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CMPS 351

Assignment 7

In [1]:
```
import numpy as np
from numpy import linalg as la
from scipy.optimize import linprog as lp
import matplotlib.pyplot as plt
import scipy.fftpack
import random
import cvxpy as cvx
```

LP Solution

Lagrangian

$$L(x, \lambda) = 5x_1 + 8x_2 + \lambda_1(-x_1 + 2) + \lambda_2(-x_1 - 2x_2 + 5) + \lambda_3(-2x_1 - 5x_2 + 8)$$

Primal Problem

$$\min_x 5x_1 + 8x_2$$

$$\begin{aligned} \textit{subject to} \qquad -x_1 + x_3 + 2 = 0 \\ -x_1 - 2x_2 + x_4 + 5 = 0 \\ -2x_1 - 5x_2 + x_5 + 8 = 0 \\ x_i \geq 0 \ \forall i: 1 \to 5 \end{aligned}$$

Primal Problem Matrix Notation

$$\min_x [5 \quad 8 \quad 0 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\textit{subject to} \quad \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 1 & 0 \\ -2 & -5 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} -2 \\ -5 \\ -8 \end{bmatrix}$$
$$x \geq 0$$

Dual Problem

$$\max_x [-2 \quad -5 \quad -8]\nu$$

$$subject\ to \quad \begin{bmatrix} -1 & -1 & -2 \\ 0 & -2 & -5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \nu + \lambda = \begin{bmatrix} 5 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\lambda \geq 0$$

KKT Conditions

In [120]:
```
def kkt(x, l):
    kkt = []
    kkt.append(-x[0] <= -2)
    kkt.append(-x[0] - 2*x[1] <= -5)
    kkt.append(-2*x[0] -5*x[1] <= -8)
    kkt.append((l >= 0).all())
    kkt.append(l[0]*(-x[0] + 2) == 0)
    kkt.append(l[1]*(-x[0] - 2*x[1] + 5) == 0)
    kkt.append(l[2]*(-2*x[0] -5*x[1] + 8) == 0)
    return kkt
```

In [124]:
```
x = np.array([2, 1.5])
l = np.array([1, 4, 0])
kkt(x, l)
```

Out[124]:
```
[True, True, True, True, True, True, True]
```

In [129]:
```
p = np.array([5, 8])
b = np.array([-2, -5, -8])

duality_gap = p@x + b@l
print(duality_gap)
```

```
0.0
```

The duality gap is equal to zero, both problems have the same optimal value

Perturbation

Considering a perturbation of the right hand side of the first constraint to -1.9. We are given l[0] as 1. Therefor we expect a change of +0.1 in the constraint to cause a change of -0.1 in the optimal value which will be 21.9

Considering a perturbation of the right hand side of the third constraint to -8.1. We are given l[1] as 0. Therefor we expect a change of +0.1 in the constraint to cause a no change in the optimal value which will still be 22.0

Contact Problem in 1D

In [7]:
```
k = np.array([1, 10, 2])
l = 1
w = 0.2
```

In [8]:
```
def f_obj(x):
    return 0.5*k[0]*(x[0]**2) + 0.5*k[1]*((x[1] - x[0])**2) + 0.5*k[2]*((l - x[1]
```

In [9]:
```
x = cvx.Variable(2)
```

In [10]:
```
def constraints_(x):
    c = []
    c.append(x[0] >= 0.5*w)
    c.append(x[1] - x[0] >= w)
    c.append(l - x[1] >= 0.5*w)
    return c
```

In [11]:
```
constraints = constraints_(x)
obj = cvx.Minimize(f_obj(x))
prob = cvx.Problem(obj, constraints)
prob.solve()
```

Out[11]:
```
0.41333333333333331
```

In [12]:
```
x.value
```

Out[12]:
```
array([0.53333333, 0.73333333])
```

In [13]:
```
for c in constraints:
    print(c.dual_value)
```

```
0.0
1.4666666666666655
0.0
```

These multipliers represent the sesitivity of the objective function to a change in the width of the blocks

