field). 1. In particular: If a stem field contains 2 roots of  $P$  then  $\exists!$  Automorphism taking one root into another.

2. If  $E$  stem field then  $[E : K] = \deg P$

3. If  $[E : K] = \deg P$  and  $E$  contains a root of  $P$  then  $E$  is a stem field

4. If  $E$  is not a stem field but contains root of  $P$  then  $[E : K] > \deg P$   
(???)

### 2.1.2 Some irreducibility criteria

**Corollary 2.4.**  $P \in K[X]$  is irreducible over  $K$  if and only if it does not have a root in Field extension  $L$  of  $K$  of such that  $[L : K] \leq \frac{n}{2}$ , where  $n = \deg P$ .

*Proof.*  $\Rightarrow$ : If  $P$  is not irreducible then it has a polynomial  $Q$  that divides  $P$  and  $\deg Q \leq \frac{n}{2}$  ( $P = RQ$  and if  $\deg Q > \frac{n}{2}$  then we can take  $R$  as  $Q$ ). The Stem field  $L$  for  $Q$  exists and its degree is  $\deg Q \leq \frac{n}{2}$ .  $L$  should have root of  $Q$  (as soon as root of  $P$ ) by definition.

$\Leftarrow$ : If  $P$  has a root  $\alpha$  in  $L$  then  $\exists P_{\min}(\alpha, K)$  with degree  $\leq \frac{n}{2} < n$  (because  $[L : K] \leq \frac{n}{2}$ ) that divides  $P$  (see lemma 1.7) i.e.  $P$  become reducible. □

**Corollary 2.5.**  $P \in K[X]$  irreducible with  $\deg P = n$ . Let  $L$  be an extension of  $K$  such that  $[L : K] = m$ . If  $\gcd(n, m) = 1$  then  $P$  is irreducible over  $L$ .

*Proof.* If it is not a case and  $\exists Q$  such that  $Q \mid P$  in  $L[X]$ . Let  $M$  be a Stem field of  $Q$  over  $L$ .

So we have  $K \subset L \subset M = L(\alpha)$ .  $M$  is a stem field that  $[M : L] = \deg Q = d < n$ . Thus  $[M : K] = md$

Lets  $K(\alpha)$  is a stem field of  $P$  over  $K$  then  $[K(\alpha) : K] = \deg P = n$ .

$K(\alpha) \subseteq M$  and therefore  $n \mid md$  thus using  $\gcd(m, n) = 1$  one can get that  $n \mid d$  but this is impossible because  $d < n$ . □

## 2.2 Splitting field

**Definition 2.6** (Splitting field). *Let  $P \in K[X]$ . The splitting field of  $P$  over  $K$  is an extension  $L$  where  $P$  is split (i.e. is a product of linear factors) and roots of  $P$  generate  $L$*

**Theorem 2.7** (About splitting fields). 1. *Splitting field  $L$  exists and  $[L : K] \leq d!$ , where  $d = \deg P$ .*

2. *If  $L$  and  $M$  are 2 splitting fields then  $\exists \phi : L \cong M$  (an Isomorphism). But the Isomorphism is not necessary to be unique.*

*Proof.* Lets prove by induction on  $d$ . The first case ( $d = 1$ ) is trivial the  $K$  itself is the splitting field. Now assume  $d > 1$  and that the theorem is valid for any polynomial of degree  $< d$  over any field  $K$ . Let  $Q$  be any irreducible factor of  $P$ . We can create a Stem field  $L_1 = K(\alpha)$  for  $Q$  that will be also a Stem field for  $P$ .

Over  $L_1$  we have  $P = (x - \alpha)R$ , where  $R$  is a polynomial with  $\deg R = d - 1$ . We know (see remark 2.3) that there exists a Splitting field  $L$  for  $R$  over  $L_1$  and its degree:  $[L : L_1] \leq (d - 1)!$  We have  $K \subset L_1 \subset L$ . The  $L$  will be a splitting field for original polynomial  $P$ . Its degree (by The multiplicativity formula for degrees) is  $\leq d \cdot (d - 1)! = d!$ .

Uniqueness: Let  $L$  and  $M$  are 2 splitting fields. Let  $\beta$  is a root of  $Q$  (irreducible factor of  $P$ ) in  $M$ . We have 2 stem fields:  $L_1 = K(\alpha)$  and  $M_1 = K(\beta)$ . Proposition 2.2 says as that

$$\exists \phi : L_1 = K(\alpha) \cong K(\beta) = M_1,$$

such that  $\phi(\alpha) = \beta$ .

Over  $M_1$  we have  $P = (x - \beta)S$ , where  $S = \phi(R)$ <sup>1</sup>

$M$  is splitting field for  $S$  over  $K(\beta) = M_1$ .  $M$  is also  $L_1$ -algebra (via the Isomorphism  $\phi$ ) and as such it's a splitting field for  $R$  over  $L_1$ . As soon as  $[L : L_1] = [M : M_1]$  the  $M/L_1 \cong L/L_1$  because the  $L_1$ -algebras with the same dimension are isomorphic (see lemma 0.19). Therefore we have an  $L_1 = K(\alpha)$  Isomorphism  $L \cong M$  and therefore  $K$  Isomorphism  $L \cong M$ .  $\square$

**Remark 2.8.** *The Isomorphism is not unique. A splitting field can have many Automorphism and this is in fact the subject of Galois theory.*

---

<sup>1</sup> We have  $\phi : K(\alpha) \rightarrow K(\beta)$ . The  $\phi : K \rightarrow K$  because  $K \subset K(\alpha)$  as well as  $K \subset K(\beta)$ . Therefore  $\phi(P) = P$  because  $P \in K[X]$ . Thus

$$P = (x - \beta)S = \phi(P) = \phi((x - \alpha)R) = (x - \beta)\phi(R)$$

and  $S = \phi(R)$ .

## 2.3 An example. Algebraic closure

### 2.3.1 An example of automorphism

**Example 2.3.1** ( $x^3 - 2$  over  $\mathbb{Q}$ ). Let us have the following polynomial  $x^3 - 2$  over  $\mathbb{Q}$ . It has the following roots:  $\sqrt[3]{2}, j\sqrt[3]{2}$  and  $j^2\sqrt[3]{2}$ , where  $j = e^{\frac{2\pi i}{3}}$ . Splitting field is the following  $L = \mathbb{Q}(\sqrt[3]{2}, j)$ . Let us find Automorphisms of the field.



