Time Series and their Applications

Homework Exercise 2

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Question 1

In this exercise, we use the theorem stating there exists a stationary solution for a process $\{X_t\}$ if $\Phi_p(z) \neq 0$ for all $|z|= 1$. Next, $\{X_t\}$ is causal w.r.t. $\{Z_t\}$ if $\Phi_p(z) \neq 0$ for all $|z|\leq 1$ and $\{X_t\}$ is invertible w.r.t. $\{Z_t\}$ if $\Theta_p(z) \neq 0$ for all $|z|\leq 1$.

A

$$X_t - 1.3X_{t-1} + 0.4X_{t-2} = Z_t$$

Notice that $\Phi(z) = 1 - 1.3z + 0.4z^2$ and the corresponding roots are

$$z^*_{1,2} = \frac{1.3 \pm \sqrt{1.3^2 - 1.6}}{0.8}.$$

So $z^*_1 = \frac{5}{4}$ and $z^*_2 = 2$. Since both roots are larger than 1, the process is causal with respect to $\{Z_t\}$. Since $\Theta(z) = 1$ for all z, the process is invertible w.r.t. $\{Z_t\}$. Moreover a stationary solution exists and $\{X_t\}$ can be rewritten as

$$\begin{aligned}
\Phi(L)X_t &= Z_t \\
X_t &= \Phi^{-1}(L)Z_t \\
X_t &= \frac{1}{1-\frac{4}{5}L}\frac{1}{1-\frac{1}{2}L}Z_t \\
X_t &= \left(\sum_{i\geq 0}\left(\frac{4}{5}\right)^i L^i\right)\left(\sum_{i\geq 0}\left(\frac{1}{2}\right)^i L^i\right) Z_t.
\end{aligned}$$

B

$$X_t = Z_t - 2Z_{t-1}$$

The autoregressive polynomial Φ equals $\Phi(z) = 1$ for all z, so $\{X_t\}$ is causal with respect to $\{Z_t\}$. The moving-average polynomial Θ equals $\Theta(z) = 1 - 2z$ and has root $z^* = \frac{1}{2}$. Therefore the process is not invertible with respect to $\{Z_t\}$. The process is already written in it's stationary solution.

C

$$X_t - 0.8X_{t-1} + 0.15X_{t-2} = Z_t - 0.3Z_{t-1}$$

The autoregressive polynomial Φ has roots

$$z^*_{1,2} = \frac{0.8 \pm \sqrt{0.8^2 - 0.6}}{0.3}.$$

Hence $z^*_1 = 2$ and $z^*_2 = 3\frac{1}{3}$ and the process is causal with respect to $\{Z_t\}$. The moving-average polynomial $\Theta(z) = 1 - 0.3z$ has root $z^* = 3\frac{1}{3}$ too. So, the process has a stationary solution and because of the cancelling common roots, we have

$$
\begin{aligned}
\left(1-\frac{1}{2}L\right)\left(1-\frac{3}{10}L\right)X_t &= \left(1-\frac{3}{10}L\right)Z_t \\
\left(1-\frac{1}{2}L\right)X_t &= Z_t \\
X_t &= \frac{1}{1-\frac{1}{2}L}Z_t \\
X_t &= \sum_{s\geq 0}\left(\frac{1}{2}\right)^s Z_{t-s}.
\end{aligned}
$$

D

$$X_t - X_{t-1} + 0.5X_{t-2} = Z_t - Z_{t-1}$$

The autoregressive polynomial Φ has roots

$$z^*_{1,2} = 1 \pm \sqrt{1^2 - 2} = 1 \pm i.$$

Since $|z^*_{1,2}| = \sqrt{1^2 + 1^2} = \sqrt{2} > 1$ the process is causal with respect to $\{Z_t\}$. However $\Theta(z) = 1 - z$ has root $z^* = 1$ and the process is therefore not invertible with respect to $\{Z_t\}$.

