Time Series and their Applications

Homework Exercise 4

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Date: May 24, 2024

Question 1

A

Notice that, conditional on $\{Z_s : s < t\}$, the value of h_t is known, because it depends on the parameters α_j, $j = 0, \ldots, q$ and on $\{Z_s : t - q \leq s < t\}$. Hence,

$$\begin{aligned}\mathbb{E}\left[R_t | \{Z_s : s < t\}\right] &= \mathbb{E}\left[\mu + \delta h_t + Z_t | \{Z_s : s < t\}\right] \\ &= \mu + \delta \mathbb{E}\left[h_t | \{Z_s : s < t\}\right] + \mathbb{E}\left[\sqrt{h_t} v_t | \{Z_s : s < t\}\right] \\ &= \mu + \delta h_t + \sqrt{h_t}\mathbb{E}\left[v_t | \{Z_s : s < t\}\right] = \mu + \delta h_t\end{aligned}$$

because v_t is standard normal distributed and independent of $\{Z_s : s < t\}$.

Due to this independence and the fact that h_t is known conditional on $\{Z_s : s < t\}$, we also have

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left(R_t | \{Z_s : s < t\}\right) &= \mathbb{V}\mathrm{ar}\left(\mu + \delta h_t + Z_t | \{Z_s : s < t\}\right) \\ &= \mathbb{V}\mathrm{ar}\left(Z_t | \{Z_s : s < t\}\right) \\ &= h_t \mathbb{V}\mathrm{ar}\left(v_t | \{Z_s : s < t\}\right) = h_t.\end{aligned}$$

Notice that an higher expected excess return $\mathbb{E}\left[R_t | \{Z_s : s < t\}\right]$ means that h_t is larger. As a result, the variance $\mathbb{V}\mathrm{ar}\left(R_t | \{Z_s : s < t\}\right)$ increases and hence the risk for the investor. We conclude that this model reflects the idea of classical asset pricing, stating that excess returns are proportional to the risk.

B

Suppose $\mathbb{E}[Z_t^2] = 1$, then

$$\mathbb{E}\left[R_t\right] = \mu + \delta \mathbb{E}\left[h_t\right] + \mathbb{E}\left[Z_t\right].$$

Notice that $\mathbb{E}\left[Z_t^2\right] = \mathbb{E}\left[h_t v_t^2\right]$. Since h_t depends on $\{Z_s : s < t\}$ and v_t is independent of $\{Z_s : s < t\}$, v_t is also independent of h_t and we have $1 = \mathbb{E}\left[Z_t^2\right] = \mathbb{E}\left[h_t\right]\mathbb{E}\left[v_t^2\right] = \mathbb{E}\left[h_t\right]$. With this we see

$$\mathbb{E}\left[R_t\right] = \mu + \delta + \mathbb{E}\left[Z_t\right] = \mu + \delta + \mathbb{E}[\sqrt{h_t}]\mathbb{E}\left[v_t\right] = \mu + \delta$$

and therefore $\mathbb{E}\left[R_t\right] = 0$ if and only if $\mu = -\delta$, where δ is strictly positive and hence μ strictly negative.

C

Assume that $q = 1$ for the following parts. Then $h_t = \alpha_0 + \alpha_1 Z_{t-1}^2$ and for $s > 0$,

$$\begin{aligned}h_t &= \alpha_0 + \alpha_1 h_{t-1} v_{t-1}^2 \\ &= \alpha_0 + \alpha_1 \left(\alpha_0 + \alpha_1 h_{t-2} v_{t-2}^2\right) v_{t-1}^2 \\ &= \alpha_0 + \alpha_0 \alpha_1 v_{t-1}^2 + \alpha_1^2 h_{t-2} v_{t-1}^2 v_{t-2}^2 \\ &= \alpha_0 + \alpha_0 \alpha_1 v_{t-1}^2 + \alpha_1^2 \left(\alpha_0 + \alpha_1 h_{t-3} v_{t-3}^2\right) v_{t-1}^2 v_{t-2}^2 \\ &= \alpha_0 + \alpha_0 \alpha_1 v_{t-1}^2 + \alpha_0 \alpha_1^2 v_{t-1}^2 v_{t-2}^2 + \alpha_1^3 h_{t-3} v_{t-1}^2 v_{t-2}^2 v_{t-3}^2 \\ &= \vdots \qquad \vdots \qquad \vdots \\ &= \alpha_0 + \alpha_0 \sum_{i=1}^{s-1} \alpha_1^i \prod_{j=1}^{i} v_{t-j}^2 + \alpha_1^s h_{t-s} \prod_{i=1}^{s} v_{t-i}^2.\end{aligned}$$

