CHAPTER 9

# Generalized Polygons

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## Introduction

Generalized polygons were introduced by Tits [1959] in his celebrated work 'Sur la trialité et certains groupes qui s'en déduisent'. The classical examples arise from groups with a BN-pair of rank 2, for which the Weyl group is a dihedral group. For finite thick generalized $n$-gons Feit and Higman [1964] show that $n = 2, 3, 4, 6$ or 8. Moreover several restrictions on the parameters of finite generalized $n$-gons hold. In the infinite case we do not have a Feit and Higman theorem, and here 'free' generalized $n$-gons may be constructed in much the same way as free projective planes (see Chapter 13). Generalized 2-gons are trivial structures, and the thick generalized 3-gons are the projective planes. Since the projective planes are considered in Chapters 2, 4 and 5, we restrict ourselves to generalized $n$-gons with $n \geqslant 4$. As much of the literature concerns the finite case, most of this chapter is on finite generalized quadrangles, hexagons and octagons. Moreover, since only a few classes of finite thick generalized hexagons and octagons are known, most of the papers concern finite generalized quadrangles. The main results, up to 1983, on this subject are contained in Payne and Thas [1984].

In the first two sections we introduce some basic notions on finite generalized polygons and also state important restrictions on the parameters, including the Feit–Higman theorem. Section 3 contains a description of the classical finite generalized polygons, in particular the only known finite thick generalized hexagons and octagons. All known nonclassical finite generalized quadrangles are considered in the next section; here the basic constructions are due to Kantor, Payne, Thas and Tits. Also some isomorphisms between the known generalized quadrangles are discussed. Section 5 is on the uniqueness of generalized polygons with small parameters. In Section 6 we consider ovoids, spreads, polarities and subpolygons of finite generalized polygons. These objects will appear to be useful tools in some characterizations of generalized polygons. Next all generalized quadrangles embedded in finite projective and affine spaces are classified. In the projective case the complete classification is due to Buekenhout and Lefèvre, in the affine case to Thas. In contrast with the projective case, five nontrivial 'sporadic' examples arise in the finite affine case. In Section 8, the largest one, important combinatorial characterizations of the finite classical generalized quadrangles and hexagons are stated; in the quadrangle case most of these theorems are due to Thas, in the hexagon case to Ronan. Automorphisms of finite generalized polygons are considered in Section 9. In the first part elation and translation generalized quadrangles are introduced, the second part contains characterizations of finite classical generalized polygons by automorphisms. The main theorem states that a finite thick generalized $n$-gon, $n \geqslant 4$, is Moufang if and only if it is classical or dual classical. It was observed by Tits that this theorem easily follows from the classification of finite split BN-pairs of rank 2 by Fong and Seitz. The section concludes with a theorem of Pasini on epimorphisms between finite thick generalized $n$-gons, $n > 2$. In the last section we say some words about infinite generalized polygons. In particular we state the result of Tits on Moufang polygons, and mention that all infinite generalized quadrangles embedded in a projective space were determined by Dienst.

Since most of the demonstrations are long, complicated and technical, we decided to not include proofs.

## 1. Finite generalized polygons

### 1.1. *Graphs*

A finite *graph* $G = (X, E)$ consists of a finite set $X$ $(\neq \varnothing)$ with $n$ elements and a set $E$ of unordered pairs of $X$. The elements of $X$ are called the *vertices* of the graph $G$, while the elements of $E$ are called the *edges*. A sequence of distinct vertices $x_0, x_1, \ldots, x_m$ of $G$ is a *path* of *length* $m$ between $x_0$ and $x_m$ if $\{x_{i-1}, x_i\}$ is an edge for $i = 1, 2, \ldots, m$. The *distance* $d(x, y)$ between two vertices $x$ and $y$ is the length of the shortest path between $x$ and $y$ ($d(x, y) = \infty$ if $G$ is disconnected and $x$ and $y$ are in distinct components of $G$). The *diameter* of $G$ is the largest distance in $G$. A sequence of vertices $x_0, x_1, \ldots, x_m$ is a *circuit* of *length* $m$ if $x_1, x_2, \ldots, x_m$ are distinct, $x_0 = x_m$, $m > 2$ and $\{x_{i-1}, x_i\}$ is an edge for $i = 1, 2, \ldots, m$. The *girth* of $G$ is the length of the shortest circuit in $G$. The graph $G = (X, E)$ is *bipartite* if

$$X = X_1 \cup X_2, \quad X_1 \cap X_2 = \varnothing, \quad X_1 \neq \varnothing \neq X_2,$$

and every edge has one vertex in $X_1$ and one vertex in $X_2$.

