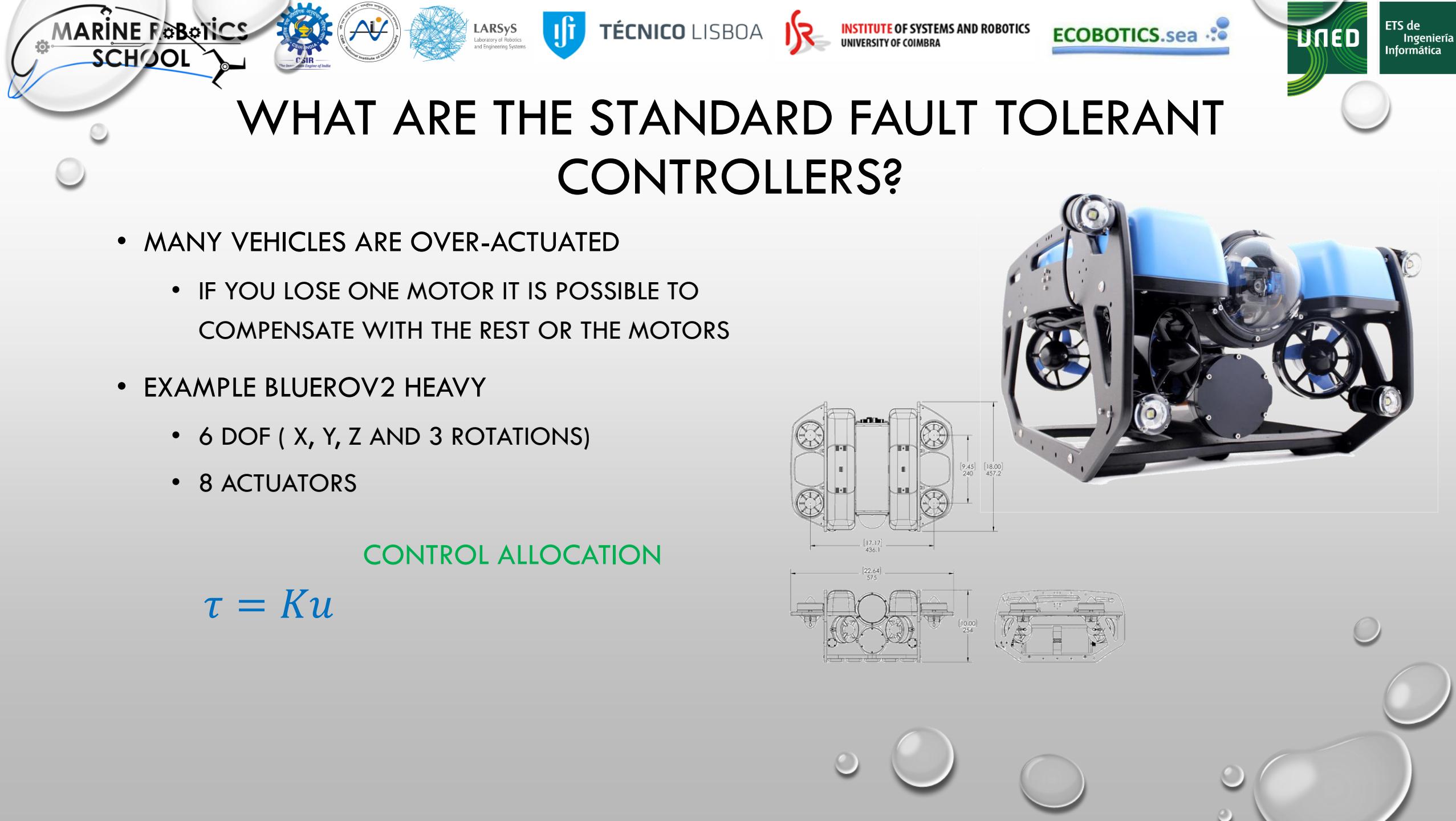


ENERGY-EFFICIENT HOMING CONTROL OF UNDERACTUATED MARINE VEHICLES UNDER FAULTY ACTUATOR CONDITIONS

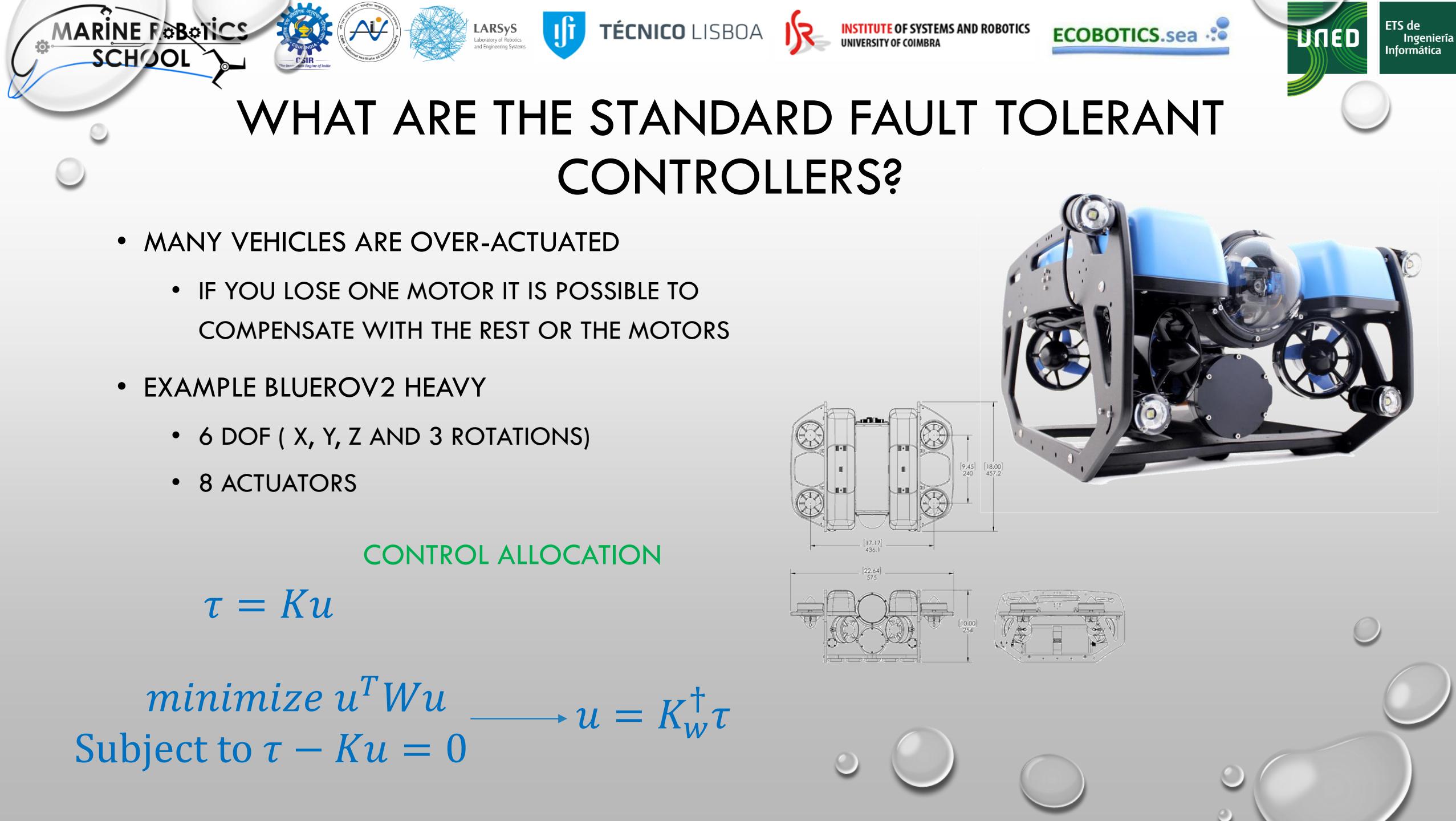
DICTINO CHAOS GARCÍA



- MANY VEHICLES ARE OVER-ACTUATED
 - IF YOU LOSE ONE MOTOR IT IS POSSIBLE TO COMPENSATE WITH THE REST OR THE MOTORS
- EXAMPLE BLUEROV2 HEAVY
 - 6 DOF (X, Y, Z AND 3 ROTATIONS)
 - 8 ACTUATORS

CONTROL ALLOCATION

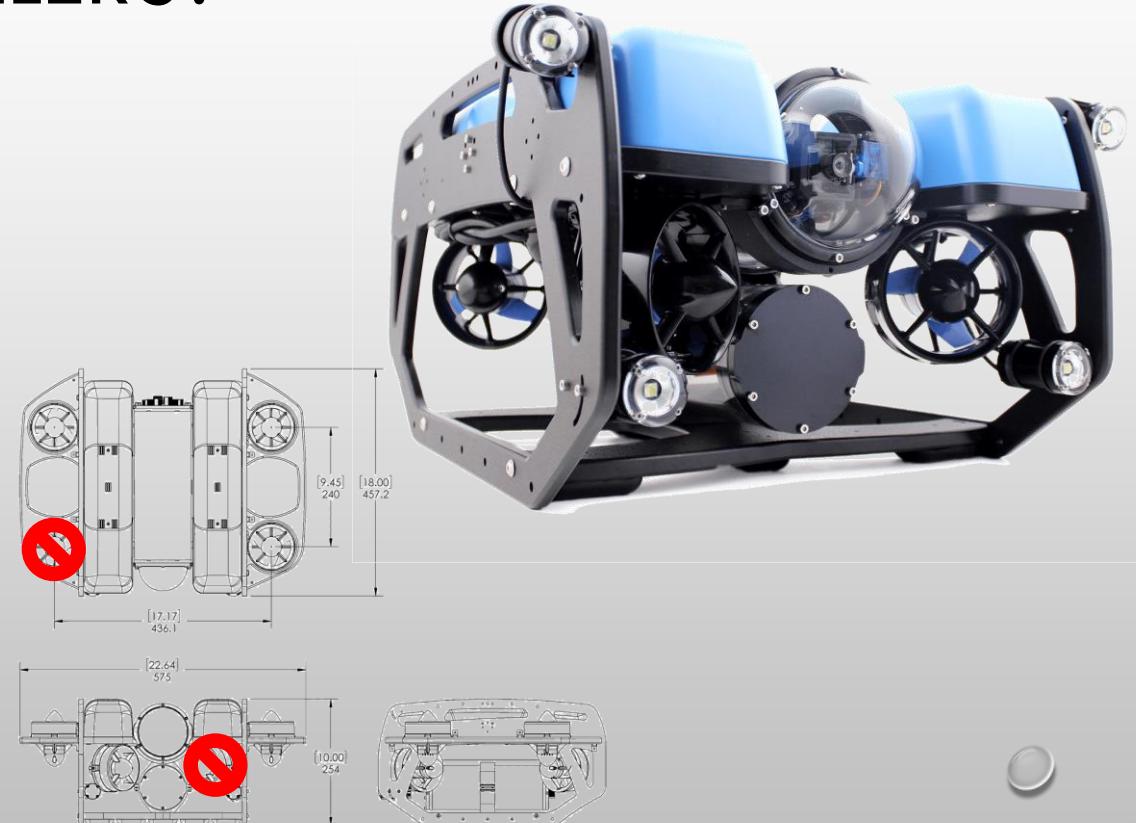
$$\tau = Ku$$

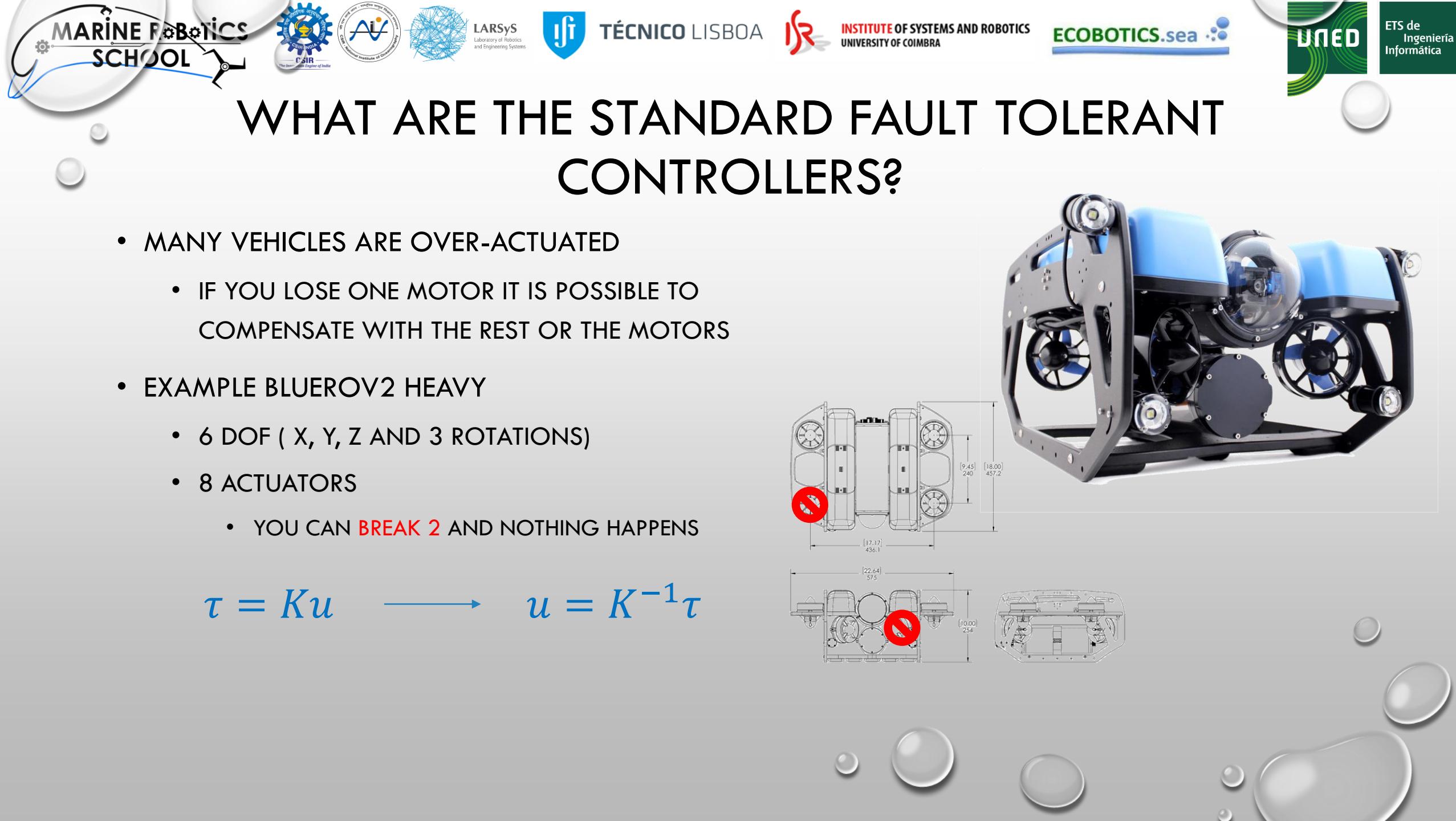


WHAT ARE THE STANDARD FAULT TOLERANT CONTROLLERS?

- MANY VEHICLES ARE OVER-ACTUATED
 - IF YOU LOSE ONE MOTOR IT IS POSSIBLE TO COMPENSATE WITH THE REST OR THE MOTORS
- EXAMPLE BLUEROV2 HEAVY
 - 6 DOF (X, Y, Z AND 3 ROTATIONS)
 - 8 ACTUATORS
 - YOU CAN **BREAK 2** AND NOTHING HAPPENS

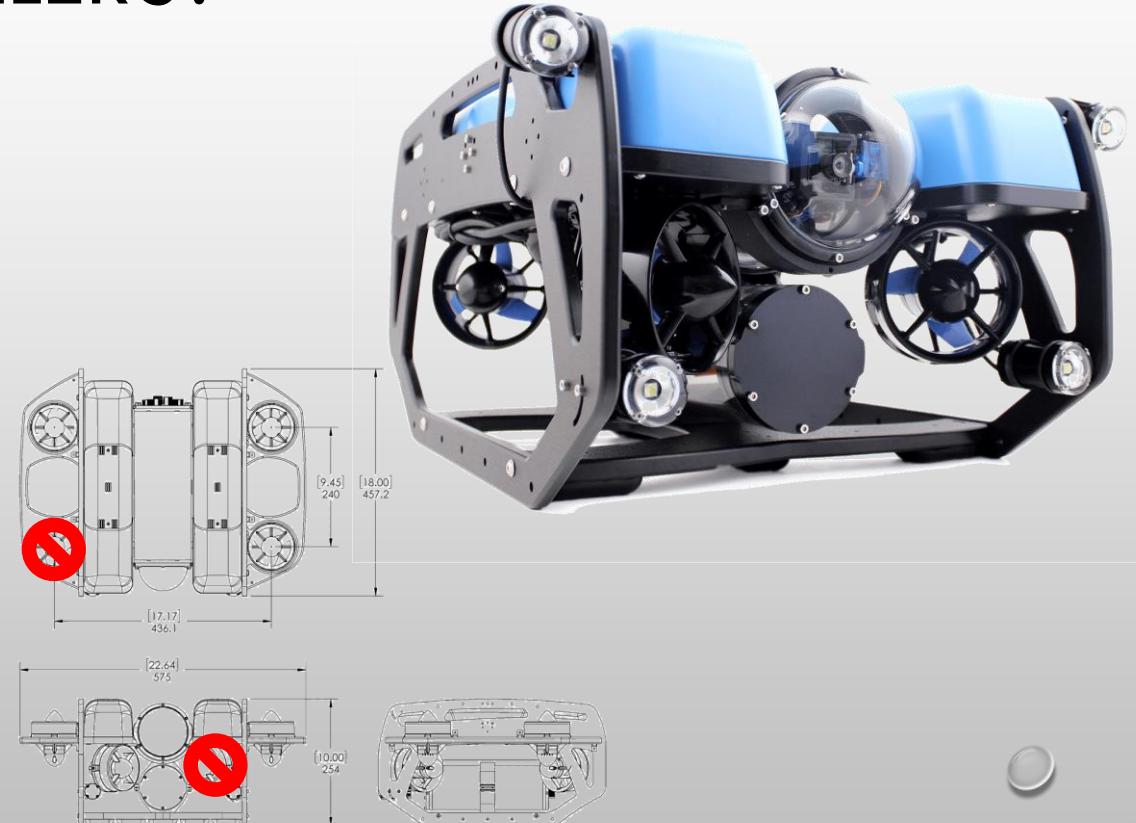
$$\tau = Ku$$





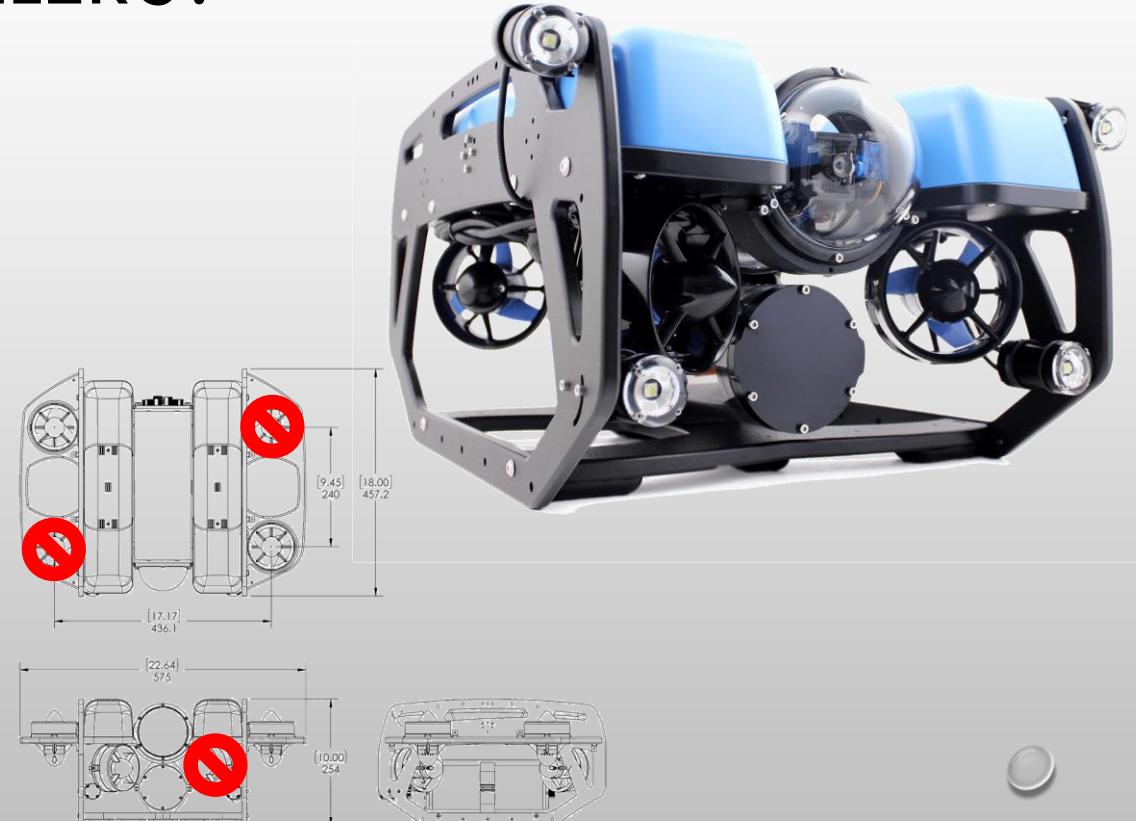
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- EXAMPLE BLUEROV2 HEAVY
 - 6 DOF (X, Y, Z AND 3 ROTATIONS)
 - 8 ACTUATORS
 - YOU CAN **BREAK 2** AND NOTHING HAPPENS

$$\tau = Ku \longrightarrow u = K^{-1}\tau$$



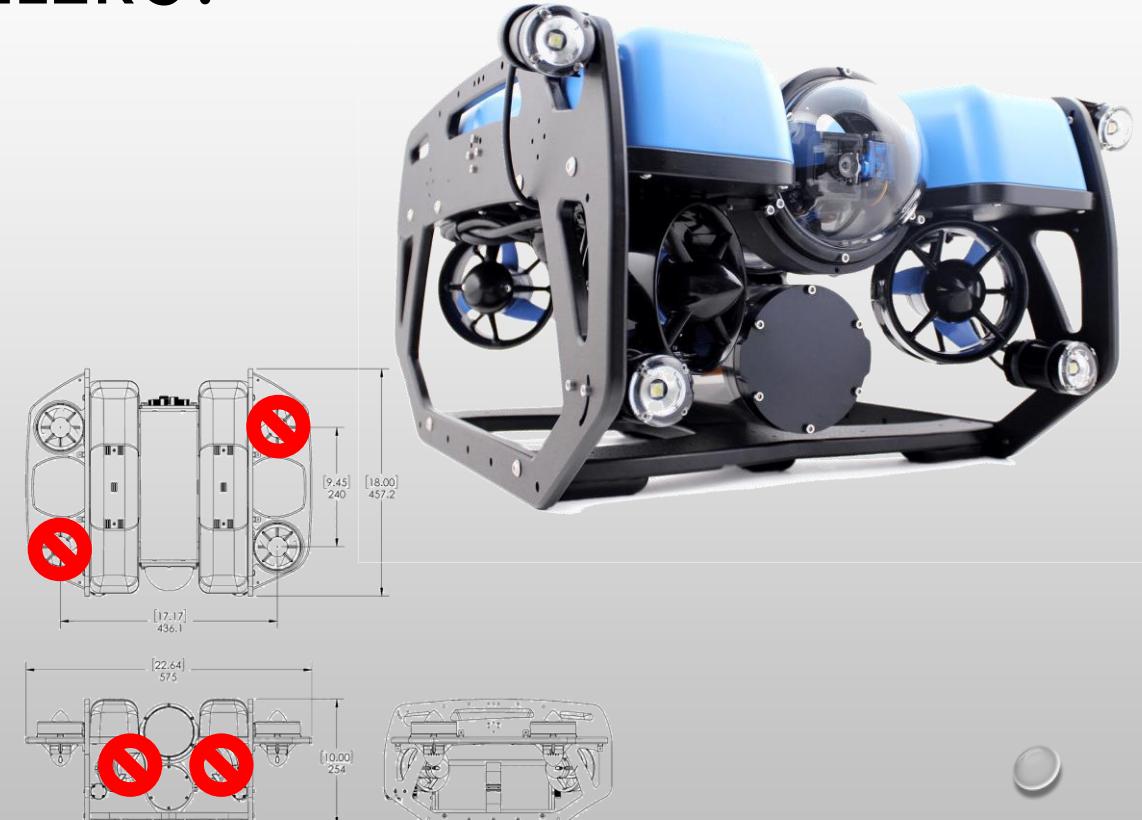
WHAT ARE THE STANDARD FAULT TOLERANT CONTROLLERS?

- EXAMPLE BLUEROV2 HEAVY
 - 6 DOF (X, Y, Z AND 3 ROTATIONS)
 - 8 ACTUATORS
 - YOU CAN BREAK 2 AND NOTHING HAPPENS
- AND IF YOU **LOSE** 3...
 - WE ARE **IN TROUBLE** (UNDERACTUATED)
 - BUT UNDERACTUATED VEHICLES ARE **STILL USEFUL**
 - WE LOSE CONTROL ON ROLL/PITCH
 - BUT WE CAN STEER AND MOVE



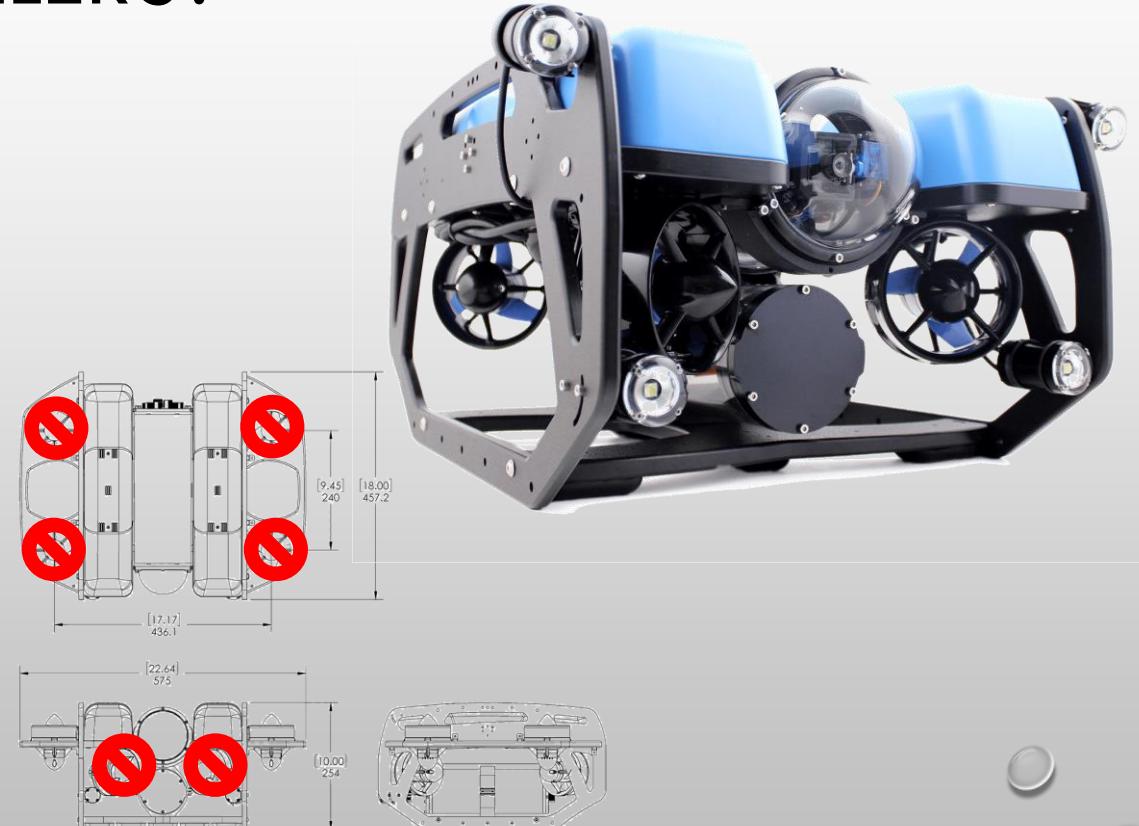
WHAT ARE THE STANDARD FAULT TOLERANT CONTROLLERS?

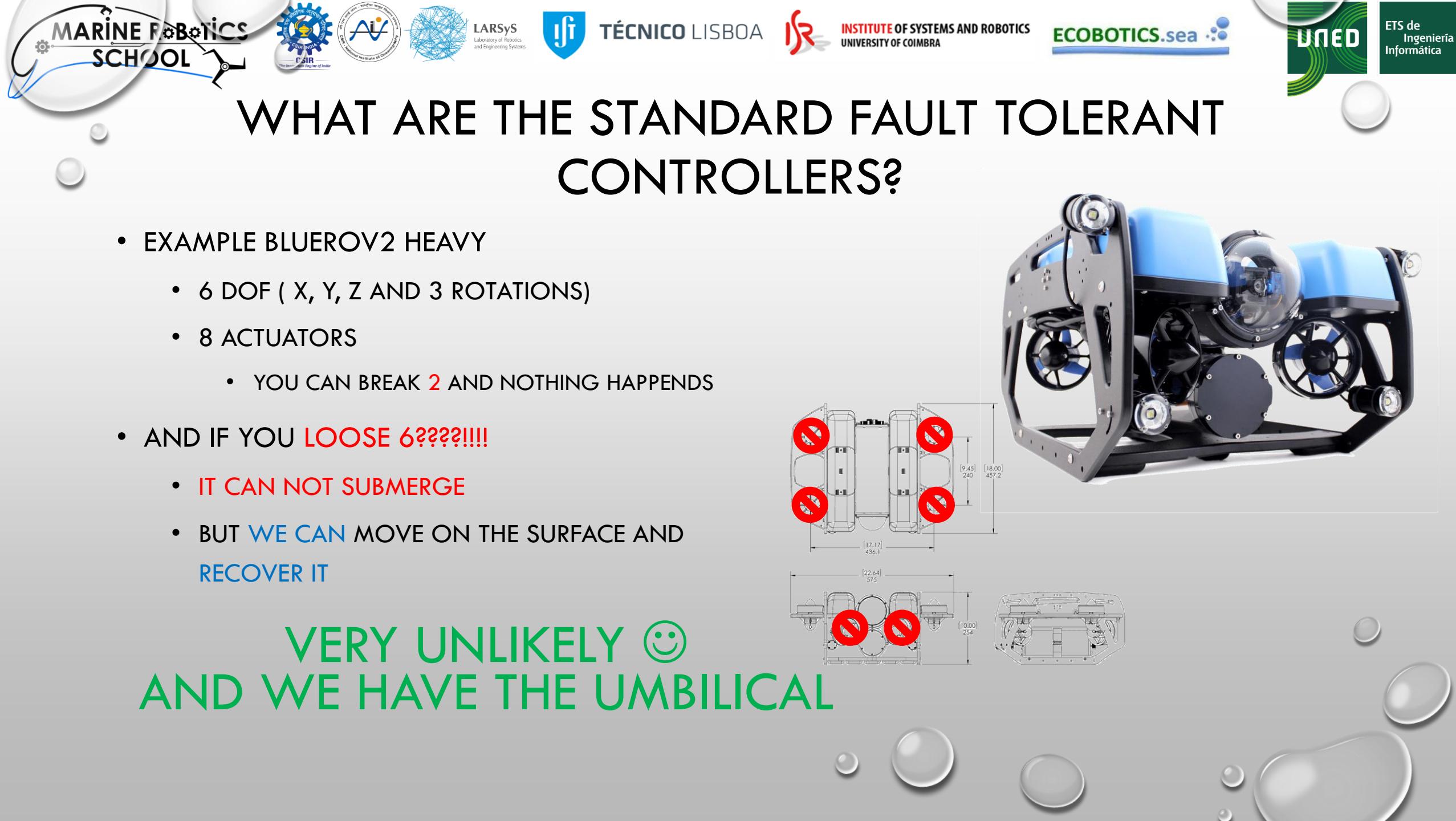
- EXAMPLE BLUEROV2 HEAVY
 - 6 DOF (X, Y, Z AND 3 ROTATIONS)
 - 8 ACTUATORS
 - YOU CAN BREAK 2 AND NOTHING HAPPENS
- AND IF YOU **LOSE 4!!!!**
 - WE ARE IN TROUBLE (UNDERACTUATED)
 - STILL USEFUL!!!
 - WE LOSE CONTROL ON ROLL/PITCH
 - WE CAN'T MOVE LATERALLY BUT STILL TURN AND MOVE FORWARD...



WHAT ARE THE STANDARD FAULT TOLERANT CONTROLLERS?

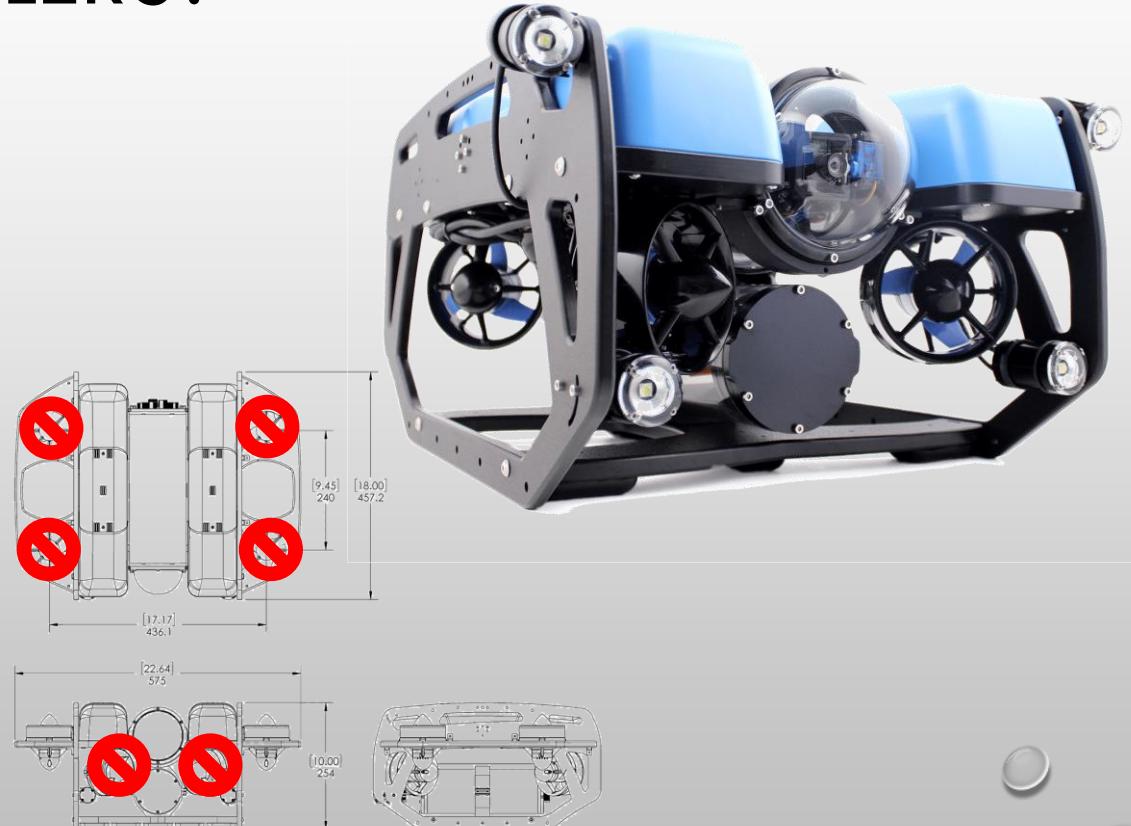
- EXAMPLE BLUEROV2 HEAVY
 - 6 DOF (X, Y, Z AND 3 ROTATIONS)
 - 8 ACTUATORS
 - YOU CAN BREAK 2 AND NOTHING HAPPENS
- AND IF YOU **LOSE 6????!!!!**
 - IT CAN NOT SUBMERGE
 - BUT **WE CAN** MOVE ON THE SURFACE AND RECOVER IT





WHAT ARE THE STANDARD FAULT TOLERANT CONTROLLERS?

- EXAMPLE BLUEROV2 HEAVY
 - 6 DOF (X, Y, Z AND 3 ROTATIONS)
 - 8 ACTUATORS
 - YOU CAN BREAK 2 AND NOTHING HAPPENS
- AND IF YOU **LOSE 6????!!!!**
 - IT CAN NOT SUBMERGE
 - BUT **WE CAN** MOVE ON THE SURFACE AND RECOVER IT



VERY UNLIKELY 😊
AND WE HAVE THE UMBILICAL

WHERE IS THE LIMIT?



MEDUSA VEHICLE

- UNDERACTUATED VEHICLE
 - 4 THRUSTERS
 - 2 FOR DEPTH
 - 2 FOR PLANAR MOTION
 - OPERATES LIKE A “3+1” DOF
 - ROLL AND PITCH ARE STABLE (DON’T NEED ACTUATION)
 - BUT CAN’T ACCELERATE SIDEWAYS
 - POSITIVE BUOYANCY



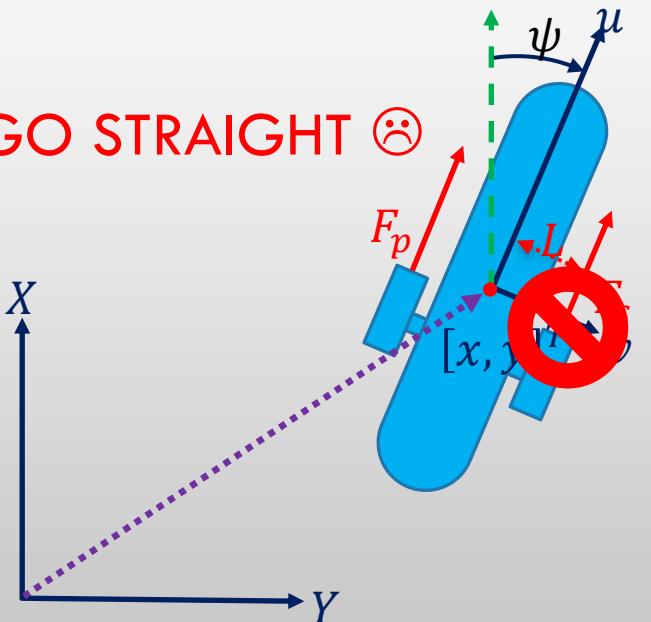
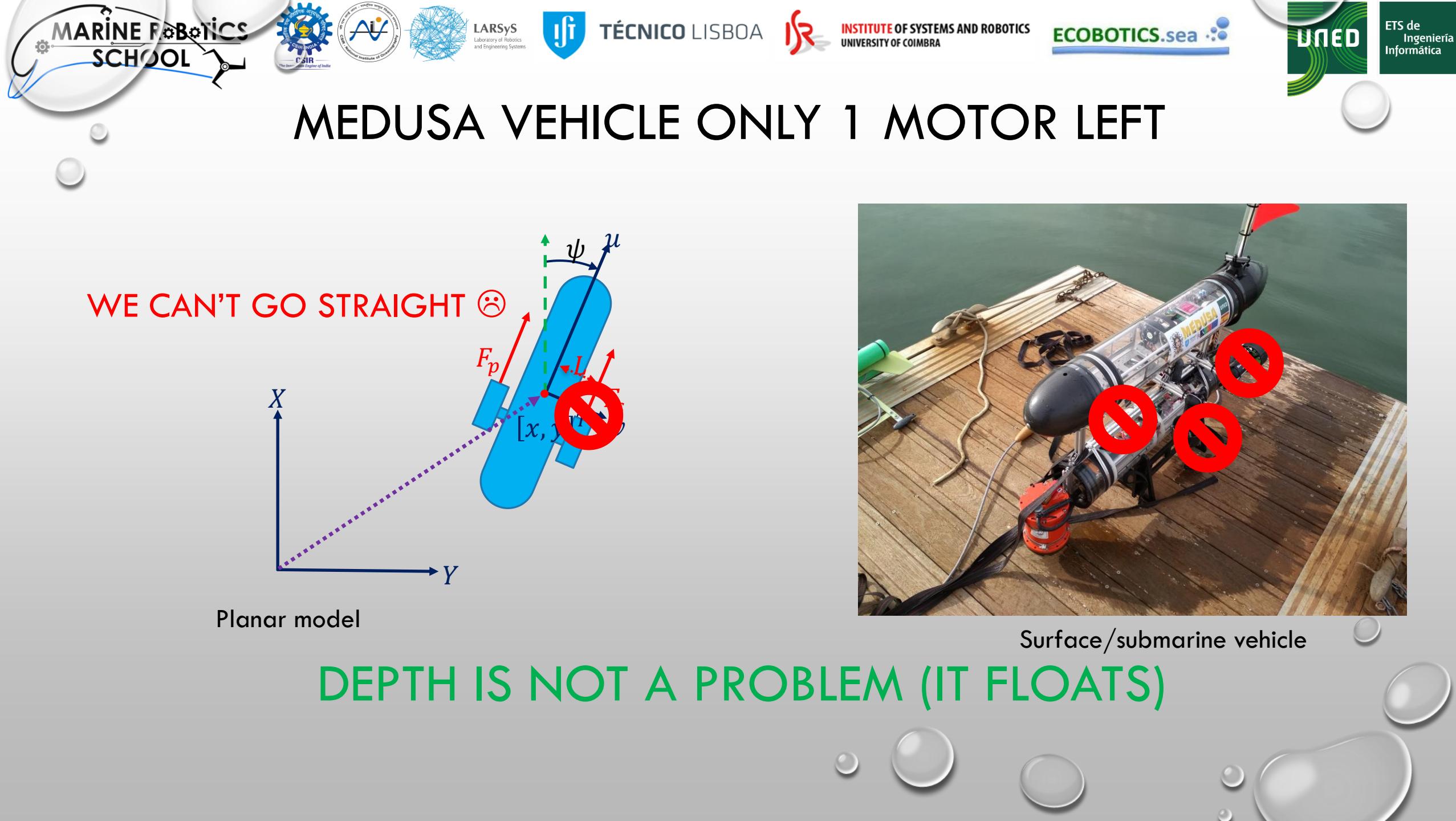
Surface/submarine vehicle

MEDUSA VEHICLE

**IF ONE MOTOR FAILS, WE ARE
IN TROUBLE, AND THIS IS NOT
UNLIKELY!!!**



Surface/submarine vehicle



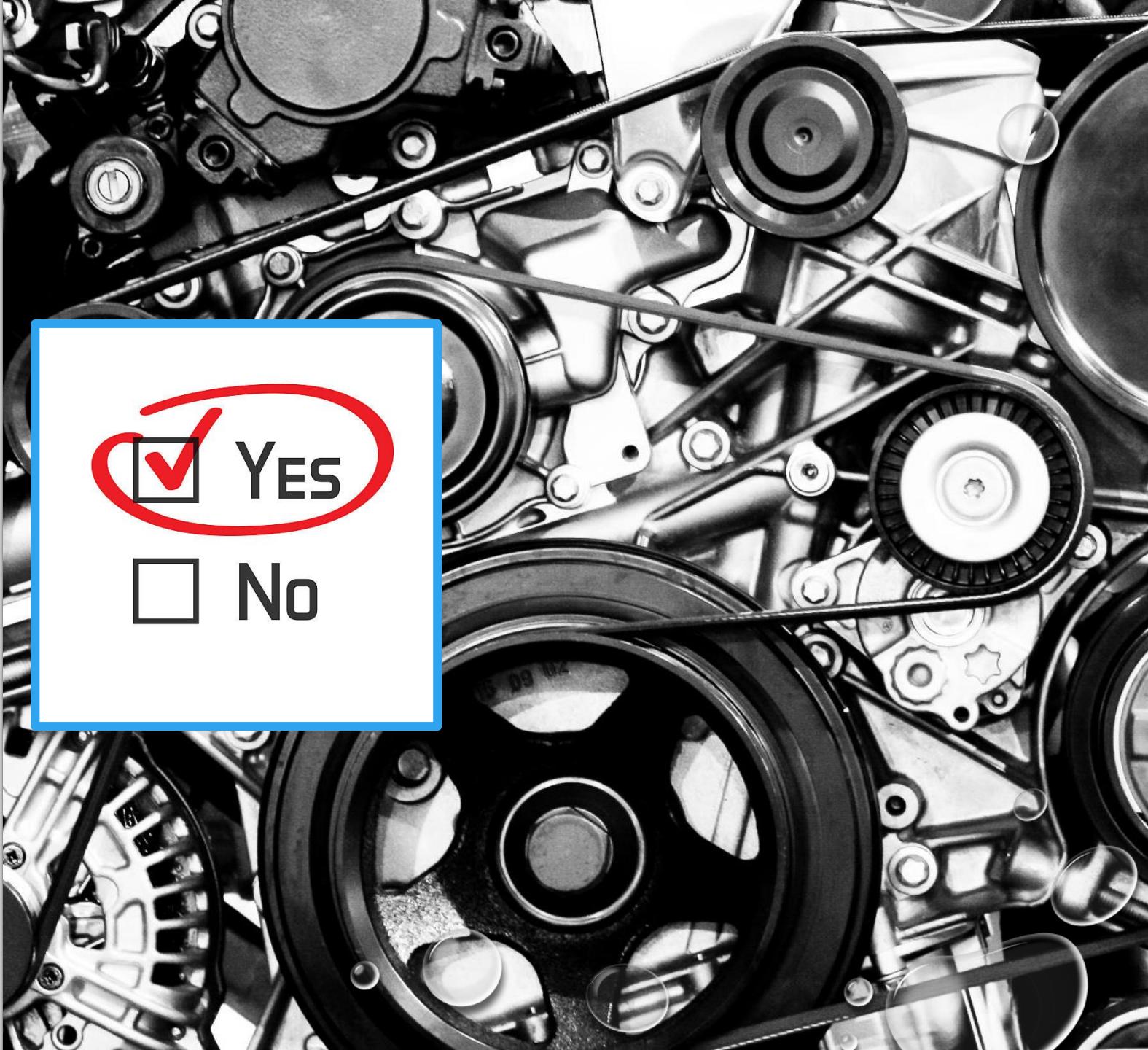
STANDARD RECOVERY SOLUTION!



In memorial to Professor JM de la Cruz

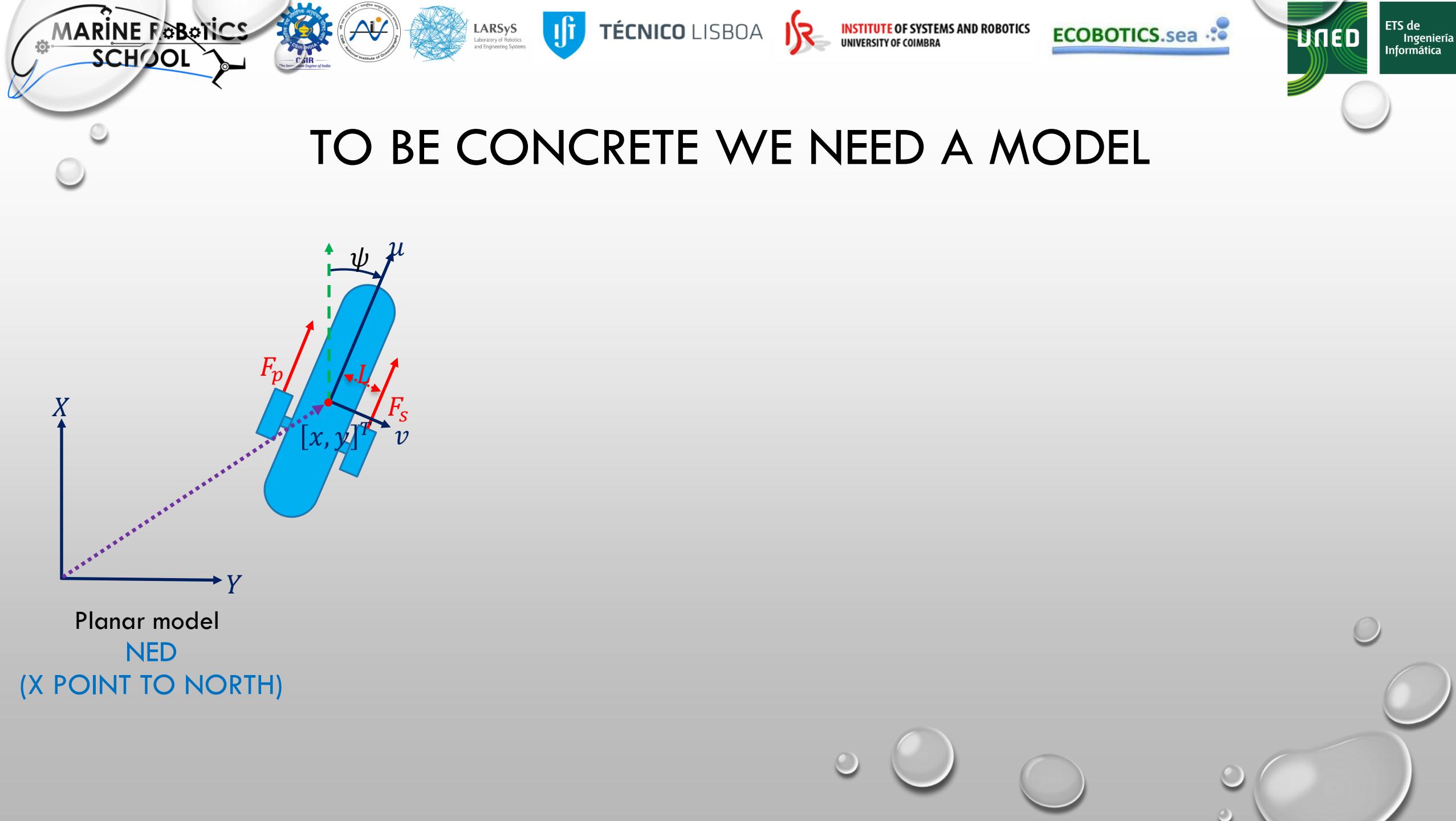
IS THERE ANOTHER WAY?

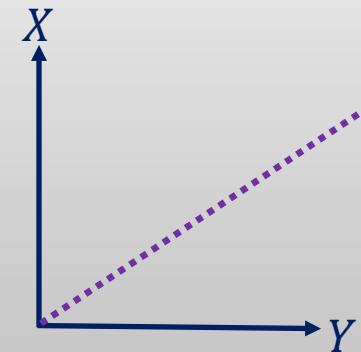
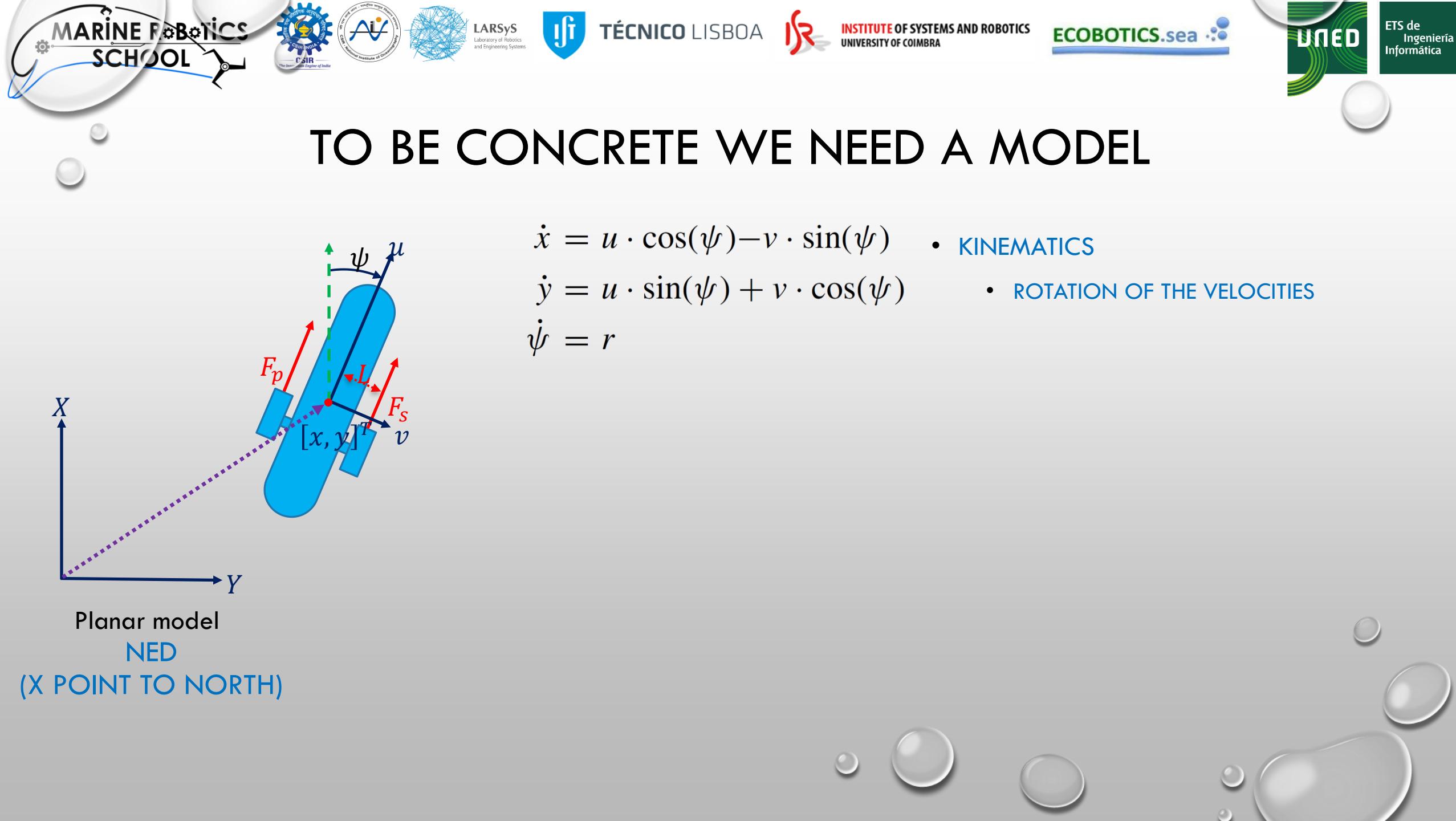
- FAST POLL
- WHO THINKS IT IS POSSIBLE
TO CONTROL THE VEHICLE
WITH ONLY ONE MOTOR?



YES

No





Planar model

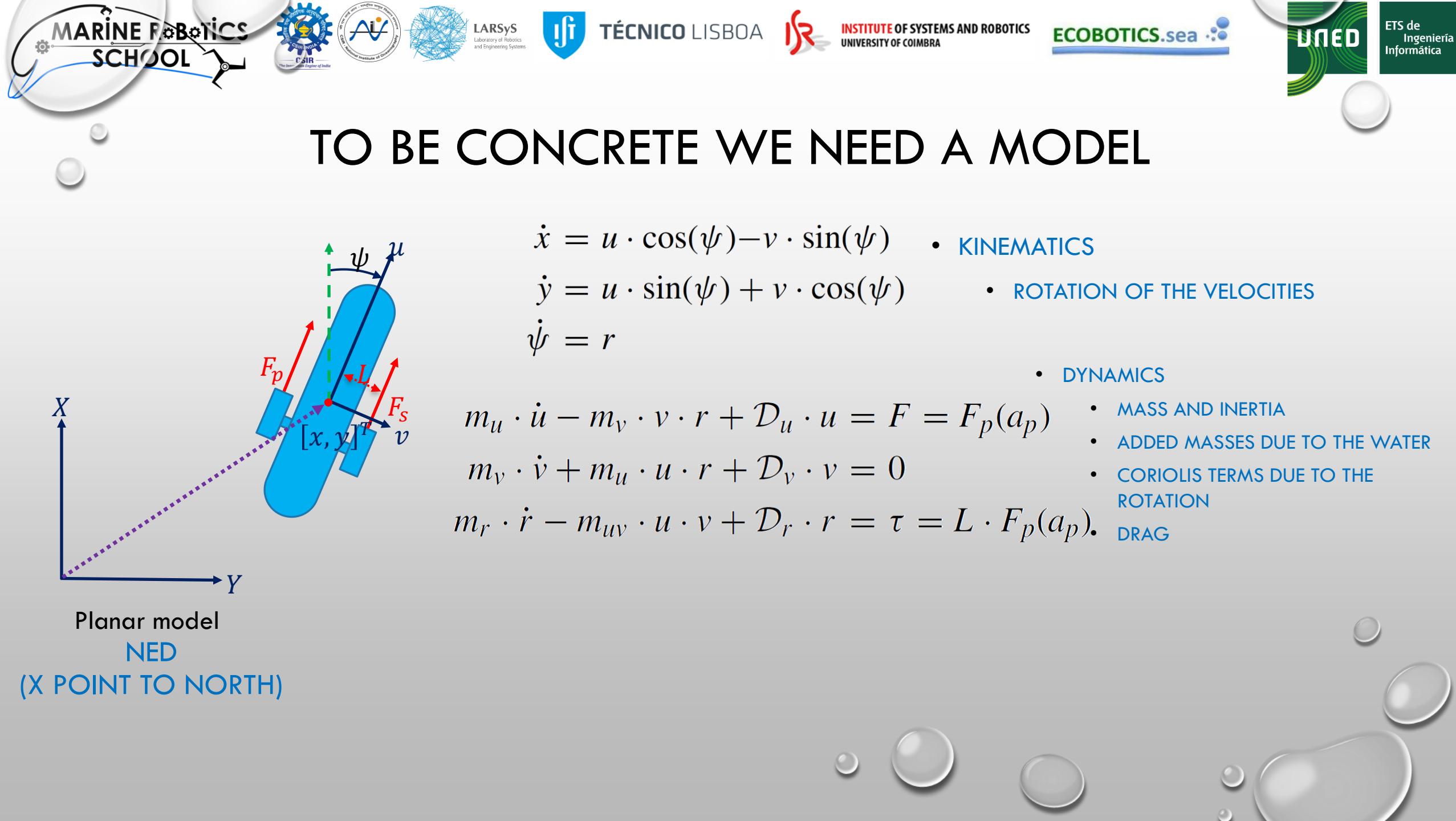
NED

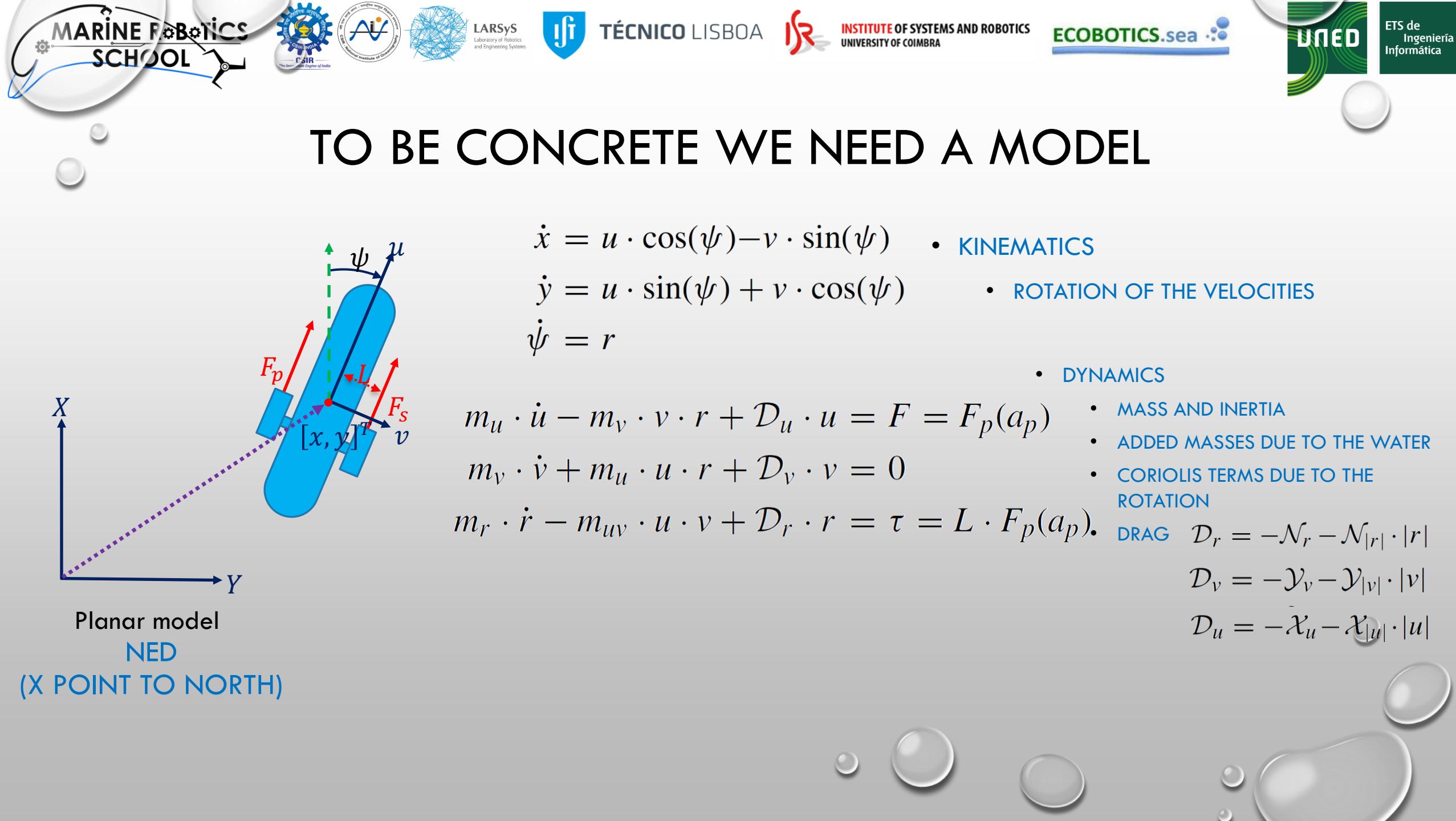
(X POINT TO NORTH)

$$\begin{aligned}\dot{x} &= u \cdot \cos(\psi) - v \cdot \sin(\psi) \\ \dot{y} &= u \cdot \sin(\psi) + v \cdot \cos(\psi) \\ \dot{\psi} &= r\end{aligned}$$

• KINEMATICS

• ROTATION OF THE VELOCITIES





TO BE CONCRETE WE NEED A MODEL

Planar model
NED
(X POINT TO NORTH)

- KINEMATICS
- ROTATION OF THE VELOCITIES
- DYNAMICS
- MASS AND INERTIA
- ADDED MASSES DUE TO THE WATER
- CORIOLIS TERMS DUE TO THE ROTATION
- DRAG $\mathcal{D}_r = -\mathcal{N}_r - \mathcal{N}_{|r|} \cdot |r|$
- $\mathcal{D}_v = -\mathcal{Y}_v - \mathcal{Y}_{|v|} \cdot |v|$
- $\mathcal{D}_u = -\mathcal{X}_u - \mathcal{X}_{|u|} \cdot |u|$

$$\dot{x} = u \cdot \cos(\psi) - v \cdot \sin(\psi)$$

$$\dot{y} = u \cdot \sin(\psi) + v \cdot \cos(\psi)$$

$$\dot{\psi} = r$$

$$m_u \cdot \dot{u} - m_v \cdot v \cdot r + \mathcal{D}_u \cdot u = F = F_p(a_p)$$

$$m_v \cdot \dot{v} + m_u \cdot u \cdot r + \mathcal{D}_v \cdot v = 0$$

$$m_r \cdot \dot{r} - m_{uv} \cdot u \cdot v + \mathcal{D}_r \cdot r = \tau = L \cdot F_p(a_p).$$

- ACTUATORS MODEL
- FORCE IS PROPORTIONAL TO THE SQUARE OF THE REVOLUTION

$$F_s(a_s) = \mathcal{K} \cdot |a_s| \cdot a_s$$

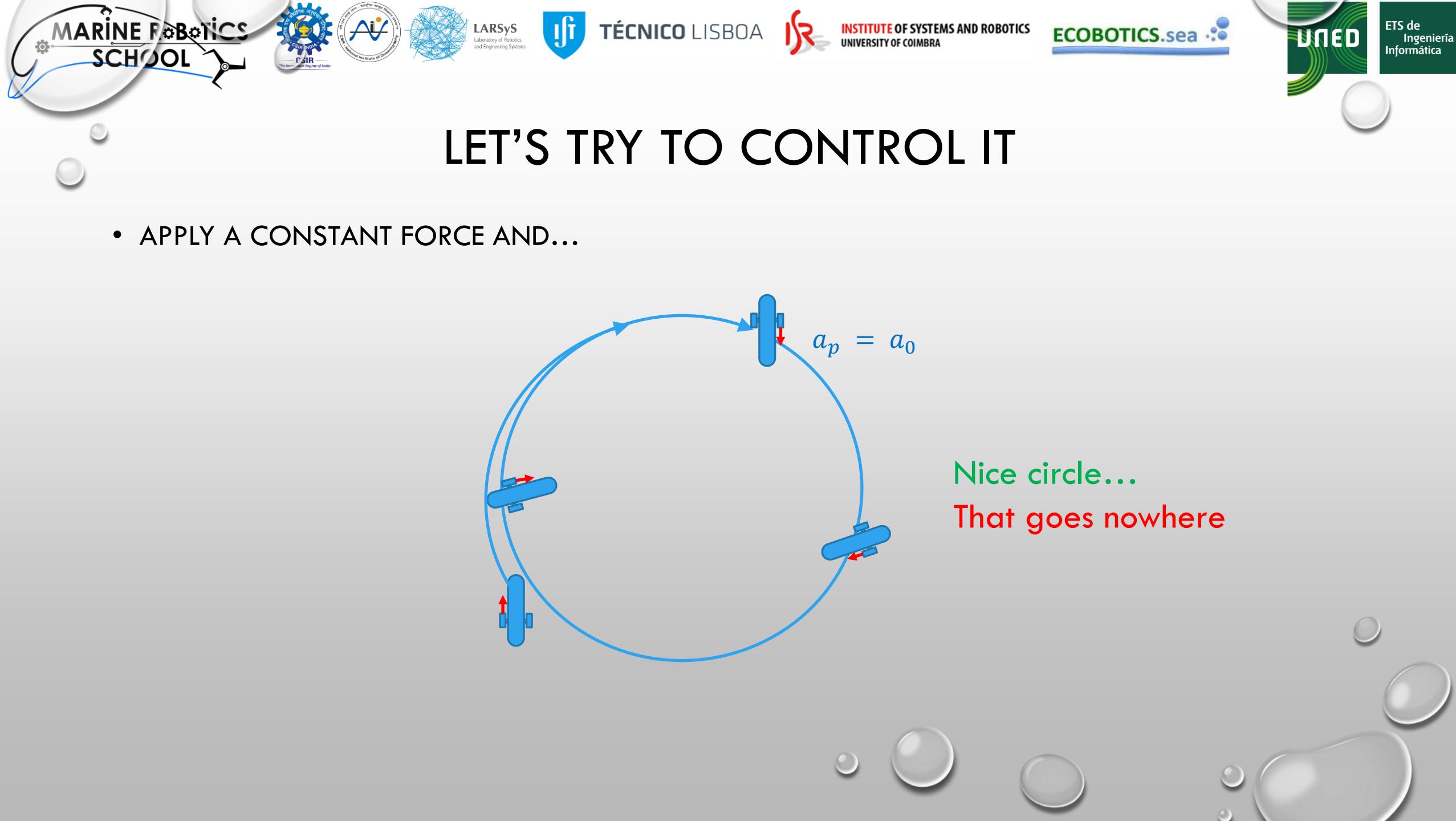
$$F_p(a_p) = \mathcal{K} \cdot |a_p| \cdot a_p$$



LET'S TRY TO CONTROL IT

- APPLY A CONSTANT FORCE AND...

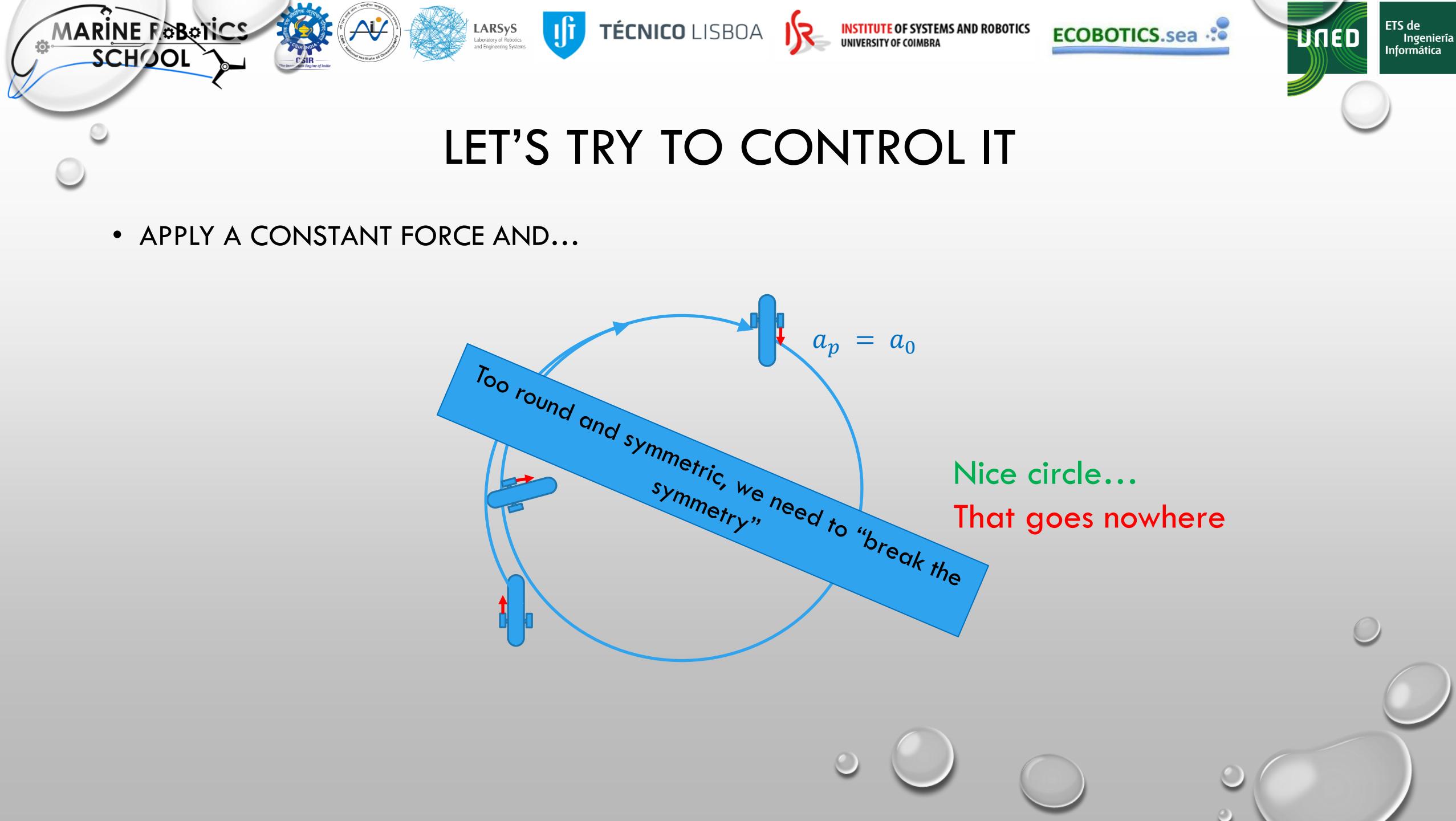
WHAT DO YOU THINK IT WILL HAPPEN?



LET'S TRY TO CONTROL IT

- APPLY A CONSTANT FORCE AND...

Nice circle...
That goes nowhere





LARSys
Laboratory of Robotics
and Engineering Systems



TÉCNICO LISBOA

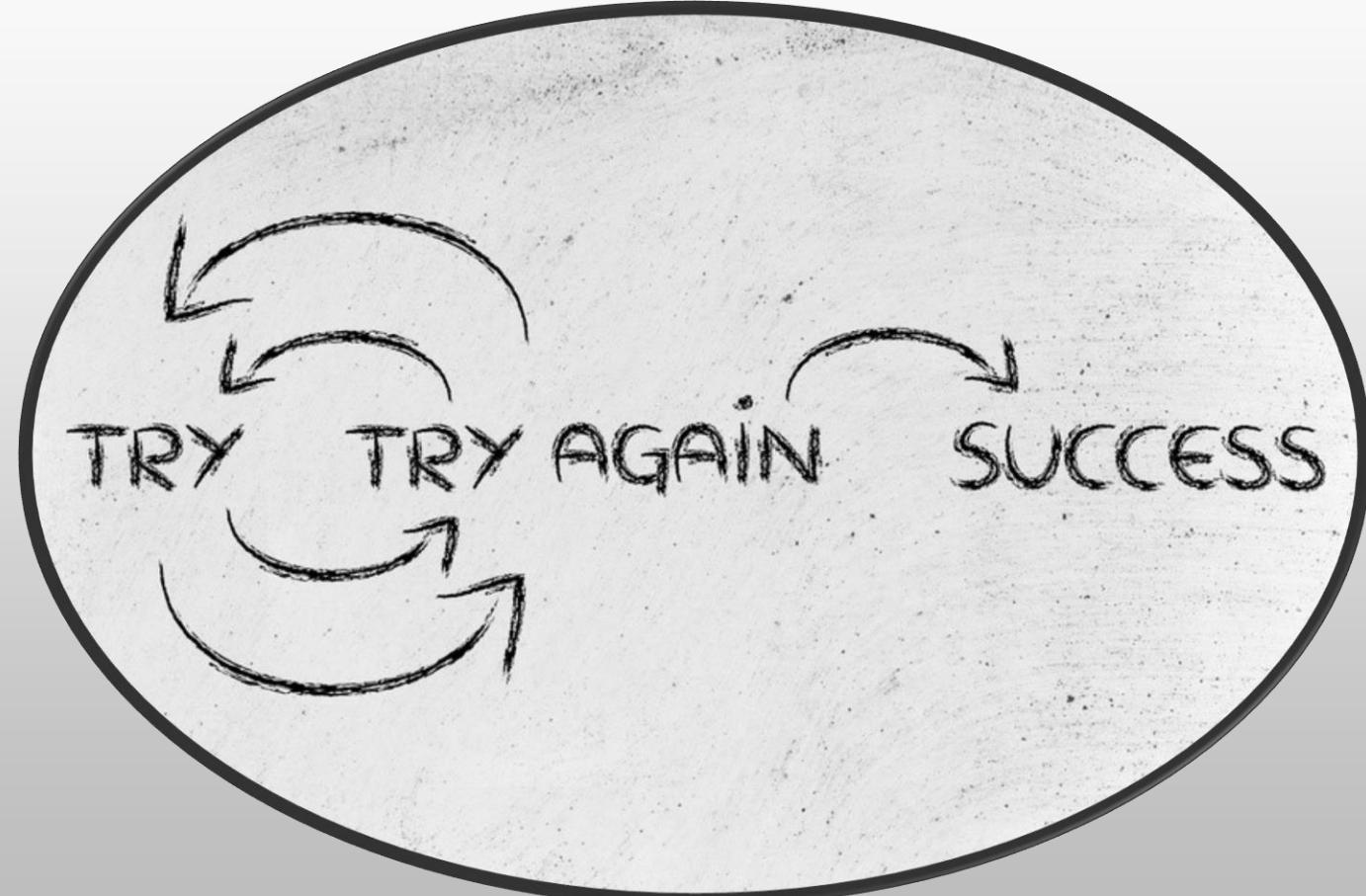


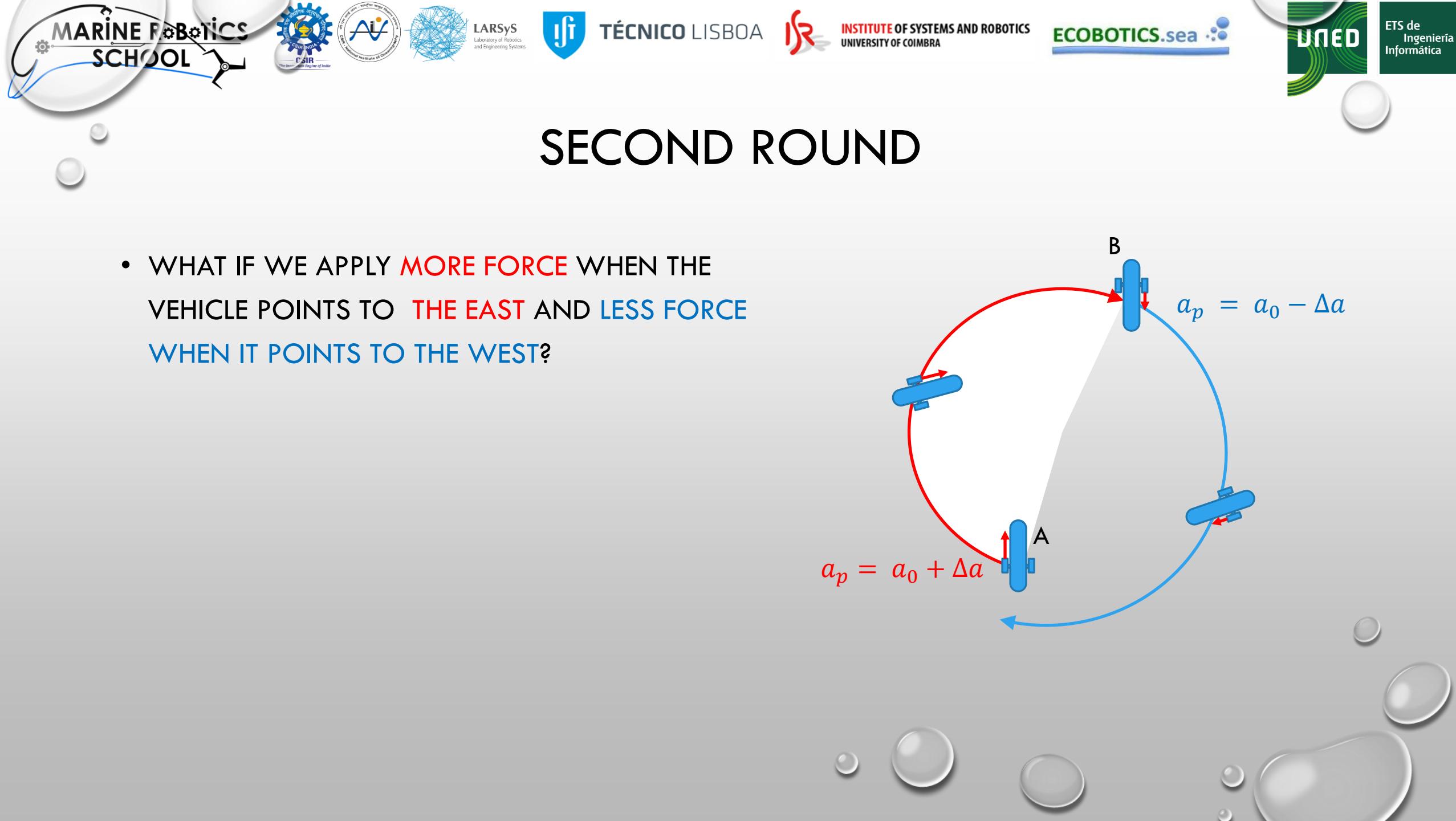
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UNIVERSITY OF COIMBRA



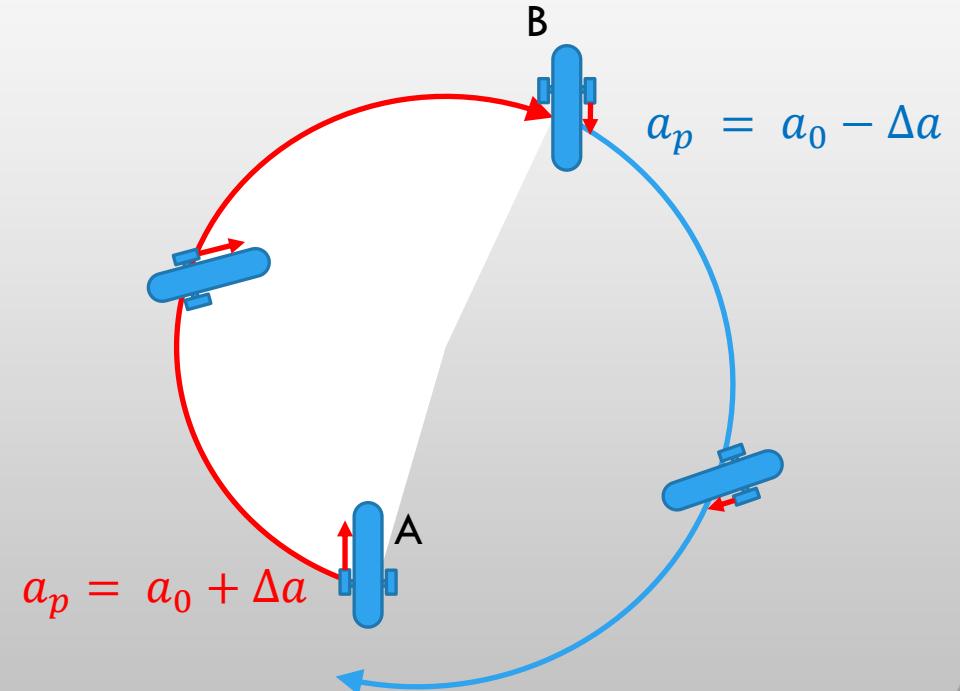
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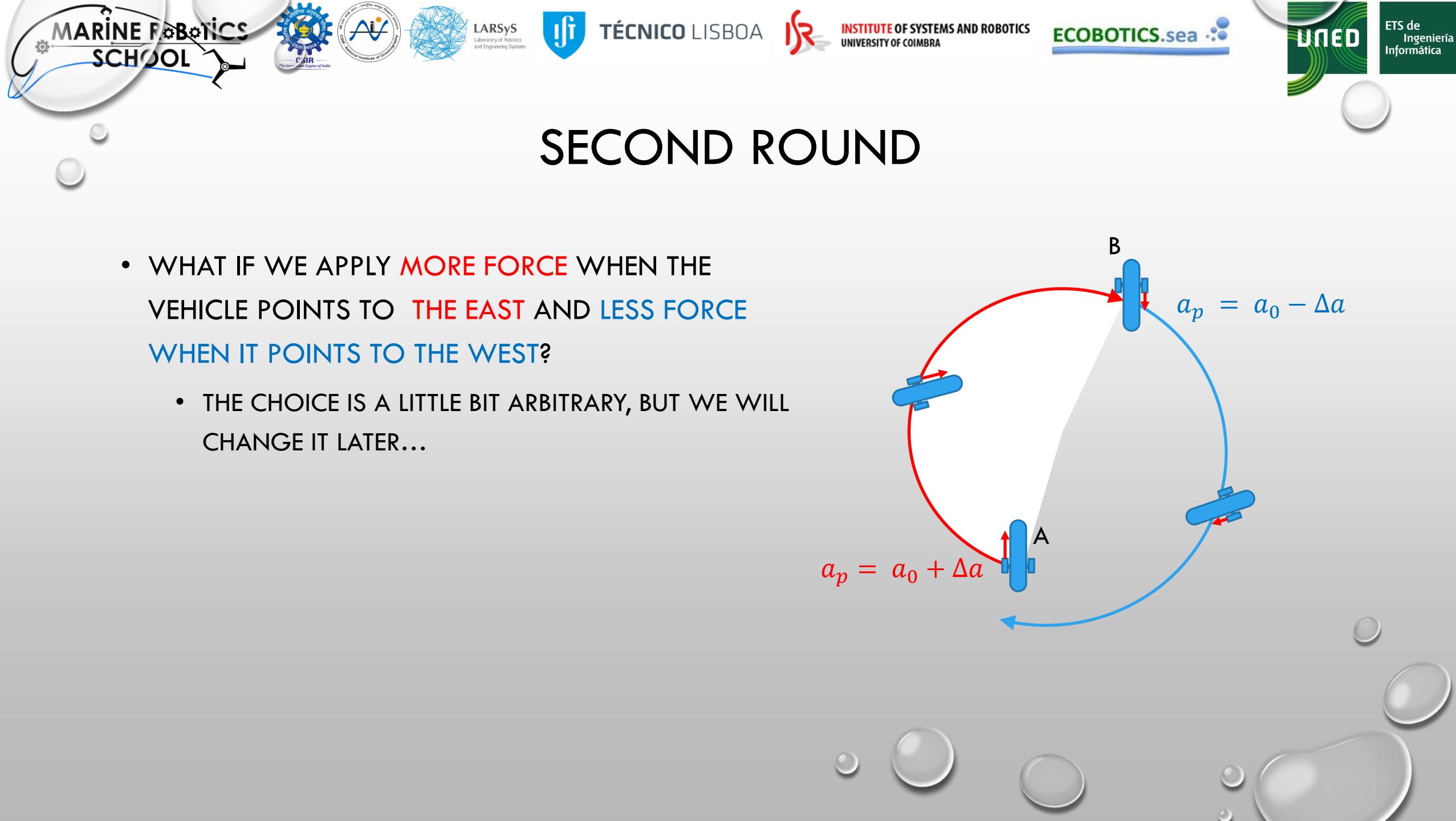
DON'T SURRENDER





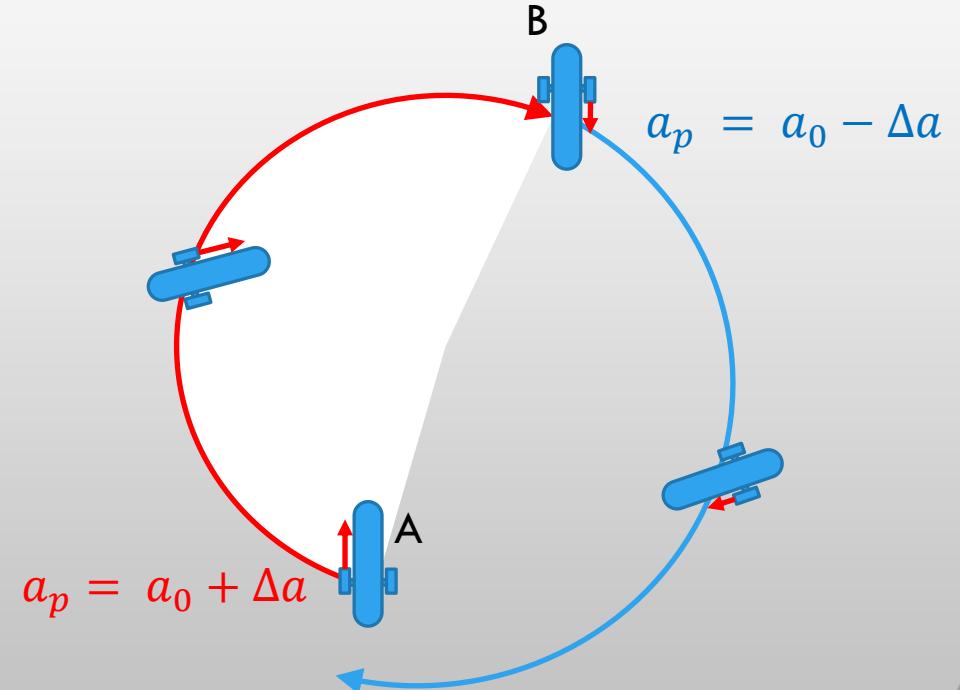
- WHAT IF WE APPLY **MORE FORCE** WHEN THE VEHICLE POINTS TO **THE EAST** AND **LESS FORCE** WHEN IT POINTS TO THE **WEST**?

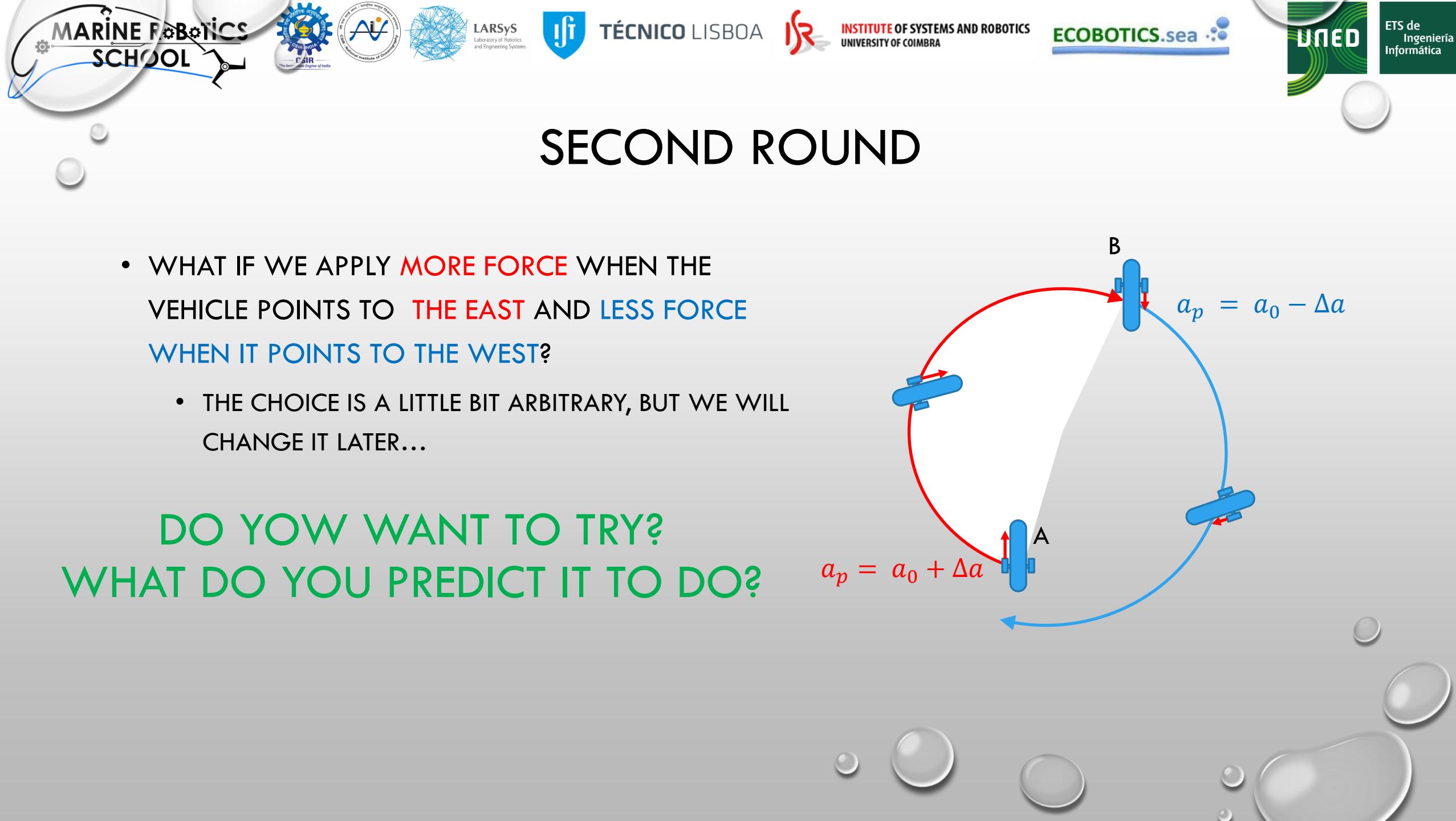




SECOND ROUND

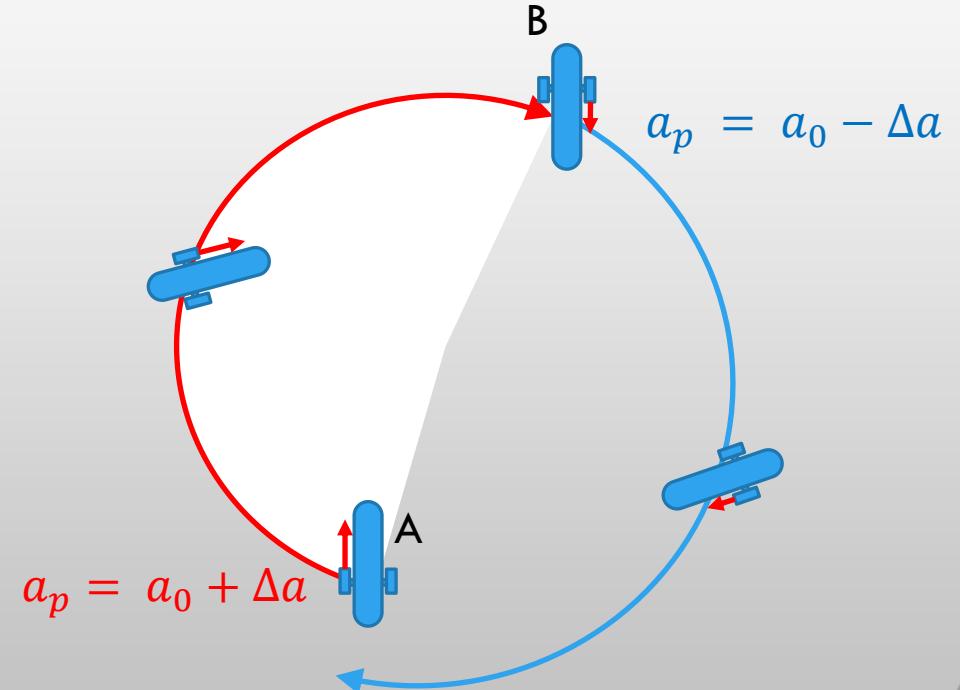
- WHAT IF WE APPLY **MORE FORCE** WHEN THE VEHICLE POINTS TO **THE EAST** AND **LESS FORCE** WHEN IT POINTS TO THE **WEST**?
 - THE CHOICE IS A LITTLE BIT ARBITRARY, BUT WE WILL CHANGE IT LATER...





- WHAT IF WE APPLY **MORE FORCE** WHEN THE VEHICLE POINTS TO **THE EAST** AND **LESS FORCE** WHEN IT POINTS TO THE **WEST**?
 - THE CHOICE IS A LITTLE BIT ARBITRARY, BUT WE WILL CHANGE IT LATER...

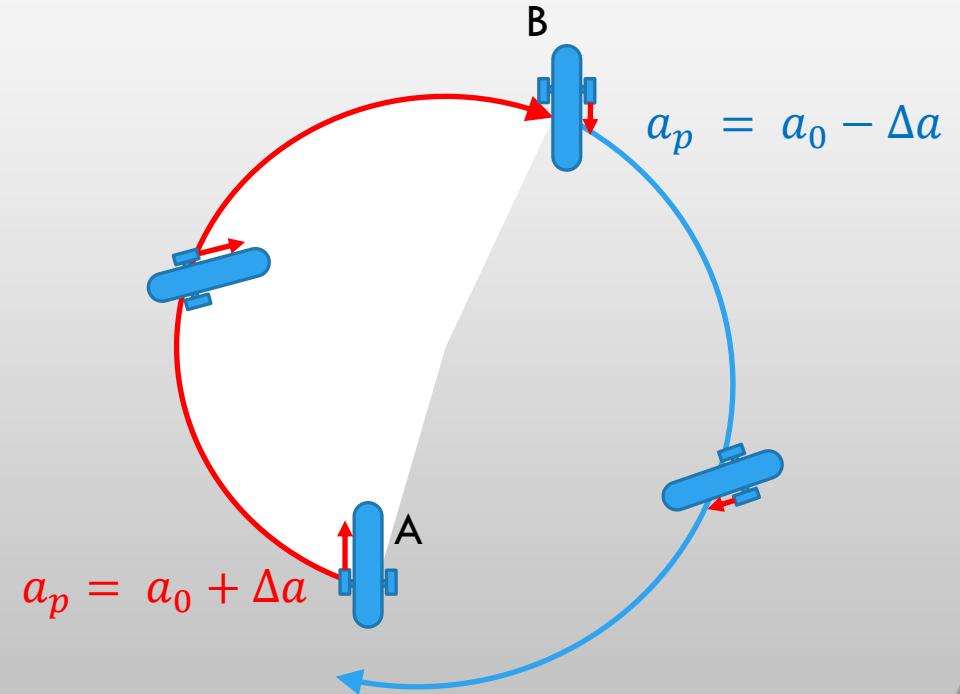
DO YOU WANT TO TRY?
WHAT DO YOU PREDICT IT TO DO?

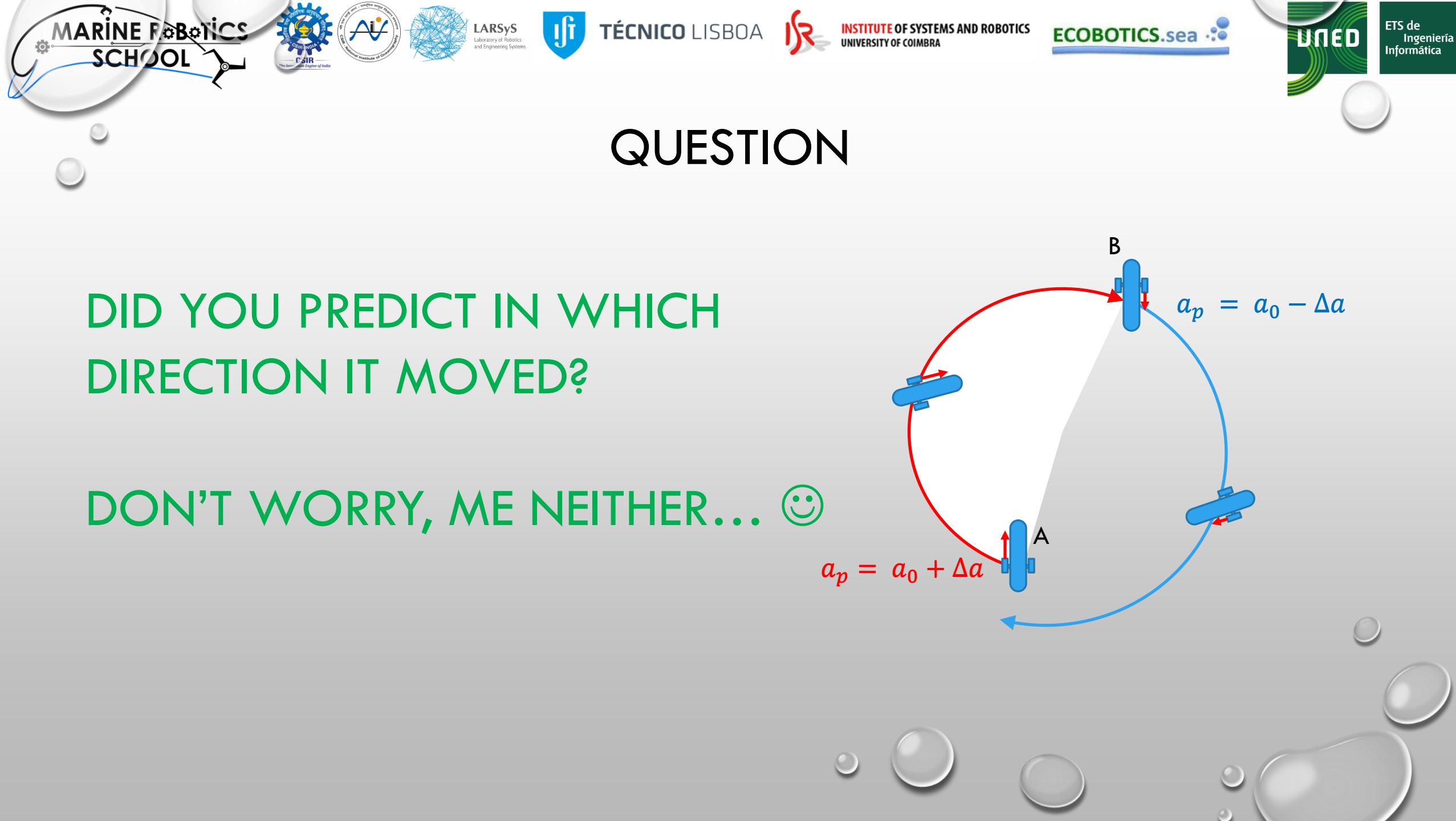


SECOND ROUND

- WHAT IF WE APPLY **MORE FORCE** WHEN THE VEHICLE POINTS TO **THE EAST** AND **LESS FORCE** WHEN IT POINTS TO THE **WEST**?
 - THE CHOICE IS A LITTLE BIT ARBITRARY, BUT WE WILL CHANGE IT LATER...
- NOW **IT MOVES**

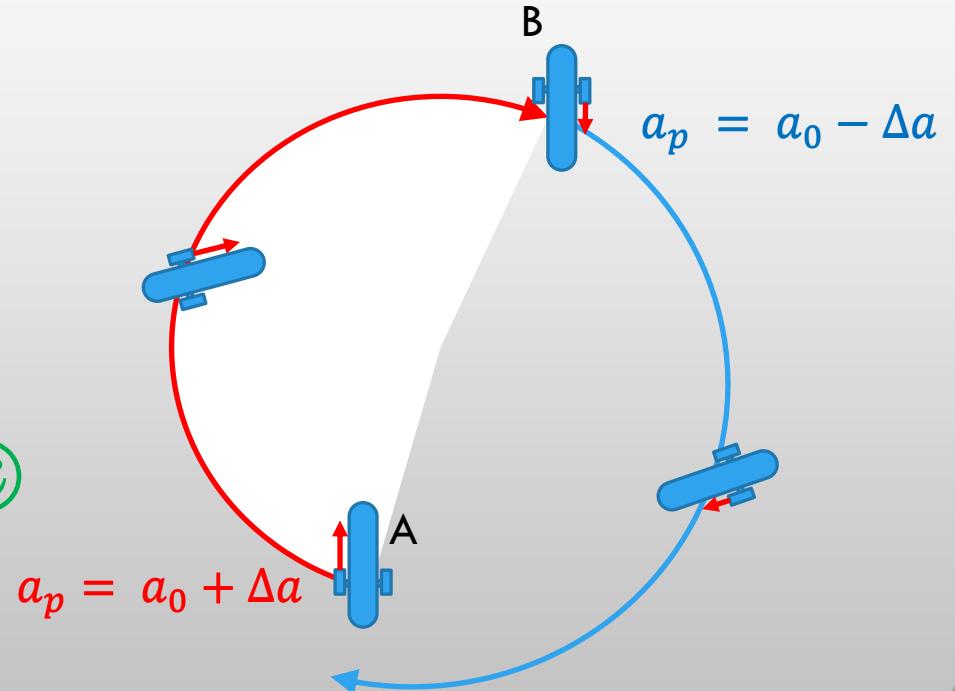
$$a_p = a_0 + \Delta a \cdot \text{sign}(\sin(\psi))$$





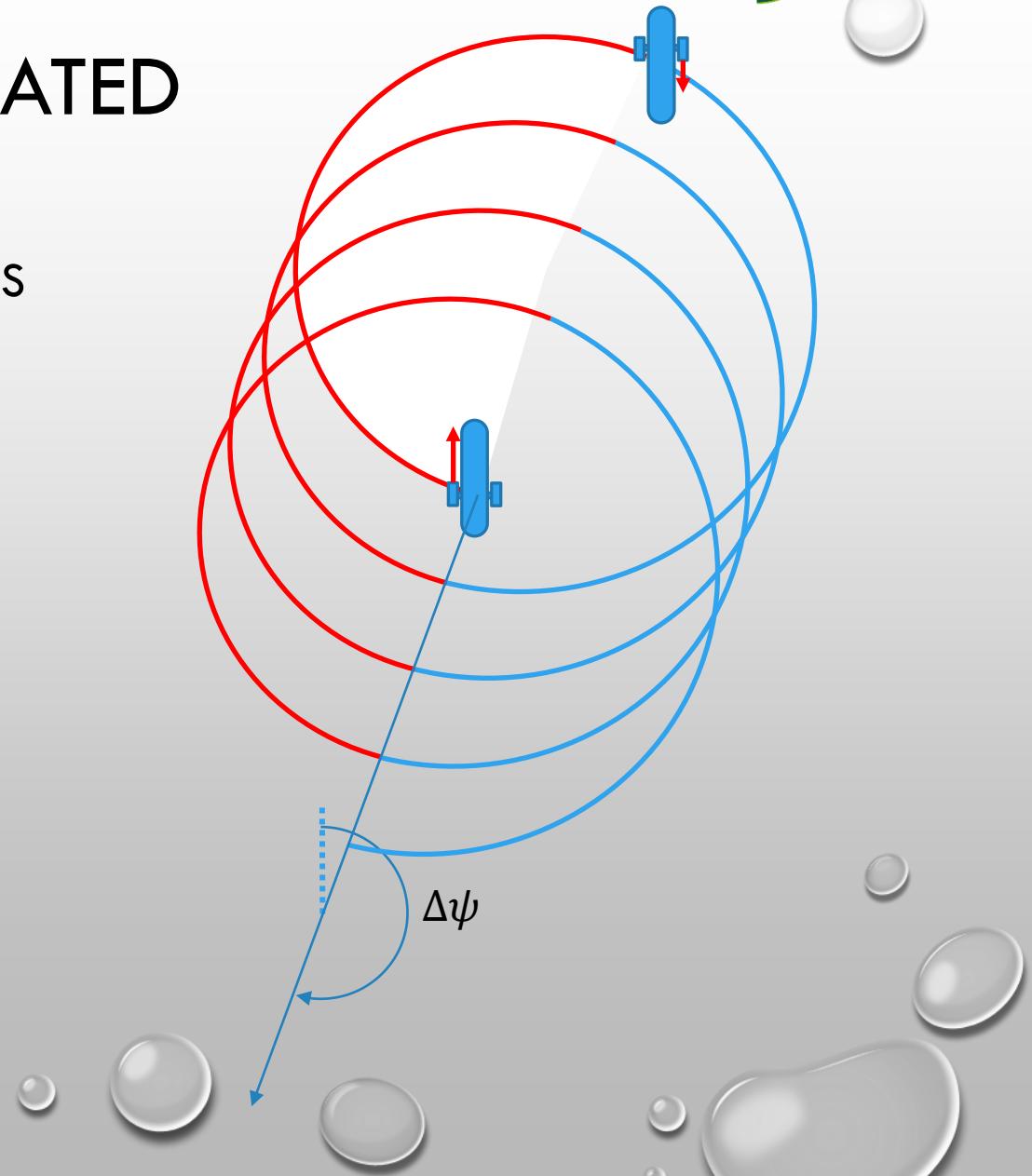
DID YOU PREDICT IN WHICH DIRECTION IT MOVED?

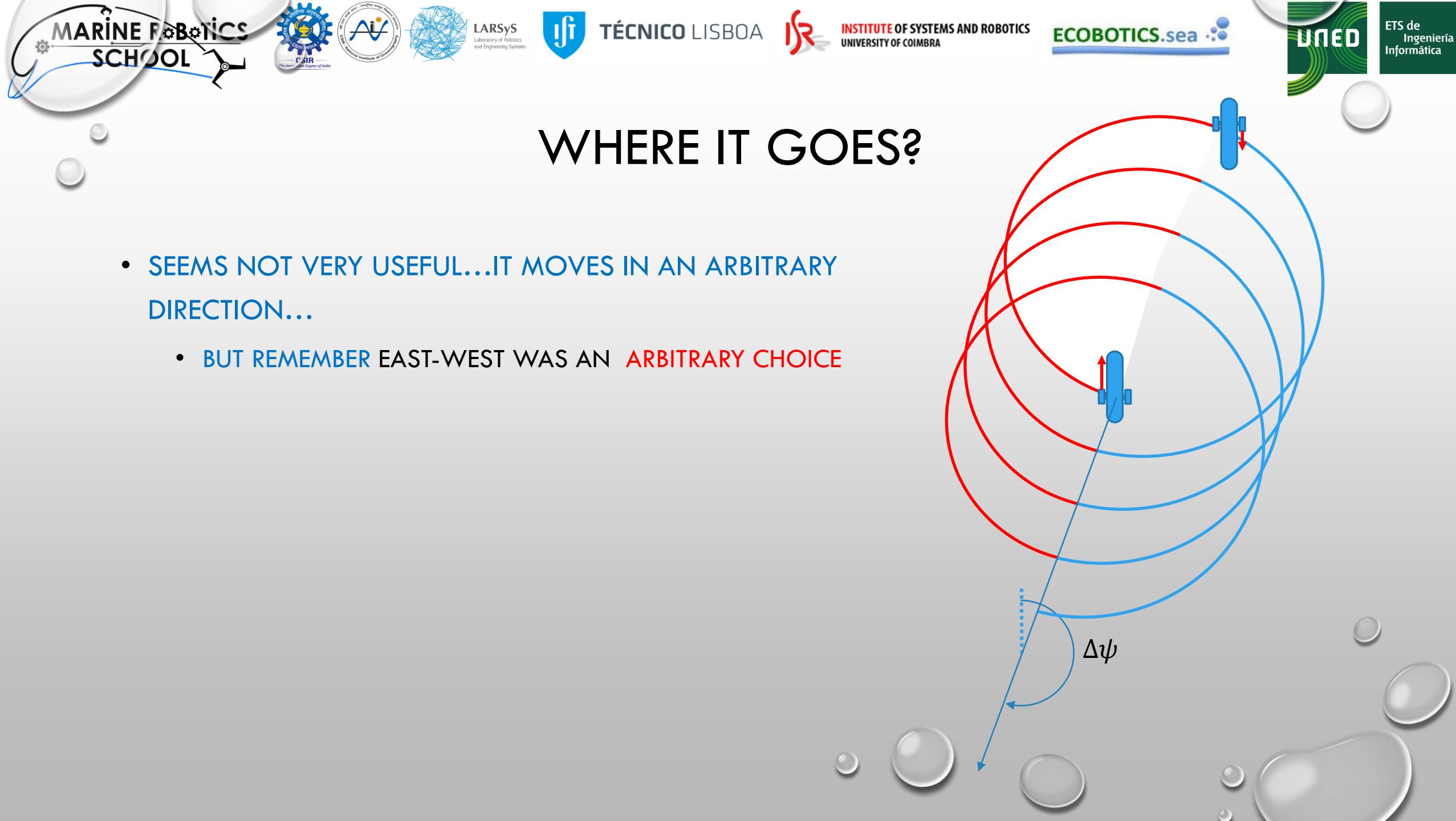
DON'T WORRY, ME NEITHER... ☺



IT'S COMPLICATED

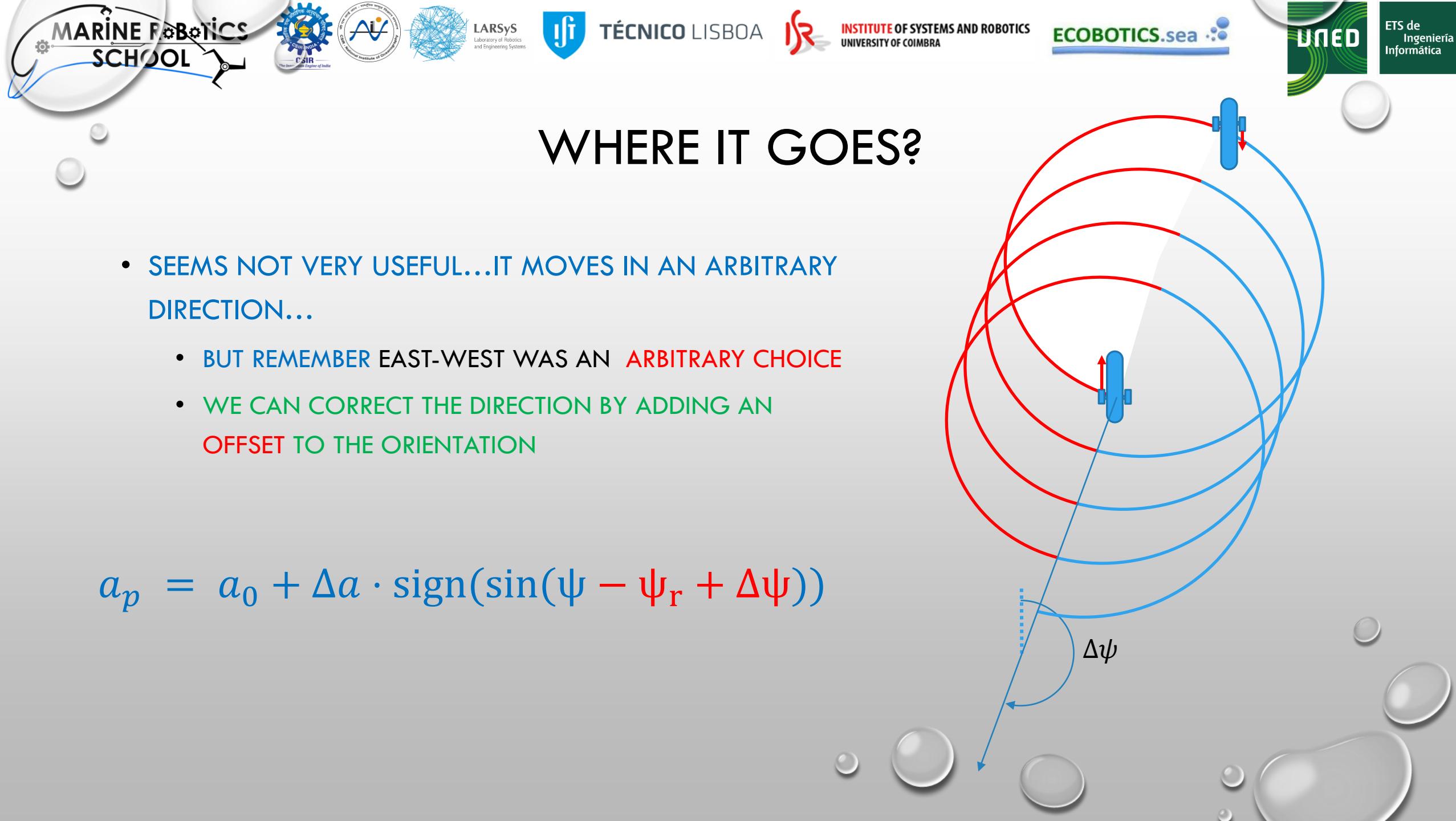
- THE DIRECTION OF THE AVERAGE MOVEMENT DEPENDS ON MANY THINGS IN A COMPLEX WAY...
 - BUT THE IMPORTANT THING IS THAT IT MOVES AND ...
IT IS EASY TO MEASURE

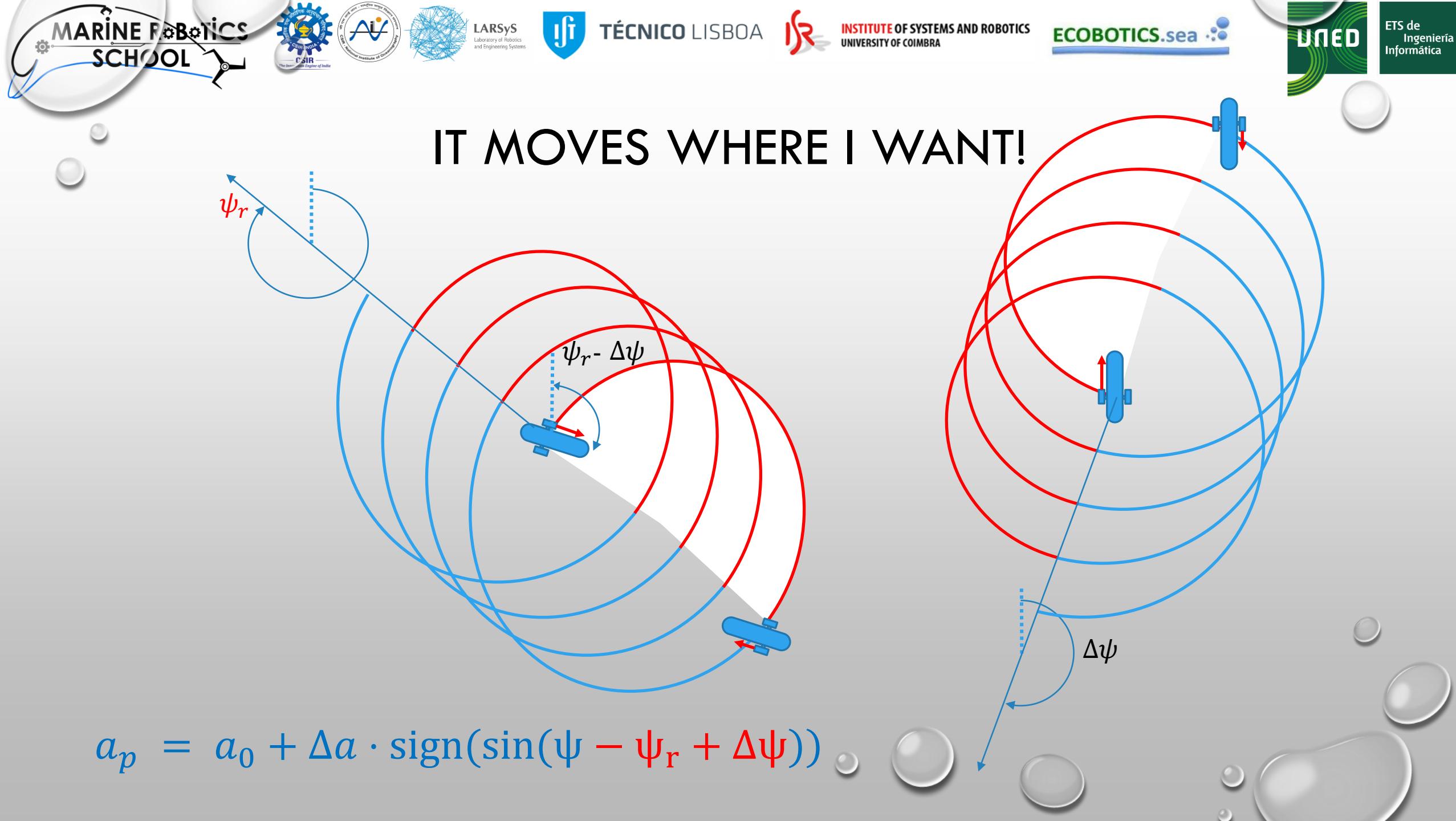


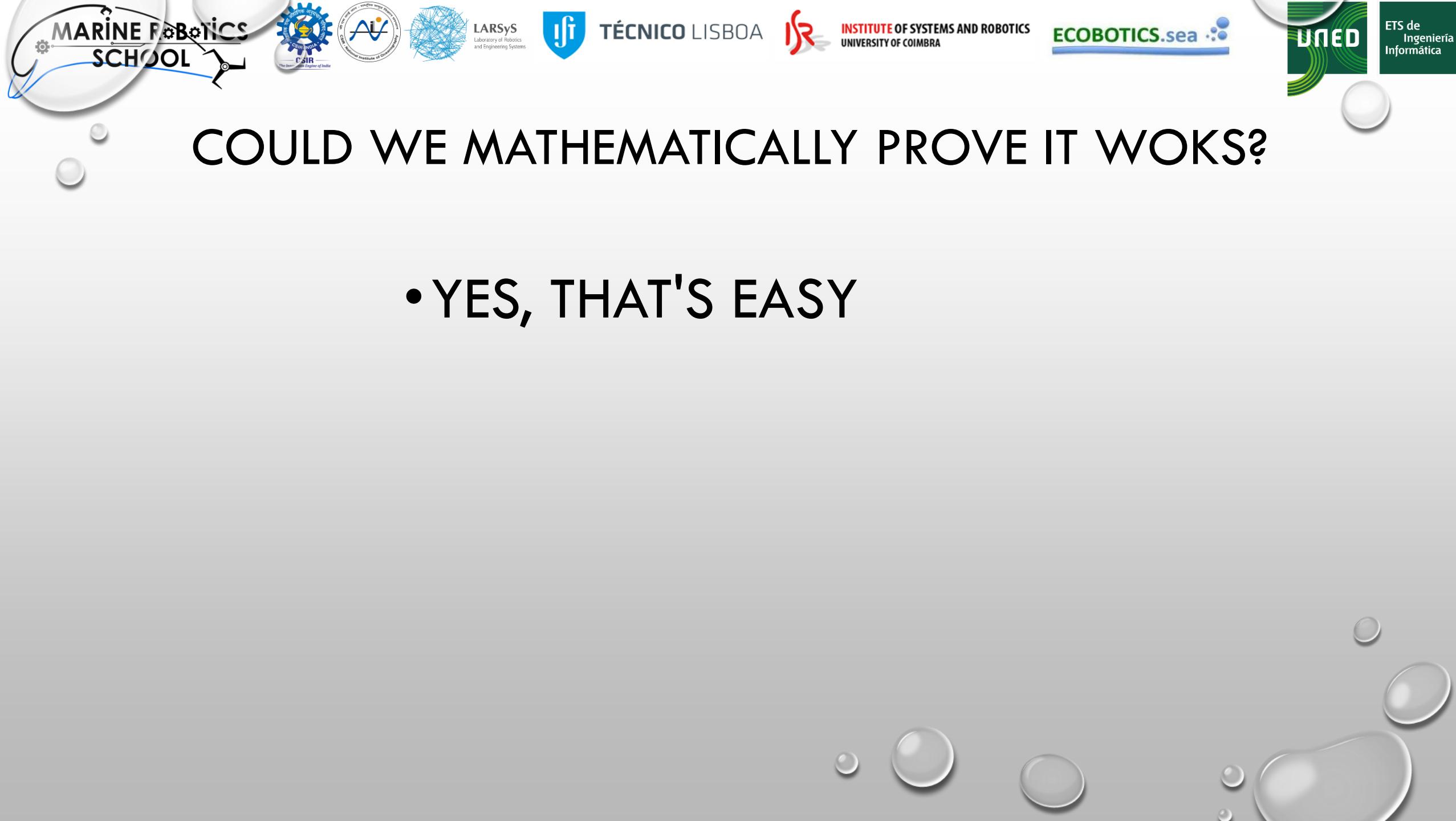


- SEEMS NOT VERY USEFUL...IT MOVES IN AN ARBITRARY DIRECTION...

- BUT REMEMBER EAST-WEST WAS AN **ARBITRARY CHOICE**







COULD WE MATHEMATICALLY PROVE IT WORKS?

- YES, THAT'S EASY

TABLE 1. Symbols and Meaning.

Symbol	Meaning
$\bar{X} = [x, y]^T, \bar{X}_s$	XY-plane position and reference position
x_t, y_t	Transitory positions
$\psi, \dot{\psi}_t, \ddot{\psi}$	Orientation, transitory orientation and fixed point orientation of the vehicle
ψ_f	Limit of the transitory orientation: $\lim_{t \rightarrow \infty} \psi_t = \psi_f$
ε_ψ	Transitory orientation error: $\psi_t - \psi_f$
$x' y' \psi'$	Positions and orientation under a reference frame rotation of angle ρ
$\nu = [u, v, r]^T$	Body-frame vehicle's specific surge, sway and yaw rate
v_{min}, v_{max}	Minimum and maximum yaw rate
$\nu_s, \nu_e, \nu_a \pm$	Body-frame transitory vehicle's speeds, equilibrium vehicle's speeds for $a_s = a_0$ and for $a_p = a_0 \pm \Delta a$, respectively
$\nu^* = [u^*, v^*, r^*]$	Fixed point speeds under control law (11)
$\bar{V} = [\bar{x}, \bar{y}]^T, \bar{V}$	XY plane velocity vector and average speed
V, V_{max}, V_c	Speed, maximum speed and current speed
V_{xf}, V_{yf}, V_f	Average final speeds X-axis speed, Y-axis speed, and total speed, respectively.
V_{x1}, R_e	Stationary equilibrium speed and turning radii when a constant control action a_o is applied
a_s, a_p	Starboard and port thrusters control actions
a_o, a_m	Constant and maximum controller action
F_s, F_p	Starboard and port forces
τ, F	Torque and total force
L	Distance of the thrusters to the symmetry axis
m_u, m_w, m_{uv}, m_r	Mass and inertia constants
D_u, D_w, D_r	Drag coefficients
K	Proportional constant of the thrusters
$X_s, X_{[u]}, Y_v, Y_{[v]}$	Hydrodynamic coefficients
$N_r, N_{[r]}$	Constant perturbation applied to the control action
Δa	Orientation offset, and its empirical estimation
$\Delta \psi, \Delta \dot{\psi}$	Desired average reference direction of movement
ψ_r	Time, transient time and time period
t, t_s, T	Times at which the control action switches
t_n	Times at which the control action switches in the fixed point trajectory
t_{end}	Time at which the the control law stops switching
Δt_n	Interval of time between to consecutive switches
Δ_1, Δ_2	$t_{n+1} - t_n$
$[\Delta x_n, \Delta y_n]^T$	Limits as $n \rightarrow \infty$ of Δt_{2n} and Δt_{2n+1} , respectively
δ_n, δ_m	Distance covered between two consecutive control switches
d_n	Difference between ψ_r in two consecutive switches and bound of $ \delta_n $, respectively
d_{lim}	Distance from the vehicle's position to the target point at t_n
D_n, D_{max}	Limit distance to the target such that if $d_n \leq d_{lim}$ then $d_{n+1} < d_n$
ν_n, ν'_n	Distance between two consecutive switching positions and maximum "diameter" of the trajectory
V	Velocity at switching time t_n starting from two different initial conditions
$c_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$	Lyapunov function
β	Positive constants
$f(\mathbf{x}), g(\mathbf{x})$	Function of class KL
$\gamma(t), \delta(t), g(t)$	Vector functions of the state
$\gamma^{(t)}$	Scalar functions of time
$\xi_1, \xi_2, \xi_x, \xi_y$	Vector function of time
g_m	Transitory functions of time
	Bound of function $g(t_m)$

FIGURE 2. Graphical representation of the forces applied to the AUV by the thrusters, AUV speeds in the body-frame and AUV position in the inertial coordinate frame.

where \dot{x} is the speed along X-axis and \dot{y} is the speed along the Y-axis. Finally ψ is the angle that the vehicle orientation forms with the X-axis and its derivative $\dot{\psi} = r$ is the yaw rate. Then, in the absence of ocean currents, the kinematic model of the vehicle yields

$$\dot{x} = u \cdot \cos(\psi) - v \cdot \sin(\psi) \quad (1)$$

$$\dot{y} = u \cdot \sin(\psi) + v \cdot \cos(\psi) \quad (2)$$

$$\dot{\psi} = r \quad (3)$$

As mentioned above, in the 2D scenario considered and under normal operation, the vehicle is controlled by two frontal thrusters whose inputs are a_s and a_p , which are normalized reference commands for the internal controller of the propellers in the range [-100 100], i.e., the angular speed reference for the starboard and port thrusters, respectively. Then, each propeller produces a force that is proportional to the square of its own angular speed, which is positive (push forward) when the control action is positive, and negative (push backward) when the control action is negative. Thus, the forces F_s and F_p produced by each thruster in response to the control actions are:

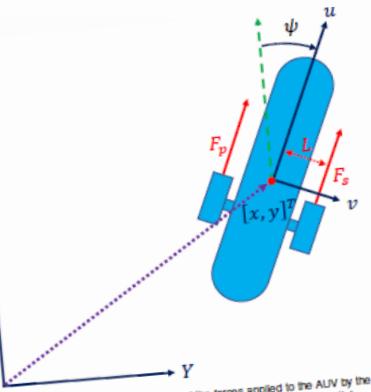
$$F_s(a_s) = K \cdot |a_s| \cdot a_s$$

$$F_p(a_p) = K \cdot |a_p| \cdot a_p$$

where K is the proportional constant which relates control actions to thrust. Then the total force is the summation of the force provided by each of the thrusters, $F = F_p(a_p) + F_s(a_s)$, which also produces the torque $\tau = L \cdot (F_p(a_p) - F_s(a_s))$, being L the distance of each of the thrusters to the symmetry axis of the vehicle.

MATHEMATICALLY PROVE IT WORKS?

THAT'S EASY



VOLUME 4, 2016



TÉCNICO LISBOA



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e_ψ	Transitory orientation error: $\psi - \psi_f$
$x' y' \psi'$	Positions and orientation under a reference rotation of angle ρ
$v = [u, v, r]^T$	Body-frame vehicle's specific surge, sway ar rate
r_{min}, r_{max}	Minimum and maximum yaw rate
ν_e, ν_e, ν_e^\pm	Body-frame transitory vehicle's speeds, minimum vehicle's speeds for $a_p = a_0$ and $a_0 \pm \Delta a$, respectively
$\nu^* = [u^*, v^*, r^*]$	Fixed point speeds under control law (11)
$V = [\dot{x}, \dot{y}]^T, V$	XY plane velocity vector and average sp
V, V_{max}, V_c	Speed, maximum speed and current sp
V_x, V_y, V_f	Average final speeds X-axis speed, Y-a and total speed, respectively.
V_e, R_e	Stationary equilibrium speed and tu
a_s, a_p	when a constant control action a_p is applied
a_e, a_m	Starboard and port thrusters control act
F_s, F_p	Constant and maximum controller act
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L	Torque and total force
m_a, m_s, m_u, m_r	Distance of the thrusters to the symm
D_a, D_s, D_u, D_r	Mass and inertia constants
K	Drag coefficients
$X_a, X_{[a]}, Y_v, Y_{[v]}$	Proportional constant of the thruster
$N_r, N_{[r]}$	Hydrodynamic coefficients
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ψ_r, t_s, T	Desired avg reference directi
t, t_s, T	Time at which the control acti
t_n	Times at which the control acti
t^*	fixed point trajectory
t_{end}	Time at which the control is
Δt_n	Interval of time between to ce
Δ_1, Δ_2	$t_{n+1} - t_n$
$[\Delta x_n, \Delta y_n]^T$	Limits as $n \rightarrow \infty$ of Δt_{2n} a
δ_n, δ_m	Distance covered between two switche
d_n	Difference between ψ_r and $r(t)$
d_{lim}	switches and bound of $ \delta_n $.
D_n, D_{max}	Distance from the vehicle's
ν_n, ν'_n	point at t_n
ν	Limit distance to the target
$c_1, \alpha_1, \alpha_2, \alpha, \varepsilon_1$	then $d_{n+1} < d_n$
β	Distance between two con
$f(\mathbf{x}), g(\mathbf{x})$	tions and maximum "dian
$\gamma(t), \delta(t), g(t)$	Velocities at switching ti
$\zeta^{(t)}$	different initial condition
g_m	Lyapunov function
	Positive constants
	Function of class KL
	Vector functions of the
	Scalar functions of tim
	Vector function of tim
	Transitory functions o
	Bound of function g

VOLUME 4, 2016

A. AVERAGE VELOCITY CONTROL

In the situation where one of the horizontal thrusters is not working, the AUV should be controlled and driven to a desired safety point. Then, a control action such that the vehicle moves, in average, towards a desired direction must be defined. It is important to notice that, as the single control action a_p is used and it impacts directly on both the surge speed u and the yaw rate r , it is not possible to make the vehicle follow a straight path with a certain desired reference surge speed and orientation, as it would be expected if both control actions a_s and a_p could be applied.

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$$m_u \cdot \dot{u} - m_v \cdot v \cdot r + D_u \cdot u = F = F_p(a_p) \quad (4)$$

$$m_v \cdot \dot{v} + m_u \cdot u \cdot r + D_v \cdot v = 0 \quad (5)$$

$$m_r \cdot \dot{r} - m_u \cdot u \cdot v + D_r \cdot r = \tau = L \cdot F_p(a_p) \quad (6)$$

where m_u , m_v and m_r are the mass and inertia constants (mass of the vehicle plus added masses that arise from the interaction with the surrounding water), and $m_{uv} = m_u - m_v$. The Coriolis terms $v \cdot r$, $u \cdot r$, and $u \cdot v$ are caused by the fact that the body frame is rotating.

Finally, the terms $D_u = -\mathcal{X}_u - \mathcal{X}_{[u]} \cdot |u|$, $D_v = -\mathcal{Y}_v - \mathcal{Y}_{[v]} \cdot |v|$, and $D_r = -\mathcal{N}_r - \mathcal{N}_{[r]} \cdot |r|$ are caused by the dissipating forces of the water, where $\mathcal{X}_u, \mathcal{X}_{[u]}, \mathcal{Y}_{[v]}, \mathcal{N}_r, \mathcal{N}_{[r]}$, are the hydrodynamic coefficients, which are negative and then D_u , D_v and D_r are positive.

Therefore, the objective is to design a control action over a_p so that the system (1)-(6) converges to a neighbourhood of the origin from any initial condition, i.e., the AUV can be driven with the use of a single thruster to a desired (recovery) area.

III. CONTROL DESIGN

The AUV velocity and position control are derived for the case in which a critical thruster failure occurs and only one thruster is available. Moreover, this remaining available thruster may also have a limited set of actions due to possible additional damage.

Notice that all the state variables $x(t)$, $y(t)$, $\psi(t)$, $u(t)$, $v(t)$ and $r(t)$ are time-dependent functions. Nevertheless, in the following, for the sake of clarity and with an abuse of notation, the explicit indication that the state variables are time-dependent will be omitted when there is no ambiguity.

This fact can be seen by noticing that a constant orientation ψ implies that $r = 0$, by Eq. (3). Thus, (5) becomes $m_v \cdot \dot{v} + D_v \cdot v = 0$, which has a global asymptotically stable equilibrium point at $v = 0$. The latter is straightforward to prove by considering the Lyapunov function $V = v^2/2$, where $\dot{V} = -\frac{D_v}{m_v} \cdot v^2 < 0$.

However, Eq. (6) becomes $-m_{uv} \cdot u \cdot v = \tau = L \cdot F_p(a_p)$, where the left hand side tends to 0 as $t \rightarrow \infty$, so $F_p(a_p) \rightarrow 0$. Therefore, $u \rightarrow 0$ too, since (4) reduces to $m_u \cdot \dot{u} + D_u \cdot u = 0$ which is also globally asymptotically stable, and it is not possible to converge to a sustainable forward velocity with constant orientation.

At this point, it is important to analyze the AUV behaviour when a constant control action $a_p = a_0$ is applied. In this situation, the equilibrium solution of (4)-(6) is $\nu_e = [u_e, v_e, r_e]^T$:

$$-m_v \cdot v_e + D_u \cdot u_e = F_p(a_0) \quad (7)$$

$$m_u \cdot u_e + r_e \cdot D_v \cdot v_e = 0 \quad (8)$$

$$-m_{uv} \cdot u_e \cdot v_e + D_r \cdot r_e = L \cdot F_p(a_0) \quad (9)$$

Let's suppose that this equilibrium point is globally asymptotically and exponentially stable. In a normal situation, this is usually true since vehicles are designed to have a stable dynamics for constant control actions (the existence of such points will be shown later in Lemma 4).

The yaw rate equation in the time domain can be written as $r(t) = r_e + r_t$, where r_t is a transitory term that, due to the exponential convergence, can be bounded as $|r_t| = |r(t) - r_e| \leq \|\nu(t) - \nu_e\| \leq c_1 \|\nu(0) - \nu_e\| e^{-c_2 t}$, being c_1 and c_2 two positive constants. Thus, the yaw angle can be defined as $\psi(t) = \int_0^t (r_e + r_t) = r_e t + \int_0^t r_t = r_e t + \psi_t$, where ψ_t is a transitory term that converges to a fixed value ψ_f as $t \rightarrow \infty$, $\psi_{t \rightarrow \infty} = \int_0^{\infty} r_t = \psi_f$. This transitory yaw angle ψ_t can be bounded as $|\psi_t| \leq \int_0^{\infty} |r_t| \leq \int_0^{\infty} c_1 \|\nu(0) - \nu_e\| e^{-c_2 t} = \frac{c_1}{c_2} \|\nu(0) - \nu_e\|$. Thus, the average angular speed yields

$$\bar{r} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (r_e + r_t) ds = r_e + \lim_{t \rightarrow \infty} \frac{\psi_t}{t} = r_e.$$

The same analysis holds for $u(t)$ and $v(t)$, considering u_t and v_t the transitory terms of the surge speed and sway speed, respectively.

Therefore, from Eq. (1), the average velocity over the X-axis of the inertial reference frame can be computed as:

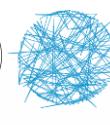
$$\bar{x} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t ((u_e + u_t) \cdot \cos(r_e \cdot s + \psi_t) - (v_e + v_t) \cdot \sin(r_e \cdot s + \psi_t)) ds \quad (10)$$

By letting $e_\psi = \psi_t - \psi_f$, the first term of the integral in (10) can be rewritten as:

$$\begin{aligned} \cos(r_e \cdot s + \psi_t) &= \cos(r_e \cdot s + \psi_f + e_\psi) \\ &= \cos(r_e \cdot s + \psi_f) \cos(e_\psi) - \sin(r_e \cdot s + \psi_f) \sin(e_\psi) \\ &= \cos(r_e \cdot s + \psi_f) + \cos(r_e \cdot s + \psi_f)(\cos(e_\psi) - 1) \\ &\quad - \sin(r_e \cdot s + \psi_f) \sin(e_\psi) = \cos(r_e \cdot s + \psi_f) + \xi_1(s) \end{aligned}$$

CALLY PROVE IT WORKS?

EASY



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$\bar{X} = [x, y]^T, \bar{X}_e$	XY-plane position and reference position
x_t, y_t, ψ_t, ψ^*	Transitory positions Orientation, transitory orientation and fixed p orientation of the vehicle
ψ_f	Orientation error: $\psi - \psi_f$
ε_ψ	Position and orientation under a reference!
$x' y' \psi'$	rotation of angle ρ
$\nu = [u, v, r]^T$	Body-frame vehicle's specific surge, sway ar ate
r_{min}, r_{max}	Minimum and maximum yaw rate
$v_{x,t}, v_{y,t}, v_z$	Body-frame transitory vehicle's speeds, imum vehicle's speeds for $a_p = a_0$ and $a_0 \pm \Delta a$, respectively
$v^* = [u^*, v^*, r^*]$	Fixed point speeds under control law (11)
$V = [x, y]^T, V_e$	XY plane velocity vector and average sp Speed, maximum speed and current spe
$V_{x,max}, V_{y,max}, V_z$	Average final speeds X-axis speed, Y-a and total speed, respectively.
$V_{x,e}, V_{y,e}$	Stationary equilibrium speed and tu when a constant control action a_p is ac
a_s, a_p	Starboard and port thrusters control ac Constant and maximum controller ac
a_{max}, F_p	Starboard and port forces
τ, F	Torque and total force
L	Distance of the thrusters to the symm
m_u, m_v, m_w, m_r, m_r	Mass and inertia constants
D_u, D_v, D_r	Drag coefficients
K	Proportional constant of the thruster
$X_e, X_{[u]}, Y_e, Y_{[v]}$	Hydrodynamic coefficients
$N_r, N_{[r]}$	Constant perturbation applied to th Orientation offset, and its empiric
Δa	Desired average reference directi
$\Delta\psi, \Delta\dot{\psi}$	Time, transient time and time per
ψ_r, t, t_e, T	Times at which the control acti Time at which the control acti
t_n, t^*	fixed point trajectory
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Δt_n	$t_{n+1} - t_n$
Δ_1, Δ_2	Limits as $n \rightarrow \infty$ of Δt_{2n} a tively
$[\Delta x_n, \Delta y_n]^T$	Distance covered between two switches
δ_n, δ_m	Difference between ψ_r i switches and bound of $ \delta_n $
d_n	Distance from the vehicle's point at t_n
d_{lim}	Limit distance to the target then $d_{n+1} < d_n$
D_n, D_{max}	Distance between two con tions and maximum "diam
ν_n, ν'_n	Velocities at switching ti different initial condition
γ	Lyapunov function
β	Positive constants
$f(\mathbf{x}), g(\mathbf{x})$	Function of class KL
$\gamma(t), \delta(t), g(t)$	Vector functions of tim
$\gamma'(t)$	Scalar function of tim
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Finally, the terms $D_u = -\mathcal{X}_u - \mathcal{Y}_{[u]} \cdot |u|$, $D_v = -\mathcal{Y}_v - \mathcal{Y}_{[v]} \cdot |v|$, and $D_r = -\mathcal{N}_r - \mathcal{N}_{[r]} \cdot |r|$ are caused by the dissipating forces of the water, where $\mathcal{X}_u, \mathcal{Y}_{[u]}, \mathcal{Y}_{[v]}, \mathcal{N}_r, \mathcal{N}_{[r]}$, are the hydrodynamic coefficients, which are negative and then D_u , D_v and D_r are positive.

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This fact can be seen by noticing that a constant $a_p = a_0$ implies that $r = 0$ in (6), where $\xi_1(s)$ tends exponentially to zero as $s \rightarrow \infty$ since ψ_t tends exponentially to ψ_f , so $e\psi \rightarrow 0$. Following similar computations, the second term in (10) yields $\sin(r_e \cdot s + \psi_t) = \sin(r_e \cdot s + \psi_f) + \xi_2(s)$, where $\xi_2(s)$ also tends exponentially to zero as $s \rightarrow \infty$, and thus, the average velocity over the X-axis of the inertial reference frame becomes:

$$\begin{aligned} \bar{v}_e &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t ((u_e + u_t) \cos(r_e \cdot s + \psi_t) \\ &\quad - (v_e + v_t) \sin(r_e \cdot s + \psi_t)) ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (u_e \cos(r_e \cdot s + \psi_f) - v_e \sin(r_e \cdot s + \psi_f)) ds \\ &\quad + \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (u_t \cos(r_e \cdot s + \psi_f) - v_t \sin(r_e \cdot s + \psi_f)) ds \\ &\quad + \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (u_e + u_t) \xi_1(s) + (v_e + v_t) \xi_2(s) ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (u_e \cos(r_e \cdot s + \psi_f) - v_e \sin(r_e \cdot s + \psi_f)) ds \end{aligned}$$

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By letting $e\psi :=$
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 $\cos(r_e \cdot s + \psi_t) =$
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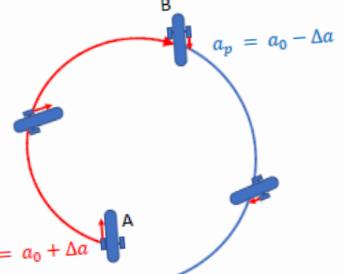


FIGURE 3. Behaviour of control law (11).

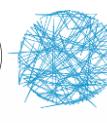
exactly to the North, which is the frontier at which the vehicle changes its orientation from West to East (point A of Figure 3), and a_p switches from $a_p = a_0 - \Delta a$ to $a_p = a_0 + \Delta a$, following the red path of Figure 3. Once ψ reaches $(2n+1)\pi$, the vehicle points exactly to the South (point B of Figure 3) and the control law switches again to $a_p = a_0 - \Delta a$, describing the blue path. Since the control action is different in both parts of the path (red and blue) the trajectory does not close, i.e. it is not a circumference. Thus, the vehicle moves in "some" direction, which depends on the angle or condition to switch the control law. This pattern is repeated periodically producing an average movement of the vehicle on a certain course.

In order to prove the above statement, and before starting the stability analysis of the control law proposed, we must resort to some technical results:

Lemma 1. Consider system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ such that \mathbf{x}_e is an exponentially stable equilibrium point of the system and the first derivative of \mathbf{f} is bounded and Lipschitz on \mathbf{x} in a domain $\|\mathbf{x} - \mathbf{x}_e\| < c_0$, where c_0 is a positive constant. Let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ be two solutions of the differential equation with initial conditions $\mathbf{x}_1(t_0)$ and $\mathbf{x}_2(t_0)$, respectively. Then $\|\mathbf{x}_2(t) - \mathbf{x}_1(t)\| < c_1 e^{-c_2(t-t_0)} \|\mathbf{x}_2(t_0) - \mathbf{x}_1(t_0)\|$ for $t > t_0$, where c_1 and c_2 two positive constants.

Proof. This is a simple application of Theorem 9.1 from [38] for the unperturbed case $\mathbf{g}(\mathbf{x}) = 0$ in equation 9.6 of [38], when, in addition, the system is autonomous. \square

Lemma 2. Consider system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, a_p)$ continuously differentiable and locally Lipschitz in a domain $\|\mathbf{x}\| < c_3 + t_1$, where c_3 is a positive constant and ε_1 is a small positive constant. Suppose also that the solutions of the system are globally uniformly ultimately bounded (GUUB) with bound c_4 for $0 \leq a_p \leq a_m$. Then, $\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(t_0)\|, t - t_0) + c_4 a_m$ for some positive constant c_4 and a function β of class KL , i.e. β is a non decreasing function of its first argument,



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$\psi, \dot{\psi}, \psi^*$	Orientation, transitory orientation and fixed ψ orientation of the vehicle
a_s	Limit of the transitory orientation: $\lim_{t \rightarrow \infty} a_s$
ψ_f	Transitory orientation error: $\psi - \psi_f$
ε_ψ	Positions and orientation under a reference!
$x' y' \psi'$	Orientation of angle ρ
$\nu = [u, v]^T$	Body-frame vehicle's specific surge, sway and yaw rates
r_{min}, r_{max}	Minimum and maximum yaw rate
$v_{x,t}, v_{y,t}, \omega_{x,t}, \omega_{y,t}$	Body-frame transitory vehicle's speeds, angular vehicle's speeds for $a_p = a_0$ and $a_p \pm \Delta a$, respectively
$\nu^* = [u^*, v^*, r^*]^T$	Fixed point speeds under control law (11)
$V = [x, v]^T, V$	XY plane velocity vector and average speed
$V_{x,max}, V_{y,max}, V_{z,max}$	Speed, maximum speed and current speed
$V_{x,f}, V_{y,f}, V_{f}$	Average final speeds X-axis speed, Y-axis speed and total speed, respectively
$V_{x,t}, R_e$	Stationary equilibrium speed and yaw rate
a_s, a_p	when a constant control action a_p is applied
a_{σ, a_m}	Starboard and port thrusters control actuator
F_x, F_y	Constant and maximum controller actuator
τ, F	Starboard and port forces
L	Distance of the thrusters to the symmetry axis and inertia constants
m_u, m_v, m_w, m_r	Mass and inertia constants
D_u, D_v, D_r	Drag coefficients
K	Proportional constant of the thruster
$X_a, X_{[a]}, Y_a, Y_{[a]}$	Hydrodynamic coefficients
$N_r, N_{[r]}$	Constant perturbation applied to the orientation offset, and its empirical value
Δa	Desired average reference direction
$\Delta \psi, \Delta \dot{\psi}$	Orientation offset, and its empirical value
ψ_r	Time at which the control action begins
t, t_s, T	Time at which the control action ends
t_n	Time at which the control action is fixed point trajectory
t^*	Time at which the control action is transitory
t_{end}	Time at which the control action is bounded
Δt_n	Interval of time between t_n and t_{n+1}
Δ_1, Δ_2	Limits as $n \rightarrow \infty$ of Δt_{2n} and Δt_{2n+1}
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D_n, D_{max}	then $d_{n+1} < d_n$
ν_n, ν'_n	Distance between two conditions and maximum "dian"
V	Velocity at switching time
$c_1, \alpha_1, \alpha_2, \alpha, \varepsilon_1$	different initial condition
$\beta(x), g(x)$	Lyapunov function
$\gamma(t), \delta(t), g(t)$	Positive constants
$\gamma^{(t)}, \xi^{(t)}$	Function of class KL
$\xi_1, \xi_2, \xi_x, \xi_y$	Vector functions of the scalar functions of time
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Finally, the terms $D_u = -X_u - X_{[u]} \cdot |u|$, $D_v = -Y_v - Y_{[v]} \cdot |v|$, and $D_r = -N_r - N_{[r]} \cdot |r|$ are caused by the dissipating forces of the water, where $X_u, X_{[u]}, Y_u, Y_{[u]}, N_r, N_{[r]}$, are the hydrodynamic coefficients, which are negative and then D_u , D_v and D_r are positive.

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This fact can be seen by noticing that a constant ψ implies that $r = 0$ and $\dot{r} = 0$, where $\xi_1(s)$ tends exponentially to zero as $s \rightarrow \infty$ since ψ tends exponentially to ψ_f , so $\dot{\psi} \rightarrow 0$. Following similar computations, the second term in (10) yields $\sin(r_e s + \psi_t) = \sin(r_e s + \psi_f) + \xi_2(s)$, where $\xi_2(s)$ also tends exponentially to zero as $s \rightarrow \infty$, and thus, the average velocity over the X-axis of the inertial reference frame becomes:

$$\begin{aligned} \bar{x} &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t ((u_e + u_t) \cos(r_e s + \psi_t) \\ &\quad - (v_e + v_t) \sin(r_e s + \psi_t)) ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (u_e \cos(r_e s + \psi_f) - v_e \sin(r_e s + \psi_f)) ds \\ &\quad + \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (u_t \cos(r_e s + \psi_f) - v_t \sin(r_e s + \psi_f)) ds \\ &\quad + \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (u_e + u_t) \xi_1(s) + (v_e + v_t) \xi_2(s) ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t (u_e \cos(r_e s + \psi_f) - v_e \sin(r_e s + \psi_f)) ds \end{aligned}$$

Let's asymptotically stable dyadic point

The y -axis is to the exponential of the system is bounded over a period are also bounded. The same analysis

frame. This result has an intuitive explanation when enough time has passed, the vehicle's constant speed $V_e = \sqrt{u_e^2 + v_e^2}$ and constant r_e . Thus, the trajectory will be a circle and the average advance velocity is zero.

In order to obtain an non-zero average

symmetry of the circular trajectory

constant input control action is not suitable

control law is proposed:

$$a_p = a_0 + \Delta a \cdot \sin'$$

The same as and v_e the transversely respectively.

Therefore, fix axis of the inert

$$\bar{x} = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t$$

By letting $\varepsilon_\psi =$ (10) can be rewritten

$$\begin{aligned} \cos(r_e s + \psi_t) &= \cos(r_e s + \psi_f) + \xi_2(s) \\ &= \cos(r_e s + \psi_f) - \sin(r_e s + \psi_f) \end{aligned}$$

The control law works as follows:

turns clockwise ($r > 0$) describing an arc trajectory (for Δa small) and pointing to the circular path because $u > 0$ and $v < 0$. Therefore, the velocity vector of the vehicle, which is tangent to the trajectory, points to port. When $\psi = 2\pi$ the vehicle points

and decreasing on the second argument, such that $\beta \rightarrow 0$ as $t \rightarrow \infty$.

Proof. The solutions of the system are GUUB with bound c_2 so, for each initial condition there exists a function β_1 of class KL such that $\|\bar{x}(t)\| \leq \beta_1(\|\bar{x}(t_0)\|, t - t_0) + c_3$ for $t > t_0$. Since $\beta_1 \rightarrow 0$ as $t \rightarrow \infty$, there exists a time $t_s = t_s(\|\bar{x}(t_0)\|)$ such that $\beta_1(\|\bar{x}(t_0)\|, t_s - t_0) < \varepsilon_1$. Then for $t \geq t_s$, $\|\bar{x}(t)\| \leq c_3 + \varepsilon_1$.

Thus, before time t_s , as $a_p \leq a_m$, we know that \bar{x} lies in the region where f is Lipschitz, and following the same arguments of the proof of Lemma 4.6 of [38], we obtain that $\|\bar{x}(t)\| \leq \beta_2(\|\bar{x}(t_s)\|, t - t_s) + c_4 a_m$ for $t \geq t_s$ and β_2 of class KL . Therefore, all the vanishing terms can be bounded by a single KL function β and the condition of the theorem holds for all $t \geq t_0$.

Lemma 3. Consider system $\dot{\nu} = f(\nu, a_p)$ defined by equations (1)-(6), where $\nu = [u, v, r]$. Let $\nu_e = [u_e, v_e, r_e]$ be the equilibrium solution of the system for $a_p = a_0 > 0$, then $u_e, r_e > 0$, $v_e < 0$ and the Jacobian $\frac{\partial f}{\partial \nu}$ is continuous and Lipschitz in a neighbourhood of ν_e .

Proof. Due to the presence of absolute values in the drag terms D_u , D_v and D_r on model (4)-(6), the Jacobian matrix of the system is not differentiable when u , v or r are zero. In the rest of domain the system is smooth. Let's prove by contradiction that for $a_0 > 0$, the equilibrium solutions are such that $u_e \neq 0$, $v_e \neq 0$ and $r_e \neq 0$.

Suppose that $u_e = 0$, then (8) becomes $D_v(v_e) \cdot v_e = 0$, which is a contradiction, and demonstrates that $u_e \neq 0$. It implies that, given (7), $F_p(a_0) = 0$ and thus, $v_e = 0$. Notice that this average speed is zero because the of exponentially decreasing functions are bounded over a period are also bounded. The same analysis

frame. This result has an intuitive explanation when enough time has passed, the vehicle's constant speed $V_e = \sqrt{u_e^2 + v_e^2}$ and constant r_e . Thus, the trajectory will be a circle and the average advance velocity is zero.

Finally, suppose that $r_e = 0$. Given (8), it yields $v_e = 0$, which is again a contradiction and $r_e \neq 0$.

To check that the signs of the speeds are correct, consider the linearization of (7) and (9) around the equilibrium point

$$|\nu(t)| \leq \sqrt{\frac{c_6}{c_2}} \|\nu(t_0)\| e^{-\frac{c_6}{2c_2}(t-t_0)}$$

$$+ \frac{c_6(1+L)}{2c_2} F_p(a_m) \int_{t_0}^t e^{-\frac{c_6}{2c_2}(t-s)} ds$$

$$\leq \sqrt{\frac{c_6}{c_2}} \|\nu(t_0)\| e^{-\frac{c_6}{2c_2}(t-t_0)}$$

$$+ \frac{c_6^2(1+L)}{c_2 c_7} F_p(a_m) \left[1 - e^{-\frac{c_6}{2c_2}(t-t_0)} \right]$$

$$\leq c_1 \|\nu(t_0)\| e^{-c_2(t-t_0)} + \frac{c_6^2(1+L)}{c_2 c_7} F_p(a_m)$$

This proves that the convergence to a neighbourhood of the origin is exponential, so letting $a_p = a_0$ small enough, the linearization of the system could be as close as desired to the continuous. Then, the trajectory converges exponentially to the attractive region of the linear system, and the equilibrium point is globally exponentially stable.

Finally, $r_e(a_0) \rightarrow 0$ as $a_0 \rightarrow 0$, so $c_1 e^{-\frac{c_6}{2c_2}(t-t_0)} \rightarrow c_1 e^{-\infty} = 0$. Thus, a_0 can be chosen so that $c_1 e^{-\frac{c_6}{2c_2}(t-t_0)} < 1$.

Now we have the necessary tools to analyse the behaviour of the control law (11).

Lemma 4. Consider the system $\dot{\nu} = f(\nu, a_p)$ defined by equations (1)-(6). There exists an a_m such that if $0 \leq a_p \leq a_m$, then the equilibrium point ν_e for $a_p = a_0$ is globally exponentially stable, i.e., there exist positive constants c_1 and c_2 such that $\|\nu(t)\| \leq c_1 e^{-c_2(t-t_0)} \|\nu(t_0) - \nu_e\|$ for $t > t_0$.

Lemma 5. Consider system (1)-(6) under control action

(11), then it is possible to choose a_0 and Δa such that

the velocities of the system converge to an exponentially

stable limit cycle of sustained oscillation and, in addition the

average velocity of the system also converges exponentially

to a stable fixed value $[\bar{x}, \bar{y}]^T \rightarrow [\bar{V}_{x,f}, \bar{V}_{y,f}]^T \neq 0$.

MARINE ROBOTICS SCHOOL

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TABLE 1. Symbols and Meaning.

Symbol	Meaning
$X = [x, y]^T, X_s$	XY-plane position and π
x_t, y_t	Transitory positions
ψ, ψ_t, ψ^*	Orientalion, transitory orientation of the vehicle
Δa	Limit of the transitor
ψ_f	Transitory orientation
x^*, y^*, ψ^*	Positions and orientation of the vehicle
$\nu = [u, v, r]^T$	Body-frame vehicle
r_{min}, r_{max}	rate
$\nu_t, \nu_e, \nu_r, \nu_e^*$	Minimum and maximum vehicle
$\nu^* = [u^*, v^*, r^*]$	Fixed point
$V = [x, v, r]^T, V$	XY plane
$V_x, V_{x,f}, V_r, V_{x,f}, V_{y,f}, V_{y,f}$	Speed, m and rotation
V_e, R_e	Station when odd
a_s, a_p	Cor. and a_m
a_o, a_m	Body-frame vehicle
F_s, F_p	Star
F, L	St
m_u, m_v, m_w, m_r	Y-axis of the inertial frame
D_u, D_v, D_r	Vehicle
K	constant
$X_s, X_{n+1}, Y_s, Y_{n+1}$	Mean value theorem
N_s, N_{n+1}	between t_{n+1} and t_{n+1}
Δa	with constant c_8 , we get $\ f(\nu(t), a_0 + \Delta a) - f(\nu(t_{n+1}), a_0 + \Delta a)\ = \ f(\nu(t), a_0 + \Delta a) - 0\ \leq c_8 \ \nu - \nu_e^*\ \leq c_8 \Delta a$. Thus $\ \nu'(t_{n+1}) - \nu'(t_{n+1})\ \leq \ \nu'_{n+1} - t_{n+1}\ c_8 \Delta a$, and, considering again Lemma 1 with (16) and (17);
Δt_n	switches of the control action, i.e., $\Delta t_n = t_{n+1} - t_n$, and $r_t = r - r_e^+$, which implies
Δt_n	where Δt_n is the time interval between two consecutive switches of the control action, i.e., $\Delta t_n = t_{n+1} - t_n$, and $r_t = r - r_e^+$, which implies
Δt_n	Finally, by the mean value theorem (see [40] Ch.11 Theorem 4), $\frac{\nu'(t_{n+1}) - \nu'(t_{n+1})}{t_{n+1} - t_{n+1}} = f(\nu(t), a_0 + \Delta a)$ for some t between t_{n+1} and t_{n+1} . Furthermore, as f is locally Lipschitz with constant c_8 , we get $\ f(\nu(t), a_0 + \Delta a) - f(\nu'(t_{n+1}), a_0 + \Delta a)\ = \ f(\nu(t), a_0 + \Delta a) - 0\ \leq c_8 \ \nu - \nu_e^*\ \leq c_8 \Delta a$. Thus $\ \nu'(t_{n+1}) - \nu'(t_{n+1})\ \leq \ \nu'_{n+1} - t_{n+1}\ c_8 \Delta a$, and, considering again Lemma 1 with (16) and (17);
Δt_n	Taking into account the exponential convergence
Δt_n	At switching time t_{n+1} the system reaches the state ν_{n+1} . Consider now another trajectory that starts at ν'_n at the same starting time t_n and ends on state $\nu'_{n+1} = \nu'(t'_{n+1})$ at a slightly different switching time t'_{n+1} , then:
$\nu'_{n+1} - \nu_{n+1} = \nu'(t'_{n+1}) - \nu'(t_{n+1})$	$= \nu'(t_{n+1}) - \nu(t_{n+1}) + \nu'(t_{n+1}) - \nu'(t_{n+1})$
$\nu'_{n+1} - \nu_{n+1}$	$= \nu'(t_{n+1}) - \nu(t_{n+1}) + \nu'(t_{n+1}) - \nu'(t_{n+1})$

Proof. As shown by Lemma 4, given a control action $a_p = a_0$ and a bounded disturbance Δa small enough so that $0 < a_0 - \Delta a < a_0 < a_0 + \Delta a < a_m$, then, the system (4)-(6) has globally exponentially stable equilibrium points at $\nu_e = [u_e, v_e, r_e]$, $\nu_e^+ = [u_e^+, v_e^+, r_e^+]$ and $\nu_e^- = [u_e^-, v_e^-, r_e^-]$ for $[u_e, v_e, r_e]$, $\nu_e^+ = [u_e^+, v_e^+, r_e^+]$ and $\nu_e^- = [u_e^-, v_e^-, r_e^-]$ for $[u_e, v_e, r_e]$.

In addition, all the three equilibrium points satisfy $c_1 e^{-\frac{r_e}{r_a}} < 1$. Lemma 3 shows that the solutions are GUUB and the system is locally Lipschitz in any bounded region, thus, Lemma 2 implies that $\nu(t)$ will converge to a neighbourhood of ν_e than can be made arbitrary small. Therefore, without loss of generality, we can start the analysis at time t_0 such that $\|\nu - \nu_e\| \leq c_4 \Delta a$, where c_4 is the positive constant of Lemma 2.

The yaw rate equation can be expressed as $r(t) = r_e^+ + r_t$, where r_t is a transitory function that tends to zero when a_p is constant. Using the exponential convergence, and the fact that $r_{min} \leq r(t) \leq r_{max}$ where $r_{min} = \min(r_e^+, r_e^-) - c_1 c_4 \Delta a$ and $r_{max} = \max(r_e^+, r_e^-) + c_1 c_4 \Delta a$, being both positive if Δa is small enough.

Since $r(t)$ is bounded and positive, then $\psi(t)$ is strictly increasing and there exists a sequence of times t_n such that at which the control action a_p switches from $a_0 - \Delta a$ to $a_0 + \Delta a$ when n is even, or from $a_0 + \Delta a$ to $a_0 - \Delta a$ when n is odd.

Now, consider the trajectory that starts at $\nu_n = \nu(t_n)$, being t_n an even switching time. During some time the control action $a_p = a_0 + \Delta a$ will remain constant, and the angle ψ will increase with $\psi(t_n) = n\pi$, for $n = 1, 2, \dots$. These are precisely the times at which the control action a_p switches from $a_0 - \Delta a$ to $a_0 + \Delta a$ when n is even, or from $a_0 + \Delta a$ to $a_0 - \Delta a$ when n is odd.

Finally, by the mean value theorem (see [40] Ch.11 Theorem 4), $\frac{\nu'(t_{n+1}) - \nu'(t_{n+1})}{t_{n+1} - t_{n+1}} = f(\nu(t), a_0 + \Delta a)$ for some t between t_{n+1} and t_{n+1} . Furthermore, as f is locally Lipschitz with constant c_8 , we get $\|f(\nu(t), a_0 + \Delta a) - f(\nu'(t_{n+1}), a_0 + \Delta a)\| = \|f(\nu(t), a_0 + \Delta a) - 0\| \leq c_8 \|\nu - \nu_e^*\| \leq c_8 \Delta a$. Thus $\|\nu'(t_{n+1}) - \nu'(t_{n+1})\| \leq \|\nu'_{n+1} - t_{n+1}\| c_8 \Delta a$, and, considering again Lemma 1 with (16) and (17);

Within (11) it is that the vehicle is one of the two possible trajectories around the origin. The sign of ν_e is the same as the sign of ν_{n+1} , with $n = 1, 2, \dots$. The sign of (11) is the same as the sign of (11) and, on any other hand, the sign of (11) is the same as the sign of (11). Within (11) the vehicle is one of the two possible trajectories around the origin. The sign of ν_e is the same as the sign of ν_{n+1} , with $n = 1, 2, \dots$. The sign of (11) is the same as the sign of (11) and, on any other hand, the sign of (11) is the same as the sign of (11).

A similar analysis can be carried out for the transition from t_{n+1} to t_{n+2} , so finally:

$$\|\nu'_{n+2} - \nu_{n+2}\| \leq \alpha_2 \|\nu'_{n+1} - \nu_{n+1}\| \leq \alpha_2 \alpha_1 \|\nu'_n - \nu_n\| = \alpha \|\nu'_n - \nu_n\|$$

where $\alpha_2 = (c_1 e^{-c_2 \Delta t_{n+1}} + \frac{2c_1 c_4 \Delta a}{r_{min}})$. Note that the factor in the term $\frac{2c_1 c_4 \Delta a}{r_{min}}$ is consequence of the fact that in this case both $|t'_{n+1} - t_{n+1}|$ and $|t'_{n+2} - t_{n+1}|$ need to be bounded, as in (17). The constant α in (19) can be made less than one by selecting Δa small enough, i.e. if $\Delta a \rightarrow 0$,

and decreasing on the second argument, such that $\beta \rightarrow 0$ as $t \rightarrow \infty$.

Proof. The solutions of the system are GUUB with bound c_2 so, for each initial condition there exists a function β_1 of class KL such that $\|\nu(t)\| \leq \beta_1(\|\nu(t_0)\|, t - t_0) + c_2$ for $t > t_0$. Since $\beta_1 \rightarrow 0$ as $t \rightarrow \infty$, there exists a time $t_s = t_s(\|\nu(t_0)\|)$ such that $\beta_1(\|\nu(t_0)\|, t_s - t_0) < \varepsilon_1$. Then for $t \geq t_s$, $\|\nu(t)\| \leq c_2 + \varepsilon_1$.

Thus, before time t_s , as $a_p \leq a_m$, we know that ν lies in the region where f is Lipschitz, and following the same arguments of the proof of Lemma 4.6 of [38], we obtain that $\|\nu(t)\| \leq \beta_2(\|\nu(t_s)\|, t - t_s) + c_4 a_m$ for $t \geq t_s$ and β_2 of class KL . Therefore, all the vanishing terms can be bounded by a single KL function β and the condition of the theorem holds for all $t \geq t_0$.

Lemma 3. Consider system $\dot{\nu} = f(\nu, a_p)$ defined by equations (1)-(6), where $\nu = [u, v, r]$. Let $\nu_e = [u_e, v_e, r_e]$ be the equilibrium solution of the system for $a_p = a_0 > 0$, then $v_e, r_e > 0$, $u_e < 0$ and the Jacobian $\frac{\partial f}{\partial \nu}$ is continuous and Lipschitz in a neighbourhood of ν_e .

Proof. Due to the presence of absolute values in the drag terms D_u, D_v and D_r on model (4)-(6), the Jacobian matrix of the system is not differentiable when u, v or r are zero. In the rest of domain the system is smooth. Let's prove by contradiction that for $a_0 > 0$, the equilibrium solutions are such that $u_e \neq 0, v_e \neq 0$ and $r_e \neq 0$.

Suppose that $u_e = 0$, then (8) becomes $D_v(v_e) \cdot v_e = 0$ and thus, $v_e = 0$. It implies that, given (7), $F_p(a_0) = 0$ and $a_0 = 0$, which is a contradiction, and demonstrates that $u_e \neq 0$.

Now suppose that $v_e = 0$. Then, (8) becomes $D_v(u_e) \cdot u_e = 0$, which implies that $r_e = 0$ because it is already known that $r_e \neq 0$. However, given (9), $L \cdot F_p(a_0) = 0$, which is also a contradiction, and then $v_e \neq 0$.

Finally, suppose that $r_e = 0$. Given (8), it yields $v_e = 0$, which is again a contradiction and $r_e \neq 0$.

To check that the signs of the speeds are correct, consider the linearization of (7) and (9) around the equilibrium point

$$\|\nu(t)\| \leq \sqrt{\frac{c_6}{c_2}} \|\nu(t_0)\| e^{-\frac{c_6}{2c_2}(t-t_0)} + \frac{c_6(1+L)}{2c_2} F_p(a_m) \int_{t_0}^t e^{-\frac{c_6}{2c_2}(t-s)} ds \leq \sqrt{\frac{c_6}{c_2}} \|\nu(t_0)\| e^{-\frac{c_6}{2c_2}(t-t_0)} + \frac{c_6^2(1+L)}{c_2 c_7} F_p(a_m) \left[1 - e^{-\frac{c_6}{2c_2}(t-t_0)} \right] \leq c_1 \|\nu(t_0)\| e^{-c_2(t-t_0)} + \frac{c_6^2(1+L)}{c_2 c_7} F_p(a_m)$$

This proves that the convergence to a neighbourhood of the origin is exponential, so letting $a_p = a_0$ small enough, the linearization of the system could be as close as desired to the continuous. Then, the trajectory converges exponentially to the attractive region of the linear system, and the equilibrium point is globally exponentially stable.

Finally, $r_e(a_0) \rightarrow 0$ as $a_0 \rightarrow 0$, so $c_1 e^{-\frac{r_e}{r_a}} \rightarrow c_1 e^{-\infty} = 0$. Thus, a_0 can be chosen so that $c_1 e^{-\frac{r_e}{r_a}} < 1$.

Now we have the necessary tools to analyse the behaviour of the control law (11).

Lemma 4. Consider the system $\dot{\nu} = f(\nu, a_p)$ defined by equations (1)-(6). There exists an a_m such that if $0 \leq a_p \leq a_m$, then the equilibrium point ν_e for $a_p = a_0$ is globally exponentially stable, i.e. there exist positive constants c_1 and c_2 such that $\|\nu(t)\| \leq c_1 e^{-c_2(t-t_0)} \|\nu(t_0) - \nu_e\|$ for $t > t_0$. Moreover $c_1 e^{-\frac{r_e}{r_a}} < 1$.

Proof. Consider the Lyapunov function $V = \frac{1}{2}(m_u u^2 + m_v v^2 + m_r r^2)$ which represents the kinetic energy of the vehicle and the surrounding water. Then, $c_2 \|\nu\|^2 \leq V \leq c_5 \|\nu\|^2$ where $c_5 = \frac{1}{2} \min(m_u, m_v, m_r)$. The derivative of V becomes:

$$\dot{V} = -D_u u^2 - D_v v^2 - D_r r^2 + u F_p(a_p) + r L F_p(a_p)$$

$$\leq -|X_u| u^2 - |Y_v| v^2 - |N_r| r^2 + \|u\|(1+L) F_p(a_p)$$

$$\leq -\min(|X_u|, |Y_v|, |N_r|) \|\nu\|^2 + \|\nu\|(1+L) F_p(a_p)$$

$$\leq -c_7 \|\nu\|^2 + \|\nu\|(1+L) F_p(a_p)$$

From theorem 4.10 of [38], the system (4)-(6) is globally exponentially stable when $a_p = 0$. Furthermore, if a_p is bounded, i.e., $0 \leq a_p \leq a_m$, then (14) is negative if $\|\nu\| \geq \frac{(1+L) F_p(a_m)}{c_7}$ and $c_7 = \min(|X_u|, |Y_v|, |N_r|)$. Therefore, the system is Input to State Stable (ISS) with respect to a_p , by theorem 4.19 of [38]. In addition, since $\|\frac{\partial \nu}{\partial a}\| \leq 2c_6 \|\nu\|$, and considering the special case of $\gamma(t) = 0$ and $\delta(t) = (1+L) F_p(a_m)$ in Lemma 9.4 of [38], we obtain:

$$c_2 \|\nu(t)\| \leq \sqrt{\frac{c_6}{c_2}} \|\nu(t_0)\| e^{-\frac{c_6}{2c_2}(t-t_0)} + \frac{c_6(1+L)}{2c_2} F_p(a_m) \int_{t_0}^t e^{-\frac{c_6}{2c_2}(t-s)} ds \leq \sqrt{\frac{c_6}{c_2}} \|\nu(t_0)\| e^{-\frac{c_6}{2c_2}(t-t_0)} + \frac{c_6^2(1+L)}{c_2 c_7} F_p(a_m) \left[1 - e^{-\frac{c_6}{2c_2}(t-t_0)} \right] \leq c_1 \|\nu(t_0)\| e^{-c_2(t-t_0)} + \frac{c_6^2(1+L)}{c_2 c_7} F_p(a_m)$$

Within (11) it is that the vehicle is one of the two possible trajectories around the origin. The sign of ν_e is the same as the sign of ν_{n+1} , with $n = 1, 2, \dots$. The sign of (11) is the same as the sign of (11) and, on any other hand, the sign of (11) is the same as the sign of (11).

Lemma 5. Consider system (1)-(6) under control action (11), then it is possible to choose a_0 and Δa such that the velocities of the system converge to an exponentially stable limit cycle of sustained oscillation and, in addition the average velocity of the system also converges exponentially to a stable fixed value $[\dot{x}, \dot{y}]^T \rightarrow [V_{x,f}, V_{y,f}]^T \neq 0$.

TABLE 1. Symbols and Meaning.

Symbol	Meaning
$\mathbf{X} = [x, y]^T, \mathbf{X}_s$	XY-plane position and orientation
x_t, y_t, ψ_t, ψ^*	Transitory positions and orientation of the vehicle
$\dot{\psi}_f$	Limit of the transitory orientation of the vehicle
x^*, y^*, ψ^*	Position and orientation of the vehicle
$\nu = [u_e, v_e, r_e]^T$	Body-frame velocity
r_{min}, r_{max}	Minimum and maximum vehicle radius
$\nu_e^\pm = [u_e^\pm, v_e^\pm, r_e^\pm]$	Fixed points
$\mathbf{V} = [x, y]^T, \mathbf{V}_s$	XY plane speed
V_x, V_y, V_z	Average speed, m and total
V_{xf}, V_{yf}, V_{zf}	Station when
V_{xe}, R_e	Station when odd
a_o, a_p	Now, consider the trajectory that starts at $\nu_e = \nu(t_n)$, being t_n an even switching time. During some time the control action $a_p = a_0 + \Delta a$ will remain constant, and the angle ψ will increase from $\psi(t_n) = n\pi$ to $\psi(t_{n+1}) = (n+1)\pi$, so:
a_o, a_m	where Δt_n is the time interval between two consecutive switches of the control action, i.e., $\Delta t_n = t_{n+1} - t_n$, and $r_t = r - r_e^+$, which implies
F_s, F_p	$\pi - \int_{t_n}^{t_{n+1}} r_t ds \leq \Delta t_n \leq \pi + \int_{t_n}^{t_{n+1}} r_t ds$
τ, L	Taking into account the exponential convergence
m_u, m_v, m_w, m_r	$\pi - \frac{c_1 c_4 \Delta a}{r_e^+} \leq \Delta t_n \leq \pi + \frac{c_1 c_4 \Delta a}{r_e^+}$
D_{ν_e}, D_ψ, D_r	At switching time t_{n+1} the system reaches the state ν_{n+1} . Consider now another trajectory that starts at ν'_e at the same starting time t_n and ends on state $\nu'_{n+1} = \nu'(t'_{n+1})$ at a slightly different switching time t'_{n+1} , then:
K	$\nu'_{n+1} - \nu_{n+1} = \nu'(t'_{n+1}) - \nu(t_{n+1})$
$X_e, X_{(u)}, Y_e, Y_{(v)}$	$= \nu'(t_{n+1}) - \nu(t_{n+1}) + \nu'(t'_{n+1}) - \nu'(t_{n+1})$
$N_e, N_{(u)}, N_{(v)}$	(16)
Δa	VOLUME 4, 2016

Proof. As shown by Lemma 4, given a control action $a_p = a_0 + \Delta a$ and a bounded disturbance Δa small enough so that $0 < a_0 - \Delta a < a_0 < a_0 + \Delta a < a_m$, then, the system (4)-(6) has globally exponentially stable equilibrium points at $\nu_e = [\nu_e^+, \nu_e^-, r_e^+]$ and $\nu_e^- = [\nu_e^-, \nu_e^+, r_e^-]$ and $\nu_e^+ = [u_e^+, v_e^+, r_e^+]$ and $\nu_e^- = [u_e^-, v_e^-, r_e^-]$. As (15) also shows, $\nu_e^+ = \nu_e^- = \nu_e^+$ and $\nu_e^- = \nu_e^-$.

Lemma 3 shows that the solutions $\nu(t)$ then $c_1 e^{-c_2(\Delta t_{n+1} + \Delta t_n)} \rightarrow c_1^2 e^{-c_2 T} \leq c_1^2 e^{-\frac{c_2 \pi}{r_e^+}}$ addition, all the three equilibrium points

Equation (19) shows that the transition from two consecutive even switching times $t_{2n} \rightarrow t_{2n+2}$ is an exponential contraction mapping from the set $\|\nu - \nu_e\| < c_4 \Delta a$ to itself, and thus a fixed point ν^* exists such that if $\nu_{2n} = \nu^*(t_{2n})$ then $\nu_{2n+2} = \nu_{2n}$ by the Contraction Mapping theorem (Theorem B.1 of [38]).

This further implies that ν_{2n} tends exponentially fast to zero, where r_t is a transitory function that tends to zero. Using the exponential convergence, and since $\nu^*(t_{2n}) \leq \nu_e^- \leq c_4 \Delta a$, it yields $|r_t| \leq c_4 \Delta a$, and there $r_{min} \leq r_t \leq r_{max}$ where $r_{min} = \min(r_e^+, r_e^-) - c_4 \Delta a$ and $r_{max} = \max(r_e^+, r_e^-) + c_4 \Delta a$, being both positive.

Since $r(t)$ is bounded and positive, then $\psi(t)$ is strictly increasing, and there exists a sequence of times t_n such that $\psi(t_n) = n\pi$, for $n = 1, 2, \dots$. These are precisely the times at which the control action a_p switches from $a_0 - \Delta a$ to $a_0 + \Delta a$ when n is even, or from $a_0 + \Delta a$ to $a_0 - \Delta a$ when n is odd.

To conclude the proof, the average velocity can be computed following the same procedure of (10):

$$\begin{aligned} \dot{x} &= \frac{1}{T} \int_0^T (u^*(t) \cos(\psi^*(t)) - v^*(t) \sin(\psi^*(t))) dt \\ &= \frac{1}{T} \int_0^{2\pi} \left(\frac{u^*(\psi)}{r^*(\psi)} \cos(\psi) - \frac{v^*(\psi)}{r^*(\psi)} \sin(\psi) \right) d\psi = V_{xf} \end{aligned} \quad (20)$$

$$\begin{aligned} \dot{y} &= \frac{1}{T} \int_0^T (u^*(t) \sin(\psi^*(t)) + v^*(t) \cos(\psi^*(t))) dt \\ &= \frac{1}{T} \int_0^{2\pi} \left(\frac{u^*(\psi)}{r^*(\psi)} \sin(\psi) + \frac{v^*(\psi)}{r^*(\psi)} \cos(\psi) \right) d\psi = V_{yf} \end{aligned} \quad (21)$$

where the change of variables $\psi^*(t) = \psi$, $d\psi = r^*(\psi) dt$ has been applied. To verify that the average velocity can be chosen to be different from zero with a proper selection of a_0 consider the limit case in which a_0 is very small and $\Delta a \ll a_0$.

On the one hand, from (7) and (9) and expanding them up to the second order in a_0 and Δa , we obtain $\nu_e^\pm \approx \frac{1}{r_e^+} F_p(a_0 \pm \Delta a)$ and $r_e^\pm \approx \frac{1}{|X_e|} F_p(a_0 \pm \Delta a)$. Thus, a similar analysis can be done for $t_{n+1} \rightarrow t_{n+2}$, so finally:

$$\|\nu'_{n+2} - \nu_{n+2}\| \leq a_2, \quad \text{or } \frac{u^*(\psi)}{r^*(\psi)} \cos(\psi) \text{ is zero.}$$

On the other hand, from (8), $v_e^\pm \approx -\frac{m_u u_e^\pm r_e^\pm}{|Y_e|}$, so

$$\text{where } a_2 = (c_1 e^{-c_2 \Delta t_{n+1}} + \frac{c_2}{r_e^+}) = -\frac{m_u u_e^\pm}{|Y_e|} + \frac{2Km_u}{|X_e|} a_0 \Delta a. \text{ This implies}$$

that $\frac{u^*(\psi)}{r^*(\psi)}$ is a function that converges to two different values at $\psi = 0$ and $\psi = \pi$. Furthermore, the convergence to the steady state is very fast since the linearization of (4) for small enough velocities is $m_u \dot{u} + |X_e| u = F_p(a_0)$, thus $u^*(t)$ converges exponentially to u_e^\pm with rate $\frac{m_u}{|X_e|}$. Therefore, $u^*(\psi)$ converges exponentially fast to u_e^\pm with, at least, the same applies to $r^*(t)$, thus $\frac{u^*(\psi)}{r^*(\psi)}$ could be replaced in the integral by its steady state value $\frac{u_e^\pm}{r_e^\pm}$.

factor in the term $\frac{2c_1 c_4 \Delta a}{r_e^+}$ is $c_1 c_4 \Delta a / r_e^+$, which is less than one by selecting Δa small, in this case both $|t'_{n+1} - t_{n+1}| / \Delta t_n$ can be bounded, as in (17). The constant

$$\begin{aligned} \dot{x} &\approx \frac{1}{T} \int_0^{2\pi} \frac{v^*(\psi)}{r^*(\psi)} \sin(\psi) d\psi \\ &\approx \frac{1}{T} \int_0^{\pi} \frac{v_e^+}{r_e^+} \sin(\psi) d\psi + \frac{1}{T} \int_{\pi}^{2\pi} \frac{v_e^-}{r_e^-} \sin(\psi) d\psi \\ &= \frac{2}{T} \left(\frac{v_e^+ - v_e^-}{r_e^+ - r_e^-} \right) \approx -\frac{8Km_u}{T|Y_e|X_e|} a_0 \Delta a \neq 0 \end{aligned}$$

? on the second argument, such that $\beta \rightarrow 0$ as $t \rightarrow t_0$. Moreover $c_1 e^{-c_2(t-t_0)} \|\nu(t_0) - \nu_e\| < 1$.

Proof. Consider the Lyapunov function $V = \frac{1}{2}(m_u u^2 + m_v v^2 + m_w w^2)$ which represents the kinetic energy of the vehicle and the surrounding water. Then, $c_2 \|\nu\|^2 \leq V \leq c_3 \|\nu\|^2$ where $c_2 = \frac{1}{2} \min(m_u, m_v, m_w)$ and $c_3 = \frac{1}{2} \max(m_u, m_v, m_w)$. The derivative of V becomes:

$$\begin{aligned} \dot{V} &= -D_u u^2 - D_v v^2 - D_w w^2 + u F_p(a_p) + r L F_p(a_p) \\ &\leq -|X_e| u^2 - |Y_e| v^2 - |Z_e| w^2 + \|\nu\|(1+L) F_p(a_p) \\ &\leq -\min(|X_e|, |Y_e|, |Z_e|) \|\nu\|^2 + \|\nu\|(1+L) F_p(a_p) \\ &\leq -c_7 \|\nu\|^2 + \|\nu\|(1+L) F_p(a_p) \end{aligned} \quad (14)$$

From theorem 4.10 of [38], the system (4)-(6) is exponentially stable when $a_p = 0$. Furthermore, if $a_p \geq \frac{(1+L) F_p(a_m)}{c_7}$ and $c_7 = \min(|X_e|, |Y_e|, |Z_e|)$. Therefore, the system is Input to State Stable (ISS) with respect to a_p by theorem 4.19 of [38]. In addition, since $\frac{\|\dot{\nu}\|}{\|\nu\|} \leq 2c_6 \|\nu\|$, and considering the special case of $\gamma(t) = 0$ and $\delta(t) = 0$ in Lemma 9.4 of [38], we obtain:

$$\begin{aligned} \|\nu(t)\| &\leq \sqrt{\frac{c_6}{c_7}} \|\nu(t_0)\| e^{-\frac{c_7}{2c_6}(t-t_0)} \\ &\quad + \frac{c_6(1+L)}{2c_7} F_p(a_m) \int_{t_0}^t e^{-\frac{c_7}{2c_6}(t-s)} ds \\ &\leq \sqrt{\frac{c_6}{c_7}} \|\nu(t_0)\| e^{-\frac{c_7}{2c_6}(t-t_0)} \\ &\quad + \frac{c_6^2(1+L)}{c_7} F_p(a_m) \left[1 - e^{-\frac{c_7}{2c_6}(t-t_0)} \right] \end{aligned}$$

$$\leq c_1 \|\nu(t_0)\| e^{-c_2(t-t_0)} + \frac{c_6^2(1+L)}{c_7 c_7} F_p(a_m)$$

where $\Delta \psi$ is the direction of the average velocity when (11) is applied, i.e. $\Delta \psi = \text{atan}(2V_{yf}, V_{xf})$.

Therefore, control law (22) switches when $\psi = \psi_r - \Delta \psi + 2\pi$ and $\psi = \psi_r - \Delta \psi + (2n+1)\pi$, as shown in Figure 4, and the vehicle moves in direction ψ_r instead of moving in direction $\Delta \psi$ with respect to the X-axis. Doing this way, the trajectory obtained with (22) is rotated an angle $\psi_r - \Delta \psi$ with respect to the trajectory obtained with (11), and the final average velocity points towards $\Delta \psi + \psi_r - \Delta \psi = \psi_r$.

On the one hand, from (7) and (9) and expanding them up to the second order in a_0 and Δa , we obtain $\nu_e^\pm \approx \frac{1}{r_e^+} F_p(a_0 \pm \Delta a)$ and $r_e^\pm \approx \frac{1}{|X_e|} F_p(a_0 \pm \Delta a)$. Thus, a similar analysis can be done for $t_{n+1} \rightarrow t_{n+2}$, so finally:

$$\|\nu'_{n+2} - \nu_{n+2}\| \leq a_2, \quad \text{or } \frac{u^*(\psi)}{r^*(\psi)}$$

is zero.

On the other hand, from (8), $v_e^\pm \approx -\frac{m_u u_e^\pm r_e^\pm}{|Y_e|}$, so

$$\text{where } a_2 = (c_1 e^{-c_2 \Delta t_{n+1}} + \frac{c_2}{r_e^+}) = -\frac{m_u u_e^\pm}{|Y_e|} + \frac{2Km_u}{|X_e|} a_0 \Delta a. \text{ This implies}$$

that $\frac{u^*(\psi)}{r^*(\psi)}$ is a function that converges to two different values at $\psi = 0$ and $\psi = \pi$. Furthermore, the convergence to the steady state is very fast since the linearization of (4) for small enough velocities is $m_u \dot{u} + |X_e| u = F_p(a_0)$, thus $u^*(t)$ converges exponentially to u_e^\pm with rate $\frac{m_u}{|X_e|}$. Therefore, $u^*(\psi)$ converges exponentially fast to u_e^\pm with, at least, the same applies to $r^*(t)$, thus $\frac{u^*(\psi)}{r^*(\psi)}$ could be replaced in the integral by its steady state value $\frac{u_e^\pm}{r_e^\pm}$.

This proves that the convergence to a neighbourhood of the origin is exponential, so letting $a_p = a_0$ small enough, the trajectory of the system could be as close as desired to the origin around the origin (since the Jacobian matrix is non-zero). Then, the trajectory converges exponentially to the region of the linear system, and the equilibrium is partially exponentially stable.

Finally, as $a_0 \rightarrow 0$ as $a_0 \rightarrow 0$, so $c_1 e^{-c_2(t-t_0)} \rightarrow c_1 e^{-\infty} = 1$.

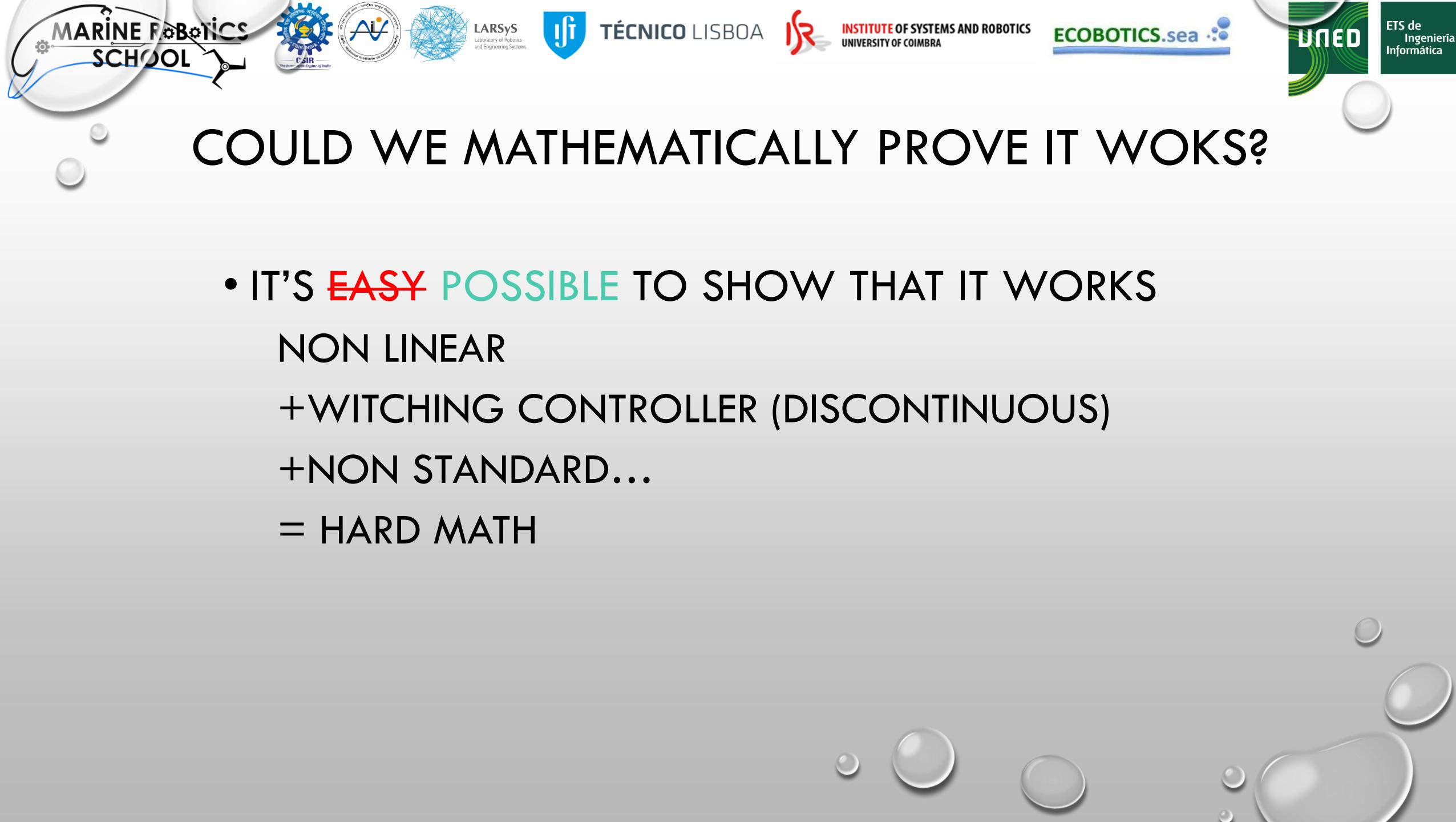
Theorem 1. Consider system (1)-(6) and control action (22), then a_0 , Δa and $\Delta \psi$ can be chosen such that the average speed of the vehicle converges exponentially to a constant speed $V_f > 0$ in the direction of ψ_r .

be necessary tools to analyse the behaviour

of system (1)-(6) under control action

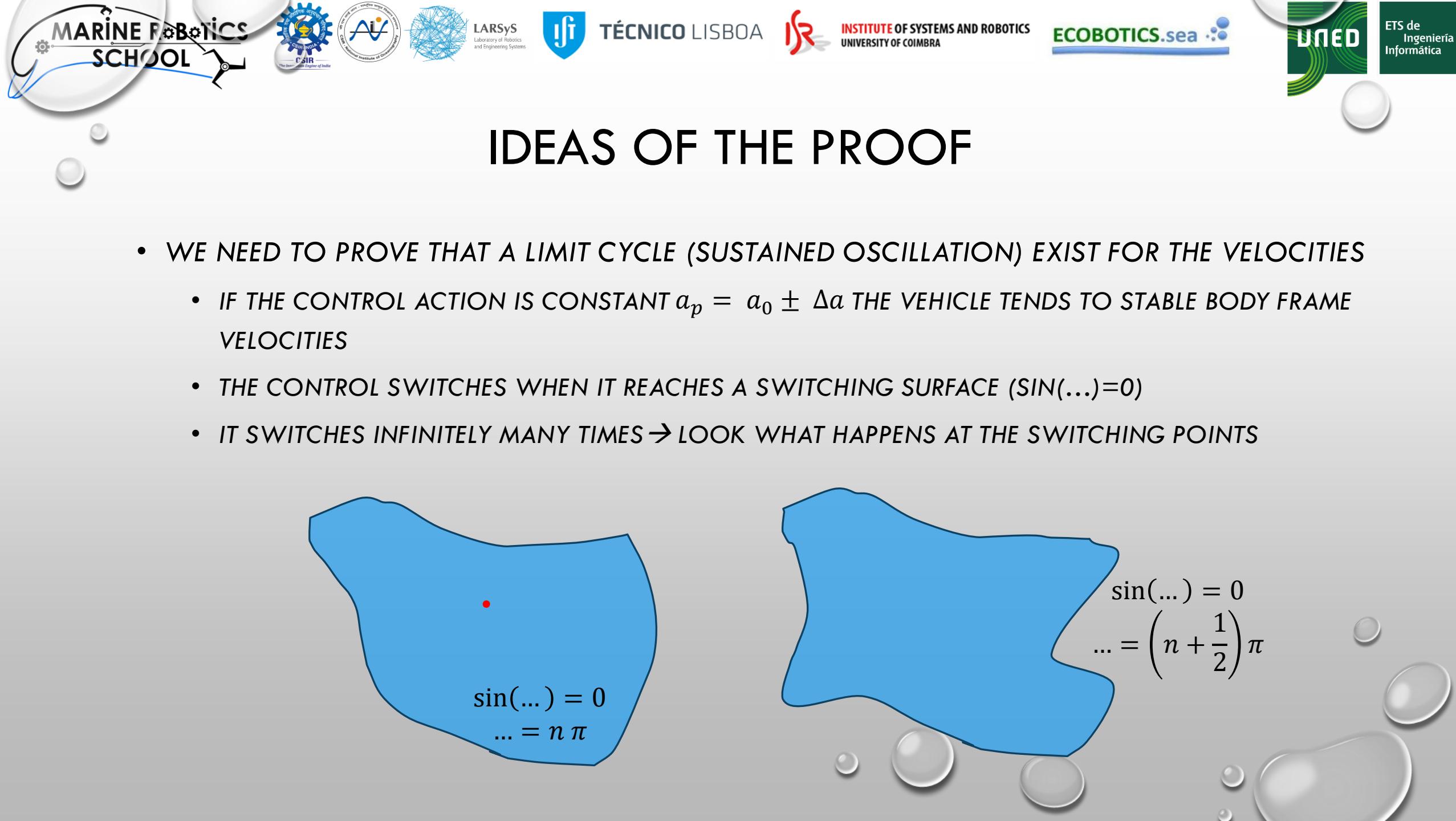
to choose a_0 and Δa such that

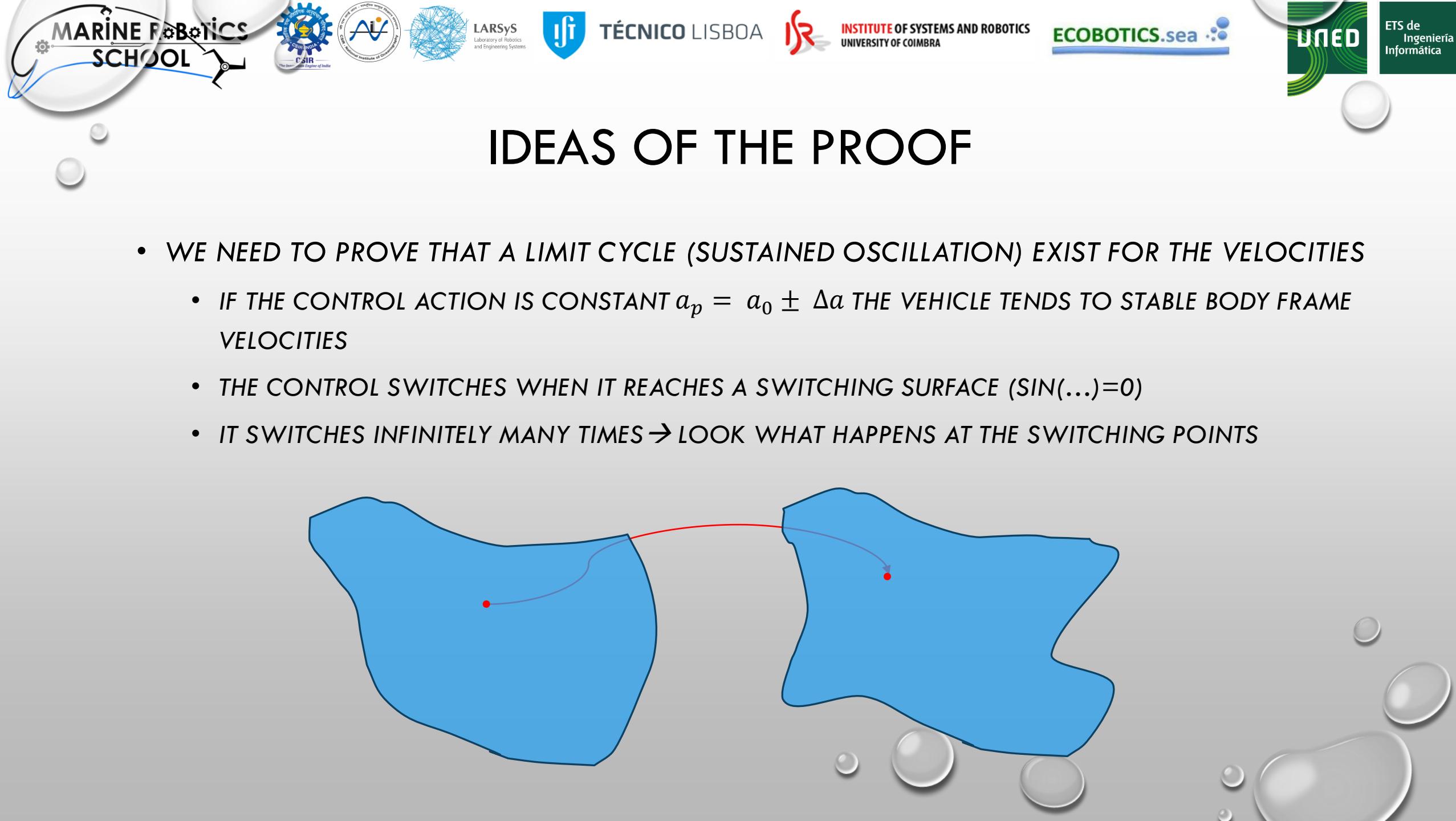
the system converges to an exponentially sustained oscillation and, in addition the fixed value $[\dot{x}, \dot{y}]^T \rightarrow [V_{xf}, V_{yf}]^T \neq 0$.



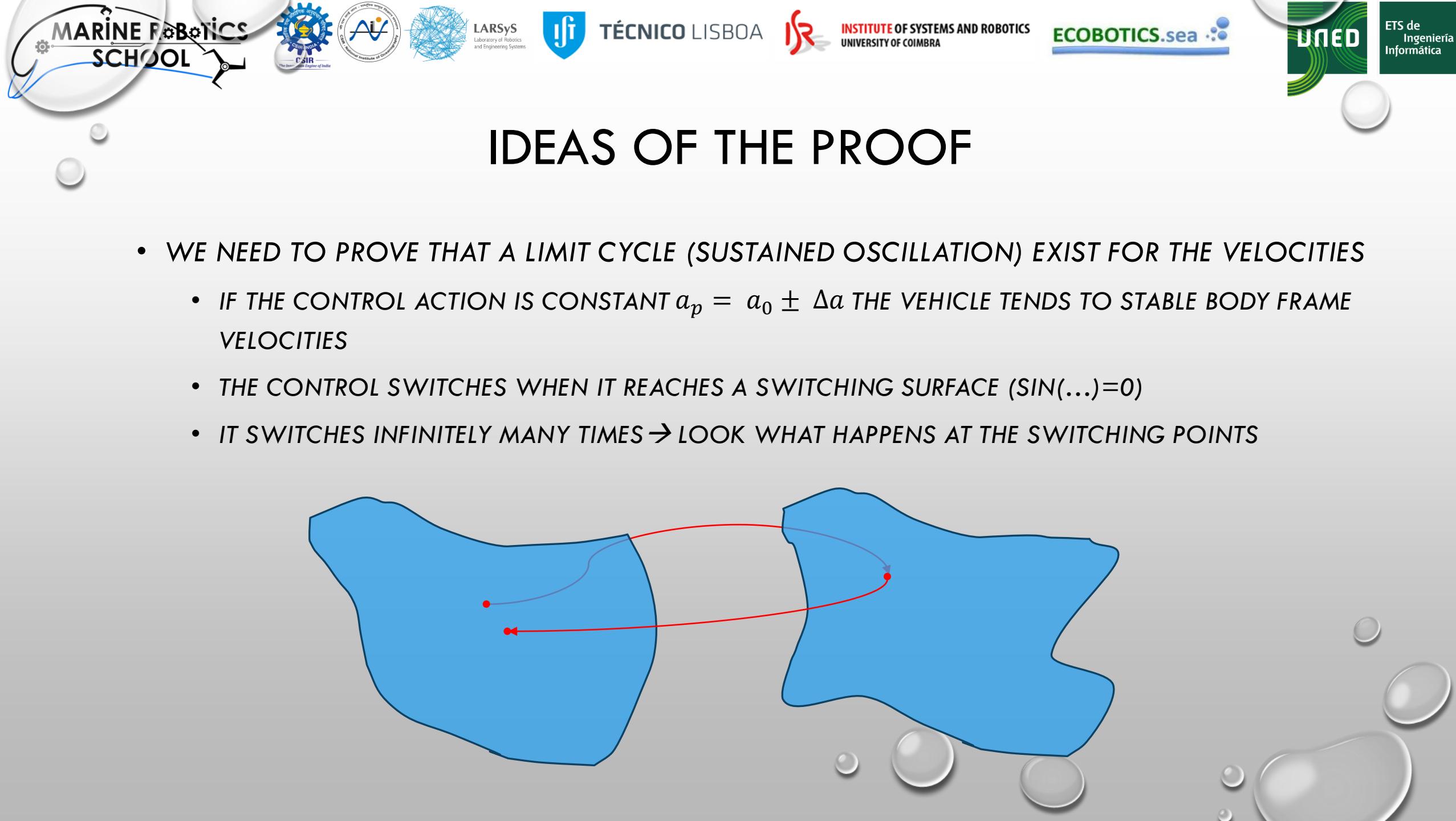
COULD WE MATHEMATICALLY PROVE IT WORKS?

- IT'S ~~EASY~~ POSSIBLE TO SHOW THAT IT WORKS
- NON LINEAR
- +WITCHING CONTROLLER (DISCONTINUOUS)
- +NON STANDARD...
- = HARD MATH

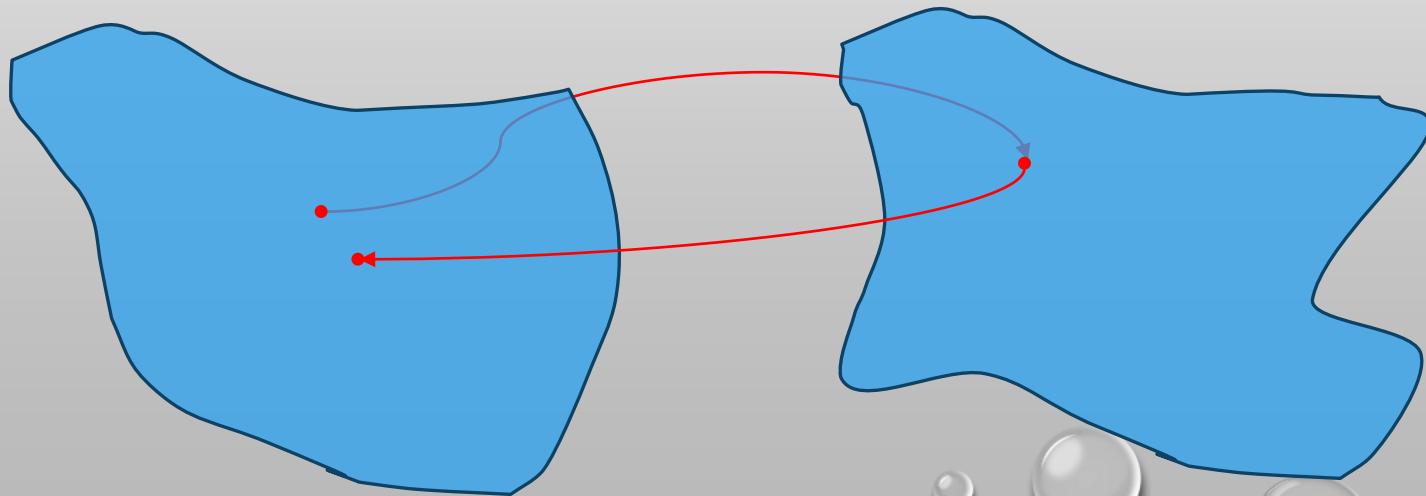


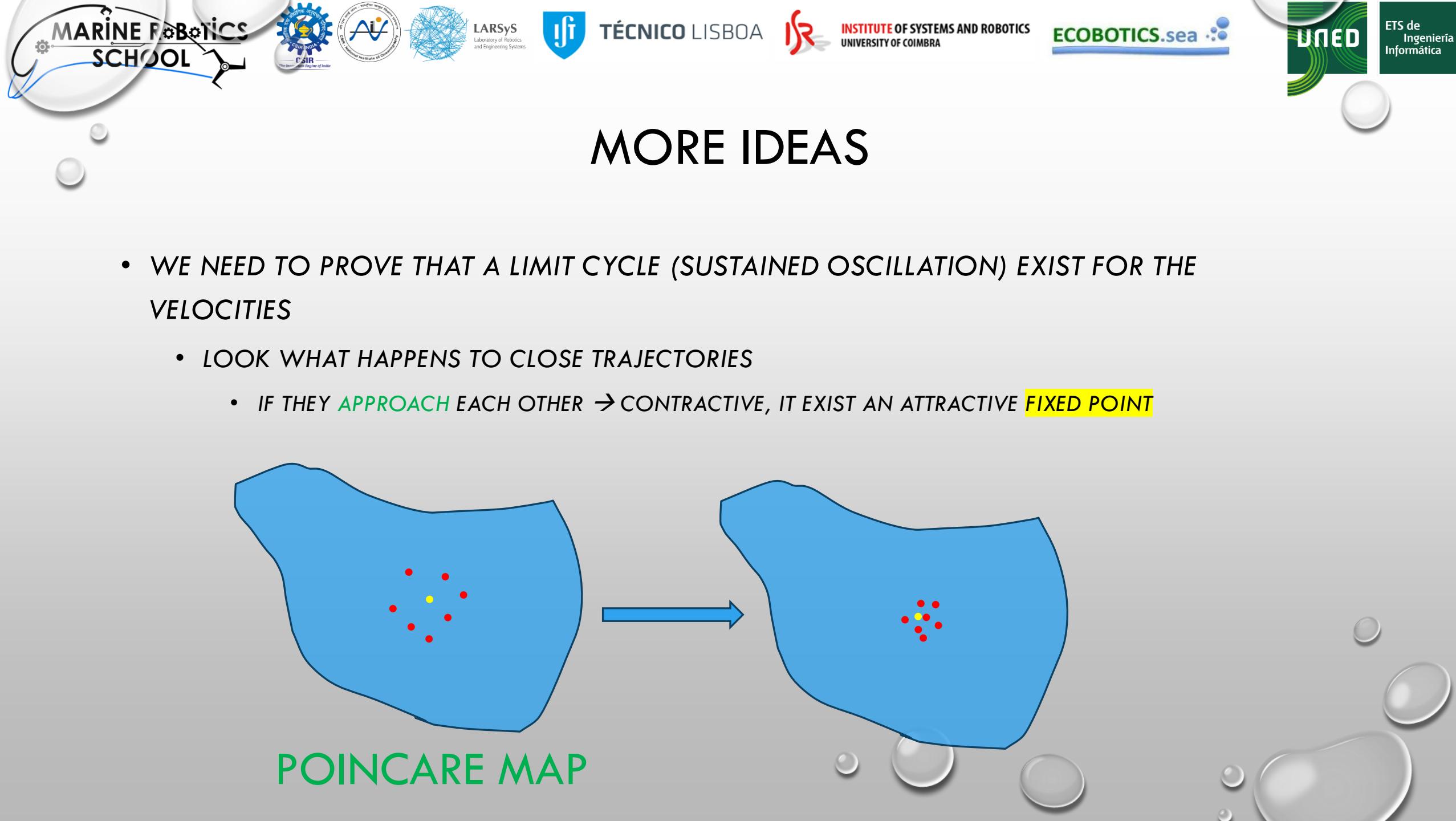


- WE NEED TO PROVE THAT A LIMIT CYCLE (SUSTAINED OSCILLATION) EXIST FOR THE VELOCITIES
 - IF THE CONTROL ACTION IS CONSTANT $a_p = a_0 \pm \Delta a$ THE VEHICLE TENDS TO STABLE BODY FRAME VELOCITIES
 - THE CONTROL SWITCHES WHEN IT REACHES A SWITCHING SURFACE ($\sin(\dots)=0$)
 - IT SWITCHES INFINITELY MANY TIMES → LOOK WHAT HAPPENS AT THE SWITCHING POINTS

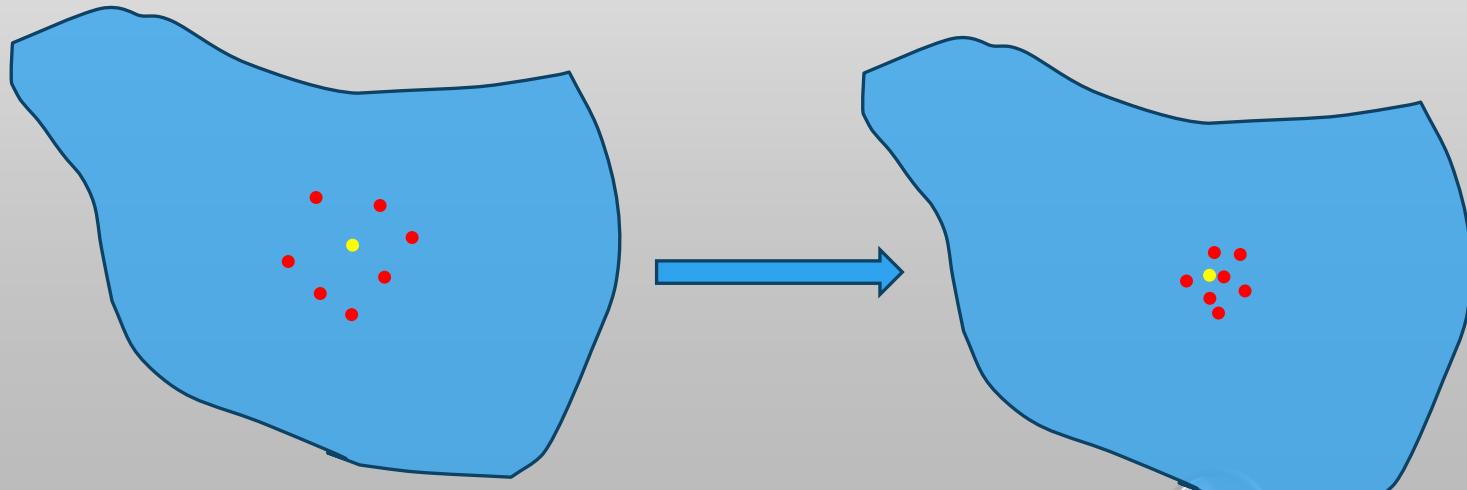


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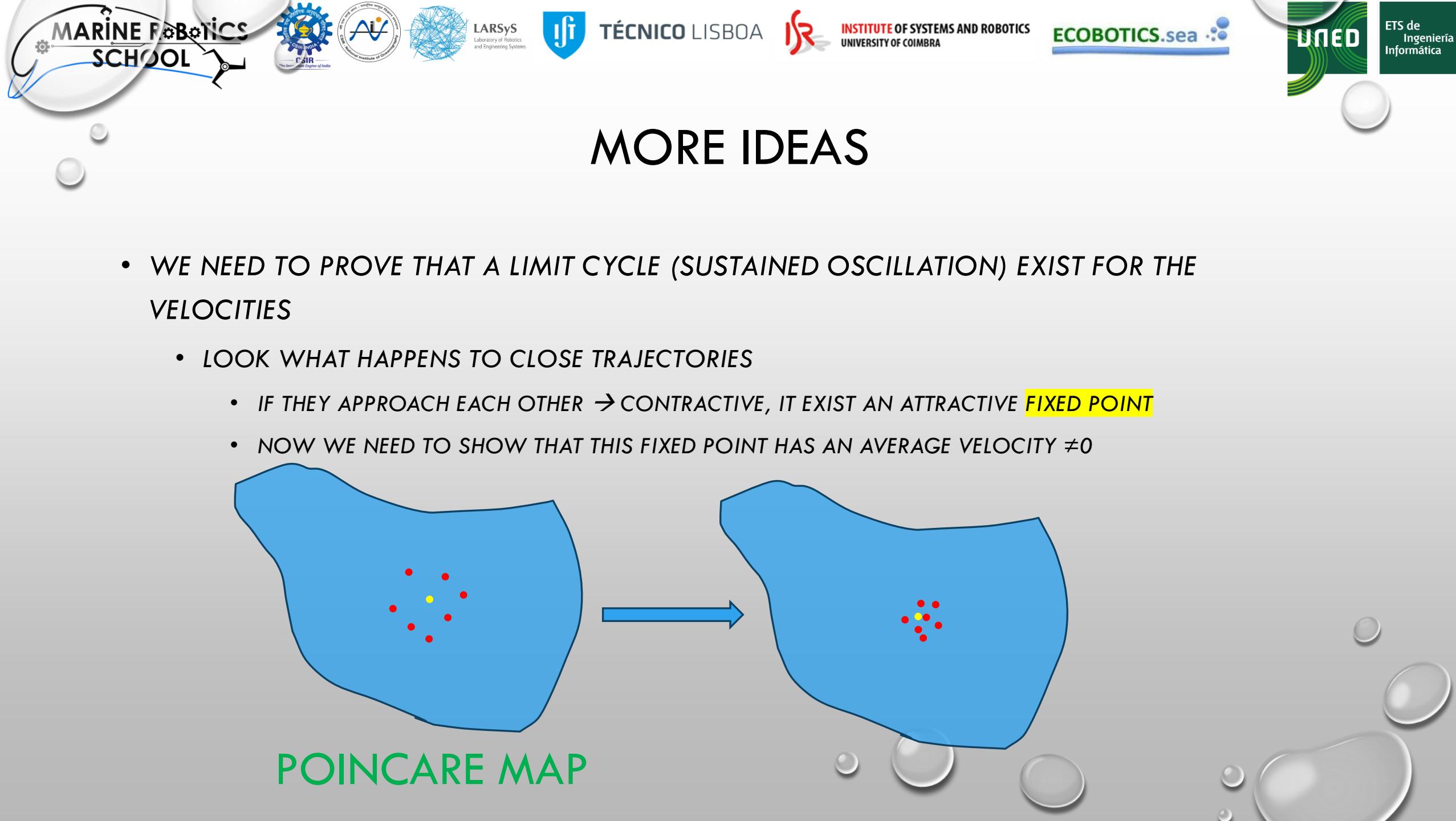




- WE NEED TO PROVE THAT A LIMIT CYCLE (SUSTAINED OSCILLATION) EXIST FOR THE VELOCITIES
 - LOOK WHAT HAPPENS TO CLOSE TRAJECTORIES
 - IF THEY APPROACH EACH OTHER → CONTRACTIVE, IT EXIST AN ATTRACTIVE FIXED POINT



POINCARÉ MAP

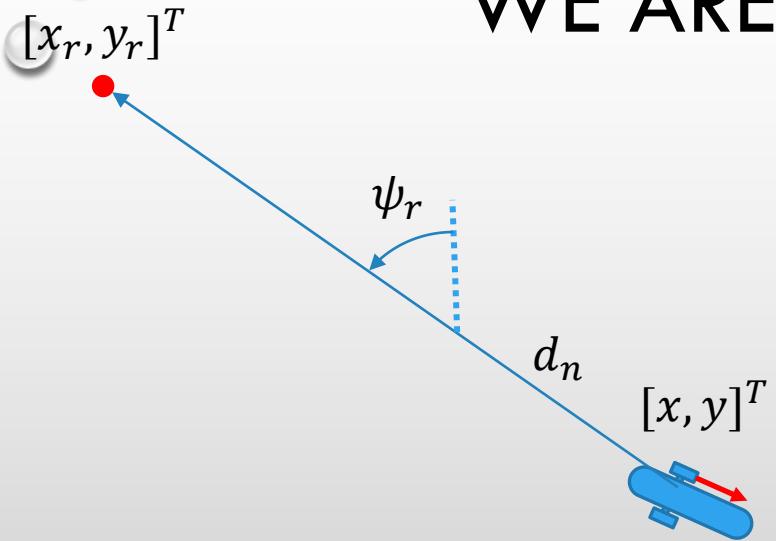


BUT WE WANT TO GO HOME!

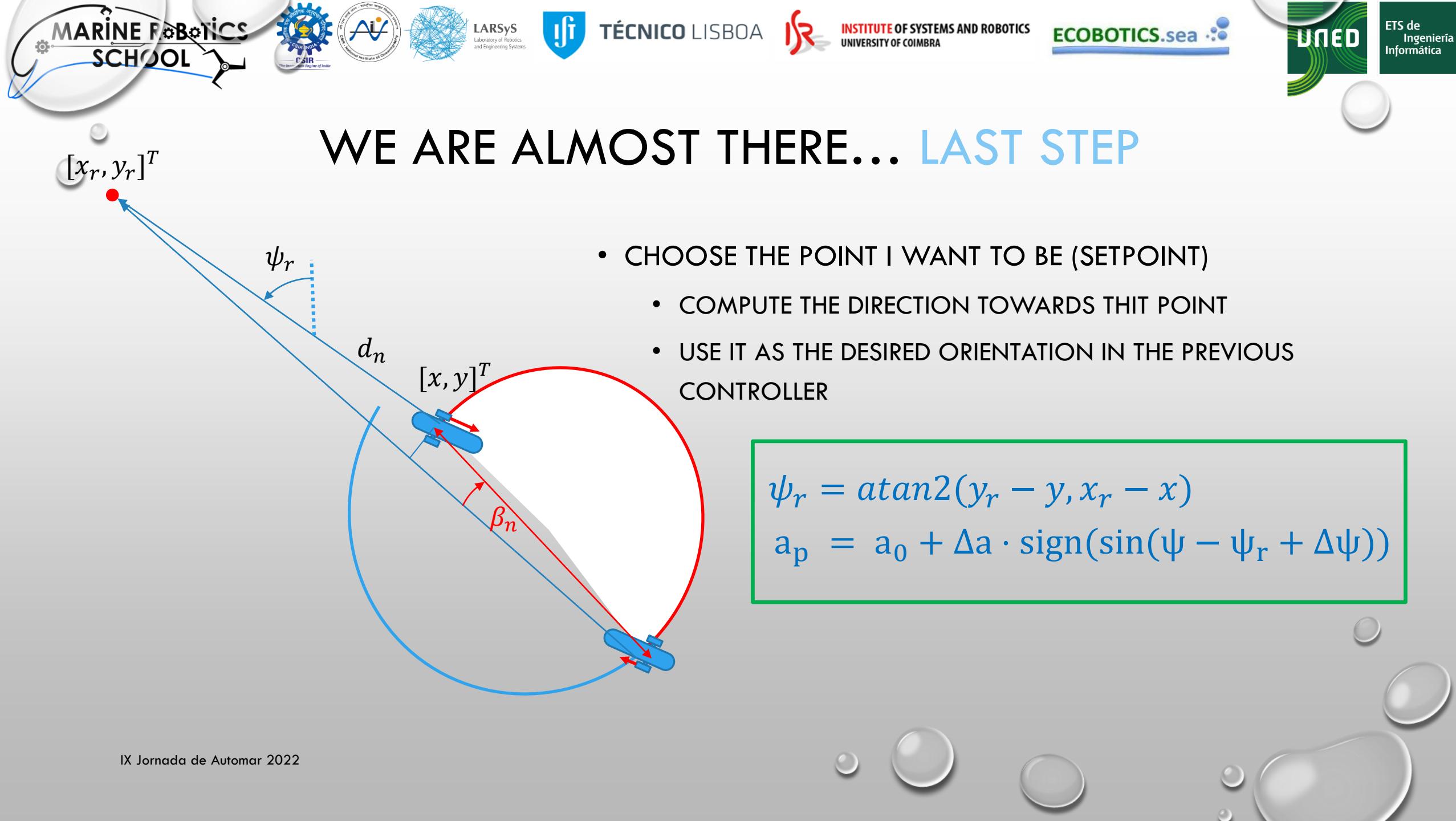


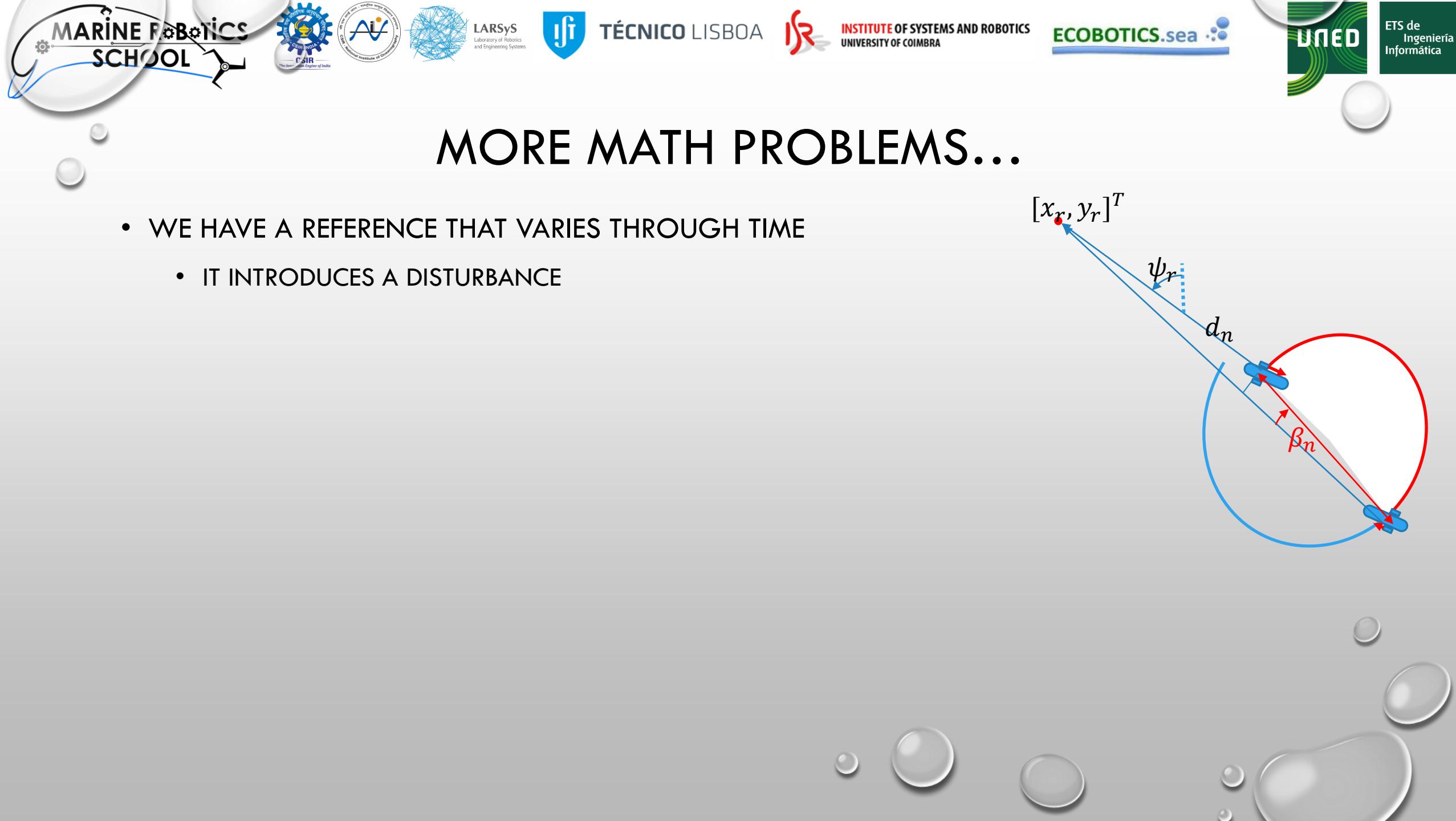
WE ARE ALMOST THERE... LAST STEP

- CHOOSE THE POINT I WANT TO BE (SETPOINT)
- COMPUTE THE DIRECTION TOWARDS THIT POINT



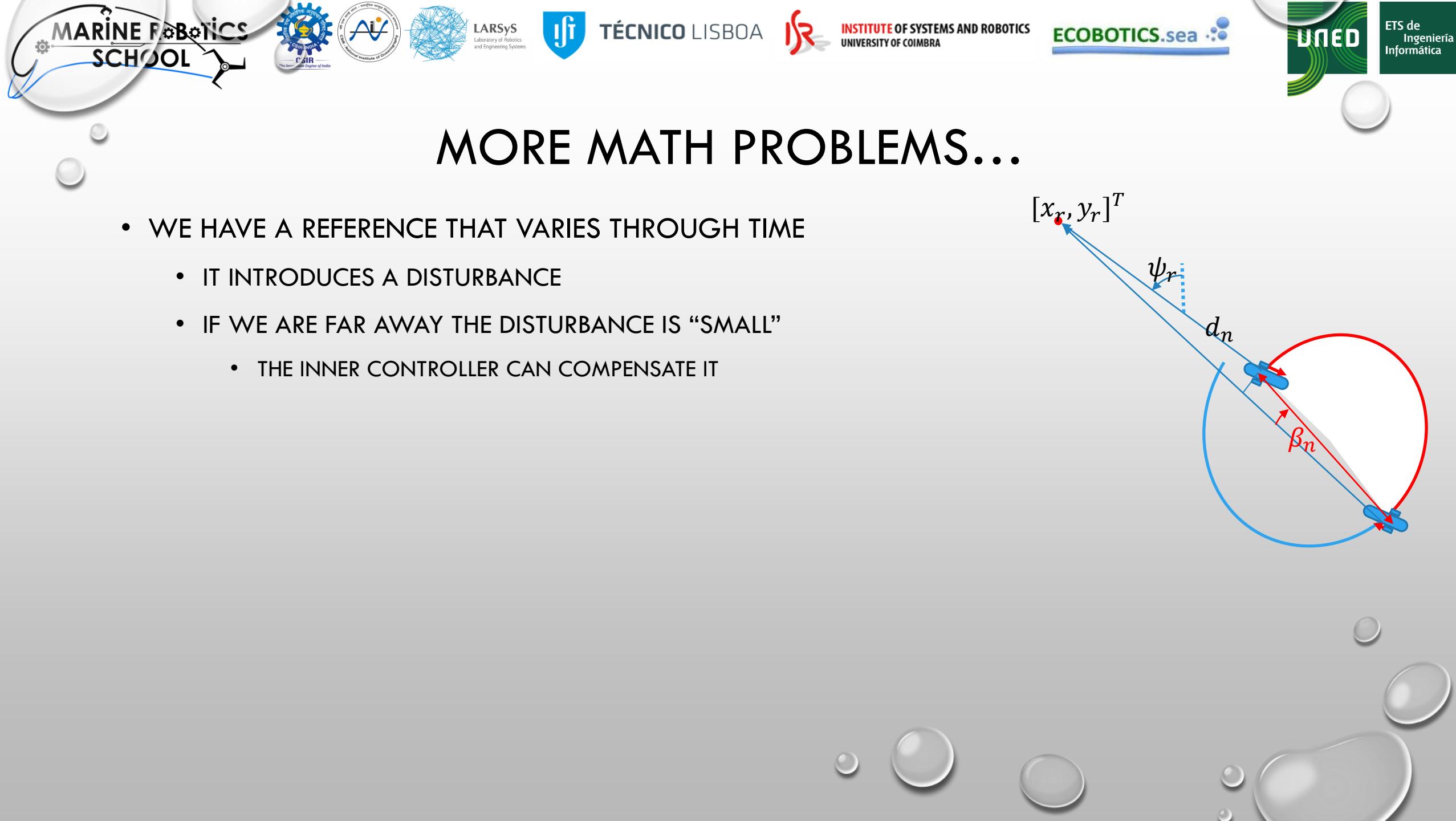
$$\psi_r = \text{atan2}(y_r - y, x_r - x)$$





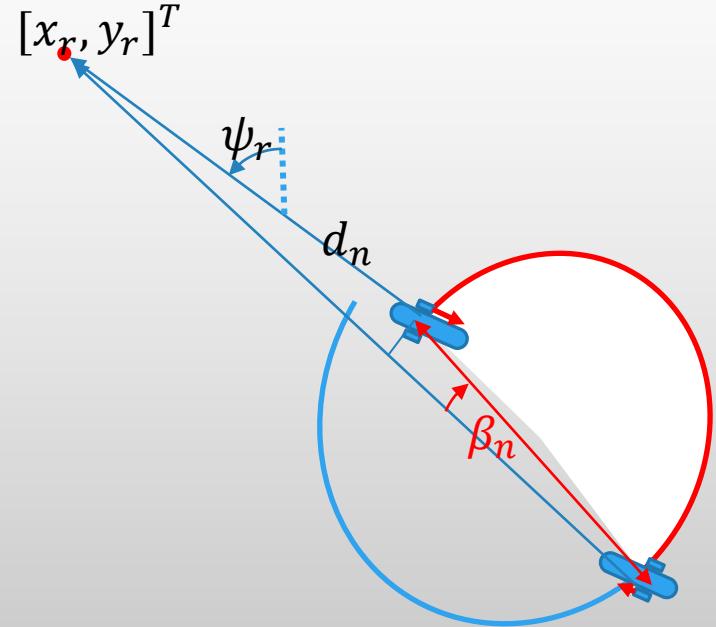
MORE MATH PROBLEMS...

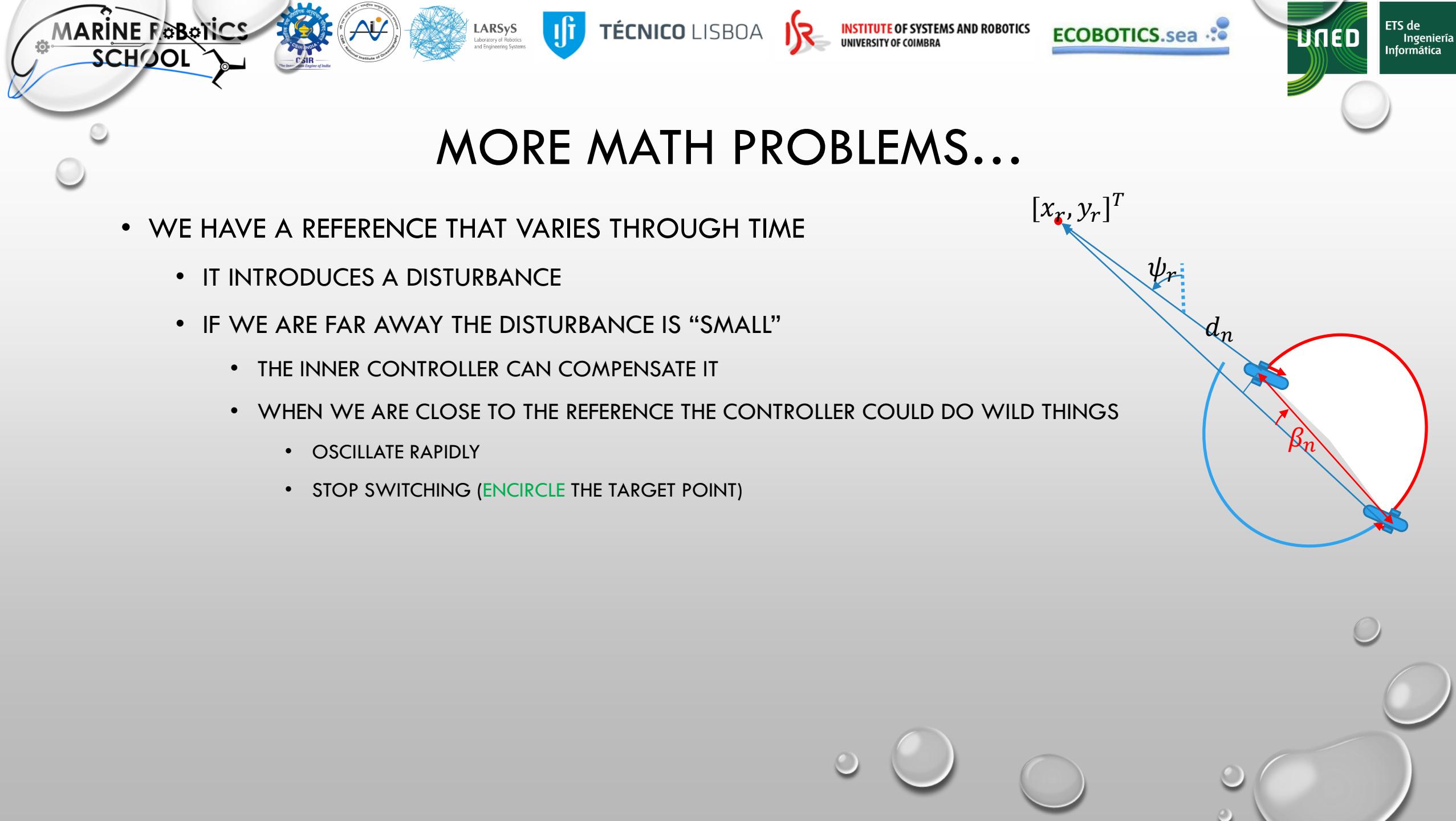
- WE HAVE A REFERENCE THAT VARIES THROUGH TIME
 - IT INTRODUCES A DISTURBANCE



MORE MATH PROBLEMS...

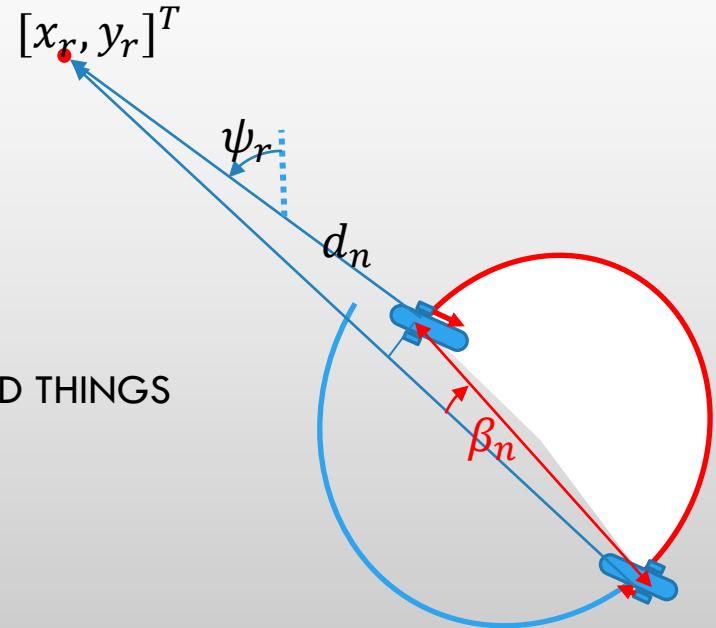
- WE HAVE A REFERENCE THAT VARIES THROUGH TIME
 - IT INTRODUCES A DISTURBANCE
 - IF WE ARE FAR AWAY THE DISTURBANCE IS “SMALL”
 - THE INNER CONTROLLER CAN COMPENSATE IT

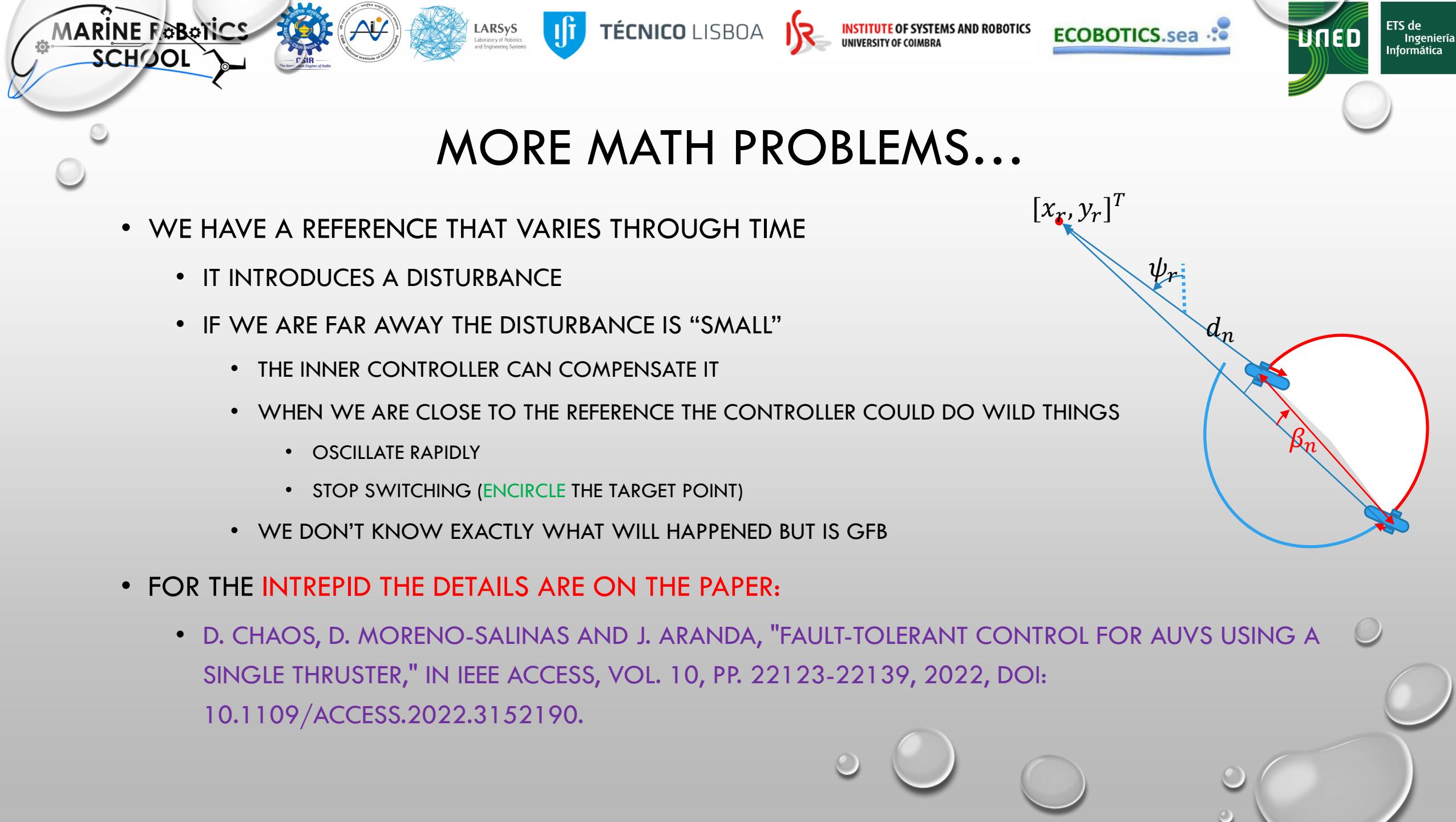




MORE MATH PROBLEMS...

- WE HAVE A REFERENCE THAT VARIES THROUGH TIME
 - IT INTRODUCES A DISTURBANCE
 - IF WE ARE FAR AWAY THE DISTURBANCE IS “SMALL”
 - THE INNER CONTROLLER CAN COMPENSATE IT
 - WHEN WE ARE CLOSE TO THE REFERENCE THE CONTROLLER COULD DO WILD THINGS
 - OSCILLATE RAPIDLY
 - STOP SWITCHING (**ENCIRCLE** THE TARGET POINT)



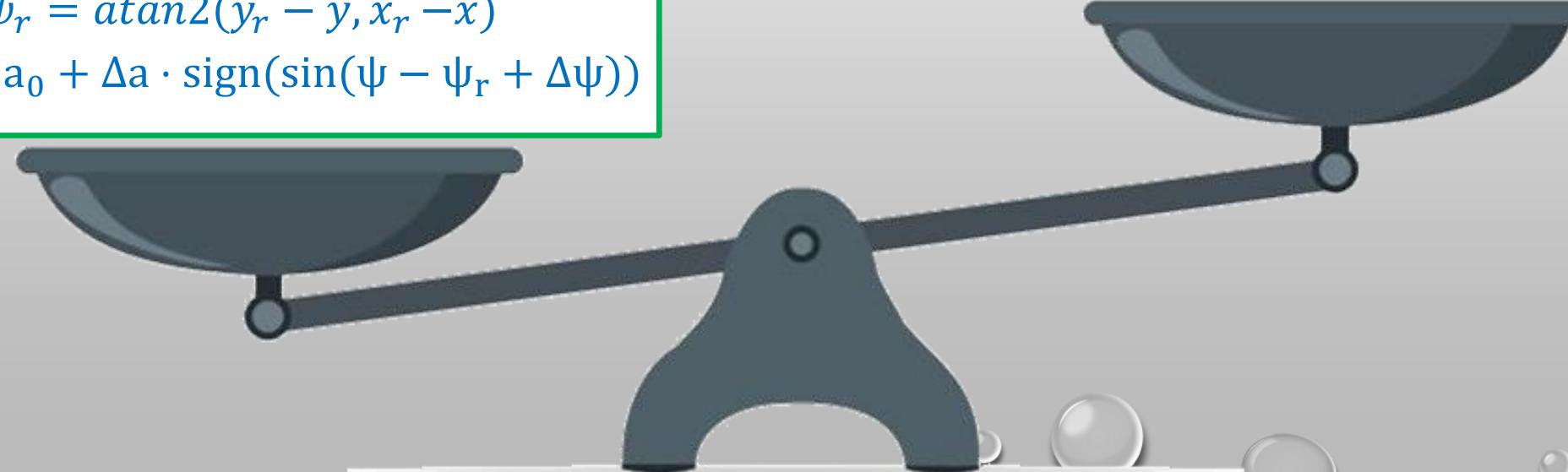
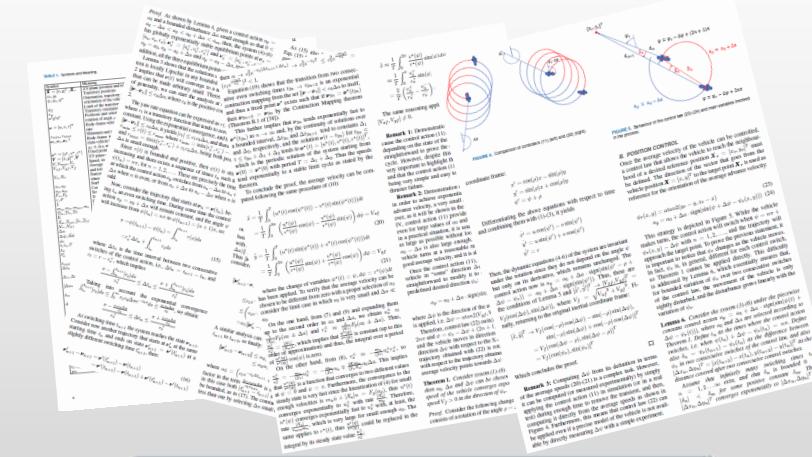


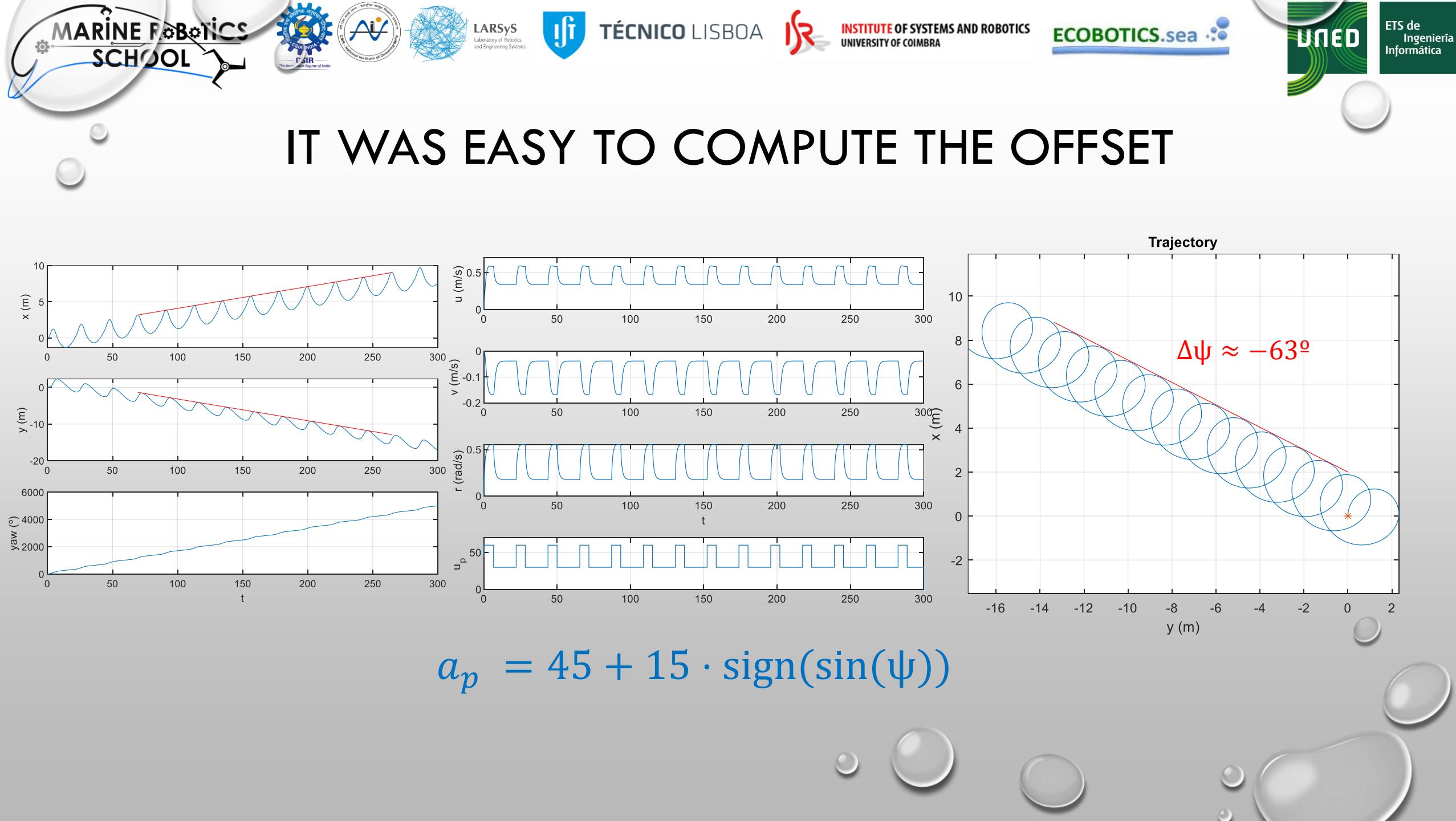
- WE HAVE A REFERENCE THAT VARIES THROUGH TIME
 - IT INTRODUCES A DISTURBANCE
 - IF WE ARE FAR AWAY THE DISTURBANCE IS “SMALL”
 - THE INNER CONTROLLER CAN COMPENSATE IT
 - WHEN WE ARE CLOSE TO THE REFERENCE THE CONTROLLER COULD DO WILD THINGS
 - OSCILLATE RAPIDLY
 - STOP SWITCHING (ENCIRCLE THE TARGET POINT)
 - WE DON’T KNOW EXACTLY WHAT WILL HAPPEN BUT IS GFB
- FOR THE INTREPID THE DETAILS ARE ON THE PAPER:
 - D. CHAOS, D. MORENO-SALINAS AND J. ARANDA, "FAULT-TOLERANT CONTROL FOR AUVS USING A SINGLE THRUSTER," IN IEEE ACCESS, VOL. 10, PP. 22123-22139, 2022, DOI: 10.1109/ACCESS.2022.3152190.

REMEMBER, THE IMPORTANT THING IS THE CONTROLLER

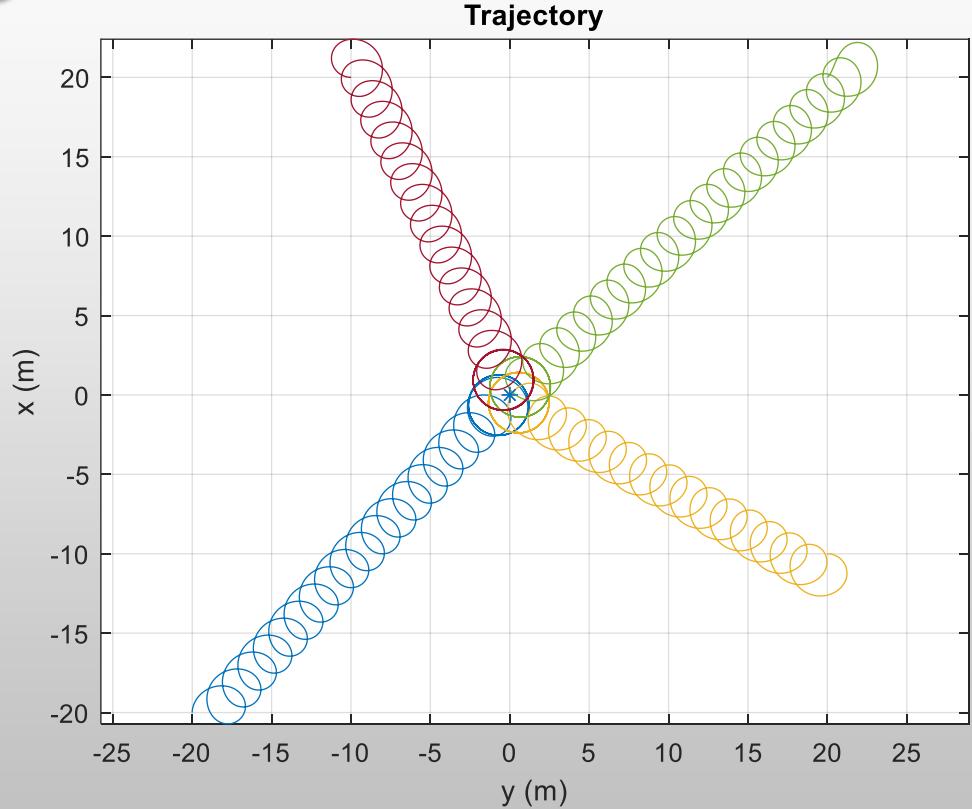
$$\psi_r = \text{atan2}(y_r - y, x_r - x)$$

$$a_p = a_0 + \Delta a \cdot \text{sign}(\sin(\psi - \psi_r + \Delta\psi))$$





ONCE THE CORRECTION IS DONE

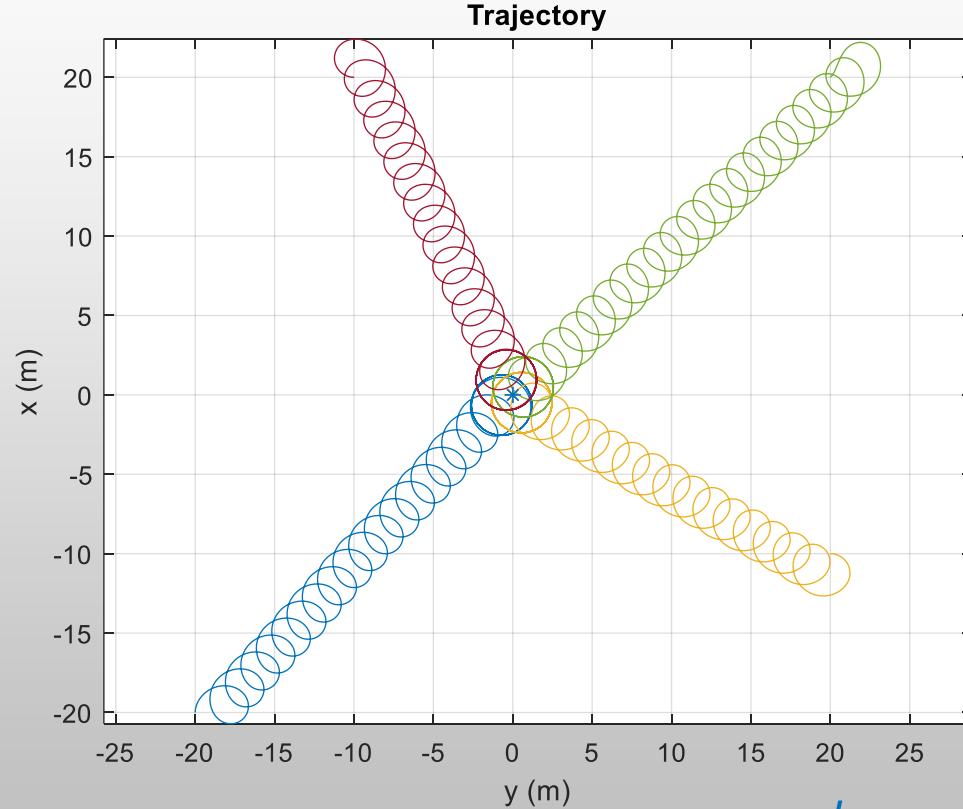


- IT GOES “STRAIGHT” TO THE POINT AND SURROUNDS IT
 - IT STOPS SWITCHING
 - THIS IS WHAT USUALLY HAPPENS
- IT IS ROBUST? WHAT HAPPENS IF THE OFFSET IS WRONG?

$$\psi_r = \text{atan}2(y_r - y, x_r - x)$$

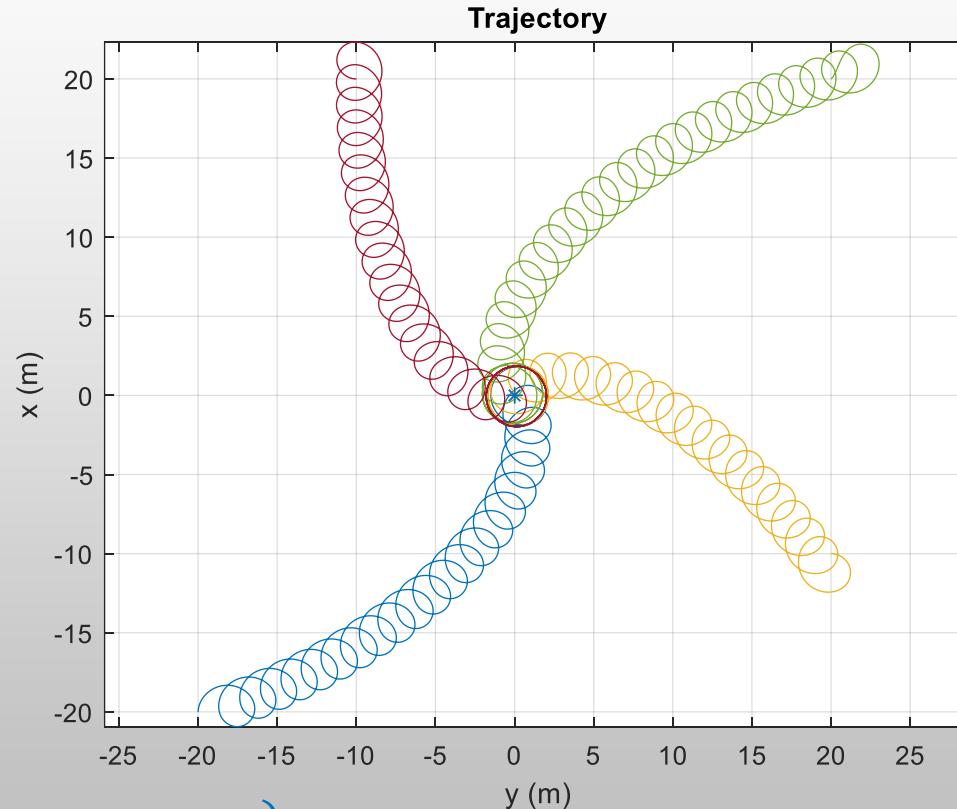
$$a_p = 45 + 15 \cdot \text{sign}(\sin(\psi - \psi_r + 63 \cdot \pi / 180))$$

IT STILL WORKS



$$\psi_r = \text{atan2}(y_r - y, x_r - x)$$

$$a_p = 45 + 15 \cdot \text{sign}(\sin(\psi - \psi_r - 63 \cdot \pi/180))$$



$$a_p = 45 + 15 \cdot \text{sign}(\sin(\psi - \psi_r - \pi/2))$$

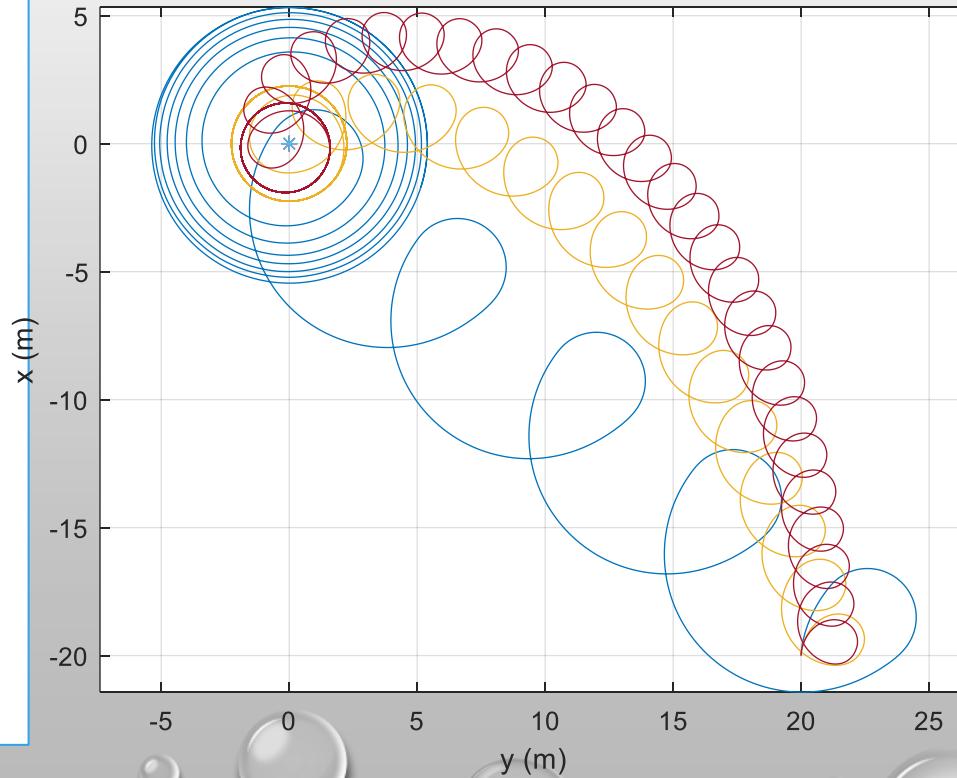
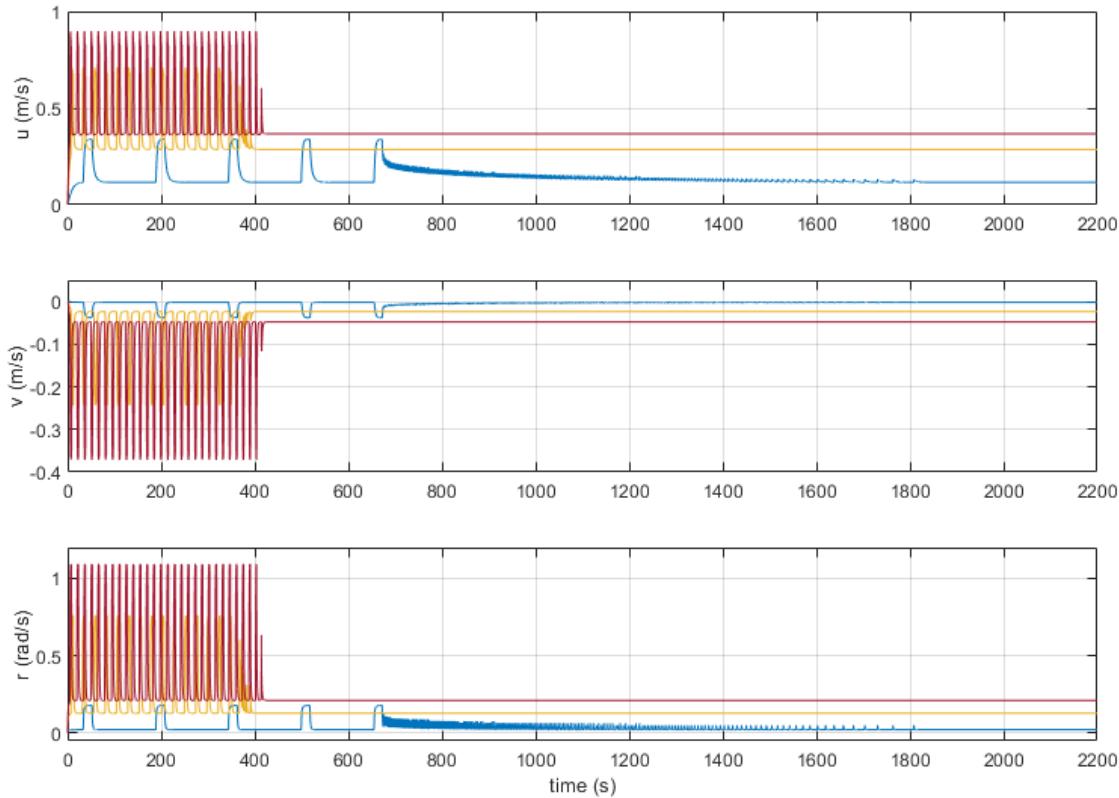
WHAT IF THE CONSTANTS ARE NOT WELL TUNED?

- WORSE RESPONSE
- SLOW CONVERGENCE
- BUT... “IT WORKS”

$$a_0 = 20,40,66$$

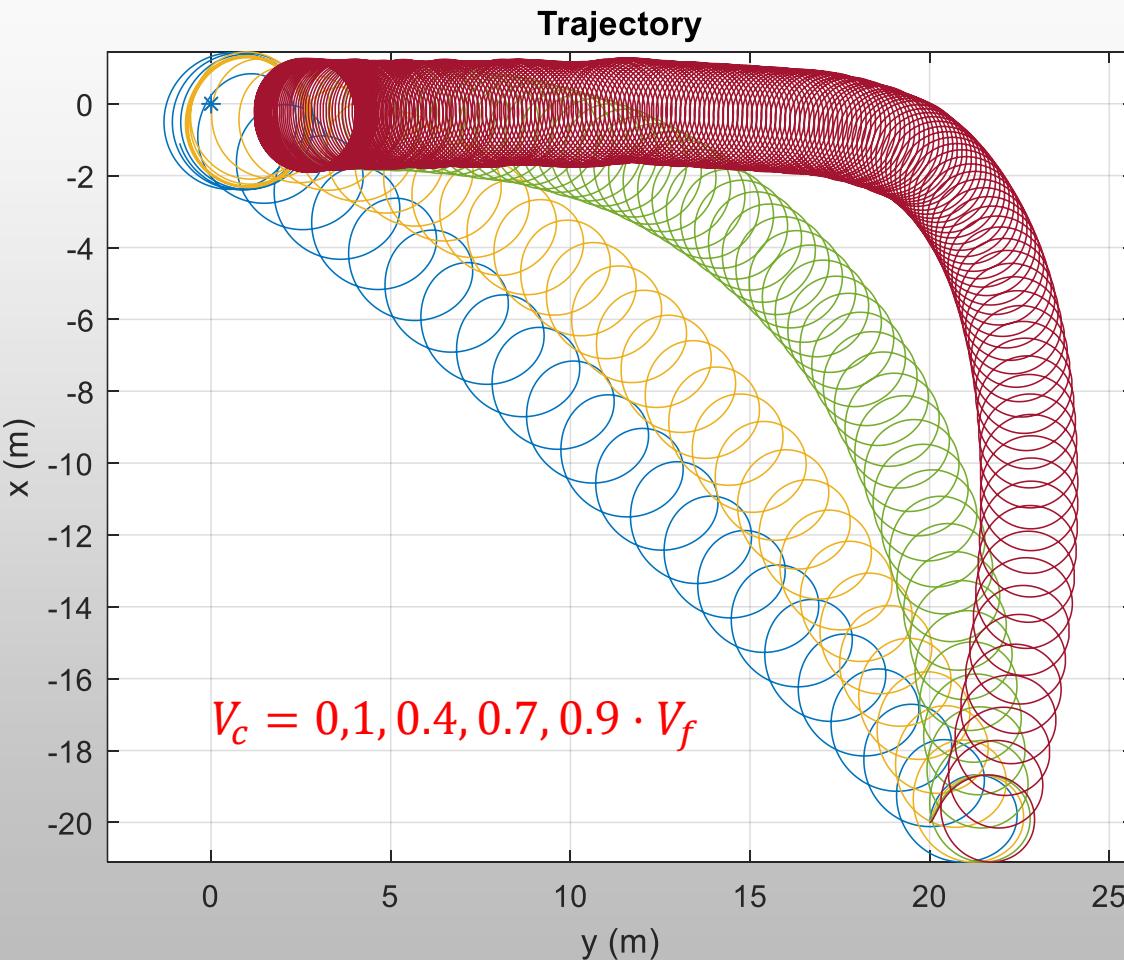
$$\Delta a = a_0/2$$

$$\Delta \psi = -90$$



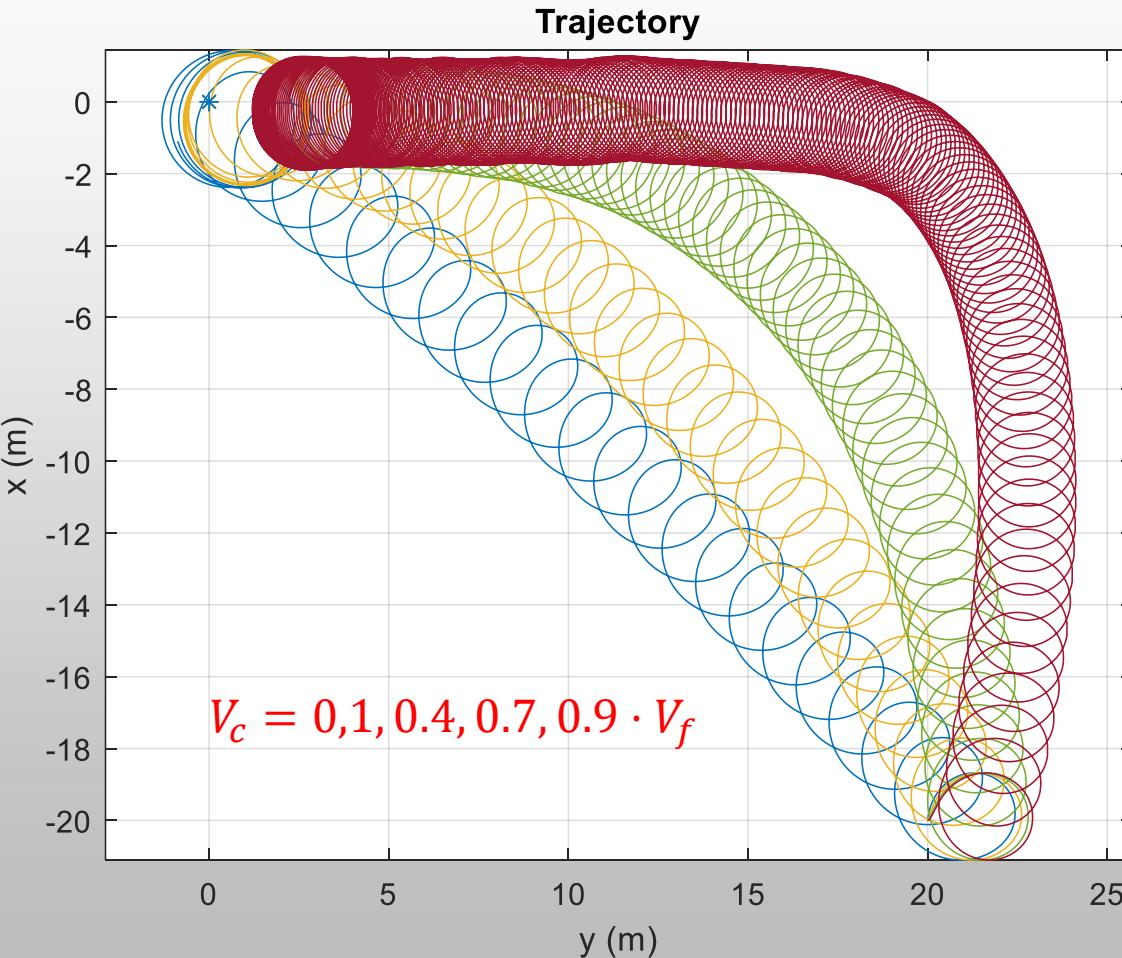
AND IF WE HAVE CURRENTS?

- IF THE SPEED OF THE CURRENT V_c IS LESS THAT THE AVERAGE VELOCITY V_f IT CAN FIGHT THE CURRENT.



AND IF WE HAVE CURRENTS?

- IF THE SPEED OF THE CURRENT V_c IS LESS THAT THE AVERAGE VELOCITY V_f IT CAN FIGHT THE CURRENT.
- PROBLEM, V_f IS TINY (6.6cm/s)

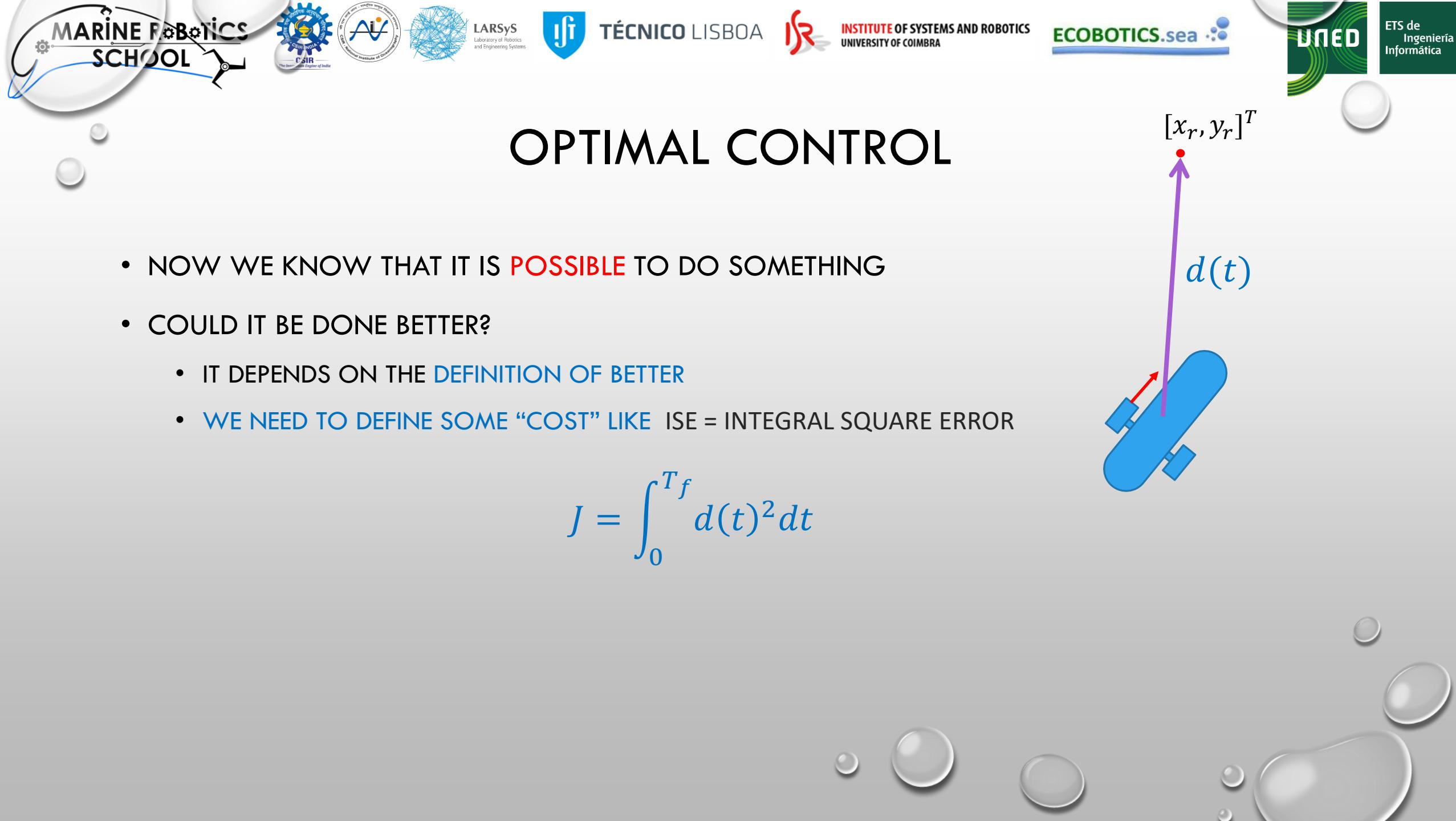


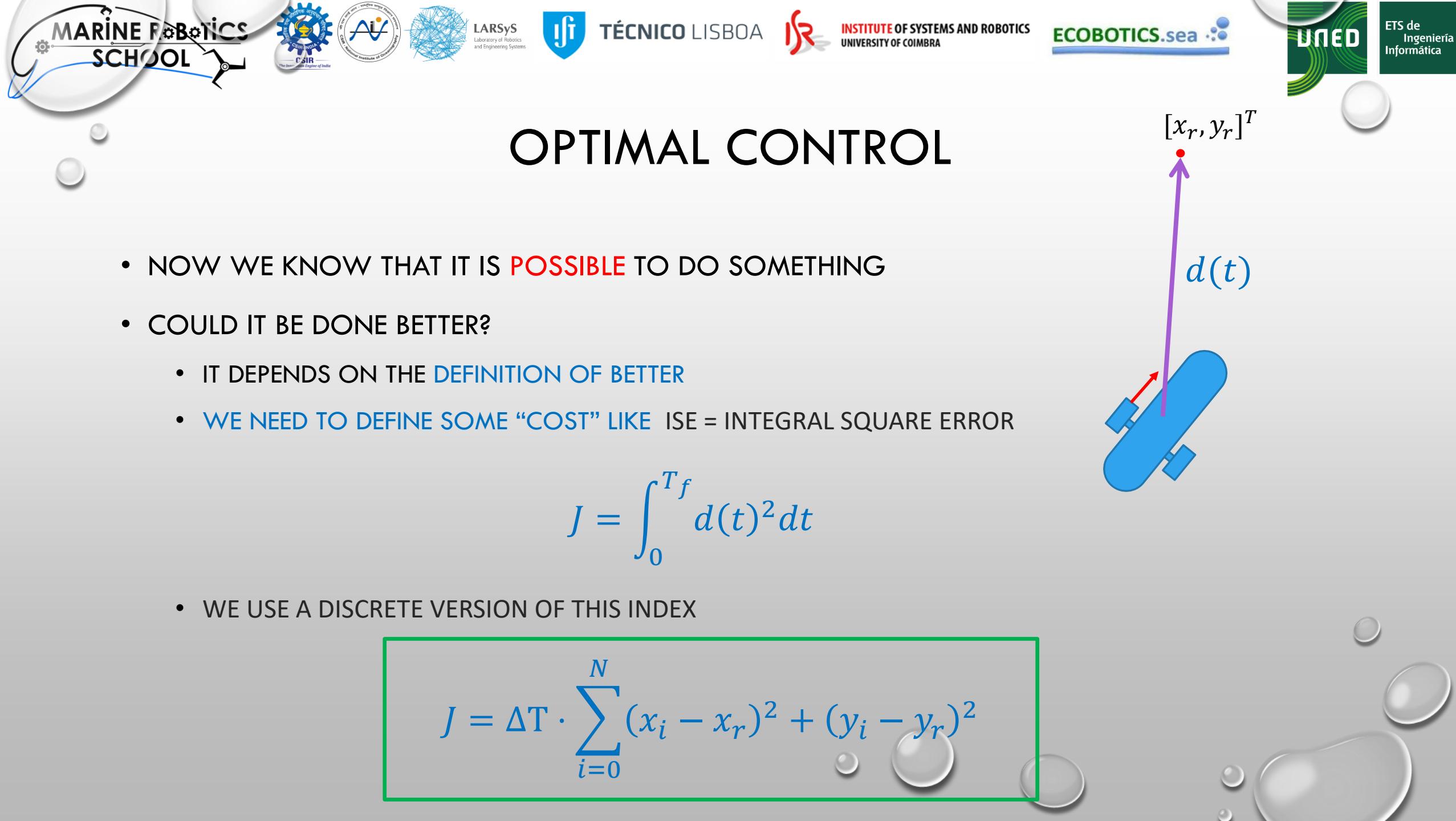
CAN WE DO IT BETTER?





- NOW WE KNOW THAT IT IS **POSSIBLE** TO DO SOMETHING
- COULD IT BE DONE BETTER?
 - IT DEPENDS ON THE **DEFINITION** OF BETTER

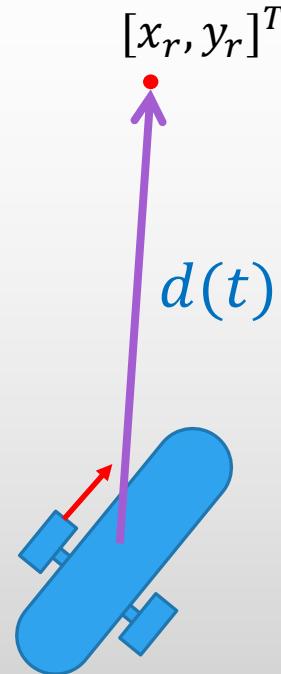






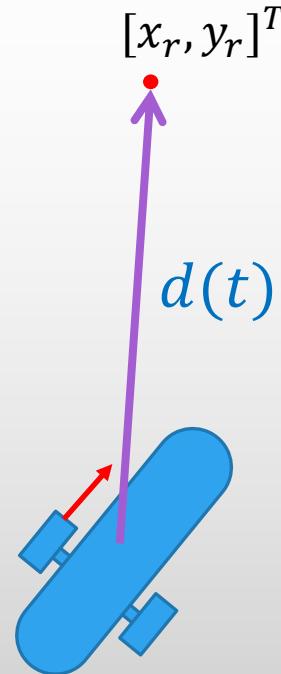
OPTIMAL CONTROL

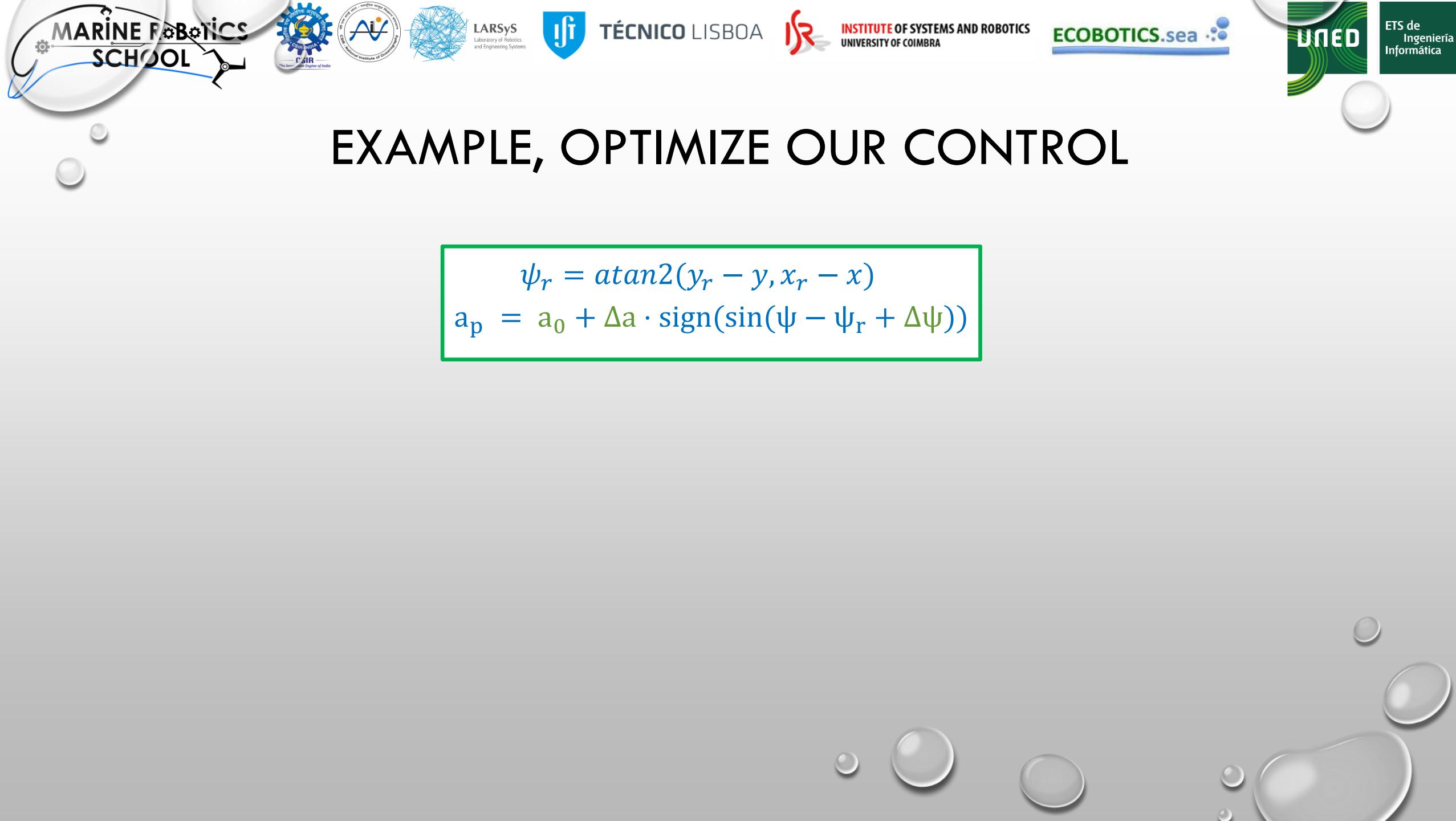
- SO, THE IDEA IS
 - 1) PARAMETRIZE THE CONTROL ACTION WITH PARAMETERS VECTOR θ





- SO, THE IDEA IS
 - 1) PARAMETRIZE THE CONTROL ACTION WITH PARAMETERS VECTOR θ
 - 2) TUNE UP θ TO GET THE MINIMUM VALUE OF THE COST J





EXAMPLE, OPTIMIZE OUR CONTROL

$$\psi_r = \text{atan2}(y_r - y, x_r - x)$$

$$a_p = a_0 + \Delta a \cdot \text{sign}(\sin(\psi - \psi_r + \Delta\psi))$$

EXAMPLE, OPTIMIZE OUR CONTROL

$$\begin{aligned}\psi_r &= \text{atan}2(y_r - y, x_r - x) \\ a_p &= a_0 + \Delta a \cdot \text{sign}(\sin(\psi - \psi_r + \Delta\psi))\end{aligned}$$

$$\theta = [a_0, \Delta a, \Delta\psi]^T$$

EXAMPLE, OPTIMIZE OUR CONTROL

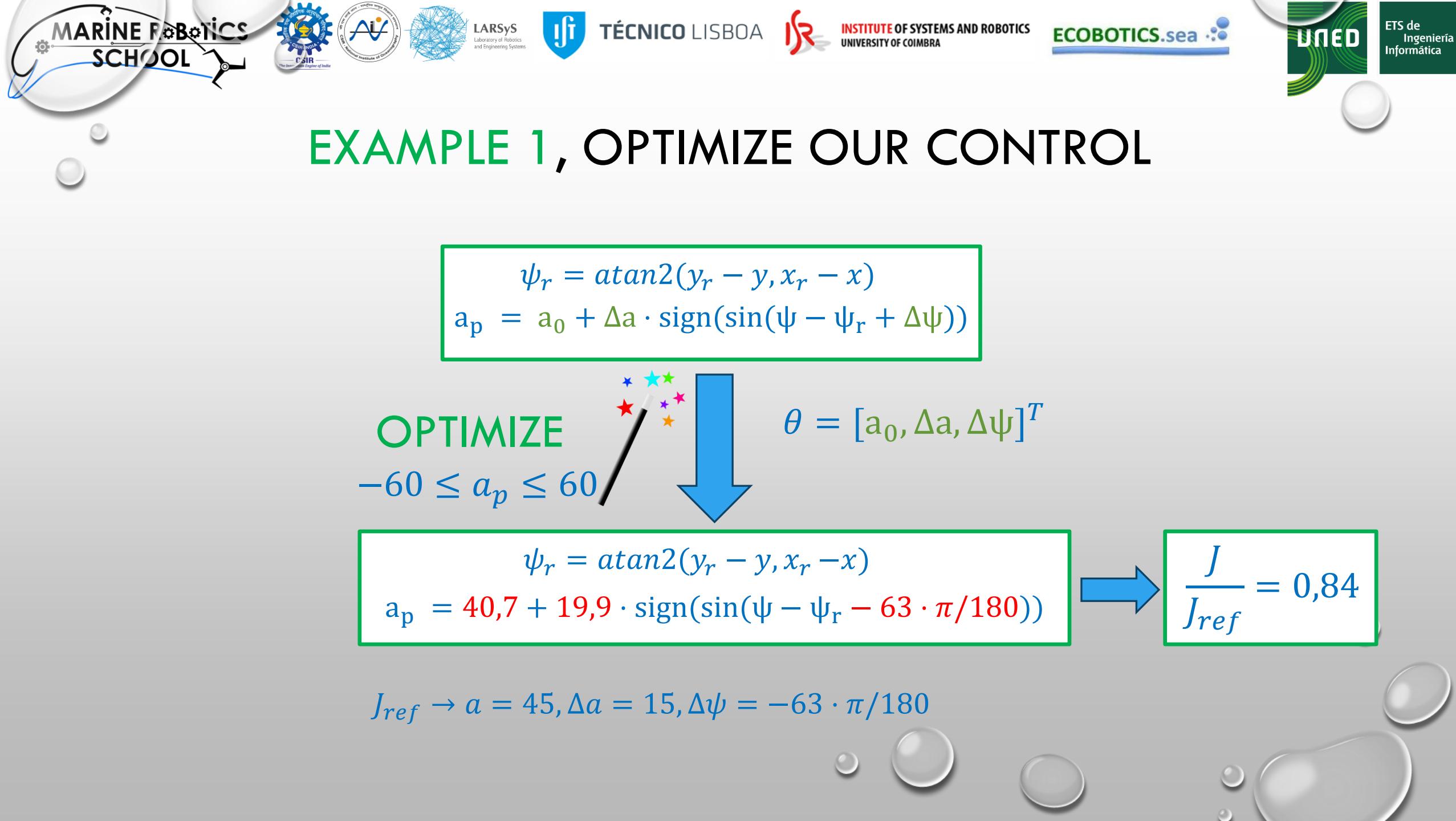
$$\begin{aligned}\psi_r &= \text{atan}2(y_r - y, x_r - x) \\ a_p &= a_0 + \Delta a \cdot \text{sign}(\sin(\psi - \psi_r + \Delta\psi))\end{aligned}$$

OPTIMIZE

$$-60 \leq a_p \leq 60$$



$$\theta = [a_0, \Delta a, \Delta\psi]^T$$



EXAMPLE 2, THE BEST POSSIBLE CONTROL FROM A → B

- START AT SOME INITIAL CONDITION (A)

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 - APPLY THE FIRST CONTROL ACTION
 - RECOMPUTE THE CONTROL IN THE NEXT TIME STEM
 - LIKE GPS NAVIGATOR RECOMPUTES THE OPTIMAL TRAJECTORY

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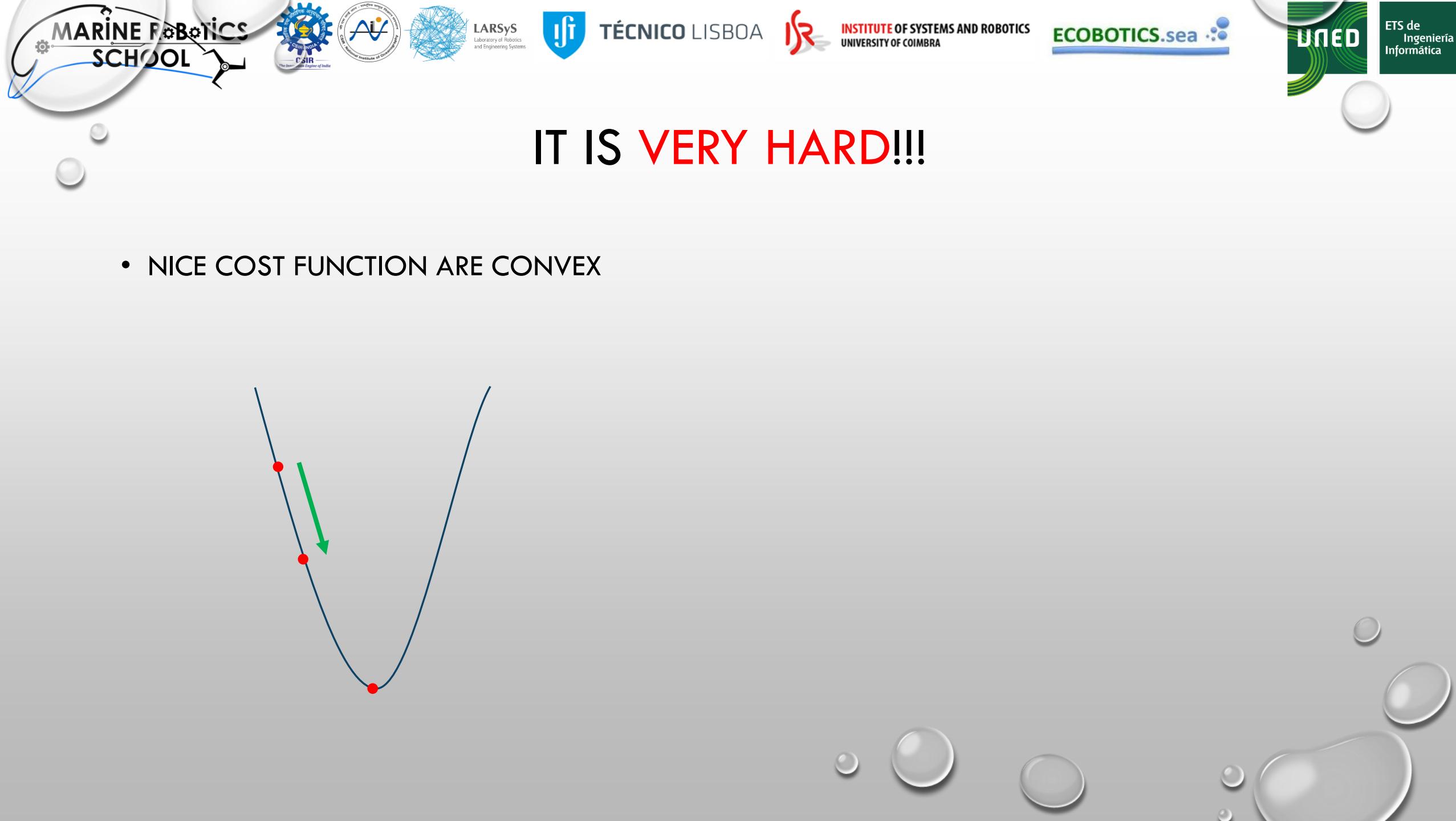
- START AT SOME INITIAL CONDITION (A)
- PARAMETRIZE THE CONTROLLER SPECIFYING THE CONTROL ACTIONS AS A FUNCTION OF SAMPLING TIME
- JUST OPTIMIZE
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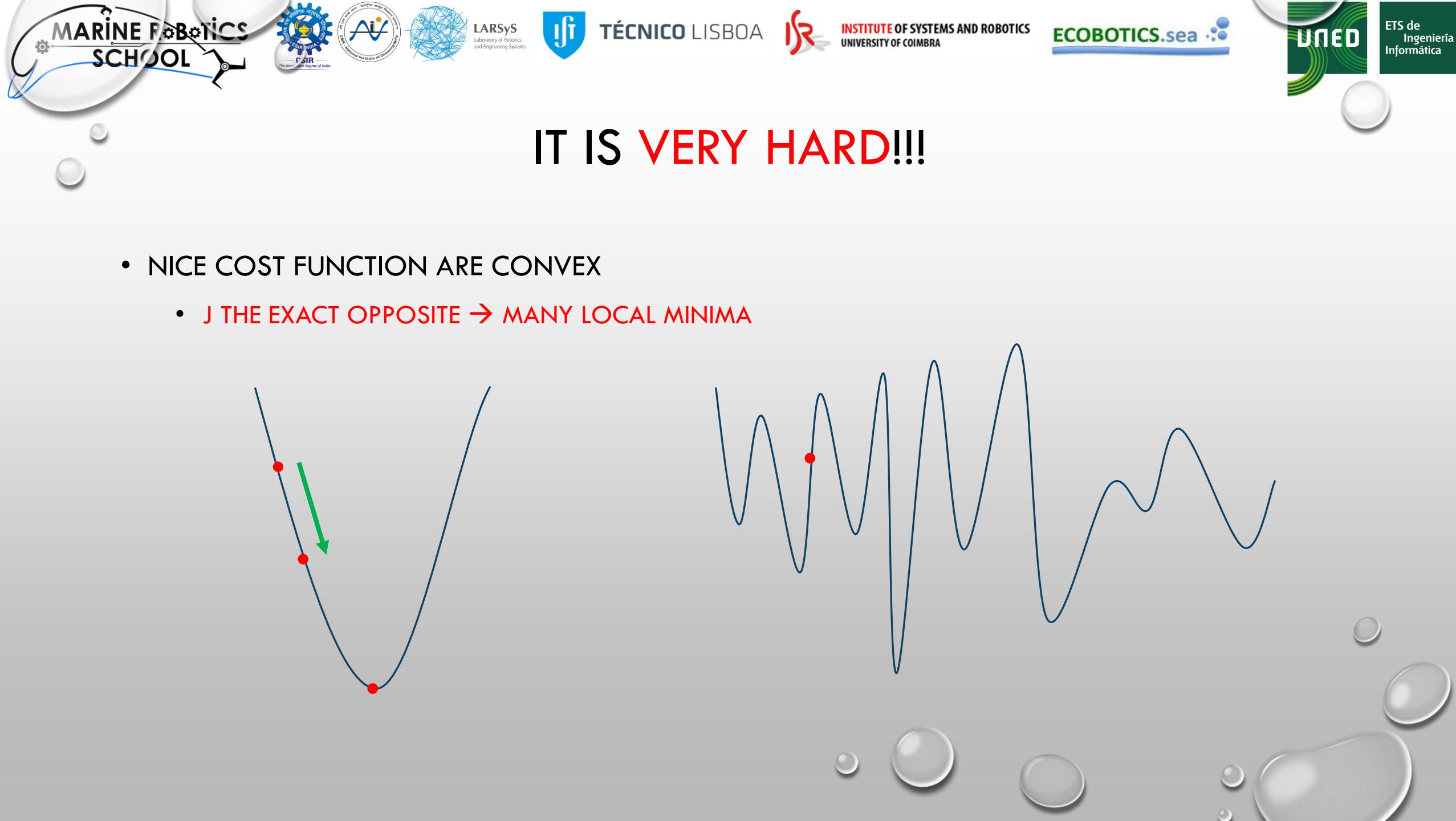
- NICE COST FUNCTION ARE CONVEX

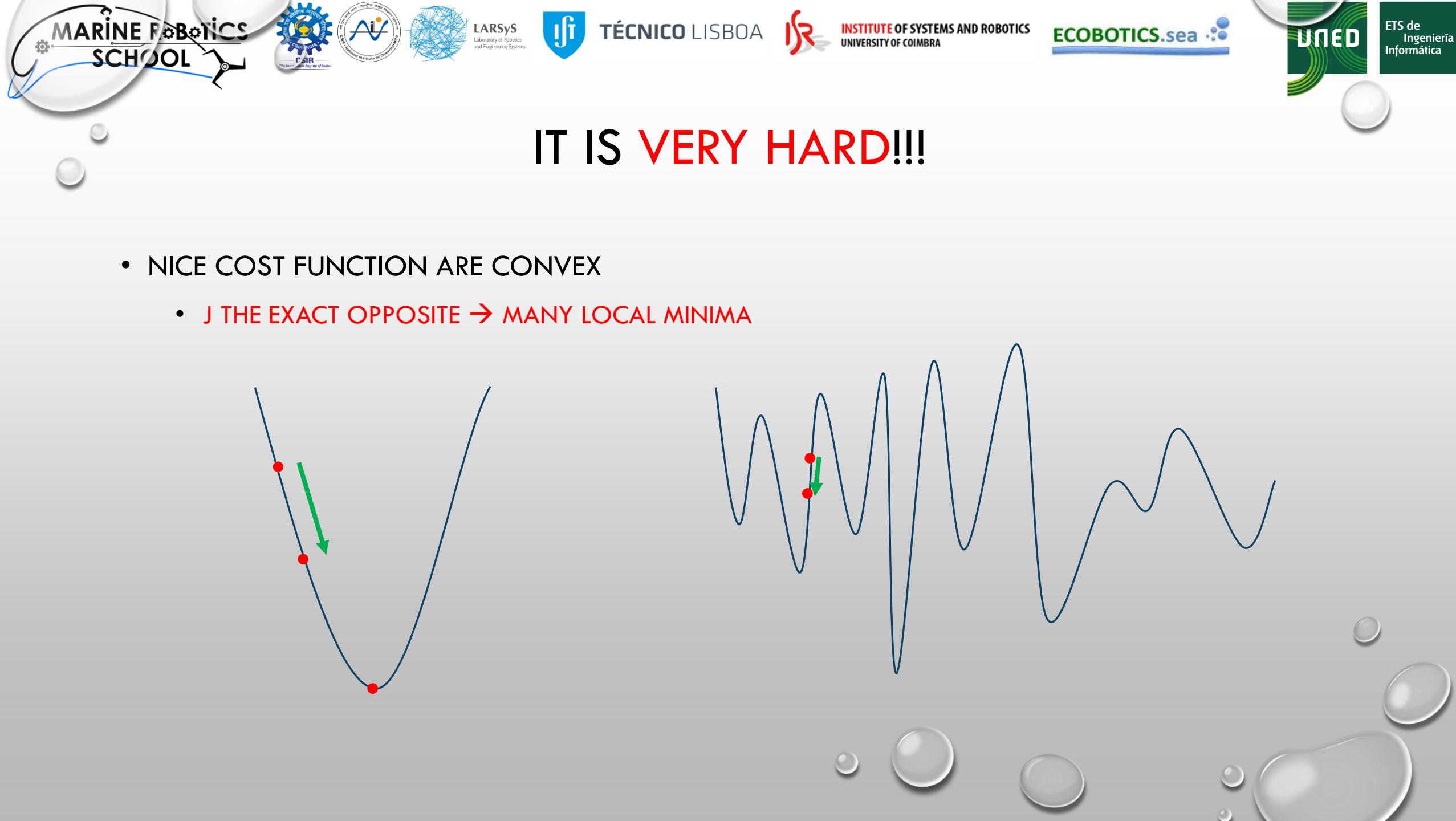


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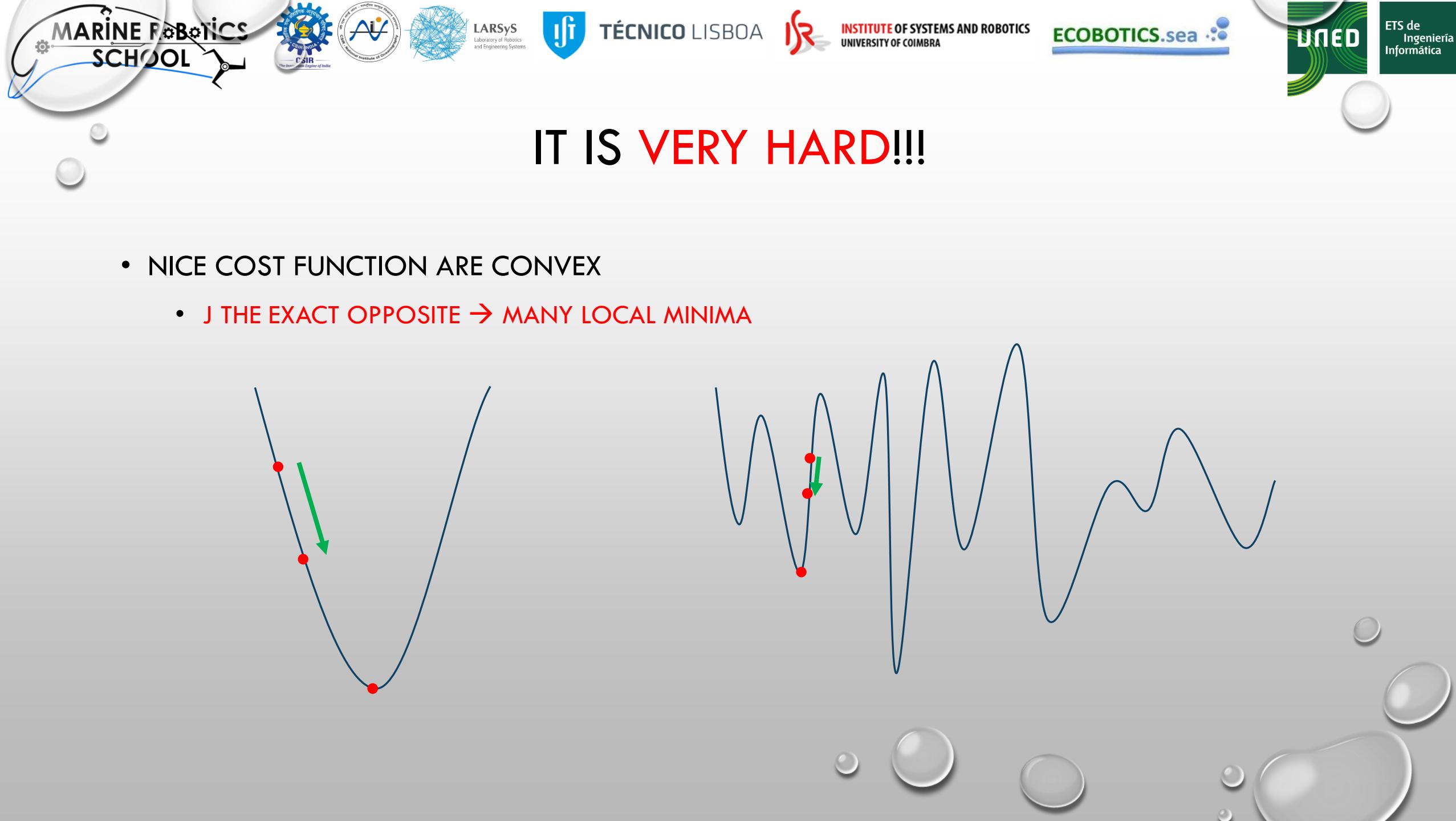


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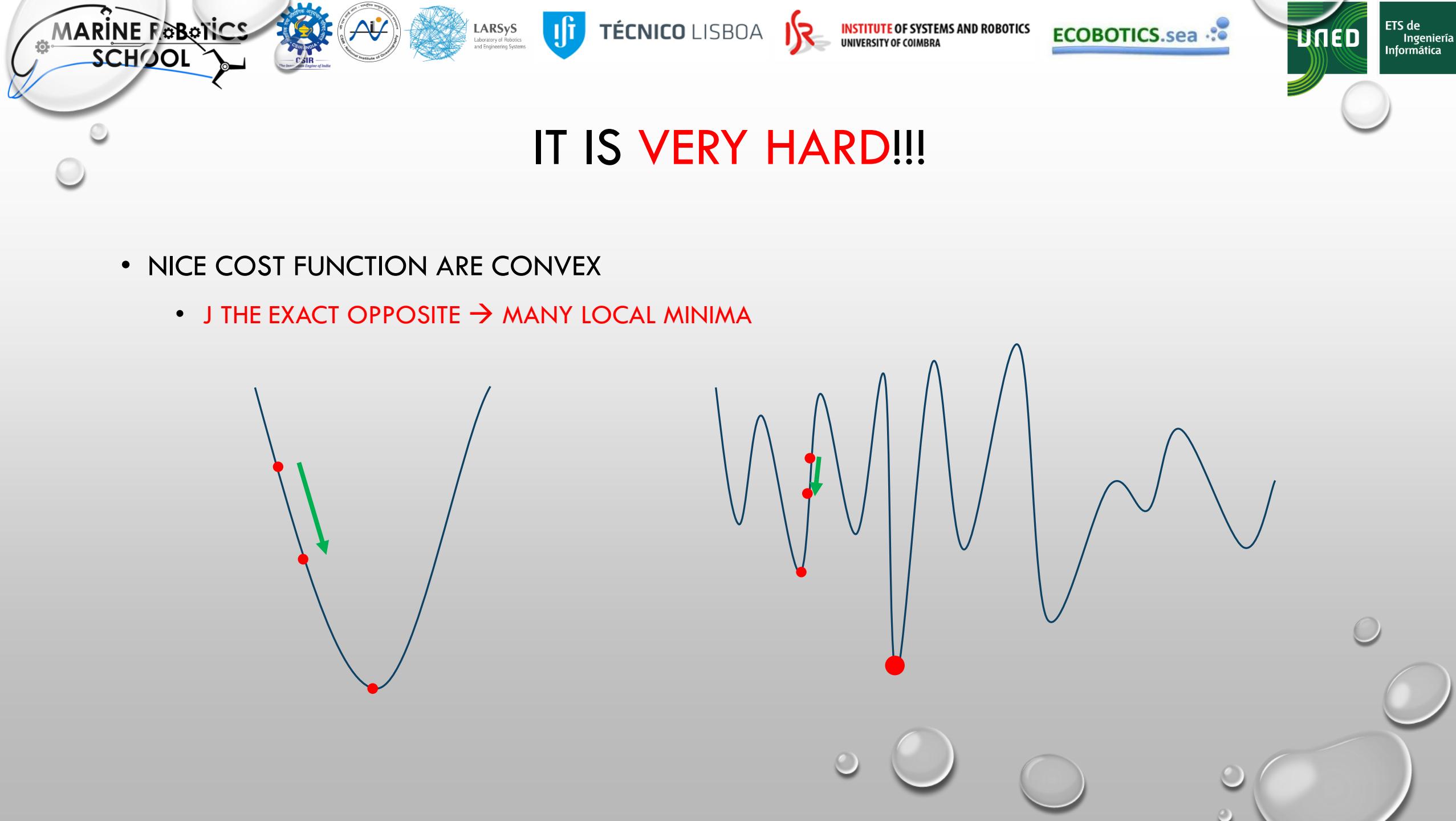




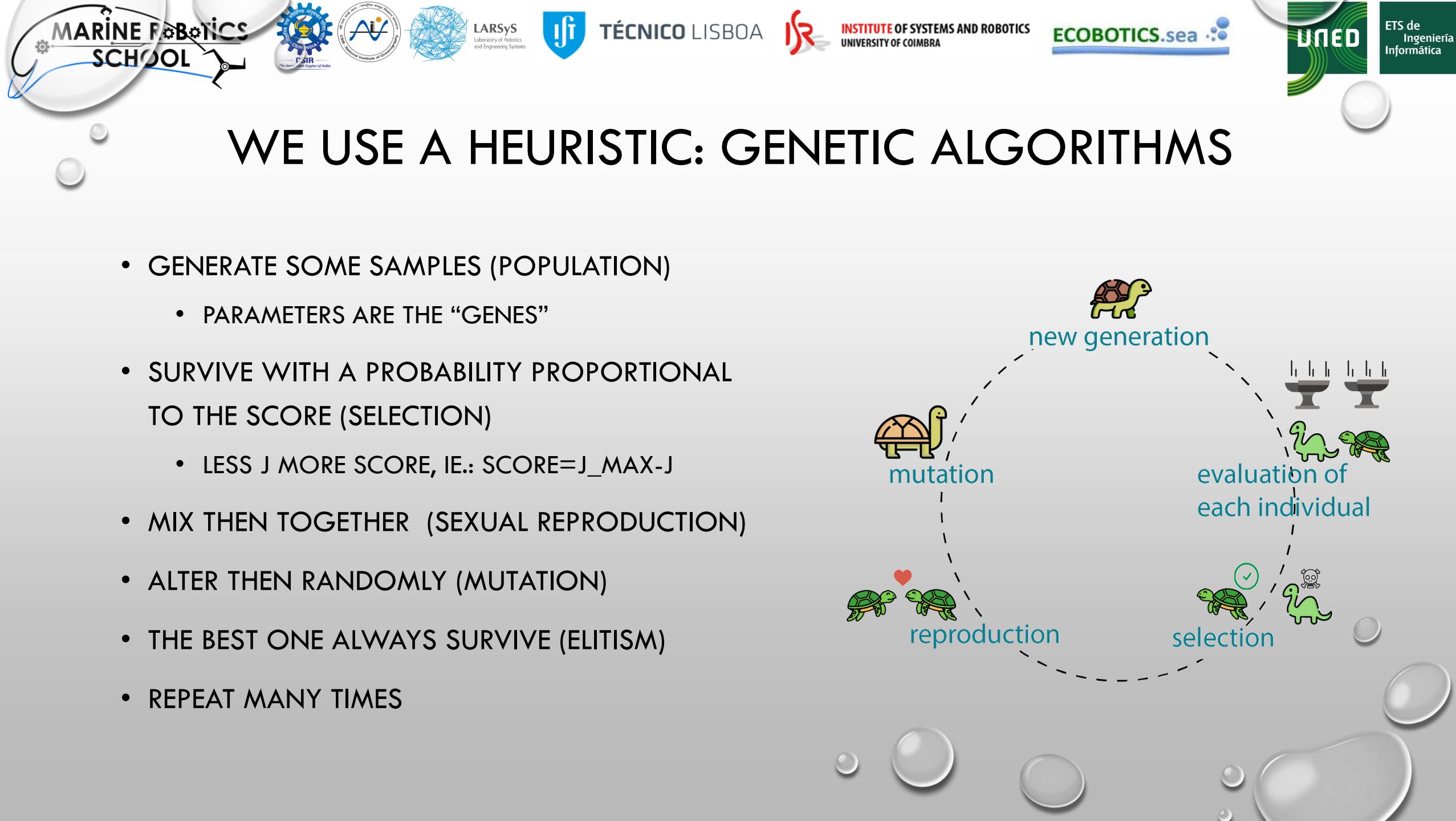
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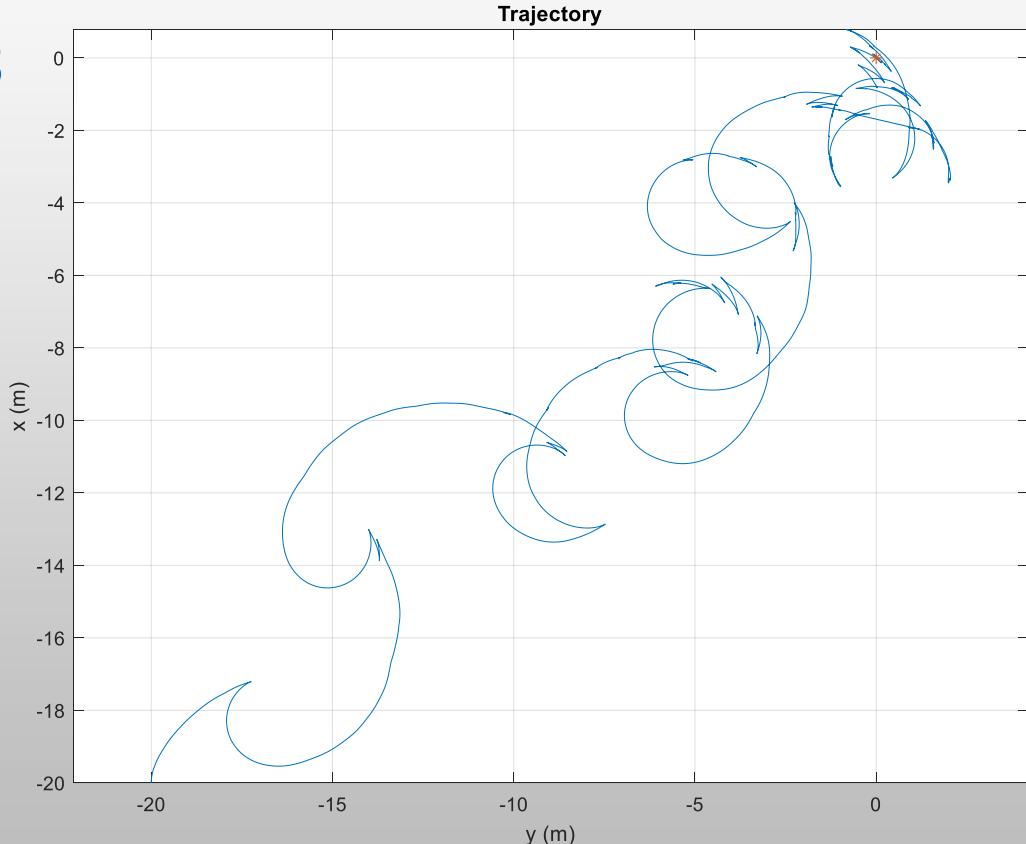
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 - J THE EXACT OPPOSITE → MANY LOCAL MINIMA



“JUST” DO IT

- FIRST “GOOD RESULT” 29N
- BROKEN IN 10 TIME WINDOWS
- INDEPENDENTLY OPTIMIZED
- OPTIMIZATION TOOK MANY HOURS
 - IMPOSSIBLE TO USE IN REAL TIME...

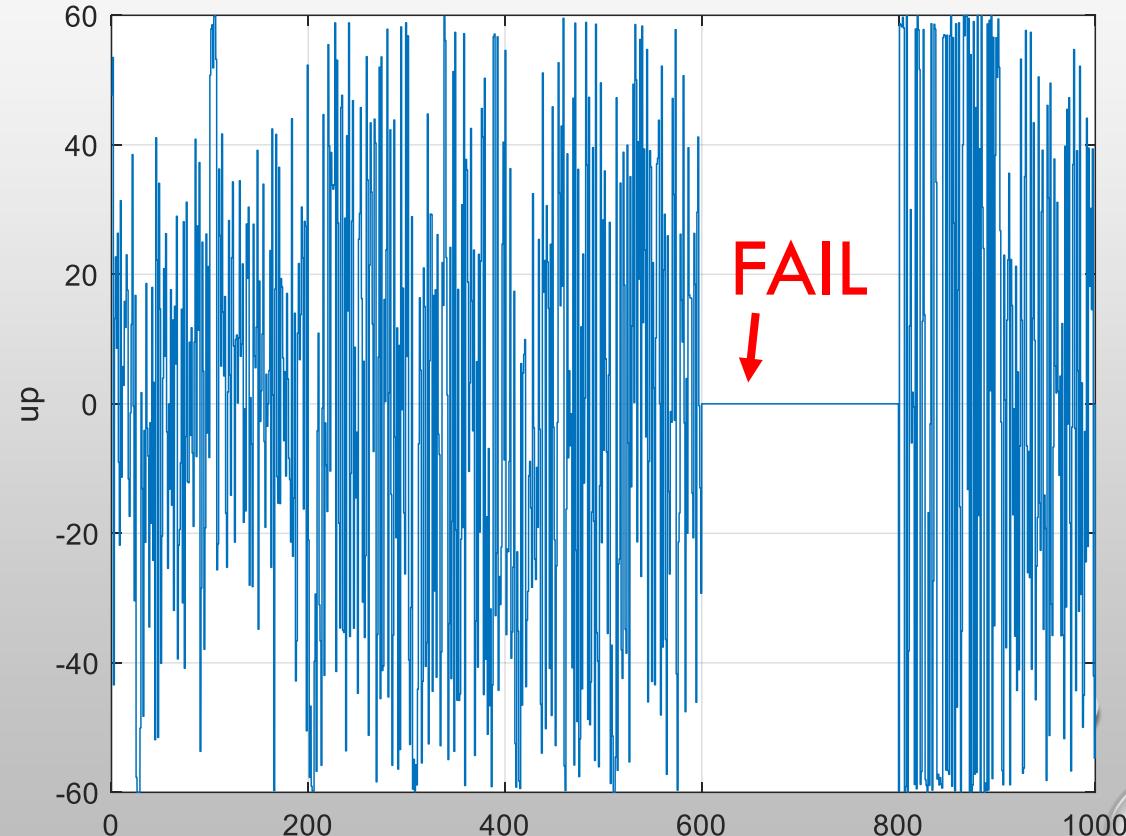
$$\frac{J}{J_{ref}} = 1,03$$

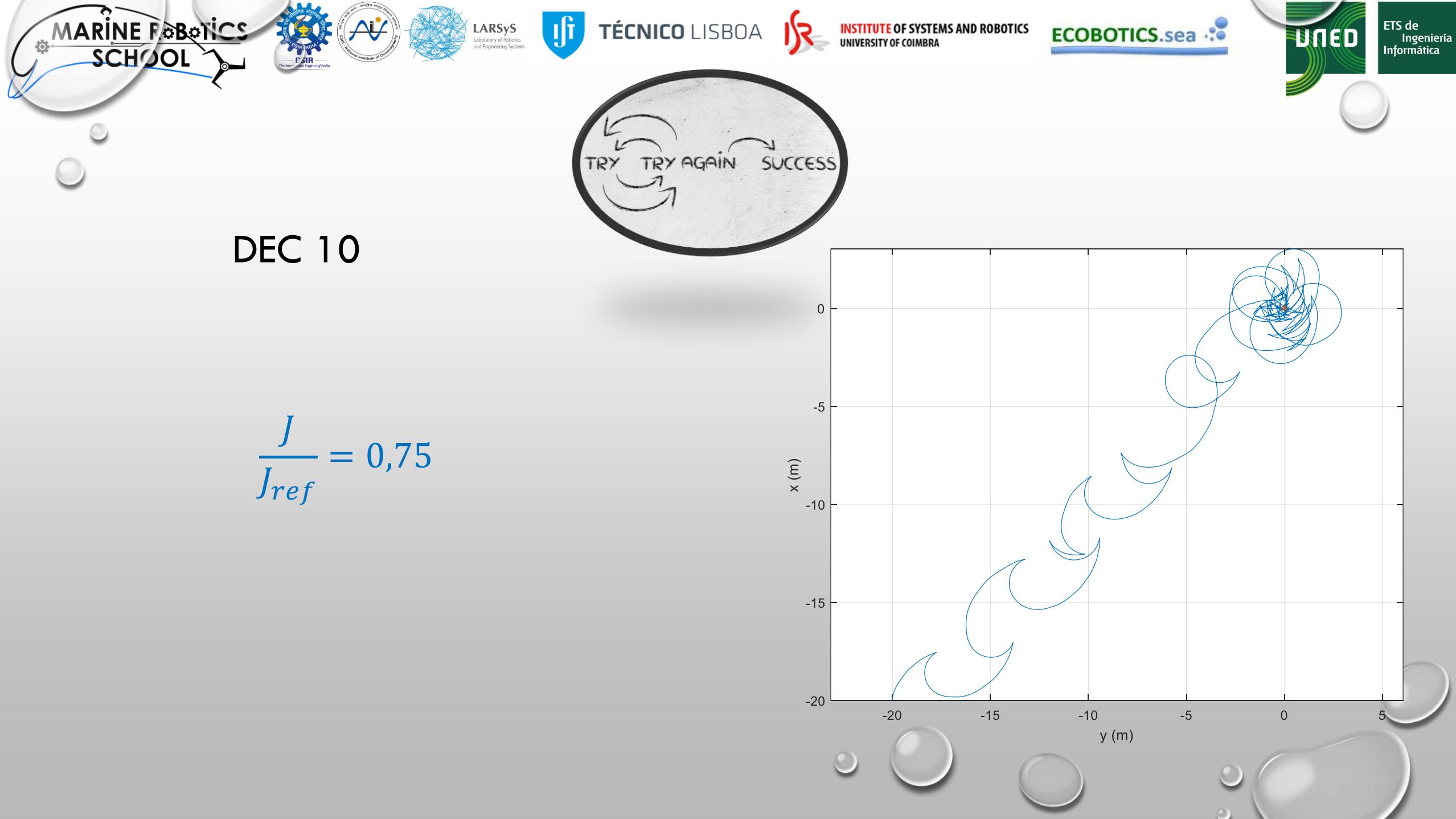


“JUST” DO IT

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- BROKEN IN 10 TIME WINDOWS
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- CONTROL ACTION IS UGLY

$$\frac{J}{J_{ref}} = 1,03$$





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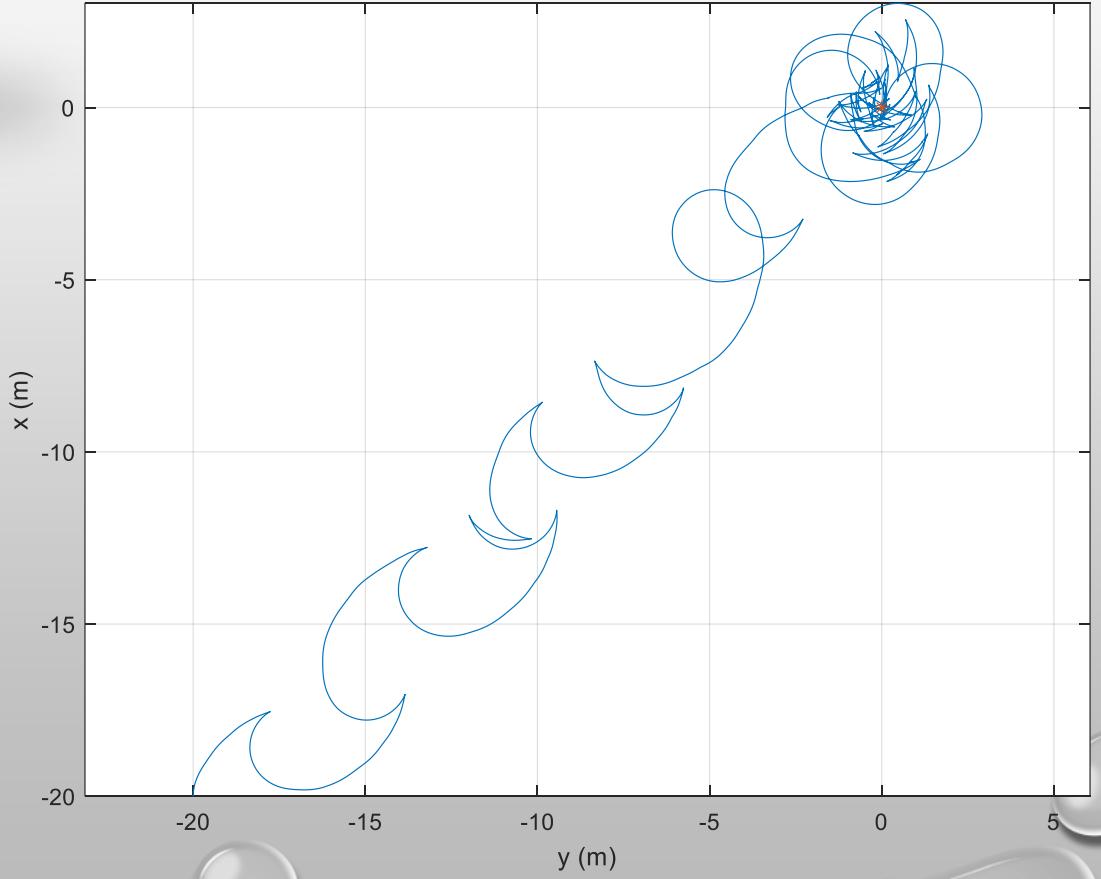
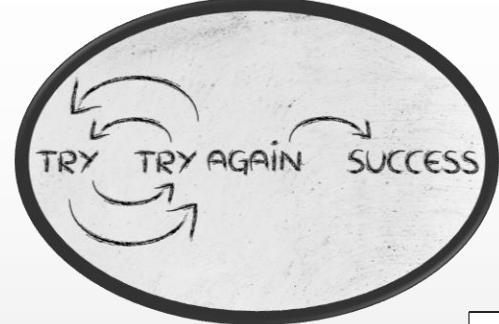
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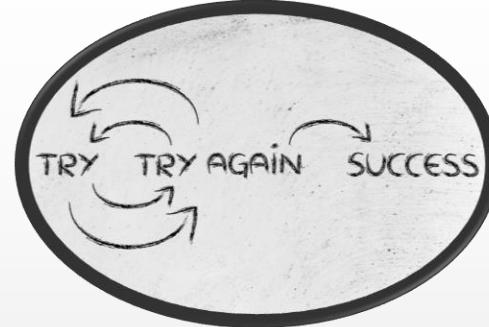


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DEC 10

$$\frac{J}{J_{ref}} = 0,75$$

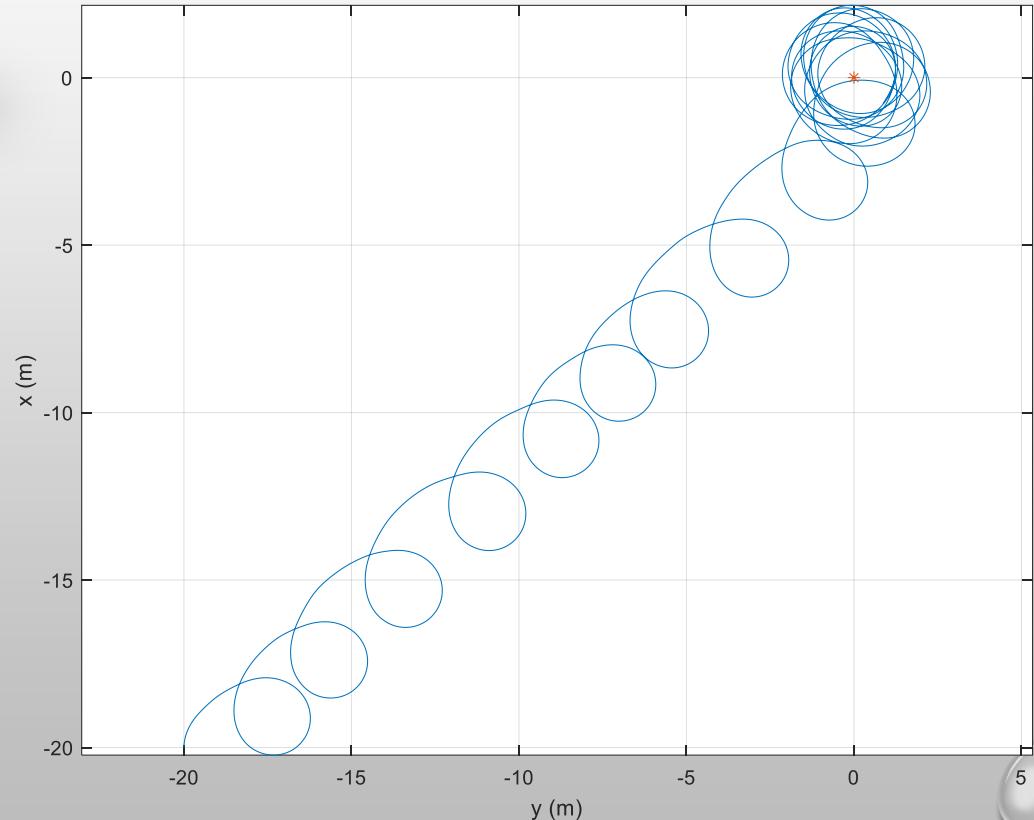




- THE BEST DEC 22
- SPIRAL LIKE
 - NOT QUALITATIVELY FOR THE INITIAL CONTROLLER

$$\frac{J}{J_{ref}} = 0,7429$$

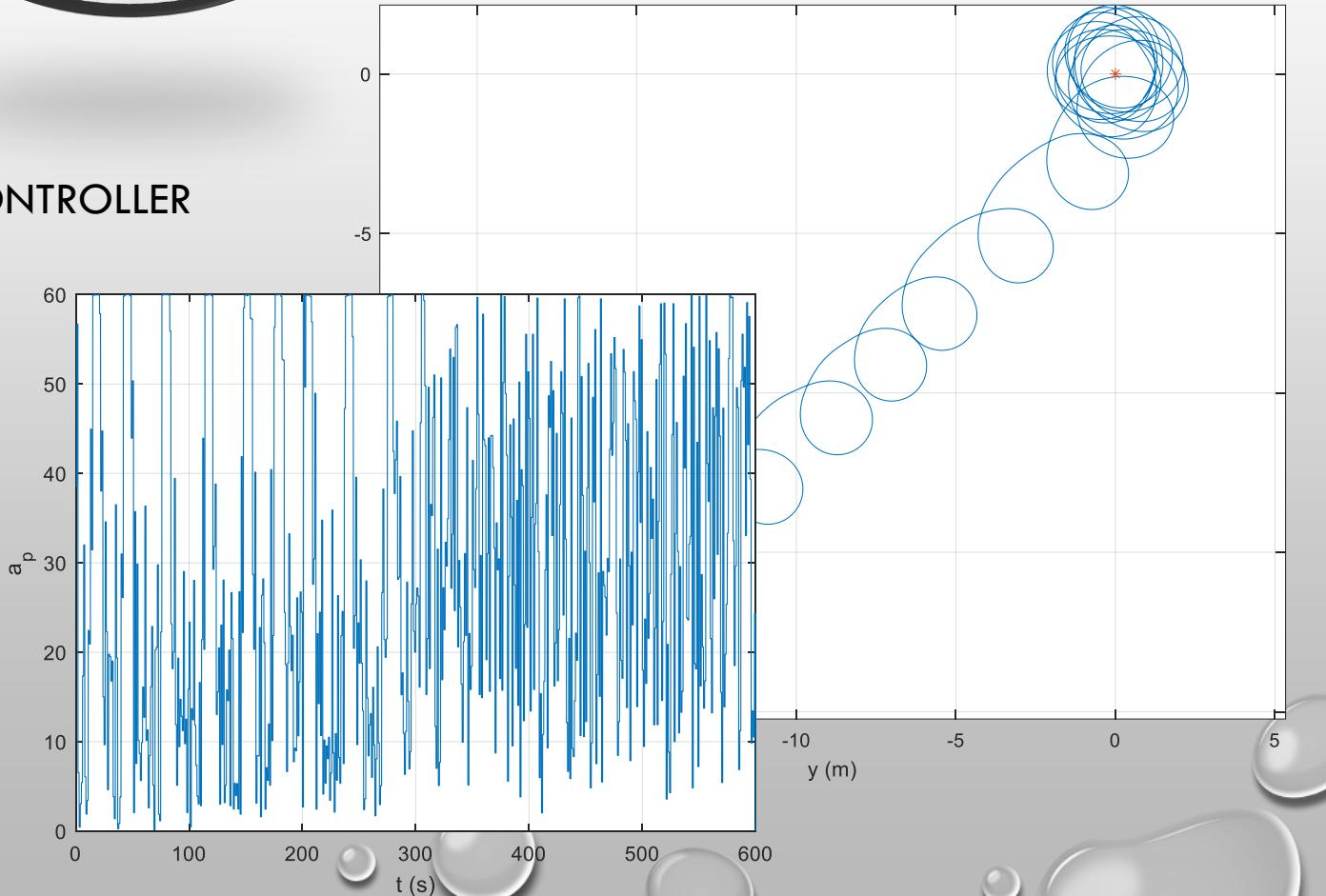
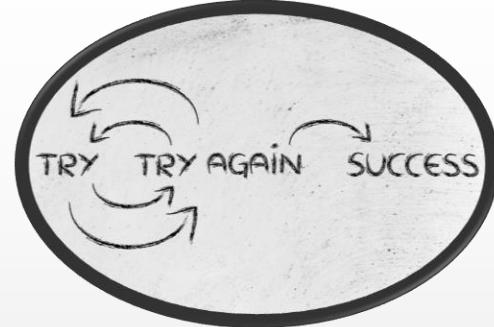
- CONTROL ACTION IS UGLY

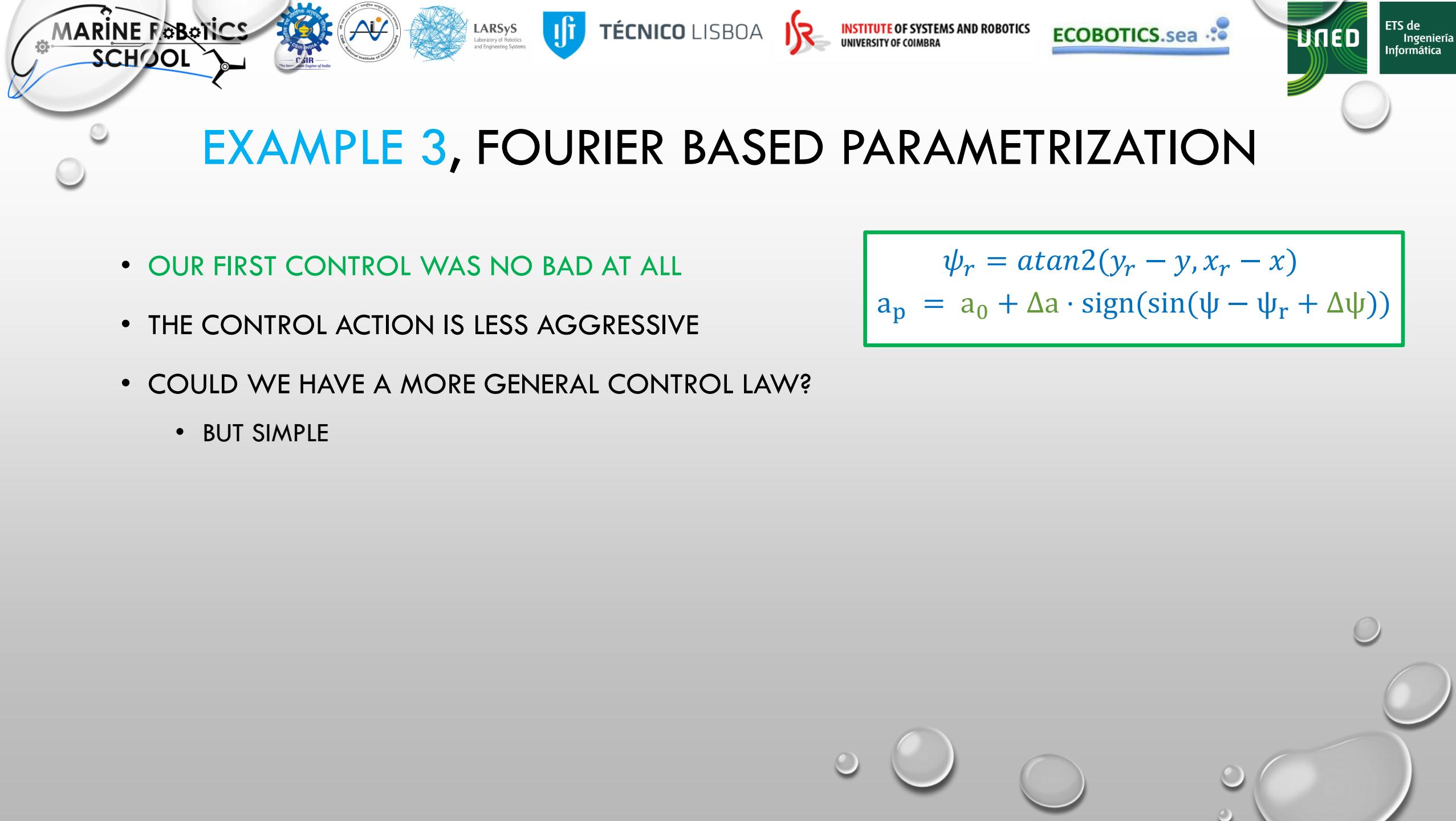


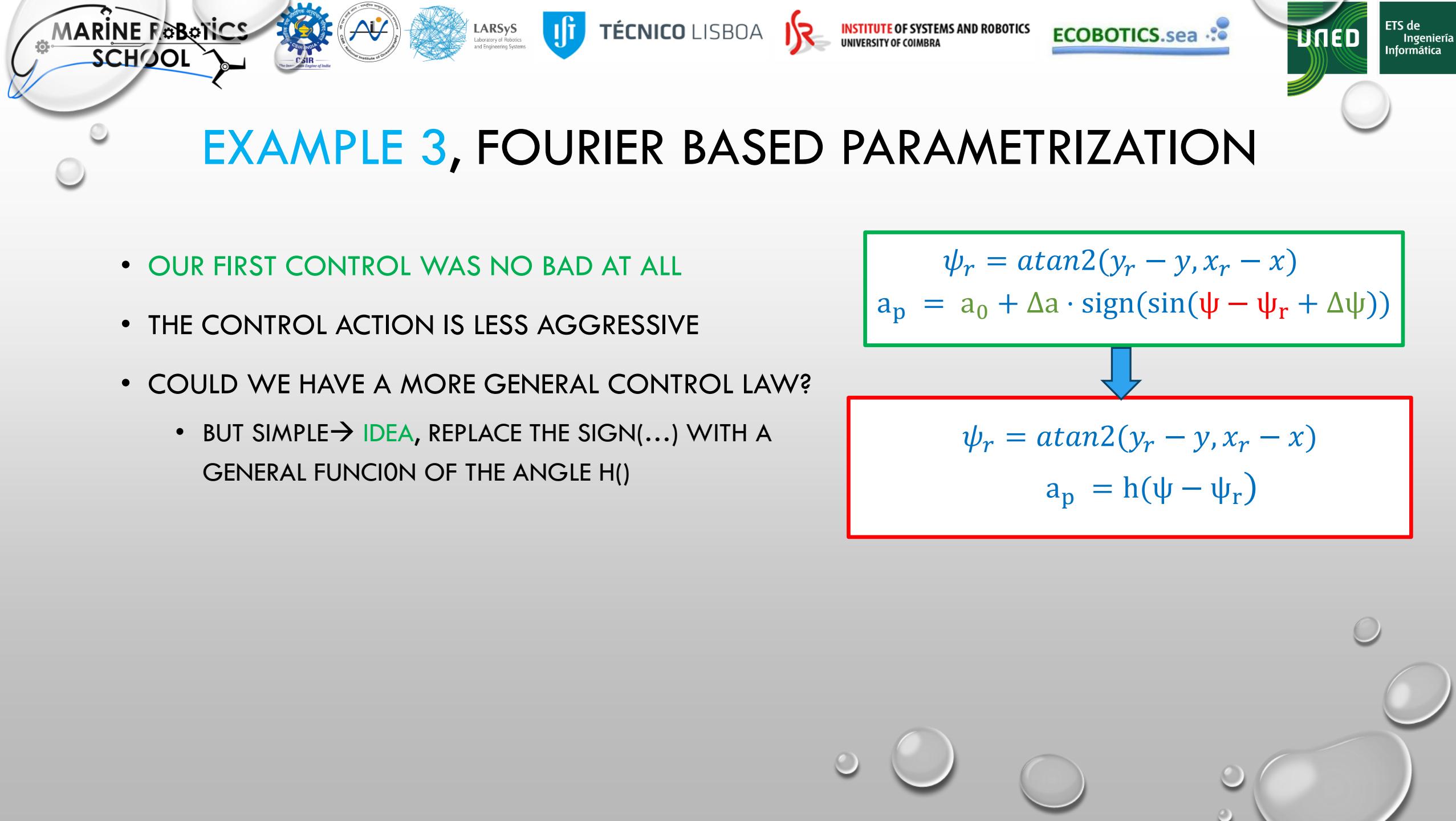
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EXAMPLE 3, FOURIER BASED PARAMETRIZATION

- OUR FIRST CONTROL WAS NO BAD AT ALL
- THE CONTROL ACTION IS LESS AGGRESSIVE
- COULD WE HAVE A MORE GENERAL CONTROL LAW?
 - BUT SIMPLE → IDEA, REPLACE THE SIGN(...) WITH A GENERAL FUNCION OF THE ANGLE H()
 - PARAMETRIZED BY A FOURIER SERIES

$$\psi_r = \text{atan2}(y_r - y, x_r - x)$$

$$a_p = a_0 + \Delta a \cdot \text{sign}(\sin(\psi - \psi_r + \Delta\psi))$$



$$\psi_r = \text{atan2}(y_r - y, x_r - x)$$

$$a_p = h(\psi - \psi_r)$$

$$h(\alpha) = \frac{b_0}{2} + \sum_{i=1}^N b_i \cos(j\alpha) + c_j \sin(j\alpha)$$

$$\theta = [b_0, \dots, b_n, c_1, \dots, c_n]^T$$










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 - PROBLEM:** IT IS DIFFICULT TO ENSURE THAT THE CONTROL ACTION IN THE **SAFE REGION** OF THE ACTUATOR

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 - SOLUTION: SATURATE THE CONTROL LAW

$$\psi_r = \text{atan2}(y_r - y, x_r - x)$$

$$a_p = \max(a_{\min}, \min(h(\psi - \psi_r), a_{\max}))$$

$$h(\alpha) = \frac{b_0}{2} + \sum_{i=1}^N b_j \cos(j\alpha) + c_j \sin(j\alpha)$$

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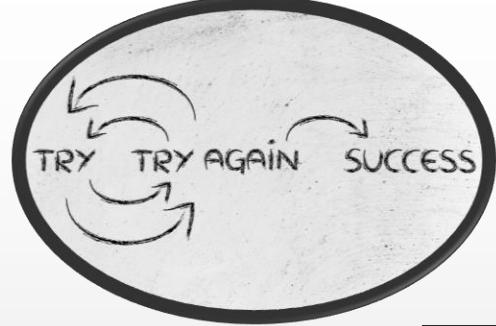
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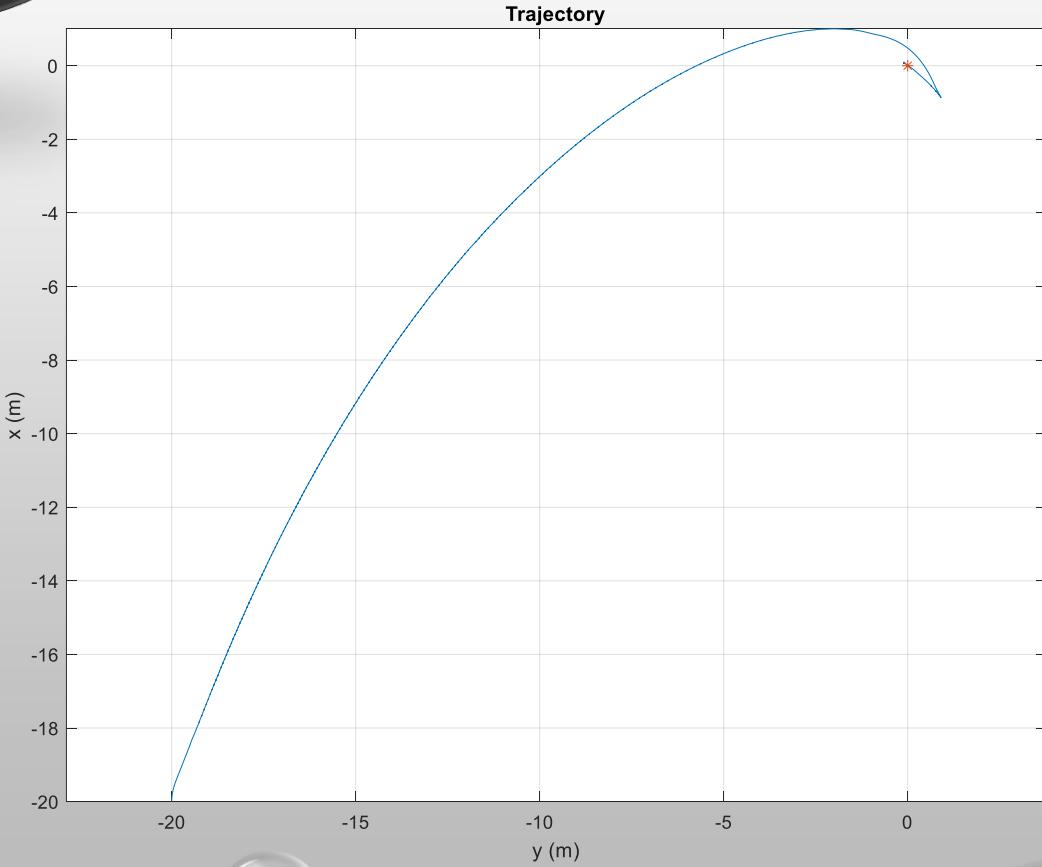
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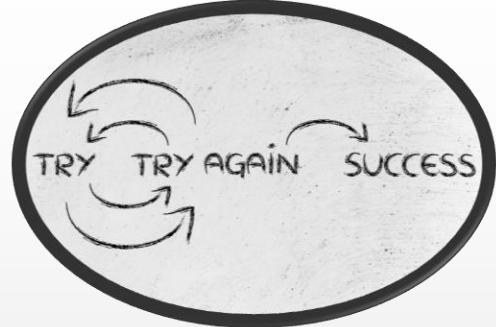
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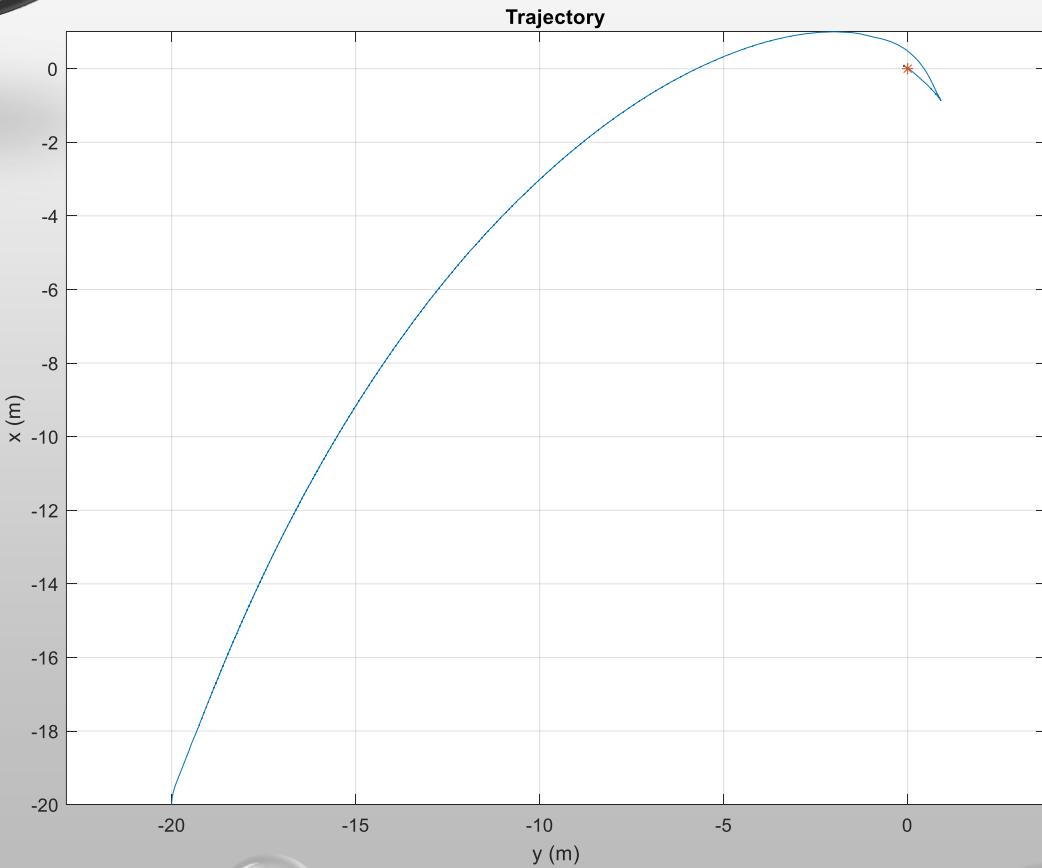


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$$\frac{J}{J_{ref}} = 2,37 \quad \text{:(}$$





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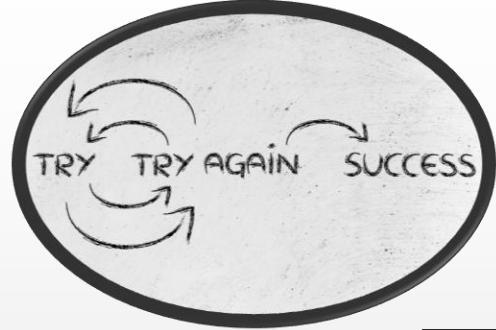
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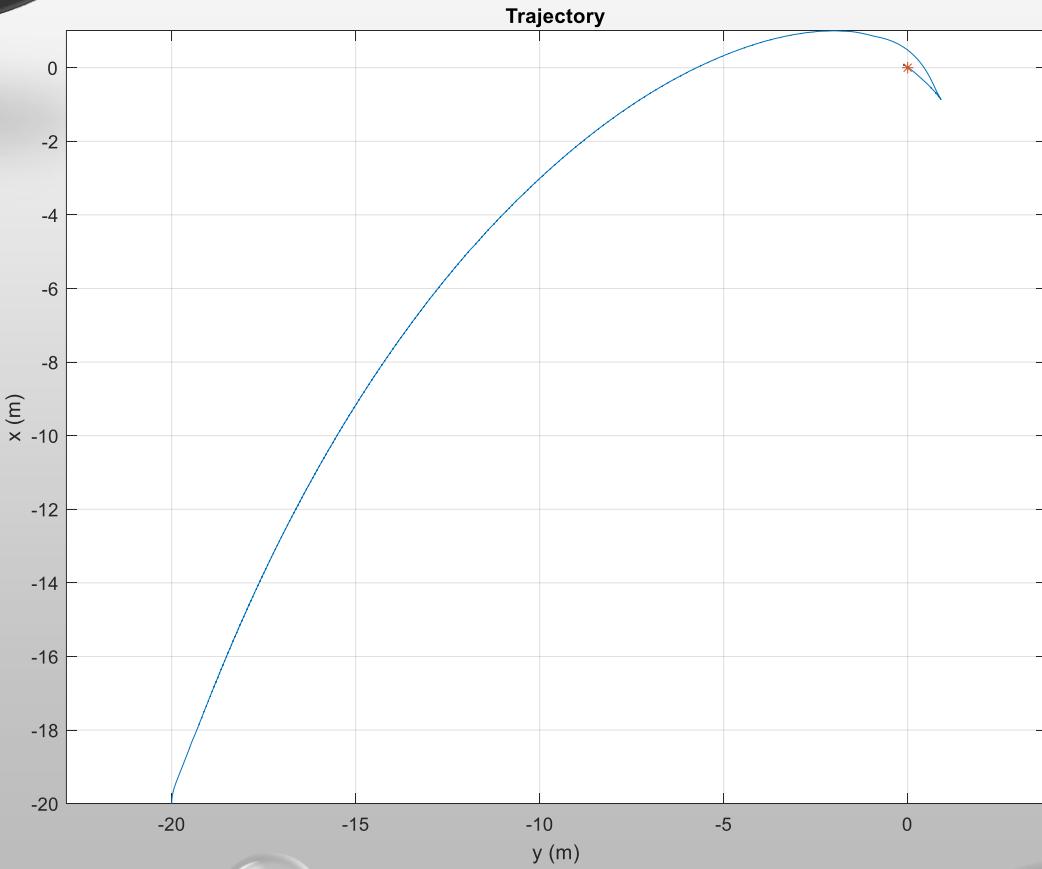


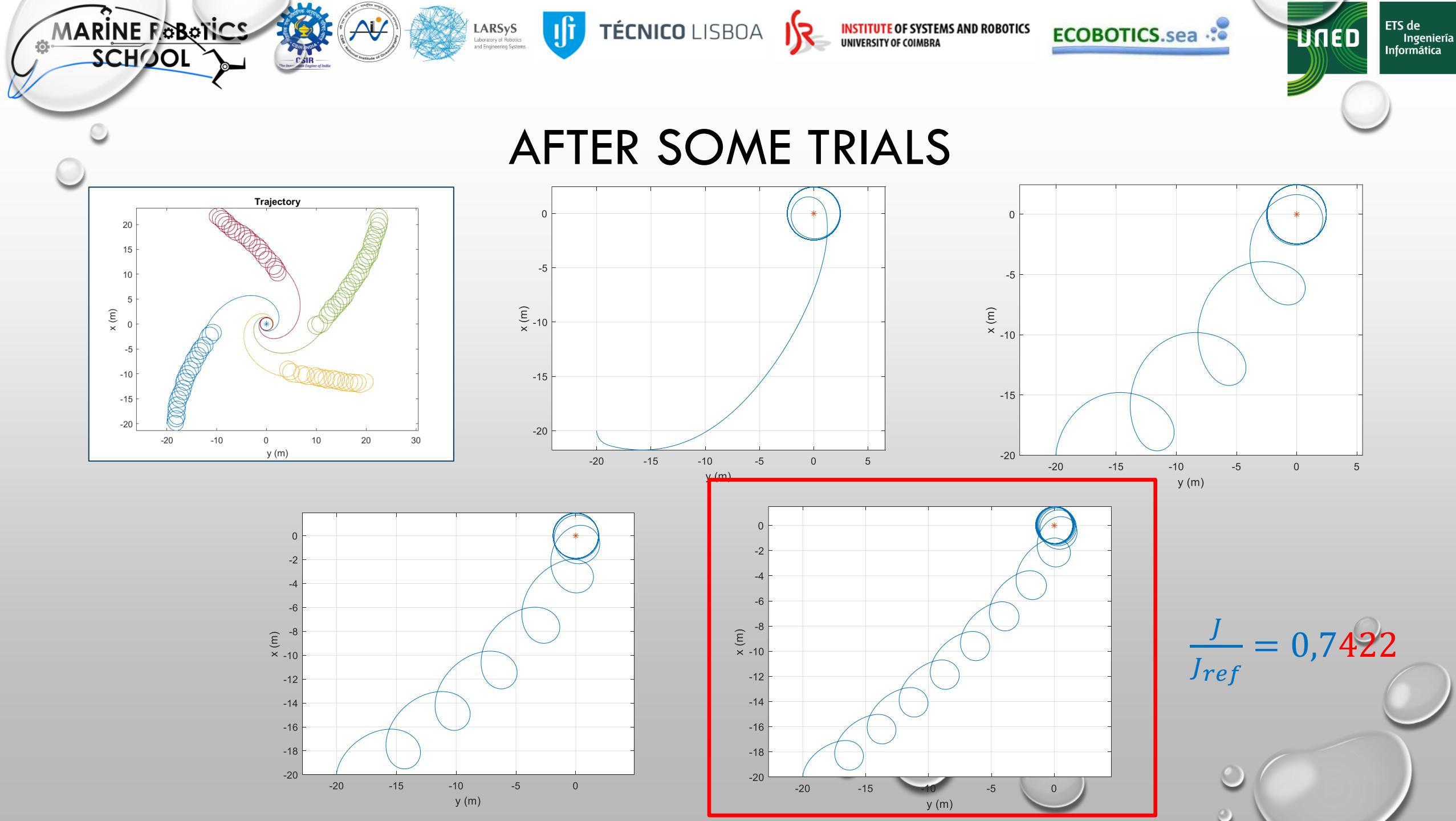
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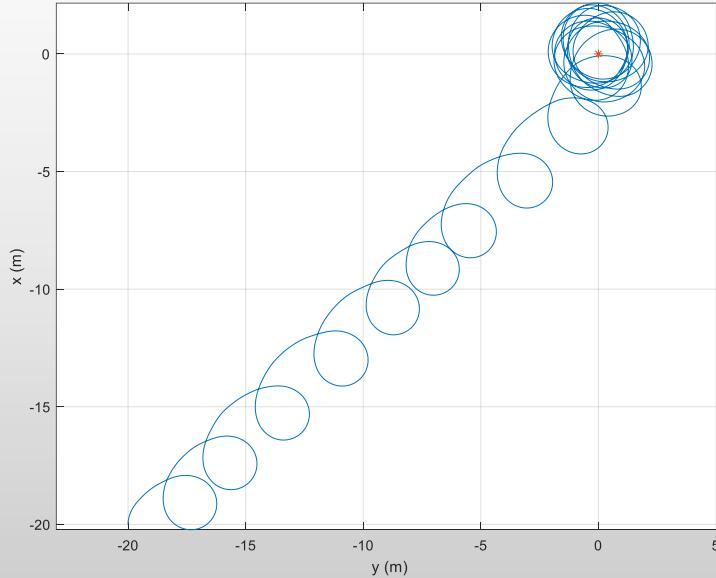
$$\frac{J}{J_{ref}} = 2,37 \quad \text{:(}$$





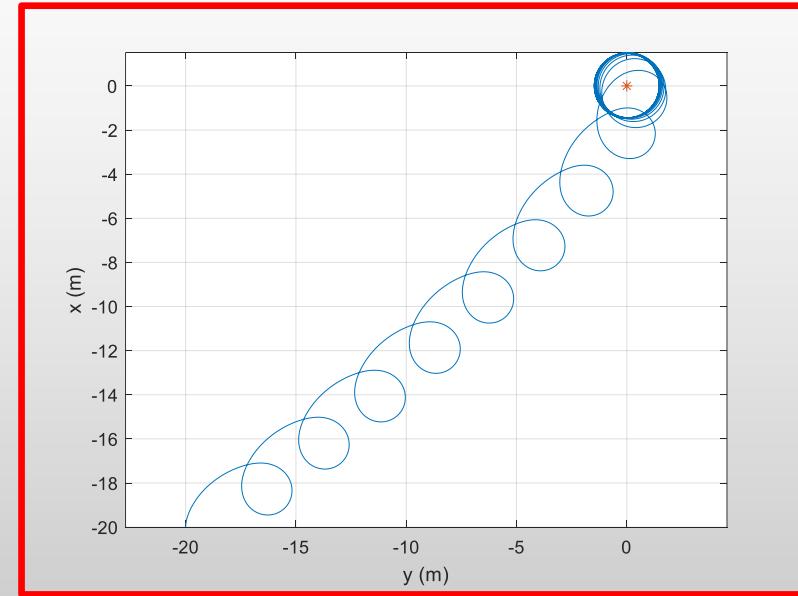
COMPARISON

“BEST POSSIBLE”



$$\frac{J}{J_{ref}} = 0,7429$$

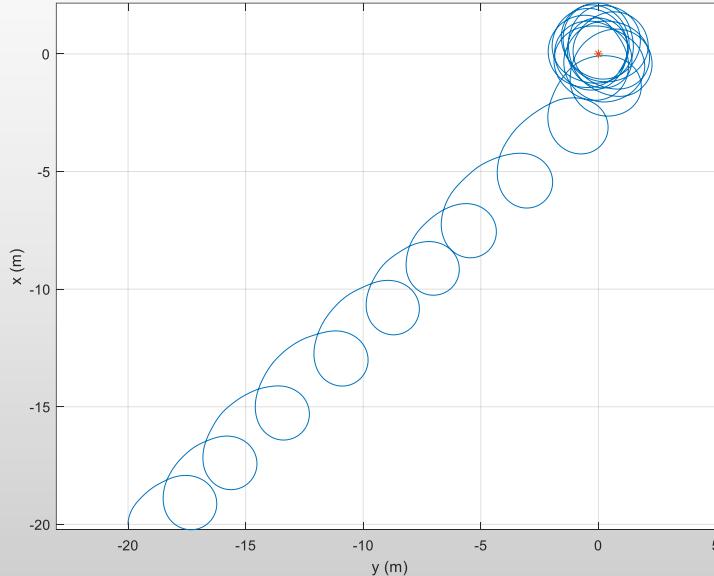
FOURIER



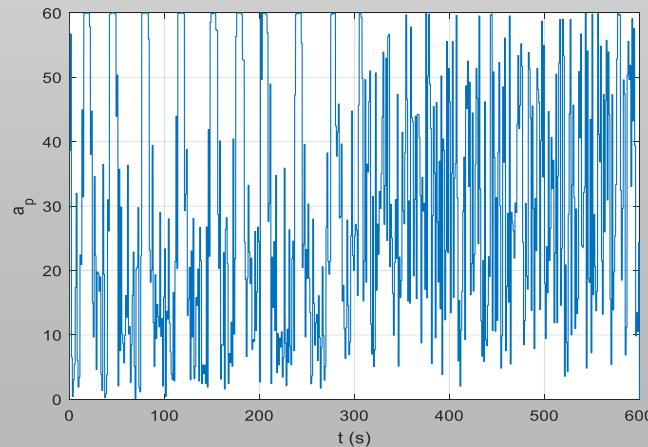
$$\frac{J}{J_{ref}} = 0,7422$$

COMPARISON

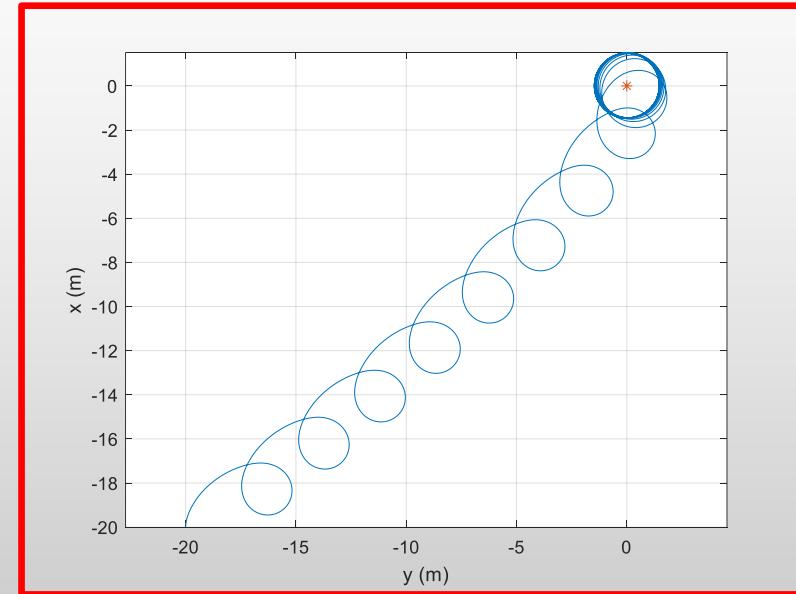
“BEST POSSIBLE”



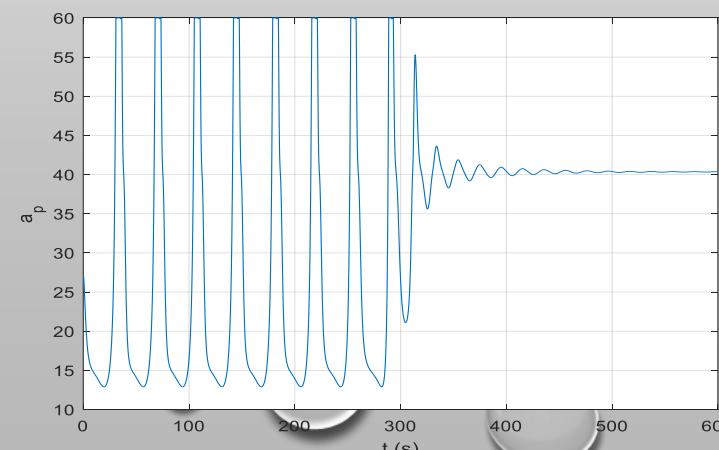
$$\frac{J}{J_{ref}} = 0,7429$$



FOURIER

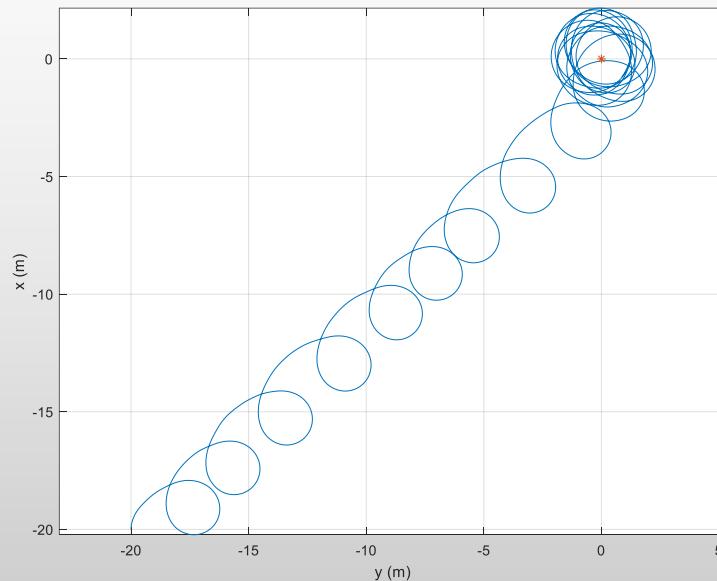


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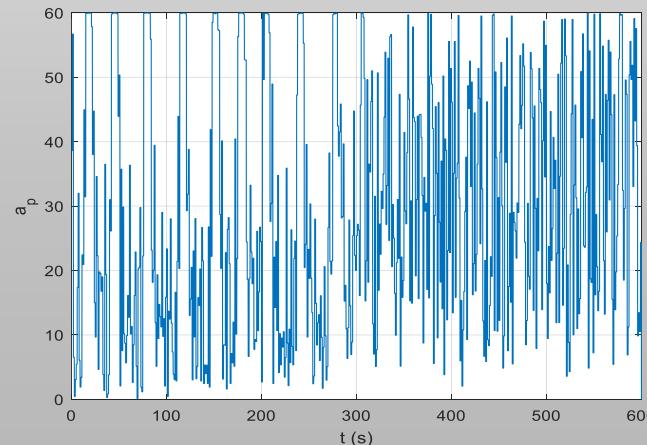


COMPARISON

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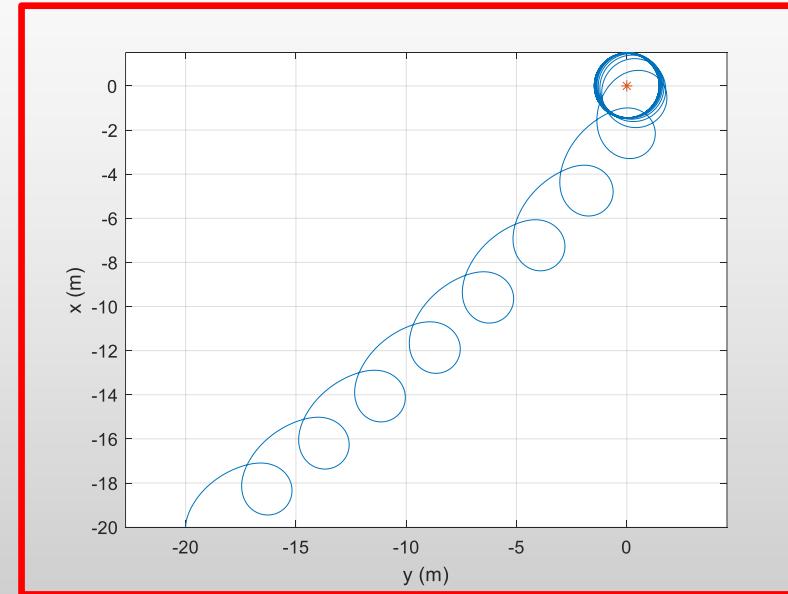


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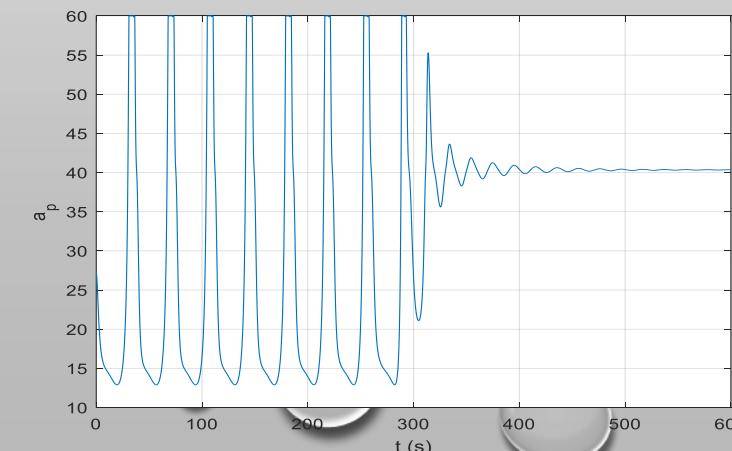


HOURS TO COMPUTE 😞
BAD FOR THE MOTOR 😞

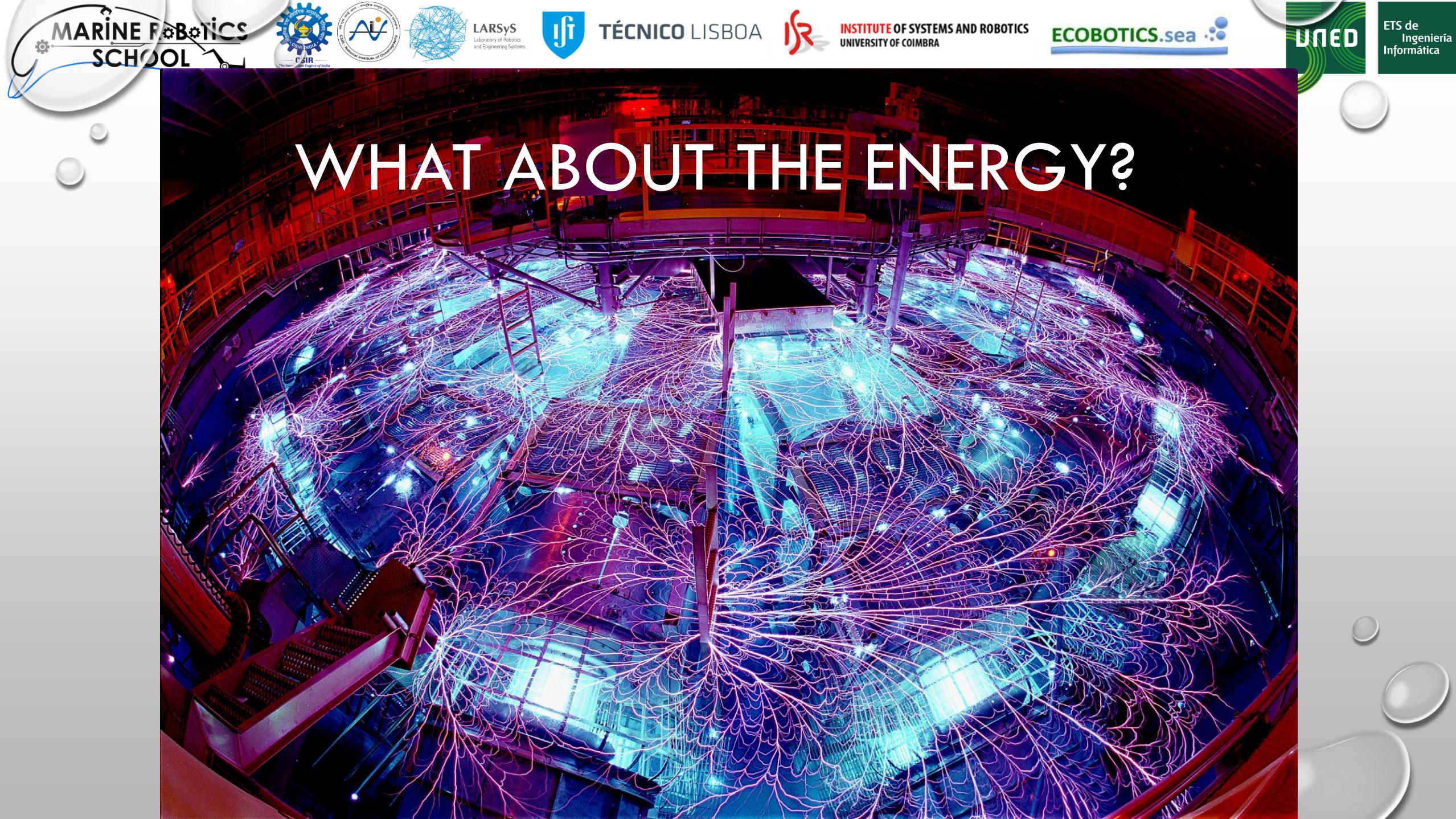
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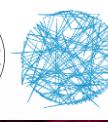
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REAL TIME 😊
NICE AND SMOOTH 😊



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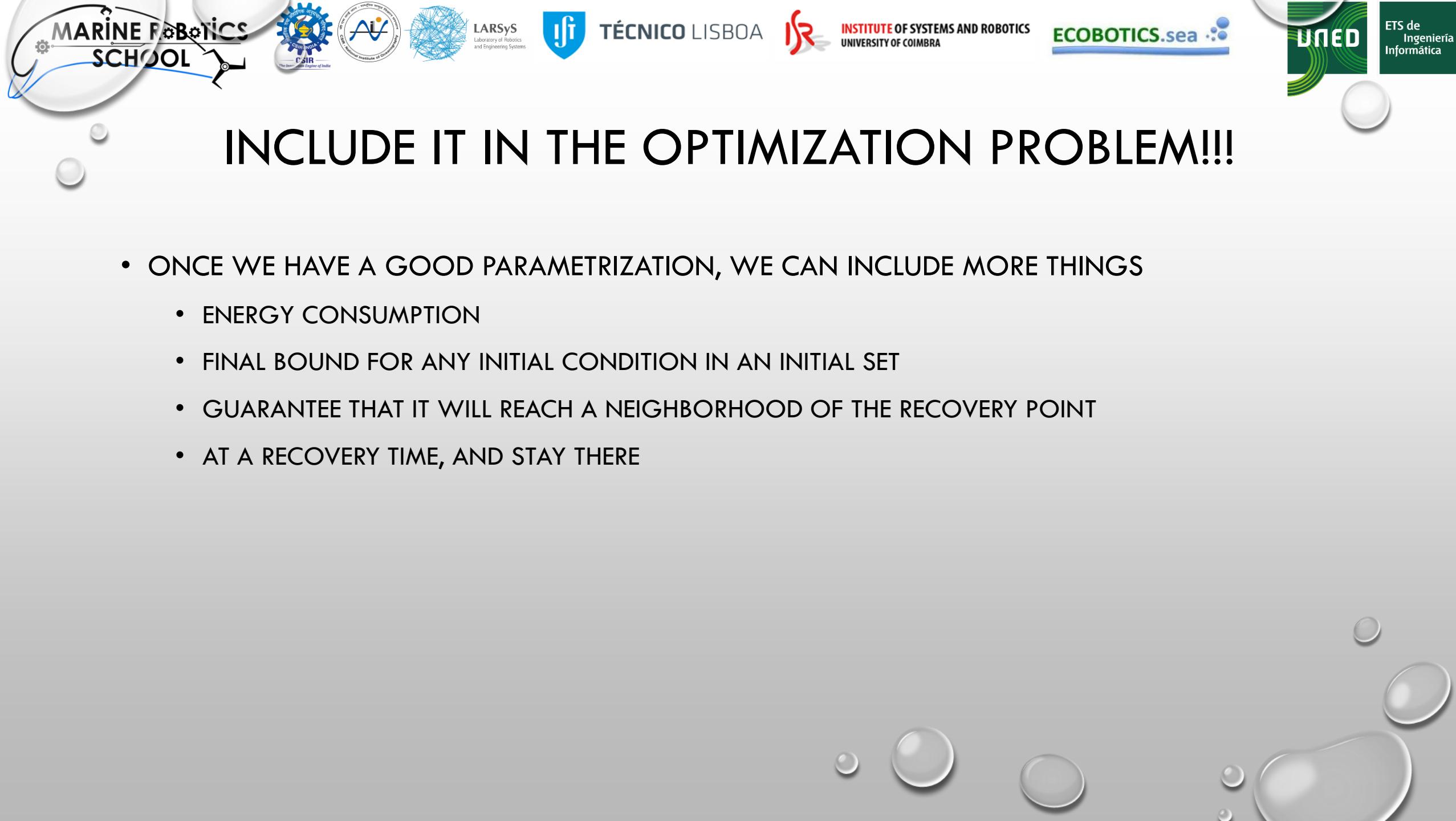
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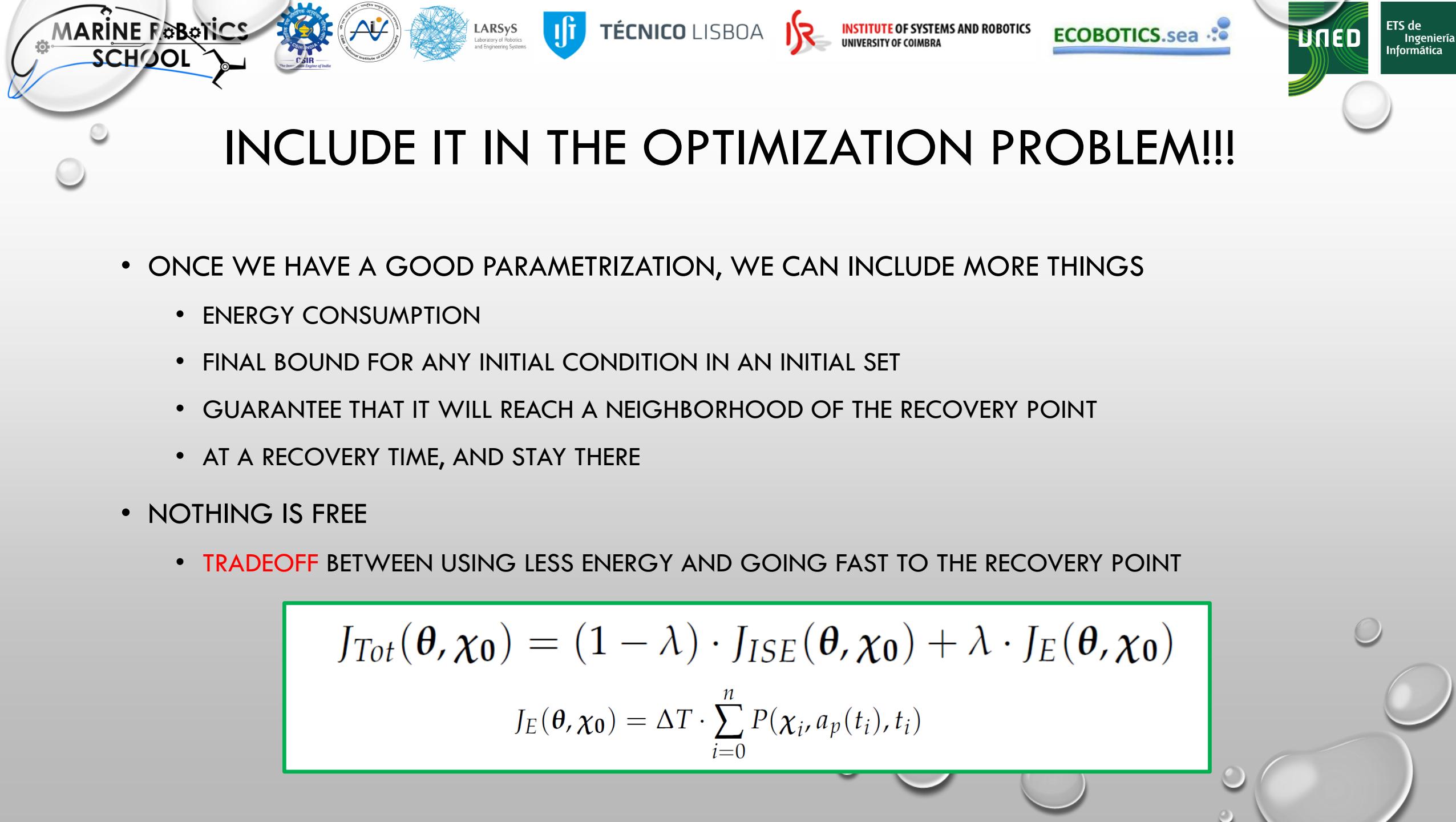
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- ONCE WE HAVE A GOOD PARAMETRIZATION, WE CAN INCLUDE MORE THINGS
 - ENERGY CONSUMPTION
 - FINAL BOUND FOR ANY INITIAL CONDITION IN AN INITIAL SET
 - GUARANTEE THAT IT WILL REACH A NEIGHBORHOOD OF THE RECOVERY POINT
 - AT A RECOVERY TIME, AND STAY THERE

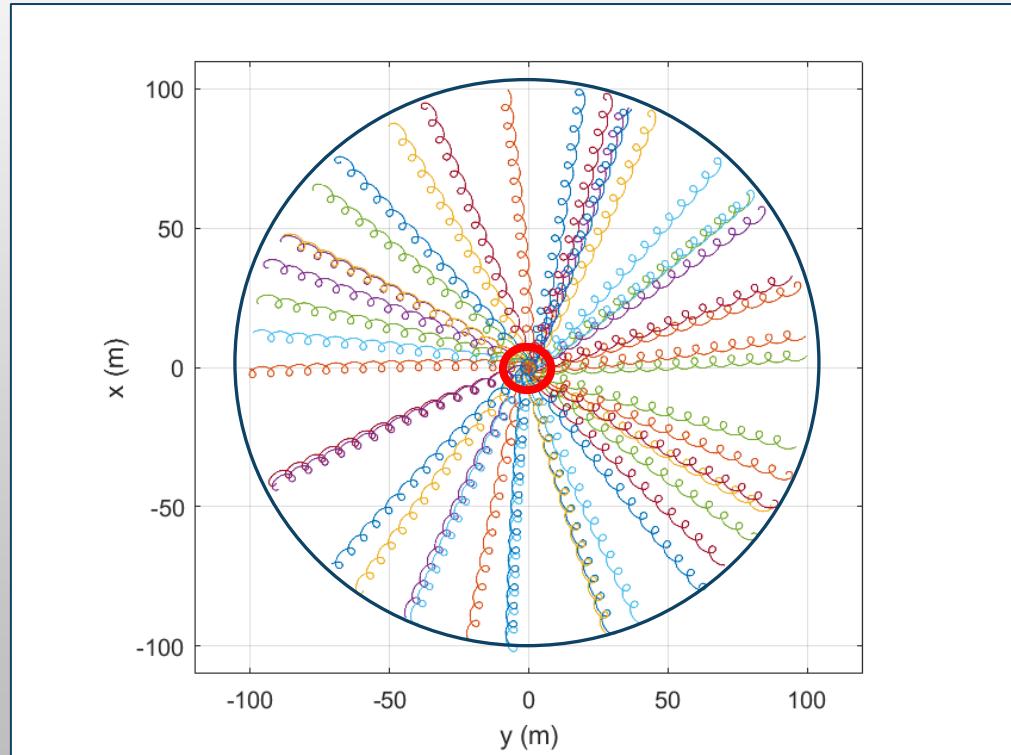


- ONCE WE HAVE A GOOD PARAMETRIZATION, WE CAN INCLUDE MORE THINGS
 - ENERGY CONSUMPTION
 - FINAL BOUND FOR ANY INITIAL CONDITION IN AN INITIAL SET
 - GUARANTEE THAT IT WILL REACH A NEIGHBORHOOD OF THE RECOVERY POINT
 - AT A RECOVERY TIME, AND STAY THERE
- NOTHING IS FREE
 - TRADEOFF BETWEEN USING LESS ENERGY AND GOING FAST TO THE RECOVERY POINT

$$J_{Tot}(\theta, \chi_0) = (1 - \lambda) \cdot J_{ISE}(\theta, \chi_0) + \lambda \cdot J_E(\theta, \chi_0)$$

$$J_E(\theta, \chi_0) = \Delta T \cdot \sum_{i=0}^n P(\chi_i, a_p(t_i), t_i)$$

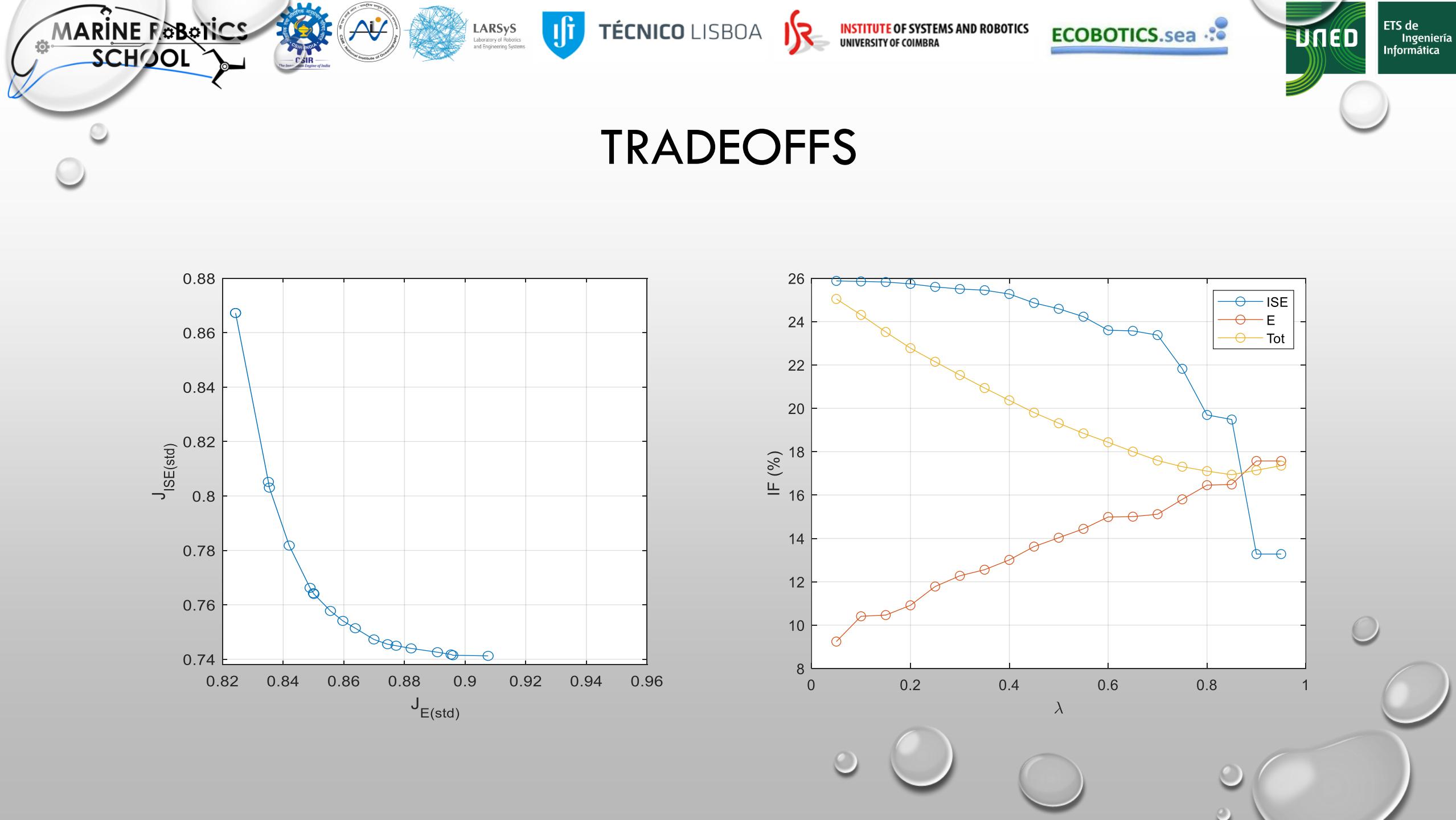
MIN-MAX PROBLEM (SOLVE FOR THE WORST CASE)



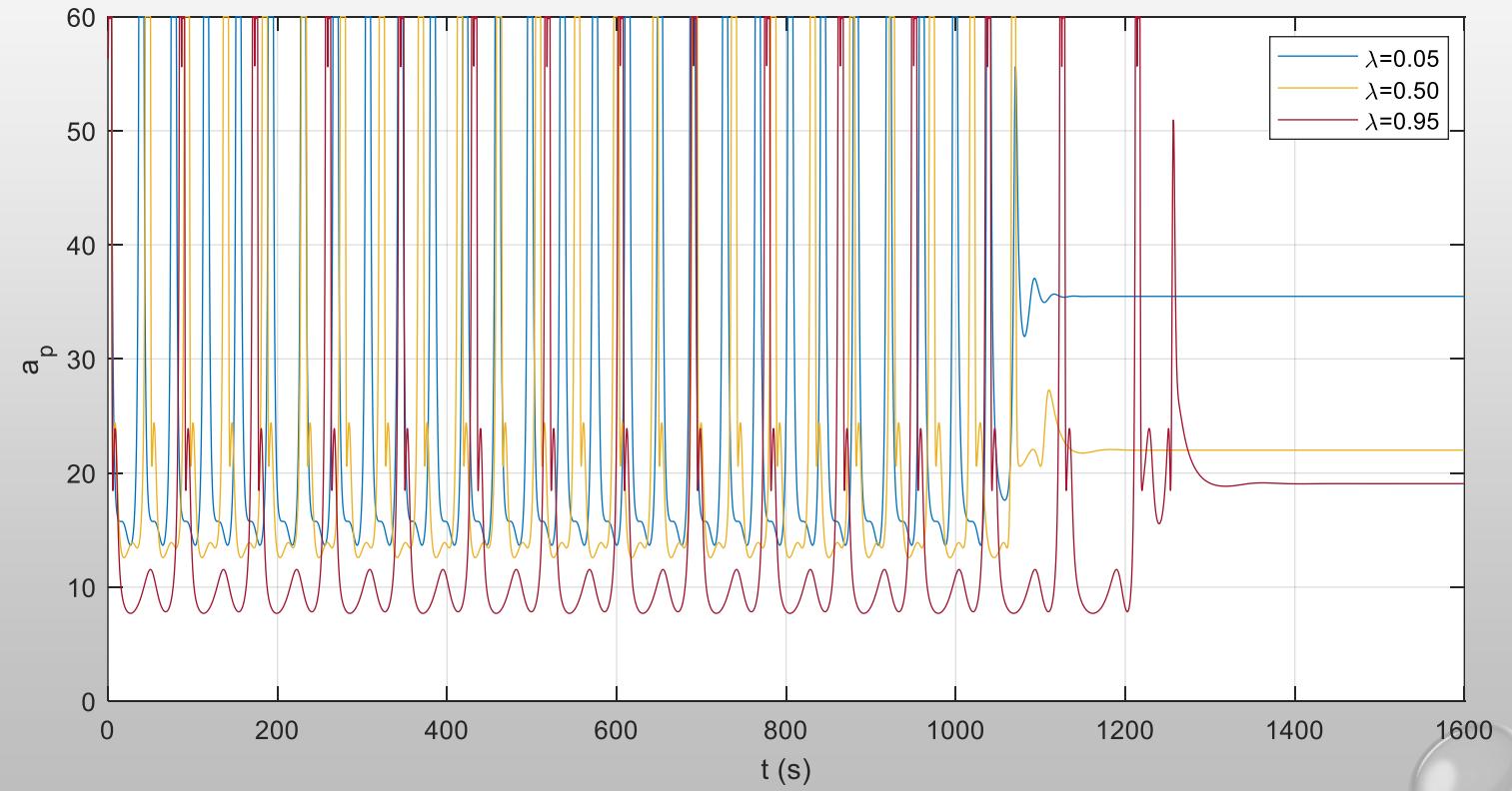
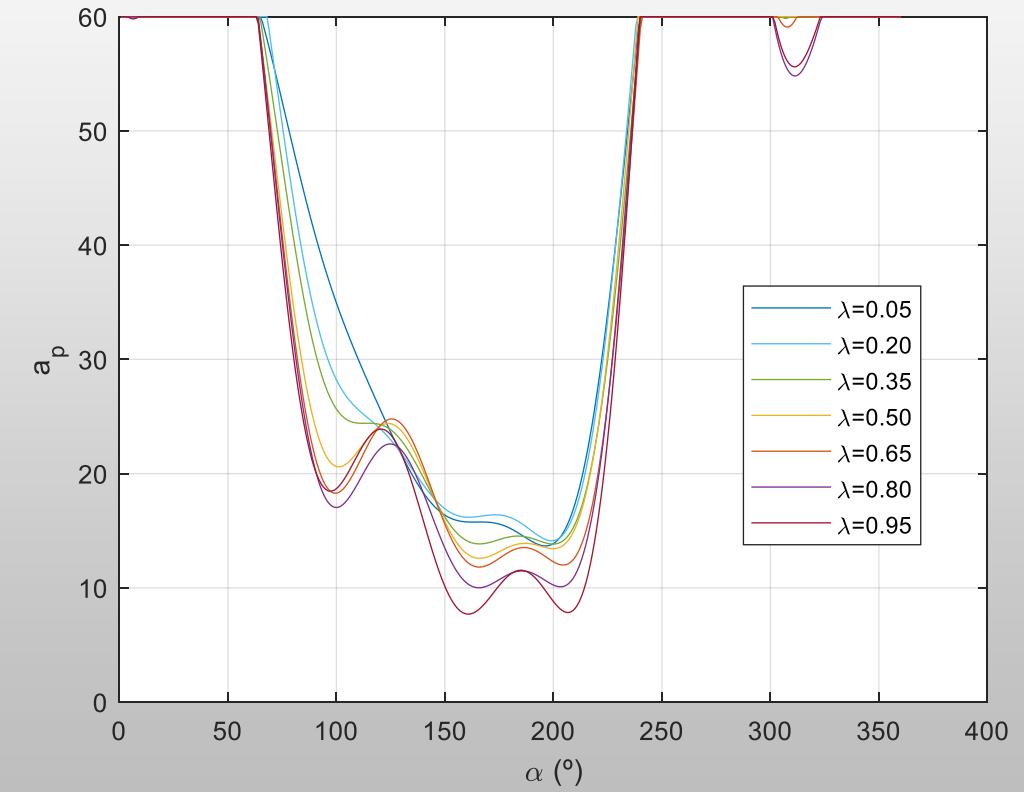
$$\theta^* = \arg \min_{\theta} \max_{x_0 \in \Omega} J_{Tot}(\theta, x_0)$$

- INITIAL SET $\Omega = \left\{ \begin{array}{l} x^2 + y^2 \leq R_0^2 \\ \nu = 0 \end{array} \right\}$
- REACH A NEIGHBORHOOD OF THE RECOVERY POINT AT A RECOVERY TIME, AND STAY THERE

$$(x(t) - x_r)^2 + (y(t) - y_r)^2 < R_f^2 \quad \forall t_f \leq t$$



CONTROL ACTION





THANK YOU VERY MUCH FOR YOUR
ATTENTION!

QUESTIONS?