Time Series and their Applications

Homework Exercise 1

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Question 1

Let $Z_t \sim WN(0, \sigma^2)$. Determine whether the following processes are strictly/weakly stationary

A

$X_t = \xi + Z_t + Z_{t-1},\ t \in \mathbb{Z}$ with $\xi \sim N(0,1)$ independent of Z_t

$\mu_X(t) = E[X_t] = E[\xi + Z_t + Z_{t-1}] = E[\xi] + E[Z_t] + E[Z_{t-1}] = 0 + 0 + 0 = 0$

To calculate $\gamma_X(t,s)$, we use that $cov(Z_t, Z_s) = 0$ for $t \neq s$

$$
\begin{aligned}
\gamma_X(t, t+h) &= cov[X_t, X_{t+h}] = cov[\xi + Z_t + Z_{t-1},\ \xi + Z_{t+h} + Z_{t+h-1}] \\
&= cov[\xi,\xi] + cov[\xi, Z_{t+h}] + cov[\xi, Z_{t+h-1}] + cov[Z_t, \xi] + cov[Z_t, Z_{t+h}] + cov[Z_t, Z_{t+h-1}] \\
&+ cov[Z_{t-1}, \xi] + cov[Z_{t-1}, Z_{t+h}] + cov[Z_{t-1}, Z_{t+h-1}] \\
&= var[\xi] + cov[Z_t, Z_{t+h}] + cov[Z_t, Z_{t+h-1}] + cov[Z_{t-1}, Z_{t+h}] + cov[Z_{t-1}, Z_{t+h-1}] \\
&= \begin{cases} 1 + 2\sigma^2 & \text{if } h = 0 \\ 1 + \sigma^2 & \text{if } h = -1, 1 \\ 1 & \text{otherwise} \end{cases}
\end{aligned}
$$

So $\mu_X(t)$ does not depend on t and $\gamma_X(t, t+h)$ does not depend on t for every value of h, hence the process is weakly stationary.

Notice that ξ has a distribution independent of t. The same can be said about Z_t and Z_{t-1}. Therefore, the sum of distributions of ξ, Z_t and Z_{t-1} is independent of t. In other words, the distribution of X_t is independent of t. This means that X_t and X_{t+h} will be equal in distribution. Thus the process is also strictly stationary.

B

$X_t = t^{-\frac{1}{2}} \sum_{s=1}^{t} Z_s,\ t \in \mathbb{N}$

$\mu_X(t) = E[t^{-\frac{1}{2}} \sum_{s=1}^{t} Z_s] = t^{-\frac{1}{2}} \sum_{s=1}^{t} E[Z_s] = t^{-\frac{1}{2}} \sum_{s=1}^{t} 0 = 0$

$$
\begin{aligned}
\gamma_X(t,t+h) = cov[X_t, X_{t+h}] &= cov\left[t^{-\frac{1}{2}}\sum_{s=1}^{t} Z_s,\ (t+h)^{-\frac{1}{2}}\sum_{s=1}^{t+h} Z_s\right] \\
&= t^{-\frac{1}{2}}(t+h)^{-\frac{1}{2}} cov\left[\sum_{s=1}^{t} Z_s, \sum_{s=1}^{t+h} Z_s\right] \\
&= \frac{1}{\sqrt{t}}\frac{1}{\sqrt{t+h}} cov[Z_1 + Z_2 + ... + Z_t, Z_1 + Z_2 + ... + Z_t + ... + Z_{t+h}] \\
&= \frac{1}{\sqrt{t}}\frac{1}{\sqrt{t+h}} (var[Z_1] + var[Z_2] + ... + var[Z_t]) \\
&= \frac{1}{\sqrt{t}}\frac{1}{\sqrt{t+h}} \cdot t \cdot var[Z_1] \\
&= \sqrt{t}\frac{1}{\sqrt{t+h}}\sigma^2 \\
&= \sigma^2\sqrt{\frac{t}{t+h}}
\end{aligned}
$$

Although $\mu_X(t)$ does not depend on t, $\gamma_X(t, t+h)$ depends on t and so the process is *not* weakly stationary. Also, the value of t influences the distribution of X_t as it determines how many white noise processes are summed up and it determines the factor with which the total sum is multiplied. Therefore, the process is also *not* strictly stationary.