As soon as  $L$  is a stem field for  $\mathbb{Q}(j)$  and for  $\mathbb{Q}(\sqrt[3]{2})$  then 2 types of automorphism exist:

1.  $\mathbb{Q}(\sqrt[3]{2})$  Automorphism. We have  $x^2 + x + 1$  as  $P_{\min}(j, \mathbb{Q}(\sqrt[3]{2}))$ . The polynomial has 2 roots:  $j$  and  $j^2$  and there is an Automorphism that exchanges the root. Let us call it  $\tau$
2.  $\mathbb{Q}(j)$  Automorphism. In this case the automorphism of exchanging  $\sqrt[3]{2}$  and  $j\sqrt[3]{2}$ .<sup>2</sup> Let us call it  $\sigma$

The group of automorphism of  $L$   $\text{Aut}(L/K)$  is embedded into permutation group of 3 elements  $S_3$  (see example 0.1.2):

$$\text{Aut}(L/K) \hookrightarrow S_3.$$

It's embedded because the automorphism exchanges the roots of  $x^3 - 2$ . Moreover

$$\text{Aut}(L/K) = S_3,$$

because  $\sigma$  and  $\tau$  generates  $S_3$  because

- $\sigma: \sqrt[3]{2} \rightarrow j\sqrt[3]{2} \rightarrow j^2\sqrt[3]{2} \rightarrow \sqrt[3]{2}$ . This is a circle.

---

<sup>2</sup> ??? The minimal polynomial is  $x^3 - 2$  there and thus we have 3 roots:  $\sqrt[3]{2}, j\sqrt[3]{2}$  and  $j^2\sqrt[3]{2}$

- $\tau$  - it keeps  $\sqrt[3]{2}$  and exchanges  $j$  and  $j^2$ :  $\sqrt[3]{2}j \leftrightarrow \sqrt[3]{2}j^2$  (???). This is a transposition.

Lets also look at  $\mathbb{Q}(\sqrt[3]{2})$ . The question is the following: how many Homomorphisms to  $L = \mathbb{Q}(\sqrt[3]{2}, j)$  do we have. As we know

$$L = \mathbb{Q}(\sqrt[3]{2}, j) = \mathbb{Q}(\sqrt[3]{2}, j\sqrt[3]{2}, j^2\sqrt[3]{2}),$$

i.e.  $\sqrt[3]{2}$  can be switched with one of the roots:  $\sqrt[3]{2}, j\sqrt[3]{2}, j^2\sqrt[3]{2}$  and each permutation is a homomorphism. To demonstrate it lets look at the following permutation  $\sqrt[3]{2} \leftrightarrow j\sqrt[3]{2}$ . We have a unique Isomorphism

$$\mathbb{Q}(\sqrt[3]{2}) \rightarrow \mathbb{Q}(j\sqrt[3]{2}) \subset L.$$

i.e. we have a homomorphism  $\mathbb{Q}(\sqrt[3]{2}) \rightarrow L$  associated with the following permutation:  $\sqrt[3]{2} \leftrightarrow j\sqrt[3]{2}$

### 2.3.2 Algebraic closure

**Definition 2.9** (Algebraically closed field).  $K$  is algebraically closed if any non constant polynomial  $P \in K[X]$  has a root in  $K$  or in other words if any  $P \in K[X]$  splits

**Example 2.3.2** ( $\mathbb{C}$ ).  $\mathbb{C}$  is an Algebraically closed field. This will be proved later.

**Definition 2.10** (Algebraic closure). An algebraic closure of  $K$  is a field  $L$  that is Algebraically closed field and Algebraic extension over  $K$ .

**Theorem 2.11** (About Algebraic closure). Any field  $K$  has an Algebraic closure

*Proof.* Lets discuss the strategy of the prove. First construct  $K_1$  such that  $\forall P \in K[X]$  has a root in  $K_1$ . There is not a victory because  $K_1$  can introduce new coefficients and polynomials that can be irreducible over  $K_1$ . Then construct  $K_2$  such that  $\forall P \in K_1[X]$  has a root in  $K_2$  and so forth. As result we will have

$$K \subset K_1 \subset K_2 \subset \dots \subset K_n \subset \dots$$

Take  $\bar{K} = \cup_i K_i$  and we claim that  $\bar{K}$  is algebraically closed. Really  $\forall P \in \bar{K}[X] \exists j : P \in K_j[X]$  thus it has a root in  $K_{j+1}$  and as result in  $\bar{K}$ .

Now how can we construct  $K_1$ . Let  $S$  be a set of all irreducible  $P \in K[X]$ . Let  $A = K[(X_p)_{p \in S}]$  - multi-variable (one variable  $X_p$  for each  $p \in S$ ) polynomial ring.

Let  $I \subset A$  is an Ideal generated by  $P(X_p) \forall p \in S$ .<sup>3</sup> We claim that  $I$  is a Proper ideal i.e.  $I \neq A$ . If not then we can write

$$1_A = \sum_i^n \lambda_i P_i(X_{p_i}), \quad (2.1)$$

where  $\lambda_i \in A$  and the sum is the finite. As soon as the sum is finite then I can take the product of the polynomials in the sum:  $P = \prod_i^n P_i$  and I can create a Splitting field  $L$  for the polynomial  $P$  over  $K$ .<sup>4</sup>

$A$  is a polynomial ring and it's very easy produce a homomorphism between polynomial algebra and any other algebra. Therefore there is a homomorphism between rings  $A$  and  $L$  such that  $\phi : A \rightarrow L$  where  $X_{p_i} \rightarrow \alpha_i$  if  $P = P_i$  and  $X_{p_i} \rightarrow 0$  otherwise. From (2.1) we have

$$\phi(1_A) = \sum_i^n \lambda_i \phi(P_i(X_{p_i})) = \sum_i^n \lambda_i P_i(\alpha_i) = 0$$

that is impossible.

Fact: Any Proper ideal  $I \subset A$  is contained in the Maximal ideal  $m$  and  $A/m$  is a field.

Thus I can take  $K_1 = A/m$  and continue in the same way to construct  $K_2, K_3, \dots, K_n, \dots$   $\square$

### 2.3.3 Ideals in a ring

The ring is commutative, associative with unity. Any Proper ideal is in a Maximal ideal. This is a consequence of what one calls Zorn's lemma

**Definition 2.12** (Chain). Let  $\mathcal{P}$  is a partially ordered set ( $\leq$  is the order relation).  $\mathcal{C} \subset \mathcal{P}$  is a chain if  $\forall \alpha, \beta \in \mathcal{C}$  exists a relation between  $\alpha$  and  $\beta$  i.e.  $\alpha \leq \beta$  or  $\beta \leq \alpha$ .

