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# Spatial Rigid Body Dynamics using Dual Quaternion Components

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## Abstract

The equations of motion of cooperating robot systems are obtained by connecting the individual equations of motion for each arm and the workpiece using the constraint equations of the closed chain. Dual quaternions have been shown to provide a convenient algebraic representation for these constraints. This paper derives the equations of motion for a rigid body whose position is defined by the eight dual quaternion coordinates. Because a rigid body has six degrees of freedom, the use of dual quaternion coordinates requires two additional differential constraint equations. The result is a set of ten differential equations prescribing the movement of the body. Use of these equations is demonstrated through a planar example of a double pendulum.

## Introduction

This paper presents a technique for solving spatial rigid body dynamics problems using the dual quaternion components that comprise the Image Space (see Ravani and Roth, 1984). The dynamic model for an unconstrained spatial rigid body with applied forces and moments is derived. They can act as the basis for the solution of any rigid body dynamics problem. The accuracy of the model is validated by using it to describe the motion of a double pendulum. This example serves as the basis for a model of a workpiece held by a 2R planar robot. A future goal of this research is to determine a convenient method of solving the dynamics of a workpiece held by cooperating spatial robots. Rodriguez (1989) has proposed a solution that uses forward recursive dynamics to solve for the motion of the robot manipulators propelling the workpiece. This method is convenient for solution of the arm dynamics. It is difficult to apply to the constrained workpiece. Using the Image Space transformation offers a potentially fruitful model for the dynamics of the workpiece. The Image Space transformation gives geometric shape to physical constraints (Ge and McCarthy, 1990). The constraints are then expressible in a simple algebraic form.

The kinematic properties of dual quaternions have been studied and presented in Study(1903), Blashke and Muller (1956), Yang (1964), Bottema and Roth (1979), and Ravani and Roth. Dual quaternions have been used in robotics to define kinematic constraints that allow algebraic descriptions of the joint space obstacles in the workspace of a robot (see Ge and McCarthy, 1989, and Dooley and McCarthy, 1990). Payandeh and Goldenberg (1987) solved the inverse kinematics problem for a 6 DOF robot arm using dual quaternions. Kim and Kumar (1990) extend these results to n-DOF arms.

Little work, however, has been done using dual quaternion components to solve dynamics problems. Yang and Freudenstein (1964) analyzed the properties of spatial mechanisms using dual quaternions. Their analysis includes a static force analysis, but does not examine the dynamics problem. Rotational quaternions have been used to solve spacecraft dynamics problems (see Kane, et. al., 1983). Haug, 1989, and Nikravesh, 1988, use rotational quaternions to develop dynamics models for mechanical systems. None of these works uses dual quaternions to formulate the dynamics problem. This paper gives the first formulation of a general dynamics problem using dual quaternion components.

# 1 Spatial Displacements

A spatial displacement can be described by a six-dimensional rigid transformation from a moving reference frame, $M'$, to a fixed reference frame, F. (See Figure 1.) The transformation can be described by a rotation matrix, $[A]$, and a translation vector, $\mathbf{d} = (d_x, d_y, d_z)$.

## 1.1 The Image Space Transformation

Ravani and Roth showed that a spatial displacement can be represented by a geometric transformation into an eight dimensional Image Space,

$$\begin{aligned} \mathrm{Q} &= (\mathrm{q}, \mathrm{q_o}) = \\ &(Q_1 i + Q_2 j + Q_3 k + Q_4) + \\ &\epsilon(Q_5 i + Q_6 j + Q_7 k + Q_8). \end{aligned}$$

where the symbols $i$, $j$, and $k$ are quaternion units, and $\epsilon$ is the dual unit and has the property $\epsilon^2 = 0$. The first four terms in the dual quaternion are the Euler parameters for the rotation matrix [A] and are denoted by:

$$\begin{array}{ll} Q_1 = s_x \sin\frac{\gamma}{2}, & Q_2 = s_y \sin\frac{\gamma}{2}, \\ Q_3 = s_z \sin\frac{\gamma}{2}, & Q_4 = \cos\frac{\gamma}{2}, \end{array} \tag{1}$$

where $\mathbf{s} = (s_x, s_y, s_z)$ is the axis of rotation and $\gamma$ is the rotation angle. The second four parameters combine the translation with the Euler Parameters:

$$\left\{ \begin{array}{c} Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{array} \right\} = \frac{1}{2} \left[ \begin{array}{cccc} 0 & -d_z & d_y & d_x \\ d_z & 0 & -d_x & d_y \\ -d_y & d_x & 0 & d_z \\ -d_x & -d_y & -d_z & 0 \end{array} \right] \left\{ \begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{array} \right\}. \tag{2}$$

