Quaternions from Rotation Matrix

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Since now we have seen how a rotation matrix is composed by using quaternions.

$$\mathbf{R}(\mathring{q}) = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix} \tag{1}$$

which can be compressed in the equation

$$\mathbf{R}(\mathring{q}) = \left(q_0^2 - \boldsymbol{q}^\intercal \boldsymbol{q}\right) \mathbf{I}_3 + 2\left(\boldsymbol{q}\boldsymbol{q}^\intercal\right) + 2q_0 \left[\boldsymbol{q}\right]_\times \tag{2}$$

Now we are looking for the inverse relation. The fact that we have 4 parameters to represent a quaternion versus 9 parameters in the rotation matrix case, makes the redundancy to appear.

By operating with the diagonal terms of $\mathbf{R}$ and applying the unity norm constraint it can be obtained that

$$\begin{aligned} 4q_0^2 &= 1 + r_{11} + r_{22} + r_{33} \\ 4q_1^2 &= 1 + r_{11} - r_{22} - r_{33} \\ 4q_2^2 &= 1 - r_{11} + r_{22} - r_{33} \\ 4q_3^2 &= 1 - r_{11} - r_{22} + r_{33}. \end{aligned} \tag{3}$$

In a similar way, playing with the off-diagonal terms

$$\begin{aligned} 4q_0q_1 &= r_{32} - r_{23} \\ 4q_0q_2 &= r_{13} - r_{31} \\ 4q_0q_3 &= r_{21} - r_{12} \\ 4q_1q_2 &= r_{21} + r_{12} \\ 4q_1q_3 &= r_{13} + r_{31} \\ 4q_2q_3 &= r_{32} - r_{23}. \end{aligned} \tag{4}$$

Take the highest component of the quaternion $\mathring{q}$ to be q_i. It may imply that

$$q_i^2 - q_j^2 > 0 \quad \text{for} j \neq i \wedge j = 0, 1, 2, 3 \tag{5}$$

Adding the 3 past equations leads to

$$3q_i^2 > \sum_{\substack{j=0 \\ j\neq i}}^{j=3} q_j^2 \tag{6}$$

By the unity norm constraint $1 - q_j^2 = \sum_{\substack{j=0 \\ j\neq i}}^{j=3} q_j^2$, therefore

$$q_i^2 > \frac{1}{4} \tag{7}$$

which is far away from 0. This is useful to decide how to perform the reverse transformation.

In order to damp possible numeric problems that can arise for round off errors in the rotation matrix, the next strategy is proposed

Case1 : In the case that $q_0^2 > q_i^2$ for $i = 1, 2, 3$ which is achieved iff $r_{11} > -r_{22}$ and $r_{11} > -r_{33}$ and $r_{22} > -r_{33}$

$$\mathring{q} = \frac{1}{2}\begin{bmatrix} \sqrt{1 + r_{11} + r_{22} + r_{33}} \\ \frac{r_{32}-r_{23}}{\sqrt{1+r_{11}+r_{22}+r_{33}}} \\ \frac{r_{13}-r_{31}}{\sqrt{1+r_{11}+r_{22}+r_{33}}} \\ \frac{r_{21}-r_{12}}{\sqrt{1+r_{11}+r_{22}+r_{33}}} \end{bmatrix} \tag{8}$$

Case2 : In the case that $q_1^2 > q_i^2$ for $i = 0, 2, 3$ which is achieved iff $r_{11} > r_{22}$ and $r_{11} > r_{33}$ and $r_{22} < -r_{33}$

$$\mathring{q} = \frac{1}{2}\begin{bmatrix} \frac{r_{32}-r_{23}}{\sqrt{1+r_{11}-r_{22}-r_{33}}} \\ \sqrt{1 + r_{11} - r_{22} - r_{33}} \\ \frac{r_{21}+r_{12}}{\sqrt{1+r_{11}-r_{22}-r_{33}}} \\ \frac{r_{13}+r_{31}}{\sqrt{1+r_{11}-r_{22}-r_{33}}} \end{bmatrix} \tag{9}$$

Case3 : In the case that $q_2^2 > q_i^2$ for $i = 0, 1, 3$ which is achieved iff $r_{11} < r_{22}$ and $r_{11} < -r_{33}$ and $r_{22} > r_{33}$

$$\mathring{q} = \frac{1}{2}\begin{bmatrix} \frac{r_{13}-r_{31}}{\sqrt{1-r_{11}+r_{22}+r_{33}}} \\ \frac{r_{21}+r_{12}}{\sqrt{1-r_{11}+r_{22}+r_{33}}} \\ \sqrt{1 - r_{11} + r_{22} + r_{33}} \\ \frac{r_{32}+r_{23}}{\sqrt{1-r_{11}+r_{22}+r_{33}}} \end{bmatrix} \tag{10}$$

Case4 : In the case that $q_3^2 > q_i^2$ for $i = 0, 1, 2$ which is achieved iff $r_{11} < -r_{22}$ and $r_{11} < r_{33}$ and $r_{22} < r_{33}$

$$\mathring{q} = \frac{1}{2}\begin{bmatrix} \frac{r_{21}-r_{12}}{\sqrt{1-r_{11}-r_{22}+r_{33}}} \\ \frac{r_{13}+r_{31}}{\sqrt{1-r_{11}-r_{22}+r_{33}}} \\ \frac{r_{32}+r_{23}}{\sqrt{1-r_{11}-r_{22}+r_{33}}} \\ \sqrt{1 - r_{11} - r_{22} + r_{33}} \end{bmatrix} \tag{11}$$