Notice again that v_t is independent of Z_s for $s < t$ and $\mathbb{E}[Z_s] = \mathbb{E}[\sqrt{h_t}]\mathbb{E}[v_t] = 0$. As a result for $s > 0$,

$$\begin{aligned}
\mathbb{C}\text{ov}\left(Z_{t-s}, h_t\right) &= \mathbb{E}\left[Z_{t-s}h_t\right] - \mathbb{E}\left[Z_{t-s}\right]\mathbb{E}\left[h_t\right] \\
&= \mathbb{E}\left[Z_{t-s}h_t\right] \\
&= \alpha_0\mathbb{E}\left[Z_{t-s}\right] + \alpha_0\sum_{i=1}^{s-1}\alpha_1^i\mathbb{E}\left[Z_{t-s}\right]\prod_{j=1}^{i}\mathbb{E}[v_{t-j}^2] + \alpha_1^s\mathbb{E}[Z_{t-s}h_{t-s}v_{t-s}^2]\prod_{i=1}^{s-1}\mathbb{E}[v_{t-i}^2] \\
&= \alpha_1^s\mathbb{E}[Z_{t-s}h_{t-s}v_{t-s}^2] \\
&= \alpha_1^s\mathbb{E}[Z_{t-s}^3]
\end{aligned}$$

which is zero, because the distribution of Z_t is symmetric around zero since $v_t \overset{d}{=} -v_t$. We conclude that Z_{t-s} is uncorrelated with h_t for $s > 0$.

For $s \leq 0$, the calculation is shorter, because v_t is independent of h_s for $s \leq t$. Hence,

$$\mathbb{C}\text{ov}\left(Z_{t-s}, h_t\right) = \mathbb{E}[v_{t-s}\sqrt{h_{t-s}}h_t] - \mathbb{E}[v_{t-s}\sqrt{h_{t-s}}]\mathbb{E}[h_t] = \mathbb{E}[v_{t-s}]\mathbb{E}[\sqrt{h_{t-s}}h_t] - \mathbb{E}[v_{t-s}]\mathbb{E}[\sqrt{h_{t-s}}]\mathbb{E}[h_t] = 0.$$

We conclude that Z_{t-s} is uncorrelated with h_t for any $s \in \mathbb{Z}$.

D

Assume $\alpha_1 < 1/\sqrt{3}$ Notice first that

$$\mathbb{E}\left[Z_t^2\right] = \mathbb{E}\left[h_t\right]\mathbb{E}\left[v_t^2\right] = \alpha_0 + \alpha_1\mathbb{E}\left[Z_{t-1}^2\right].$$

From the lecture notes, we know $\{Z_t\}$ is stationary. Therefore, write $\mu_{Z^2} = \mathbb{E}\left[Z_t^2\right]$ and this gives

$$\mu_{Z^2} = \alpha_0 + \alpha_1\mu_{Z^2}$$

and thus

$$\mu_{Z^2} = \mathbb{E}\left[Z_t^2\right] = \frac{\alpha_0}{1-\alpha_1}$$

where dividing by $1-\alpha_1$ is possible, because $\alpha_1 < 1/\sqrt{3}$.

Next write $\mu_{Z^4} = \mathbb{E}\left[Z_t^4\right]$ (possible because of stationarity), then

$$\begin{aligned}
\mu_{Z^4} &= \mathbb{E}\left[v_t^4\left(\alpha_0 + \alpha_1 Z_{t-1}^2\right)^2\right] \\
&= \mathbb{E}\left[v_t^4\right]\mathbb{E}\left[\alpha_0^2 + 2\alpha_0\alpha_1 Z_{t-1}^2 + \alpha_1^2 Z_{t-1}^4\right] \\
&= 3\left(\alpha_0^2 + 2\alpha_0\alpha_1\mu_{Z^2} + \alpha_1^2\mu_{Z^4}\right).
\end{aligned}$$

From this we deduce

$$\mu_{Z^4} = \frac{3\left(\alpha_0^2 + \frac{2\alpha_0^2\alpha_1}{1-\alpha_1}\right)}{1-3\alpha_1^2} = \frac{3\alpha_0^2 - 3\alpha_0^2\alpha_1 + 6\alpha_0^2\alpha_1}{(1-3\alpha_1^2)(1-\alpha_1)} = \frac{3\alpha_0^2(1+\alpha_1)}{(1-3\alpha_1^2)(1-\alpha_1)}$$

where dividing by $1 - 3\alpha_1^2$ is possible, because $\alpha_1 < 1/\sqrt{3}$.

Taking μ_{Z^2} and μ_{Z^4} together yields

$$\begin{aligned}
\mathbb{V}\text{ar}\left(Z_t^2\right) &= \mathbb{E}\left[Z_t^4\right] - \left(\mathbb{E}\left[Z_t^2\right]\right)^2 \\
&= \mu_{Z^4} - \mu_{Z^2}^2 \\
&= \frac{3\alpha_0^2(1+\alpha_1)}{(1-3\alpha_1^2)(1-\alpha_1)} - \frac{\alpha_0^2}{(1-\alpha_1)^2} \\
&= \frac{3\alpha_0^2(1+\alpha_1)(1-\alpha_1) - \alpha_0^2(1-3\alpha_1^2)}{(1-3\alpha_1^2)(1-\alpha_1)^2} \\
&= \frac{3\alpha_0^2 - 3\alpha_0^2\alpha_1^2 - \alpha_0^2 + 3\alpha_0^2\alpha_1^2}{(1-3\alpha_1^2)(1-\alpha_1)^2} \\
&= \frac{2\alpha_0^2}{(1-3\alpha_1^2)(1-\alpha_1)^2}
\end{aligned}$$