C

$X_t = 1 + (-1)^t Z_t$, $Z_t \sim N(0, \sigma^2)$, so in other words:

$$X_t = \begin{cases} 1 + Z_t & \text{if } t \text{ is even} \\ 1 - Z_t & \text{if } t \text{ is odd} \end{cases}$$

If t is even:

$\mu_X(t) = E[X_t] = E[1 + Z_t] = 1 + E[Z_t] = 1$, since $Z_t \sim N(0, \sigma^2)$

If t is odd:

$\mu_X(t) = E[X_t] = E[1 - Z_t] = 1 - E[Z_t] = 1$, since $Z_t \sim N(0, \sigma^2)$

So no matter if t is even or odd, in both cases we have $\mu_X(t) = 1$

$\gamma_X(t+h, t) = cov[X_{t+h}, X_t]$

If both t and $t+h$ are even, then $|t+h-t|$ is even and:

$cov[X_{t+h}, X_t] = cov[1 + Z_t, 1 + Z_t] = cov[Z_t, Z_t] = var[Z_t] = \sigma^2$

If both t and $t+h$ are odd, then $|t+h-t|$ is even and:

$cov[X_{t+h}, X_t] = cov[1 - Z_t, 1 - Z_t] = cov[-Z_t, -Z_t] = -cov[Z_t, -Z_t] = cov[Z_t, Z_t] = var[Z_t] = \sigma^2$

If either t even and $t+h$ odd or t odd and $t+h$ even, $|t+h-t|$ is odd and:

$cov[X_{t+h}, X_t] = cov[1 + Z_t, 1 - Z_t] = cov[Z_t, -Z_t] = -cov[Z_t, Z_t] = -var[Z_t] = -\sigma^2$

So $\gamma_X(t, t+h) = \begin{cases} \sigma^2 & \text{if } |t+h-t| \text{ is even} \\ -\sigma^2 & \text{if } |t+h-t| \text{ is odd} \end{cases}$

Which is equivalent to:

So $\gamma_X(t, t+h) = \begin{cases} \sigma^2 & \text{if } |h| \text{ is even} \\ -\sigma^2 & \text{if } |h| \text{ is odd} \end{cases}$

In conclusion, $\mu_X(t)$ does not depend on t and $\gamma_X(t, t+h)$ does not depend on t for every value of h, hence the process is weakly stationary.

To prove strict stationarity, we prove that X_t has the same distribution for each value of t.
Let $Z_t \sim N(0, \sigma^2)$ and $X_t = 1 + (-1)^t Z_t$.
If t is even: $X_t = 1 + Z_t \sim N(1, \sigma^2)$
If t is odd: $X_t = 1 - Z_t \sim N(1, \sigma^2)$, since $-Z_t \sim N(-1 \cdot 0,\ (-1)^2 \cdot \sigma^2) \sim N(0, \sigma^2)$ so X_t becomes $1 + N(0, \sigma^2) \sim N(1, \sigma^2)$).

So no matter the value for t, $X_t \sim N(1, \sigma^2)$. Therefore, X_t equals X_{t+h} in distribution and the process is strictly stationary

D

$X_t = (-1)^t Z_t$ where $Z_t = \begin{cases} -\frac{2}{3} & \text{with prob. } \frac{1}{3} \\ \frac{1}{3} & \text{with prob. } \frac{2}{3} \end{cases}$

Rewriting gives

$X_t = \begin{cases} Z_t & \text{if } t \text{ is even} \\ -Z_t & \text{if } t \text{ is odd} \end{cases}$

We can easily calculate that $E[Z_t] = -\frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = 0$ and

$$var[Z_t] = (-\frac{2}{3} - 0)^2 \cdot \frac{1}{3} + (\frac{1}{3} - 0)^2 \cdot \frac{2}{3} = \frac{4}{9} \cdot \frac{1}{3} + \frac{1}{9} \cdot \frac{2}{3} = \frac{4}{27} + \frac{2}{27} = \frac{2}{9}$$

