

Definitions

Finite difference:

$$f(x + \Delta x) = f(x) + \Delta f(x)$$

Limit of a sequence:

$$|a - S_n| < \epsilon$$

$$\lim_{n \rightarrow \infty} S_n = a$$

where $a = N_\epsilon$ such that $|a - S_n| < \epsilon$

TODO explain better

Limit of a function:

$$|f(x) - l| \leq \epsilon$$

$$\lim_{x \rightarrow x_0} f(x) = l$$

TODO explain better

Infinite sum:

$$\sum_{n=0}^{\infty} a_n = \lim_{n \rightarrow \infty} \sum_{n=0}^{\infty} a_n$$

Discrete derivative:

$$\frac{\Delta f}{\Delta x} = \frac{f(x_n + \Delta x) - f(x_n)}{\Delta x}$$

Derivative:

$$\frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Asymptotic relation:

Sequences:

$$\lim_{n \rightarrow \infty} \frac{A_n}{B_n} = 1 \text{ means } A \sim B$$

Functions:

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 1, \text{ then } f(x) \sim g(x).$$

Discrete integral:

$$\sum_{n=0}^N f(x_n) \cdot \Delta x$$

Integral:

$$\int_a^b f(x) dx = \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} f(x_n) \Delta x = \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} f(x_n) (x_{n+1} - x_n)$$

$$\Delta x = \frac{b-a}{N}$$

Definite integral with relation to a differential:

$$\int_a^b f(x) dg(x) = \lim_{N \rightarrow \infty} \sum_{n=1}^{N-1} f(x_n) \Delta g(x_n) = \lim_{N \rightarrow \infty} \sum_{n=1}^{N-1} f(x_n) (g(x_{n+1}) - g(x_n))$$

Fundamental theorem of calculus:

$$f(x) dx = \frac{dF(x)}{dx} dx = dF(x)$$

Critical points:

1. Compute the derivative
2. Find the points where the derivative = 0
3. Compute the second derivative
4. Find the value at the point in the second derivative
5. If the value > 0 then minimum, if < 0, maximum

Rules

Derivative rules:

Product rule:

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}f(x)$$

Example:

$$\frac{d}{dx}(x^3 \cdot \sin(x)) = x^3 \cdot \frac{d}{dx}\sin(x) + (\sin(x)) \cdot \frac{d}{dx}x^3 = x^3 \cdot \cos(x) + 3x^2 \cdot \sin(x)$$

Chain rule:

$$\frac{dh(x)}{dx} = \frac{df(u(x))}{du} \cdot \frac{du(x)}{dx}$$

Example:

$$\frac{d}{dx} \ln(x^2 + 1) = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot 2x = \frac{1}{x^2 + 1} \cdot 2x = \frac{2x}{x^2 + 1}$$

Quotient rule:

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x) \cdot \frac{d}{dx}f(x) - f(x) \cdot \frac{d}{dx}g(x)}{(g(x))^2}$$

Example:

$$\frac{d}{dx} \frac{2+3x}{x+1} = \frac{(x+1) \cdot 3 - (2+3x) \cdot 1}{(x+1)^2} = \frac{1}{(x+1)^2}$$

Integral rules:

Integration by parts:

$$\int_a^b g(x) \frac{df(x)}{dx} dx = (f(b)g(b) - f(a)g(a)) - \int_a^b f(x) \frac{dg(x)}{dx} dx$$

Example:

$$\int_0^1 x e^x dx = \int_0^1 x \frac{de^x}{dx} dx = (e^1 \cdot 1 - e^0 \cdot 0) - \int_0^1 e^x \frac{dx}{dx} dx = e - \int_0^1 e^x dx = e - (e^1 - e^0) = e - e + 1 = 1$$

Integration by substitution:

$$\int_a^b f(g(x)) \left(\frac{dg(x)}{dx} \right) dx = \int_a^b f(g(x)) dg(x)$$

Example:

TODO

ABC formula:

$$ax^2 + bx + c = 0 \iff x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Examples

Answers in the expected notation:

Limits:

$$\lim_{n \rightarrow \infty} \frac{3n^3 + 1}{4n^3 + n^3 + n} = \lim_{n \rightarrow \infty} \frac{3n^3}{4n^3} = \lim_{n \rightarrow \infty} \frac{3}{4} = \frac{3}{4} \text{ or: } \lim_{x \rightarrow 0} \frac{2x^6 + 3x^2 - 6x}{7x^6 - 6x} \sim \frac{-6x}{-6x} = 1$$