E

Define γ_R as the ACVF of $\{R_t\}$, then

$$\gamma_R(t+s,t) = \mathbb{C}\text{ov}\left(\delta h_{t+s} + Z_{t+s}, \delta h_t + Z_t\right).$$

First consider the case for $s \neq 0$. From **C** we know that both h_{t+s} and Z_t, and h_t and Z_{t+s} are uncorrelated. Besides, Z_t is uncorrelated with Z_{t+s}, because for $s > 0$, we have

$$\begin{aligned}
\mathbb{C}\text{ov}\left(Z_{t+s}, Z_t\right) &= \mathbb{E}\left[Z_{t+s}\sqrt{h_t}v_t\right] - \mathbb{E}\left[Z_{t+s}\right]\mathbb{E}\left[\sqrt{h_t}v_t\right] \\
&= \mathbb{E}\left[Z_{t+s}\sqrt{h_t}\right]\mathbb{E}[v_t] - \mathbb{E}\left[Z_{t+s}\right]\mathbb{E}\left[\sqrt{h_t}\right]\mathbb{E}[v_t] = 0
\end{aligned}$$

where we used that v_t is independent of h_t and Z_{t+s}, and that v_t follows a standard normal distribution. The same idea holds for $s < 0$.

$$\begin{aligned}
\mathbb{C}\text{ov}\left(Z_{t+s}, Z_t\right) &= \mathbb{E}\left[\sqrt{h_{t+s}}v_{t+s}Z_t\right] - \mathbb{E}\left[\sqrt{h_{t+s}}v_{t+s}\right]\mathbb{E}\left[Z_t\right] \\
&= \mathbb{E}\left[\sqrt{h_{t+s}}Z_t\right]\mathbb{E}[v_{t+s}] - \mathbb{E}\left[\sqrt{h_{t+s}}\right]\mathbb{E}\left[Z_t\right]\mathbb{E}[v_{t+s}] = 0
\end{aligned}$$

Hence, we have

$$\gamma_R(t+s,t) = \mathbb{C}\text{ov}\left(\delta h_{t+s} + Z_{t+s}, \delta h_t + Z_t\right) = \delta^2\mathbb{C}\text{ov}\left(h_{t+s}, h_t\right) = \delta^2\alpha_1^2\mathbb{C}\text{ov}\left(Z_{t+s-1}^2, Z_{t-1}^2\right)$$

which is zero too, which becomes clear from the same kind of reasoning as for $\mathbb{C}\text{ov}\left(Z_{t+s}, Z_t\right)$. Therefore, we conclude for $s \neq 0$ that $\gamma_R(t+s,t) = 0$.

For $s = 0$, we have

$$\begin{aligned}
\gamma_R(t,t) &= \mathbb{C}\text{ov}\left(\delta h_t + Z_t, \delta h_t + Z_t\right) \\
&= \delta^2\mathbb{V}\text{ar}\left(h_t\right) + 2\delta\mathbb{C}\text{ov}\left(h_t, Z_t\right) + \mathbb{V}\text{ar}\left(Z_t\right) \\
&= \delta^2\gamma_{Z^2}(0) + \mu_{Z^2} \\
&= \delta^2\gamma_{Z^2}(0) + \frac{\alpha_0}{1-\alpha_1}
\end{aligned}$$

where we write $\gamma_{Z^2}(0)$ for $\mathbb{V}\text{ar}\left(h_t\right) = \mathbb{E}\left[h_t^2\right] - \left(\mathbb{E}\left[h_t\right]\right)^2 = \mathbb{V}\text{ar}\left(Z_t^2\right)$.

Question 2

A

Using $\mathcal{Z}_0 = [Z_0, ..., Z_{1-q}]^T$ we can calculate h_1, which is dependent on $\alpha_0, ..., \alpha_q$ and $\mathcal{Z}_0$. Then we use this value to calculate Z_1. Using this new value of Z_1 we can calculate h_2 and then Z_2 with the new value of h_2. Then we can repeat this process to calculate all values of Z_t and h_t for $t = 1, ..., T$.

B

Since the fact that $v_t \sim \mathcal{N}(0,1)$ and $Z_t = \sqrt{h_t}v_t$, we can conclude that $\frac{Z_t}{\sqrt{h_t}} \sim \mathcal{N}(0,1)$, thus $\frac{R_t-\mu-\delta*h_t}{\sqrt{h_t}} \sim \mathcal{N}(0,1)$. Then we can calculate the conditional density:

$$f(R_t|\mathcal{Z}_0) = \frac{1}{\sqrt{h_t}}\phi(\frac{R_t - \mu - \delta * h_t}{\sqrt{h_t}})$$

$$f(z_{p+1}, ..., z_T|\mathcal{Z}_0) = \prod_{t=q+1}^{T} \frac{1}{\sqrt{h_t}}\phi(\frac{R_t - \mu - \delta * h_t}{\sqrt{h_t}})$$

Then the log-likelihood function can be written as:

$$\begin{aligned}
ln(L) &= ln(\prod_{t=q+1}^{T} \frac{1}{\sqrt{h_t}}\phi(\frac{R_t - \mu - \delta * h_t}{\sqrt{h_t}})) \\
&= \sum_{t=q+1}^{T} ln(\frac{1}{\sqrt{h_t}} * \frac{1}{\sqrt{2\pi}} * e^{-0.5(\frac{R_t-\mu-\delta*h_t}{\sqrt{h_t}})^2}) \\
&= \sum_{t=q+1}^{T} ln(h_t)^{-0.5} + ln(2\pi)^{-0.5} - 0.5\frac{(R_t - \mu - \delta * h_t)^2}{h_t} \\
&= -\frac{T-q}{2}ln(2\pi) - 0.5\sum_{t=q+1}^{T} ln(h_t) - 0.5\sum_{t=q+1}^{T} \frac{(R_t - \mu - \delta * h_t)^2}{h_t}
\end{aligned}$$

A natural choice for Z_0 is $Z_0 = 0$. This is because at the start, the impact of the volatility is not there and the mean of $v_t = 0$.

