Time Series and their Applications

Homework Exercise 3

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Date: May 17, 2024

Question 1

A

First we determine the mean function μ_Y of $\{Y_t\}$,

$$\mu_Y(t) = \mathbb{E}[Y_t] = \mathbb{E}[X_t] + \mathbb{E}[W_t] = \mathbb{E}[X_t] \tag{1}$$

because $\{W_t\}$ has mean zero. Since $\{X_t\}$ is an AR(1) process, $X_t = \phi X_{t-1} + Z_t$, we have

$$\mathbb{E}[X_t] = \phi\mathbb{E}[X_{t-1}] + \mathbb{E}[Z_t] = \phi\mathbb{E}[X_{t-1}]$$

Since the AR(1) process $\{X_t\}$ is stationary, denote $\mu_X = \mathbb{E}[X_t] = \mathbb{E}[X_{t-1}]$. Then

$$\mu_X = \phi\mu_X$$

for all $t \in \mathbb{Z}$. Since $\phi \neq 0$ and $|\phi|< 1$, we conclude $\mathbb{E}[X_t] = 0$ for all $t \in \mathbb{Z}$. Hence, with Equation 1 becomes clear that $\mu_Y = 0$.

Next we determine the autocovariance function γ_Y of $\{Y_t\}$,

$$\mathbb{C}\text{ov}\left(Y_{t+h}, Y_t\right) = \mathbb{C}\text{ov}\left(X_{t+h}, X_t\right) + \mathbb{C}\text{ov}\left(X_{t+h}, W_t\right) + \mathbb{C}\text{ov}\left(W_{t+h}, X_t\right) + \mathbb{C}\text{ov}\left(W_{t+h}, W_t\right). \tag{2}$$

We determine each of the four elements seperately. First,

$$\begin{aligned}
\mathbb{C}\text{ov}\left(X_{t+h}, X_t\right) &= \mathbb{E}\left[X_{t+h}X_t\right] - \mathbb{E}[X_{t+h}]\mathbb{E}[X_t] \\
&= \mathbb{E}\left[\left(\phi X_{t+h-1} + Z_{t+h}\right)X_t\right] \\
&= \phi\mathbb{E}\left[X_{t+h-1}X_t\right] + \mathbb{E}\left[Z_{t+h}X_t\right] \\
&= \phi\mathbb{C}\text{ov}\left(X_{t+h-1}, X_t\right).
\end{aligned}$$

where $\mathbb{E}\left[Z_{t+h}X_t\right] = 0$ for $h \geq 1$. From which we deduce the relation for $h \geq 1$,

$$\gamma_X(t+h, t) = \phi\gamma_X(t+h-1, t) = \cdots = \phi^h\gamma_X(t, t) \tag{3}$$

Furthermore, we know

$$\gamma_X(t, t) = \mathbb{V}\text{ar}\left(X_t\right) = \phi^2\mathbb{V}\text{ar}\left(X_{t-1}\right) + \sigma_z^2$$

because of the independence of Z_{t-1} and X_t. Since the AR(1) process $\{X_t\}$ is stationary, denote $\sigma_x^2 = \mathbb{V}\text{ar}\left(X_t\right) = \mathbb{V}\text{ar}\left(X_{t-1}\right)$. This gives $\sigma_x^2 = \phi^2\sigma_x^2 + \sigma_z^2$ and hence

$$\gamma_X(t, t) = \sigma_x^2 = \frac{\sigma_z^2}{1 - \phi^2}$$

where dividing by ϕ^2 is possible, because $\phi \neq 0$. From this we obtain for $h \geq 1$

$$\gamma_X(t+h, t) = \phi^h\gamma_X(t, t) = \phi^h\frac{\sigma_z^2}{1 - \phi^2}.$$

For $h = -1$, notice that

$$\gamma_X(t-1, t) = \mathbb{C}\text{ov}\left(X_{t-1}, X_t\right) = \mathbb{C}\text{ov}\left(X_{t-1}, \phi X_{t-1} + Z_t\right) = \phi\mathbb{C}\text{ov}\left(X_{t-1}, X_{t-1}\right) = \phi\gamma_X(t, t).$$