Derivatives:

$$f(x) = x^3 \cdot \cos(x^2 + 3x)$$

$$\frac{df(x)}{dx} = 3x^2 \cdot (\cos(x^2 + 3x)) + x^3 \cdot (-\sin(x^2 + 3x)) \cdot 2x + 3 = x^2(3\cos(x^2 + 3x) - x \sin(x^2 + 3x) \cdot 2x + 3)$$

Derivatives by definition:

$$\frac{df(x^2)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x\Delta x}{\Delta x} = 2x$$

$$\frac{df(\frac{1}{x})}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} = \lim_{\Delta x \rightarrow 0} -\frac{\frac{\Delta x}{x(\Delta x+x)}}{\Delta x} = \lim_{\Delta x \rightarrow 0} -\frac{\Delta x}{x(\Delta x+x)\Delta x} = \lim_{\Delta x \rightarrow 0} -\frac{1}{x(\Delta x+x)} = -\frac{1}{x^2}$$

$$\frac{df(e^x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{e^{x+\Delta x} - e^x}{1} = \lim_{\Delta x \rightarrow 0} e^{x+\Delta x} = e^x$$

$$\frac{df(e^{x+1})}{dx} = \lim_{\Delta x \rightarrow 0} \frac{e^{x+\Delta x+1} - e^{x+1}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{e^{x+1}e^{\Delta x} - e^{x+1}}{\Delta x} = e^{x+1} \cdot \lim_{\Delta x \rightarrow 0} \frac{e^{\Delta x} - 1}{\Delta x} = e^{x+1} \cdot \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x} = e^{x+1} \cdot 1 = e^{x+1}$$

$$\frac{df(e^{2x})}{dx} = \lim_{\Delta x \rightarrow 0} \frac{e^{2(x+\Delta x)} - e^{2x}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2e^{2(\Delta x+x)}}{1} = \lim_{\Delta x \rightarrow 0} 2e^{2(\Delta x+x)} = \lim_{\Delta x \rightarrow 0} 2e^{2(\Delta x+x)} = 2e^{2x}$$

$$\frac{df(x^3)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta x^2 + 2\Delta x + x^2 + \Delta x + x^2 + x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{3x^2 + 3\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{3x^2 + 3\Delta x + \Delta x^2}{\Delta x} = 3x^2$$

Integrals by definition:

$$\int_0^b x^2 dx = \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} x_n^2 \Delta x$$

$$\text{where: } \Delta x = \frac{b-0}{N} = \frac{b}{N} \text{ and } x_n = n \cdot \Delta x = \frac{n \cdot b}{N}$$

$$\text{rewrite: } \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} x_n^2 \cdot \Delta x = \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} \left(\frac{n \cdot b}{N} \right)^2 \cdot \frac{b}{N} = \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} \frac{n^2 \cdot b^3}{N^3} = \lim_{N \rightarrow \infty} \frac{b^3}{N^3} \sum_{n=0}^{N-1} n^2$$

$$\text{we use the following sum: } \sum_{n=0}^{N-1} n^2 = \frac{(N-1)N(2N-1)}{6}$$

$$\text{our sum becomes: } \lim_{N \rightarrow \infty} \frac{b^3}{N^3} \cdot \frac{(N-1)N(2N-1)}{6} = \lim_{N \rightarrow \infty} b^3 \cdot \frac{2 - \frac{3}{N} + \frac{1}{N^2}}{6}$$

$$\text{take the limit: } \int_0^b x^2 dx = \lim_{N \rightarrow \infty} b^3 \cdot \frac{2 - \frac{3}{N} + \frac{1}{N^2}}{6} = b^3 \cdot \frac{2-0+0}{6} = \frac{1}{3} b^3$$

$$\text{thus: } \int_0^b x^2 dx = \frac{1}{3} b^3$$

Integration by parts:

Example:

$$\int_{-1}^1 (x e^x + x^3) dx = \int_{-1}^1 x e^x dx + \int_{-1}^1 x^3 dx$$

$$= (x e^x - e^x + \frac{1}{4} x^4) - (x e^x - e^x + \frac{1}{4} x^4)$$

$$= (1 \cdot e^1 - e^1 + \frac{1}{4} \cdot 1^4) - (-1 \cdot e^{-1} - e^{-1} + \frac{1}{4} \cdot (-1)^4) = \frac{2}{e}$$

Integration by substitution: