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CMPS 351

Assignment 2

In [1]:

```
import numpy as np
from numpy import linalg as la
import matplotlib.pyplot as plt
```

Rosenbrock

In [2]:

```
def rosenbrock(x):
    f = 10 * (x[1] - x[0]**2)**2 + (1 - x[0])**2
    return f
```

Rosenbrock Gradient

In [3]:

```
def rosenbrock_gradient(x):
    g = np.zeros(2)
    g[0] = -40*x[0]*(x[1] - x[0]**2) + 2*x[0] - 2
    g[1] = 20*x[1] - 20*x[0]**2
    return g
```

Rosenbrock Hessian

In [4]:

```
def rosenbrock_hessian(x):
    h = np.array([np.zeros(2), np.zeros(2)])
    h[0, 0] = 120*x[0]**2 - 40*x[1] + 2
    h[0, 1] = -40*x[0]
    h[1, 0] = -40*x[0]
    h[1, 1] = 20
    return h
```

Backtrack Line Search

In [5]:

```
def backtrack_linesearch(f, gk, pk, xk, alpha = 0.01, beta = 0.6):
    t = 1
    while ( f(xk + t*pk) > f(xk) + alpha*t*gk@pk):
        t *= beta
    return t
```

Newton Backtrack Line Search

In [6]:
```
def newton_backtrack(f, grad, hess, x0, tol = 1e-5):
    x = x0
    history = np.array([x0])
    while (la.norm(grad(x)) > tol):
        p = la.solve(hess(x), -grad(x))
        t = backtrack_linesearch(f, grad(x), p, x)
        x += t * p
        history = np.vstack((history, x))
    return x, history
```

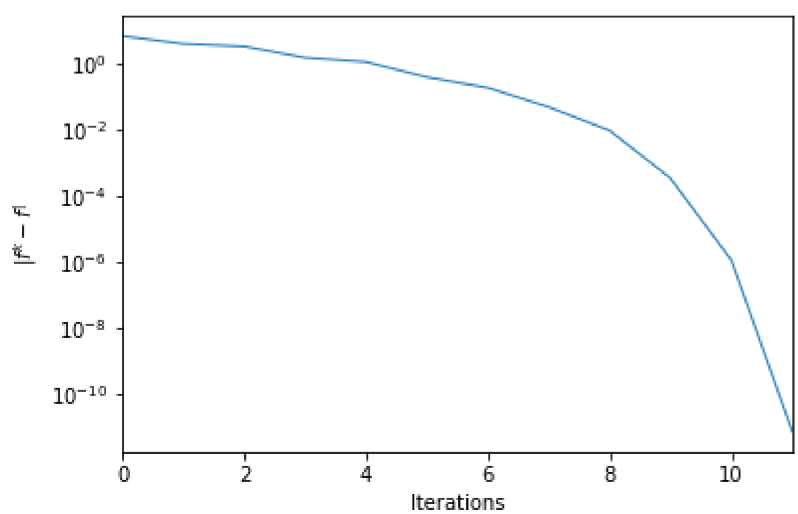
Plotting the performance

In [7]:
```
x0 = np.array([-1.2, 1.0])
xstar, history = newton_backtrack(rosenbrock, rosenbrock_gradient, rosenbrock

nsteps = history.shape[0]
fhist = np.zeros(nsteps)

for i in range(nsteps):
    fhist[i] = rosenbrock(history[i,:])

plt.figure()
plt.autoscale(enable=True, axis='x', tight=True)
plt.semilogy(np.arange(0, nsteps), fhist, linewidth=1)
plt.xlabel('Iterations')
plt.ylabel(r'$|f^k - f^|$')
plt.show()
```



Near the minimum, the algorithm converges quatratically

$$\left|f^{k+1} - f^*\right| < C\left|f^k - f^*\right|^2$$

Steepest descent converged linearly on the plot, and took over 1000 iterations to reach the local minimum. Newton's method converged in 11 iterations and made more progress the closer it got to the minimum

Plotting the contours

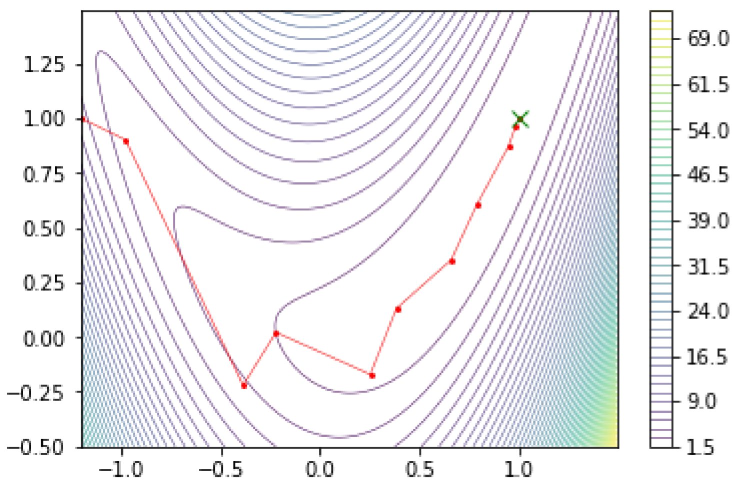
In [8]:
```
x0 = np.arange(-1.2, 1.5, 0.01)
x1 = np.arange(-0.5, 1.5, 0.01)

F = np.zeros((x1.shape[0], x0.shape[0]))

for i in range(F.shape[0]):
    for j in range(F.shape[1]):
        x = [x0[j], x1[i]]
        F[i, j] = rosenbrock(x)

plt.figure('Contours')
plt.plot(history[:,0], history[:,1], linewidth=0.5, color='red', marker='o',
plt.plot([1],[1], color='green', marker='x', markersize=8)
plt.contour(x0, x1, F, 50, linewidths=0.5)
#plt.axis('equal')
plt.colorbar()
```

Out[8]:
```
<matplotlib.colorbar.Colorbar at 0x2372a7f4fd0>
```



Function in R^100

$$f(x) = -\sum_{i=1}^{500} \log(1 - a_i^t x) - \sum_{i=1}^{100} \log(1 - x_i^2)$$

In [9]:
```
def f(a, x):
    try:
        f1 = 0
        for at in a:
            f1 += np.log10(1 - at@x)
        f2 = np.sum(np.log10(1 - x**2))
        return - f1 - f2
    except:
        return np.nan
```

R^100 Gradient

$$f(x)\frac{\partial}{\partial x_j} = \sum_{i=1}^{500} \frac{a_{ij}}{\ln(10)\left(1 - a_i^t x\right)} + \frac{2x_j}{\ln{(10)}(1 - x_j^2)}$$

In [10]:
```
def f_gradient(a, x):
    g = np.zeros(100)
    for j in range(100):
        term1 = 0
        for i in range(500):
            term1 += a[i, j]/(np.log(10)*(1 - a[i]@x))
        g[j] = term1 + 2*x[j]/(np.log(10)*(1 - x[j]**2))
    return g
```

R^100 Hessian

$$f(x)\frac{\partial^2}{\partial x_j \partial x_k} = \begin{cases} \displaystyle\sum_{i=1}^{500} \frac{a_{ij}a_{ik}}{\ln{(10)}(1 - a_i^t x)^2} & where\ k \neq j \\ \displaystyle\sum_{i=1}^{500} \frac{a_{ij}^2}{\ln{(10)}(1 - a_i^t x)^2} + \frac{2 + 2x_j^2}{\ln{(10)}(1 - x_j^2)^2} & where\ k = j \end{cases}$$

In [11]:
```
def f_hessian(a, x):
    h = np.zeros([100,100])
    for k in range(100):
        for j in range(100):
            term1 = 0
            for i in range(500):
                term1 += a[i, j]*a[i, k]/(np.log(10)*(1 - a[i]@x)**2)
            if k == j:
                h[k, j] = term1
            else:
                h[k, j] = term1 + (2 + 2*x[j]**2)/(np.log(10)*(1 - x[j]**2)**
    return h
```

The system of equations needed to be solved at every iteration can be written as

$$\nabla^2 f(x) \cdot x = -\nabla f(x)$$

Modified Backtrack Line Search

In [12]:
```
def backtrack_linesearch_mod(f, gk, pk, xk, a, alpha = 0.01, beta = 0.6):
    t = 1
    while (np.isnan(f(a, xk + t*pk)) or f(a, xk + t*pk) > f(a, xk) + alpha*t*
        t *= beta
    return t
```

Modified Newton Backtrack

In [13]:
```python
def newton_backtrack_mod(f, grad, hess, x0, a, tol = 1e-5):
    x = x0
    history = np.array([x0])
    while (la.norm(grad(a, x)) > tol):
        p = la.solve(hess(a, x), -grad(a, x))
        t = backtrack_linesearch_mod(f, grad(a, x), p, x, a)
        x += t * p
        #print(la.norm(grad(a, x)))
        history = np.vstack((history, x))
    return x, history
```

Plotting Performance in R^100

In [14]:
```python
x0 = np.array(np.zeros(100))
a = np.random.randn(500, 100)
xstar, history = newton_backtrack_mod(f, f_gradient, f_hessian, x0, a)
```

```
C:\Users\Danny\Anaconda3\lib\site-packages\ipykernel_launcher.py:5: RuntimeWarn
ing: invalid value encountered in log10
  """
```

In [15]:
```python
nsteps = history.shape[0]
fhist = np.zeros(nsteps)
fstar = f(a,xstar)

for i in range(nsteps):
    fhist[i] = f(a, history[i,:])
    #print(fhist[i])

plt.figure()
plt.autoscale(enable=True, axis='x', tight=True)
plt.semilogy(np.arange(0, nsteps), abs(fhist - fstar), linewidth=1)
plt.xlabel('Iterations')
plt.ylabel(r'$|f^k - f^|$')
plt.show()
```