Faculty of Arts and Sciences
Department of Computer Science
CMPS 351 – Numerical Optimization
Assignment 10 – Due Apr 1, 2019

Background reading / reference material:

- Inequality-constrained problems: chapter 11 from reference 2.

Reminder:

- Exam will take place on Wednesday April 10 at 6:30pm in Bliss 203.

1. Central path. The goal of this exercise is to show that the barrier method generates a sequence of iterates along the central path. Consider the convex problem:

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & Ax = b \\ & f_i(x) \leq 0, \quad i = 1, \cdots, m \end{aligned} \tag{1}$$

- Write the KKT conditions for its log barrier problem

$$\begin{aligned} \text{minimize} \quad & f_0(x) - \tau \sum_{i=1}^{m} \log(-f_i(x)) \\ \text{subject to} \quad & Ax = b \end{aligned}$$

- Show that the KKT conditions for a point on the central path of (1) can be written in the form above.

2. Interior Point Method for LP. We saw in class that an inequality constrained LP

$$\begin{aligned} \text{minimize} \quad & p^t x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{2}$$

may be solved by following a "central path" in the interior of the feasible region. The central path is defined as the set of points where the complementarity conditions are equally violated (all $\lambda_i x_i = \tau$).

- Checking whether or not the problem has a feasible solution may be done by solving the equality-constrained problem of the previous assignment. Explain how. When that solution exists, it may be used as the initial point for solving (2).
- Every step along the central path may be computed by solving an equality constrained problem. Write the form of the equality constrained problem and show that its KKT conditions are the same conditions defining the central path.
- The path-following method may then be viewed as a sequence of equality-constrained problems, each with a decreasing value of the parameter τ ($\tau^{l+1} = \tau^l/\mu$). Implement such a strategy:

```
def interiorLP(p, A, b, x0, tol):
```

  to solve an LP to a tolerance `tol` on the duality gap. The function should return the values of the primal and dual variables at optimality, and a history of the iterates.
- Test your implementation on the given data (with $A_{100\times150}$). Starting from the same initial feasible point, how many total (inner) Newton iterations are needed to decrease the duality gap to 10^{-6} with $\mu = 2$ and with $\mu = 10$. Comment.

3. Interior point barrier method. Implement a basic log-barrier interior point method for solving inequality-constrained problems as described in class and in section 11.3 for solving the convex quadratic program:

$$\begin{array}{ll} \text{minimize} & 1/2x^tHx + g^tx \\ \text{subject to} & Ax - b \leq 0 \end{array}$$

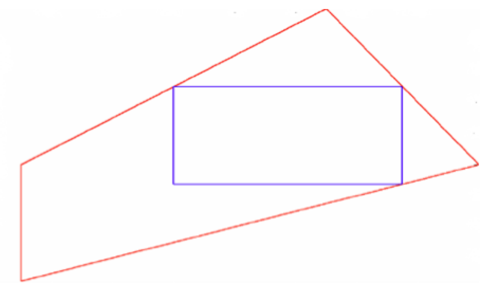
- Write a routine to generate an initial feasible point, or determine that the problem in infeasible.
- Generate random data (symmetric positive definite H) and use them to test the performance of the method by plotting duality gap vs number of newton iterations (use $n = 200$, $m = 100$).
- Is there a variation on the basic method that does not require nested iteration?

4. Inverse barrier. The barrier method we described in class solves inequality-constrained problems using a sequence of minimization problems of the form $\frac{1}{\tau} f_0(x) + \phi(x)$ subject to equality-only constraints, where τ parameterizes the central path. The log-barrier function we used was defined as $\phi(x) = -\sum_{i=1}^m \log(-f_i(x))$.

Other choices for barriers are possible such the inverse barrier $-\sum_{i=1}^m -1/f_i(x)$. In this problem we consider the use of this inverse barrier on a problem with no equality constraints.

- Which function is minimized along the central path $x^*(t)$ corresponding to the inverse barrier.
- Show how to construct a dual feasible λ from $x^*(t)$. Find the associated duality gap.
- Write (in pseudo-code only) an algorithm for solving the problem with the inverse barrier.
- Write the gradient and Hessian of the inverse barrier and use them to describe the computations involved at every inner iteration of the method.

5. Largest Enclosed Rectangle. Consider the problem of finding the largest-volume rectangle enclosed in a polygon (problem 11.22 on page 629 of Ref 2)



- Formulate an inequality constrained optimization problem in terms of the coordinates of the lower-left and upper right corners (or something equivalent). Use an appropriate routine or your code above to solve the problem for the following polygon $Ax \leq b$, where:

  $$A = [[0, -1], [2, -4], [2, 1], [-4, 4], [-4, 0]] \text{ and } b = 1.$$

  Hint. The simplest, but non-scalable, formulation involves checking the four corners of the rectangle.
- As formulated above, the problem involves an exponential number of constraints in n (the number of the vertices of the R^n rectangle grows as 2^n). Formulate the problem in a way that avoids this combinatorial explosion.