C

Using the Matlab format provided, we found the following estimates and sample errors:

	μ	δ	α_0	α_1	α_2
true values	-0.1	1	0.05	0.4	0.1
estimates	-0.0576	0.4865	0.0513	0.3681	0.1068
sample standard deviation	0.0433	0.0524	0.0076	0.0522	0.0256
estimated asymptotic standard errors	0.2179	0.3553	0.0096	0.0989	0.0747

We see that the true values and the estimates are quite close for most estimators. Only the estimator for δ is a bit of. The estimated asymptotic standard errors are for most estimators larger then the sample standard deviation.

Question 3

A

Assume $p = q = 1$. Notice that, conditional on $Z_s^2 : s < t$, both h_t and h_{t-1} are known. With the independence of $Z_s^2 : s < t$ from v_t follows

$$\begin{aligned}
\mathbb{E}\left[Z_t^2 | Z_{t-1}^2, Z_{t-2}^2, \dots\right] &= \mathbb{E}\left[h_t v_t^2 | Z_{t-1}^2, Z_{t-2}^2, \dots\right] \\
&= \mathbb{E}\left[v_t^2\right] \mathbb{E}\left[h_t | Z_{t-1}^2, Z_{t-2}^2, \dots\right] \\
&= \mathbb{E}\left[\alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1} | Z_{t-1}^2, Z_{t-2}^2, \dots\right] \\
&= \alpha_0 + \alpha_1 Z_{t-1}^2 + \beta_1 h_{t-1} = h_t.
\end{aligned}$$

B

Assume $\mathbb{E}\left[Z_t^4\right] < \infty$ and define $W_t := Z_t^2 - h_t$. To prove $\{W_t\}$ is white noise, we show that $\{W_t\}$ is stationary, $\mathbb{E}[W_t] = 0$, and $\gamma_W(t+s, t) = \sigma^2 1_{s=0}$ for γ_W the ACVF of $\{W_t\}$ and some parameter σ^2.

Notice that $\{Z_t\}$ is stationary and hence $\{Z_t^2\}$ and $\{h_t\}$ are stationary. Therefore $\{W_t\}$ is stationary too.

Next, since v_t and h_t are independent, notice that $\mathbb{E}[Z_t] = \mathbb{E}[\sqrt{h_t}]\mathbb{E}[v_t] = 0$. Moreover, $\mathbb{E}[Z_t^2] = \mathbb{E}[h_t]\mathbb{E}[v_t^2] = \mathbb{E}[h_t]$. Therefore, $\mathbb{E}[h_t] = 1$ and

$$\mathbb{E}[W_t] = \mathbb{E}[Z_t^2] - \mathbb{E}[h_t] = \mathbb{E}[h_t] - \mathbb{E}[h_t] = 0.$$

Last, we prove $\gamma_W(t+s, t) = \sigma^2 1_{s=0}$. First consider $s > 0$. Since v_t and Z_s are independent for $s < t$, $\mathbb{E}[Z_{t+s}^2 h_t] = \mathbb{E}[v_{t+s}^2]\mathbb{E}[h_{t+s} h_t] = \mathbb{E}[h_{t+s} h_t]$ and $\mathbb{E}[Z_{t+s}^2 Z_t^2] = \mathbb{E}[h_{t+s} Z_t^2]$. As a result,

$$\begin{aligned}
\gamma_W(t+s, t) &= \mathbb{C}\mathrm{ov}\left(Z_{t+s}^2 - h_{t+s}, Z_t^2 - h_t\right) \\
&= \mathbb{C}\mathrm{ov}\left(Z_{t+s}^2, Z_t^2\right) - \mathbb{C}\mathrm{ov}\left(Z_{t+s}^2, h_t\right) - \mathbb{C}\mathrm{ov}\left(h_{t+s}, Z_t^2\right) + \mathbb{C}\mathrm{ov}\left(h_{t+s}, h_t\right) \\
&= \mathbb{E}[Z_{t+s}^2 Z_t^2] - \mathbb{E}[Z_{t+s}^2]\mathbb{E}[Z_t^2] - \left(\mathbb{E}[Z_{t+s}^2 h_t] - \mathbb{E}[Z_{t+s}^2]\mathbb{E}[h_t]\right) \\
&- \left(\mathbb{E}[Z_t^2 h_{t+s}] - \mathbb{E}[Z_t^2]\mathbb{E}[h_{t+s}]\right) + \mathbb{E}[h_{t+s} h_t] - \mathbb{E}[h_{t+s}]\mathbb{E}[h_t] \\
&= \mathbb{E}[h_{t+s} Z_t^2] - 1 - \left(\mathbb{E}[Z_{t+s}^2 h_t] - \mathbb{E}[h_t]\right) \\
&- \left(\mathbb{E}[Z_t^2 h_{t+s}] - \mathbb{E}[h_{t+s}]\right) + \mathbb{E}[h_{t+s} h_t] - \mathbb{E}[h_{t+s}]\mathbb{E}[h_t] \\
&= -1 + \mathbb{E}[h_t] + \mathbb{E}[h_{t+s}] - \mathbb{E}[h_{t+s}]\mathbb{E}[h_t]
\end{aligned}$$