### 1.2. *Graphs defined by finite incidence structures*

Let $S = (P, B, I)$ be a finite incidence structure with $P$ $(\neq \varnothing)$ the set of all *points* of $S$, $B$ $(\neq \varnothing)$ the set of all *blocks* of $S$, and $I$ a symmetric point-block *incidence relation*[1]. If any two distinct points are incident with at most one block, then the blocks usually are called *lines*. The *point graph* of $S$ is the graph with vertex set $P$, two vertices being adjacent if and only if they are incident with a common block; the *block graph* of $S$ is the graph with vertex set $B$, two vertices being adjacent if and only if they are incident with a common point. Now assume that $P \cap B = \varnothing$. Then the *incidence graph* of $S$ has vertex set $P \cup B$, where two vertices $x$ and $y$ are adjacent if and only if either $x \in P$ and $y \in B$, or $x \in B$ and $y \in P$, and $xIy$. Clearly the incidence graph of $S$ is bipartite.

Assume that any two distinct points of $S$ are incident with at most one common element of $B$, that any point of $S$ is incident with $t + 1$ $(t \geqslant 0)$ lines of $S$, that any line of $S$ is incident with $s + 1$ $(s \geqslant 0)$ points of $S$. Further, let $P = \{x_1, x_2, \ldots, x_v\}$, $B = \{L_1, L_2, \ldots, L_b\}$, and let $N$ be the corresponding incidence matrix of $S$, where rows are indexed by points and columns by lines. Then

$$NN^T - (t + 1)I$$

is the corresponding adjacency matrix of the point graph of $S$,

$$N^T N - (s + 1)I$$

is the corresponding adjacency matrix of the line graph of $S$, and

$$\begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}$$

is the corresponding adjacency matrix of the incidence graph of $S$.

[1] In Chapter 3, this is called a *rank* 2 *geometry*.

**1.3.** *Finite generalized polygons*

A *generalized n-gon of order* $(s,t)$, $n \geqslant 2$, $s > 0$, $t > 0$, is a 1-$(v, s+1, t+1)$ design $S = (P, B, I)$ whose incidence graph has girth $2n$ and diameter $n$. A *generalized polygon* is a generalized $n$-gon for some $n$. If $s = t$, $S$ is said to have *order* $s$. Generalized polygons were introduced by Tits [1959] in his celebrated work on triality. We emphasize that the generalized polygons form a particular class of buildings; see Chapter 11.

There is a point-line duality for generalized polygons (of order $(s,t)$) for which in any definition or theorem the words 'point' and 'line' are interchanged and the parameters $s$ and $t$ are interchanged. Normally, assume without further notice that the dual of a given theorem or definition has also been given.

Clearly in a generalized 2-gon, i.e. a generalized *digon*, every point is incident with every line, and a generalized $n$-gon with $s = t = 1$ is an ordinary $n$-gon. Generalized $n$-gons with $s > 1$ and $t > 1$ are called *thick*. It is not difficult to verify that a generalized 3-gon, i.e. a *generalized triangle*, of order $(s,t)$ is either a triangle or a projective plane of order $s$, $s > 1$. Thus for a generalized triangle one always has $s = t$.

A generalized 4-gon, i.e. a *generalized quadrangle*, of order $(s,t)$ is easily seen to be any incidence structure of points and lines satisfying:

(i) each point is incident with $1+t$ lines ($t \geqslant 1$) and two distinct points are incident with at most one line;

(ii) each line is incident with $1+s$ points ($s \geqslant 1$) and two distinct lines are incident with at most one point;

(iii) if $x$ is a point and $L$ is a line not incident with $x$, then there is a unique pair $(y, M) \in P \times B$ for which $xIMIyIL$.

