HW2

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1 Problem 1

1.1 Subproblem 1

Proof.
Firstly we can write $\hat{\sigma}_a^2$ and $\hat{\sigma}_b^2$ as following:

$$\begin{bmatrix} \hat{\sigma}_a^2 \\ \hat{\sigma}_b^2 \end{bmatrix} = \frac{1}{nq}\sum_{k=1}^{n} \begin{bmatrix} \sum_{j=1}^{q}(\varepsilon^2_{q(k-1)+j}) \\ (\sum_{j=1}^{q}\varepsilon_{q(k-1)+j})^2 \end{bmatrix}$$

Since $\{\varepsilon_{qk}\}$ is an iid sequence, $\{\sum_{j=1}^{q}(\varepsilon^2_{q(k-1)+j})\}$,$\{(\sum_{j=1}^{q}\varepsilon_{q(k-1)+j})^2\}$ are also iid sequences. Which enables us to use central limit theorem to obtain the asymptotic distribution.
The expectation is easy to obtain:

$$\mathbb{E}\begin{bmatrix} \sum_{j=1}^{q}(\varepsilon^2_{q(k-1)+j}) \\ (\sum_{j=1}^{q}\varepsilon_{q(k-1)+j})^2 \end{bmatrix} = \begin{bmatrix} q\sigma^2 \\ q\sigma^2 \end{bmatrix}$$

The variance can be obtained as below:

$$\begin{aligned} &Var(\sum_{j=1}^{q}(\varepsilon^2_{q(k-1)+j})) \\ =&\sum_{j=1}^{q}(Var(\varepsilon^2_{q(k-1)+j})) \\ =&q(A-1)\sigma^4 \end{aligned}$$

(Here I use A to represent $\mathbb{E}x^4$, x is subject to standard normal distribution, in fact $A=3$;)

$$\begin{aligned} &Var((\sum_{j=1}^{q}\varepsilon_{q(k-1)+j})^2) \\ =&qA\sigma^4 + 6*C_q^2\sigma^4 - q^2*\sigma^4 \\ =&(Aq+2q^2-3q)\sigma^4 \end{aligned}$$

and

$$\begin{aligned} &Cov((\sum_{j=1}^{q}\varepsilon_{q(k-1)+j})^2, \sum_{j=1}^{q}(\varepsilon^2_{q(k-1)+j})) \\ =&Var(\sum_{j=1}^{q}(\varepsilon^2_{q(k-1)+j})) + Cov(\sum_{i=1}^{q}\sum_{j\neq i,j=1}^{q}(\varepsilon_{q(k-1)+i}\varepsilon_{q(k-1)+j}), \sum_{j=1}^{q}(\varepsilon^2_{q(k-1)+j})) \\ =&q(A-1)\sigma^4 \end{aligned}$$

Finally we get:

$$Var\left(\left[\begin{array}{c}\sum_{j=1}^{q}(\varepsilon_{q(k-1)+j}^2)\\(\sum_{j=1}^{q}\varepsilon_{q(k-1)+j})^2\end{array}\right]\right)=\left[\begin{array}{cc}2q\sigma^4 & 2q\sigma^4\\2q\sigma^4 & 2q^2\sigma^4\end{array}\right]$$

Therefore, using CLT,

$$\sqrt{n}\left(\left[\begin{array}{c}\hat{\sigma}_a^2\\\hat{\sigma}_b^2\end{array}\right]-\left[\begin{array}{c}\sigma^2\\\sigma^2\end{array}\right]\right)\rightarrow_d \mathbb{N}\left(0,\left[\begin{array}{cc}\frac{2\sigma^4}{q} & \frac{2\sigma^4}{q}\\\frac{2\sigma^4}{q} & 2\sigma^4\end{array}\right]\right)$$

With the Delta method, representing $J_r=\frac{\hat{\sigma}_b^2}{\hat{\sigma}_a^2}-1$ *by* $J_r=h(\hat{\sigma}_a^2,\hat{\sigma}_b^2)$*,we have:*

$$\sqrt{n}(h(\hat{\beta})-h(\beta))\rightarrow_d \mathbb{N}\left(0,\nabla h^T(\beta)D\nabla h(\beta)\right.$$

where $\hat{\beta}=(\hat{\sigma}_a^2,\hat{\sigma}_b^2)^T$*,* $\beta=\vec{0}$*, and* $D=\left[\begin{array}{cc}\frac{2\sigma^4}{q} & \frac{2\sigma^4}{q}\\\frac{2\sigma^4}{q} & 2\sigma^4\end{array}\right]$,

Namely, the distribution of J_r *is,*

$$\sqrt{n}\,(J_r-0)\rightarrow_d \mathbb{N}\left(0,\left[\begin{array}{cc}-\frac{1}{\sigma^2} & \frac{1}{\sigma^2}\end{array}\right]\left[\begin{array}{cc}\frac{A-\sigma^4}{2} & \frac{A-\sigma^4}{2}\\\frac{A-\sigma^4}{2} & \frac{A+\sigma^4}{2}\end{array}\right]\left[\begin{array}{c}-\frac{1}{\sigma^2}\\\frac{1}{\sigma^2}\end{array}\right]\right)=\mathbb{N}(0,1)$$