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# Adaptive Position and Attitude-Tracking Controller for Satellite Proximity Operations Using Dual Quaternions

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**This paper proposes a nonlinear adaptive position and attitude-tracking controller for satellite proximity operations between a target and a chaser satellite. The controller requires no information about the mass and inertia matrix of the chaser satellite and takes into account the gravitational acceleration, the gravity-gradient torque, the perturbing acceleration due to Earth's oblateness, and constant (but otherwise unknown) disturbance forces and torques. Sufficient conditions to identify the mass and inertia matrix of the chaser satellite are also given. The controller is shown to ensure almost global asymptotical stability of the translational and rotational position and velocity tracking errors. Unit dual quaternions are used to simultaneously represent the absolute and relative attitude and position of the target and chaser satellites. The analogies between quaternions and dual quaternions are explored in the development of the controller.**

## Nomenclature

| | | |
|---|---|---|
| $\bar{a}_g^B$ | = | gravitational acceleration expressed in the body frame |
| $\bar{a}_{J_2}^I$ | = | perturbing acceleration due to Earth's oblateness expressed in the inertial frame |
| $\bar{f}^B$ | = | total external force vector applied to the body expressed in the body frame |
| $\hat{f}^B$ | = | total external dual force applied to the body about its center of mass expressed in body coordinates |
| $\hat{f}_c^B$ | = | dual control force expressed in body coordinates |
| $\hat{f}_d^B$ | = | dual disturbance force expressed in body coordinates |
| $\widehat{\hat{f}_d^B}$ | = | estimate of the dual disturbance force |
| $\mathbb{H}$ | = | set of quaternions |
| $\mathbb{H}_d$ | = | set of dual quaternions |
| $\mathbb{H}_d^r$ | = | set of dual scalar quaternions with zero dual part |
| $\mathbb{H}^s$ | = | set of scalar quaternions |
| $\mathbb{H}_d^s$ | = | set of dual scalar quaternions |
| $\mathbb{H}^u$ | = | set of unit quaternions |
| $\mathbb{H}_d^u$ | = | set of unit dual quaternions |
| $\mathbb{H}^v$ | = | set of vector quaternions |
| $\mathbb{H}_d^v$ | = | set of dual vector quaternions |
| $h(.,.)$ | = | function $h$: $\mathbb{H}_d^v \times \mathbb{H}_d^v \to \mathbb{R}^7$ |
| $\bar{I}^B$ | = | inertia matrix of the chaser satellite |
| $I_n$ | = | identity matrix of size $n$ |
| $K_p$, $K_d$, $K_i$, $K_j$, $K_{I_{ij}}$, $K_m$ | = | control gains |
| $M^B$ | = | dual inertia matrix |
| $\widehat{M^B}$ | = | estimate of the dual inertia matrix |
| $m$ | = | mass of chaser satellite |
| $q_{Y/Z}$ | = | unit quaternion from the $z$ frame to the $y$ frame |
| $\hat{q}_{Y/Z}$ | = | unit dual quaternion from the $z$ frame to the $y$ frame |
| $R_e$ | = | Earth's mean equatorial radius |
| $\bar{r}_{Y/Z}^X$ | = | translation vector from the origin of the $z$ frame to the origin of the $y$ frame expressed in the $x$ frame |
| $\bar{v}_{Y/Z}^X$ | = | linear velocity of the $y$ frame with respect to the $z$ frame expressed in the $x$ frame |
| $W(\cdot)$ | = | function $W$: $[0,\infty) \to \mathbb{R}^{8\times 7}$ |
| $\Delta\hat{f}_d^B$ | = | dual disturbance force estimation error |
| $\Delta M^B$ | = | dual matrix estimation error |
| $\epsilon$ | = | dual unit |
| $\mu$ | = | Earth's gravitational parameter |
| $\bar{\tau}^B$ | = | total external moment vector applied to the body about its center of mass expressed in the body frame |
| $\bar{\tau}_{\nabla g}^B$ | = | gravity-gradient torque expressed in the body frame |
| $\bar{\omega}_{Y/Z}^X$ | = | angular velocity of the $y$ frame with respect to the $z$ frame expressed in the $x$ frame |
| $\hat{\omega}_{Y/Z}^X$ | = | dual velocity of the $y$ frame with respect to the $z$ frame expressed in the $x$ frame |
| $\mathbf{0}$ | = | quaternion $(\bar{0}, 0)$ |
| $\hat{\mathbf{0}}$ | = | dual quaternion $\mathbf{0} + \epsilon\mathbf{0}$ |
| $\mathbf{1}$ | = | quaternion $(\bar{0}, 1)$ |
| $\hat{\mathbf{1}}$ | = | dual quaternion $\mathbf{1} + \epsilon\mathbf{0}$ |
| $*$ | = | matrix-quaternion multiplication |
| $\star$ | = | matrix-dual-quaternion multiplication |

## Introduction

SEVERAL agencies and organizations around the world are investigating satellite proximity operations as an enabling technology for several space missions such as on-orbit satellite inspection, health monitoring, surveillance, servicing, refueling, and optical interferometry [1–4]. One of the biggest challenges introduced by this technology is the need to simultaneously and accurately track both time-varying relative position and attitude references trajectories to avoid collisions between the satellites and achieve mission objectives.

The problem of deriving control laws for satellite proximity operations has a long history. For example, in [4], nonlinear control and adaptation laws were designed ensuring almost global asymptotic convergence of the position and attitude errors, despite the presence of unknown mass and inertia parameters, using the vectrix formalism. (As is usually done in literature, the terminology "almost" globally asymptotically stable controller will be used to designate controllers that are asymptotically stable over an open and dense set. It has been

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shown that this is the best one can achieve with a continuous controller for the rotational motion, because the special group of rotation matrices SO(3) is a compact manifold [5].) However, the controller in [4] is a very high-order dynamic compensator, which limits its applicability, especially for satellites with limited onboard computational resources. In [3], three different nonlinear position and attitude controllers for spacecraft formation flying are presented. All these controllers require full knowledge of the mass and inertia matrix of the satellite. In [6], a relative position and attitude-tracking controller that requires no linear and angular velocity measurements and no mass and inertia matrix information is presented. However, as explained in [7], if the reference trajectory is not sufficiently exciting, this controller cannot guarantee that the relative position and attitude errors will converge to zero. In [1], an adaptive terminal sliding-mode "pose" (i.e., position and attitude) tracking controller is proposed, based on dual quaternions, that does not require full knowledge of the mass and inertia matrix of the spacecraft. This controller takes into account the gravitational acceleration, the gravity-gradient torque, and constant (but otherwise unknown) disturbance forces and torques, but not the perturbing acceleration due to Earth's oblateness. In addition, the convergence region of the controller is not specified in [1] and no conditions for identifying the mass and inertia matrix of the spacecraft are given. Moreover, this controller requires a priori knowledge of upper bounds on the mass, on the maximum eigenvalue of the inertia matrix, on the constant but otherwise unknown disturbance forces and torques, on the desired relative linear and angular velocity between the spacecraft and their first derivative, on the linear and angular velocity of the chaser spacecraft with respect to the inertial frame, and on the position of the chaser spacecraft with respect to the inertial frame.

Similar to [1], this paper proposes an adaptive pose-tracking controller based on dual quaternions [8]. However, unlike [1], the controller proposed in this paper does not require a priori knowledge of any upper bounds on the system parameters or states. Another contribution of this paper with respect to [1] is the consideration of the perturbing acceleration due to Earth's oblateness, which is typically the largest perturbing acceleration on a satellite below Geosynchronous Earth Orbit (GEO) [9]. Moreover, unlike [1], the controller proposed in this paper is proven to ensure almost global asymptotical stability of the linear and rotational position and velocity tracking errors. With respect to [4], the controller proposed in this paper has only as many states as unknown parameters and, hence, requires less computational resources. A final contribution of this paper with respect to existing literature is the definition of sufficient conditions for both mass and inertia matrix identification. Although these conditions are not needed for convergence, they can be useful to design maneuvers to identify these parameters, if needed (e.g., after a docking maneuver, after the deployment of antennas or solar panels, etc.).

The use of dual quaternions, in addition to the insight it provides to derive the controller, it also allows one to write the controller in a compact form. Owing to their numerous advantages in providing a "natural" representation of the combined translational and rotational kinematics, dual quaternions have been successfully applied to inertial navigation [10], control of rigid-body kinematics [11,12] and dynamics [1,8,13–18], inverse kinematic analysis [19,20], computer vision [21,22], and animation [23]. Dual quaternions are an extension of classical quaternions and, as already mentioned, they provide a compact way to represent the attitude and position of a rigid body. Dual quaternions are actually closely related to Chasles's theorem, which states that the general displacement of a rigid body can be represented by a rotation about an axis (called the "screw axis") and a translation along that axis, creating a screw-like motion [10]. Compared with other representations of this screw-like motion, such as dual orthogonal $3 \times 3$ matrices, dual special unitary $2 \times 2$ matrices, and dual Pauli spin matrices, dual quaternions have been found to be the most efficient representation to perform basic pose transformations in terms of storage requirements and number of operations [24]. Under the same metrics, dual quaternions have also been found to be more efficient than $4 \times 4$ homogeneous matrix transformations and Rodriguez parameters/translation vector pairs for solving the direct kinematic problem in robotics [25]. Moreover, dual quaternions allow attitude and position controllers to be written as a single control law. It has also been shown that they automatically take into account the natural coupling between the rotational and translational motions [12,14]. However, the most useful property of dual quaternions is that the combined translational and rotational kinematic and dynamic equations of motion written in terms of dual quaternions have the same form as the rotational-only kinematic and dynamic equations of motion written in terms of quaternions [17]. This appealing property has been recently used in [8,16,17], where it was shown that it is possible to extend an attitude controller with some desirable properties into a combined position and attitude controller with equivalent desirable properties, by often simply substituting quaternions with dual quaternions in the attitude-only quaternion control law and corresponding Lyapunov function. This paper extends the results presented in [26] to include position-tracking and mass identification.

The development of combined position and attitude controllers from existing attitude controllers has some advantages over techniques based on the special Euclidean group SE(3), where rotations are represented directly by rotation matrices [27–29]. In the latter, asymptotically stability of the combined rotational and translational motion is proven by either defining two different error functions for the position and attitude error [29] or, in two steps, by first proving the asymptotical stability of the rotational motion before the asymptotical stability of the translational motion can be proven [27] (recall that the translational motion depends on the rotational motion). The use of dual quaternions to describe the kinematics allows the use of a single error function, the "error dual quaternion" (defined by analogy to the classical rotation error quaternion) to represent the combined position and attitude error. As a result, the asymptotic stability of the combined rotational and translational motion is proven in a single step by using a Lyapunov function with the same form as the Lyapunov function used to prove the asymptotic stability of the rotational-only controller. On the other hand, whereas quaternions produce two closed-loop equilibrium points (because quaternions cover SO(3) twice [30], both representing the identity rotation matrix), the use of rotation matrices produces a minimum of four closed-loop equilibrium points [27,28], only one of which is the identity rotation matrix. On the downside, dual quaternions inherit the so-called unwinding phenomenon from classical quaternions [5]. Solutions for this problem are well known and some are suggested in this paper. Finally, note that full knowledge of the mass and inertia matrix are required to implement the tracking controllers proposed in [27,28].

The paper is organized as follows. In Sec. II, unit quaternions and unit dual quaternions are introduced. Then, the relative kinematic and dynamic equations of motion for satellite proximity operations written in terms of dual quaternions are derived in Sec. III. In Sec. IV, the adaptive attitude and position tracking controller for satellite proximity operations is deduced and proved to ensure almost global asymptotical stability of the translational and rotational position and velocity tracking errors. Then, sufficient conditions for mass and inertia matrix identification are given in Sec. V. Finally, the proposed controller is analyzed and validated through numerical examples in Sec. VI.

## Mathematical Preliminaries

Most of the mathematical preliminaries in this section can be found in [8,16,17]. For the benefit of the reader, the main properties of quaternions and dual quaternions, which are essential for the results presented in this paper, are summarized here.