The Image Space resides in $\mathbf{R^8}$, but a spatial displacement can be described with six coordinates. So the Image Space actually occupies a six-dimensional manifold within $\mathbf{R^8}$. From Equation 1 and Equation 2, the constraints are

$$Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 - 1 = 0, \tag{3}$$
$$Q_1 Q_5 + Q_2 Q_6 + Q_3 Q_7 + Q_4 Q_8 = 0. \tag{4}$$

A convenient property of the Image Space transformation is that the composition of two successive

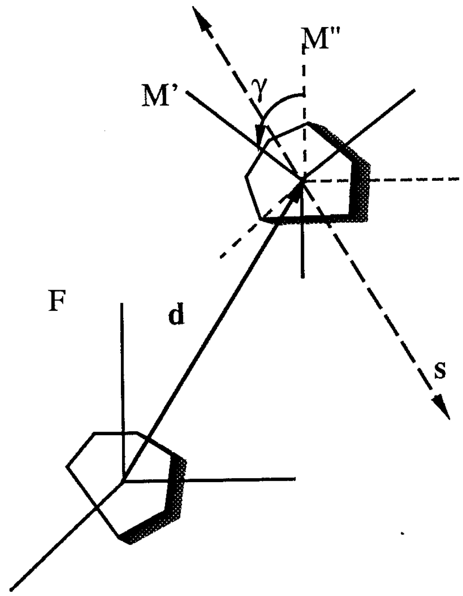


Figure 1: A spatial displacement of a rigid body.

displacements, $\mathbf{G} = (\mathbf{g}, \mathbf{g^o})$ and $\mathbf{H} = (\mathbf{h}, \mathbf{h^o})$ can be calculated as a product (see McCarthy 1990):

$$\mathbf{GH} = [G^+]\mathbf{H} = \left[ \begin{array}{cc} [g^+] & [0] \\ [g^{0+}] & [g^+] \end{array} \right] \left\{ \begin{array}{c} \mathbf{h} \\ \mathbf{h}^0 \end{array} \right\}$$
$$\mathbf{GH} = [H^-]\mathbf{G} = \left[ \begin{array}{cc} [h^-] & [0] \\ [h^{0-}] & [h^-] \end{array} \right] \left\{ \begin{array}{c} \mathbf{g} \\ \mathbf{g}^0 \end{array} \right\} \tag{5}$$

where

$$[g^+] = \left[ \begin{array}{cccc} g_4 & -g_3 & g_2 & g_1 \\ g_3 & g_4 & -g_1 & g_2 \\ -g_2 & g_1 & g_4 & g_3 \\ -g_1 & -g_2 & -g_3 & g_4 \end{array} \right], \tag{6}$$

$$[h^-] = \left[ \begin{array}{cccc} h_4 & h_3 & -h_2 & h_1 \\ -h_3 & h_4 & h_1 & h_2 \\ h_2 & -h_1 & h_4 & h_3 \\ -h_1 & -h_2 & -h_3 & h_4 \end{array} \right] \tag{7}$$

are $4 \times 4$ matrices constructed from the vectors $\mathbf{g}$ and $\mathbf{h}$ respectively, and $[g^{0+}]$, $[h^{0-}]$ are constructed in the same way as $[g^+]$, $[h^-]$ from $\mathbf{g}^0$ and $\mathbf{h}^0$ respectively. The matrix [0] is the $4 \times 4$ zero matrix.

## 1.2 The Inverse of the Transformation

To write the equations of motion for a rigid body transformation, it is necessary to write the kinematic properties in terms of the dual quaternion coordinates. From Equation 2 the translation, $\mathbf{d} = (d_x, d_y, d_z, 0)$, can be recovered from the dual quaternion by the equations

$$\mathbf{d} = 2[G]\mathbf{q_O} = 2[G_o]\mathbf{q}, \tag{8}$$

where

$$[G] = [q^-]^T,$$

where $[q^-]$ is defined as in Equation 7, and

$$[G_o] = \begin{bmatrix} -Q_8 & Q_7 & -Q_6 & Q_5 \\ -Q_7 & -Q_8 & Q_5 & Q_6 \\ Q_6 & -Q_5 & -Q_8 & Q_7 \\ Q_5 & Q_6 & Q_7 & Q_8 \end{bmatrix}.$$

The velocity, $\mathbf{v}$, is the derivative of Equation 8,

$$\mathbf{v} = \dot{\mathbf{d}} = 2[G]\dot{\mathbf{q_O}} + 2[G_o]\dot{\mathbf{q}}. \tag{9}$$

The acceleration, $\mathbf{a}$, is the second derivative of Equation 8,

$$\mathbf{a} = \ddot{\mathbf{d}} = 2[G]\ddot{\mathbf{q_O}} + 2[G_o]\ddot{\mathbf{q}} + 4[\dot{G}_o]\dot{\mathbf{q}}. \tag{10}$$

The angular velocity, $\omega' = (\omega'_x, \omega'_y, \omega'_z, 0)$, as seen in the moving frame, $M'$, is

$$\omega' = [G]\dot{\mathbf{q}}. \tag{11}$$

For computational purposes the angular velocity can also be written as a skew symmetric matrix, $W$, where

$$[W] = [G][\dot{G}]^T. \tag{12}$$

The angular acceleration, $\alpha$, is found by taking the derivative of Equation 11. Its components are

$$\dot{\omega'} = [G]\ddot{\mathbf{q}}. \tag{13}$$

# 2 Equations of Motion

In this section, the equations of motion for an unconstrained rigid body are derived. We anticipate using the results to solve for the motion of a workpiece acted on by cooperating robots. Dual quaternion representation is a convenient method to analyze the constrained workpiece, because kinematic constraints have simple algebraic forms when transformed into the Image Space. (See Ge and McCarthy, 1990.)

The equations of motion will be derived using the method of generalized forces presented in Kane and Levinson. This method is convenient to apply to dual quaternions because of its straightforward application to any set of generalized coordinates. Before formulating the dynamics with dual quaternions, we will summarize the method of generalized forces.

## 2.1 Generalized Forces

The first step in writing the equations of motion is to select a set of generalized coordinates, $x_i$, and generalized speeds, $u_i$. The velocity of the system can be written as a combination of the generalized speeds,

$$\mathbf{v} = \sum_{i=1}^{n} \mathbf{\Psi_i} u_i, \tag{14}$$

where $\mathbf{\Psi_i}$ is defined as the $i$th partial velocity for the system. In a similar manner, the angular velocity can be written as a combination of the generalized speeds,

$$\omega = \sum_{i=1}^{n} \mathbf{\Omega_i} u_i, \tag{15}$$

where $\mathbf{\Omega_i}$ is defined as the $i$th partial angular velocity for the system.

The partial velocities and partial angular velocities are used to formulate the generalized forces and generalized inertia forces that comprise the equations of motion. The $i$th generalized force is defined as

$$F_i = \mathbf{R} \bullet \mathbf{\Psi_i} + \mathbf{M} \bullet \mathbf{\Omega_i}, \tag{16}$$

where $\mathbf{R}$ and $\mathbf{M}$ are the resultant applied force and applied moment acting on the rigid body. The $i$th generalized inertia force is defined as

$$F_i^\star = -m\mathbf{a} \bullet \mathbf{\Psi_i} - ([\mathbf{J}']\dot{\omega'} + \omega' \times [\mathbf{J}']\omega') \bullet \mathbf{\Omega_i}, \tag{17}$$

where m is the mass and $[\mathbf{J}']$ is the inertia matrix of the rigid body. Kane and Levinson show that Equation 16 and Equation 17 are related by the equation

$$F_i + F_i^\star = 0. \tag{18}$$

This equation defines the equations of motion for the generalized coordinates, $\mathbf{x}$.

## 2.2 Application to Dual Quaternions

To apply Equation 18 to a rigid body whose motion is described using dual quaternions, define the dual quaternion coordinates as the generalized coordinates and their derivatives $(\dot{Q}_i)$ as the generalized speeds.

Then, apply Equation 14 to the velocity given by Equation 9 to determine the partial velocities. Define the $4 \times 8$ matrix, $[\mathbf{\Psi}] = [\mathbf{\Psi_1} \cdots \mathbf{\Psi_8}]$, then

$$[\mathbf{\Psi}] = 2 \left[ \ [G_o] \quad [G] \ \right], \tag{19}$$

where $[G]$ and $[G_o]$ are defined in Equation 8. Apply Equation 15 to the angular velocity given by Equation 11 to determine the partial angular velocities. Similarly, define a $4 \times 8$ matrix, $[\mathbf{\Omega}]$, where

$$[\mathbf{\Omega}] = 2 \left[ \ [G] \quad [0] \ \right]. \tag{20}$$

These partial velocities and partial angular velocities can be used to write the generalized forces by appying Equation 16. If $\mathbf{R} = (R_x, R_y, R_z, 0)$ and $\mathbf{n}' = (n'_x, n'_y, n'_z, 0)$, the generalized forces are

$$\mathbf{F} = 2 \begin{bmatrix} [G_o]^T & [G]^T \\ [G]^T & [0] \end{bmatrix} \begin{Bmatrix} \mathbf{R} \\ \mathbf{n}' \end{Bmatrix}. \tag{21}$$

The generalized inertia forces are found similarly. Use the inertia matrix attached to the moving frame, $M'$, and expand it to a $4 \times 4$ matrix by adding a row and column of zeroes. Define $\chi = [J']\dot{\omega}' + \omega' \times [J']\omega'$. Then, the generalized inertia forces are

$$\mathbf{F}^* = -2 \begin{bmatrix} [G_o]^T & [G]^T \\ [G]^T & [0] \end{bmatrix} \begin{Bmatrix} m\ddot{\mathbf{d}} \\ \chi \end{Bmatrix}, \tag{22}$$

This equation can be written in terms of the generalized coordinates by substituting for $\ddot{d}$, $\omega'$, and $\dot{\omega}'$ with the terms in Equation 10 through Equation 13. With these substitutions, Equation 18 becomes

$$[M]\ddot{\mathbf{Q}} = \mathbf{F} + h(\mathbf{Q}, \dot{\mathbf{Q}}), \tag{23}$$

where

$$[M] = \begin{bmatrix} 4m[\Delta_o] + 4[G]^T[J'][G] & 4m[G_o]^T[G] \\ 4m[G]^T[G_o] & 4m[I_4] \end{bmatrix} \tag{24}$$

where $[I_4]$ is the $4 \times 4$ identity matrix and

$$[\Delta_o] = [I_4] \sum_{i=5}^{8} Q_i^2,$$

and where

$$h(\mathbf{Q}, \dot{\mathbf{Q}}) = \begin{Bmatrix} 8[\dot{G}][J'][\dot{G}]^T \mathbf{q} - 8m[G_o]^T[\dot{G}_o]\dot{\mathbf{q}} \\ -8m[G]^T[\dot{G}]\dot{\mathbf{q}}_\mathbf{o} \end{Bmatrix}. \tag{25}$$

Equation 23 provides a basis for solving any spatial rigid body dynamics problem.

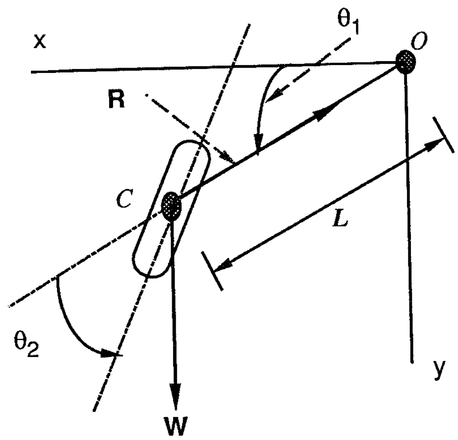


Figure 2: The double pendulum.

# 3 Example

To show that Equation 23 accurately models the dynamic properties of a rigid body, we solve a pertinent, yet relatively simple problem. The planar double pendulum, shown in Figure 2, was chosen because it is relatively simple, yet nontrivial; with some modification, it can serve as the basis for a model of a rigid body propelled by a planar robot (see Figure 3); and, by externally applying only the gravitational force, the accuracy of the solution can be verified using Conservation of Energy.

## 3.1 Planar Quaternions

Planar quaternions are a subset of dual quaternions. Motion constrained to the $xy$-plane introduces four additional constraints

$$Q_1 = Q_2 = Q_7 = Q_8 = 0. \tag{26}$$

These constraints leave four nontrivial equations of motion, $i = 3, \ldots, 6$. Equation 26 trivially satisfies Equation 4 and simplifies Equation 3 to

$$Q_3^2 + Q_4^2 - 1 = 0. \tag{27}$$

## 3.2 The Double Pendulum

The double pendulum has two links, see Figure 2. The first link is massless. Its length is $L$. It rotates $\theta_1$ about the fixed point, $O$. The second link has

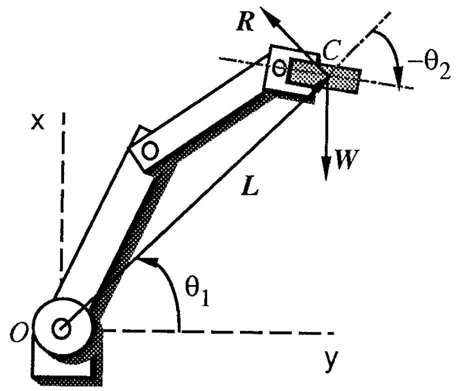


Figure 3: An open chain planar robot holding a workpiece.

mass, m, and moment of inertia, $j'$. It rotates $\theta_2$ about the center of mass, $C$.

The position of the second link is obtained by a rotation, $\theta_1$, a translation, $L$, along the x-axis, and a rotation, $\theta_2$. Applying Equation 1, Equation 2 and Equation 5 to this set of displacements yields the dual quaternion for the second link

$$\begin{aligned} Q_3 &= \sin\frac{\theta_1+\theta_2}{2},\ Q_4 = \cos\frac{\theta_1+\theta_2}{2}, \\ Q_5 &= \frac{L}{2}\cos\frac{\theta_1-\theta_2}{2}, \\ Q_6 &= \frac{L}{2}\sin\frac{\theta_1-\theta_2}{2}. \end{aligned} \tag{28}$$

These components introduce a sixth constraint equation,

$$Q_5^2 + Q_6^2 - (\frac{L}{2})^2 = 0. \tag{29}$$

### 3.3 Solution

To solve for the motion of the rigid body, differentiate the constraints, Equation 27 and Equation 29, twice and attach them to the equations of motion. The motion is then defined by

$$\begin{bmatrix} [M] & [\Phi]^T \\ [\Phi] & [0] \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{Q}} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} + h(\mathbf{Q}, \dot{\mathbf{Q}}) \\ \gamma \end{Bmatrix}. \tag{30}$$

$\mathbf{Q} = (Q_3, Q_4, Q_5, Q_6)$ in the above equation. The mass matrix is

$$[M] = \begin{bmatrix} mL^2[I_2] + 4[g]^T[j'][g] & 4m[g_o]^T[g] \\ 4m[g]^T[g_o] & 4m[I_2] \end{bmatrix}, \tag{31}$$

where

$$[g] = \begin{bmatrix} Q_4 & -Q_3 \\ Q_3 & Q_4 \end{bmatrix},$$

$$[g_o] = \begin{bmatrix} -Q_6 & Q_5 \\ Q_5 & Q_6 \end{bmatrix},$$

$$[j'] = \begin{bmatrix} j' & 0 \\ 0 & 0 \end{bmatrix},$$

and $[I_2]$ is the $2 \times 2$ identity matrix. The constraint matrix is

$$[\Phi] = \begin{bmatrix} Q_3 & Q_4 & 0 & 0 \\ 0 & 0 & Q_5 & Q_6 \end{bmatrix}. \tag{32}$$

The force is

$$\mathbf{F} = 2W\{Q_5, Q_6, Q_3, Q_4\}^T.$$

The vector

$$\gamma = -\left\{\dot{Q}_3^{\,2} + \dot{Q}_4^{\,2}, \dot{Q}_5^{\,2} + \dot{Q}_6^{\,2}\right\}^T.$$

And, the vector

$$h(\mathbf{Q}, \dot{\mathbf{Q}}) = \begin{Bmatrix} 8[\dot{g}][j'][\dot{g}]^T\mathbf{q} - 8m[g_o]^T[\dot{g}_o]\dot{\mathbf{q}} \\ -8m[g]^T[\dot{g}]\dot{\mathbf{q}}_{\mathbf{O}} \end{Bmatrix}.$$

Integration and calculation of the energy of the system proves the accuracy of the equations of motion. Since the only externally applied force is gravity, the energy remains constant throughout the motion. Our simulations showed that energy indeed was conserved for motion of the double pendulum including $360^o$ rotations of both links. A second method of validating the accuracy of the results was to apply Newton's Laws directly and compare the two results. For every set of initial conditions, the results of the two simulations matched closely. All errors were smaller than the expected accuracy of the numeric integration.

## Conclusion

In this paper we have presented the formulation of a completely general rigid body dynamics problem using the dual quaternion components of the Image Space. The accuracy of this formulation was validated by using this technique to solve a problem that has a verfiable solution. This technique has both disadvantages and advantages. As seen in Equation 23 the equations of motion can be complicated. A second disadvantage is that the physical significance of the variables is not intuitively apparent. For instance, using a more conventional set of coordinates, the variable $\lambda$ would be the reaction force. In our example $\lambda$ is linearly related to the reaction force. Nevertheless, Kane and Levinson's method allows us to write equations of motion despite this drawback. One advantage of dual quaternion coordinates is the ease with which the system kinematics can be obtained. This ease stems from the multiplicative properties (Equation 5) of the image space. A second advantage is the simple form of the constraint equations (for example, Equation 29). We anticipate that these advantages will enable us to directly solve more complicated problems involving closed chains.


## Acknowlegement

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