With the same kind of reasoning as for $h \geq 1$ follows that $\gamma_X(t+h, t) = \phi^{|h|}\gamma_X(t, t)$ for $h \leq -1$. Therefore we conclude together with Equation 3, the first term of Equation 2 is

$$\mathbb{C}\text{ov}\left(X_{t+h}, X_t\right) = \gamma_X(h) = \phi^{|h|}\frac{\sigma_z^2}{1 - \phi^2}.$$

The second term equals

$$\mathbb{C}\text{ov}\left(X_{t+h}, W_t\right) = \mathbb{E}\left[X_{t+h}W_t\right] - \mathbb{E}\left[X_{t+h}\right]\mathbb{E}\left[W_t\right] = \mathbb{E}\left[X_{t+h}W_t\right] = 0$$

for all h. Applying the same reasoning yields for the third term

$$\mathbb{C}\text{ov}\left(W_{t+h}, X_t\right) = \mathbb{E}\left[W_{t+h}X_t\right] - \mathbb{E}\left[W_{t+h}\right]\mathbb{E}\left[X_t\right] = 0.$$

For the fourth term, it holds $\mathbb{C}\text{ov}\left(W_{t+h}, W_t\right) = \sigma_w^2$ for $h = 0$ and zero otherwise. Taking all the terms together gives,

$$\gamma_Y(t+h,t) = \begin{cases} \sigma_z^2 \frac{1}{1-\phi^2} + \sigma_w^2 & \text{if} \quad h = 0, \\ \phi^{|h|}\sigma_z^2 \frac{1}{1-\phi^2} & \text{otherwise} \end{cases}$$

which is time-independent. Since $\{Y_t\}$ also has mean zero independent of time, the process is weakly stationary.

The process $\{Y_t\}$ is strictly stationary too, because we know from the first lecture notes that the AR(1) process $\{X_t\}$ is strictly stationary for $|\phi|< 1$. Since $W_t \sim \text{WN}(0, \sigma_w^2)$ we conclude

$$\begin{aligned} \mathbb{P}(Y_{t_1} < y_1, \ldots, Y_{t_n} < y_n) &= \mathbb{P}(X_{t_1} + W_{t_1} < y_1, \ldots, X_{t_n} + W_{t_n} < y_n) \\ &= \mathbb{P}(X_{t_1+h} + W_{t_1+h} < y_1, \ldots, X_{t_n+h} + W_{t_n+h} < y_n) \\ &= \mathbb{P}(Y_{t_1+h} < y_1, \ldots, Y_{t_n+h} < y_n) \end{aligned}$$

for each $h \in \mathbb{Z}$ and hence the process $\{Y_t\}$ is strictly stationary.

B

Define γ_U as the autocovariance function of $U_t = Y_t - \phi Y_{t-1}$. Then

$$\begin{aligned} \gamma_U(t+h,t) &= \mathbb{C}\text{ov}\left(Y_{t+h} - \phi Y_{t+h-1}, Y_t - \phi Y_{t-1}\right) \\ &= \mathbb{C}\text{ov}\left(Y_{t+h}, Y_t\right) - \phi\mathbb{C}\text{ov}\left(Y_{t+h}, Y_{t-1}\right) - \phi\mathbb{C}\text{ov}\left(Y_{t+h-1}, Y_t\right) + \phi^2\mathbb{C}\text{ov}\left(Y_{t+h-1}, Y_{t-1}\right) \\ &= \gamma_Y(h) - \phi\gamma_Y(h+1) - \phi\gamma_Y(h-1) + \phi^2\gamma_Y(h) \\ &= (1+\phi^2)\gamma_Y(h) - \phi(\gamma_Y(h+1) + \gamma_Y(h-1)). \end{aligned}$$

We divide the answer in three different cases: $h = 0$, $|h|= 1$ and $|h|> 1$. For $h = 0$, we have

$$\begin{aligned} \gamma_U(t+0,t) &= (1+\phi^2)\gamma_Y(0) - \phi(\gamma_Y(1) + \gamma_Y(-1)) \\ &= (1+\phi^2)\left(\frac{\sigma_z^2}{1-\phi^2} + \sigma_w^2\right) - 2\phi^2\frac{\sigma_z^2}{1-\phi^2} \\ &= \frac{\sigma_z^2}{1-\phi^2}(1-\phi^2) + (1+\phi^2)\sigma_w^2 = \sigma_z^2 + (1+\phi^2)\sigma_w^2 \end{aligned}$$

which is larger than zero if we assume that $\sigma_z^2 \vee \sigma_w^2 \neq 0$.