We find the stationary solution of $\{X_t\}$ with the definition of causality. Since $\{X_t\}$ is causal with respect to $\{Z_t\}$, there exists a sequence $\{\psi_s\}$, $\psi_0 = 1$, absolutely summable, such that

$$X_t = \sum_{s\geq 0}\psi_s Z_{t-s}.$$

Besides we know the coefficients $\{\psi_s\}$ are uniquely determined by

$$\Phi(z)\Psi(z) = \Theta(z)$$

and this gives the equality

$$
\begin{aligned}
\left(1 - z + 0.5z^2\right)\left(1 + \psi_1 z + \psi_2 z^2 + \psi_3 z^3 + \dots\right) &= 1 - z \\
1 + (\psi_1 - 1)z + (\psi_2 - \psi_1 + 0.5)z^2 + (\psi_3 - \psi_2 + 0.5\psi_1)z^3 + \dots &= 1 - z.
\end{aligned}
$$

Equating the coefficients of z yields $\psi_1 = 0$ and $\psi_s = \psi_{s-1} - 0.5$ for $s \geq 2$. As a result, the stationary solution of $\{X_t\}$ is

$$X_t = Z_t + \sum_{s\geq 2}\psi_s Z_{t-s}.$$

Question 2

A

The process $\{X_t\}$ is causal with respect to $\{Z_t\}$ if the modulus of the roots of the autoregressive polynomial are larger than 1. The roots are given by

$$z^*_{1,2} = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$$

where dividing by ϕ_2 is possible, because $\phi_2 \neq 0$ from the definition of an ARMA$(2,1)$ process. As a result, the process $\{X_t\}$ is causal with respect to $\{Z_t\}$ if

$$\left|\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}\right| > 1.$$

B

If $\phi_1 = 2\phi$ and $\phi_2 = -\phi^2$, the autoregressive polynomial Φ is $\Phi(z) = 1-2\phi z+\phi^2 z^2 = (1-\phi z)^2$. Therefore there is one root $z^* = \frac{1}{\phi}$ and since the conditions of (A) hold, we have $\left|\frac{1}{\phi}\right| > 1$ and hence $|\phi|< 1$.

Next we prove the hint,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} &= \frac{\mathrm{d}}{\mathrm{d}x}1 + x + x^2 + x^3 + \dots \\ &= 1 + 2x + 3x^2 + \dots \\ &= \sum_{s\geq 0}(s+1)x^s\end{aligned}$$

for $|x|< 1$. Moreover with the hint follows

$$\begin{aligned}\Phi^{-1}(z)\Theta(z) &= \frac{1}{(1-\phi z)^2}\,(1+\theta z) \\ &= (1+\theta z)\left(1 + 2\phi z + 3(\phi z)^2 + 4(\phi z)^3 + \dots\right) \\ &= 1 + (2\phi + \theta)z + (3\phi^2 + 2\theta\phi)z^2 + (4\phi^3 + 3\theta\phi^2)z^3 + \dots \\ &= \sum_{s\geq 0}\left((s+1)\phi^s + \theta s\phi^{s-1}\right) z^s.\end{aligned}$$

From this we derive the causal representation of $\{X_t\}$

$$\begin{aligned}X_t &= \Phi^{-1}(L)\Theta(L)Z_t \\ &= \sum_{s\geq 0}\left((s+1)\phi^s + \theta s\phi^{s-1}\right) Z_{t-s} \\ &= \sum_{s\geq 0}\psi_s Z_{t-s}\end{aligned}$$

for $\psi_0 = 1$ and $\psi_s = (s+1)\phi^s + \theta s\phi^{s-1}$ for $s \geq 1$.

Question 3

A

First we determine the variance of $\tilde{Z}_t$

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\tilde{Z}_t) &= \mathbb{V}\mathrm{ar}(X_t - \phi^{-1}X_{t-1}) \\ &= \mathbb{V}\mathrm{ar}\left(\phi^{-2}Z_t + (\phi^{-2}-1)\sum_{s\geq 1}\phi^{-s}Z_{t+s}\right).\end{aligned}$$

Since Z_t and Z_s are independent for $t \neq s$, this is

$$\begin{aligned}&= \phi^{-4}\mathbb{V}\mathrm{ar}(Z_t) + (\phi^{-2}-1)^2\sum_{s\geq 1}\phi^{-2s}\mathbb{V}\mathrm{ar}(Z_{t+s}) \\ &= \phi^{-4}\sigma^2 + (\phi^{-2}-1)^2\sigma^2\sum_{s\geq 1}\phi^{-2s} \\ &= \phi^{-4}\sigma^2 + (\phi^{-2}-1)^2\sigma^2\frac{1}{\phi^2-1} \\ &= \frac{\phi^{-2}\sigma^2 - \phi^{-4}\sigma^2 + (\phi^{-2}-1)^2\sigma^2}{\phi^2-1} \\ &= \sigma^2\frac{1-\phi^{-2}}{\phi^2-1} = \sigma^2\phi^{-2}.\end{aligned}$$

Next we determine the covariance of $\tilde{Z}_t$

$$\begin{aligned}
\mathbb{C}\text{ov}\left(\tilde{Z}_t, \tilde{Z}_{t+h}\right) &= \mathbb{C}\text{ov}\left(X_t - \phi^{-1}X_{t-1}, X_{t+h} - \phi^{-1}X_{t+h-1}\right) \\
&= \mathbb{C}\text{ov}\left(X_t, X_{t+h}\right) - \phi^{-1}\mathbb{C}\text{ov}\left(X_{t-1}, X_{t+h}\right) \\
&- \phi^{-1}\mathbb{C}\text{ov}\left(X_t, X_{t+h-1}\right) + \phi^{-2}\mathbb{C}\text{ov}\left(X_{t-1}, X_{t+h-1}\right) \\
&= \gamma(h) - \phi^{-1}\gamma(h+1) - \phi^{-1}\gamma(h-1) + \phi^{-2}\gamma(h).
\end{aligned}$$

From the lecture notes we know $\gamma(h) = \frac{\sigma^2\phi^{-h}}{\phi^2-1}$, hence

$$= \frac{\sigma^2}{\phi^2-1}\left(\phi^{-h} - \phi^{-h} - \phi^{-h-2} + \phi^{-2-h}\right) = 0.$$

B

From the lecture notes we know that

$$P_T X_{T+1} = a_0 + a_T^T X_T$$

where a_T is the solution of

$$\Gamma_T a_T = \gamma_T(h).$$

From the lectures we know also that

$$\gamma(0) = \frac{\sigma^2}{\phi^2-1} \quad \text{and} \quad \gamma(h) = \frac{\sigma^2\phi^{-h}}{\phi^2-1}.$$

Hence,

$$\begin{gathered}
\Gamma_T a_T = \gamma_T(h) \\
\begin{bmatrix} \gamma(0) & \gamma(1) & \dots & \gamma(T-1) \\ \gamma(1) & \gamma(0) & \dots & \dots \\ \vdots & \vdots & \ddots & \\ \gamma(T-1) & & & \gamma(0) \end{bmatrix} \begin{bmatrix} a_T \\ a_{T-1} \\ \vdots \\ a_1 \end{bmatrix} = \begin{bmatrix} \gamma(h) \\ \gamma(h+1) \\ \vdots \\ \gamma(h+T-1) \end{bmatrix} \\
\frac{\sigma^2}{\phi^2-1}\begin{bmatrix} 1 & \phi^{-1} & \dots & \phi^{-(T-1)} \\ \phi^{-1} & 1 & \dots & \dots \\ \vdots & \vdots & \ddots & \\ \phi^{-(T-1)} & & & 1 \end{bmatrix} \begin{bmatrix} a_T \\ a_{T-1} \\ \vdots \\ a_1 \end{bmatrix} = \frac{\sigma^2}{\phi^2-1}\begin{bmatrix} \phi^{-h} \\ \phi^{-(h+1)} \\ \vdots \\ \phi^{-(h+T-1)} \end{bmatrix}
\end{gathered}$$

where the solution is $a_T = [\phi^{-h} 0 \cdots 0]^T$. As a result,

$$P_T X_{T+1} = a_0 + a_T^T X_T = \phi^{-h} X_T,$$

because $a_0 = \mu = 0$ for the process $\{X_t\}$.

C

From the lecture notes we know that the MSPE of $P_T X_{T+1}$ is

$$\begin{aligned}
\mathbb{E}\left[(X_{T+1} - P_T X_{T+1})^2\right] &= \gamma(0) - a_T^T \Gamma_T a_T \\
&= \gamma(0) - a_T^T \gamma_T(h) = \frac{\sigma^2}{\phi^2-1}\left(1 - \phi^{-2h}\right).
\end{aligned}$$

On the other hand,

$$\mathbb{E}\left[(X_{T+1} - \hat{X}_{T+1})^2\right] = \mathbb{E}\left[Z_t^2\right] = \mathbb{V}\text{ar}(Z_t) = \sigma^2.$$

Since $|\phi|> 1$, the MSPE of $P_T X_{T+1}$ converges to

$$\frac{\sigma^2}{\phi^2 - 1}$$

for $h \to \infty$. So if $\phi^2 > 2$, we have

$$\frac{\sigma^2}{\phi^2 - 1} < \sigma^2$$

and we prefer the predictor $P_T X_{T+1}$. However, for $\phi^2 < 2$, the MSPE of $P_T X_{T+1}$ is larger than the MSPE of $\hat{X}_{T+1}$ and we prefer $\hat{X}_{T+1}$. For $\phi^2 = 2$ we also prefer $\hat{X}_{T+1}$, because the MSPE is smaller for h not being arbitrarily large.

Question 4

We consider the following MA(1) process:

$$X_t = Z_t - 0.5Z_{t-1}, \quad \text{with} \quad Z_t \sim WN(0, 1)$$

From page 9 of lecture 1, we know the ACVF of a MA(1) process:

$$\gamma_X(t+h, t) = \begin{cases} (1+\theta^2)\sigma^2 = 1.25 & \text{if } h = 0 \\ \theta\sigma^2 = -0.5 & \text{if } |h|= 1 \\ 0 & \text{if } |h|> 1 \end{cases}$$

Then we have

$$\Gamma_T = \begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(2) & \gamma(3) \\ \gamma(1) & \gamma(0) & \gamma(1) & \gamma(2) \\ \gamma(2) & \gamma(1) & \gamma(0) & \gamma(1) \\ \gamma(3) & \gamma(2) & \gamma(1) & \gamma(0) \end{bmatrix} = \begin{bmatrix} 1.25 & -0.5 & 0 & 0 \\ -0.5 & 1.25 & -0.5 & 0 \\ 0 & -0.5 & 1.25 & -0.5 \\ 0 & 0 & -0.5 & 1.25 \end{bmatrix}$$

$$\gamma_T(1) = \begin{bmatrix} \gamma(1) \\ \gamma(2) \\ \gamma(3) \\ \gamma(4) \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now we can calculate $a_T = \Gamma_T^{-1}\gamma_T(1)$ using Matlab we get the following result for a_T:

$$a_T(1) = \begin{bmatrix} -0.400 \end{bmatrix} \quad a_T(2) = \begin{bmatrix} -0.4762 \\ -0.1905 \end{bmatrix} \quad a_T(3) = \begin{bmatrix} -0.4941 \\ -0.2353 \\ -0.0941 \end{bmatrix} \quad a_T(4) = \begin{bmatrix} -0.4985 \\ -0.2463 \\ -0.1173 \\ -0.0469 \end{bmatrix}$$

Here we can see that the best linear forecast $P_T X_{T+1}$ is dependent on all observations, since all values in a_T are non-zero. The MSPE is given by $MSPE = \gamma(0) - a_T(1)^T \gamma_T(1)$. Again, using Matlab, we find that the MSPEs for T = 1,2,3 and 4 are 1.05, 1.0119, 1.0029 and 1.0007 respectively.

Empirical Exercise

(1)

