HW1

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1 Problem 1

1.1 Subproblem 1

The density function of r_t is:

$$f(r_t) = \frac{e^{-\frac{(r_t-\mu_t)^2}{2\sigma_t^2}}}{\sqrt{2\pi}\sigma_t}$$

1.2 Subproblem 2

The relation ship between R_t and r_t is:

$$R_t = \exp{(r_t)} - 1$$

1.3 Subproblem 3

The expectation of R_t is:

$$\begin{aligned}
\mathbb{E}(R_t) &= \mathbb{E}[\exp(r_t) - 1] \\
&= \int_{-\infty}^{+\infty} (\exp(r_t) - 1) f(r_t) dr_t \\
&= e^{\mu_t + \frac{\sigma_t^2}{2}} - 1
\end{aligned}$$

1.4 Subproblem 4

The variance of R_t is:

$$\begin{aligned}
Var(R_t) &= \mathbb{E}(R_t^2) - (\mathbb{E}R_t)^2 \\
&= \int_{-\infty}^{+\infty} (\exp(r_t) - 1)^2 f(r_t) dr_t - (e^{\mu_t + \frac{\sigma_t^2}{2}} - 1)^2 \\
&= -2e^{\mu_t + \frac{\sigma_t^2}{2}} + e^{2\left(\mu_t + \sigma_t^2\right)} + 1 - (-2e^{\mu_t + \frac{\sigma_t^2}{2}} + e^{2\mu_t + \sigma_t^2} + 1) \\
&= e^{2\left(\mu_t + \sigma_t^2\right)} - e^{2\mu_t + \sigma_t^2}
\end{aligned}$$

1.5 Subproblem 5

Now using h_Y to express the density of Y. Since function g is strictly increasing and continuously differentiable. If $a < X < b$, then $g(a) < Y < g(b)$, their probability should be the same. Namely we have:

$$\int_a^b f_X(x)dx = \int_{g(a)}^{g(b)} h_Y(y)dy$$

Recall that $y = g(x)$, then $dy = g'(x)dx$. So:

$$\begin{aligned}\int_a^b f_X(x)dx &= \int_{g(a)}^{g(b)} h_Y(y)dy \\ &= \int_a^b h_Y(g(x))g'(x)dx\end{aligned}$$

Since this relationship should be valid for all $-\infty < a < b < +\infty$, we have:

$$f_X(x) = h_Y(g(x))g'(x)$$

Which gives that:

$$h_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}$$

1.6 Subproblem 6

Using the relationship of Subproblem 5, we have:

$$\begin{aligned}h(R_t) &= \frac{f(\ln{(R_t+1)})}{\exp(\ln{(R_t+1)})} \\ &= \frac{e^{-\frac{(\ln{(R_t+1)}-\mu_t)^2}{2\sigma_t^2}}}{(R_t+1)\sqrt{2\pi}\sigma_t}\end{aligned}$$

2 Problem 2

2.1 Subproblem 1

The expectation of $\hat{\mu}_n$ is:

$$\begin{aligned}\mathbb{E}(\hat{\mu}_n) &= \mathbb{E}(\sum_{i=1}^{n}\frac{X_i}{n}) \\ &= \sum_{i=1}^{n}\frac{\mu}{n} \\ &= \mu\end{aligned}$$

So $\hat{\mu}_n$ is an unbiased estimator.

2.2 Subproblem 2

The variance of $\hat{\mu}_n$ is:

$$\begin{aligned} Var(\hat{\mu}_n) &= Var(\sum_{i=1}^{n} \frac{X_i}{n}) \\ &= \sum_{i=1}^{n} \frac{\sigma^2}{n^2} \\ &= \frac{\sigma^2}{n} \end{aligned}$$

If $n \to \infty$, $Var(\hat{\mu}_n) \to 0$.

2.3 Subproblem 3

The definition of convergence in probability is:

Definition (Convergence in probability)
For a random variable series $\{X_n\}$*, if* $\forall \epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(|X - X_n| \geq \epsilon) = 0$.

2.4 Subproblem 4

Proof.

$$\begin{aligned} &Y_n \to_{L^2} Y \\ \Leftrightarrow &\lim_{n\to\infty} \mathbb{E}(Y_n - Y)^2 = 0 \end{aligned}$$

Therefore:

$$\begin{aligned} 0 &< \lim_{n\to\infty} Var(Y_n - Y) \\ &= \lim_{n\to\infty} \mathbb{E}(Y_n - Y)^2 - \lim_{n\to\infty} (\mathbb{E}(Y_n - Y))^2 \\ &< \lim_{n\to\infty} \mathbb{E}(Y_n - Y)^2 \\ &= 0 \end{aligned}$$

So:

$$\begin{aligned} \lim_{n\to\infty} Var(Y_n - Y) = 0 \\ \lim_{n\to\infty} (\mathbb{E}(Y_n - Y)) = 0 \end{aligned}$$

According to the Chebyshev inequality:

$$\forall \epsilon > 0,$$
$$\lim_{n\to\infty} \mathbb{P}(|Y_n - Y - 0| \geq \epsilon) \leq \frac{\lim_{n\to\infty} Var(Y_n - Y)}{\epsilon^2}$$
$$= 0$$

Since probability should be not minus,

$$\forall \epsilon > 0, \lim_{n\to\infty} \mathbb{P}(|Y_n - Y| \geq \epsilon) = 0$$

Namely, $Y_n \to_p Y$.
Q.E.D

2.5 Subproblem 5

Proof.
We have:

$$\lim_{n\to\infty} \mathbb{E}(\hat{\mu}_n) = \mu$$
$$\lim_{n\to\infty} Var(\hat{\mu}_n) = 0$$

According to the Chebyshev inequality:

$$\forall \epsilon > 0,$$
$$\lim_{n\to\infty} \mathbb{P}(|\hat{\mu}_n - \mu| \geq \epsilon) \leq \frac{\lim_{n\to\infty} Var(\hat{\mu}_n)}{\epsilon^2}$$
$$= 0$$

Since probability should be not minus,

$$\forall \epsilon > 0, \lim_{n\to\infty} \mathbb{P}(|\hat{\mu}_n - \mu| \geq \epsilon) = 0$$

Namely, $\hat{\mu}_n \to_p \mu$.
Q.E.D

This means that, when n becomes very very very large, then the probability of that the value of $\hat{\mu}_n$ largely differs from μ becomes very very very small, so we can confidently say that $m\hat{u}_n$ is close to μ.

3 Problem 3

3.1 Subproblem 1

Output The output of given code is:

```
[2,5,8,11]
```

Explanation Variable ”a” is a python list contains integers from 1 to 13, and the python function ”print()” will print the input parameter to the console/terminal/jupyter cell output/...

The input parameter, ”a[1:12:3]”, is indeed a python list, whose members is taken from ”a”. The way of taking is based on the index, ”[1:12:3]”. It means python will take members from ”a[1]” to ”a[12]”, the 2nd to the 13th member of ”a”, with a step of 3.

To sum up, ”a[1:12:3]” is a list containing ”a[1],a[4],a[7],a[10]”, namely ”2,5,8,11”. Last but not least, when you use ”print()” to print a python list, the output have ”[]” on the sides, and members are separated with ”,”.

3.2 Subproblem 2

output The output of given code is:

```
0
```

Explanation Since variable ”luck” is set to be 0, then the condition of the if sentence is true. So python will only set ”ret” to be -2. Since the value of ”total” has not been changed, when you print ”total” you will get 0 as output.