Generalized 6-gons and generalized 8-gons are usually called *generalized hexagons* and *generalized octagons*, respectively.

Let $S = (P, B, I)$ be a generalized $n$-gon of order $s$. Consider the incidence structure $S' = (P \cup B, F, I')$, where $F$ is the set of incident point-line pairs and $I'$ is the natural incidence relation. Then $S'$ is a generalized $2n$-gon of order $(1, s)$. Conversely, it can be proved easily that all generalized $n$-gons of order $(1,s)$, $s > 1$, are of this form.

Let $S = (P, B, I)$ be a generalized polygon. Given two (not necessarily distinct) points $x, y$ of $S$, we write $x \sim y$ and say that $x$ and $y$ are *collinear*, provided that there is some line $L$ for which $xILIy$. And $x \not\sim y$ means that $x$ and $y$ are not collinear. Dually, for $L, M \in B$, we write $L \sim M$ or $L \not\sim M$ according as $L$ and $M$ are *concurrent* or nonconcurrent respectively. If $x \sim y$ (respectively, $L \sim M$) we may also say that $x$ (respectively, $L$) is *orthogonal* or *perpendicular* to $y$ (respectively, $M$). The line (respectively, point) which is incident with distinct collinear points $x, y$ (respectively, distinct concurrent lines $L, M$) is denoted by $xy$ (respectively, $LM$ or $L \cap M$).

For $x \in P$ put $x^\perp = \{y \in P\colon\ y \sim x\}$, and note that $x \in x^\perp$. More generally, if $A \subset P$, $A$ 'perp' is defined by $A^\perp = \cap\{x^\perp\colon\ x \in A\}$.

Isomorphisms (or collineations), anti-isomorphisms (or correlations), automorphisms, anti-automorphisms, involutions and polarities of generalized polygons are defined in the usual way.

## 2. Restrictions on the parameters

### 2.1. *The theorem of Feit and Higman*

THEOREM 1 (Feit and Higman [1964]). *If $S = (P, B, I)$ is a generalized $n$-gon of order $(s, t)$, which is not an ordinary $n$-gon ($s = t = 1$), then there are only the following possibilities*:

(i) $n = 2$;

(ii) $n = 3$ *and* $s = t$;

(iii) $n = 4$;

(iv) $n = 6$ *and* $s = 1$ *or* $t = 1$ *or* $st$ *is a square*;

(v) $n = 8$ *and* $s = 1$ *or* $t = 1$ *or* $2st$ *is a square*;

(vi) $n = 12$ *and either* $s = 1$ *or* $t = 1$.

Let $|P| = v$ and $|B| = b$. Clearly $(s + 1)b = (t + 1)v$. In case (i) $b = t + 1$ and $v = s + 1$, and in case (ii) $v = b = s^2 + s + 1$. In the other cases $v$ and $b$ are given by the following theorem, a proof of which can be found in, e.g., Dembowski [1968].

THEOREM 2. *If $S = (P, B, I)$ is a generalized $2n$-gon of order $(s, t)$, with $|P| = v$ and $|B| = b$, then*

$$v = (1 + s)\big(1 + st + (st)^2 + \cdots + (st)^{n-1}\big)$$

*and*

$$b = (1 + t)\big(1 + st + (st)^2 + \cdots + (st)^{n-1}\big).$$

Calculating the multiplicities of the eigenvalues of the adjacency matrix of the point graph of $S$, necessary conditions for the existence of a generalized $n$-gon of order $(s, t)$ are obtained. Two examples: for $n = 4$, $s + t$ divides $st(s + 1)(t + 1)$, see, e.g., Payne and Thas [1984], for $n = 6$, $s^2 + st + t^2$ divides $s^3(s^2t^2 + st + 1)$, see, e.g., Haemers and Roos [1981].