### Quaternions

A quaternion is defined as $q = q_1 i + q_2 j + q_3 k + q_4$, where $q_1$, $q_2$, $q_3$, $q_4 \in \mathbb{R}$, and $i$, $j$, and $k$ satisfy $i^2 = j^2 = k^2 = -1$, $i = jk = -kj$, $j = ki = -ik$, and $k = ij = -ji$ [14]. A quaternion can also be represented as the ordered pair $q = (\bar{q}, q_4)$, where $\bar{q} = [\, q_1 \quad q_2 \quad q_3 \,]^\top \in \mathbb{R}^3$ is the vector part of the quaternion and $q_4 \in \mathbb{R}$ is the scalar part of the quaternion. Vector quaternions and scalar quaternions are quaternions with zero scalar part and vector part, respectively. The set of quaternions, vector quaternions, and scalar quaternions will be denoted by $\mathbb{H} = \{q\colon q = q_1 i + q_2 j + q_3 k + q_4, q_1, q_2, q_3, q_4 \in \mathbb{R}\}$, $\mathbb{H}^v = \{q \in \mathbb{H}\colon q_4 = 0\}$, and $\mathbb{H}^s = \{q \in \mathbb{H}\colon q_1 = q_2 = q_3 = 0\}$, respectively.

The basic operations on quaternions are defined as follows:

Addition:

$$a + b = (\bar{a} + \bar{b}, a_4 + b_4) \in \mathbb{H}$$

Multiplication by a scalar:

$$\lambda a = (\lambda \bar{a}, \lambda a_4) \in \mathbb{H}$$

Multiplication:

$$ab = (a_4 \bar{b} + b_4 \bar{a} + \bar{a} \times \bar{b}, a_4 b_4 - \bar{a} \cdot \bar{b}) \in \mathbb{H}$$

Conjugation:

$$a^* = (-\bar{a}, a_4) \in \mathbb{H}$$

Dot product:

$$a \cdot b = \frac{1}{2}(a^* b + b^* a) = \frac{1}{2}(a b^* + b a^*) = (\bar{0}, a_4 b_4 + \bar{a} \cdot \bar{b}) \in \mathbb{H}^s$$

Cross product:

$$a \times b = \frac{1}{2}(ab - b^* a^*) = (b_4 \bar{a} + a_4 \bar{b} + \bar{a} \times \bar{b}, 0) \in \mathbb{H}^v$$

Norm:

$$\|a\|^2 = a a^* = a^* a = a \cdot a = (\bar{0}, a_4^2 + \bar{a} \cdot \bar{a}) \in \mathbb{H}^s$$

Scalar part:

$$\mathrm{sc}(a) = (\bar{0}, a_4) \in \mathbb{H}^s$$

Vector part:

$$\mathrm{vec}(a) = (\bar{a}, 0) \in \mathbb{H}^v$$

where $a$, $b \in \mathbb{H}$, $\lambda \in \mathbb{R}$, and $\bar{0} = [0 \quad 0 \quad 0]^\mathsf{T}$. Note that the quaternion multiplication is not commutative. In this paper, the quaternions $(\bar{0}, 1)$ and $(\bar{0}, 0)$ will be denoted by $\mathbf{1}$ and $\mathbf{0}$, respectively.

The multiplication of a matrix $M \in \mathbb{R}^{4\times 4}$ with a quaternion $q \in \mathbb{H}$ will be defined as $M * q = (M_{11}\bar{q} + M_{12} q_4, M_{21}\bar{q} + M_{22} q_4) \in \mathbb{H}$, where

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$M_{11} \in \mathbb{R}^{3\times 3}$, $M_{12} \in \mathbb{R}^{3\times 1}$, $M_{21} \in \mathbb{R}^{1\times 3}$, and $M_{22} \in \mathbb{R}$. This definition is analogous to the multiplication of a $4 \times 4$ matrix with a four-dimensional vector. It can be easily shown that the following property holds from the previous definitions:

$$(M * a) \cdot b = a \cdot (M^\mathsf{T} * b), \qquad a, b \in \mathbb{H}, \qquad M \in \mathbb{R}^{4\times 4}$$

Finally, the $\mathcal{L}_\infty$ norm of a function $u\colon [0, \infty) \to \mathbb{H}$ is defined as $\|u\|_\infty = \sup_{t\geq 0}\|u(t)\|$. Moreover, $u \in \mathcal{L}_\infty$ if and only if $\|u\|_\infty < \infty$.

*Attitude Representation with Unit Quaternions*

The relative orientation of a body frame with respect to the inertial frame can be represented by the unit quaternion $q_{B/I} = (\sin(\phi/2)\bar{n}, \cos(\phi/2))$, where the body frame is said to be rotated with respect to the inertial frame about the unit vector $\bar{n}$ by an angle $\phi$. Note that $q_{B/I}$ is a unit quaternion because it belongs to the set $\mathbb{H}^u = \{q \in \mathbb{H}\colon q \cdot q = \mathbf{1}\}$. The body coordinates of a vector $\bar{v}^B$ can be calculated from the inertial coordinates of that same vector $\bar{v}^I$, and vice versa, via $v^B = q_{B/I}^* v^I q_{B/I}$ and $v^I = q_{B/I} v^B q_{B/I}^*$, where $v^B = (\bar{v}^B, 0)$ and $v^I = (\bar{v}^I, 0)$.

*Quaternion Representation of the Rotational Kinematic Equations*

The rotational kinematic equations of the body frame and of a frame with some desired attitude, both with respect to the inertial frame and represented by the unit quaternions $q_{B/I}$ and $q_{D/I}$, respectively, are given by $\dot{q}_{B/I} = \frac{1}{2} q_{B/I}\omega_{B/I}^B = \frac{1}{2}\omega_{B/I}^I q_{B/I}$ and $\dot{q}_{D/I} = \frac{1}{2} q_{D/I}\omega_{D/I}^D = \frac{1}{2}\omega_{D/I}^I q_{D/I}$, where $\omega_{Y/Z}^X = (\bar{\omega}_{Y/Z}^X, 0)$ and $\bar{\omega}_{Y/Z}^X$ is the angular velocity of the $y$ frame with respect to the $z$ frame expressed in the $x$ frame. The error quaternion

$$q_{B/D} = q_{D/I}^* q_{B/I} \tag{1}$$

is the unit quaternion that rotates the desired frame onto the body frame. By differentiating Eq. (1), the kinematic equations of the error quaternion turn out to be

$$\dot{q}_{B/D} = \frac{1}{2} q_{B/D}\omega_{B/D}^B = \frac{1}{2}\omega_{B/D}^D q_{B/D} \tag{2}$$

where $\omega_{B/D}^B = \omega_{B/I}^B - \omega_{D/I}^B$ (and $\omega_{B/D}^D = \omega_{B/I}^D - \omega_{D/I}^D$).

**Dual Quaternions**

A dual quaternion is defined as $\hat{q} = q_r + \epsilon q_d$, where $\epsilon$ is the dual unit defined by $\epsilon^2 = 0$ and $\epsilon \neq 0$. The quaternions $q_r$, $q_d \in \mathbb{H}$ are called the "real part" and the "dual part" of the dual quaternion, respectively.

Dual vector quaternions and dual scalar quaternions are dual quaternions formed from vector quaternions (i.e., $q_r$, $q_d \in \mathbb{H}^v$) and scalar quaternions (i.e., $q_r$, $q_d \in \mathbb{H}^s$), respectively. The set of dual quaternions, dual scalar quaternions, dual vector quaternions, and dual scalar quaternions with zero dual part will be denoted by $\mathbb{H}_d = \{\hat{q}\colon \hat{q} = q_r + \epsilon q_d, q_r, q_d \in \mathbb{H}\}$, $\mathbb{H}_d^s = \{\hat{q}\colon \hat{q} = q_r + \epsilon q_d, q_r, q_d \in \mathbb{H}^s\}$, $\mathbb{H}_d^v = \{\hat{q}\colon \hat{q} = q_r + \epsilon q_d, q_r, q_d \in \mathbb{H}^v\}$, and $\mathbb{H}_d^r = \{\hat{q}\colon \hat{q} = q_r + \epsilon \mathbf{0}, q_r \in \mathbb{H}^s\}$, respectively.

The basic operations on dual quaternions are defined as follows [1,12]:

Addition:

$$\hat{a} + \hat{b} = (a_r + b_r) + \epsilon(a_d + b_d) \in \mathbb{H}_d$$

Multiplication by a scalar:

$$\lambda\hat{a} = (\lambda a_r) + \epsilon(\lambda a_d) \in \mathbb{H}_d$$

Multiplication:

$$\hat{a}\,\hat{b} = (a_r b_r) + \epsilon(a_r b_d + a_d b_r) \in \mathbb{H}_d$$

Conjugation:

$$\hat{a}^* = a_r^* + \epsilon a_d^* \in \mathbb{H}_d$$

Swap:

$$\hat{a}^s = a_d + \epsilon a_r \in \mathbb{H}_d$$

Dot product:

$$\hat{a} \cdot \hat{b} = \frac{1}{2}(\hat{a}^*\hat{b} + \hat{b}^*\hat{a}) = \frac{1}{2}(\hat{a}\hat{b}^* + \hat{b}\hat{a}^*) = a_r \cdot b_r + \epsilon(a_d \cdot b_r + a_r \cdot b_d) \in \mathbb{H}_d^s$$

Cross product:

$$\hat{a} \times \hat{b} = \frac{1}{2}(\hat{a}\,\hat{b} - \hat{b}^*\hat{a}^*) = a_r \times b_r + \epsilon(a_d \times b_r + a_r \times b_d) \in \mathbb{H}_d^v$$

Dual norm:

$$\|\hat{a}\|_d^2 = \hat{a}\hat{a}^* = \hat{a}^*\hat{a} = \hat{a}\cdot\hat{a} = (a_r \cdot a_r) + \epsilon(2a_r \cdot a_d) \in \mathbb{H}_d^s$$

Scalar part:

$$\mathrm{sc}(\hat{a}) = \mathrm{sc}(a_r) + \epsilon\,\mathrm{sc}(a_d) \in \mathbb{H}_d^s$$

Vector part:

$$\mathrm{vec}(\hat{a}) = \mathrm{vec}(a_r) + \epsilon\,\mathrm{vec}(a_d) \in \mathbb{H}_d^v$$

where $\hat{a}$, $\hat{b} \in \mathbb{H}_d$ and $\lambda \in \mathbb{R}$. Note that the dual quaternion multiplication is not commutative. In this paper, the dual quaternions $1 + \epsilon\mathbf{0}$ and $\mathbf{0} + \epsilon\mathbf{0}$ will be denoted by $\hat{\mathbf{1}}$ and $\hat{\mathbf{0}}$, respectively.

Because the dot product and dual norm of dual quaternions yield, in general, a dual number, the norm of a dual quaternion will be defined in this paper as [1,31]

$$\|\hat{a}\|^2 = \hat{a} \circ \hat{a} \in \mathbb{H}_d^r \tag{3}$$

where $\circ$ denotes the "dual quaternion circle product" given by $\hat{a} \circ \hat{b} = a_r \cdot b_r + a_d \cdot b_d \in \mathbb{H}_d^r$, where $\hat{a}, \hat{b} \in \mathbb{H}_d$. Note that the dual quaternion circle product is commutative. The definition of norm of a dual quaternion is not unique and other authors have used alternative norms, for example, based on the logarithm of the dual quaternion [15,32,33]. The attitude part of the norm of a dual quaternion given by Eq. (3) matches the quaternion norm used in [26]. Hence, the norm of a dual quaternion given by Eq. (3) is preferred in this paper because it will allow a relatively straightforward extension of the attitude-only controller presented in [26] into a combined position and attitude controller.

The multiplication of a matrix $M \in \mathbb{R}^{8\times 8}$ with a dual quaternion $\hat{q} \in \mathbb{H}_d$ will be defined as $M\star\hat{q} = (M_{11} * q_r + M_{12} * q_d) + \epsilon(M_{21} * q_r + M_{22} * q_d) \in \mathbb{H}_d$, where

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \qquad M_{11}, M_{12}, M_{21}, M_{22} \in \mathbb{R}^{4\times 4}$$

This definition is analogous to the multiplication of an $8 \times 8$ matrix with an eight-dimensional vector.

It can be shown that the following properties [17] follow from the previous definitions:

$$\hat{a} \circ (\hat{b}\,\hat{c}) = \hat{b}^s \circ (\hat{a}^s\hat{c}^*) = \hat{c}^s \circ (\hat{b}^*\hat{a}^s) \in \mathbb{R}, \qquad \hat{a}, \hat{b}, \hat{c} \in \mathbb{H}_d \tag{4}$$

$$\hat{a} \circ (\hat{b} \times \hat{c}) = \hat{b}^s \circ (\hat{c} \times \hat{a}^s) = \hat{c}^s \circ (\hat{a}^s \times \hat{b}), \qquad \hat{a}, \hat{b}, \hat{c} \in \mathbb{H}_d^v \tag{5}$$

$$\hat{a} \times \hat{a} = \hat{\mathbf{0}}, \qquad \hat{a} \in \mathbb{H}_d^v \tag{6}$$

$$\hat{a} \times \hat{b} = -\hat{b} \times \hat{a}, \qquad \hat{a}, \hat{b} \in \mathbb{H}_d^v \tag{7}$$

$$\hat{a}^s \circ \hat{b}^s = \hat{a} \circ \hat{b}, \qquad \hat{a}, \hat{b} \in \mathbb{H}_d \tag{8}$$

$$\|\hat{a}^s\| = \|\hat{a}\|, \qquad \hat{a} \in \mathbb{H}_d \tag{9}$$

$$\|\hat{a}^*\| = \|\hat{a}\|, \qquad \hat{a} \in \mathbb{H}_d \tag{10}$$

$$(M\star\hat{a}) \circ \hat{b} = \hat{a} \circ (M^\top\star\hat{b}), \qquad \hat{a}, \hat{b} \in \mathbb{H}_d, \qquad M \in \mathbb{R}^{8\times 8} \tag{11}$$

$$|\hat{a} \circ \hat{b}| \le \|\hat{a}\|\|\hat{b}\|, \qquad \hat{a}, \hat{b} \in \mathbb{H}_d \tag{12}$$

$$\|\hat{a}\,\hat{b}\| \le \sqrt{3/2}\,\|\hat{a}\|\|\hat{b}\|, \qquad \hat{a}, \hat{b} \in \mathbb{H}_d \tag{13}$$

Finally, the $\mathcal{L}_\infty$ norm of a function $\hat{u}$: $[0,\infty) \to \mathbb{H}_d$ is defined as $\|\hat{u}\|_\infty = \sup_{t\ge 0}\|\hat{u}(t)\|$. Moreover, $\hat{u} \in \mathcal{L}_\infty$ if and only if $\|\hat{u}\|_\infty < \infty$.

*Attitude and Position Representation with Unit Dual Quaternions*

The position and orientation (i.e., pose) of a body frame with respect to the inertial frame can be represented by a unit quaternion $q_{B/I} \in \mathbb{H}^u$ and by a translation vector $\bar{r}_{B/I} \in \mathbb{R}^3$. Alternatively, the pose of the body frame with respect to the inertial frame can be represented more compactly by the unit dual quaternion [10] $\hat{q}_{B/I} = q_{B/I} + \epsilon\frac{1}{2}r^I_{B/I}q_{B/I} = q_{B/I} + \epsilon\frac{1}{2}q_{B/I}r^B_{B/I}$, where $r^X_{Y/Z} = (\bar{r}^X_{Y/Z}, 0)$ and $\bar{r}^X_{Y/Z} = [x^X_{Y/Z} \quad y^X_{Y/Z} \quad z^X_{Y/Z}]^\top$ is the translation vector from the origin of the $z$ frame to the origin of the $y$ frame expressed in the $x$ frame. Note that $\hat{q}_{B/I}$ is a unit dual quaternion because it belongs to the set [16] $\mathbb{H}^u_d = \{\hat{q} \in \mathbb{H}_d\colon \hat{q}\cdot\hat{q} = \hat{q}\hat{q}^* = \hat{q}^*\hat{q} = \|\hat{q}\|_d = \hat{1}\}$.

*Lemma:* The unit dual quaternion $\hat{q}_{Y/Z} = q_{Y/Z} + \epsilon\frac{1}{2}q_{Y/Z}r^Y_{Y/Z} \in \mathcal{L}_\infty$, if and only if $r^Y_{Y/Z} \in \mathcal{L}_\infty$.

*Proof:* If $\hat{q}_{Y/Z} \in \mathcal{L}_\infty$, then $q_{Y/Z}r^Y_{Y/Z} \in \mathcal{L}_\infty$. Note that the unit quaternion $q_{Y/Z} \in \mathcal{L}_\infty$ by definition. Moreover, since $\|q_{Y/Z}r^Y_{Y/Z}\| = \|r^Y_{Y/Z}\|$, this also implies that $r^Y_{Y/Z} \in \mathcal{L}_\infty$. On the other hand, it is trivial to see that, if $q_{Y/Z}$, $r^Y_{Y/Z} \in \mathcal{L}_\infty$, then $\hat{q}_{Y/Z} = q_{Y/Z} + \epsilon(1/2)q_{Y/Z}r^Y_{Y/Z} \in \mathcal{L}_\infty$ as well. $\square$

*Dual Quaternion Representation of the Rotational and Translational Kinematic Equations*

The kinematic equations of the body frame and of a frame with some desired position and attitude, both with respect to the inertial frame and represented by the unit dual quaternions $\hat{q}_{B/I}$ and $\hat{q}_{D/I} = q_{D/I} + \epsilon(1/2)r^I_{D/I}q_{D/I} = q_{D/I} + \epsilon(1/2)q_{D/I}r^D_{D/I}$, respectively, are given by [10]

$$\begin{aligned} \dot{\hat{q}}_{B/I} &= \frac{1}{2}\hat{\omega}^I_{B/I}\hat{q}_{B/I} = \frac{1}{2}\hat{q}_{B/I}\hat{\omega}^B_{B/I} \quad \text{and} \\ \dot{\hat{q}}_{D/I} &= \frac{1}{2}\hat{\omega}^I_{D/I}\hat{q}_{D/I} = \frac{1}{2}\hat{q}_{D/I}\hat{\omega}^D_{D/I} \end{aligned} \tag{14}$$

where $\hat{\omega}^X_{Y/Z}$ is the "dual velocity" of the $y$ frame with respect to the $z$ frame expressed in the $x$ frame, so that $\hat{\omega}^X_{Y/Z} = \omega^X_{Y/Z} + \epsilon(v^X_{Y/Z} + \omega^X_{Y/Z} \times r^X_{X/Y})$, $v^X_{Y/Z} = (\bar{v}^X_{Y/Z}, 0)$, and $\bar{v}^X_{Y/Z}$ is the linear velocity of the $y$ frame with respect to the $z$ frame expressed in the $x$ frame.

By direct analogy to Eq. (1), the "dual error quaternion" [1,14,15,17] is defined as

$$\hat{q}_{B/D} \triangleq \hat{q}^*_{D/I}\hat{q}_{B/I} = q_{B/D} + \epsilon\frac{1}{2}q_{B/D}r^B_{B/D} \tag{15}$$

where $r^B_{B/D} = r^B_{B/I} - r^B_{D/I}$. As illustrated in Fig. 1, the dual error quaternion represents the rotation $q_{B/D}$ and the translation $r^B_{B/D}$ necessary to align the desired frame with the body frame. It can be shown [16] that $\hat{q}_{B/D}$ is a unit dual quaternion. By differentiating Eq. (15) and using Eq. (14), the kinematic equations of the dual error quaternion turn out to be [1,15,17]

$$\dot{\hat{q}}_{B/D} = \frac{1}{2}\hat{q}_{B/D}\hat{\omega}^B_{B/D} = \frac{1}{2}\hat{\omega}^D_{B/D}\hat{q}_{B/D} \tag{16}$$

where $\hat{\omega}^B_{B/D} = \hat{\omega}^B_{B/I} - \hat{\omega}^B_{D/I}$ is the "dual relative velocity" between the body frame and the desired frame expressed in the body frame. Note that $\hat{\omega}^B_{D/I} = \hat{q}^*_{B/D}\hat{\omega}^D_{D/I}\hat{q}_{B/D}$ and $\hat{\omega}^D_{B/I} = \hat{q}_{B/D}\hat{\omega}^B_{B/I}\hat{q}^*_{B/D}$. Note also that the kinematic equations of the dual error quaternion (16) and of the error quaternion (2) have the same form.

## Dual Quaternion Representation of the Relative Dynamic Equations for Satellite Proximity Operations

The dual quaternion representation of the rigid-body dynamic equations, assuming constant (or slowly varying) mass and inertia matrix, is given by [1,17]

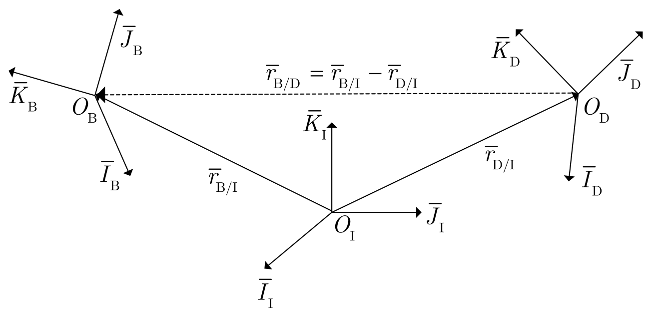


**Fig. 1 Relation between frames.**

$$
\begin{aligned}
(\dot{\hat{\omega}}^B_{B/D})^s = (M^B)^{-1}\star(\hat{f}^B - (\hat{\omega}^B_{B/D} + \hat{\omega}^B_{D/I}) \\
\times (M^B\star((\hat{\omega}^B_{B/D})^s + (\hat{\omega}^B_{D/I})^s)) \\
- M^B\star(\hat{q}^*_{B/D}\dot{\hat{\omega}}^D_{D/I}\hat{q}_{B/D})^s - M^B\star(\hat{\omega}^B_{D/I}\times\hat{\omega}^B_{B/D})^s)
\end{aligned} \tag{17}
$$

where $\hat{f}^B = f^B + \epsilon\tau^B$ is the total external "dual force" applied to the body about its center of mass expressed in body coordinates, $f^B = (\bar{f}^B, 0)$, $\bar{f}^B$ is the total external force vector applied to the body, $\tau^B = (\bar{\tau}^B, 0)$, and $\bar{\tau}^B$ is the total external moment vector applied to the body about its center of mass. Finally, $M^B \in \mathbb{R}^{8\times8}$ is the "dual inertia matrix" [16] defined as

$$
M^B = \begin{bmatrix} mI_3 & 0_{3\times1} & 0_{3\times3} & 0_{3\times1} \\ 0_{1\times3} & 1 & 0_{1\times3} & 0 \\ 0_{3\times3} & 0_{3\times1} & \bar{I}^B & 0_{3\times1} \\ 0_{1\times3} & 0 & 0_{1\times3} & 1 \end{bmatrix}, \qquad I^B = \begin{bmatrix} \bar{I}^B & 0_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix},
$$

$$
\bar{I}^B = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{bmatrix} \tag{18}
$$

where $\bar{I}^B \in \mathbb{R}^{3\times3}$ is the mass moment of inertia of the body about its center of mass written in the body frame, and $m$ is the mass of the body. Note that the dual inertia matrix is symmetric. Also note the similarity between the dual quaternion representation of the combined rotational and translational dynamic equations given by Eq. (17) and the quaternion representation of the rotational(-only) dynamic equations given by [1]

$$
\begin{aligned}
\dot{\omega}^B_{B/D} = (I^B)^{-1} * (\tau^B - (\omega^B_{B/D} + \omega^B_{D/I}) \times (I^B * (\omega^B_{B/D} + \omega^B_{D/I})) \\
- I^B * (q^*_{B/D}\dot{\omega}^D_{D/I}q_{B/D}) - I^B * (\omega^B_{D/I}\times\omega^B_{B/D}))
\end{aligned}
$$

Other dual quaternion representations of the rigid-body dynamic equations exist. These alternative representations are compared with Eq. (17) in [16].

For the case of a spacecraft in Earth orbit, the total external dual force acting on the body will be decomposed in this paper as follows:

$$
\hat{f}^B = \hat{f}^B_g + \hat{f}^B_{\nabla g} + \hat{f}^B_{J_2} + \hat{f}^B_d + \hat{f}^B_c \tag{19}
$$

where $\hat{f}^B_g = m\hat{a}^B_g$, $\hat{a}^B_g = a^B_g + \epsilon\mathbf{0}$, $a^B_g = (\bar{a}^B_g, 0)$, $\bar{a}^B_g$ is the gravitational acceleration given by

$$
\bar{a}^B_g = -\mu\frac{\bar{r}^B_{B/I}}{\|\bar{r}^B_{B/I}\|^3}
$$

$\mu = 398,600.4418\ \mathrm{km}^3/\mathrm{s}^2$ is Earth's gravitational parameter [9], $\hat{f}^B_{\nabla g} = \mathbf{0} + \epsilon\tau^B_{\nabla g}$, $\tau^B_{\nabla g} = (\bar{\tau}^B_{\nabla g}, 0)$, $\bar{\tau}^B_{\nabla g}$ is the gravity-gradient torque [1] given by

$$
\bar{\tau}^B_{\nabla g} = 3\mu\frac{\bar{r}^B_{B/I}\times(\bar{I}^B\bar{r}^B_{B/I})}{\|\bar{r}^B_{B/I}\|^5}
$$

$\hat{f}^B_{J_2} = m\hat{a}^B_{J_2}$, $\hat{a}^B_{J_2} = a^B_{J_2} + \epsilon\mathbf{0}$, $a^B_{J_2} = (\bar{a}^B_{J_2}, 0)$, $\bar{a}^B_{J_2}$ is the perturbing acceleration due to Earth's oblateness [34] given by

$$
\bar{a}^I_{J_2} = -\frac{3}{2}\frac{\mu J_2 R_e^2}{\|\bar{r}^I_{B/I}\|^4}\begin{bmatrix} \left(1 - 5\left(\frac{z^I_{B/I}}{\|\bar{r}^I_{B/I}\|}\right)^2\right)\frac{x^I_{B/I}}{\|\bar{r}^I_{B/I}\|} \\ \left(1 - 5\left(\frac{z^I_{B/I}}{\|\bar{r}^I_{B/I}\|}\right)^2\right)\frac{y^I_{B/I}}{\|\bar{r}^I_{B/I}\|} \\ \left(3 - 5\left(\frac{z^I_{B/I}}{\|\bar{r}^I_{B/I}\|}\right)^2\right)\frac{z^I_{B/I}}{\|\bar{r}^I_{B/I}\|} \end{bmatrix} \tag{20}
$$

$J_2 = 0.0010826267$, $R_e = 6378.137$ km is Earth's mean equatorial radius [9], $\hat{f}^B_d = f^B_d + \epsilon\tau^B_d$ is the dual disturbance force, and $\hat{f}^B_c = f^B_c + \epsilon\tau^B_c$ is the dual control force. This paper does not explicitly take into account other disturbance forces and torques due to, for example, atmospheric drag, solar radiation, and third bodies. Instead, this paper assumes that $\hat{f}^B_d$ is a constant (or slowly varying), but otherwise unknown, dual force that captures all neglected (but small) external forces and torques. For the sake of simplicity and compactness, it is more convenient to write $\hat{f}^B_g$, $\hat{f}^B_{\nabla g}$, and $\hat{f}^B_{J_2}$ in terms of the dual inertia matrix as follows:

$$
\hat{f}^B_g = M^B\star\hat{a}^B_g, \qquad \hat{f}^B_{\nabla g} = \frac{3\mu\hat{r}^B_{B/I}}{\|\hat{r}^B_{B/I}\|^5}\times(M^B\star(\hat{r}^B_{B/I})^s),
$$

$$
\text{and} \quad \hat{f}^B_{J_2} = M^B\star\hat{a}^B_{J_2}
$$

where $\hat{r}^B_{B/I} = r^B_{B/I} + \epsilon\mathbf{0}$.

## Adaptive Position and Attitude-Tracking Controller

The main result of this paper is an adaptive pose-tracking controller for satellite proximity operations that requires no information about the mass and inertia matrix of the body. In particular, it requires no bounds on the mass and/or eigenvalues of the inertia matrix. The next theorem presents this controller and shows that it ensures almost global asymptotic stability of the linear and angular position and velocity tracking errors.

*Theorem 1:* Consider the relative kinematic and dynamic equations given by Eqs. (16) and (17). Let the dual control force be defined by the feedback control law

$$
\begin{aligned}
\hat{f}^B_c = -\widehat{M^B}\star\hat{a}^B_g - \frac{3\mu\hat{r}^B_{B/I}}{\|\hat{r}^B_{B/I}\|^5}\times(\widehat{M^B}\star(\hat{r}^B_{B/I})^s) - \widehat{M^B}\star\hat{a}^B_{J_2} - \widehat{\hat{f}}{}^B_d \\
- \mathrm{vec}(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s)) - K_d\star\hat{s}^s + \hat{\omega}^B_{B/I}\times(\widehat{M^B}\star(\hat{\omega}^B_{B/I})^s) \\
+ \widehat{M^B}\star(\hat{q}^*_{B/D}\dot{\hat{\omega}}^D_{D/I}\hat{q}_{B/D})^s + \widehat{M^B}\star(\hat{\omega}^B_{D/I}\times\hat{\omega}^B_{B/D})^s \\
- \widehat{M^B}\star\left(K_p\star\frac{\mathrm{d}}{\mathrm{d}t}(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s))\right)^s
\end{aligned} \tag{21}
$$

where

$$
\hat{s} = \hat{\omega}^B_{B/D} + (K_p\star(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s)))^s \tag{22}
$$

$$
K_p = \begin{bmatrix} K_r & 0_{4\times4} \\ 0_{4\times4} & K_q \end{bmatrix}, \qquad K_d = \begin{bmatrix} K_v & 0_{4\times4} \\ 0_{4\times4} & K_\omega \end{bmatrix} \tag{23}
$$

$$
\begin{aligned}
K_r = \begin{bmatrix} \bar{K}_r & 0_{3\times1} \\ 0_{1\times3} & 0 \end{bmatrix}, \qquad K_q = \begin{bmatrix} \bar{K}_q & 0_{3\times1} \\ 0_{1\times3} & 0 \end{bmatrix}, \\
K_v = \begin{bmatrix} \bar{K}_v & 0_{3\times1} \\ 0_{1\times3} & 0 \end{bmatrix}, \qquad K_\omega = \begin{bmatrix} \bar{K}_\omega & 0_{3\times1} \\ 0_{1\times3} & 0 \end{bmatrix}
\end{aligned} \tag{24}
$$

$\bar{K}_r,\ \bar{K}_q,\ \bar{K}_v,\ \bar{K}_\omega \in \mathbb{R}^{3\times 3}$ are positive definite matrices, $\widehat{M^B}$ is an estimate of the dual inertia matrix updated according to

$$\frac{\mathrm{d}}{\mathrm{d}t}v(\widehat{M^B}) = K_i\Bigg[h\Bigg(\hat{s}^s, -(\hat{q}^*_{B/D}\dot{\hat{\omega}}^D_{D/I}\hat{q}_{B/D})^s - (\hat{\omega}^B_{D/I}\times\hat{\omega}^B_{B/D})^s + \left(K_p\star\frac{\mathrm{d}(\hat{q}^*_{B/D}(\hat{q}^s_{B/D}-\hat{1}^s))}{\mathrm{d}t}\right)^s + \hat{a}^B_g + \hat{a}^B_{J_2}\Bigg) - h((\hat{s}\times\hat{\omega}^B_{B/I})^s, (\hat{\omega}^B_{B/I})^s) + h\left(\left(\hat{s}\times\frac{3\mu\hat{r}^B_{B/I}}{\|\hat{r}^B_{B/I}\|^5}\right)^s, (\hat{r}^B_{B/I})^s\right)\Bigg] \tag{25}$$

$K_i \in \mathbb{R}^{7\times 7}$ is a positive definite matrix, $v(M^B) = [I_{11}\quad I_{12}\quad I_{13}\quad I_{22}\quad I_{23}\quad I_{33}\quad m]^\top$ is a vectorized version of the dual inertia matrix $M^B$, the function $h\colon \mathbb{H}^v_d\times\mathbb{H}^v_d\to\mathbb{R}^7$ is defined as $\hat{a}\circ(M^B\star\hat{b}) = h(\hat{a},\hat{b})^\top v(M^B) = v(M^B)^\top h(\hat{a},\hat{b})$ or, equivalently, $h(\hat{a},\hat{b}) = [a_5b_5, a_6b_5 + a_5b_6, a_7b_5 + a_5b_7, a_6b_6, a_7b_6 + a_6b_7, a_7b_7, a_1b_1 + a_2b_2 + a_3b_3]^\top$, $\hat{f}^B_d$ is an estimate of the dual disturbance force updated according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\hat{f}^B_d} = K_j\star\hat{s}^s \tag{26}$$

$$K_j = \begin{bmatrix} K_f & 0_{4\times 4}\\ 0_{4\times 4} & K_\tau\end{bmatrix},\quad K_f = \begin{bmatrix}\bar{K}_f & 0_{3\times 1}\\ 0_{1\times 3} & 1\end{bmatrix},\quad K_\tau = \begin{bmatrix}\bar{K}_\tau & 0_{3\times 1}\\ 0_{1\times 3} & 1\end{bmatrix} \tag{27}$$

and $\bar{K}_f, \bar{K}_\tau \in \mathbb{R}^{3\times 3}$ are positive definite matrices. Assume that $\hat{q}_{D/I}$, $\hat{\omega}^D_{D/I}$, $\dot{\hat{\omega}}^D_{D/I} \in \mathcal{L}_\infty$ and $\hat{r}^B_{B/I} \neq \hat{\mathbf{0}}$. Then, for all initial conditions, $\lim_{t\to\infty}\hat{q}_{B/D} = \pm\hat{\mathbf{1}}$ (i.e., $\lim_{t\to\infty}q_{B/D} = \pm\mathbf{1}$ and $\lim_{t\to\infty}r^B_{B/D} = \mathbf{0}$), $\lim_{t\to\infty}\hat{\omega}^B_{B/D} = \hat{\mathbf{0}}$ (i.e., $\lim_{t\to\infty}\omega^B_{B/D} = \mathbf{0}$ and $\lim_{t\to\infty}v^B_{B/D} = \mathbf{0}$), and $v(\widehat{M^B}),\ \widehat{\hat{f}^B_d} \in \mathcal{L}_\infty$.

*Proof:* First, define the dual inertia matrix and dual disturbance force estimation errors as

$$\Delta M^B = \widehat{M^B} - M^B \quad\text{and}\quad \Delta\hat{f}^B_d = \widehat{\hat{f}^B_d} - \hat{f}^B_d \tag{28}$$

respectively. Note that $\hat{q}_{B/D} = \pm\hat{\mathbf{1}}$, $\hat{s} = \hat{\mathbf{0}}$, $v(\Delta M^B) = 0_{7\times 1}$, and $\Delta\hat{f}^B_d = \hat{\mathbf{0}}$ are the equilibrium conditions of the closed-loop system formed by Eqs. (16), (17), (19), (25), and (26). Consider now the following candidate Lyapunov function for the equilibrium point $(\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta\hat{f}^B_d) = (+\hat{\mathbf{1}}, \hat{\mathbf{0}}, 0_{7\times 1}, \hat{\mathbf{0}})$:

$$V(\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta\hat{f}^B_d) = (\hat{q}_{B/D} - \hat{\mathbf{1}})\circ(\hat{q}_{B/D} - \hat{\mathbf{1}}) + \frac{1}{2}\hat{s}^s\circ(M^B\star\hat{s}^s) + \frac{1}{2}v(\Delta M^B)^\top K_i^{-1}v(\Delta M^B) + \frac{1}{2}\Delta\hat{f}^B_d\circ(K_j^{-1}\star\Delta\hat{f}^B_d) \tag{29}$$

Note that $V$ is a valid candidate Lyapunov function since $V(\hat{q}_{B/D} = \hat{\mathbf{1}}, \hat{s} = \hat{\mathbf{0}}, v(\Delta M^B) = 0_{7\times 1}, \Delta\hat{f}^B_d = \hat{\mathbf{0}}) = 0$ and $V(\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta\hat{f}^B_d) > 0$ for all $(\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta\hat{f}^B_d) \in \mathbb{H}^u_d\times\mathbb{H}^v_d\times\mathbb{R}^7\times\mathbb{H}^v_d \setminus \{\hat{\mathbf{1}}, \hat{\mathbf{0}}, 0_{7\times 1}, \hat{\mathbf{0}}\}$. Note also that the real part of the first three terms of the Lyapunov function are equal to the Lyapunov function used in [26]. The time derivative of $V$ is equal to

$$\dot{V} = 2(\hat{q}_{B/D} - \hat{\mathbf{1}})\circ\dot{\hat{q}}_{B/D} + \hat{s}^s\circ(M^B\star\dot{\hat{s}}^s) + v(\Delta M^B)^\top K_i^{-1}\frac{\mathrm{d}}{\mathrm{d}t}v(\Delta M^B) + \Delta\hat{f}^B_d\circ\left(K_j^{-1}\star\frac{\mathrm{d}}{\mathrm{d}t}\Delta\hat{f}^B_d\right)$$

Then, since from Eq. (16), $\hat{\omega}^B_{B/D} = 2\hat{q}^*_{B/D}\dot{\hat{q}}_{B/D}$, Eq. (22) can be rewritten as $\dot{\hat{q}}_{B/D} = \frac{1}{2}\hat{q}_{B/D}\hat{s} - \frac{1}{2}\hat{q}_{B/D}(K_p\star(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s)))^s$, which can then be plugged into $\dot{V}$, together with the time derivative of Eq. (22), to yield

$$\dot{V} = (\hat{q}_{B/D} - \hat{\mathbf{1}})\circ(\hat{q}_{B/D}\hat{s} - \hat{q}_{B/D}(K_p\star(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s)))^s) + \hat{s}^s\circ(M^B\star(\dot{\hat{\omega}}^B_{B/D})^s) + \hat{s}^s\circ\left(M^B\star\left(K_p\star\frac{\mathrm{d}(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s))}{\mathrm{d}t}\right)^s\right) + v(\Delta M^B)^\top K_i^{-1}\frac{\mathrm{d}}{\mathrm{d}t}v(\Delta M^B) + \Delta\hat{f}^B_d\circ\left(K_j^{-1}\star\frac{\mathrm{d}}{\mathrm{d}t}\Delta\hat{f}^B_d\right) \tag{30}$$

Applying Eq. (4), inserting Eq. (17), and using $\hat{\omega}^B_{B/D} + \hat{\omega}^B_{D/I} = \hat{\omega}^B_{B/I}$ yields

$$\dot{V} = \hat{s}^s\circ(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s)) - (K_p\star(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s))) \circ(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s)) + \hat{s}^s\circ(\hat{f}^B - \hat{\omega}^B_{B/I}\times(M^B\star(\hat{\omega}^B_{B/I})^s) - M^B\star(\hat{q}^*_{B/D}\dot{\hat{\omega}}^D_{D/I}\hat{q}_{B/D})^s - M^B\star(\hat{\omega}^B_{D/I}\times\hat{\omega}^B_{B/D})^s) + \hat{s}^s\circ\left(M^B\star\left(K_p\star\frac{\mathrm{d}(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s))}{\mathrm{d}t}\right)^s\right) + v(\Delta M^B)^\top K_i^{-1}\frac{\mathrm{d}}{\mathrm{d}t}v(\Delta M^B) + \Delta\hat{f}^B_d\circ\left(K_j^{-1}\star\frac{\mathrm{d}}{\mathrm{d}t}\Delta\hat{f}^B_d\right) \tag{31}$$

Introducing the feedback control law given by Eq. (21) and using Eqs. (5) and (8), and the commutativity of the dual quaternion circle product yields

$$\dot{V} = -(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s))\circ(K_p\star(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s))) + \hat{s}^s\circ\Bigg(\hat{\omega}^B_{B/I}\times(\Delta M^B\star(\hat{\omega}^B_{B/I})^s) + \Delta M^B\star(\hat{q}^*_{B/D}\dot{\hat{\omega}}^D_{D/I}\hat{q}_{B/D})^s + \Delta M^B\star(\hat{\omega}^B_{D/I}\times\hat{\omega}^B_{B/D})^s - \Delta M^B\star\left(K_p\star\frac{\mathrm{d}(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s))}{\mathrm{d}t}\right)^s - \Delta M^B\star\hat{a}^B_g - \frac{3\mu\hat{r}^B_{B/I}}{\|\hat{r}^B_{B/I}\|^5}\times(\Delta M^B\star(\hat{r}^B_{B/I})^s) - \Delta M^B\star\hat{a}^B_{J_2} - \Delta\hat{f}^B_d\Bigg) - \hat{s}^s\circ(K_d\star\hat{s}^s) + v(\Delta M^B)^\top K_i^{-1}\frac{\mathrm{d}}{\mathrm{d}t}v(\Delta M^B) + \Delta\hat{f}^B_d\circ\left(K_j^{-1}\star\frac{\mathrm{d}}{\mathrm{d}t}\Delta\hat{f}^B_d\right) \tag{32}$$

or

$$\dot{V} = -(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s))\circ(K_p\star(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s))) + (\hat{s}\times\hat{\omega}^B_{B/I})^s \circ(\Delta M^B\star(\hat{\omega}^B_{B/I})^s) + \hat{s}^s\circ(\Delta M^B\star(\hat{q}^*_{B/D}\dot{\hat{\omega}}^D_{D/I}\hat{q}_{B/D})^s + \Delta M^B\star(\hat{\omega}^B_{D/I}\times\hat{\omega}^B_{B/D})^s - \Delta M^B\star\left(K_p\star\frac{\mathrm{d}(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s))}{\mathrm{d}t}\right)^s - \Delta M^B\star\hat{a}^B_g - \Delta M^B\star\hat{a}^B_{J_2} - \Delta\hat{f}^B_d) - \left(\hat{s}\times\frac{3\mu\hat{r}^B_{B/I}}{\|\hat{r}^B_{B/I}\|^5}\right)^s \circ(\Delta M^B\star(\hat{r}^B_{B/I})^s) - \hat{s}^s\circ(K_d\star\hat{s}^s) + v(\Delta M^B)^\top K_i^{-1}\frac{\mathrm{d}}{\mathrm{d}t}v(\Delta M^B) + \Delta\hat{f}^B_d\circ\left(K_j^{-1}\star\frac{\mathrm{d}}{\mathrm{d}t}\Delta\hat{f}^B_d\right) \tag{33}$$

Therefore, if $(\mathrm{d}/\mathrm{d}t)v(\Delta M^B)$ is defined as in Eq. (25) and $(\mathrm{d}/\mathrm{d}t)\Delta\hat{f}^B_d$ is defined as in Eq. (26), it follows that $\dot{V} = -(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s))\circ(K_p\star(\hat{q}^*_{B/D}(\hat{q}^s_{B/D} - \hat{\mathbf{1}}^s))) - \hat{s}^s\circ(K_d\star\hat{s}^s) \le 0$, for all $(\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta\hat{f}^B_d) \in \mathbb{H}^u_d\times\mathbb{H}^v_d\times\mathbb{R}^7\times\mathbb{H}^v_d \setminus \{\hat{\mathbf{1}}, \hat{\mathbf{0}}, 0_{7\times 1}, \hat{\mathbf{0}}\}$. Hence, the equilibrium point $(\hat{q}_{B/D}, \hat{s}, v(\Delta M^B), \Delta\hat{f}^B_d) = (+\hat{\mathbf{1}}, \hat{\mathbf{0}}, 0_{7\times 1}, \hat{\mathbf{0}})$ is uniformly stable and uniformly bounded [i.e., $\hat{q}_{B/D}$, $\hat{s}$, $v(\Delta M^B)$, $\Delta\hat{f}^B_d \in \mathcal{L}_\infty$]. Moreover, from Eqs. (22) and (28), this also means that

$$\hat{\omega}^B_{B/D},\qquad v(\widehat{M^B}),\qquad \widehat{\hat{f}^B_d} \in \mathcal{L}_\infty$$

Since $V \geq 0$ and $\dot{V} \leq 0$, $\lim_{t\to\infty} V(t)$ exists and is finite. Hence,

$$\lim_{t\to\infty} \int_0^t \dot{V}(\tau)\,\mathrm{d}\tau = \lim_{t\to\infty} V(t) - V(0)$$

also exists and is finite. Since

$$\hat{q}_{B/D}, \quad \hat{s}, \quad v(\Delta M^B), \quad \Delta\hat{f}_d^B, \quad \hat{\omega}_{B/D}^B, \quad v(\widehat{M^B}), \quad \hat{\hat{f}}{}_d^B, \quad \dot{\hat{\omega}}_{D/I}^D, \quad \hat{\omega}_{D/I}^B, \quad \hat{q}_{D/I} \in \mathcal{L}_\infty$$

and $\hat{r}_{B/I}^B \neq \hat{\mathbf{0}}$, then from Eqs. (16), (17), and (21), and from Lemma 1, $\hat{r}_{B/I}^B$, $\dot{\hat{q}}_{B/D}$, $\hat{f}^B$, $\dot{\hat{\omega}}_{B/D}^B$, $\dot{\hat{s}} \in \mathcal{L}_\infty$. Hence, by Barbalat's lemma, $\mathrm{vec}(\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s)) \to \hat{\mathbf{0}}$ and $\hat{s} \to \hat{\mathbf{0}}$ as $t \to \infty$. In [16], it is shown that $\mathrm{vec}(\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s)) \to \hat{\mathbf{0}}$ is equivalent to $\hat{q}_{B/D} \to \pm\hat{1}$. Finally, calculating the limit as $t \to \infty$ of both sides of Eq. (22) yields $\hat{\omega}_{B/D}^B \to \hat{\mathbf{0}}$. □

The following remarks are in order.

*Remark 1:* Theorem 1 states that $\hat{q}_{B/D}$ converges to either $+\hat{1}$ or $-\hat{1}$. Note that $\hat{q}_{B/D} = +\hat{1}$ and $\hat{q}_{B/D} = -\hat{1}$ represent the same physical relative position and attitude between frames, and so either equilibrium is acceptable. However, this can lead to the so-called unwinding phenomenon where a large rotation (greater than 180 deg) is performed, despite the fact that a smaller rotation to the equilibrium (less than 180 deg) exits. This problem of quaternions is well documented and possible solutions exist in literature [1,5,12,35].

*Remark 2:* The terms $\widehat{M^B}\star\hat{a}_g^B$,

$$\frac{3\mu\hat{r}_{B/I}^B}{\|\hat{r}_{B/I}^B\|^5}\times(\widehat{M^B}\star(\hat{r}_{B/I}^B)^s), \qquad \widehat{M^B}\star\hat{a}_{J_2}^B, \quad \text{and} \quad \hat{\hat{f}}{}_d^B$$

of the control law given by Eq. (21) are estimates of the gravitational force, gravity-gradient torque, perturbing force due to Earth's oblateness, and dual disturbance force calculated using the estimated mass and inertia matrix. These terms can be thought of as an approximate cancelation of these forces and torques. The remaining terms of the control law are a result of the rigid-body dynamic equations of motion [8]. As shown in [16], the term $\mathrm{vec}(\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s))$ is equal to $(1/2)r_{B/D}^B + \epsilon\mathrm{vec}(q_{B/D})$ and, hence, is the feedback of the relative position error and of the vector part of the relative attitude error. The term $K_d\star\hat{s}^s$ can be thought of as a damping term, where $\hat{s}$ takes the place of $\hat{\omega}_{B/D}^B$. The terms $\hat{\omega}_{B/I}^B\times(\widehat{M^B}\star(\hat{\omega}_{B/I}^B)^s)$, $\widehat{M^B}\star(\hat{q}_{B/D}^*\dot{\hat{\omega}}_{D/I}^D\hat{q}_{B/D})^s$, and $\widehat{M^B}\star(\hat{\omega}_{D/I}^B\times\hat{\omega}_{B/D}^B)^s$ are a direct cancelation of identical terms in Eq. (17) with the true mass and inertia matrix replaced by their estimates. Finally, the term

$$\widehat{M^B}\star\left(K_p\star\frac{\mathrm{d}}{\mathrm{d}t}(\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s))\right)^s$$

is a result of using $\hat{s}$ instead of $\hat{\omega}_{B/D}^B$ in the damping term and, ultimately, guarantees that the pose error will converge to zero even if the reference motion is not sufficiently exciting, unlike in [6].

*Remark 3:* Apart from the terms due to the gravity field, the dual part of the control law given by Eq. (21) is

$$\begin{aligned}\tau^B &= \mathrm{vec}(q_{B/D}) - K_\omega * \omega_{B/D}^B - (K_\omega K_q) * q_{B/D} + \omega_{B/I}^B \\ &\quad \times(\widehat{I^B} * \omega_{B/I}^B) + \widehat{I^B} * (q_{B/D}^*\dot{\omega}_{D/I}^D q_{B/D}) \\ &\quad + \widehat{I^B} * (\omega_{D/I}^B \times \omega_{B/D}^B) - (\widehat{I^B}K_q) * \frac{\mathrm{d}}{\mathrm{d}t}(q_{B/D})\end{aligned} \tag{34}$$

where $\widehat{I^B}$ is an estimate of the inertia matrix of the rigid body. This law is identical to the adaptive attitude(-only) tracking law proposed in [26].

*Remark 4:* It can be easily shown that the model-dependent version of the control law given by Eq. (21), where the estimates of the dual inertia matrix and dual disturbance force are replaced by its true values, that is,

$$\begin{aligned}\hat{f}_c^B &= -M^B\star\hat{a}_g^B - \frac{3\mu\hat{r}_{B/I}^B}{\|\hat{r}_{B/I}^B\|^5}\times(M^B\star(\hat{r}_{B/I}^B)^s) - M^B\star\hat{a}_{J_2}^B - \hat{f}_d^B \\ &\quad - \mathrm{vec}(\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s)) - K_d\star\hat{s}^s + \hat{\omega}_{B/I}^B\times(M^B\star(\hat{\omega}_{B/I}^B)^s) \\ &\quad + M^B\star(\hat{q}_{B/D}^*\dot{\hat{\omega}}_{D/I}^D\hat{q}_{B/D})^s + M^B\star(\hat{\omega}_{D/I}^B\times\hat{\omega}_{B/D}^B)^s \\ &\quad - M^B\star\left(K_p\star\frac{\mathrm{d}}{\mathrm{d}t}(\hat{q}_{B/D}^*(\hat{q}_{B/D}^s - \hat{1}^s))\right)^s\end{aligned} \tag{35}$$

still guarantees that, for all initial conditions, $\lim_{t\to\infty}\hat{q}_{B/D} = \pm\hat{1}$ and $\lim_{t\to\infty}\hat{\omega}_{B/D}^B = \hat{\mathbf{0}}$.

## Sufficient Conditions for Mass and Inertia Matrix Identification

In this section, sufficient conditions on the reference pose (i.e., the reference position and attitude) are given that guarantee that the estimate of the dual inertia matrix will converge to the true dual inertia matrix. Note, however, that the result presented in Theorem 1 does not depend on the convergence of this estimate. In other words, the controller proposed in Theorem 1 guarantees almost global asymptotical stability of the linear and angular position and velocity tracking errors even without estimate convergence. Nevertheless, identification of the mass and inertia matrix of the satellite might be important, for example, for fuel consumption estimation, for calculation of reentry trajectories and terminal velocities, for state estimation, for fault-detecting-and-isolation systems, and for docking/undocking scenarios.

*Proposition 1:* Let the dual disturbance force be exactly known or estimated so that $\hat{\hat{f}}{}_d^B$ can be replaced by $\hat{f}_d^B$ in Eq. (21). Moreover, assume that $\hat{q}_{D/I}$, $\hat{\omega}_{D/I}^D$, $\dot{\hat{\omega}}_{D/I}^D$, $\ddot{\hat{\omega}}_{D/I}^D \in \mathcal{L}_\infty$, $\hat{r}_{B/I}^B \neq \hat{\mathbf{0}}$, and $\hat{q}_{D/I}$ is periodic. Furthermore, let $W$: $[0,\infty) \to \mathbb{R}^{8\times 7}$ be defined as

$$\begin{aligned}W(t)v(\Delta M^B) &= \hat{\omega}_{D/I}^D(t)\times(\Delta M^B\star(\hat{\omega}_{D/I}^D(t))^s) + \Delta M^B\star(\dot{\hat{\omega}}_{D/I}^D(t))^s \\ &\quad - \Delta M^B\star\hat{a}_g^D - \frac{3\mu\hat{r}_{D/I}^D}{\|\hat{r}_{D/I}^D\|^5}\times(\Delta M^B\star(\hat{r}_{D/I}^D)^s) - \Delta M^B\star\hat{a}_{J_2}^D\end{aligned} \tag{36}$$

or, equivalently, $W(t) = W_{rb}(t) + W_g(t) + W_{\nabla g}(t) + W_{J_2}(t)$, where

$$W_{rb}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dot{u}+qw-rv \\ 0 & 0 & 0 & 0 & 0 & 0 & \dot{v}-pw+ru \\ 0 & 0 & 0 & 0 & 0 & 0 & \dot{w}+pv-qu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dot{p} & \dot{q}-pr & \dot{r}+pq & -qr & q^2-r^2 & qr & 0 \\ pr & \dot{p}+qr & -p^2+r^2 & \dot{q} & \dot{r}-pq & -pr & 0 \\ -pq & p^2-q^2 & \dot{p}-qr & pq & \dot{q}+pr & \dot{r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{37}$$

$$W_g(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mu x}{\|\bar{r}^D_{D/I}\|^3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mu y}{\|\bar{r}^D_{D/I}\|^3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\mu z}{\|\bar{r}^D_{D/I}\|^3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad W_{J_2}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{38}$$

$$W_{\nabla g}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3\mu xz}{\|\bar{r}^D_{D/I}\|^5} & -\frac{3\mu xy}{\|\bar{r}^D_{D/I}\|^5} & \frac{3\mu yz}{\|\bar{r}^D_{D/I}\|^5} & -\frac{3\mu(y^2-z^2)}{\|\bar{r}^D_{D/I}\|^5} & -\frac{3\mu yz}{\|\bar{r}^D_{D/I}\|^5} & 0 \\ -\frac{3\mu xz}{\|\bar{r}^D_{D/I}\|^5} & -\frac{3\mu yz}{\|\bar{r}^D_{D/I}\|^5} & -\frac{3\mu(z^2-x^2)}{\|\bar{r}^D_{D/I}\|^5} & 0 & \frac{3\mu xy}{\|\bar{r}^D_{D/I}\|^5} & \frac{3\mu xz}{\|\bar{r}^D_{D/I}\|^5} & 0 \\ \frac{3\mu xy}{\|\bar{r}^D_{D/I}\|^5} & -\frac{3\mu(x^2-y^2)}{\|\bar{r}^D_{D/I}\|^5} & \frac{3\mu yz}{\|\bar{r}^D_{D/I}\|^5} & -\frac{3\mu xy}{\|\bar{r}^D_{D/I}\|^5} & -\frac{3\mu xz}{\|\bar{r}^D_{D/I}\|^5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{39}$$

where $\bar{v}^D_{D/I} = [u \quad v \quad w]^\top$, $\bar{\omega}^D_{D/I} = [p \quad q \quad r]^\top$, $\bar{r}^D_{D/I} = [x \quad y \quad z]^\top$, and $\bar{a}^D_{J_2} = [a_1 \quad a_2 \quad a_3]^\top$. Let also $0 \le t_1 \le t_2 \le \ldots \le t_n$ be such that

$$\operatorname{rank}\begin{bmatrix} W(t_1) \\ \vdots \\ W(t_n) \end{bmatrix} = 7 \tag{40}$$

Then, under the control law given by Eq. (21), $\lim_{t\to\infty} \widehat{M^B} = M^B$.

*Proof.* The proof starts by proving that $\lim_{t\to\infty}\dot{\hat{\omega}}^B_{B/D} = \hat{\mathbf{0}}$. (Note that $\lim_{t\to\infty}\hat{\omega}^B_{B/D} = \hat{\mathbf{0}}$ does not, in general, imply that $\lim_{t\to\infty}\dot{\hat{\omega}}^B_{B/D} = \hat{\mathbf{0}}$.) Note that

$$\lim_{t\to\infty} \int_0^t \dot{\hat{\omega}}^B_{B/D}(\tau)\,\mathrm{d}\tau = \lim_{t\to\infty}\hat{\omega}^B_{B/D}(t) - \hat{\omega}^B_{B/D}(0) = -\hat{\omega}^B_{B/D}(0)$$

exists and is finite. Furthermore, since $\hat{q}_{D/I}$, $\hat{\omega}^D_{D/I}$, $\dot{\hat{\omega}}^D_{D/I}$, $\ddot{\hat{\omega}}^D_{D/I}$, $\dot{\hat{q}}_{B/D}$, $\ddot{\hat{q}}_{B/D}$, $\dot{\hat{\omega}}^B_{B/D}$,

$$\frac{\mathrm{d}v(\widehat{M^B})}{\mathrm{d}t}, \qquad \frac{\mathrm{d}\widehat{f^B_d}}{\mathrm{d}t} \in \mathcal{L}_\infty$$

and $\hat{r}^B_{B/I} \neq \hat{\mathbf{0}}$, it follows that $\ddot{\hat{\omega}}^B_{B/D} \in \mathcal{L}_\infty$ by differentiating Eq. (17). Hence, by Barbalat's lemma, $\lim_{t\to\infty}\dot{\hat{\omega}}^B_{B/D} = \hat{\mathbf{0}}$. Now, calculate the limit as $t \to \infty$ of both sides of Eq. (17). Next, substitute the dual control force by Eq. (21) and replace $\widehat{\hat{f}^B_d}$ by $\hat{f}^B_d$ in Eq. (21) (note that $\hat{f}^B_d$ is assumed to be known). Finally, using the fact that, according to Theorem 1, $\lim_{t\to\infty}\hat{\omega}^B_{B/D} = \hat{\mathbf{0}}$ and $\lim_{t\to\infty}\hat{q}_{B/D} = \pm\hat{\mathbf{1}}$ (in other words, in the limit, the body frame and the desired frame have the same origin and orientation) yields

$$\lim_{t\to\infty}\Big(\hat{\omega}^D_{D/I} \times (\Delta M^B \star (\hat{\omega}^D_{D/I})^s) + \Delta M^B \star (\dot{\hat{\omega}}^D_{D/I})^s - \Delta M^B \star \hat{a}^D_g - \frac{3\mu\hat{r}^D_{D/I}}{\|\hat{r}^D_{D/I}\|^5} \times (\Delta M^B \star (\hat{r}^D_{D/I})^s) - \Delta M^B \star \hat{a}^D_{J_2}\Big) = \hat{\mathbf{0}} \tag{41}$$

Moreover, note that if $\hat{q}_{D/I}$ is periodic with period $T$, so are $\dot{\hat{q}}_{D/I}$, $\hat{\omega}^D_{D/I}$, $\dot{\hat{\omega}}^D_{D/I}$, $\hat{r}^D_{D/I}$, $\hat{a}^D_g$, $\hat{a}^D_{J_2}$, and $W(t)$. Finally, noting that

$$\lim_{t\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}v(\widehat{M^B}) = 0_{7\times 1}$$

from Eq. (25) and Theorem 1, under the conditions of Proposition 1, Eq. (41) implies that $\lim_{t\to\infty}v(\Delta M^B) = 0_{7\times 1}$ or, equivalently, $\lim_{t\to\infty}\widehat{M^B} = M^B$. □

*Remark 5:* In practice, the true dual disturbance force $\hat{f}^B_d$ is never known. Moreover, there is no guarantee that the estimate of the dual disturbance force will converge to its true value. Hence, in practice, the estimate of the mass and inertia matrix of the spacecraft will only be as good as the estimate of the dual disturbance force.

*Remark 6:* An alternative, and more general, sufficient condition than Eq. (40) for dual inertia matrix identification, which does not require $\hat{q}_{D/I}$ to be periodic, is that the $7 \times 7$ matrix

$$\int_t^{t+T_2} W^\top(t)W(t)\,\mathrm{d}t$$

is positive definite for all $t \ge T_1$ for some $T_1 \ge 0$ and $T_2 > 0$ [36,37].

## Simulation Results

In this section, two examples are considered. In the first example, the controller proposed in this paper is applied to a conceivable satellite proximity operations scenario, where a chaser satellite approaches, circumnavigates, and docks with a target satellite. In the second example, the mass and inertia matrix of a satellite are identified using the controller.

### Satellite Proximity Operations

In this example, the versatility of the controller is demonstrated by using it, in sequence, to approach, circumnavigate, and dock with a target satellite, while always pointing at it.

Four reference frames are defined: the inertial frame, the target frame, the desired frame, and the body frame. The inertial frame is the Earth-centered-inertial (ECI) frame. The body frame is some frame fixed to the chaser satellite and centered at its center of mass. The target frame and the desired frame are defined as

$$\bar{I}_T = \frac{\bar{r}_{T/I}}{\|\bar{r}_{T/I}\|}, \qquad \bar{J}_T = \bar{K}_T \times \bar{I}_T, \qquad \bar{K}_T = \frac{\bar{\omega}_{T/I}}{\|\bar{\omega}_{T/I}\|}$$

and

$$\bar{I}_D = \frac{\bar{r}_{D/T}}{\|\bar{r}_{D/T}\|}, \qquad \bar{J}_D = \bar{K}_D \times \bar{I}_D, \qquad \bar{K}_D \| \bar{K}_T$$

respectively, where

$$\bar{\omega}_{T/I} = \frac{\bar{r}_{T/I} \times \bar{v}_{T/I}}{\|\bar{r}_{T/I}\|^2}$$

is calculated from the orbital angular momentum of the target spacecraft with respect to the inertial frame given by $\bar{h}_{T/I} = m\|\bar{r}_{T/I}\|^2\bar{\omega}_{T/I} = \bar{r}_{T/I} \times m\bar{v}_{T/I}$. The target satellite is assumed to be fixed to the target frame. The objective of the control law is to superimpose the body frame to the desired frame. The relationship between the different frames is illustrated in Fig. 2.

The target spacecraft is assumed to be in a highly eccentric Molniya orbit with orbital elements given in Table 1 and nadir pointing. The relative motion of the desired frame with respect to the target frame is divided into the following three phases:

1) Phase 1 is straight line approach along $\bar{J}_T$ from −30 to −20 m at a constant speed of 0.025 m/s. In other words, during this phase, $\bar{\omega}^T_{D/T} = [0, 0, 0]^\top$ rad/s and $\bar{v}^T_{D/T} = [0, 0.025, 0]^\top$ m/s, with initial condition $\bar{r}^T_{D/T} = [0, -30, 0]^\top$ m.

2) Phase 2 is a circular circumnavigation around the target satellite with a radius of 20 m in the $\bar{J}_T - \bar{K}_T$ plane (so that the chaser satellite does not cross the nadir direction of the target satellite) and with constant angular speed equal to the mean motion of the

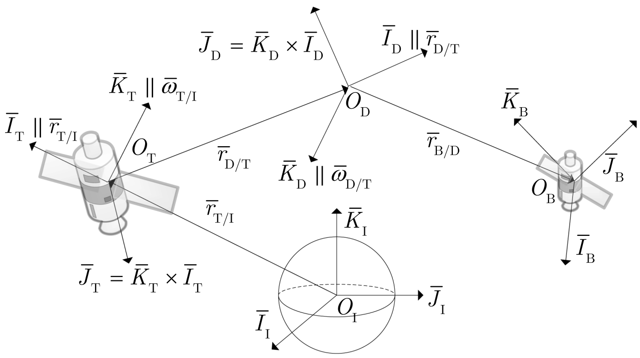

Fig. 2 Reference frames.

target satellite. In other words, during this phase, $\bar{\omega}^T_{D/T} = [-n, 0, 0]^\top$ rad/s, $\bar{v}^T_{D/T} = [0, -a_e n \sin(nt), b_e n \cos(nt)]^\top$ m/s, and $a_e = b_e = 20$ m, where $n = \sqrt{\mu/a^3}$ is the mean motion of the target satellite (assuming no perturbation due to Earth's oblateness) and $a$ is the semimajor axis of the target satellite (assuming no perturbation due to Earth's oblateness).

3) Phase 3 is a straight-line docking along $\bar{J}_T$ from $-20$ m to contact at a constant speed of 0.025 m/s. In other words, during this phase, $\bar{\omega}^T_{D/T} = [0, 0, 0]^\top$ rad/s and $\bar{v}^T_{D/T} = [0, 0.025, 0]^\top$ m/s.

The linear velocity of the target satellite with respect to the inertial frame is calculated by numerically integrating the gravitational acceleration and also the perturbing acceleration due to Earth's oblateness acting on the target satellite. On the other hand, the angular acceleration of the target satellite with respect to the inertial frame is calculated analytically through

$$\alpha^I_{T/I} = \dot{\omega}^I_{T/I} = \frac{(r^I_{T/I} \times a^I_{T/I})\|r^I_{T/I}\|^2 - (r^I_{T/I} \times v^I_{T/I})2(r^I_{T/I} \cdot v^I_{T/I})}{\|r^I_{T/I}\|^4} \tag{42}$$

Note that the perturbation due to Earth's oblateness changes the direction of the target's angular velocity with respect to the inertial frame. However, this change is relatively small in this scenario due to the critical inclination of the Molniya orbit. The rotational and translational kinematic equations of the target frame with respect to the inertial frame and of the desired frame with respect to the target frame are written in terms of dual quaternions as in Eqs. (14) and (16), respectively.

The control law given by Eq. (21) is a function of $\hat{\omega}^D_{D/I}$ and $\dot{\hat{\omega}}^D_{D/I}$. These variables are calculated in terms of dual quaternions as follows:

$$\hat{\omega}^D_{D/I} = \hat{\omega}^D_{T/I} + \hat{\omega}^D_{D/T} = \hat{q}^*_{D/I}\hat{\omega}^I_{T/I}\hat{q}_{D/I} + \hat{q}^*_{D/T}\hat{\omega}^T_{D/T}\hat{q}_{D/T} \tag{43}$$

$$\dot{\hat{\omega}}^D_{D/I} = \hat{q}^*_{D/I}\hat{\alpha}^I_{T/I}\hat{q}_{D/I} - \hat{\omega}^D_{D/I} \times \hat{\omega}^D_{T/I} + \hat{q}^*_{D/T}\hat{\alpha}^T_{D/T}\hat{q}_{D/T} \tag{44}$$

where $\hat{\alpha}^T_{D/T} = \dot{\hat{\omega}}^T_{D/T} = \alpha^T_{D/T} + \epsilon(a^T_{D/T} - \alpha^T_{D/T} \times r^T_{D/T} - \omega^T_{D/T} \times v^T_{D/T})$ and $\hat{\alpha}^I_{T/I} = \dot{\hat{\omega}}^I_{T/I} = \alpha^I_{T/I} + \epsilon(a^I_{T/I} - \alpha^I_{T/I} \times r^I_{T/I} - \omega^I_{T/I} \times v^I_{T/I})$. Equation (44) is calculated by differentiating Eq. (43) and using the dual quaternion counterpart of the classical transport theorem [17]. Note that, instead of calculating $\hat{\omega}^D_{D/I}$ and $\dot{\hat{\omega}}^D_{D/I}$ in terms of dual quaternions, one could instead calculate $\omega^D_{D/I}$, $\dot{\omega}^D_{D/I}$, $v^D_{D/I}$, and $\dot{v}^D_{D/I}$ using the traditional equations for a point moving with respect to a rotating rigid body. However, this would require the calculation of four parameters instead of just two and significant more work to calculate $v^D_{D/I}$ and $\dot{v}^D_{D/I}$, whose expressions are coupled with the rotational motion. Thus, Eqs. (43) and (44) are another good example of the benefits in terms of compactness and simplicity of using dual quaternions.

Table 1 Orbital elements of target satellite

| | Molniya orbit | GEO |
|---|---|---|
| Perigee altitude, km | 813.2 | 35,786 |
| Eccentricity, – | 0.7 | 0 |
| Inclination, deg | 63.4 | 0 |
| Argument of perigee, deg | 270 | 0 |
| RAAN, deg | 329.6 | 0 |
| True anomaly, deg | 180 | 0 |

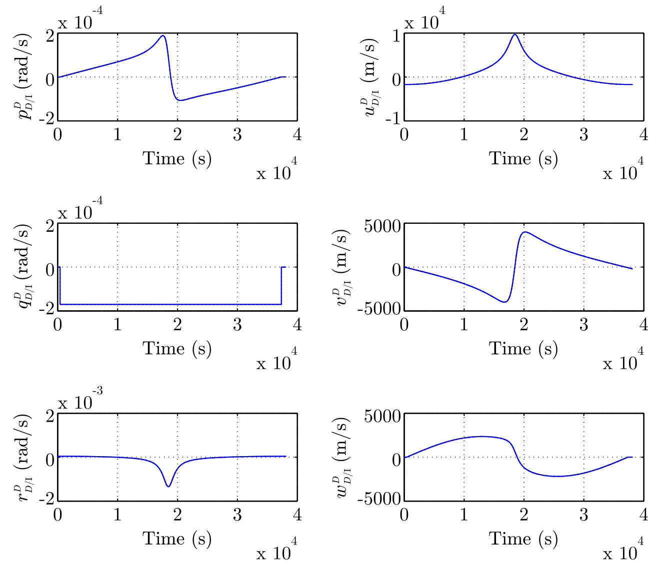

Fig. 3 Reference motion.

The inertia matrix and mass of the chaser satellite are assumed to be [1]

$$\bar{I}^B = \begin{bmatrix} 22 & 0.2 & 0.5 \\ 0.2 & 20 & 0.4 \\ 0.5 & 0.4 & 23 \end{bmatrix} \text{ kg} \cdot \text{m}^2$$

and $m = 100$ kg, respectively. The constant disturbance force and torque acting on the chaser satellite are set to $\bar{f}^B_d = [0.005, 0.005, 0.005]^\top$ N and $\bar{\tau}^B_d = [0.005, 0.005, 0.005]^\top$ N $\cdot$ m, respectively. The origin of the body frame (coincident with the center of mass of the chaser satellite) is positioned relatively to the origin of the desired frame at $\bar{r}^B_{B/D} = [2, 2, 2]^\top$ m. The initial error quaternion, relative linear velocity, and relative angular velocity of the body frame with respect to the desired frame are set to $q_{B/D} = [q_{B/D1} \;\; q_{B/D2} \;\; q_{B/D3} \;\; q_{B/D4}]^\top = [0.4618, 0.1917, 0.7999, 0.3320]^\top$, $\bar{v}^B_{B/D} = [u^B_{B/D} \;\; v^B_{B/D} \;\; w^B_{B/D}]^\top = [0.1, 0.1, 0.1]^\top$ m/s, and $\bar{\omega}^B_{B/D} = [p^B_{B/D} \;\; q^B_{B/D} \;\; r^B_{B/D}]^\top = [0.1, 0.1, 0.1]^\top$ rad/s, respectively.

The initial estimates for the mass, inertia matrix, and dual disturbance force are set to zero, whereas the control gains are chosen to be $\bar{K}_r = 0.05I_3$, $\bar{K}_q = 0.25I_3$, $\bar{K}_v = 15I_3$, $\bar{K}_w = 15I_3$, $K_i = \mathrm{diag}([K_{I_{11}}, K_{I_{12}}, K_{I_{13}}, K_{I_{22}}, K_{I_{23}}, K_{I_{33}}, K_m])$, $K_{I_{11}}$, $K_{I_{12}}$, $K_{I_{13}}$, $K_{I_{22}}$, $K_{I_{23}} = K_{I_{33}} = 100$, $K_m = 1$, $\bar{K}_f = 0.8I_3$, and $\bar{K}_\tau = 0.8I_3$.

Figure 3 shows the linear and angular velocity of the desired frame with respect to the inertial frame expressed in the desired frame for the complete maneuver. These signals form the reference for the controller.

Figure 4 shows the initial transient response and the transient response between phases 1 and 2 of the position and attitude of the body frame with respect to the desired frame using the controller given by Eq. (21) (adaptive) and the controller given by Eq. (35) (nonadaptive). Note that the transition between phases 1 and 2 occurs at 400 s. The transient response between phases 2 and 3 is similar and, thus, is not shown here. Both controllers successfully cancel the relative position and attitude errors at the beginning of the maneuver and between phases. These latter are due to the fact that $\bar{\omega}^T_{D/T}$ and $\bar{v}^T_{D/T}$ are discontinuous between phases. In other words, between

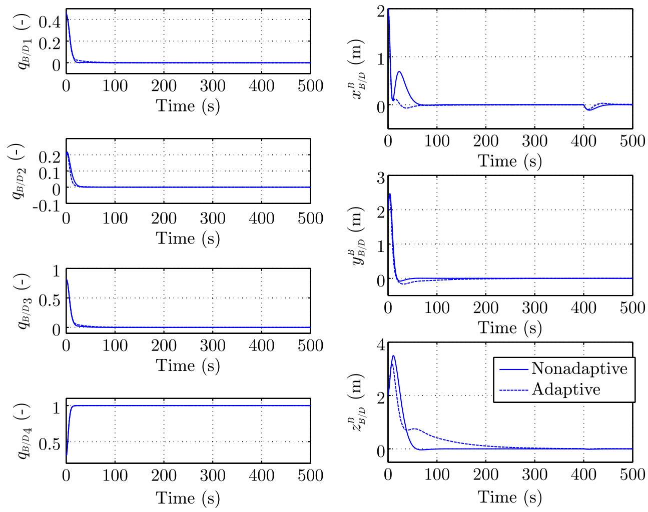


**Fig. 4 Relative attitude and position expressed in the body frame.**

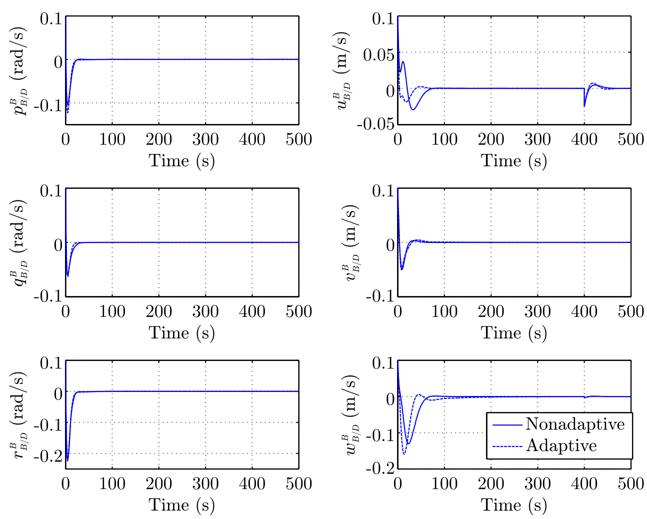


**Fig. 5 Relative linear and angular velocity expressed in the body frame.**

phases $\hat{\omega}^D_{D/I} \notin \mathcal{L}_\infty$, which instantaneously violates the conditions of Theorem 1.