If t is even:
$\mu_X(t) = E[X_t] = E[Z_t] = 0$
If t is odd:
$\mu_X(t) = E[X_t] = E[-Z_t] = -1 \cdot E[Z_t] = 0$

So no matter if t is even or odd, in both cases we have $\mu_X(t) = 0$

$\gamma_X(t+h, t) = cov[X_{t+h}, X_t]$

If both t and $t+h$ are even, then $|t+h-t|$ is even and:
$cov[X_{t+h}, X_t] = cov[Z_t, Z_t] = var[Z_t] = \frac{2}{9}$
If both t and $t+h$ are odd, then $|t+h-t|$ is even and:
$cov[X_{t+h}, X_t] = cov[-Z_t, -Z_t] = -cov[Z_t, -Z_t] = cov[Z_t, Z_t] = var[Z_t] = \frac{2}{9}$
If either t even and $t+h$ odd or t odd and $t+h$ even, $|t+h-t|$ is odd and:
$cov[X_{t+h}, X_t] = cov[Z_t, -Z_t] = -cov[Z_t, Z_t] = -var[Z_t] = -\frac{2}{9}$

So $\gamma_X(t, t+h) = \begin{cases} \frac{2}{9} & \text{if } |t+h-t| \text{ is even} \\ -\frac{2}{9} & \text{if } |t+h-t| \text{ is odd} \end{cases}$

Which is equivalent to:

So $\gamma_X(t, t+h) = \begin{cases} \frac{2}{9} & \text{if } |h| \text{ is even} \\ -\frac{2}{9} & \text{if } |h| \text{ is odd} \end{cases}$

In conclusion, $\mu_X(t)$ does not depend on t and $\gamma_X(t, t+h)$ does not depend on t for every value of h, hence the process is weakly stationary.

Since $-Z_t = \begin{cases} \frac{2}{3} & \text{with prob. } \frac{1}{3} \\ -\frac{1}{3} & \text{with prob. } \frac{2}{3} \end{cases}$ clearly is a different discrete distribution than Z_t, the distribution of X_t depends on t (i.e. whether t is even or odd). This means that X_t is not necessarily equal in distribution to X_{t+h} as either t or $t+h$ may be even (odd) while the other may be odd (even), leading to different distributions. Therefore, the process is not *not* strictly stationary.

E

$X_t = Y_t^2$ where Y_t is a stationary Gaussian process.

The process $\{X_t\}$ is strictly stationary, because we have

$$\begin{aligned}\mathbb{P}(X_1 \leq x_1, \ldots, X_t \leq x_t) &= \mathbb{P}(Y_1^2 \leq x_1, \ldots, Y_t^2 \leq x_t) \\ &= \mathbb{P}(-\sqrt{x_1} \leq Y_1 \leq \sqrt{x_1}, \ldots, -\sqrt{x_t} \leq Y_t \leq \sqrt{x_t}).\end{aligned}$$

From stationarity follows

$$\begin{aligned}&= \mathbb{P}(-\sqrt{x_1} \leq Y_{1+h} \leq \sqrt{x_1}, \ldots, -\sqrt{x_t} \leq Y_{t+h} \leq \sqrt{x_t}) \\ &= \mathbb{P}(Y_{1+h}^2 \leq x_1, \ldots, Y_{t+h}^2 \leq x_t) \\ &= \mathbb{P}(X_{1+h} \leq x_1, \ldots, X_{t+h} \leq x_t)\end{aligned}$$

and therefore the process $\{X_t\}$ is strictly stationary.

Consider next $\mathbb{E}[X_1^2] = \mathbb{E}[Y_1^4]$. We know that $\{Y_t\}$ is Gaussian and therefore only depends on the first and second moments, which are finite because of the stationarity. As a result, $\mathbb{E}[X_1^2] = \mathbb{E}[Y_1^4]$ is finite and we conclude with the strict stationarity, that $\{X_t\}$ is weakly stationary too.