### 2.2. *The inequalities of Higman, and Haemers and Roos*

THEOREM 3 (Higman [1974]). *If $S$ is a thick generalized quadrangle or octagon of order $(s, t)$, then $t \leqslant s^2$, and dually $s \leqslant t^2$.*

THEOREM 4 (Haemers and Roos [1981]). *If $S$ is a thick generalized hexagon of order $(s, t)$, then $t \leqslant s^3$, and dually $s \leqslant t^3$.*

For generalized quadrangles attaining the bound, we have the following theorem.

THEOREM 5 (Bose and Shrikhande [1972]). *Let $S$ be a thick generalized quadrangle of order $(s,t)$. Then $t = s^2$ if and only if for each triple $\{x,y,z\}$ of pairwise noncollinear points there is a constant number of points collinear with $x$, $y$ and $z$. This constant number of points is $s+1$.*

Let $S = (P,B,I)$ be a generalized hexagon of order $(s,t)$, and let $d$ denote the distance in the incidence graph of $S$. For a line $L$ and points $x$ and $y$ define

$$p_{ijk}(L,x,y) = \left|\left\{z \in P\colon\ d(z,L) = 2i+1,\ d(z,x) = 2j,\ d(z,y) = 2k\right\}\right|,$$

for $i = 0,1,2$ and $j,k = 0,1,2,3$. If $d(x,y) \leqslant 4$, then the configuration *induced* by $L$, $x$ and $y$ is the substructure of $S$ formed by the points and the lines, which are on the shortest path between $L$ and $x$, $L$ and $y$, $x$ and $y$; if $d(x,y) = 6$, then the configuration *induced* by $L$, $x$ and $y$ is the substructure of $S$ formed by the points and the lines, which are on the shortest path between $L$ and $x$, $L$ and $y$. For generalized hexagons attaining the bound in Theorem 4, we have the following theorem.

THEOREM 6 (Haemers [1979]). *If a generalized hexagon has order $(s,s^3)$, then*

$$p_{ijk}(L,x,y) = p_{ijk}(L',x',y') \quad \textit{for all } i,j,k$$

*if there is an isomorphism $\theta$ of the configuration induced by $L$, $x$ and $y$ onto the configuration induced by $L'$, $x'$ and $y'$, with $L^\theta = L'$, $x^\theta = x'$ and $y^\theta = y'$.*

## 3. The classical finite generalized polygons

### 3.1. *The classical finite projective planes*

A projective plane of order $s$ is called *classical* if it is the (Desarguesian) projective plane PG$(2,s)$ over the Galois field GF$(s)$.

### 3.2. *The classical finite generalized quadrangles*

We give a brief description of three families of generalized quadrangles, known as the *classical generalized quadrangles,* all of which are associated with classical groups and were first recognized as generalized quadrangles by Tits; see, e.g., Dembowski [1968].

(i) Consider a nonsingular quadric $Q^+$ (respectively, $Q$ and $Q^-$) of Witt index 2, i.e. of projective index 1, in the projective space PG$(3,q)$ (respectively, PG$(4,q)$ and PG$(5,q)$). Then the points of the quadric together with the lines of the quadric form a generalized quadrangle with parameters$Q^+(3,q)Q(4,q)$

$$s = q,\ t = 1,\ v = (q+1)^2,\ b = 2(q+1), \quad \text{for } Q^+(3,q),$$

$$s = t = q,\ v = b = (q+1)(q^2+1), \quad \text{for } Q(4,q),$$

$$s = q,\ t = q^2,\ v = (q+1)(q^3+1),\ b = (q^2+1)(q^3+1), \quad \text{for } Q^-(5,q).$$

$Q^-(5,q)$. Since $Q^+(3,q)$ has $t = 1$, its structure is trivial. Further, recall that these quadrics have the following canonical equations:

$$Q^+\colon\ X_0X_1 + X_2X_3 = 0,$$

$$Q\colon\ X_0^2 + X_1X_2 + X_3X_4 = 0,$$

$$Q^-\colon\ F(X_0, X_1) + X_2X_3 + X_4X_5 = 0,$$

where $F$ is an irreducible homogeneous polynomial in $X_0, X_1$ over GF($q$).

(ii) Let $H$ be a nonsingular Hermitian variety of the projective space PG($d, q^2$), $d = 3$ or 4. Then the points of $H$, together with the lines on $H$, form a generalized quadrangle $H(d, q^2)$ with parameters:

$$s = q^2,\ t = q,\ v = (q^2+1)(q^3+1),\ b = (q+1)(q^3+1), \quad \text{when } d = 3,$$

$$s = q^2,\ t = q^3,\ v = (q^2+1)(q^5+1),\ b = (q^3+1)(q^5+1), \quad \text{when } d = 4.$$

Remember that $H$ has the canonical equation

$$X_0^{q+1} + X_1^{q+1} + \cdots + X_d^{q+1} = 0.$$

(iii) The points of PG($3, q$), together with the totally isotropic lines with respect to a symplectic polarity, form a generalized quadrangle $W_3(q)$, shortly $W(q)$, with parameters

$$s = t = q,\ v = b = (q+1)(q^2+1).$$

Remember that the lines of $W_3(q)$ are the elements of a general linear complex of lines of PG($3, q$), see, e.g., Hirschfeld [1985], and that a symplectic polarity of PG($3, q$) has the following canonical bilinear form:

$$X_0Y_1 - X_1Y_0 + X_2Y_3 - X_3Y_2 = 0.$$

THEOREM 1 (Payne and Thas [1984]).

(a) *$Q(4, q)$ is isomorphic to the dual of $W(q)$. Moreover, $Q(4, q)$ (or $W(q)$) is self-dual if and only if $q$ is even.*

(b) *$Q^-(5, q)$ is isomorphic to the dual of $H(3, q^2)$.*

The polarities of $W(q)$ are discussed in Section 3.2 of Chapter 7.

**3.3.** *The classical finite generalized hexagons*

Let $Q^+$ be a hyperbolic quadric in PG$(7, q)$ with systems $U$ and $V$ of generators. A *triality* is a permutation $\tau$ of order three of $Q^+ \cup U \cup V$ such that

$$(Q^+)^\tau = U, \qquad U^\tau = V, \qquad V^\tau = Q^+$$

and which preserves *incidence*, where incidence is defined as follows:
(a) a point is incident with a solid (3-space) if it lies in the solid,
(b) two points are incident if the line joining them lies on $Q^+$,
(c) two solids of the same system of $Q^+$ are incident if they meet in a line, and
(d) two solids of distinct systems of $Q^+$ are incident if they meet in a plane.
See also Section 8 of Chapter 2.

For $x$ a point of $Q^+$, the solid $x^\tau$ is the *trial solid* of $x$. If $xy$ is a line of $Q^+$, then so are $x^\tau \cap y^\tau$ and $x^{\tau^2} \cap y^{\tau^2}$; these are cases of generators of the same system intersecting in a line. The point $x$ is *self-conjugate* if it lies in its own trial solid $x^\tau$. If $x$ and $y$ are self-conjugate points and if $y \in x^\tau$, then $xy = x^\tau \cap y^\tau$ and is called a *fixed line*.

Tits [1959] showed that any triality with some self-conjugate point corresponds to a collineation of PG$(2, q)$, of order dividing three, of one of the following types:

$\mathrm{I}_\sigma$: $x'_0 = x_0^\sigma,\ x'_1 = x_1^\sigma,\ x'_2 = x_2^\sigma$;
$\mathrm{I}_{\mathrm{id}}$: $x'_0 = x_0,\ x'_1 = x_1,\ x'_2 = x_2$;
II: $x'_0 = x_1,\ x'_1 = x_2,\ x'_2 = x_0$.

Here, $\sigma$ is a field automorphism of order three, and so exists only when $q$ is a cube. Since the characteristic polynomial of II is $X^3 - 1$, its properties vary considerably according to whether $q \equiv 0, 1, -1 \pmod 3$.

Let $\tau$ be a triality with some self-conjugate point. The set of self-conjugate points is denoted by $P$ and the set of fixed lines by $B$.