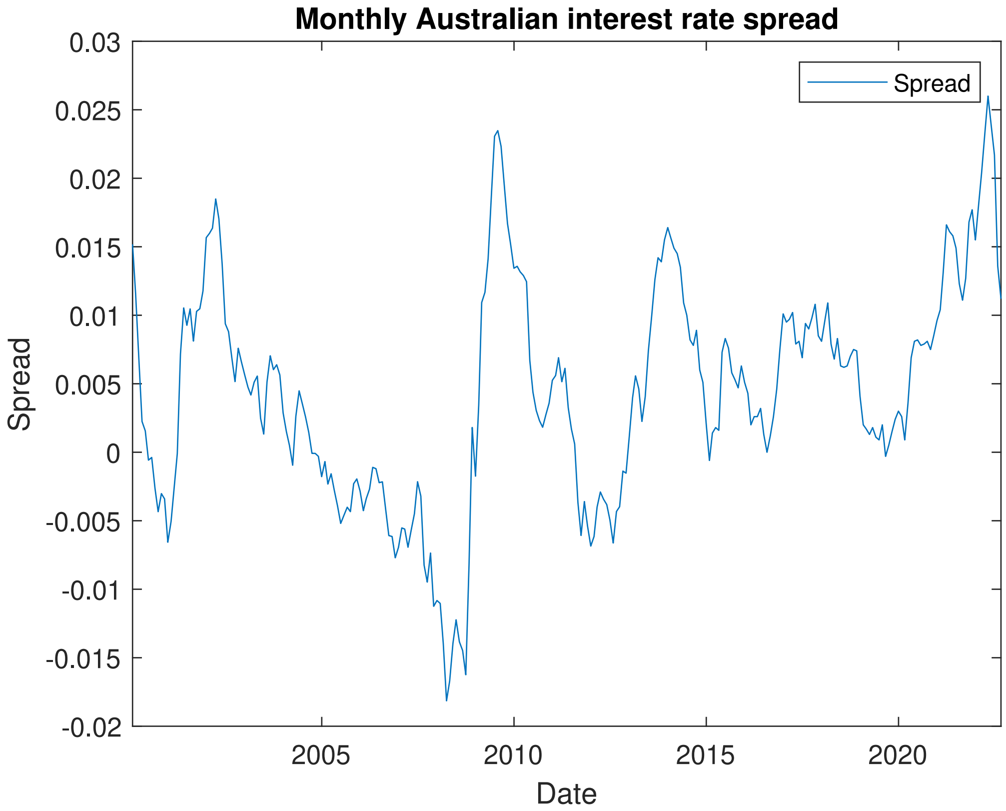

Figure 1: Interest rate spread in subsample

For the series to look like a stationary process, the average must be independent of the time and the autocovariance between two observations should only depend on the time jump, not the moment in time itself. From Figure 1 it would seem that the average indeed does not depend on time, but Figure 2 suggests that the autocovariance does depend on the moment in time. Therefore, the series does not look like a stationary process.

(2)

Please see the Matlab file for the code

(3)

We abbreviate Theoretical ACF as T-ACF and Sample ACF as S-ACF. Table 1 is visually represented in Figure 3. We conclude that for "the first few" ACF's, the sample and theoretical ACF's are approximately the same. However, the larger the value of h, the larger the difference in theoretical and sample ACF becomes. Therefore, the model definitely captures some features of the series, but we think it does not capture the features of the series *sufficiently* well.

(4)

Please see the Matlab file for the code

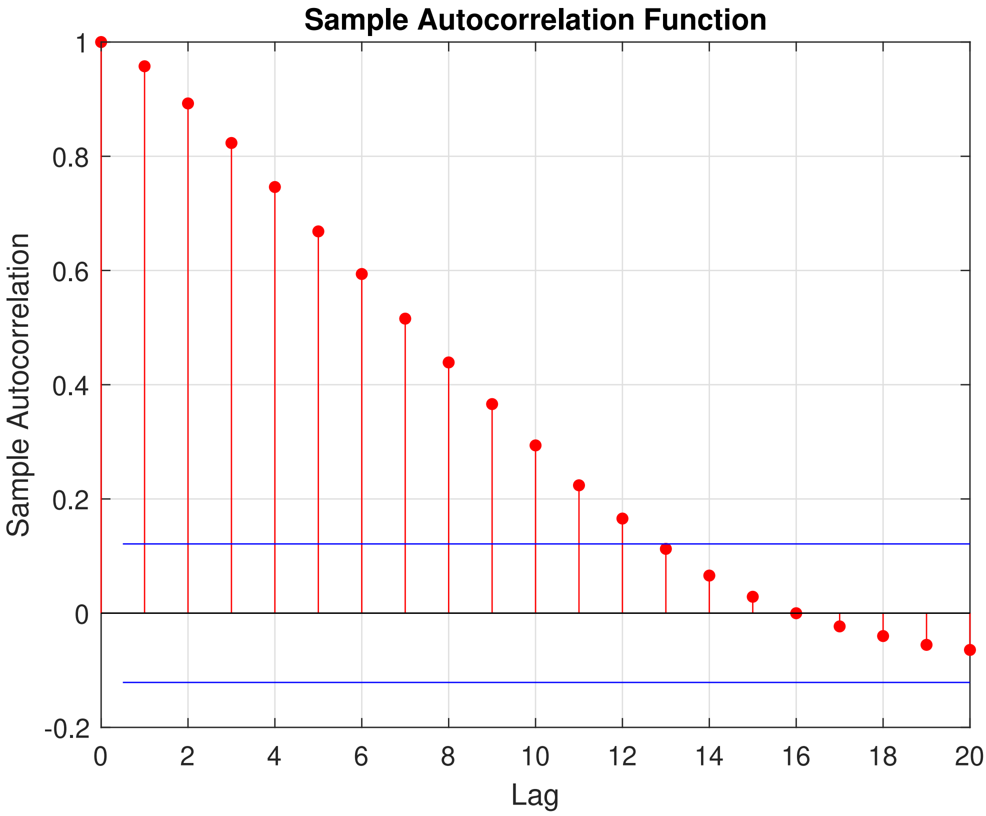

Figure 2: Sample ACF of the subsample

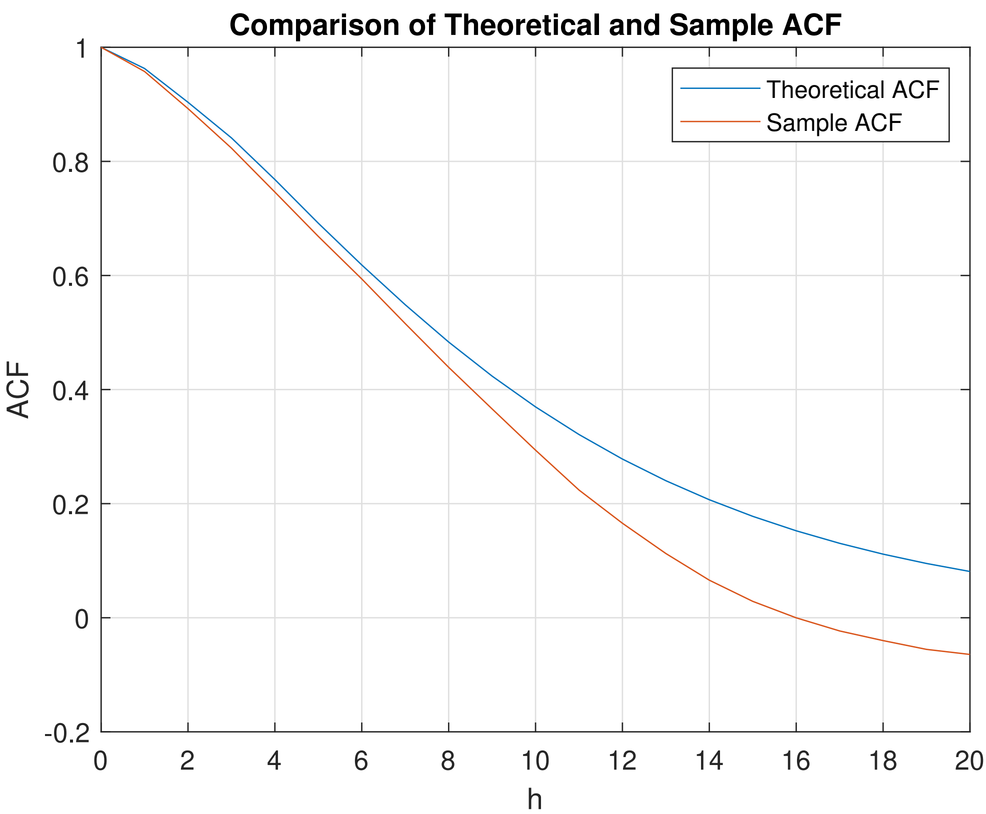

Figure 3: Theoretical ACF against Sample ACF

h	T-ACF	S-ACF	h	T-ACF	S-ACF	h	T-ACF	S-ACF
0	1	1	7	0.5487	0.5157	14	0.2069	0.0658
1	0.9629	0.9576	8	0.4833	0.4390	15	0.1778	0.0287
2	0.9036	0.8925	9	0.4236	0.3661	16	0.1524	-0.0001
3	0.8412	0.8233	10	0.3696	0.2938	17	0.1305	-0.0232
4	0.7682	0.7462	11	0.3212	0.2239	18	0.1115	-0.0401
5	0.6916	0.6685	12	0.2782	0.1656	19	0.0951	-0.0555
6	0.6184	0.5939	13	0.2402	0.1126	20	0.0811	-0.0644

Table 1: Theoretical and Sample ACF values

(5)

h	Prediction	Lower confidence band	Upper confidence band	Realization
1	0.0143	0.0140	0.0145	0.0099
2	0.0118	0.0114	0.0122	0.0097
3	0.0102	0.0097	0.0107	0.0065
4	0.0092	0.0086	0.0098	0.0040
5	0.0082	0.0075	0.0089	0.0030
6	0.0074	0.0067	0.0081	0.0025
7	0.0068	0.0060	0.0076	-0.0017

Table 2: Predictions, Confidence Bands and Realizations (visualized in Figure 4)

Comparing the predicted values of the interest rate with the realized values, we conclude that the predictions are not very accurate. The realizations even do not lie within the 95% confidence bands of the predictions. Therefore, the only reasonable conclusion is that the prediction accuracy is very poor

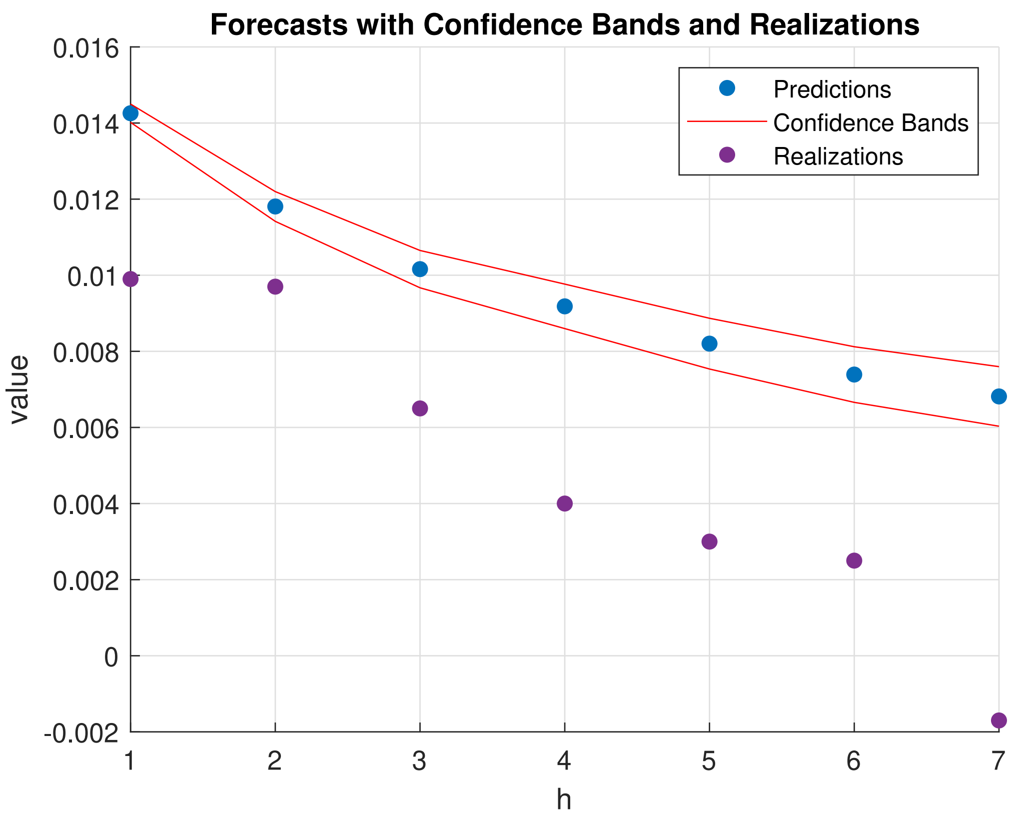

Figure 4: Visual representation of Table 2