where the stationary process $\{h_t\}$ leads to $\mathbb{E}[h_t] = \mathbb{E}[h_{t+s}] = 1$. Therefore

$$= -1 + 1 + 1 - 1 = 0.$$

For $s < 0$, the same idea of reasoning holds, therefore we conclude $\gamma_W(t+s, t) = 0$ for $s \neq 0$.

Consider $s = 0$. Then,

$$\begin{aligned}
\gamma_W(t, t) &= \mathbb{C}\mathrm{ov}\left(Z_t^2 - h_t, Z_t^2 - h_t\right) \\
&= \mathbb{E}[Z_t^4] - 1 - 2\mathbb{C}\mathrm{ov}\left(Z_t^2, h_t\right) + \gamma_h(0)
\end{aligned}$$

for γ_h the ACVF of the stationary process $\{h_t\}$. Notice that $\mathbb{C}\mathrm{ov}\left(Z_t^2, h_t\right)$ is on hand equal to $\mathbb{E}[h_t v_t^2 h_t] - \mathbb{E}[h_t v_t^2]\mathbb{E}[h_t] = \gamma_h(0)$ and on the other hand to $\mathbb{E}[h_t^2] - \mathbb{E}[h_t]\mathbb{E}[h_t] = \mathbb{E}[h_t^2] - 1$. Therefore

$$\gamma_W(t, t) = \mathbb{E}[Z_t^4] - 1 - 2\gamma_h(0) + \gamma_h(0) = \mathbb{E}[Z_t^4] - \mathbb{E}[h_t^2].$$

Because v_t is both normally distributed, $\mathbb{E}[v_t^4] = 3$. From $\mathbb{E}[Z_t^4] = \mathbb{E}[v_t^4]\mathbb{E}[h_t^2]$, we derive then $\mathbb{E}[Z_t^4] = 3\mathbb{E}[h_t^2]$. We conclude

$$\gamma_W(t,t) = \mathbb{E}[Z_t^4] - \mathbb{E}[h_t^2] = 2\mathbb{E}[h_t^2] = 2\gamma_h(0)$$

where $\gamma_h(0)$ is not zero, because $\{h_t\}$ is a white noise process. Therefore we have proven all the properties of a white noise process and conclude with the definition that $\{W_t\}$ is white noise.

C

Since $h_t = Z_t^2 - W_t$, we first show that

$$h_t = a_0 + \sum_{i=1}^{m} a_i Z_{t-i}^2 + \sum_{j=1}^{q} b_j W_{t-j}$$

for some parameters $a_0, \cdots, a_m$, $b_1, \ldots, b_q$ and $m = \max\{p, q\}$.

Define l as $l := \min\{p, q\}$ and use $h_t = Z_t^2 - W_t$ to come up with

$$\begin{aligned} h_t &= \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j} \\ &= \alpha_0 + \sum_{i=1}^{p} \alpha_i Z_{t-i}^2 + \sum_{j=1}^{q} \beta_j \left(Z_{t-j}^2 - W_{t-j} \right) \end{aligned}$$

Consider the case $p \geq q$ and hence $m = p$,

$$h_t = \alpha_0 + \sum_{i=1}^{l} (\alpha_i + \beta_i) Z_{t-i}^2 + \sum_{i=l+1}^{m} \alpha_i Z_{t-i}^2 - \sum_{j=1}^{q} \beta_j W_{t-j}.$$

However, for the case $p < q$, we have

$$h_t = \alpha_0 + \sum_{i=1}^{l} (\alpha_i + \beta_i) Z_{t-i}^2 + \sum_{i=l+1}^{m} \beta_i Z_{t-i}^2 - \sum_{j=1}^{q} \beta_j W_{t-j}.$$

Therefore define b_j as $b_j := -\beta_j$ for $j = 1, \ldots, q$. Next, define $a_0 := \alpha_0$ and $a_i = \alpha_i + \beta_i$ for $i = 1, \ldots, l$. Last, for the case $p \geq q$, define $a_i = \alpha_i$ for $i = l+1, \ldots, m$ and define $a_i = \beta_i$ for $i = l+1, \ldots, m$ for the case $p < q$. As a result,

$$\begin{aligned} h_t &= a_0 + \sum_{i=1}^{l} a_i Z_{t-i}^2 + \sum_{i=l+1}^{m} a_i Z_{t-i}^2 + \sum_{j=1}^{q} b_j W_{t-j} \\ &= a_0 + \sum_{i=1}^{m} a_i Z_{t-i}^2 + \sum_{j=1}^{q} b_j W_{t-j}. \end{aligned}$$

From **B** we already know $\{Z_t^2\}$ is stationary. Next to this, $\{W_t\}$ is white noise and $\alpha_p, \beta_q > 0$ from the definition of an GARCH(p,q) process. Hence $a_m, b_j > 0$ and with the definition of an ARMA process follows that $\{Z_t^2\}$ is an ARMA(m,q) process depending on m lags of Z_t^2, $Z_{t-1}^2, \ldots, Z_{t-m}^2$ and on q lags of W_t, $W_{t-1}, \ldots, W_{t-q}$.