For $|h|= 1$, we have that either $\gamma_Y(h+1)$ or $\gamma_Y(h-1)$ equals $\frac{\phi^2\sigma_z^2}{1-\phi^2}$. As a result, the autocovariance function γ_U is

$$\begin{aligned} (1+\phi^2)\frac{\phi\sigma_z^2}{1-\phi^2} - \phi\frac{\phi^2\sigma_z^2}{1-\phi^2} &= \\ \frac{\sigma_z^2}{1-\phi^2}\left(\phi + \phi^3 - \phi^3\right) &= \frac{\phi\sigma_z^2}{1-\phi^2} \end{aligned}$$

which is not zero if $\sigma_z^2 \neq 0$.

For $|h|> 1$, consider first $h > 1$. Then $|h|= |h-1|+1$ and $|h+1|+1 = |h|+2$. On the other hand, for $h < -1$, we have $|h|= |h+1|+1$ and $|h-1|+1 = |h|+2$. From this we conclude

$$\begin{aligned}\gamma_U(t+h,t) &= (1+\phi^2)\gamma_Y(h) - \phi(\gamma_Y(h+1)+\gamma_Y(h-1)) \\ &= \frac{\sigma_z^2}{1-\phi^2}\left(\phi^{|h|} - \phi^{|h+1|+1} - \phi^{|h-1|+1} + \phi^{|h|+2}\right) = 0\end{aligned}$$

for all $|h|> 1$. Therefore we conclude that the process $\{U_t\}$ is 1-correlated.

C

From the q-correlation definition, we know with **B** that $\{Y_t - \phi Y_{t-1}\}$ can be represented as an MA(1) process. Hence it can be written as

$$Y_t - \phi Y_{t-1} = \mu + \xi_t + \theta\xi_{t-1}$$

and therefore also as an ARMA(1,1) process

$$Y_t = \phi Y_{t-1} + \mu + \xi_t + \theta\xi_{t-1}$$

where $\xi_t \sim \mathrm{WN}(0, \sigma_\xi^2)$ and the parameters μ, θ and σ_ξ^2 are unknown. We assume $\phi = 0.5$, $\sigma_w^2 = \sigma_z^2 = 1$ and find these three parameters.

First notice that

$$\mathbb{E}[Y_t] = 0.5\mathbb{E}[Y_{t-1}] + \mu.$$

We know from **A** that $\{Y_t\}$ is stationary and $\mu_Y = \mathbb{E}[Y_t] = \mathbb{E}[Y_{t-1}] = 0$. Hence μ is zero too.

Next we compare the autocovariance functions of $U_t = \xi_t + \theta\xi_{t-1}$ and $U_t = Y_t - 0.5Y_{t-1}$ for $h = 0$ and $h = 1$ to come up with a system of two equations and two unknowns θ, σ_ξ^2. For $h = 0$ we have on one hand,

$$\mathbb{V}\mathrm{ar}\left(\xi_t + \theta\xi_{t-1}\right) = \mathbb{V}\mathrm{ar}\left(\xi_t\right) + \theta^2\mathbb{V}\mathrm{ar}\left(\xi_{t-1}\right) = (1+\theta^2)\sigma_\xi^2$$

and on the other hand with **B**

$$\mathbb{V}\mathrm{ar}\left(Y_t - 0.5Y_{t-1}\right) = \sigma_z^2 + (1+\phi^2)\sigma_w^2 = 2.25.$$

For $h = 1$, we have on one hand

$$\gamma_U(1) = \mathbb{C}\mathrm{ov}\left(\xi_{t+1} + \theta\xi_t, \xi_t + \theta\xi_{t-1}\right) = \theta\sigma_\xi^2$$

and on the other hand with **B**

$$\gamma_U(1) = \frac{\phi\sigma_z^2}{1-\phi^2} = \frac{2}{3}.$$

Therefore we have the system of equations

$$\begin{aligned}(1+\theta^2)\sigma_\xi^2 &= 2.25 \\ \theta\sigma_\xi^2 &= \frac{2}{3}\end{aligned}$$

with invertible solution (i.e. $|\theta|> 1$), $\theta = 3.05$ and $\sigma_\xi^2 = 0.22$.