Question 2

We consider the random walk with drift model:

$$X_t = \delta + X_{t-1} + Z_t, \quad t \in \mathbb{N}, \quad X_0 = 0, \quad Z_t \sim WN(0, \sigma^2)$$

A

$$\begin{aligned}X_t &= \delta + X_{t-1} + Z_t = \delta + (\delta + X_{t-2} + Z_{t-1}) + Z_t = \delta + \delta + (\delta + X_{t-3} + Z_{t-2}) + Z_t + Z_{t-1} \\ &= ... = \delta(t-1) + (\delta + X_0 + Z_1) + Z_t + Z_{t-1} + ... + Z_2 = \delta t + \sum_{s=1}^{t} Z_s + X_0 = \delta t + \sum_{s=1}^{t} Z_s\end{aligned}$$

B

$$\mu_X(t) = \mathbb{E}(\delta t + \sum_{s=1}^{t} Z_s) = \delta t + \sum_{s=1}^{t} \mathbb{E}(Z_s) = \delta t + \sum_{s=1}^{t} 0 = \delta t$$

$$\gamma_X(t,s) = Cov(X_t, X_s) = \mathbb{E}\left[(X_t - \mu_X(t))(X_s - \mu_X(s))\right] = \mathbb{E}\left[(\delta t + \sum_{i=1}^{t} Z_i - \delta t)(\delta s + \sum_{i=1}^{s} Z_i - \delta s)\right]$$

$$= \mathbb{E}\left[(\sum_{i=1}^{t} Z_i)(\sum_{i=1}^{s} Z_i)\right] = Cov((\sum_{i=1}^{t} Z_i), (\sum_{i=1}^{s} Z_i)) + \mathbb{E}(\sum_{i=1}^{t} Z_i) * \mathbb{E}(\sum_{i=1}^{s} Z_i) = Var(Z_1) + Var(Z_2)$$

$$+ Var(Z_3) + \ldots + Var(Z_t) + 0 = t\sigma^2 \text{ for } t \leq s$$

Via the same reasoning, we conclude that $\gamma_X(t,s) = s\sigma^2$ for $t > s$
The process is not stationary, since $\mu_X(t)$ and $\gamma_X(t,s)$ are dependent on t.

C

$$Corr(X_{t+1}, X_t) = Corr(\delta + X_t + Z_{t+1}, X_t) = \frac{Cov(\delta + X_t + Z_{t+1}, X_t)}{\sqrt{Var(\delta + X_t + Z_{t+1})Var(X_t)}}$$

$$= \frac{Cov(Z_{t+1}, X_t) + Var(X_t)}{\sqrt{\sigma^2(t+1)\sigma^2 t}} = \frac{Cov(Z_{t+1}, \delta t + \sum_{s=1}^{t} Z_s) + \sigma^2 t}{\sigma^2\sqrt{t^2+t}}$$

$$= \frac{\mathbb{E}\left[(Z_{t+1})(\delta t + \sum_{s=1}^{t} Z_s - \delta t)\right] + \sigma^2 t}{\sigma^2\sqrt{t^2+t}} = \frac{0 + \sigma^2 t}{\sigma^2\sqrt{t^2+t}} = \frac{t}{\sqrt{t^2+t}}$$

$$= \frac{\frac{t}{t}}{\frac{\sqrt{t^2+t}}{t}} = \frac{1}{\sqrt{\frac{t^2+t}{t^2}}} = \frac{1}{\sqrt{1+t^{-1}}} \xrightarrow{t\to\infty} \frac{1}{\sqrt{1+0}} = 1$$

When a correlation is 1, it implies that there is a perfect increasing linear relationship. Thus when $t \to \infty$, this result implies that we can predict with very high accuracy the value of X_{t+1} given the value of X_t as the points will lie on a straight line.