In [14]:
```
c = np.array([2, 4])
```

In [15]:
```
def f_obj_deform(x, w_):
    return 0.5*k[0]*(x[0]**2) + 0.5*k[1]*((x[1] - x[0])**2) + 0.5*k[2]*((l - x[1]
           + (1/2*c[0])*((w_[0] + w_[1] - w)**2) \
           + (1/2*c[1])*((w_[2] + w_[3] - w)**2)
```

In [16]:
```
x = cvx.Variable(2)
w_ = cvx.Variable(4)
```

In [17]:
```
def constraints_(x, w_):
    c = []
    c.append(x[0] >= w_[0])
    c.append(x[1] - x[0] >= w_[1] + w_[2])
    c.append(l - x[1] >= w_[3])
    c.append(w_[0] <= 0.1)
    c.append(w_[1] <= 0.1)
    c.append(w_[2] <= 0.1)
    c.append(w_[3] <= 0.1)
    c.append(w_[0] >= 0)
    c.append(w_[1] >= 0)
    c.append(w_[2] >= 0)
    c.append(w_[3] >= 0)
    return c
```

In [18]:
```
constraints = constraints_(x, w_)
obj = cvx.Minimize(f_obj_deform(x, w_))
prob = cvx.Problem(obj, constraints)
prob.solve()
```

Out[18]:
```
0.32370370370370366
```

In [19]:
```
x.value
```

Out[19]:
```
array([0.61481481, 0.69259259])
```

In [20]:
```
w_.value
```

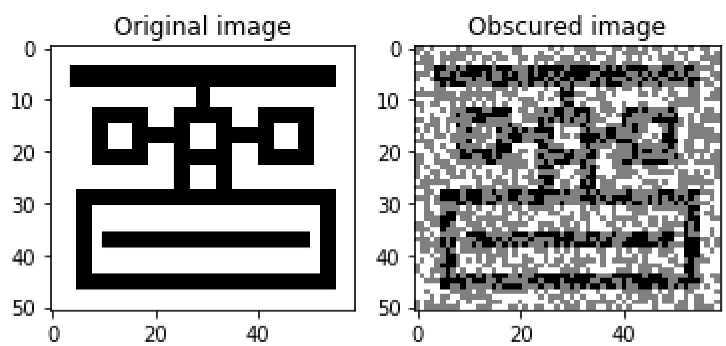
Out[20]:
```
array([0.1       , 0.01851852, 0.05925926, 0.1       ])
```

In [21]:
```
for c in constraints:
    print(c.dual_value)
```

```
0.0
0.1629629629629629
0.0
0.1629629629629629
0.0
0.0
0.1629629629629629
0.0
0.0
0.0
0.0
```

Image Reconstruction Revisited

In [2]:
```
# Read a sample image
U0 = plt.imread('bwicon.png')
m, n = U0.shape

# Create 50% mask of known pixels and use it to obscure the original
np.random.seed(7592)                    # seed the randonm number generator (for rep
unknown = np.random.rand(m,n) < 0.5
U1 = U0*(1-unknown) + 0.5 *unknown



# Display images
plt.figure(1)
plt.subplot(1, 2, 1)
plt.imshow(U0, cmap='gray')
plt.title('Original image')
plt.subplot(1, 2, 2)
plt.imshow(U1, cmap='gray')
plt.title('Obscured image')
plt.show()
```



In [3]:
```
ux, uy = U1.shape
```

In [4]:
```
def L1_norm(u):
    s = 0
    for i in range(1, ux):
        for j in range(1, uy):
            s += abs(u[i][j] - u[i-1][j]) + abs((u[i][j] - u[i][j-1]))
    return s
```

In [5]:
```
B = np.zeros([ux*uy, ux*uy])
for i in range(uy, ux*uy):
    if i%uy != 0:
        B[i][i] = 1
        B[i][i-1] = -1
```

In [6]:
```
C = np.zeros([ux*uy, ux*uy])
for i in range(uy, ux*uy):
    if i%uy != 0:
        C[i][i] = 1
        C[i][i-uy] = -1
```

