Faculty of Arts and Sciences
Department of Computer Science
CMPS 351 – Numerical Optimization
Assignment 7 – Due Mar 11, 2019

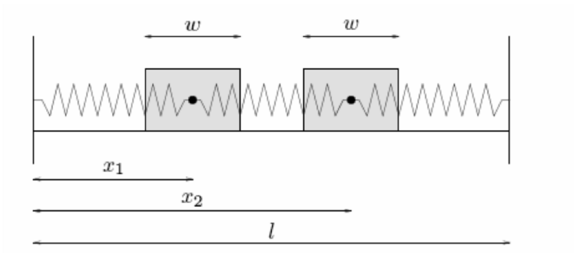
1. LP solution. Consider the following linear program.

$$\begin{aligned} \text{minimize} \quad & 5x_1 + 8x_2 \\ \text{subject to} \quad & -x_1 \le -2 \\ & -x_1 - 2x_2 \le -5 \\ & -2x_1 - 5x_2 \le -8 \end{aligned}$$

- Write the Lagrangian of this problem, and the Lagrange dual LP.
- Verify that the primal/dual point $x = [2.0\ 1.5]^t$, $\lambda = [1\ 4\ 0]^t$ is an optimal solution by verifying the KKT conditions.
- Compute the duality gap at the primal/dual point above. Comment.
- Consider a perturbation of the right hand side of the first constraint to -1.9. Estimate the new optimal value of the objective.
- Consider a perturbation of the right hand side of the third constraint to -8.1. Estimate the new optimal value of the objective.

2. Contact problem in 1D. Contact problems that arise in mechanics can be formulated as optimization problems. In fact, the understanding of mechanics problems was one of Lagrange's original motivations for the study of optimization. As an example consider the following set of rigid blocks connected by deformable springs.

- Using the model described on p. 247 of reference 2, find the equilibrium position of the blocks for the following values of the spring stiffnesses $k_1 = 1; k_2 = 10; k_3 = 2$ (units are force per length). Use $l = 1; w = 0.2$.
- What is the significance of the multipliers in this problem?
- Consider an extension to this problem where the blocks are deformable, and their deformation is defined by the elastic energy $1/2c_i(\Delta w_i)^2$, where Δw_i is the change in width of the i-th block. The problem becomes one of minimizing the sum of potential energies of the springs and the blocks. Formulate and solve the problem for the following values $c_1 = 2; c_2 = 4$.



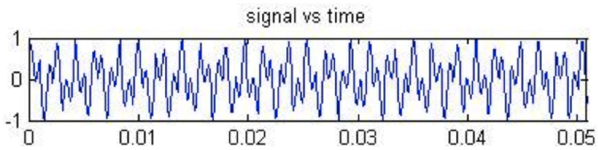
3. Image reconstruction revisited. Redo the reconstruction problem of last homework using a *total variation* roughness measure, which is an L_1 norm of the form:

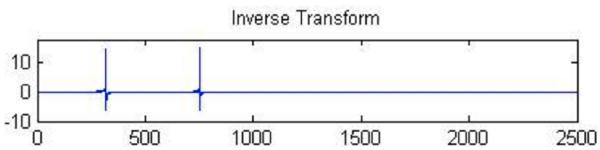
$$\sum_{i=2}^{M} \sum_{j=2}^{N} |U_{ij} - U_{i-1,j}| + |U_{ij} - U_{i,j-1}|$$

This objective is piecewise linear and therefore has no useful pointwise Hessian information and does not even have a unique gradient at some points. Formulate this L_1 problem as a linear optimization problem and find its solution.

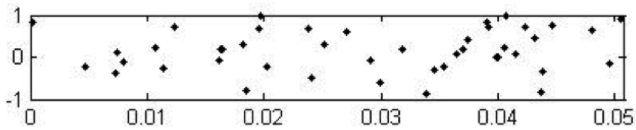
4. Compressed sensing. Consider the following expression $\frac{1}{2}\left(\sin(2\pi\, 697t)+\sin(2\pi\, 1209t)\right)$, representing a signal as a function of time. If you have background in signal processing, you may know that we can construct the frequency content of the discretized signal, and conversely generate the signal from its frequency contents, using the discrete Fourier transform and its inverse. We can also use other Fourier-related transforms such as the discrete cosine transform. Here's how one might do such reconstructions:

```
# signal
n = 2500
fs = 5e4  # sampling frequency
t = np.arange(n) / fs
y = (np.sin(2*np.pi*697*t) + np.sin(2*np.pi*1209*t)) /2.0
# Transform matrix
D = scipy.fftpack.dct(np.eye(n), norm='ortho').transpose()
# Frequency content from signal (inverse transform), idct(y)
x = np.linalg.solve(D, y)
# Reconstructed signal from frequency content, dct(x)
y = D @ x;
```





The above computation assumes we have complete data, i.e. the whole right hand side vector is known. The interesting question that is often asked is: can we reconstruct the complete signal (or equivalently its frequency content) if we have a small sample of the data. Lets say we only know 10% of the entries in the data as shown in the figure below. Our system of equations is no longer $D_{n\times n}x_{n\times 1}=y_{n\times 1}$ but a highly under-determined set of equations $A_{m\times n}x=b_{m\times 1}$, $m \ll n$. Can we find x?



A standard formulation of this problem asks for the fewest number of individual frequencies that best reconstruct the signal (this is justified by the fact that real signals are not white-noise). Unfortunately, asking for the sparsest vector (smallest number of non-zero entries in) that fits the data is known to be a very difficult problem computationally (NP-complete). In an insightful work a few years ago, T. Tao and colleagues showed that one could minimize the L_1 norm of the vector x instead of the number of non-zeros in it and generally obtain the same solution, or a high-quality approximation of it. In this problem, we explore this solution to the problem. Sample data is posted on moodle.

- Formulate the problem as a least squares problem to minimize the L_2 norm of x

$$\begin{aligned} &\text{minimize} \quad \|x\|_2^2=\sum_{i=1}^{n} x_i^2 \\ &\text{subject to} \quad Ax=b \end{aligned}$$

  You may solve this quadratic optimization problem by writing the KKT conditions and solving them. Comment on the cost of obtaining the solution and on its quality.

- Formulate the problem as the problem of minimizing the L_1 norm of x

$$\begin{aligned} &\text{minimize} \quad \|x\|_1=\sum_{i=1}^{n} |x_i| \\ &\text{subject to} \quad Ax=b \end{aligned}$$

  Express this problem as an LP and solve it. Comment on the cost of obtaining the solution and on its quality. Comment on the quality of the solution you obtain.