Angular Momentum Operator Derivation

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Abstract

We present a simple derivation of the famous angular momentum ladder operators $\hat{J}_{\pm}$. Starting from basic Lie algebra commutator identities and the canonical angular momentum commutator relationships, we show that the ladder operators can be simply constructed in a straightforward manner using the basic canonical angular momentum operator commutators and some algebraic tricks. Ultimately, all that is needed is elementary algebra and some knowledge of certain commutation relations and so called ladder operators which arise from Lie group theory. The full form for all the ladder operators is derived using the orbital angular momentum $\hat{L}$ as reference to construct the formula. It is to be noted this derivation applies to both $\hat{S}$ spin angular momentum and $\hat{J}$ total angular momentum as well.

1 Background Theory

1.1 Ladder Operator Commutator

In most introductory quantum mechanical textbooks, the angular momentum ladder operators are simply given as definitions. That is to say, they are treated as axioms upon which the rest of the theory regarding orbital angular momentum $\hat{L}$ and spin angular momentum $\hat{S}$ are derived from. The definition given for the ladder operators for the orbital angular momentum is the following [1]:

$$\hat{L}_{\pm} = \hat{L}_x \pm i\hat{L}_y, \tag{1}$$

where the plus sign represents the raising operator and the minus sign is the lowering operator. A similar form for the spin angular momentum ladder operator is:

$$\hat{S}_{\pm} = \hat{S}_x \pm i\hat{S}_y. \tag{2}$$

In books such as [1] Griffiths the definition is taken to be a fact that is not derived. One may then question why such a form satisfies the criteria for a ladder operator. As such, our starting point will be to answer that question: how exactly do we define the function of a ladder operator? What do we expect the ladder operator to do and how can we formulate the ladder operators so as to satisfy said requirement? The structure, as we will soon see, relies heavily on Lie algebra theory. Although we are essentially using Lie algebras to perform this derivation, it is not necessary to go into too much detail to get to this proof. In fact, we will demonstrate that all that is required is some simple algebra supplemented by some commutator identities that arise from Lie algebras.

Starting with $\hat{L}_z$, we will define the associated eigenfunctions and eigenvalues of the z-axis angular momentum operator as:

$$\hat{L}_z|\Gamma\rangle = \lambda|\Gamma\rangle, \tag{3}$$

where λ is the eigenvalue and $|\Gamma\rangle$ is the associated eigenfunction. We would like to define the ***raising*** operator for angular momentum to be $\hat{L}_+$. This operator has the ability to raise the eigenvalue λ by some constant c when applied to the eigenfunction $|\Gamma\rangle$ [1] associated to that eigenvalue. In other words:

$$\hat{L}_z(\hat{L}_+|\Gamma\rangle) = (\lambda + c)(\hat{L}_+|\Gamma\rangle). \tag{4}$$

Equation 4 is what we expect for the raising operator to do when we apply it to a certain eigenfunction of $\hat{L}_z$. As mentioned previously, the crutch of the proof for the structure of the ladder operators lies in Lie algebras and commutators. We will then define the commutator relationship between $\hat{L}_z$ and $\hat{L}_+$. Starting off:

$$[\hat{L}_z, \hat{L}_+] = \hat{L}_z\hat{L}_+ - \hat{L}_+\hat{L}_z, \tag{5}$$

we will then apply an eigenfunction $|\Gamma\rangle$ in the usual way commutator relationships are treated. That is to say we do the following with equation 5:

$$[\hat{L}_z, \hat{L}_+]|\Gamma\rangle = \hat{L}_z(\hat{L}_+|\Gamma\rangle) - \hat{L}_+(\hat{L}_z|\Gamma\rangle). \tag{6}$$

We then insert equation 4 into equation 6, and change around the last term on the right hand side and get the following result:

$$[\hat{L}_z, \hat{L}_+]|\Gamma\rangle + \hat{L}_+(\hat{L}_z|\Gamma\rangle) = (\lambda + c)(\hat{L}_+|\Gamma\rangle), \tag{7}$$

which can be simplified even further to the following:

$$[\hat{L}_z, \hat{L}_+]|\Gamma\rangle + \lambda(\hat{L}_+|\Gamma\rangle) = (\lambda + c)(\hat{L}_+|\Gamma\rangle). \tag{8}$$

It is easy to observe that in order for both sides of equation 8 to hold true, the commutator relationship between $\hat{L}_z$ and $\hat{L}_+$ must take the following form:

$$[\hat{L}_z, \hat{L}_+] = +c\hat{L}_+. \tag{9}$$

Note that an exact similar treatment for the ***lowering*** operator applies; where instead we expect that the lowering operator decrease the λ eigenvalue by some constant c. We will get $[\hat{L}_z, \hat{L}_+] = -c\hat{L}_-$ instead.

We summarize the above to the following important commutator relationship:

$$[\hat{L}_z, \hat{L}_\pm] = \pm c\hat{L}_\pm. \tag{10}$$

Equation 10 completely satisfies the requirement and function we expect from ladder operators. The purpose of the ladder operators is to add or subtract some constant c from the eigenvalue of the $\hat{L}_z$ eigenfunction which led us to the above 10 relationship.

1.2 Lie Algebra Commutator Identities

The important foundation for angular momentum theory in quantum mechanics rests on the following commutator relationship between the 3 different components x, y, z of the angular momentum operator $\hat{L}$:

$$[\hat{L}_i, \hat{L}_j] = i\hbar\epsilon_{ijk}\hat{L}_k, \tag{11}$$

where ϵ_{ijk} is the Levi-Civita symbol. The following is an important commutator identity that arises from Lie algebra:

$$[A, B + C] = [A, B] + [A, C] \tag{12}$$

We have thus all the identities and relationships we need to start the derivation for the ladder operators.

2 Derivation

With the above background tools handled, we start off with equation 10 again but we only focus on the raising operator aspect for now:

$$[\hat{L}_z, \hat{L}_+] = c\hat{L}_+$$

In general we seek a form that satisfies $[n, X] = \gamma X$ for some constant γ and a raising operator X. This form corresponds to equation 10. Consider the commutator identity 12. We shall use equation 11 to formulate the following sum:

$$[\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_y] = i\hbar\hat{L}_y - i\hbar\hat{L}_x = \hbar(i\hat{L}_y - i\hat{L}_x) \tag{13}$$

We can notice a familiarity in the equation 13. We have found our constant c to be in fact equal to $\hbar$. However we haven't quite managed to get the form on the far right side to match what we expect in the form of $[n, X] = \gamma X$. This is because using identity 12 for equation 13 we actually get:

$$[\hat{L}_z, \hat{L}_x] + [\hat{L}_z, \hat{L}_y] = [\hat{L}_z, (\hat{L}_x + \hat{L}_y)] \neq \hbar(i\hat{L}_y - i\hat{L}_x) \tag{14}$$

However this result assures us that we simply need to pick the right coefficients for a linear combination of $\hat{L}_x$ and $\hat{L}_y$ in order to get a commutator relationship $[n, X] = \gamma X$ for the raising operator. So in essence, $\hat{L}_+$ can be represented as a linear combination of the x-axis angular momentum component and the y-axis one. Our job would then be to determine coefficients a and b in $\hat{L}_+ = a\hat{L}_x + b\hat{L}_y$. We have to solve the following equation:

$$[\hat{L}_z, a\hat{L}_x + b\hat{L}_y] = a[\hat{L}_z, \hat{L}_x] + b[\hat{L}_z, \hat{L}_y]$$

Simplifying even further we see that we get what we found earlier but with coefficients involved, and we have to equate the second term of the left hand side commutator to the linear combination of the operators we have on the right hand side. As such we find the equation we have to solve to be the following:

$$[\hat{L}_z, a\hat{L}_x + b\hat{L}_y] = \hbar(ia\hat{L}_y - ib\hat{L}_x) \tag{15}$$

To find the coefficients a and b for the raising operator, we essentially have to solve the following equation:

$$a\hat{L}_x + b\hat{L}_y = ia\hat{L}_y - ib\hat{L}_x \tag{16}$$

Equation 16 is a very basic system of linear equations which solves for unknown variables a and b. It is easy to conclude that $a = 1$ and $b = i$. The finality of all these procedures is that we arrive to the raising ladder operator $\hat{L}_+$ as a function of the x and y angular momentum operator components:

$$[\hat{L}_z, \hat{L}_x + i\hat{L}_y] = \hbar(\hat{L}_x + i\hat{L}_y) \tag{17}$$

All of the above can be applied to the lowering operator $\hat{L}_-$. The result and the procedure is identical to what we have performed so far, we simply have a difference in signs in certain places. Namely:

$$[\hat{L}_z, \hat{L}_\pm] = \pm\hbar\hat{L}_\pm = \pm\hbar(\hat{L}_x \pm i\hat{L}_y) \tag{18}$$

As mentioned earlier on in the article, such formalism can be applied to all the angular momentum ladder operators. So if the total angular momentum is defined by [2]:

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}, \tag{19}$$

we can also conclude that:

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y \tag{20}$$

References

[1] D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics 3rd edition*, (Cambridge University Press, Cambridge, UK, 2018).

[2] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics 2nd edition*, (Addison-Wesley, San Fransisco, USA, 2011).