D

Assume $p \geq 1$ and $\alpha_1 > 0$. Define $V_t := \alpha_1 W_{t-1}$ and $b_i^{'}$ as $b_i^{'} = \frac{\alpha_{i+1}}{\alpha_1}$ for $i = 1, \ldots, p-1$. (Dividing by α_1 is possible, because $\alpha_1 > 0$.) Notice that

$$\sum_{i=1}^{p} \alpha_i W_{t-i} = V_t + \sum_{i=2}^{p} \alpha_i W_{t-i} = V_t + \sum_{i=1}^{p-1} \alpha_{i+1} W_{t-1-i} = V_t + \sum_{i=1}^{p-1} b_i^{'} V_{t-i}.$$

Combining this equality with **C** gives

$$\begin{aligned}
h_t &= a_0 + \sum_{i=1}^{m} a_i Z_{t-i}^2 + \sum_{j=1}^{q} b_j W_{t-j} \\
&= a_0 + \sum_{i=1}^{m} a_i \left(h_{t-i} + W_{t-i}\right) + \sum_{j=1}^{q} b_j W_{t-j}
\end{aligned}$$

First consider the case where $p \geq q$. Using the definitions of a_i and b_j gives

$$\begin{aligned}
h_t &= a_0 + \sum_{i=1}^{p} a_i \left(h_{t-i} + W_{t-i}\right) + \sum_{j=1}^{q} b_j W_{t-j} \\
&= a_0 + \sum_{i=1}^{p} a_i h_{t-i} + \sum_{i=1}^{q} (a_i + b_i) W_{t-i} + \sum_{i=q+1}^{p} a_i W_{t-i} \\
&= a_0 + \sum_{i=1}^{p} a_i h_{t-i} + \sum_{i=1}^{q} \alpha_i W_{t-i} + \sum_{i=q+1}^{p} \alpha_i W_{t-i} \\
&= a_0 + \sum_{i=1}^{p} a_i h_{t-i} + \sum_{i=1}^{p} \alpha_i W_{t-i} \\
&= a_0 + \sum_{i=1}^{p} a_i h_{t-i} + V_t + \sum_{i=1}^{p-1} b_i^{'} V_{t-i}.
\end{aligned}$$

On the other hand, consider $p < q$,

$$\begin{aligned}
h_t &= a_0 + \sum_{i=1}^{q} a_i \left(h_{t-i} + W_{t-i}\right) + \sum_{j=1}^{q} b_j W_{t-j} \\
&= a_0 + \sum_{i=1}^{q} a_i h_{t-i} + \sum_{i=1}^{p} (a_i + b_i) W_{t-i} + \sum_{i=p+1}^{q} (a_i + b_i) W_{t-i} \\
&= a_0 + \sum_{i=1}^{q} a_i h_{t-i} + \sum_{i=1}^{p} \alpha_i W_{t-i} \\
&= a_0 + \sum_{i=1}^{q} a_i h_{t-i} + V_t + \sum_{i=1}^{p-1} b_i^{'} V_{t-i}.
\end{aligned}$$

Define $a_0^{'} := a_0$ and $a_i^{'} := a_i$ for $i = 1, \ldots, m$. Then

$$h_t = a_0^{'} + \sum_{i=1}^{m} a_i^{'} h_{t-i} + V_t + \sum_{i=1}^{p-1} b_i^{'} V_{t-i}.$$

Since $a_m, \alpha_p > 0$, we have $a_m^{'}, b_i^{'} > 0$. Moreover h_t depends on m lags of h_t, $h_{t-1}, \ldots, h_{t-m}$ and on $p-1$ lags of V_t, $V_{t-1}, \ldots, V_{t-p+1}$. So $\{h_t\}$ is an ARMA $(m, p-1)$ process if $\{V_t\}$ is white noise.

Notice that $\{V_t\}$ is white noise, because $\{W_t\}$ is white noise (**B**). So $V_t = \alpha_1 W_{t-1}$ is stationary and

$$\mathbb{E}\left[V_t\right] = \alpha_1 \mathbb{E}\left[W_{t-1}\right] = 0$$

and

$$\mathbb{C}\mathrm{ov}\left(V_{t+s}, V_t\right) = \alpha_1^2 \gamma_W(s) = 2\alpha_1^2 \gamma_h(0) 1_{s=0}.$$

We conclude that $\{V_t\}$ is white noise and therefore $\{h_t\}$ is an ARMA $(m, p-1)$ process.

Empirical Exercise

(1)

Just like in the last assignment, we vary both p and q between 0 and 5. Again, we choose the best model based on the AIC and we conclude that the model that best is the ARMA(4,4) model.