**Lemma 2.13** (Zorn). If any non-empty Chain  $\mathcal{C}$  in a non-empty set  $\mathcal{P}$  has an upper bound (that is  $M \in \mathcal{P}$  such that  $M \geq x, \forall x \in \mathcal{C}$ ) then  $\mathcal{P}$  has a maximal element.

---

<sup>3</sup>  $I = \sum_i \lambda_i P_i(X_{p_i})$ , where  $\lambda_i \in A$

<sup>4</sup>  $\alpha_i$  is a root of  $P_i$

We can use Zorn lemma to prove that any proper ideal is in a Maximal ideal.

Let  $\mathcal{P}$  is the set of proper ideals in  $A$  containing  $I$ . The set is not empty because it has at least one element  $I$ . Any Chain  $\mathcal{C} = \{I_\alpha\}$  <sup>5</sup> has an upper bound: it's  $\cup_\alpha I_\alpha$  (exercise that the union is an ideal). So  $\mathcal{P}$  has a maximal element  $m$  and  $I \subset m$ .

If we take a Quotient ring by maximal ideal it's always a field otherwise it will have a proper ideal:  $\exists a \in A/m$  such that  $(a)$  is a proper ideal and its pre-image in  $\pi : A \rightarrow A/m$  should strictly contain  $m$  <sup>6</sup>.

## 2.4 Extension of homomorphisms. Uniqueness of algebraic closure

Some summary about just proved existence of algebraic closure. There exists  $\bar{K} = \cup_{i=1}^\infty K_i$  - algebraic closure of  $K$ , where

$$K \subset K_1 \subset K_2 \subset \dots \subset K_{i-1} \subset K_i \subset \dots$$

$K_i$  is a field where each polynomial  $P \in K_{i-1}$  has a root. The field  $K_i$  is Quotient ring of huge polynomial ring  $K_{i-1}[X]$  by a suitable Maximal ideal that is got by means of Zorn lemma.

Another question is the closure unique? The answer is yes. We start the proof with the following theorem

**Theorem 2.14** (About extension of homomorphism). *Let  $K \subset L \subset M$  - Algebraic extension.  $K \subset \Omega$ , where  $\Omega$  - Algebraic closure of  $K$ .  $\forall \phi : L \rightarrow \Omega$  extends to  $\tilde{\phi} : M \rightarrow \Omega$*

*Proof.* Apply Zorn lemma to the following set (of pairs)

$$\mathcal{E} = \{(N, \psi) : L \subset N \subset M, \psi \text{ extends } \phi\}$$

$\mathcal{E}$  is non empty because  $(L, \phi) \in \mathcal{E}$ .

The set  $\mathcal{E}$  is partially ordered by the following relation ( $\leq$ ):

$$(N, \psi) \leq (N', \psi'),$$

if  $N \subseteq N'$  and  $\psi'/N = \psi$  ( $\psi'$  extends  $\psi$ ). Any Chain  $(N_\alpha, \psi_\alpha)$  has an upper bound  $(N, \psi)$ , where  $N = \cup_\alpha N_\alpha$  - field, sub extension of  $M$ .  $\psi$  defined in the following way: for  $x \in N_\alpha$   $\psi(x) = \psi_\alpha(x)$ .

<sup>5</sup> The order is the following  $I_\alpha \leq I_\beta$  if  $I_\alpha \subset I_\beta$

<sup>6</sup> ??? i.e.  $m$  is not a maximal ideal in the case.

Thus  $\mathcal{E}$  has a maximal element that we denote by  $(N_0, \psi_0)$ .

Lets suppose that  $N_0 \neq M$ , i.e.  $N_0 \subsetneq M$ . Now it's very easy to get a contradiction. Lets take  $x \in M \setminus N_0$  and consider Minimal polynomial  $P_{min}(x, N_0)$ . It should have a root  $\alpha \in \Omega$ . Now we extend  $N_0$  to  $N_0(x)$  and define  $\psi'$  on  $N_0(x)$  as follows:  $\forall y \in N_0 : \psi'(y) = \psi_0(y)$  and  $\psi'(x) = \alpha$ . Thus we was able to find an element of the chain that is greater than maximal. Therefore our assumption about  $N_0 \neq M$  was incorrect and we can conclude that  $N_0 = M$  and therefore  $\tilde{\phi} = \psi_0$ .  $\square$

**Corollary 2.15** (About algebraic closure isomorphism). *If  $\Delta$  and  $\Delta'$  are 2 algebraic closures of  $K$  then they are isomorphic as  $K$ -algebras.*

*Proof.* Using theorem 2.14 one can assume  $L = K$ ,  $M = \Delta'$  and  $\Omega = \Delta$  i.e. we have

$$K \subset K \subset \Delta'$$

in this case homomorphism  $K \rightarrow \Delta$  can be extended to  $\Delta' \rightarrow \Delta$  i.e. there exists a homomorphism (i.e. Injection) from  $\Delta'$  to  $\Delta$ .

If we assume  $M = \Delta$  and  $\Omega = \Delta$  then there exists a homomorphism (i.e. Injection) from  $\Delta$  to  $\Delta'$ . The Injection is also Surjection in another direction:  $\Delta' \rightarrow \Delta$  and as result we have Isomorphism  $\Delta' \rightarrow \Delta$   $\square$



# Chapter 3

## Finite fields. Separability, perfect fields

We recall the construction and basic properties of finite fields. We prove that the multiplicative group of a finite field is cyclic, and that the automorphism group of a finite field is cyclic generated by the Frobenius map. We introduce the notions of separable (resp. purely inseparable) elements, extensions, degree. We briefly discuss perfect fields.

### 3.1 An example (of extension)s. Finite fields

**Corollary 3.1.** *Algebraic closure of  $K$  is unique up to Isomorphism of  $K$ -algebras*<sup>1</sup>

**Corollary 3.2.** *Any Algebraic extension of  $K$  is embedded into the Algebraic closure*<sup>2</sup>

**Example 3.1.1** (Of extension of homomorphism). *Let  $K = \mathbb{Q}$  and  $\overline{\mathbb{Q}}$  is the Algebraic closure of  $K$ . For instance we can consider  $\overline{\mathbb{Q}} \subset \mathbb{C}$ .*<sup>3</sup>

Let

$$L = \mathbb{Q}(\sqrt{2}) = \mathbb{Q}[X] / (X^2 - 2),$$

$\alpha$  is a class of  $X$  in  $L$ .  $L$  has 2 Embeddings into  $\overline{\mathbb{Q}}$

1.  $\phi_1 : \alpha \rightarrow \sqrt{2}$
2.  $\phi_2 : \alpha \rightarrow -\sqrt{2}$

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<sup>1</sup> There is a redefinition of corollary 2.15.