D

The process $\tilde{X}_t = 0.5X_t - \delta = 0.5X_{t-1} + 0.5Z_t$ is a transformation of of X_t. We can rewrite this process to the following:

$$\tilde{X}_t = \sum_{s=0}^{t-1} 0.5^{s+1} Z_{t-s}$$

Then we get the following mean function and ACVF:

$$\mu_{\tilde{X}}(t) = \sum_{s=0}^{t-1} 0.5^{s+1}\mathbb{E}(Z_{t-s}) = 0$$

$$\gamma_{\tilde{X}}(1) = Cov(X_{t-1}, X_t) = Cov(X_{t-1}, 0.5X_{t-1} + 0.5Z_t) = 0.5Cov(X_{t-1}, X_{t-1}) = 0.5\gamma_{\tilde{X}}(0)$$

$$\gamma_{\tilde{X}}(2) = Cov(X_{t-2}, X_t) = Cov(X_{t-2}, 0.5X_{t-1} + 0.5Z_t) = 0.5Cov(X_{t-2}, X_{t-1}) = 0.5\gamma_{\tilde{X}}(1) = 0.5^2\gamma_{\tilde{X}}(0)$$

$$\vdots$$

$$\gamma_{\tilde{X}}(h) = 0.5^h\gamma_{\tilde{X}}(0)$$

In order to account for negative values of h, we should take the absolute value of h. And using that:

$$\gamma_{\tilde{X}}(0) = Var(X_t) = Var(0.5X_{t-1} + 0.5Z_t) = 0.5^2 Var(X_t) + 0.5^2 Var(Z_t)$$
$$\leftrightarrow Var(X_t) = \frac{0.5^2}{1-0.5^2} Var(Z_t) = \frac{\sigma^2}{3}$$

Thus both the mean function and ACVF are independent of t, and thus is stationary.

Question 4

The naïve estimator $\tilde{v}$ of the long-run variance of $\{X_t\}$ is defined as

$$\tilde{v} = \hat{\gamma}_X(0) + 2\sum_{h=1}^{T-1} \hat{\gamma}_X(h)$$

where

$$\hat{\gamma}_X(h) = \frac{1}{T}\sum_{t=1}^{T-h}(X_t - \bar{X}_T)(X_{t+h} - \bar{X}_T).$$

From this definition follows

$$\hat{\gamma}_X(0) = \frac{1}{T}\sum_{t=1}^{T}(X_t - \bar{X}_T)^2$$

and hence

$$\tilde{v} = \frac{1}{T}\sum_{t=1}^{T}(X_t - \bar{X}_T)^2 + 2\sum_{h=1}^{T-1}\frac{1}{T}\sum_{t=1}^{T-h}(X_t - \bar{X}_T)(X_{t+h} - \bar{X}_T).$$

Define $Y_t := X_t - \bar{X}_T$, then

$$\begin{aligned}
\tilde{v} &= \frac{1}{T}\sum_{t=1}^{T} Y_t^2 + \frac{2}{T}\sum_{h=1}^{T-1}\sum_{t=1}^{T-h} Y_t Y_{t+h} \\
&= \frac{1}{T}\sum_{t=1}^{T} Y_t^2 + \frac{2}{T}\left(\sum_{t=1}^{T-1} Y_t Y_{t+1} + \sum_{t=1}^{T-2} Y_t Y_{t+2} + \cdots + \sum_{t=1}^{T-(T-1)} Y_t Y_{T-1}\right) \\
&= \frac{1}{T}\sum_{t=1}^{T} Y_t^2 + \frac{2}{T}\left((Y_1Y_2 + \cdots + Y_{T-1}Y_T) + (Y_1Y_3 + \cdots + Y_{T-2}Y_T) + \cdots + (Y_1Y_{T-1})\right) \\
&= \frac{1}{T}\Big((Y_1^2 + 2Y_1Y_2 + 2Y_1Y_3 + \cdots + 2Y_1Y_T) + (Y_2^2 + 2Y_2Y_3 + 2Y_2Y_4 + \cdots + 2Y_2Y_T) + \cdots + (Y_T^2)\Big) \\
&= \frac{1}{T}(Y_1 + \cdots + Y_T)^2
\end{aligned}$$

As a result,

$$\tilde{v} = \frac{1}{T}\left(\sum_{t=1}^{T} Y_t\right)^2.$$

Notice that from the definition of Y_t and $\bar{X}_T$ follows

$$\sum_{t=1}^{T} Y_t = Y_1 + \cdots + Y_T = X_1 - \bar{X}_T + \cdots + X_T - \bar{X}_T = X_1 + \cdots + X_T - T\bar{X}_T = 0$$

and therefore we conclude

$$\tilde{v} = \frac{1}{T}0^2 = 0.$$

Question 5

We consider the AR(1) process,

$$X_t = \phi X_{t-1} + Z_t, \quad t \in \mathbb{N}, \quad |\phi| < 1, \quad X_0 \sim N(0, \frac{\sigma^2}{1-\phi^2}) \text{ independent of } Z_t \sim WN(0, \sigma^2)$$

A

$$\begin{aligned}
X_t &= \phi X_{t-1} + Z_t \\
X_{t+1} &= \phi X_t + Z_{t+1} \\
X_{t+2} &= \phi(\phi X_t + Z_{t+1}) = \phi^2 X_t + \phi Z_{t+1} + Z_{t+2} \\
X_{t+3} &= \phi(\phi^2 X_t + \phi Z_{t+1} + Z_{t+2}) + Z_{t+3} = \phi^3 X_t + \phi^2 Z_{t+1} + \phi Z_{t+2} + Z_{t+3} \\
&\vdots \\
X_{t+h} &= \phi^h X_t + \sum_{s=0}^{t-1} \phi^s Z_{t+h-s}
\end{aligned}$$

$$\begin{aligned}
Var(X_t) = Var(\phi X_{t-1} + Z_t) = \phi^2 Var(X_{t-1}) + Var(Z_t) &= \phi^2 Var(X_t) + Var(Z_t) \\
\leftrightarrow Var(X_t)(1-\phi^2) &= \sigma^2 \\
\leftrightarrow Var(X_t) &= \frac{\sigma^2}{1-\phi^2}
\end{aligned}$$

Using this we can calculate the ACVF of X_t:

$$\begin{aligned}
\gamma_X(h) = Cov(X_t, X_{t+h}) &= Cov(X_t, \phi^h X_t + \sum_{s=0}^{t-1} \phi^s Z_{t+h-s}) = Cov(X_t, \phi^h X_t) + Cov(X_t, \sum_{s=0}^{t-1} \phi^s Z_{t+h-s}) \\
&= Cov(X_t, \phi^h X_t) = \phi^h Var(X_t) = \phi^h \frac{\sigma^2}{1-\phi^2}
\end{aligned}$$

However, to also take negative values of h into account, we need to take the absolute value:

$$\gamma_X(h) = \phi^{|h|} \frac{\sigma^2}{1-\phi^2}$$

B

According to the Theorem on page 6 of the Lecture notes of lecture 2, we have the following:

Suppose that X_t is a linear process of the form

$$X_t = \mu + \sum_{s=-\infty}^{\infty} \psi_s Z_{t-s}$$

with $Z_t \sim IID(0, \sigma^2), \sum_{s=-\infty}^{\infty} |\psi_s| < \infty$, and $\sum_{s=\infty}^{\infty} s\psi_s^2 < \infty$. Then, for each $h \geq 1$,

$$\sqrt{T}(\hat{\rho}_X(h) - \rho_X(h)) \xrightarrow{d} N(0, V_h),$$

where the asymptotic variance V_h is given by

$$V_h = \sum_{s=1}^{\infty} (\rho_X(s+h) + \rho_X(s-h) - 2\rho_X(s)\rho_X(h))^2$$

Now we have the process $X_t = \sum_{s=0}^{\infty} \phi^s Z_{t-s}$ with $|\phi|< 1$ and $Z_t \sim IID(0, \sigma^2)$ and the ACVF from part A. Thus we get:

$$\sum_{s=0}^{\infty} |\phi^s| = \frac{1}{1-\phi} < \infty$$
$$\sum_{s=0}^{\infty} s\phi^{2s} = \sum_{s=1}^{\infty} \phi^{2s} + \sum_{s=2}^{\infty} \phi^{2s} + \sum_{s=3}^{\infty} \phi^{2s} + \ldots = \phi^2 \sum_{s=0}^{\infty} \phi^{2s} + \phi^4 \sum_{s=0}^{\infty} \phi^{2s} + \phi^6 \sum_{s=0}^{\infty} \phi^{2s} + \ldots$$
$$= (\phi^2 + \phi^4 + \phi^6 + \ldots) \sum_{s=0}^{\infty} \phi^{2s} = \frac{1}{1-\phi^2} \sum_{s=1}^{\infty} \phi^{2s} = \frac{\phi^2}{1-\phi^2} \sum_{s=0}^{\infty} \phi^{2s} = \frac{\phi^2}{(1-\phi^2)^2} < \infty$$

To calculate the summations above, we used the sum of geometric series and the fact that $|\phi|< 1$. The ACF is given by:

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \frac{\phi^{|h|}\sigma^2}{1-\phi^2} \cdot \frac{1-\phi^2}{\phi^0 \sigma^2} = \phi^{|h|}$$

Then the asymptotic variance V_1 is equal to

$$V_1 = \sum_{s=1}^{\infty} (\rho_X(s+1) + \rho_X(s-1) - 2\rho_X(s)\rho_X(1))^2 = \sum_{s=1}^{\infty} (\phi^{s+1} + \phi^{s-1} - 2\phi^s\phi)^2$$
$$= \sum_{s=1}^{\infty} (\phi^{s-1} - \phi^{s+1})^2 = \sum_{s=1}^{\infty} \phi^{2s-2} + \sum_{s=1}^{\infty} \phi^{2s+2} - 2\sum_{s=1}^{\infty} \phi^{2s} = \sum_{s=0}^{\infty} \phi^{2s} + \phi^4 \sum_{s=0}^{\infty} \phi^{2s} - 2\phi^2 \sum_{s=0}^{\infty} \phi^{2s}$$
$$= \frac{1}{1-\phi^2}(1 - 2\phi^2 + \phi^4) = \frac{(1-\phi^2)^2}{1-\phi^2} = 1 - \phi^2 = 1 - \rho_X^2(1)$$

Thus, using the theorem, we get:

$$\sqrt{T}(\hat{\rho}_X(1) - \rho_X(1)) \xrightarrow{d} N(0, 1 - \rho_X^2(1))$$

Which is equivalent to:

$$\frac{\sqrt{T}(\hat{\rho}_X(1) - \rho_X(1))}{\sqrt{1 - \rho_X^2(1)}} \xrightarrow{d} N(0, 1)$$

C

The $(1-\alpha)\%$confidence interval of ϕ $(= \hat{\rho}_x(1))$ is given by:

$$CI = \left[\hat{\rho}_x(1) - z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{v}}{T}}, \hat{\rho}_x(1) + z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{v}}{T}}\right]$$

where $z_{1-\frac{\alpha}{2}}$ is the $1 - \frac{\alpha}{2}$ quantile of the standard normal distribution. Using the fact that $\hat{\rho}_x(1)$ $= 0.64$, $T = 100$, and $z_{1-\frac{\alpha}{2}} = z_0.975 = 1.96$, we get:

$$CI = \left[\hat{\rho}_x(1) - z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{v}}{T}}, \hat{\rho}_x(1) + z_{1-\frac{\alpha}{2}}\sqrt{\frac{\hat{v}}{T}}\right] = \left[\hat{\rho}_x(1) - z_{1-\frac{\alpha}{2}}\sqrt{\frac{1-\hat{\rho}_x^2(1)}{T}}, \hat{\rho}_x(1) + z_{1-\frac{\alpha}{2}}\sqrt{\frac{1-\hat{\rho}_x^2(1)}{T}}\right]$$

$$= \left[0.64 - 1.96\sqrt{\frac{1-0.64^2}{100}}, 0.64 + 1.96\sqrt{\frac{1-0.64^2}{100}}\right] = [0.49, 0.79]$$

Empirical Exercise

(1)

Using the given Matlab functions in the exercise, we arrive at a mean of 0.0010 and a variance of 0.0007. The values for the ACFs $\rho_X(h)$ are given in Table 1. Notice that the sign of the ACFs is alternating for many lags. Since the variance is always positive, this indicates that a negative covariance is often followed by a positive covariance for the next lag and vice versa. Next, the ACF seems to decrease for larger lags.

h	ACF	h	ACF	h	ACF
0	1	7	0.0280	14	0.0007
1	-0.0382	8	-0.0795	15	0.0246
2	0.0435	9	0.0267	16	0.0466
3	-0.0483	10	-0.0437	17	-0.0068
4	0.0313	11	0.0277	18	0.0225
5	0.0488	12	-0.0195	19	-0.0647
6	-0.0491	13	0.0087	20	0.0003

Table 1: ACF values

(2)

We arrive at a p-value of $0.6386 > 0.05$. This means that we do not reject the null hypothesis that all autocorrelation functions are 0. There is no statistically significant autocorrelation present in the returns.

(3)

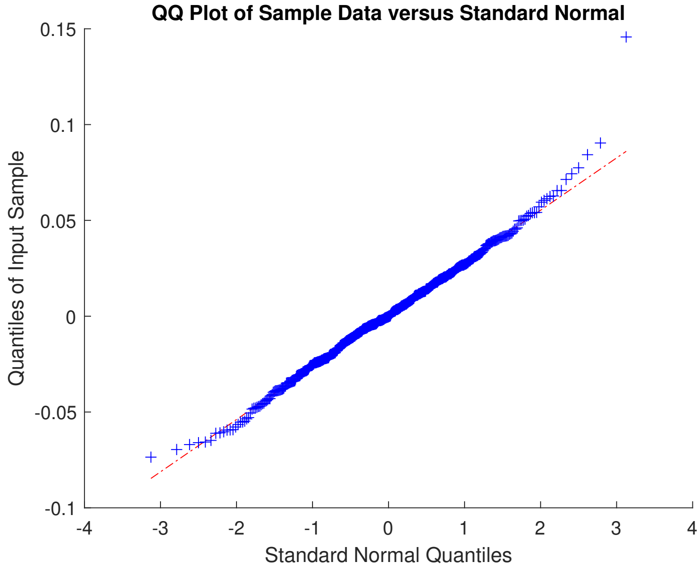

Figure 1: Quantiles of return compared to quantile of normal distribution

Although there are a few outliers in the plot, especially for larger standard normal quantiles, most quantiles of the plotted quantiles coincide with the standard normal quantiles. Hence the series appears to be the realization of Gaussian white noise.

(4)

Now the p-value $\approx 0.0000 < 0.05$. This means that we reject the null hypothesis that $\rho_X(1) = ... = \rho_X(20) = 0$ for the absolute differences. Hence there is statistically significant autocorrelation present in the absolute values of the returns.

(5)

We calculated the R statistic to be $R = 53874$. We also concluded that $\mu_R = \frac{T\cdot(T-1)}{4} = 39900$ and $\sigma_R^2 = \frac{2\cdot(T-1)\cdot(2T+5)}{72} \approx 1784416.67$. Since it was indicated that large values of $R - \mu_R$ indicate the presence of an increasing trend, we constructed the following hypotheses and selection criteria.

$$H_0 : R - \mu_R = 0$$
$$H_1 : R - \mu_R \neq 0$$

We used the following test statistic

$$Z = \frac{R-\mu_R-0}{\sigma_R/\sqrt{n}}$$

And we reject the null hypothesis when $|Z| > 1.96$, i.e. a two-sided test at the 5% significance level. From the data we calculated $|Z| \approx 209.22 > 1.96$, hence we reject the null hypothesis that there is no linear trend in the data.

(6)

We fit a linear trend to the data, which is determined to be of the form $y = 1.3306 + 0.0032 \cdot x$ after which the corresponding residuals are calculated. We apply the same logic as in exercise (5) [1], where we now calculate the new statistic $R_{\text{residuals}} = 40124$ and test whether it is significantly different from zero using the same test as in (5). $Z_{\text{residual}} = 3.3537 > 1.96$, meaning that we again reject the null hypothesis that there is no linear trend in the data. Therefore, the conclusion does not change, although we would have expected that there would *not* be a linear trend in the data based on a plot of the residuals

(7)

In the previous exercise, we concluded that there is a linear trend in the residuals. Based on this, the series cannot be the realization of an i.i.d. process, as it would require independent errors. The linear trend in the errors indicates that the errors are not independent.

[1] Notice that the parameters μ_R and σ_R^2 do not change for the residuals