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CMPS 351

Assignment 5

In [1]:
```
import numpy as np
from numpy import linalg as la
import matplotlib.pyplot as plt
from scipy.optimize import linprog as lp
```

Exercise in R^3

Lagrangian

In [2]:
```
def lagrangian(x, l):
    f = x[0]*x[1] + x[1]*x[2]
    h1 = l[0]*(x[0]**2 + x[1]**2 - 2)
    h2 = l[1]*(x[0]**2 + x[2]**2 - 2)
    return f + h1 + h2
```

Objective Function

In [3]:
```
def lagrangian_grad(x, l):
    g = np.zeros(5)
    g[0] = x[1] + 2*l[0]*x[0] + 2*l[1]*x[0]
    g[1] = x[0] + x[2] + 2*l[0]*x[1]
    g[2] = x[1] + 2*l[1]*x[2]
    g[3] = x[0]**2 + x[1]**2 -2
    g[4] = x[0]**2 + x[2]**2 -2
    return g
```

Gradient

In [4]:
```
def lagrangian_hess(x, l):
    h = np.zeros([5, 5])
    h[0] = np.array([2*l[0] + 2*l[1], 1, 0, 2*x[0], 2*x[0]])
    h[1] = np.array([1, 2*l[0], 1, 2*x[1], 0])
    h[2] = np.array([0, 1, 2*l[1], 0, 2*x[2]])
    h[3] = np.array([2*x[0], 2*x[1], 0, 0, 0])
    h[4] = np.array([2*x[0], 0, 2*x[2], 0, 0])
    return h
```

Globally Convergent Newton

In [5]:
```
def backtrack_linesearch(f, gk, pk, xk, alpha = 0.01, beta = 0.6):
    t = 1
    while ( la.norm(f((xk + t*pk)[:-2], (xk + t*pk)[3:]))/2 > la.norm(f(xk[:-2], 
        t *= beta
    return t
```

In [6]:
```
def newton_method(x0, tol = 1e-6):
    x = x0
    history = np.array([x0])
    while (la.norm(lagrangian_grad(x[:-2], x[3:])) >= tol):
        p = la.solve(lagrangian_hess(x[:-2], x[3:]), -lagrangian_grad(x[:-2], x[3
        t = backtrack_linesearch(lagrangian_grad, lagrangian_grad(x[:-2], x[3:]),
        x += t * p
        history = np.append(history, [x])
        #print(la.norm(lagrangian_grad(x[:-2], x[3:])))
    return x, history
```

In [7]:
```
x0 = np.array([0.1, 0.0000001, 0.0000001, 1, 1])
xstar, history = newton_method(x0)
print(xstar)
```

```
[ 1.30656297 -0.5411961  -0.5411961   0.70710679 -0.50000001]
```

Significance of the Lagrange Multipliers

The Taylor series of the constraint gives us

$h_i(x^k + \Delta x) = h(x^k) + \nabla h_i(x^k)^t \Delta x$

The modified constraint h[i] is shifted by e

$h_i(x^k + \Delta x) - h(x^k) = \nabla h_i(x^k)^t \Delta x$

$h_i(x^k + \Delta x) - h(x^k) = \epsilon$

$\nabla h_j(x) \Delta x = 0 \, for \, all \, i \neq j$

For a feasible direction, we need to move on the surface perpendicular to the gradients of the constraints

