Estimating the Heston Model

Programming I project

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Contents

1 User Guide

This code is implemented in Python and is largely based on the work of Stauffenegger & Fleissner (2019). The folder contains the following documents: this guide, the Python script named heston_model_program.py and the file requirements.txt

This code does not require modification to execute except for the required library indicated in the requirements.txt file. By default, it runs using standard parameters that are already embedded in the program. The time to maturity is fixed, and the code selects the sixth option closest to the maturity date. By altering these parameters, it is possible to generate new results with the code. The parameters that we can modified are:

- **ticker**
- **r** = risk free rate
- **mu** = mean rate of drift
- **sigma** = Initial volatility at time t
- **edate** = expiration date
- **q** = dividend yield
- **kappa** = speed of mean revertion
- **theta** = mean level of variance
- **volvol** = volatility of the volatility
- **rho** = correlation

The program computes option prices using the Heston characteristic function shown in the paper of Cui et al. (2017) and the COS-FFT (Fang & Oosterlee 2008) method. Initially, calculations are performed with the parameters specified at the start of the execution. Subsequently, the code conducts parameter optimization through a differential evolution algorithm. This optimization adjusts the parameters by minimizing the sum of squared errors of the function. Once the parameters are optimized, the code runs a Monte Carlo simulation, generating ten price paths for Apple stock. These paths are modeled as two correlated Brownian motions. Using the average of these simulated prices, the program determines the option price.

2 Economic Problem

2.1 Summary

The economic problem that the Heston Model aims to address is the inability of the Black-Scholes model to accurately model and price options in a realistic market. The Black-Scholes model operates under stringent assumptions, and the more restrictive these assumptions, the less general the model becomes, leading to significant pricing errors. The Heston Model relaxes some of the Black-Scholes assumptions and seeks to provide a more accurate pricing mechanism for options. (Heston 1993).

2.2 From Deterministic to Stochastic Volatility

The Black-Scholes model makes a strong assumption about price volatilities, positing that all asset volatilities remain constant over time. Additionally, the Black-Scholes model does not account for volatility clustering (periods of high volatility followed by high volatility, and vice versa). The Heston model, by incorporating a mean-reverting volatility process, offers a more holistic perspective by considering stochastic volatility. (Heston 1993) The real-world economic problems addressed by the Heston Model include:

- **Volatility Clustering:** The model assumes that volatility will eventually revert to its mean in the long term (a mean-reverting process) while allowing for short-term persistence of volatility. This behavior aligns with observed clustering in real-world markets. (Schoutens et al. 2004).
- **Leverage Effect:** The Heston model includes a correlation between the asset price and its volatility through the parameter ρ (the correlation between two Brownian motions). A negative ρ captures the leverage effect, where a decline in asset price leads to an increase in volatility, reflecting typical financial market behavior. (Heston 1993).
- **Smile and Skew Effects:** By incorporating stochastic volatility and the correlation between price and volatility, the Heston model captures changes in volatility across different maturities and strike prices. Stochastic volatility ensures that volatility is dynamic and dependent on the underlying asset price, reproducing the volatility smile and skew observed in real markets. The mean-reverting variance enhances realism in the volatility structure. (Cui et al. 2017).

3 Mathematical Methods

3.1 Stock Price Process

The Heston model describes the evolution of the stock price S_t and its variance v_t under a risk-neutral measure Q:

$$dS_t = \mu S_t\, dt + \sqrt{v_t} S_t\, dW_{1,t}^P$$

where:

$$\begin{aligned} S_t &: \text{Price of the underlying asset at time } t, \\ \mu &: \text{Price process drift}, \\ v_t &: \text{Instantaneous variance (stochastic)}, \\ dW_{1,t}^P &: \text{Brownian motion of the asset price.} \end{aligned}$$

This equation combines:

1. A deterministic growth term, $\mu S_t dt$, capturing the expected price change.
2. A stochastic term, $\sqrt{v_t} S_t dW_{1,t}^P$, introducing randomness proportional to both the current price and volatility.

The stochastic term accounts for market uncertainty, with variance v_t evolving dynamically. (Heston 1993).

3.2 Variance Process

$$dv_t = \kappa(\theta - v_t)\,dt + \xi\sqrt{v_t}\,dW^P_{2,t}$$

where:

$$\begin{aligned} \kappa &: \text{Speed of mean reversion,} \\ \theta &: \text{Long-term variance mean,} \\ \xi &: \text{Variance volatility,} \\ dW^P_{2,t} &: \text{Brownian motion of variance.} \end{aligned}$$

3.3 Correlation Between Brownian Motions

$$dW^P_{1,t}\,dW^P_{2,t} = \rho\,dt$$

where:

ρ : Correlation between Brownian motions (typically negative for stocks).