(1) If $\tau$ is of type $\mathrm{I}_{\mathrm{id}}$, the set $P$ is a nonsingular quadric, the section of $Q^+$ by a hyperplane. Further, $(P, B, I)$, with $I$ the natural incidence, is a generalized hexagon of order $q$. It is denoted by $H(q)$.

(2) If $\tau$ is of type $\mathrm{I}_\sigma$, then $q$ is a cube and $(P, B, I)$, with $I$ the natural incidence, is a generalized hexagon of order $(q, \sqrt[3]{q})$. It is denoted by $H(q, \sqrt[3]{q})$.

(3) If $\tau$ is of type II and $q \equiv 1 \pmod 3$, then $(P, B, I)$, with $I$ the natural incidence, is a generalized hexagon of order $(q, 1)$.

(4) If $\tau$ is of type II and $q \equiv -1 \pmod 3$, then $|P| = q^3 + 1$ and $|B| = 0$. Each solid of $U \cup V$ has exactly one point in common with $P$, and so $P$ is an ovoid of $Q^+$ (cf. Section 9 of Chapter 7).

(5) If $\tau$ is of type II and $q \equiv 0 \pmod 3$, then $B$ consists of all lines of a generalized hexagon $H(q)$ of type (1) which meet a fixed line of $H(q)$, and $P$ consists of all points on these lines. Here $|B| = q^2 + q + 1$ and $|P| = q^3 + q^2 + q + 1$.

The generalized hexagons of type (1) and (2) are called the *classical generalized hexagons*. Together with their duals these are the only known generalized hexagons of order $(s, t)$ with $s > 1$ and $t > 1$. Dickson's group $\mathsf{G}_2(q)$ and the triality group ${}^3\mathsf{D}_4(\sqrt[3]{q})$ act as automorphism groups on $H(q)$ and $H(q, \sqrt[3]{q})$, respectively (see Tits [1959]).

Let $Q$ be the nonsingular quadric of PG$(6, q)$ with equation

$$X_3^2 = X_0X_4 + X_1X_5 + X_2X_6.$$

Tits [1959] showed that $H(q)$ is isomorphic to the incidence structure formed by all the points of $Q$ and by those lines on $Q$ whose *Grassmann coordinates* (for the definition of Grassmann coordinates, see Hodge and Pedoe [1947]), satisfy

$$\begin{aligned} &p_{34} = p_{12}, \qquad p_{35} = p_{20}, \qquad p_{36} = p_{01}, \\ &p_{03} = p_{56}, \qquad p_{13} = p_{64}, \qquad p_{23} = p_{45}. \end{aligned} \tag{1}$$

Further, these lines are also the lines of PG$(6, q)$ whose Grassmann coordinates satisfy (1) together with

$$p_{04} + p_{15} + p_{26} = 0.$$

REMARK. We refer to Cameron and Kantor [1979] for another construction of the generalized hexagon $H(q)$. See also Bader and Lunardon [1993] for a purely geometrical and very elegant construction of $H(q)$, $q$ odd and $q \neq 3^h$.

For a proof of the following theorem we refer to Tits [1960].

THEOREM 2. *The generalized hexagon $H(q)$ is self-dual if and only if $q$ is a power of* 3; *it admits a polarity if and only if $q = 3^{2h+1}$, $h \geqslant 0$.*

### **3.4.** *The classical finite generalized octagons*

Consider the Ree group ${}^2\mathsf{F}_4(q)$, with $q = 2^{2h+1}$, $h \geqslant 0$. Let $M$ be a minimal parabolic subgroup of ${}^2\mathsf{F}_4(q)$ and let $P_1$, $P_2$ be the maximal parabolic subgroups of ${}^2\mathsf{F}_4(q)$, with $|P_1| > |P_2|$, containing $M$ (for the definition of parabolic subgroup, see, e.g., Carter [1972]). Further, let

$$P = \{xP_1\colon\ x \in {}^2\mathsf{F}_4(q)\}, \qquad B = \{yP_2\colon\ y \in {}^2\mathsf{F}_4(q)\},$$

and call $xP_1$ and $yP_2$ incident if and only if $xP_1 \cap yP_2 \neq \varnothing$. