Quantitative Finance Assignment

by group 17:

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Chapter 1

Question 1

Consider the following option on the AEX-index. If you buy the option, at time $t = 0$, you get the following payoff at expiration date $T = 5$ (years):

$$C_T = max\left(K - \frac{1}{T}\sum_{t=1}^{T} S_t, 0\right)$$

where $K = 740$ and where S denotes the level of the AEX-index.

1.1 Question A

Assume that the level of the AEX-index is described by a geometric Brownian motion. Estimate the parameters (of the geometric Brownian motion) on basis of historical data.

We utilized data concerning the AEX-index covering the period from December 15, 2021, to December 13, 2023 and put this in the csv file "`AEX_group17.csv`" that we have attached. However, we are missing several daily data points within these two years, due to the stock market not being operative on the weekends and holidays. We eventually have 515 data points to work with, so T = 515 (days). Through an analysis of this historical data, the estimated parameters for a geometric Brownian motion model were derived. First, we see that our starting value S_0 is equal to 787.02. The average return (drift) was determined to be $-2.8788 \cdot e^{-05}$, while the volatility was calculated at around 0.011168.

1.2 Question B

Assuming the Black-Scholes model for the financial market, use Monte-Carlo simulations to obtain a confidence interval for the price, at t = 0, of the option.

We have previously estimated our parameters using historical data on a daily basis. However for this question, we need the parameters to represent annual parameters and thus have to translate the the parameters. We will assume we have $515/2 \approx 257$ trading days in a year. From this we can calculate the annual volatility.

$$\sigma_{annual} = \sigma_{daily} \cdot \sqrt{\text{number of trading days}} = 0.011168 \cdot \sqrt{257} = 0.1790$$

Furthermore, the initial price of the risky asset S_0 remains the same, $S_0 = 787.02$, and we have $T = 5$ (years). Since the interest rate is not given, we looked up the risk free rate for a 5 year period starting on December 15, 2021, and we found $r = -0.567\%$ [1].

Using this data and running a Monte Carlo simulation with $n = 10,000$ number of simulations, we obtain a 95% confidence interval for the price at $t = 0$:

$$95\% \text{ confidence Interval} = (11.0268, 12.3672)$$

We see that the initial value of the stock at $t = 0$ is already above the strike price, meaning payoff of the option is likely to be zero. This explains the low values in the confidence interval.

1.3 Question C

Use the bump-and-reprice method to obtain an estimate and a confidence interval for the delta of this option at t = 0. Consider 'non-common random numbers' as well as 'common random numbers'.

Common numbers

We first used the bump-and-reprice method for the common random numbers. We run this simulation 10,000 times with a bump equal to 0.0001. From this we get several values of delta with a mean equal to: $\Delta = -0.2534$ and we obtain the following confidence interval for this delta of the option at $t = 0$:

$$95\% \text{ confidence Interval} = (-0.2607, -0.2461)$$

[1] https://www.worldgovernmentbonds.com/bond-historical-data/netherlands/5-years/

Independent numbers

After this, we looked at using the bump-and-reprice method for the non-common independent random numbers. We ran 10,000 simulations with bump $n^{-\frac{1}{4}}$ and got several values of delta with a mean equal to: $\Delta = 11.3921$. Furthermore, we obtained the following confidence interval for the delta of the option at $t = 0$:

$$95\% \text{ confidence Interval} = (-12.4110, 35.1952)$$

Chapter 2

Question 2

Consider, the Black-Scholes market with $\mu = 10\%$, $\sigma = 25\%$, $S_0 = 100$, $B_0 = 1$, and $r = 3\%$. We consider a European put option with strike price $K = 100$ and maturity $T = 2$.

2.1 Question A

Compute the delta of the option at $t = 0$. Also obtain numerical approximations by using bump-and-reprice, the pathwise method, and the likelihood ratio method.

We first calculate the delta using closed form formulas from the slides. This results in a delta equal to $\Delta = -0.3645$.

Bump-and-Reprice

The first way we used to approximate the delta of the option at $t = 0$ is the bump and reprice method. We use $10,000$ simulations with a bump of $h = 0.0001$. Using this method we obtain a delta equal to $\Delta = -0.3599$.

Pathwise Method

We also use the pathwise method to approximate the gamma at $t = 0$ and set the number of simulations to n = 10,000 and the bump to 0.0001. We now see our approximated delta is equal to $\Delta = 0.3912$. This delta vastly differs from the other ones that we have calculated so we suspect there is something wrong with our implementation, only we cannot find what it is.

Likelihood Ratio Method

Lastly, we use the Likelihood Ratio Method to approximate the delta of the option at $t = 0$. Here we again use 10,000 simulations with a bump of 0.0001 and we get

$\Delta = -0.3551$.

2.2 Question B

Compute the gamma of the option at $t = 0$. Obtain, if possible, numerical approximations by using bump-and-reprice, the pathwise method, and the likelihood ratio method.

We first calculate the gamma using closed form formulas. This results in a gamma equal to $\Gamma = 0.0106$.

Bump-and-Reprice

The first way we used to approximate the gamma of the option at $t = 0$ is the bump and reprice method. We use 10000 simulations with a 0.0001 bump. Using this method we obtain a gamma equal to $\Gamma = 0.0000$.

Pathwise Method

We cannot use the pathwise method to determine the gamma of the option. This is because the first derivative is not continuous (because of the indicator function) and therefore we cannot apply this method.

Likelihood Ratio Method

Lastly, we use the Likelihood Ratio Method to approximate the gamma of the option at $t = 0$. We use the following derivations:

$$
\begin{aligned}
\frac{\partial}{\partial S_0}\; log\; g(s, S_0) &= \frac{log(s/S_0) - (r - 0.5\sigma^2)T}{\sigma^2 T S_0} \text{ (from the slides)} \\
&= \frac{log(s) - log(S_0) - (r - 0.5\sigma^2)T}{\sigma^2 T S_0} \\
\frac{\partial^2}{\partial S_0^2}\; log\; g(s, S_0) &= \frac{-\sigma^2 T S_0 \frac{1}{S_0} - (log(s) - log(S_0) - (r - 0.5\sigma^2)T)\sigma^2 T}{\sigma^4 T^2 S_0^2} \\
&= \frac{-\sigma^2 T - (log(s) - log(S_0) - (r - 0.5\sigma^2)T)\sigma^2 T}{\sigma^4 T^2 S_0^2} \\
&= \frac{-1 - log(s) + log(S_0) + (r - 0.5\sigma^2)T)}{\sigma^2 T S_0^2}
\end{aligned}
$$

Here we again use 10,000 simulations and we get $\Gamma = -0.0049$.

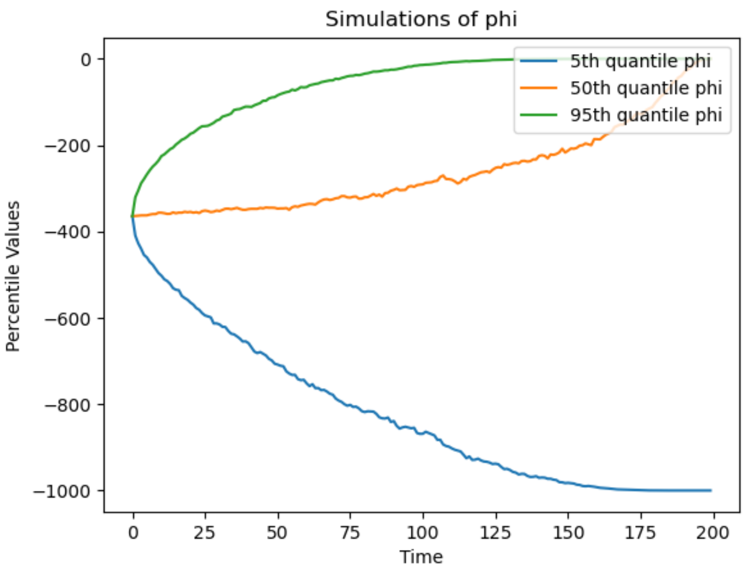
2.3 Question C

Assume that a financial institution has sold 1,000 puts and assume that these put options are not directly traded on financial markets and would like to hedge the risk.

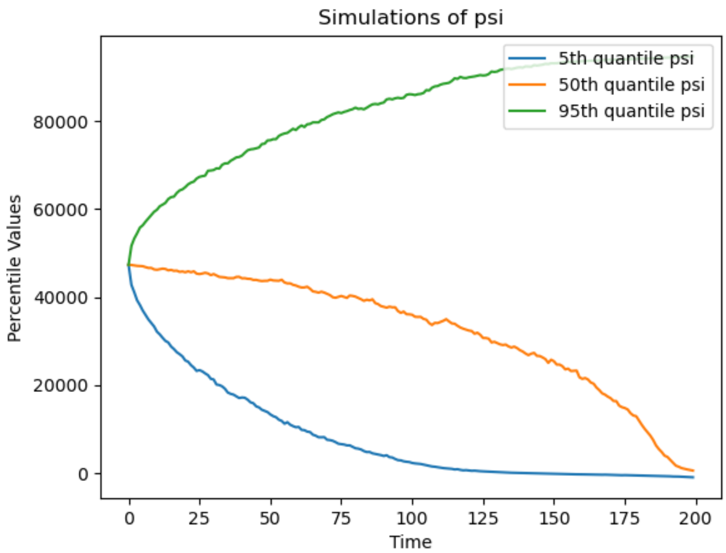
Report the precise positions in the stock and money-market account (at $t = 0$*). Next, generate at least 2,500 simulations of this strategy. Plot, for each point on the time grid, the* 5%*,* 50%*, and* 95% *quantiles of the positions in* S *and* B*. Also plot, for each point on the time grid, the* 5%*,* 50%*, and* 95% *quantiles of the gamma of the portfolio. Finally, we present the mean and standard deviation of the total portfolio value at maturity. What price would you charge for the option?*

We first look at the initial position in the stock and money-market at t = 0. We see that our initial position in the stock is equal to -364.4901, meaning that we start with a short position in the stock, and our initial position in the money-market account is equal to 47353.8913 at t = 0.

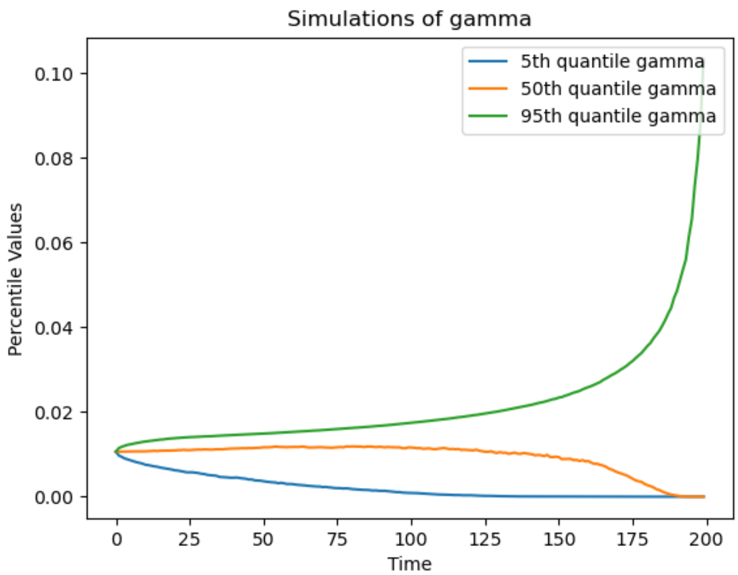
We now plot the 5%, 50% and 95% quantiles of ϕ,the positions in the stocks. In these plots, time represents the different time points. Since $T = 2$ and our increment in time is 0.01, we have 200 time points.



and we plot the 5%, 50% and 95% quantiles of ψ, the position in the money-market account.



After this, we also look at the 5%, 50% and 95% quantiles of the gamma of the portfolio.



Calculating the mean and the standard deviation of the portfolio gives us a mean equal to 10.3130 and a standard deviation equal to 872.2347 We also see the price for one put option is equal to 10.9049, and thus the price for 1,000 puts is equal to 10904.8780.

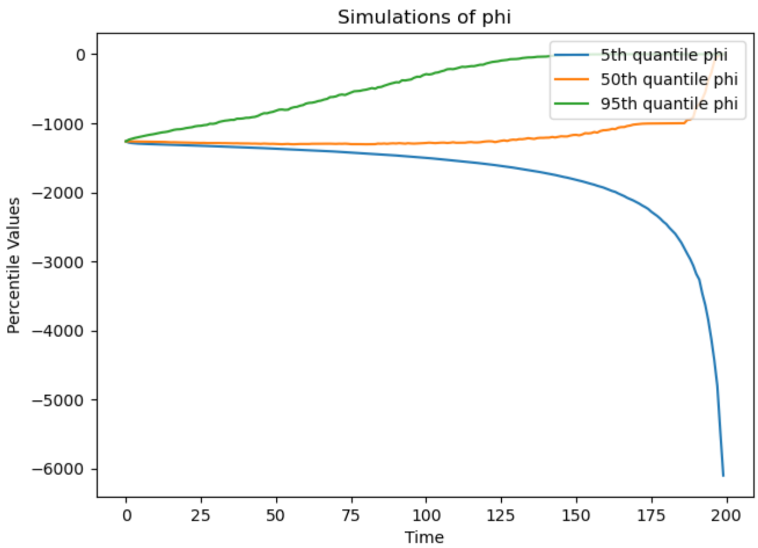
2.4 Question D

2.4.1 Question Di

Suppose now that a call option with strike price $K = 120$ and $T = 5$ is also traded. Repeat exercise (2C) using delta-gamma hedging. Interpret the results.

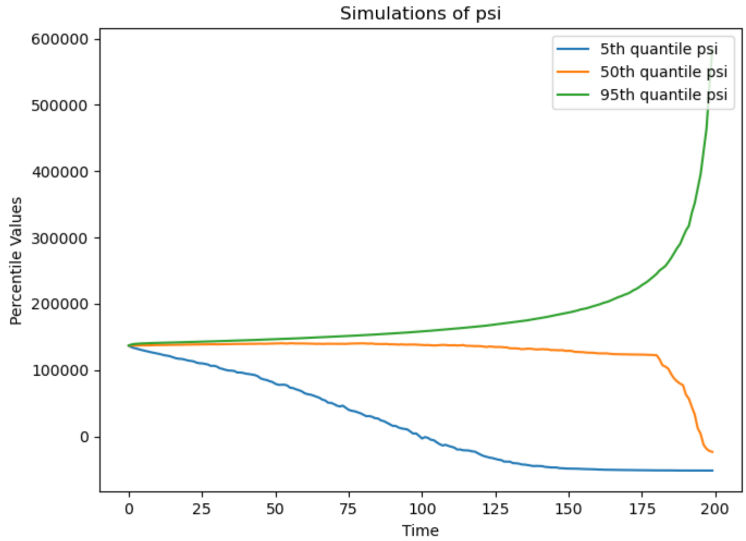
In addition to the 1000 put options, we now also have a call option. This means that we can use this call option to make our portfolio delta-gamma-neutral. The first step is to buy or sell call options in order to compensate for the gamma we had from the put options. This leads to an increase or decrease in delta, which we have to compensate for by changing the position in stocks. This shift in the stocks position does not affect the gamma.
We first look at the initial position is the stock and money-market at t = 0. We see that our initial position in the stock is equal to -1261.3936, meaning that we start with a short position in the stock, and our initial position in the money-market account is equal to 137044.2394 at t = 0.

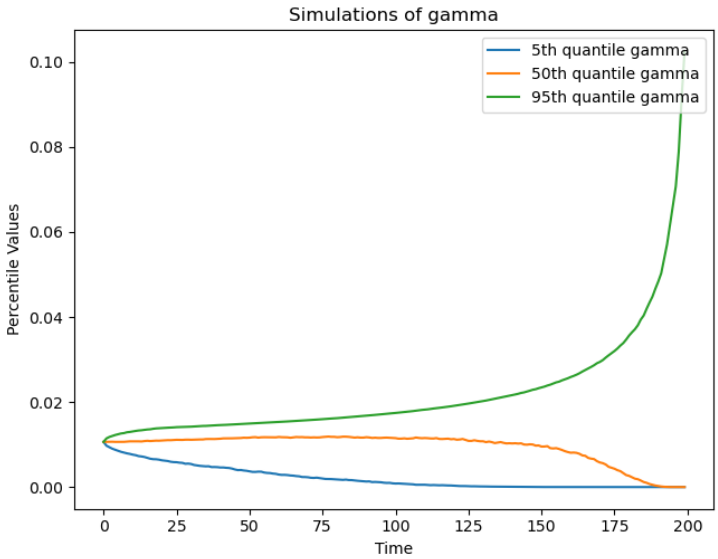
We now plot the 5%, 50% and 95% quantiles of ϕ, the positions in the stocks:



and we plot the 5%, 50% and 95% quantiles of ψ, the position in the money-market account.
After this, we also look at the 5%, 50% and 95% quantiles of the gamma of the portfolio.

Calculating the mean and the standard deviation of the portfolio gives us a mean equal to -12423.8271 and a standard deviation equal to 34101.6384. We also see the





price for one call option is equal to 20.7701, and thus the price for 1,000 puts is equal to 2077.0067.

We see for the ψ that the 95% quantile is increasing much slower, for the ϕ we notice that the 5% quantile is decreasing much slower. For γ we see similar results as before. This is unexpected as we would have expected γ to be close to 0. Although γ will never be completely 0 due to discrete-time hedging, we still expected γ to be lower. Therefore, we suspect there is something wrong with our implementation.

2.4.2 Question Dii

We were not able to come up with a solution for this exercise.