Question 2

A

Multiplying the equation $X_t = \phi X_{t-2} + Z_t + \theta Z_{t-1}$ with X_{t-h} for $h = 0, 1, 2$ yields

$$\begin{aligned} X_t^2 &= \phi X_t X_{t-2} + X_t Z_t + \theta X_t Z_{t-1} \\ X_t X_{t-1} &= \phi X_{t-1} X_{t-2} + X_{t-1} Z_t + \theta X_{t-1} Z_{t-1} \\ X_t X_{t-2} &= \phi X_{t-2}^2 + X_{t-2} Z_t + \theta X_{t-2} Z_{t-1}. \end{aligned}$$

So

$$\begin{aligned} \gamma_X(0) &= \phi \mathbb{E}[X_t X_{t-2}] + \mathbb{E}[X_t Z_t] + \theta \mathbb{E}[X_t Z_{t-1}] \\ \gamma_X(1) &= \phi \mathbb{E}[X_{t-1} X_{t-2}] + \mathbb{E}[X_{t-1} Z_t] + \theta \mathbb{E}[X_{t-1} Z_{t-1}] \\ \gamma_X(2) &= \phi \mathbb{E}[X_{t-2}^2] + \mathbb{E}[X_{t-2} Z_t] + \theta \mathbb{E}[X_{t-2} Z_{t-1}]. \end{aligned}$$

because $\mathbb{E}[X_t] = 0$, which can be deduced from $\mathbb{E}[X_t] = \phi \mathbb{E}[X_{t-2}]$ and $\phi \neq 0$.

Notice that $\mathbb{E}[X_t Z_s] = 0$ for $t \neq s$ and for $t = s$, we have

$$\mathbb{E}[X_t Z_t] = \phi \mathbb{E}[X_{t-2} Z_t] + \mathbb{E}[Z_t^2] + \theta \mathbb{E}[Z_t Z_{t-1}] = \sigma^2$$

because $Z_t \sim \mathrm{IID}(0, \sigma^2)$. Therefore we can simplify the system of equations to

$$\begin{aligned} \gamma_X(0) &= \phi \gamma_X(2) + \sigma^2 \\ \gamma_X(1) &= \phi \gamma_X(1) + \theta \sigma^2 \\ \gamma_X(2) &= \phi \gamma_X(0). \end{aligned}$$

B

Solving the second equation yields

$$\gamma_X(1) = \frac{\theta \sigma^2}{1 - \phi}$$

and combining the first and third equations gives

$$\gamma_X(0) = \phi^2 \gamma_X(0) + \sigma^2.$$

Hence,

$$\gamma_X(0) = \frac{\sigma^2}{1 - \phi^2}$$

and

$$\gamma_X(2) = \frac{\phi \sigma^2}{1 - \phi^2}.$$

C

If we estimate φ, we use the notation $\Gamma_1 = \gamma(0)$ and $\boldsymbol{\gamma}_1(1) = \gamma(1)$. Then the estimator $\hat{\varphi}$ equals

$$\hat{\varphi} = \hat{\Gamma_1}^{-1} \hat{\gamma}_1(1) = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)}.$$

If we fit the estimator to the ARMA(2,1) process, we get

$$\hat{\varphi} = \frac{\theta\sigma^2}{1-\phi}\frac{1-\phi^2}{\sigma^2} = \frac{\theta(1-\phi^2)}{1-\phi}.$$

Since $Z_t \sim \text{IID}(0, \sigma^2)$, we know

$$\sqrt{T}\left(\hat{\varphi} - \varphi\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2\Gamma_1^{-1}\right).$$

Hence

$$\hat{\varphi} \xrightarrow{d} \mathcal{N}\left(\varphi, \frac{T\sigma^2}{\gamma(0)}\right).$$

3

A

The joint conditional density of $\boldsymbol{X}_T$ given Z_0 is

$$f_{\boldsymbol{X}_T|Z_0}(\boldsymbol{x}_T, z_0) = f_{X_T,\ldots,X_1|Z_0}(\boldsymbol{x}_T, z_0).$$