$h_i(x^k + \Delta x) = h(x^k) + \nabla h_i(x^k)^t \Delta x$

Taylor series of the objective function

$f(x^k + \Delta x) = f(x^k) + \nabla f(x^k)^t \Delta x$

But the graient of the bjective function can be written as

$f(x^k + \Delta x) = f(x^k) - (A(x^k)^t \lambda)^t \Delta x$

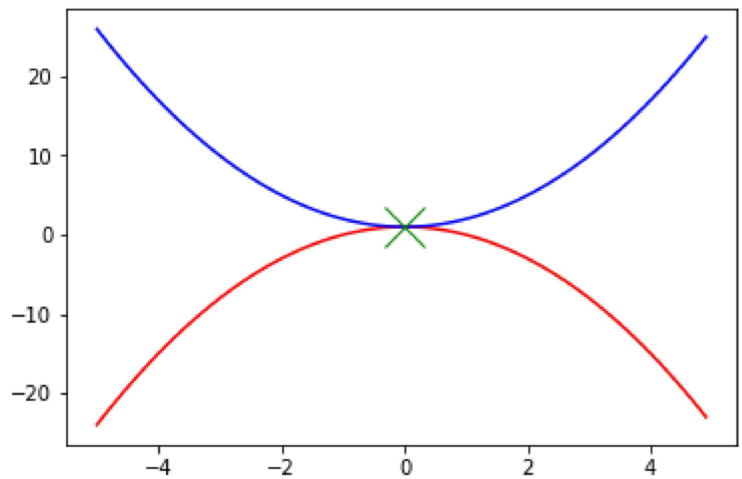
$\Delta f = -(A(x^k)^t \lambda)^t \Delta x = -\lambda \epsilon$

Nonnegativity of the Lagrange Multipliers

In [8]:
```
f1 = lambda x : 1 - x**2
f2 = lambda x : 1 + x**2

x = np.arange(-5., 5., 0.1)
plt.plot(x, f1(x), color='red')
plt.plot(x, f2(x), color='blue')
plt.plot(f1(0), marker='x', color='green', ms=20)
```

Out[8]:
```
[<matplotlib.lines.Line2D at 0x1eb4fc95e48>]
```



When the inequality contraint is inactive, lambda = 0

When the inequality constraint is active, the gradient of the objective function and the gradient of the constraint are aligned but have opposite directions. This is demonstrated in the example above with f1 and f2.

$$\nabla f_0(x) + \lambda_i \nabla f_i(x) = 0$$

Because the gradients are aligned and opposite, lambda must be positive

Necessary but not Sufficient Conditions

Lagrangian

In [9]:
```
def lag(x, l):
    return x[0] - x[1] + l*(x[0]*x[1] + 4)
```

Lagrangian Gradient

In [10]:
```
def lag_grad(x, l):
    g = np.zeros(3)
    g[0] = l*x[1] + 1
    g[1] = l*x[0] - 1
    g[2] = x[0]*x[1] + 4
    return g
```

In [11]:
```
x = np.array([2, -2])
l = 0.5
print(lag_grad(x, l))
x = np.array([-2, 2])
l = -0.5
print(lag_grad(x, l))
```

```
[0. 0. 0.]
[0. 0. 0.]
```

For x = [2, -2] and for x = [-2, 2] the first order optimality conditions are satisfied, however neither are the optimal points because x = [-4, 1] achieves a smaller value for the primal problem at f(x) = -5 and it satisfies the constraint

Healthy Snack

In [12]:
```
c = [50, 80]
A = [[-3, 0],
     [-2, -4],
     [-2,-5]]
b = [-6, -10, -8]
res = lp(c, A_ub=A, b_ub=b, options = { "disp" : True } )
```

```
Optimization terminated successfully.
         Current function value: 220.000000
         Iterations: 2
```

In [13]:
```
print(res)
```

```
     con: array([], dtype=float64)
     fun: 220.0
 message: 'Optimization terminated successfully.'
     nit: 2
   slack: array([0. , 0. , 3.5])
  status: 0
 success: True
       x: array([2. , 1.5])
```

Lagrangian:

$L = 50x_1 + 80x_2 + \lambda_1(-3x_1 + 6) + \lambda_2(-2x_1 - 4x_2 + 10) + \lambda_3(-2x_1 + -5x_2 + 8)$

Optimality conditions:

$-3\lambda_1 - 2\lambda_2 - 2\lambda_3 = -50$

$-4\lambda_2 - 5\lambda_3 = -80$

Since the third consraint is not active lambda3 is equal to 0.

Solving for lambda1 and lambda2 we get 10/3 and 20 respectively.

The lagrange multipliers represent the sensitivity towards a certain constraint. For example here we know optimality will more if we change the second constraint than if we change the third constraint. Also the changing teh first constraint wont change optimality. This is of course for small changes.