In [7]:
```
x = U1.flatten()
```

In [8]:
```
B_ub = np.zeros(4*ux*uy)
A_ub = np.zeros([4*ux*uy, 3*ux*uy])

A_ub[0:ux*uy, 0:ux*uy] = B
A_ub[ux*uy:2*ux*uy, 0:ux*uy] = -B
A_ub[0:ux*uy, ux*uy:2*ux*uy] = -np.identity(ux*uy)
A_ub[ux*uy:2*ux*uy, ux*uy:2*ux*uy] = -np.identity(ux*uy)

A_ub[2*ux*uy:3*ux*uy, 0:ux*uy] = C
A_ub[3*ux*uy:4*ux*uy, 0:ux*uy] = -C
A_ub[2*ux*uy:3*ux*uy, 2*ux*uy:3*ux*uy] = -np.identity(ux*uy)
A_ub[3*ux*uy:4*ux*uy, 2*ux*uy:3*ux*uy] = -np.identity(ux*uy)
```

In [9]:
```
A_eq = np.zeros([ux*uy,3*ux*uy])
B_eq = np.zeros([ux*uy])
unknown2 = unknown.flatten()
x = U1.flatten()
for i in range(ux*uy):
    if not unknown2[i]:
        A_eq[i][i] = 1
        B_eq[i] = x[i]
```

In [10]:
```
obj = np.zeros(3*ux*uy)
obj[ux*uy:] = np.ones(2*ux*uy)
```

In [12]:
```
res = lp(obj, A_eq=A_eq, b_eq=B_eq, A_ub=A_ub, b_ub=B_ub, options={"disp": True},
```

```
Primal Feasibility  Dual Feasibility    Duality Gap         Step             Pa
th Parameter      Objective
1.0                 1.0                 1.0                 -                1.
0                 6018.0
0.6375661631591     0.6375661631591     0.6375661631591     0.3758205487603  0.
6375661631591     3425.658411165
0.2727436078755     0.2727436078755     0.2727436078755     0.5836467223876  0.
2727436078755     1629.29439874
0.1054429048606     0.1054429048606     0.1054429048606     0.6312325037131  0.
1054429048606     994.0502499014
0.01791546604316    0.01791546604317    0.01791546604317    0.8391015706486  0.
01791546604317    710.1659140639
3.634504843227e-05  3.634504843231e-05  3.634504843226e-05  0.9997322587388  3.
634504843236e-05  656.117741571
1.818497659929e-09  1.818497680763e-09  1.818497711278e-09  0.9999499658404  1.
818497699082e-09  656.0000058909
Optimization terminated successfully.
         Current function value: 656.000006
         Iterations: 6
```

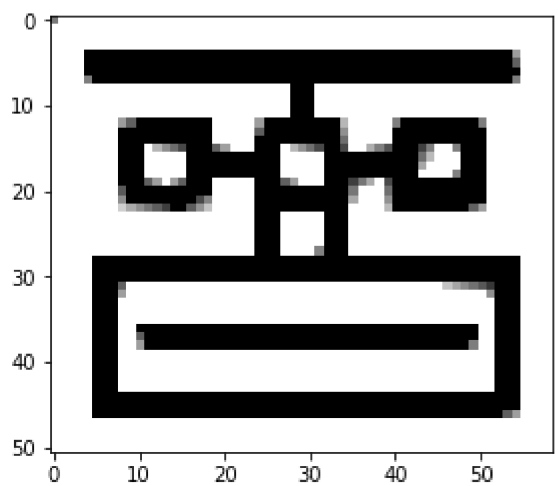
In [13]:
```
print(res)
```

```
     con: array([0., 0., 0., ..., 0., 0., 0.])
     fun: 656.0000058908898
 message: 'Optimization terminated successfully.'
     nit: 6
   slack: array([5.74883491e-10, 5.74883491e-10, 5.74883491e-10, ...,
       1.21679381e-09, 1.19136634e-09, 1.11164683e-09])
  status: 0
 success: True
       x: array([4.74686056e-01, 1.00000000e+00, 1.00000000e+00, ...,
       1.14668589e-09, 1.14231159e-09, 1.13962290e-09])
```

In [14]:
```
x = res.x[:ux*uy]
x = x.reshape([ux,uy])
```

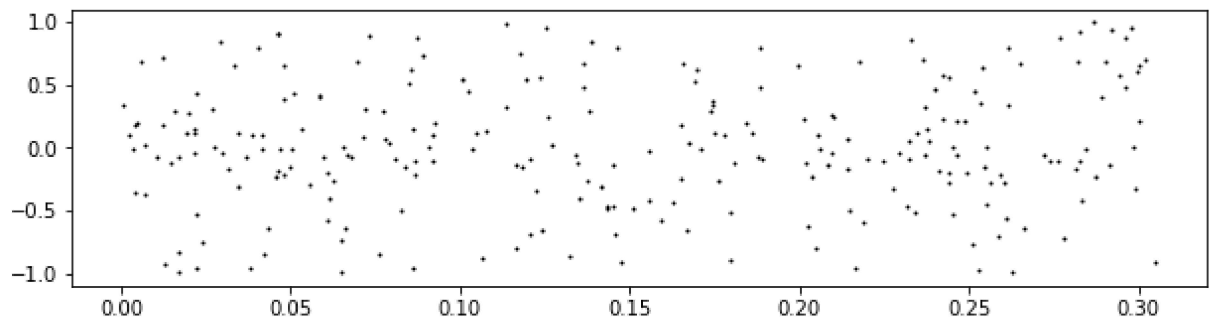
In [15]:
```
plt.imshow(x, cmap='gray')
```

Out[15]:
```
<matplotlib.image.AxesImage at 0x24a397fe358>
```



Compressed Sensing

In [25]:
```
n = 2500
m = 250
fs = 8192                              # sampling frequency
t = np.arange(n) / fs
y = ( np.sin(2*np.pi*521*t) + np.sin(2*np.pi*1233*t) ) /2.0

D = scipy.fftpack.dct(np.eye(n), norm='ortho').transpose()
k = sorted(random.sample(range(n), m)) # Pick m random points for the reconstruct
A = D[k, :]
b = y[k]

plt.figure(figsize=(10.24, 2.56))      # change aspect ratio of plot
plt.plot(t[k], b, 'ko', markersize=1)  # plot data to use to reconstruct
plt.show()
```



L2 Norm

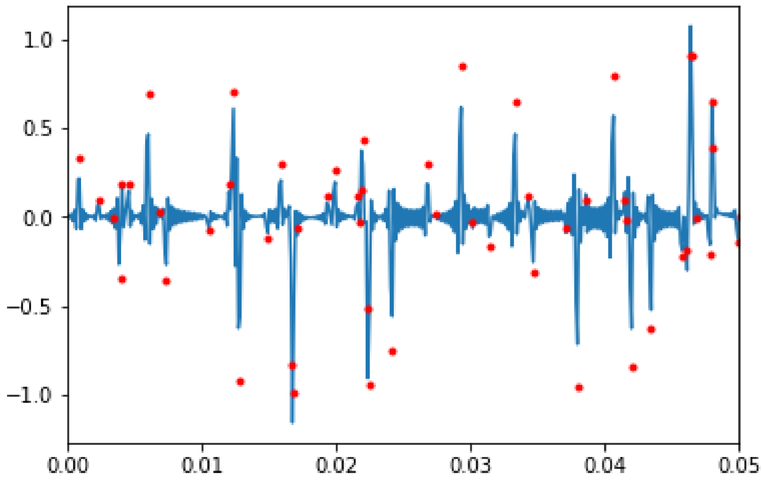
In [38]:
```
vx = cvx.Variable(n)
objective = cvx.Minimize(cvx.norm(vx, 2))
constraints = [A*vx == b]
prob = cvx.Problem(objective, constraints)
result = prob.solve(verbose=True)
```

```
C:\Users\Danny\Anaconda3\lib\site-packages\cvxpy\problems\problem.py:781: Runti
meWarning: overflow encountered in long_scalars
  if self.max_big_small_squared < big*small**2:


ECOS 2.0.4 - (C) embotech GmbH, Zurich Switzerland, 2012-15. Web: www.embotech.
com/ECOS

It     pcost       dcost      gap   pres   dres    k/t    mu     step   sigma
IR    |   BT
 0  +0.000e+00  -0.000e+00  +1e+03  2e-01  2e-06  1e+00  7e+02    ---    ---
1  1  - |  -  -
 1  +7.772e-03  +8.045e-02  +2e+01  4e-03  3e-08  9e-02  1e+01  0.9849  1e-04
3  2  2 |  0  0
 2  +4.133e+00  +4.299e+00  +4e+00  6e-04  4e-09  2e-01  2e+00  0.8491  2e-02
4  5  5 |  0  0
 3  +7.658e+00  +7.679e+00  +1e-01  2e-05  6e-11  2e-02  6e-02  0.9890  9e-03
4  6  6 |  0  0
 4  +7.767e+00  +7.767e+00  +1e-03  3e-07  9e-13  2e-04  7e-04  0.9890  1e-04
4  4  4 |  0  0
 5  +7.768e+00  +7.768e+00  +1e-05  5e-09  2e-14  3e-06  8e-06  0.9890  1e-04
4  3  3 |  0  0
 6  +7.768e+00  +7.768e+00  +1e-07  2e-10  2e-16  3e-08  9e-08  0.9890  1e-04
4  3  3 |  0  0
 7  +7.768e+00  +7.768e+00  +2e-09  3e-11  1e-17  3e-10  9e-10  0.9890  1e-04
4  2  2 |  0  0

OPTIMAL (within feastol=3.1e-11, reltol=2.0e-10, abstol=1.6e-09).
Runtime: 4.468852 seconds.
```