<sup>2</sup> seems to be a reformulation of theorem 2.14

<sup>3</sup> Really  $\overline{\mathbb{Q}} = \mathbb{A}$  - the set of all algebraic numbers, i.e. roots of polynomials  $P \in \mathbb{Q}[X]$ .

Let

$$M = \mathbb{Q}(\sqrt[4]{2}) = \mathbb{Q}[Y] / (Y^4 - 2),$$

$\beta$  is a class of  $Y$  in  $M$ .  $M$  has 4 Embeddings into  $\overline{\mathbb{Q}}$

1.  $\psi_1 : \beta \rightarrow \sqrt[4]{2}$  (extends  $\phi_1$ )
2.  $\psi_2 : \beta \rightarrow -\sqrt[4]{2}$  (extends  $\phi_1$ )
3.  $\psi_3 : \beta \rightarrow i\sqrt[4]{2}$  (extends  $\phi_2$ )
4.  $\psi_4 : \beta \rightarrow -i\sqrt[4]{2}$  (extends  $\phi_2$ )

This (“extends”) is because

$$M = L[Y] / (Y^2 - \alpha)$$

### 3.1.1 Finite fields

**Definition 3.3** (Finite field).  $K$  is a finite field if it's characteristic (see section 1.1.3)  $\text{char}K = p$ , where  $p$  - prime number

**Remark 3.4** ( $\mathbb{F}_{p^n}$ ). If  $K$  is a finite extension of  $\mathbb{F}_p$  and  $n = [K : \mathbb{F}_p]$  then number of elements of  $K$ :  $|K| = p^n$ . The following notation is also used for a finite extension of a finite field:  $\mathbb{F}_{p^n}$

**Remark 3.5** (Frobenius homomorphism). If  $\text{char}K = p$ , then exists a Homomorphism  $F_p : K \rightarrow K$  such that  $x \in K \rightarrow x^p \in K$ . Really if we consider  $(x + y)^p$  and  $(xy)^p$  then we can get  $(x + y)^p = x^p + y^p$  and  $(xy)^p = x^p y^p$ . The second property is the truth in the all fields (of course) but the first one is the special property of  $\mathbb{F}_p$  fields.

**Remark 3.6.** Also  $F_{p^n} : x \in K \rightarrow x^{p^n} \in K$  is also homomorphism (a power of Frobenius homomorphism).

## 3.2 Properties of finite fields

**Theorem 3.7.** Lets fix  $\mathbb{F}_P$  and it's Algebraic closure  $\overline{\mathbb{F}_P}$ .

The Splitting field of  $x^{p^n} - x$  has  $p^n$  elements. Conversely any field of  $p^n$  elements is a splitting field of  $x^{p^n} - x$ . Moreover there is an unique sub extension of  $\overline{\mathbb{F}_P}$  with  $p^n$  elements.

*Proof.* Note that  $F_{p^n} : x \rightarrow x^{p^n}$  is a Homomorphism (see remark 3.6) as result the following set  $\{x \mid F_{p^n}(x) = x\}$  is a sub-field containing  $\mathbb{F}_p$ .<sup>4</sup> Lets  $Q_n(X) = X^{p^n} - X$  thus the considered set consists of the root of the polynomial  $Q_n$ . The polynomial has no multiple roots because  $\gcd(Q_n, Q'_n) = 1$ .<sup>5</sup> This is because  $Q'_n \equiv 1 \pmod{p}$ .<sup>6</sup> As soon as  $Q_n$  has no multiple roots then there are  $p^n$  different roots and therefore the splitting field is the field with  $p^n$  elements.

Conversely lets  $|K| = p^n$  and  $\alpha \neq 0 \in K$ . Using the fact that the multiplication group of  $K$  has  $p^n - 1$  elements:  $|K^*| = p^n - 1$ <sup>7</sup> as result the multiplication of all the elements should give us 1:  $\alpha^{p^n-1} = 1$  or  $\alpha^{p^n} - \alpha = 0$ . Therefore  $\alpha$  is a root of  $Q_n$ . Thus the splitting field of  $Q_n$  consists of elements of  $K$ .

The uniqueness of sub-extension of  $\mathbb{F}_p$  with  $p^n$  elements is a result of uniqueness of the splitting field (see theorem 2.7).  $\square$

**Theorem 3.8.**  $\mathbb{F}_{p^d} \subset \mathbb{F}_{p^n}$  if and only if  $d \mid n$ .

*Proof.* Let  $\mathbb{F}_{p^d} \subset \mathbb{F}_{p^n}$  in this case  $\mathbb{F}_p \subset \mathbb{F}_{p^d} \subset \mathbb{F}_{p^n}$  and

$$[\mathbb{F}_{p^n} : \mathbb{F}_p] = [\mathbb{F}_{p^n} : \mathbb{F}_{p^d}] [\mathbb{F}_{p^d} : \mathbb{F}_p]$$

or  $n = x \cdot d$  i.e.  $d \mid n$

Conversely if  $d \mid n$  then  $n = x \cdot d$  or  $p^n = \prod_{i=1}^x p^d$  thus if  $x^{p^d} = x$  then

$$x^{p^n} = x^{\prod_{i=1}^x p^d} (x^{p^d})^{\prod_{i=2}^x p^d} = x^{\prod_{i=2}^x p^d} = \dots = x^{p^d} = x,$$

i.e.  $\forall \alpha \in \mathbb{F}_{p^d}$  we also have  $\alpha \in \mathbb{F}_{p^n}$  or in other notation:  $\mathbb{F}_{p^d} \subset \mathbb{F}_{p^n}$ .  $\square$

**Theorem 3.9.**  $\mathbb{F}_{p^n}$  is a Stem field and a Splitting field of any irreducible polynomial  $P \in \mathbb{F}_p$  of degree  $n$ .

*Proof.* Stem field  $K$  has to have degree  $n$  over  $\mathbb{F}_p$  i.e.  $[K : \mathbb{F}_p] = n$  i.e. it should have  $p^n$  elements and therefore  $K = \mathbb{F}_{p^n}$  (see also remark 3.4).

About Splitting field. Using the just proved result we can say that if  $\alpha$  is a root of  $P$  then  $\alpha \in \mathbb{F}_{p^n}$  thus  $Q_n(\alpha) = 0$ . Therefore  $P$  divides  $Q_n$ <sup>8</sup> and as result  $P$  splits in  $\mathbb{F}_{p^n}$ .  $\square$

<sup>4</sup> For  $x \in \mathbb{F}_p$  we have that  $x^{p-1} = 1$  (the field with fixed number of elements) and therefore  $\forall x \in \mathbb{F}_p : x^p = x$  i. e.  $F_{p^n}(x) = x$ .

<sup>5</sup> If  $Q_n$  has a multiple root  $\beta$  then it is divisible by  $(X - \beta)^2$  and the  $Q'_n$  is divisible by (at least)  $(X - \beta)$  thus the  $(X - \beta)$  should be a part of gcd.

<sup>6</sup> ???  $Q'_n = p^n X^{p^n-1} - 1 \equiv -1 \pmod{p}$

<sup>7</sup>  $K^* = K \setminus \{0\}$

<sup>8</sup> as soon as any root of  $P$  also a root of  $Q_n$

**Corollary 3.10.** *Let  $\mathcal{P}_d$  is the set of all irreducible, Monic polynomials of degree  $d$  such that  $\mathcal{P}_d \subset \mathbb{F}_p[X]$  then*

$$Q_n = \prod_{d|n} \prod_{P \in \mathcal{P}_d} P$$

*Proof.* As we just seen if  $P \in \mathcal{P}_d$  and  $d \mid n$  then  $P \mid Q_n$ .<sup>9</sup> Since all such polynomials are relatively prime of course<sup>10 11</sup> and  $Q_n$  have no multiple roots (as result no multiple factors) then

$$\prod_{d|n} \prod_{P \in \mathcal{P}_d} P \mid Q_n$$

From other side let  $R$  is an irreducible factor of  $Q_n$ .  $\alpha$  is a root of  $R$  then  $Q_n(\alpha) = 0$  thus  $\mathbb{F}_p(\alpha) \subset \mathbb{F}_{p^n}$  therefore  $\mathbb{F}_p(\alpha) = \mathbb{F}_{p^d}$  where  $d \mid n$  and as result,  $\deg R \mid n$ . Thus the polynomial should be in the product  $\prod_{d|n} \prod_{P \in \mathcal{P}_d} P$ .  $\square$

**Example 3.2.1.** *Let  $p = n = 2$ . The monic irreducible polynomials in  $\mathbb{F}_2$  whose degree divides 2 are:  $x$ ,  $x + 1$  and  $x^2 + x + 1$ . As you can see*

$$x(x+1)(x^2+x+1) = x^4 + x = x^4 - x$$

*because  $2x = 0 \pmod{2}$  or  $x = -x$ .*

### 3.3 Multiplicative group and automorphism group of a finite field

**Theorem 3.11.** *Let  $K$  be a field and  $G$  be a finite Subgroup of  $K^*$  then  $G$  is a Cyclic group*

*Proof.* Idea is to compare  $G$  and the Cyclic group  $\mathbb{Z}/N\mathbb{Z}$  where  $N = |G|$ .<sup>12</sup>

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<sup>9</sup> Since stem field is  $\mathbb{F}_{p^d} \subset \mathbb{F}_{p^n}$

<sup>10</sup> As soon as  $\mathbb{F}_p[X]$  is Unique factorization domain then any polynomial can be written as a product of irreducible elements, uniquely up to order and units this means that each  $P \in \mathcal{P}_d$  (where  $d \mid n$ ) should be in the factorization of  $Q_n$ . It should be only one time because there is no multiply roots.

<sup>11</sup> We also can say that 2 irreducible polynomial  $P_1, P_2 \in \mathbb{F}_p[X]$  should not have same roots. For example if  $\alpha$  is the same root - it cannot be in  $\mathbb{F}_p$  because in the case the polynomials will be reducible. Thus it can be only in an extension of  $\mathbb{F}_p$  from other side  $\gcd(P_1, P_2) = 1$  and therefore with Bézout's lemma one can get that  $\exists Q, R \in \mathbb{F}_p[X]$  such that  $P_1Q + P_2R = 1$  and setting  $\alpha$  into the equation leads to fail statement that  $0 = 1$ .

<sup>12</sup> We also will use the fact that any cyclic group of order  $N$  is isomorphic to  $\mathbb{Z}/N\mathbb{Z}$

### 3.3. MULTIPLICATIVE GROUP AND AUTOMORPHISM GROUP OF A FINITE FIELD 37

Let  $\psi(d)$  - is the number of elements of order  $d$  ( see also Order of element in group) in  $G$ . We need  $\psi(N) \neq 0$  <sup>13</sup> and we know that  $N = \sum \psi(d)$ .

Let also  $\phi(d)$  - is the number of elements of order  $d$  ( see also Order of element in group) in  $\mathbb{Z}/N\mathbb{Z}$ . <sup>14</sup> As  $\mathbb{Z}/N\mathbb{Z}$  contains a single (cyclic) subgroup of order  $d$  for each  $d \mid N$ . <sup>15</sup>  $\phi(d)$  is the number of generators of  $\mathbb{Z}/d\mathbb{Z}$  i.e. the number of elements between 1 and  $d - 1$  that are prime to  $d$ . We know that  $\phi(N) \neq 0$ .

We claim that either  $\psi(d) = 0$  or  $\psi(d) = \phi(d)$  <sup>16</sup> If no element of order  $d$  in  $G$  then  $\psi(d) = 0$  otherwise if  $x \in G$  has order  $d$  then  $x^d = 1$  or  $x$  is a root of the following polynomial  $x^d - 1$ . The roots of the polynomial forms a cyclic subgroup of  $G$ . So  $G$  as well as  $\mathbb{Z}/N\mathbb{Z}$  has a single cyclic subgroup of order  $d$  (which is cyclic) or no such group at all. <sup>17</sup>

If  $\psi(d) \neq 0$  then exists such a subgroup and  $\psi(d)$  is equal to the number of generators of that group or  $\phi(d)$  <sup>18</sup> In particular  $\psi(d) \leq \phi(d)$  <sup>19</sup> but there should be equality because the sum of both  $\sum \psi(d) = \sum \phi(d) = N$ . In particular  $\psi(N) \neq 0$  and we proved the theorem.  $\square$

**Corollary 3.12.** *If  $\mathbb{F}_p \subset K$  and  $[K : \mathbb{F}_p] = n$  then  $\exists \alpha$  such that  $K = \mathbb{F}_p(\alpha)$ . In particular  $\exists$  an irreducible polynomial of degree  $n$  over  $\mathbb{F}_p$  <sup>20</sup>*

*Proof.* We can take  $\alpha =$  generator of  $K^*$  <sup>21</sup>  $\square$

<sup>13</sup> In this case we will have at least one element  $x$  of order  $N$  i.e.  $N$  different elements of  $G$  is generated by the  $x$  i.e. the  $G$  is cyclic.