(2)

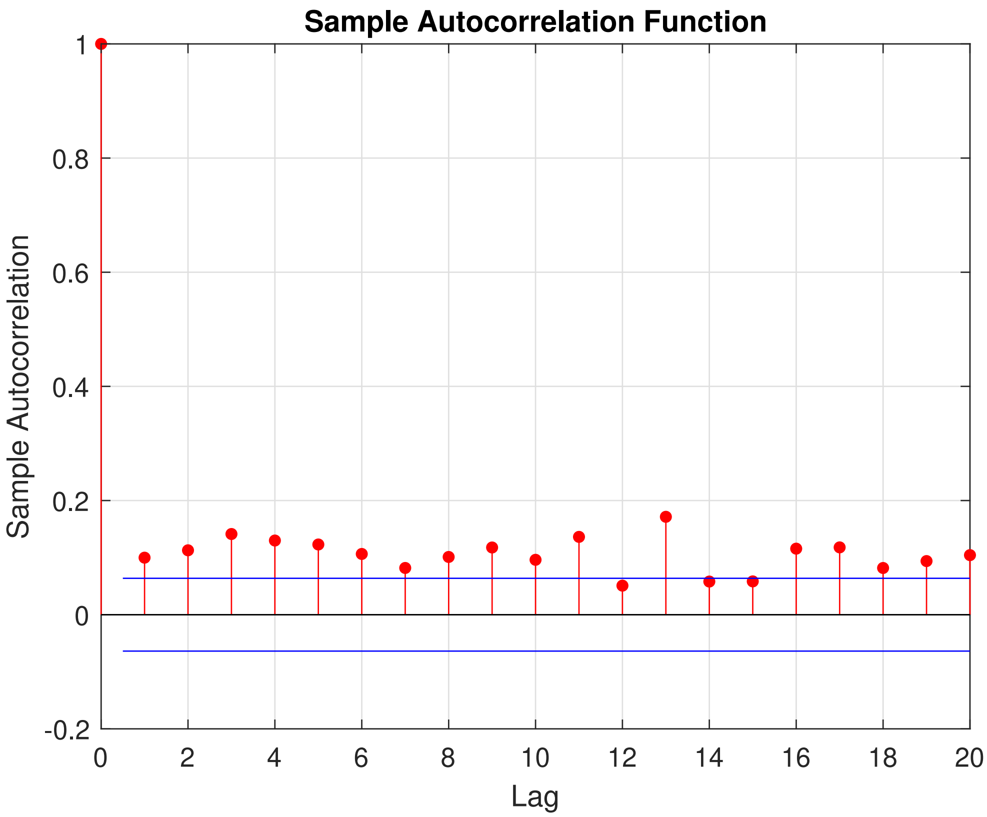

Figure 1: ACFs of the squared residuals

From Figure 1, we see that almost all ACFs lie outside of the confidence bands. Using Engle's Lagrange multiplier test, we test the null hypothesis that there are no ARCH effects in the data. In other words, we test

$$H_0 : \alpha_1 = ... = \alpha_m = 0$$

Using the Matlab *Archtest* command, we found a p-value of 0.0017. This means that, at the 5% level, we reject the null hypothesis that there are no ARCH effects in the data.

(3)

First, we took the ARMA(4,4) model from (1) as a starting point. We then specified the conditional variance of this model to be the Garch(2,2). We chose for the Garch(2,2) model as we intuitively expect it not to overfit the variance. The estimated coefficients as well as their standard errors are reported in Tables 1 and 2

	Value	Standard Error
Constant	0.0011	0.0007
AR1	-1.0847	0.1438
AR2	-0.3438	0.2601
AR3	0.5180	0.2492
AR4	0.6457	0.1233
MA1	1.0498	0.1308
MA2	0.2672	0.2328
MA3	-0.6073	0.2212
MA4	-0.7100	0.1093

Table 1: ARMA(4,4) estimated coefficients and standard errors

	Value	Standard Error
Constant	0.000005	0.000002
GARCH1	0	0.0984
GARCH2	0.7781	0.0904
ARCH1	0.0716	0.0252
ARCH2	0.1087	0.0238

Table 2: Garch(2,2) estimated coefficients and standard errors

(4)

Please see the Matlab file for the code, predictions will be visualized in (5).

(5)

Figure 2 shows the estimated conditional variance against the Proxy. We conclude that the chosen garch(2,2) model quite accurately estimates the underlying volatility. Figure 3 shows the forecasted conditional variance against the proxy. From this figure, we conclude that the prediction is somewhat accurate as it seems to capture the upward trend, but the prediction is always too low compared to the proxy.