Using the decomposition $f_{X,Y|Z}(x,y|z) = f_{X|Y,Z}(x|y,z)f_{Y|Z}(y|z)$ with $X = X_T, \ldots, X_k$ and $Y = X_k$ for $k = 1, ..., T-1$ yields

$$\begin{aligned}
f_{X_T,\ldots,X_1|Z_0}(\boldsymbol{x}_T, z_0) &= f_{X_T,\ldots,X_2|X_1,Z_0}(x_T,\ldots,x_2|x_1,z_0)f_{X_1|Z_0}(x_1|z_0) \\
&= f_{X_T,\ldots,X_3|X_2,X_1,Z_0}(x_T,\ldots,x_3|x_2,x_1,z_0)f_{X_2|X_1,Z_0}(x_2|x_1,z_0)f_{X_1|Z_0}(x_1|z_0) \\
&= \vdots \\
&= f_{X_T|X_{T-1},\ldots,X_1,Z_0}(x_T|x_{T-1}\ldots,x_1,z_0)\cdots f_{X_2|X_1,Z_0}(x_2|x_1,z_0)f_{X_1|Z_0}(x_1|z_0).
\end{aligned}$$

From the lecture notes, we know this is

$$= f_{X_T|Z_{T-1}}(x_T|z_{T-1})\cdots f_{X_2|Z_1}(x_2|z_1)f_{X_1|Z_0}(x_1|z_0) = \prod_{t=1}^{T} f_{X_t|Z_{t-1}}(x_t|z_{t-1}).$$

B

Taking the derivative of $\ln\tilde{L}(\theta, \sigma^2)$ with respect to σ gives

$$\frac{\partial \ln\tilde{L}}{\partial\sigma}(\theta,\sigma^2) = \frac{-T}{\sigma} + \frac{1}{\sigma^3}\sum_{t=1}^{T} Z_t^2.$$

If we assume that $\sigma \neq 0$, this is well-defined and setting to zero gives the maximizer

$$\hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T} Z_t^2. \tag{4}$$

Notice this gives a local maximum, because the second order derivative is

$$\frac{\partial^2 \ln\tilde{L}}{\partial\sigma^2}(\theta,\sigma^2) = \frac{T}{\sigma^2} - \frac{3}{\sigma^4}\sum_{t=1}^{T} Z_t^2.$$

So for $\hat{\sigma}^2$ as in Equation 4, this is

$$\frac{\partial^2 \ln \tilde{L}}{\partial \sigma^2}(\theta, \hat{\sigma}^2) = \frac{T^2}{\sum_{t=1}^T Z_t^2} - \frac{3T^2}{\sum_{t=1}^T Z_t^2} = \frac{-2T^2}{\sum_{t=1}^T Z_t^2} < 0$$

because T^2 and Z_t^2 are larger or equal to zero for $t = 1, \dots, T$. Hence we conclude $\hat{\sigma}^2$ belongs to a local maximum, because the second order derivative for $\hat{\sigma}^2$ is smaller than zero.

The concentrated log-likelihood function is then given by

$$\begin{aligned} \ln \tilde{L}(\theta, \hat{\sigma}^2(\theta)) &= \frac{-T}{2} \ln 2\pi - \frac{T}{2} \ln \left\{ \frac{1}{T} \sum_{t=1}^T Z_t^2 \right\} - \frac{1}{\frac{2}{T} \sum_{t=1}^T Z_t^2} \sum_{t=1}^T Z_t^2 \\ &= \frac{-T}{2} \ln 2\pi - \frac{T}{2} \ln \left\{ \frac{1}{T} \sum_{t=1}^T Z_t^2 \right\} - \frac{T}{2} \\ &= C_T - \frac{T}{2} \ln \left\{ \frac{1}{T} \sum_{t=1}^T Z_t^2 \right\} \end{aligned}$$

where $C_T = \frac{-T}{2}(\ln 2\pi + 1)$ is a constant depending on T.

C

Using the Matlab code provided, we found that the true value of $\theta = 0.6498$ and the average estimate of $\theta = 0.7017$. Thus the true value is a bit lower than the average estimate of θ.

The sample variance equals 0.00103. The asymptotic variance is given by

$$\frac{1-\theta^2}{T} = 0.00116, \tag{5}$$

which is approximately the same as the sample variance.

Empirical Exercise

(1)

Please see the provided Matlab file.

(2)

We select the model based on the AIC. This results in the ARMA(3,2) model.
Judging from Figures 1 and 2, the theoretical ACFs and PACFs capture the features of the data sufficiently well, as they often lie quite close to the sample ACFs and PACFs.

(3)

The estimated coefficients and their standard errors are given in Table 1. Next, we use diagnostic checking to test the null hypothesis: $H_0 : \rho_Z(1) = ... = \rho_Z(m) = 0$.
We first compute the residuals and then apply the Ljung-Box test to these residuals. We have to choose a reasonable value of m, up to $\frac{T}{4}$ and we chose $m = 20$ since this is the default number of autocorrelations returned. This results in a p-value of 0.1446. Therefore, at the 5% level, we do not reject the null hypothesis. This is also reflected in Figure 3 in which we see that ACFs of the estimated residuals all lie within the $\pm\frac{1.96}{\sqrt{T}}$ bounds.

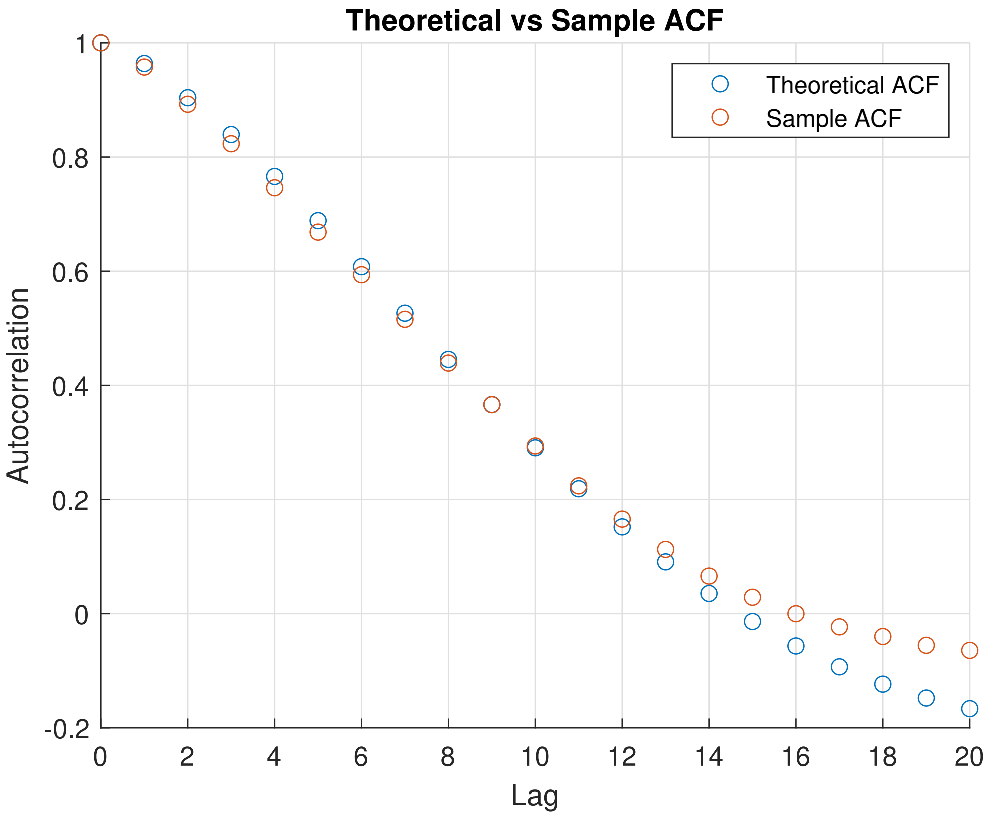

Figure 1: Theoretical vs Sample ACF

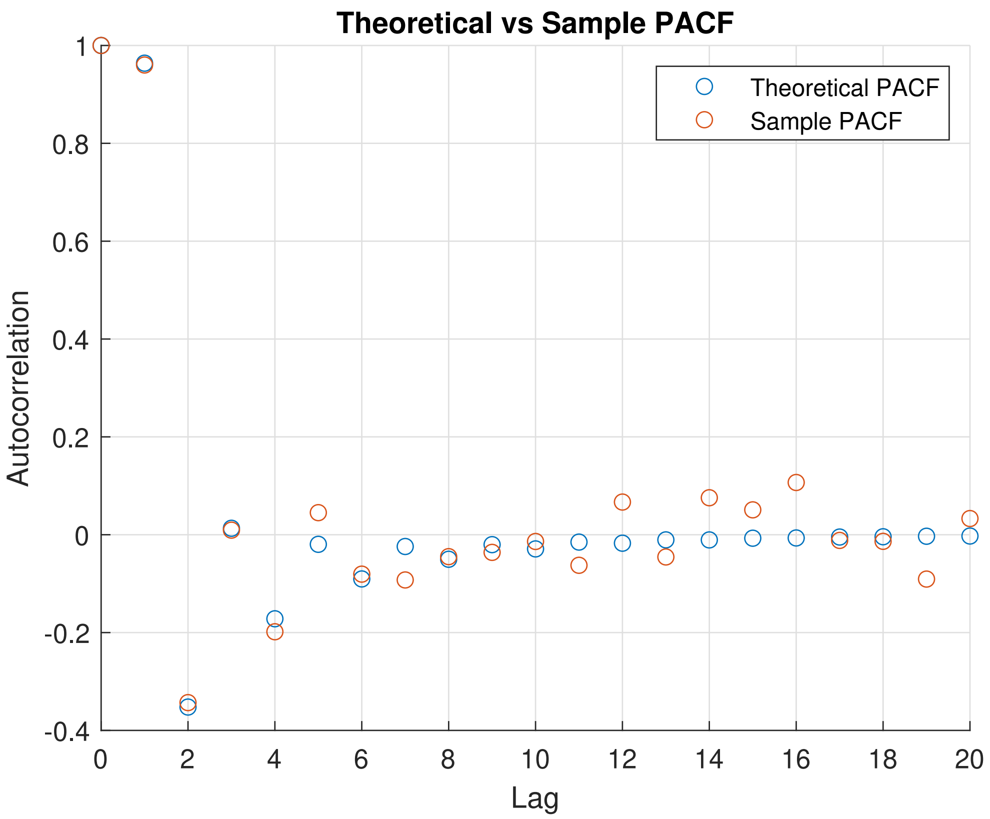

Figure 2: Theoretical vs Sample PACF

	Value	Standard Error
Constant	0.00011	0.00005
AR1	1.4588	0.17634
AR2	-0.13383	0.30018
AR3	-0.3502	0.13933
MA1	-0.15988	0.17084
MA2	-0.51553	0.1063
Variance	0.000004	0.0000005

Table 1: Estimated coefficients and Standard Errors of the ARMA(3,2) model

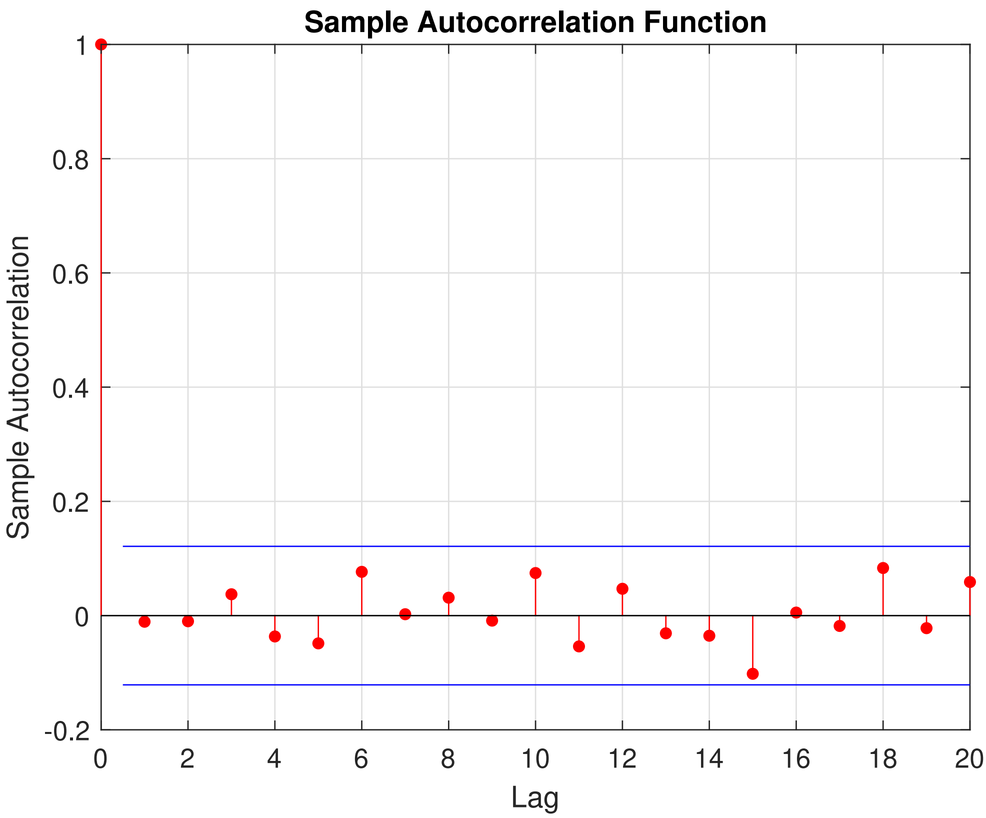

Figure 3: ACFs of the estimated residuals

(4)

Please see the provided Matlab file.

(5)

Comparing the predictions with the realized values using Table 2 and Figure 4, we see that the predictions are somewhat close to the realized values, at least closer than they were in assignment 2. However, there is still considerable room for improvement as, for example, the last prediction is quite far off compared to the realized value.

h	Prediction	Lower confidence band	Upper confidence band	Realization
1	0.0104	0.0101	0.0106	0.0099
2	0.0084	0.0080	0.0088	0.0097
3	0.0071	0.0066	0.0075	0.0065
4	0.0056	0.0051	0.0062	0.0040
5	0.0045	0.0038	0.0051	0.0030
6	0.0034	0.0027	0.0041	0.0025
7	0.0025	0.0017	0.0033	-0.0017

Table 2: Predictions, Confidence Bands and Realizations

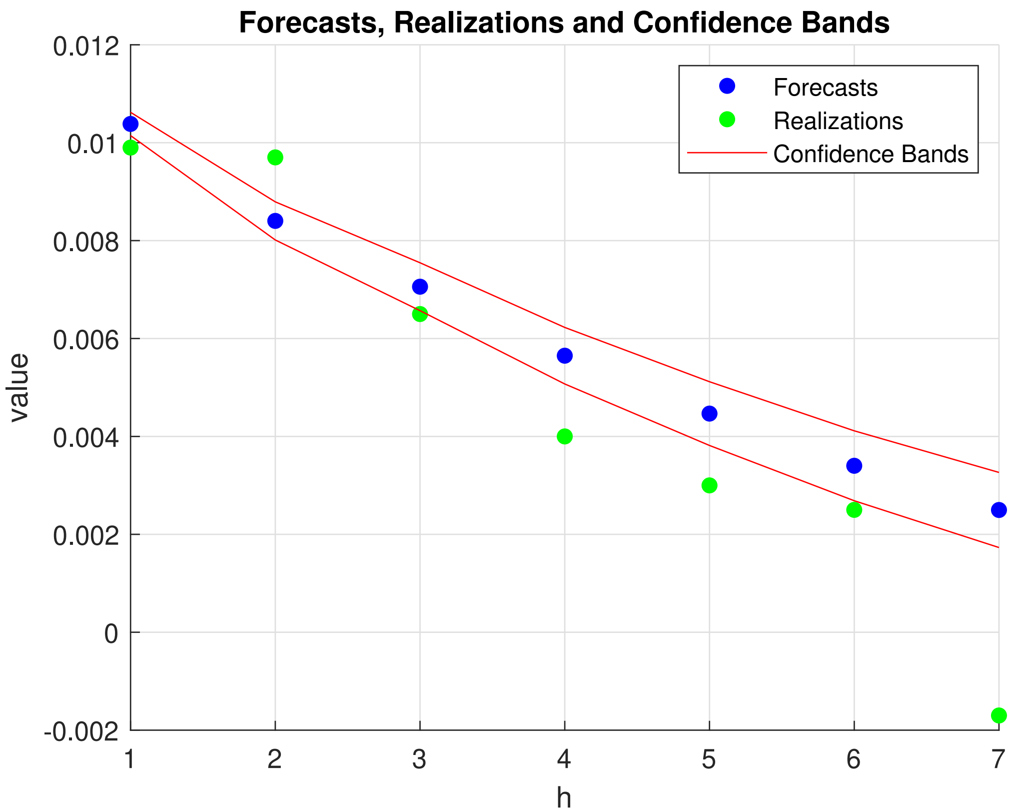

Figure 4: Predictions, Confidence Bands and Realizations