

HW1

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1 Problem 1

1.1 Subproblem 1

The density function of r_t is:

$$f(r_t) = \frac{e^{-\frac{(r_t - \mu_t)^2}{2\sigma_t^2}}}{\sqrt{2\pi}\sigma_t}$$

1.2 Subproblem 2

The relation ship between R_t and r_t is:

$$R_t = \exp(r_t) - 1$$

1.3 Subproblem 3

The expectation of R_t is:

$$\begin{aligned}\mathbb{E}(R_t) &= \mathbb{E}[\exp(r_t) - 1] \\ &= \int_{-\infty}^{+\infty} (\exp(r_t) - 1)f(r_t)dr_t \\ &= e^{\mu_t + \frac{\sigma_t^2}{2}} - 1\end{aligned}$$

1.4 Subproblem 4

The variance of R_t is:

$$\begin{aligned}Var(R_t) &= \mathbb{E}(R_t^2) - (\mathbb{E}R_t)^2 \\ &= \int_{-\infty}^{+\infty} (\exp(r_t) - 1)^2 f(r_t)dr_t - (e^{\mu_t + \frac{\sigma_t^2}{2}} - 1)^2 \\ &= -2e^{\mu_t + \frac{\sigma_t^2}{2}} + e^{2(\mu_t + \sigma_t^2)} + 1 - (-2e^{\mu_t + \frac{\sigma_t^2}{2}} + e^{2\mu_t + \sigma_t^2} + 1) \\ &= e^{2(\mu_t + \sigma_t^2)} - e^{2\mu_t + \sigma_t^2}\end{aligned}$$

1.5 Subproblem 5

Now using h_Y to express the density of Y . Since function g is strictly increasing and continuously differentiable. If $a < X < b$, then $g(a) < Y < g(b)$, their probability should be the same. Namely we have:

$$\int_a^b f_X(x)dx = \int_{g(a)}^{g(b)} h_Y(y)dy$$

Recall that $y = g(x)$, then $dy = g'(x)dx$. So:

$$\begin{aligned}\int_a^b f_X(x)dx &= \int_{g(a)}^{g(b)} h_Y(y)dy \\ &= \int_a^b h_Y(g(x))g'(x)dx\end{aligned}$$

Since this relationship should be valid for all $-\infty < a < b < +\infty$, we have:

$$f_X(x) = h_Y(g(x))g'(x)$$

Which gives that:

$$h_Y(y) = \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}$$

1.6 Subproblem 6

Using the relationship of Subproblem 5, we have:

$$\begin{aligned}h(R_t) &= \frac{f(\ln(R_t + 1))}{\exp(\ln(R_t + 1))} \\ &= \frac{e^{-\frac{(\ln(R_t + 1) - \mu_t)^2}{2\sigma_t^2}}}{(R_t + 1)\sqrt{2\pi}\sigma_t}\end{aligned}$$

2 Problem 2

2.1 Subproblem 1

The expectation of $\hat{\mu}_n$ is:

$$\begin{aligned}\mathbb{E}(\hat{\mu}_n) &= \mathbb{E}\left(\sum_{i=1}^n \frac{X_i}{n}\right) \\ &= \sum_{i=1}^n \frac{\mu}{n} \\ &= \mu\end{aligned}$$

So $\hat{\mu}_n$ is an unbiased estimator.

2.2 Subproblem 2

The variance of $\hat{\mu}_n$ is:

$$\begin{aligned}Var(\hat{\mu}_n) &= Var\left(\sum_{i=1}^n \frac{X_i}{n}\right) \\ &= \sum_{i=1}^n \frac{\sigma^2}{n^2} \\ &= \frac{\sigma^2}{n}\end{aligned}$$

If $n \rightarrow \infty$, $Var(\hat{\mu}_n) \rightarrow 0$.