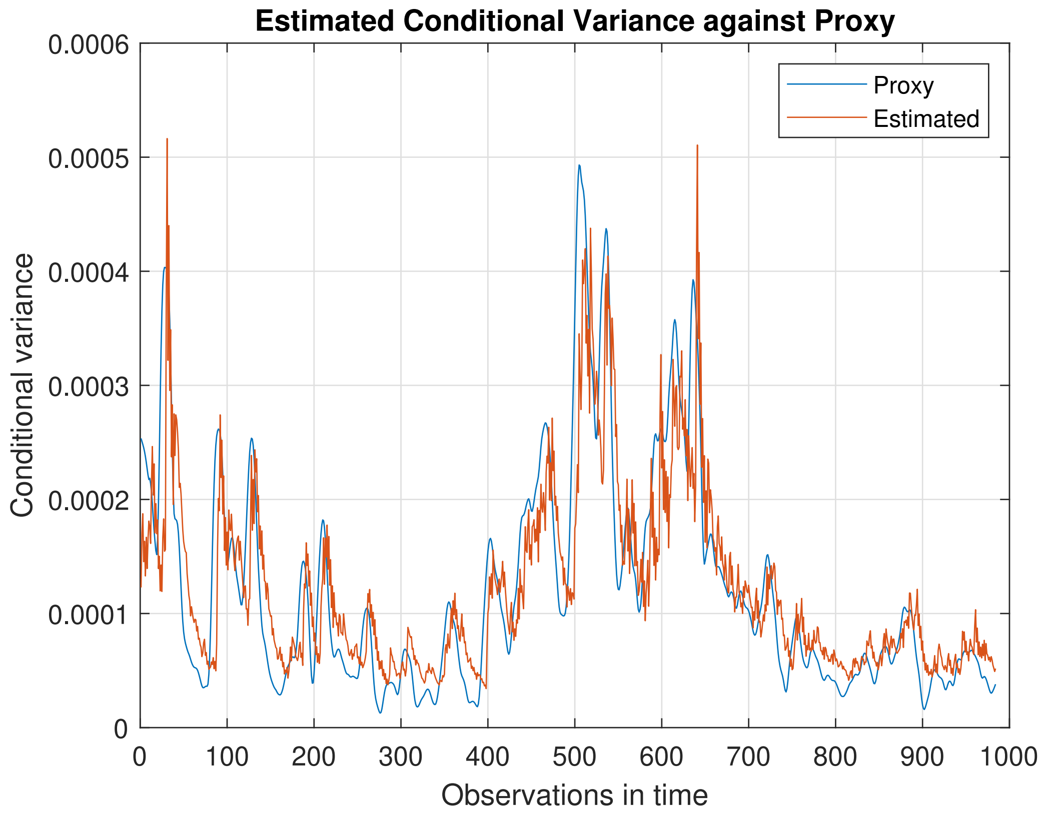

Figure 2: Estimated conditional variance against Proxy

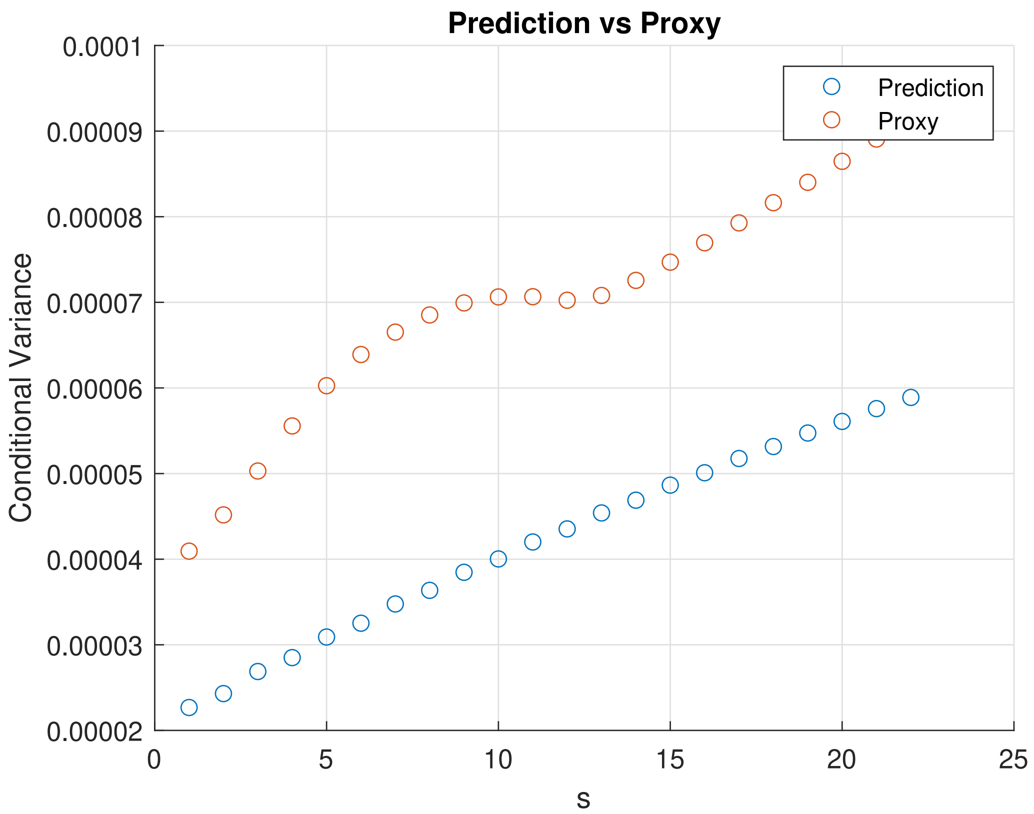

Figure 3: Prediction of conditional variance against Proxy