Figure 5 shows the relative linear and angular velocity of the body frame with respect to the desired frame for the same two cases studied in Fig. 4. Again, both controllers successfully cancel the relative linear and angular velocity errors at the beginning of the maneuver and between phases.

Figure 6 shows that, although the adaptive controller does not converge to the true mass and inertia matrix of the chaser satellite for this reference motion, nevertheless, it is still able to track the reference motion. As a matter of fact, the similarities between the responses obtained with the adaptive controller (which has no information about the true mass, inertia matrix, and dual disturbance force) and the nonadaptive controller (which knows the true mass, inertia matrix, and dual disturbance force) are quite remarkable. For this reference motion, the minimum singular value of the matrix in Eq. (40) for $t_1 = 0, t_2 \approx 3.5e^{-2}, \ldots, t_{25,131} \approx 3.8e^4$ s is calculated to be $1.3e^{-6}$ .

Figure 7 shows that, for this reference motion, even though the adaptive controller is unable to exactly identify the true dual disturbance force, it converges to values of the same order of magnitude. Note that Theorem 1 only guarantees that these estimates will be uniformly bounded. Relatively small oscillations in the estimates can be seen between phases as a result of the discontinuities in $\hat{\omega}^D_{D/I}$.

For completeness, Fig. 8 shows the control force $\bar{f}^B_c = [f^B_{c1} \quad f^B_{c2} \quad f^B_{c3}]^\top$, and the control torque $\bar{\tau}^B_c = [\tau^B_{c1} \quad \tau^B_{c2} \quad \tau^B_{c3}]^\top$, produced by the adaptive and nonadaptive controllers during the initial transient response and between phases 1 and 2. The relatively high values of control force and torque during the initial transient response are required to eliminate the initial relative position, attitude, and linear and angular velocity errors that were arbitrarily set between the body frame and the desired frame.

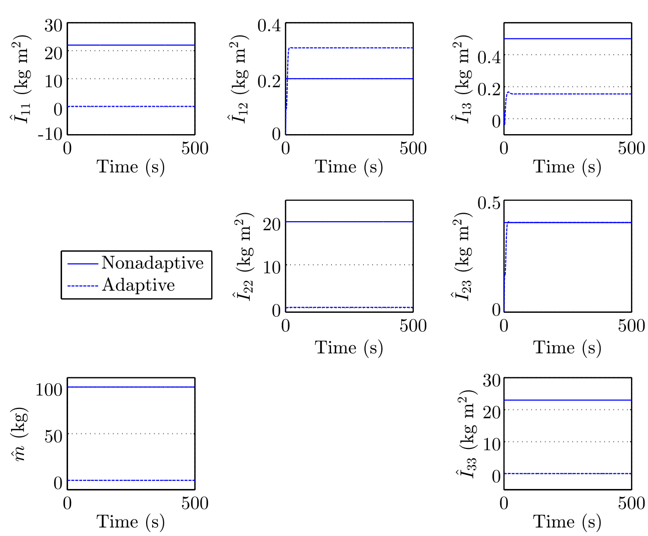


**Fig. 6 Mass and inertia matrix estimation for low-exciting reference motion.**

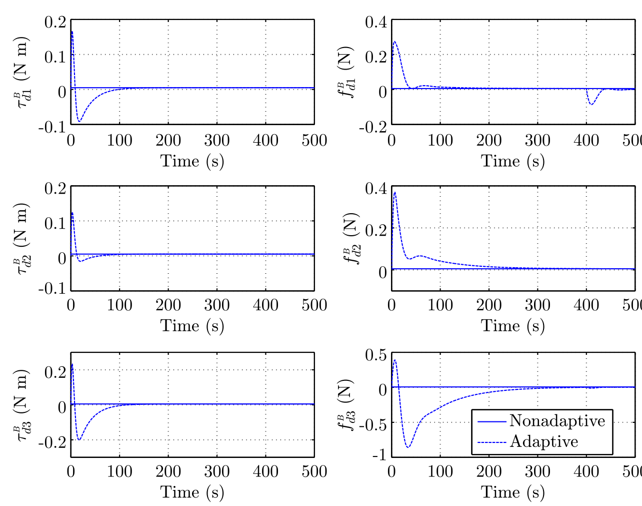


**Fig. 7 Dual disturbance force estimation.**

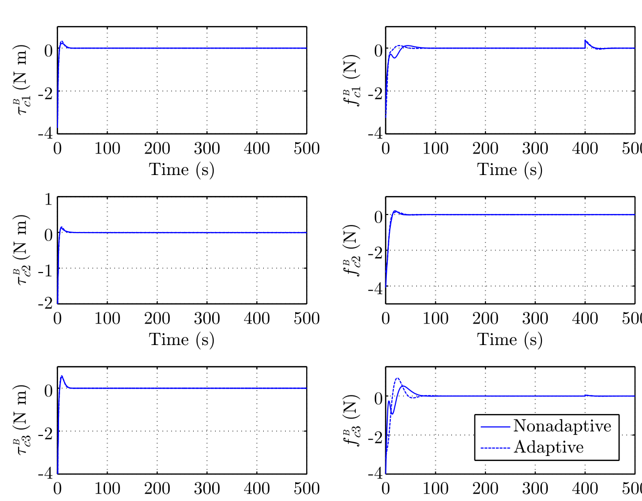


**Fig. 8 Control force and torque.**

**Mass and Inertia Matrix Identification**

In this example, the adaptive control law is used to identify the mass and inertia matrix of a satellite in GEO with orbital elements given in Table 1. In this scenario, the target frame is the unperturbed Hill frame [3] of the satellite. Note that, in this case, there is not a physical spacecraft attached to the target frame. The desired frame is defined to have the same position and orientation as the target frame at the beginning of the simulation. The inertial frame and the body frame are defined as in the previous example.

The satellite has the same mass and inertia matrix as the chaser satellite in the previous example. As assumed in Proposition 1, the dual disturbance force is assumed to be known and, in this example, equal to zero. The body frame is assumed to have the same position, attitude, linear velocity, and angular velocity as the desired frame at the beginning of the simulation. The initial estimates for the mass and inertia matrix are set to zero. With the exception of $K_m = 100$, the control gains are the same as in the previous example.

The relative motion of the desired frame with respect to the target frame is defined in Fig. 9. It is composed by a pure translation and several pure rotations designed to identify the mass and the elements of the inertia matrix in sequence, while keeping the control forces and torques within reasonable values. This reference motion was created by taking into consideration the matrix $W(t)$ and the results presented in [26]. For this reference motion, the minimum singular value of the

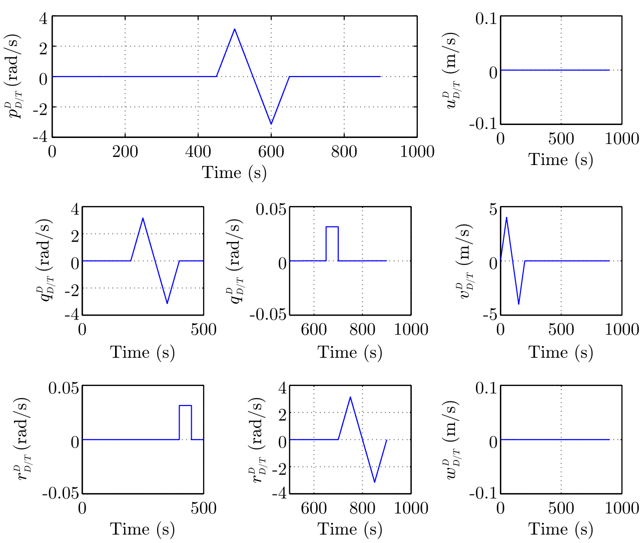


**Fig. 9 Reference motion for identification.**

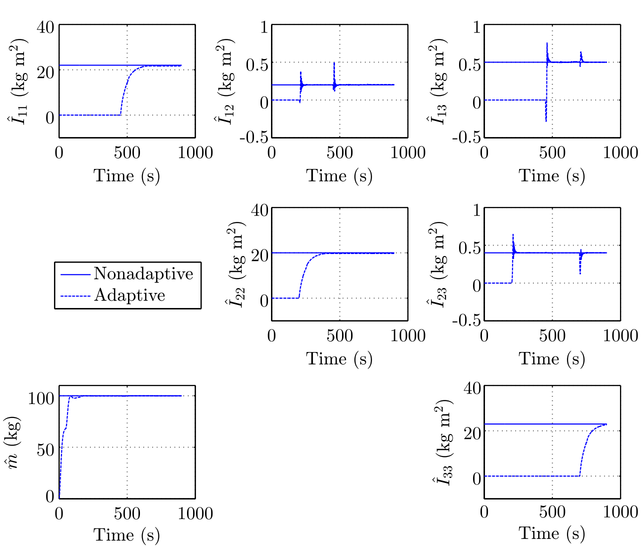


**Fig. 10 Mass and inertia matrix identification.**

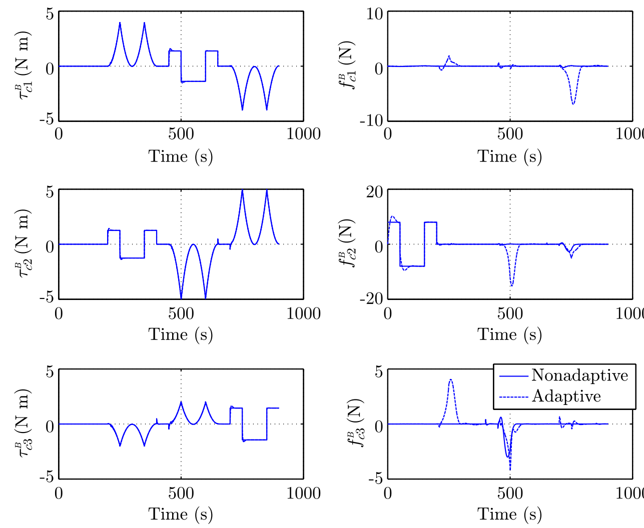


**Fig. 11 Control force and torque during identification.**

matrix in Eq. (40) for $t_1 = 0$, $t_2 \approx 1.0e^{-5}, \ldots, t_{14,014} = 900$ s is calculated to be 1.15.

The mass and inertia matrix identification is shown in Fig. 10. Note that the mass and inertia matrix are identified even though their initial estimates are zero. They are identified in sequence: $m$ is identified during the first triangle waveform (on $v^D_{D/T}$); $I_{12}$, $I_{22}$, and $I_{23}$ are identified during the second triangle waveform (on $q^D_{D/T}$); $I_{11}$ and $I_{13}$ are identified during the third triangle waveform (on $p^D_{D/T}$); and $I_{33}$ is identified during the fourth and last triangle waveform (on $r^D_{D/T}$). The associated control forces and torques are shown in Fig. 11.

## Conclusions

An adaptive tracking controller for satellite proximity operations is presented in this paper. The controller requires no information about the mass and inertia matrix of the chaser satellite and takes into account the gravitational acceleration, the perturbing acceleration due to Earth's oblateness, and the gravity-gradient torque. The controller is shown to ensure almost global asymptotical stability of the linear and angular position and velocity tracking errors, even in the presence of constant unknown disturbance forces and torques. Hence, the controller can handle large error angles and displacements. Sufficient conditions for mass and inertia matrix identification are also given. Because this controller is based on the relative nonlinear equations of motion, it can be used to asymptotically track time-varying relative position and attitude profiles with respect to, for example, tumbling target satellites in elliptical orbits or small asteroids. Moreover, because the controller requires no information about the mass and inertia matrix of the chaser satellite, it can be used when little or no information about the mass and/or inertia matrix of the chaser satellite is available. The relatively low order of the controller makes it suitable for satellites with limited onboard computational resources. One of the key contributions of this paper is that it demonstrates how dual quaternions can be used to extend existing attitude-only controllers based on quaternions having certain desirable properties (e.g., stability, adaptivity, boundedness) to position and attitude controllers having similar properties.

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