3.4 The Cui Heston Model's Characteristic Function

$$\phi(\theta; u, \tau) = \exp\left(iu\log F - \frac{\kappa\bar{v}\rho\tau iu}{\sigma} - A + \frac{2\kappa\bar{v}}{\sigma^2}D\right),$$

where:

$$F = S_0 exp^{(r-q)\tau}$$

$$A = \frac{\frac{\left(u^2+iu\right)\sinh(d\tau)}{2d}}{\sigma\cosh(d\tau) + \frac{\xi\sigma\sinh(d\tau)}{d}}$$

$$D = \ln\left(\frac{d}{\sigma} + \frac{(kappa-d)\tau}{2}\right) - \ln\left(\frac{d+\xi}{2\sigma} + \frac{d-\xi}{2\sigma}e^{-d\tau}\right)$$

The characteristic function proposed by Cui et al. (2017) is a modified representation of the characteristic function for the Heston stochastic volatility model. This version addresses numerical discontinuities encountered in earlier formulations, particularly in handling complex logarithms and square roots for long maturities. (Cui et al. 2017)

3.4.1 Comparison Between Cui Functions and Other Characteristic Functions

The original formulation by Heston (1993) introduced the characteristic function but faced numerical instabilities due to branch switching in complex power and logarithmic terms. This instability arises from multivalued functions in the complex domain, especially when crossing the negative real axis. Cui et al. (2017) resolved these issues by employing a more stable approach to handling multivalued functions, improving accuracy and stability in computations. From Figure 1, it can be observed that, for different characteristic functions, the Cui function is integrable within its domain and continuous.

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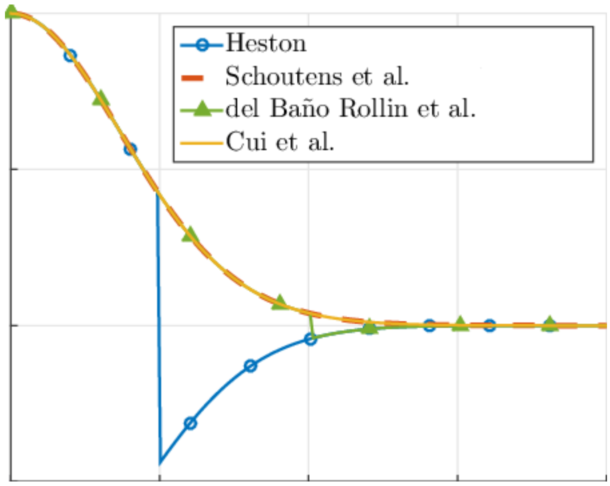

Figure 1: Comparison with others' Heston characteristic functions (Cui et al. 2017)

3.5 Truncation Range

The choice of truncation range is crucial for option pricing. Initially, our method, based on a multiple of the time-to-maturity adjusted standard deviation, yielded unsatisfactory results, particularly for low volatility. Consequently, we adopted a more sophisticated range determination technique, as outlined by Fitt et al. (2010).

3.6 Fourier Transforms

Fourier transforms are employed in the Heston model to enable efficient computation of option prices by transforming the pricing problem into the frequency domain. The model's characteristic function, derived using Fourier methods, provides a semi-closed-form solution for option pricing. This approach enhances computational speed and accuracy, particularly for complex payoff structures. (Cui et al. 2017)

3.7 Application of the Monte Carlo Method in the Heston Model

The Monte Carlo method, a computational algorithm that employs repeated random sampling to derive numerical results, is particularly well-suited for valuing complex financial derivatives for which analytical solutions are challenging or infeasible. Monte Carlo simulation is a widely used technique for valuing options by simulating the stochastic price paths of the underlying asset as governed by the Heston model. For example, the price of a European call option can be estimated by calculating the average of the discounted payoffs obtained across a large number of simulated price trajectories.

4 Key Results and Interpretation

This section presents the key findings of our analysis, emphasizing their relevance to the research objectives. The results are analyzed in the context of the proposed methodology and the existing literature, underscoring their significance and originality. Table 1 provides the values of the optimized parameters obtained from the differential evolution optimization process, implemented using the SciPy library. The optimization is made minimizing the residual minimal squared error with a value of 18557.65836.

kappa	theta	sigma	rho	volvol
2.1324827	0.13632047	0.49853898	-0.65173857	0.07954157

Table 1: Optimitazion evolution parameters results

Figure 2 shows ten price paths of Apple stock, simulated with a Monte Carlo algorithm.
Figure 3 represents the comparison between the real call market prices and the simulated call Monte Carlo prices.
To compute the Monte Carlo simulation, we used the optimized parameters found from Table 1. The

difference between the highest and lowest final simulated Apple stock price represents the volatility. The generated final stock price paths are primarily concentrated between \$210 and \$260, indicating a more accurate estimation of the Monte Carlo call option prices. From Figure 3, it can be inferred that the

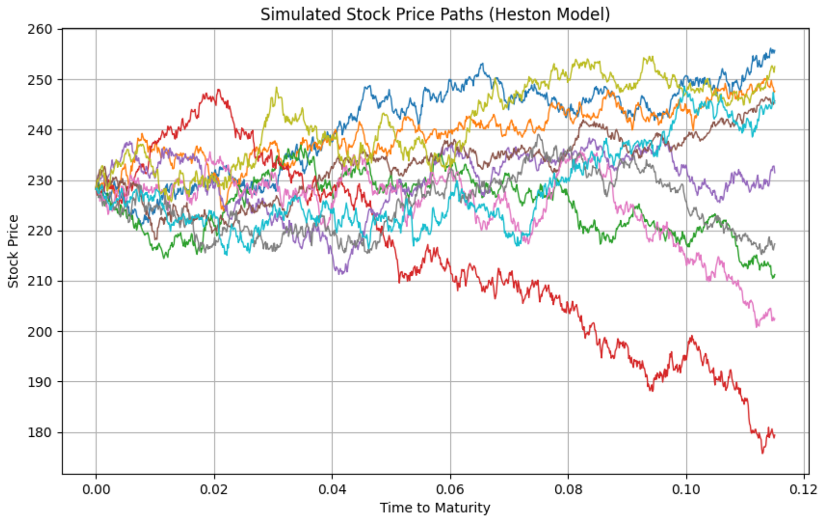

Figure 2: Monte Carlo simulation of Apple stock prices using the Heston Model. Random seed used: 15.

Monte Carlo simulation demonstrates a high level of precision with a small error, enabling a reasonably accurate prediction of option prices. However, a significant error is observed at the strike price of \$190. Improved results could be achieved with a greater number of iterations. This simulation was performed using seed 15 to ensure replicability, as the seed is not included in the provided code. Without a fixed seed, the simulation may produce different outcomes.

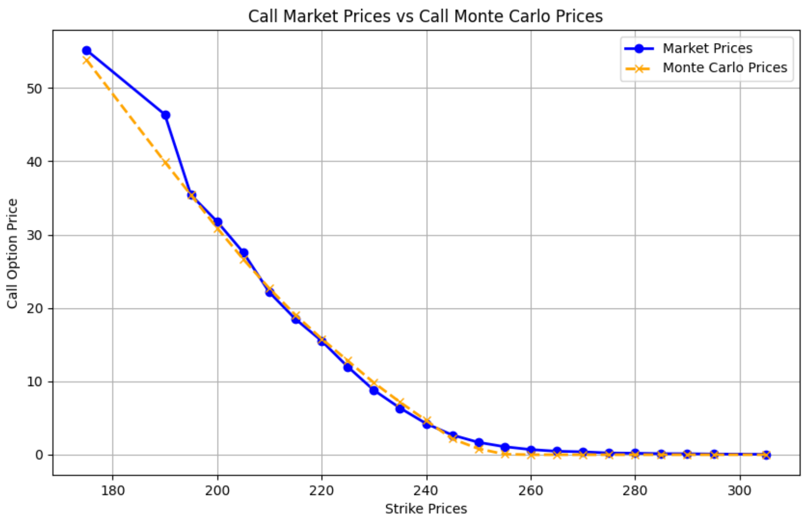

Figure 3: Comparison between call market prices and call Monte Carlo prices

In conclusion, it can be stated that the Heston model is a powerful tool for option pricing. Its implementation can be enhanced by estimating implied volatility and optimizing parameters through Monte Carlo simulations focused specifically on the parameters. However, with a shorter time horizon, the resulting prices may be less accurate due to less realistic outcomes from the Wiener processes.

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