<sup>14</sup> The function  $\phi(d)$  is also called as Euler's totient function and it counts the positive integers up to a given integer  $d$  that are relatively prime to  $d$

<sup>15</sup> The one generated by  $N/d$ . Let  $N = r \cdot d$  in the case  $x^N = 1$  there  $x$  is a  $\mathbb{Z}/N\mathbb{Z}$  group generator. From other side

$$x^N = x^{r \cdot d} = \prod_{i=1}^r x^d$$

thus  $x^d = 1$  i.e. there is a cyclic subgroup of order  $d$ .

<sup>16</sup> suffices since  $\sum \psi(d) = \sum \phi(d) = N$

<sup>17</sup> Several comments about the subgroup. There is a group because multiplication of any elements is in the set. It's cyclic because it's generated by one element. All  $x^i$  where  $i \leq d$  are different (in other case the group should have an order less than  $d$ ). Each element of the group  $x^i$  is a root of  $x^d - 1$  because  $(x^i)^d = (x^d)^i = 1^i = 1$ . And the group is unique as well as we have  $d$  different roots of  $x^d - 1$  in the group.

<sup>18</sup> Because the group is cyclic and any cyclic group is isomorphic to  $\mathbb{Z}/d\mathbb{Z}$  and as result has the same number of generators.

<sup>19</sup> because  $\psi(d) = 0$  or  $\psi(d) = \phi(d)$

<sup>20</sup> The theorem 3.9 and remark 3.4 says that the stem field for any polynomial of degree  $n$  over  $\mathbb{F}_p$  exists and there is  $\mathbb{F}_{p^n}$  and  $[\mathbb{F}_{p^n} : \mathbb{F}_p] = n$  i.e.  $K = \mathbb{F}_{p^n}$ . But we had not proved yet that an irreducible polynomial of degree  $n$  exists.

<sup>21</sup> This is because  $K^* = \langle \alpha \rangle$  i.e. any element of  $K$  except 0 can be got as a power of  $\alpha$

**Corollary 3.13.** *The group of automorphism of  $\mathbb{F}_{p^n}$  over  $\mathbb{F}_p$  is cyclic and generated by Frobenius map:  $F_p : x \rightarrow x^p$  (see remark 3.5)*

*Proof.* As we know from theorem 3.7:  $\forall x \in \mathbb{F}_{p^n} : x^{p^n} = x$  so  $F_p^n = Id$  <sup>22</sup>. From other side if  $m < n$  then  $x^{p^m} - x = 0$  has  $p^m < p^n$  roots and cannot be identity <sup>23</sup> Finally (from corollary 3.12) we have  $\mathbb{F}_{p^n} = \mathbb{F}_p(\alpha)$  where  $\alpha$  is a root of an irreducible polynomial of degree  $n$ . Thus there exists exactly  $n$  automorphisms of  $\mathbb{F}_{p^n}$ . <sup>24</sup> So

$$|Aut(\mathbb{F}_{p^n}/\mathbb{F}_p)| \leq n$$

and as we have  $n$  of them (Automorphisms) then

$$|Aut(\mathbb{F}_{p^n}/\mathbb{F}_p)| = n$$

□

### 3.4 Separable elements

Let  $E$  is a Splitting field of an irreducible polynomial  $P$ . We would like to say that it “has many Automorphisms”. What does this mean? This means the following thing: Let  $\alpha$  and  $\beta$  be 2 roots of  $P$  then we have 2 extensions  $K(\alpha) \subset E$  and  $K(\beta) \subset E$ .

There exists an Isomorphism (see proposition 2.2) over  $K$

$$\phi : K(\alpha) \rightarrow K(\beta)$$

that is also extended to an Automorphism on  $E$  (see theorem 2.14).

There is one problem with it: is that truth that an irreducible polynomial of degree  $n$  has “many” (no more than  $n$  and not single) roots.

The answer is yes if  $\text{char} K = 0$ , but not always if  $\text{char} K = p$  (where  $p$  is a prime number).  $P$  can have multiple roots in the case i.e.  $\gcd(P, P') \neq 1$ .

---

i.e. we really got  $K = \mathbb{F}_p(\alpha)$ .

??? The irreducible polynomial we can get if consider  $1, \alpha, \dots, \alpha^{n-1}$  as a basis and  $\alpha^n$  can be represented via the basis.

<sup>22</sup> because  $F_p^n : x \rightarrow x^{p^n} = x$ .

<sup>23</sup> because operates only with  $p^m$  elements i.e. not of all elements of  $\mathbb{F}_{p^n}$ .

<sup>24</sup> Each automorphism converts the root  $\alpha$  into another one of  $n$  roots of the irreducible polynomial

Why it's not a case for  $\text{char} K = 0$  - it is because  $\deg P' < \deg P$  and  $P \nmid P'$  for  $P' \neq 0$  (non constant polynomial) <sup>25</sup>

But for  $\text{char} K = p$  there can be a case when  $P' = 0$  for a non constant polynomial thus  $P \mid P'$  and as result  $\gcd(P, P') = P$ . The  $P' = 0$  i.e. is vanish if  $P = \sum a_i x^i$  and  $p \mid i$  or  $a_i = 0$ .

Let  $r = \max h$  such that  $P$  is a polynomial in  $x^{p^h}$  that is  $a_i = 0$  whenever  $p^h \nmid i$

**Proposition 3.14.** *Let  $P(X) = Q(x^{p^r})$  and  $Q' \neq 0$  i.e.  $\gcd(Q, Q') = 1$  and  $Q$  does not have multiple roots but all roots of  $P$  have multiplicity  $p^r$ .*

*Proof.* If  $\lambda$  is a root of  $P$  then  $\lambda: P(X) = (X - \lambda)R$  Thus  $\mu = \lambda^{p^r}$  is the root of  $Q$ :  $Q(Y) = (Y - \lambda^{p^r})S(Y)$  therefore

$$P(X) = (X^{p^r} - \lambda^{p^r}) S(X^{p^r}) = (X - \lambda)^{p^r} S(X^{p^r})$$

and  $\lambda$  is not a root of  $S(X^{p^r})$ . <sup>26</sup> Thus we just got that multiplicity of  $\lambda$  is  $p^r$ .  $\square$

**Definition 3.15** (Separable polynomial).  $P \in K[X]$  irreducible polynomial is called separable if  $\gcd(P, P') = 1$

**Definition 3.16** (Degree of separability).  $d_{\text{sep}}(P) = \deg Q$  (as above) <sup>27</sup>

**Definition 3.17** (Degree of inseparability).  $d_i(P) = \frac{\deg P}{\deg Q}$  ( $= p^r$  in definition 3.16)

**Definition 3.18** (Pure inseparable polynomial).  $P$  is pure inseparable if  $d_i = \deg P$ . Then  $P = X^{p^r} - a$  <sup>28</sup>

**Definition 3.19** (Separable element). Let  $L$  be an Algebraic extension of  $K$  then  $\alpha \in K$  is called separable(inseparable) if it's Minimal polynomial  $P_{\min}(\alpha, K)$  has the property. Note: the separable element is also Algebraic element because it has minimal polynomial.

<sup>25</sup> Let  $P$  has multiply roots. As soon as it's irreducible a multiply root is in an extension of  $K$ . In this case the root should be also a root for  $P'$  thus by theorem 0.30 one can get that  $P \mid P'$  in  $K[X]$  but that is impossible because  $\deg P' < \deg P$  and can be only possible if  $P' = 0$ .

<sup>26</sup> This is because  $Q$  does not have multiply roots and as result  $\mu = \lambda^{p^r}$  is not a root of  $S$  or in other words  $S(X^{p^r})_{X=\lambda} \neq 0$

<sup>27</sup> It requires some explanation compare to that one was got on the lecture video. If  $P$  is a Separable polynomial then  $d_{\text{sep}}(P) = \deg P$ . In other case  $P$  should be represented as  $P(X) = q_1(X^{p^r})$ . If  $q_1(Y)$  is separable than  $Q = q_1$  otherwise we continue and represent  $q_1(X) = q_2(X^{p^r})$ . We should stop on some  $q_r$  in this case  $Q = q_r$  and  $P(X) = Q(X^{p^r})$ . In the case  $d_{\text{sep}}(P) = \deg Q$ .

<sup>28</sup> ??? For example but not then

**Proposition 3.20** (On number of homomorphisms). *If  $\alpha$  is separable on  $K$  then the number of Homomorphisms*

$$|Hom_k(K(\alpha), \bar{K})| = \deg P_{min}(\alpha, K)$$

*in general*

$$|Hom_k(K(\alpha), \bar{K})| = d_{sep} P_{min}(\alpha, K)$$

*Proof.* It's obvious because  $d_{sep}$  is the number of distinct roots.  $\square$

### 3.5 Separable degree, separable extensions

We want to generalize the proposition 3.20 for any field extension (not necessary  $K(\alpha)$ ). Let  $L$  be a finite extension of  $K$

**Definition 3.21** (Separable degree).  $[L : K]_{sep} = |Hom_k(L, \bar{K})|$

As we know if  $L = K(\alpha)$  then Separable degree is a number of distinct roots of minimal polynomial  $P_{min}(\alpha, K)$

**Definition 3.22** (Separable extension).  $L$  is separable over  $K$  if  $[L : K]_{sep} = [L : K]$

**Definition 3.23** (Inseparable degree).

$$[L : K]_i = \frac{[L : K]}{[L : K]_{sep}}$$

**Theorem 3.24** (About separable extensions). 1. If  $K \subset L \subset M$  then  $[M : K]_{sep} = [M : L]_{sep} [L : K]_{sep}$  and  $M$  is Separable extension over  $K$  if and only if  $M$  is separable over  $L$  and  $L$  is separable over  $K$

2. The following things are equivalent

- (a)  $L$  is separable over  $K$
- (b)  $\forall \alpha \in L$   $\alpha$  Separable element over  $K$
- (c)  $L$  is generated over  $K$  by a finite number of Separable elements  
i.e.  $L = K(\alpha_1, \alpha_2, \dots, \alpha_n)$ , there  $\alpha_i$  is separable over  $K$
- (d)  $L = K(\alpha_1, \alpha_2, \dots, \alpha_n)$ , there  $\alpha_i$  is separable over  $K(\alpha_1, \alpha_2, \dots, \alpha_{i-1})$



**Remark 3.25.** *That holds if we replace separability with pure inseparability.*

*Proof.* About 1st part: If we have a Homomorphism  $\phi : L \rightarrow \bar{K}$  then it is extended to  $\tilde{\phi} : M \rightarrow \bar{K}$  (by extension theorem 2.14) it can be done with one way per each homomorphism from  $L$  to  $M$  i.e. it can be done by  $|Hom_L(M, \bar{K})|$  ways but we have

$$|Hom_L(M, \bar{K})| = |Hom_L(M, \bar{L})| = [M : L]_{sep}$$

because  $\bar{K}$  is also  $\bar{L}$  (Algebraic closure over  $L$ ) thus for the total number of homomorphisms one can get the following equations

$$\begin{aligned} [M : K]_{sep} &= |Hom_k(M, \bar{K})| = |Hom_k(L, \bar{K})| |Hom_L(M, \bar{K})| = \\ &= |Hom_k(L, \bar{K})| |Hom_L(M, \bar{L})| = [M : L]_{sep} [L : K]_{sep} \end{aligned}$$

We have the following inequality

$$[E : K]_{sep} \leq [E : K] \quad (3.1)$$

29

With the inequality (3.1) we also have

$$[M : K]_{sep} = [M : L]_{sep} [L : K]_{sep} \leq [M : L] [L : K] = [M : K]$$

The equality is possible if  $[M : L]_{sep} = [M : L]$  and  $[L : K]_{sep} = [L : K]$  i.e. if  $M$  is separable over  $L$  and  $L$  is separable over  $K$ . This finishes the proof of the first part.

About 2d part:

2a  $\Rightarrow$  2b: Part 1 implies that a separable sub extension  $K(\alpha)$  or a separable extension  $L$  is separable. <sup>30</sup>

2b  $\Rightarrow$  2c: obvious <sup>31</sup>

---

<sup>29</sup> The inequality can be proved by induction using the fact that it's true for  $K(\alpha)$  because from general case of proposition 3.20

$$|Hom_k(K(\alpha), \bar{K})| = d_{sep} P_{min}(\alpha, K) \leq \deg P_{min}(\alpha, K) = [K(\alpha) : K]$$

Then let it was proved for  $E = K(\alpha_1, \dots, \alpha_{n-1})$  and we want to prove it for  $K(\alpha_1, \dots, \alpha_{n-1}, \alpha_n) = E(\alpha_n)$ . It's easy because  $\bar{E} = \bar{K}$  and we can use the same approach as for the first induction step.

<sup>30</sup> I.e. in the case we have  $K \subset K(\alpha) \subset L$  and if  $L$  is separable then  $K(\alpha)$  is separable and as result  $\alpha$  is a Separable element because  $P_{min}(\alpha, K)$  is separable.

<sup>31</sup> We consider finite extensions (see remark 3.4) i.e. which consists of finite number of elements

2c  $\Rightarrow$  2d: We know that  $P_{\min}(\alpha_i, K(\alpha_1, \dots, \alpha_{i-1}))$  divides  $P_{\min}(\alpha_i, K)$ .  
<sup>32</sup> Thus if  $P_{\min}(\alpha_i, K)$  is separable (i.e. have distinct roots) then it's divisor  $P_{\min}(\alpha_i, K(\alpha_1, \dots, \alpha_{i-1}))$  also should have distinct roots i.e.  $\alpha_i$  is a Separable element over  $K(\alpha_1, \dots, \alpha_{i-1})$   
 2d  $\Rightarrow$  2a: Induction as above <sup>33</sup>  $\square$

What's about not finite extension? For that case we can define separable extension as follows.

**Definition 3.26** (Separable closure). *If  $L$  over  $K$  not necessary finite (but algebraic over  $K$ ) we can define*

$$L^{\text{sep}} = \{x | x \text{ separable over } K\}$$

$L^{\text{sep}}$  is a sub extension (??? from theorem 3.24) called separable closure of  $K$  over  $L$

$L$  is pure inseparable over  $L^{\text{sep}}$ .

**Remark 3.27.** 1. If  $\text{char } K = 0$  then any extension of  $K$  is separable

2. If  $\text{char } K = p$  then pure inseparable extension has degree  $p^r$  and always degree of inseparability  $[L : K]_i = p^r$

## 3.6 Perfect fields

**Definition 3.28** (Perfect field). *Let  $K$  is a field and  $\text{char } K = p > 0$ .  $K$  is perfect if Frobenius homomorphism is a Surjection*

**Example 3.6.1.** 1. Finite field is perfect because an Injection of a set into itself is always a Surjection

2. Algebraically closed fields are perfect because  $x^p - a$  has a root for any  $a$  particularly  $a = F_p(\alpha)$  <sup>34</sup>

---

<sup>32</sup> Let  $K(\alpha_1, \dots, \alpha_{i-1}) = L$  then  $K \subset L$  and  $P_{\min}(\alpha_i, K) \in L[X]$  From other side  $P_{\min}(\alpha_i, L)$  is the minimal irreducible polynomial in  $L[X]$  and any other polynomial with  $\alpha_i$  as root has to be divisible by it. see also lemma 1.7

<sup>33</sup> The first induction step is trivial:  $L = K(\alpha)$  where  $\alpha$  is separable over  $K$  in this case  $K(\alpha)$  is also separable. Now we have that  $\forall k < n$ : if  $L = K(\alpha_1, \alpha_2, \dots, \alpha_k)$ , there  $\alpha_i$  is separable over  $K(\alpha_1, \alpha_2, \dots, \alpha_{i-1})$  then  $L$  is separable over  $K$ . Thus we have  $K(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$  separable and  $\alpha_n$  is separable over  $K(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$  thus using the first part of the theorem we can conclude that  $K(\alpha_1, \alpha_2, \dots, \alpha_n)$  is also separable over  $K$

<sup>34</sup> As a polynomial in  $Y = X^p$  has a root then it also have a root for  $X$  variable.

3. *Not perfect field example.* Let  $K = \mathbb{F}_p(X)$  be a field of rational fractions in 1 variable over  $\mathbb{F}_p$ . I.e. elements of the field are  $\frac{f(X)}{g(X)}$  where  $f, g \in \mathbb{F}_p[X]$ . It's not perfect because  $\text{Im}(F_p) = \mathbb{F}_p(X^p) \neq \mathbb{F}_p(X)$

**Theorem 3.29.**  $K$  is a Perfect field if and only if all irreducible polynomial over  $K$  are separable or in other words all Algebraic extensions of  $K$  are separable.

*Proof.* Let  $K$  is perfect and  $P \in K[X]$  is an irreducible polynomial. Let also

$$P(X) = Q(X^{p^r}) = \sum_i a_i (X^{p^r})^i$$

but as soon as my field is perfect then I can extract  $p$ -root of  $a_i$  and do it repeatedly. I.e.  $\exists b_i \in K$  such that  $b_i^{p^r} = a_i$ . Therefore

$$P(X) = \sum_i b_i^{p^r} (X^{p^r})^i = \sum_i (b_i X^i)^{p^r} = \left( \sum_i b_i X^i \right)^{p^r}.$$

The polynomial is not irreducible unless  $r = 0$ <sup>35</sup> so irreducible means separable.

If  $K$  is not perfect but all irreducible polynomial are separable.  $K$  is not perfect means that  $\exists a \notin \text{Im}(F_p)$  and lets consider the following polynomial:  $X^{p^r} - a$ . It is irreducible and not separable.

About separability: in fact all roots are in  $\bar{K}$  are the same  $x$  with  $x^{p^r} = a$ <sup>36</sup> and of course  $x^{p^{r-1}} \notin K$ .<sup>37</sup>

About the polynomial is irreducible. We have already seen that in the case  $[K(x) : K] = p^r$  so the polynomial is irreducible (??? see corollary 3.12) and this finishes the proof.<sup>38</sup>  $\square$

<sup>35</sup> In other case each root will have at least multiplicity  $p^r$ .

<sup>36</sup> We have  $x^{p^r} = a$  thus polynomial  $X^{p^r} - a$  can be written as  $X^{p^r} - a = X^{p^r} - x^{p^r} = (X - x)^{p^r}$  thus  $x$  has multiplicity  $p^r$

<sup>37</sup> as soon as any power of  $x$  (little  $x$  but not the big one  $X$ )

<sup>38</sup> Because we found an irreducible polynomial that is not separable because has a root of multiplicity  $p^r$ .