2.3 Subproblem 3

The definition of convergence in probability is:

Definition (Convergence in probability)

For a random variable series $\{X_n\}$, if $\forall \epsilon > 0$, $\lim_{n \rightarrow \infty} \mathbb{P}(|X - X_n| \geq \epsilon) = 0$.

2.4 Subproblem 4

Proof.

$$\begin{aligned} & Y_n \rightarrow_{L^2} Y \\ \Leftrightarrow & \lim_{n \rightarrow \infty} \mathbb{E}(Y_n - Y)^2 = 0 \end{aligned}$$

Therefore:

$$\begin{aligned} 0 & < \lim_{n \rightarrow \infty} \text{Var}(Y_n - Y) \\ & = \lim_{n \rightarrow \infty} \mathbb{E}(Y_n - Y)^2 - \lim_{n \rightarrow \infty} (\mathbb{E}(Y_n - Y))^2 \\ & < \lim_{n \rightarrow \infty} \mathbb{E}(Y_n - Y)^2 \\ & = 0 \end{aligned}$$

So:

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{Var}(Y_n - Y) & = 0 \\ \lim_{n \rightarrow \infty} (\mathbb{E}(Y_n - Y)) & = 0 \end{aligned}$$

According to the Chebyshev inequality:

$$\begin{aligned} & \forall \epsilon > 0, \\ \lim_{n \rightarrow \infty} \mathbb{P}(|Y_n - Y - 0| \geq \epsilon) & \leq \frac{\lim_{n \rightarrow \infty} \text{Var}(Y_n - Y)}{\epsilon^2} \\ & = 0 \end{aligned}$$

Since probability should be not minus,

$$\forall \epsilon > 0, \lim_{n \rightarrow \infty} \mathbb{P}(|Y_n - Y| \geq \epsilon) = 0$$

Namely, $Y_n \rightarrow_p Y$.

Q.E.D

2.5 Subproblem 5

Proof.

We have:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{E}(\hat{\mu}_n) & = \mu \\ \lim_{n \rightarrow \infty} \text{Var}(\hat{\mu}_n) & = 0 \end{aligned}$$

According to the Chebyshev inequality:

$$\begin{aligned} & \forall \epsilon > 0, \\ \lim_{n \rightarrow \infty} \mathbb{P}(|\hat{\mu}_n - \mu| \geq \epsilon) & \leq \frac{\lim_{n \rightarrow \infty} \text{Var}(\hat{\mu}_n)}{\epsilon^2} \\ & = 0 \end{aligned}$$

Since probability should be not minus,

$$\forall \epsilon > 0, \lim_{n \rightarrow \infty} \mathbb{P}(|\hat{\mu}_n - \mu| \geq \epsilon) = 0$$

Namely, $\hat{\mu}_n \rightarrow_p \mu$.

Q.E.D

This means that, when n becomes very very very large, then the probability of that the value of $\hat{\mu}_n$ largely differs from μ becomes very very very small, so we can confidently say that $\hat{\mu}_n$ is close to μ .

3 Problem 3

3.1 Subproblem 1

Output The output of given code is:

```
1 [2, 5, 8, 11]
```

Explanation Variable "a" is a python list contains integers from 1 to 13, and the python function "print()" will print the input parameter to the console/terminal/jupyter cell output/...

The input parameter, "a[1:12:3]", is indeed a python list, whose members is taken from "a". The way of taking is based on the index, "[1:12:3]". It means python will take members from "a[1]" to "a[12]", the 2nd to the 13th member of "a", with a step of 3.

To sum up, "a[1:12:3]" is a list containing "a[1], a[4], a[7], a[10]", namely "2, 5, 8, 11". Last but not least, when you use "print()" to print a python list, the output have "[]" on the sides, and members are separated with ",".

3.2 Subproblem 2

output The output of given code is:

```
1 0
```

Explanation Since variable "luck" is set to be 0, then the condition of the if sentence is true. So python will only set "ret" to be -2. Since the value of "total" has not been changed, when you print "total" you will get 0 as output.