In [39]:
```
x = np.array(vx.value)
x = np.squeeze(x)
sig = scipy.fftpack.idct(x, norm='ortho', axis=0)
```

In [40]:
```
plt.plot(t, sig)
plt.plot(t[k], b, 'ko', markersize=3, color='red')  # plot data to use to reconst
plt.xlim((0,0.05))
```

Out[40]: (0, 0.05)



L1 Norm

In [41]:
```
vx = cvx.Variable(n)
objective = cvx.Minimize(cvx.norm(vx, 1))
constraints = [A*vx == b]
prob = cvx.Problem(objective, constraints)
result = prob.solve(verbose=True)
```

```
C:\Users\Danny\Anaconda3\lib\site-packages\cvxpy\problems\problem.py:781: Runti
meWarning: overflow encountered in long_scalars
  if self.max_big_small_squared < big*small**2:
C:\Users\Danny\Anaconda3\lib\site-packages\cvxpy\problems\problem.py:782: Runti
meWarning: overflow encountered in long_scalars
  self.max_big_small_squared = big*small**2

-----------------------------------------------------------------
           OSQP v0.5.0  -  Operator Splitting QP Solver
              (c) Bartolomeo Stellato,  Goran Banjac
        University of Oxford  -  Stanford University 2018
-----------------------------------------------------------------
problem:  variables n = 5000, constraints m = 5250
          nnz(P) + nnz(A) = 634948
settings: linear system solver = qdldl,
          eps_abs = 1.0e-04, eps_rel = 1.0e-04,
          eps_prim_inf = 1.0e-04, eps_dual_inf = 1.0e-04,
          rho = 1.00e-01 (adaptive),
          sigma = 1.00e-06, alpha = 1.60, max_iter = 10000
          check_termination: on (interval 25),
          scaling: on, scaled_termination: off
          warm start: on, polish: on

objective    pri res    dua res    rho        time
   1  -2.0000e+04   8.00e+00   1.39e+05   1.00e-01   4.61e-01s
 200   1.0758e+02   1.17e-02   4.31e-03   1.00e-01   1.28e+00s
 400   1.0819e+02   4.08e-03   1.10e-03   1.00e-01   1.88e+00s
 600   1.0834e+02   2.19e-03   5.39e-04   1.00e-01   2.44e+00s
 800   1.0838e+02   1.14e-03   4.00e-04   1.00e-01   3.00e+00s
1000   1.0839e+02   7.74e-04   2.10e-04   1.00e-01   3.54e+00s
1025   1.0840e+02   8.46e-04   1.88e-04   1.00e-01   3.60e+00s

status:               solved
solution polish:      unsuccessful
number of iterations: 1025
optimal objective:    108.3985
run time:             3.94e+00s
optimal rho estimate: 3.58e-02
```

In [42]:
```
x = np.array(vx.value)
x = np.squeeze(x)
sig = scipy.fftpack.idct(x, norm='ortho', axis=0)
```

In [43]:
```
plt.plot(t, sig)
plt.plot(t[k], b, 'ko', markersize=3, color='red')  # plot data to use to reconst
plt.xlim((0,0.05))
```

Out[43]: (0, 0.05)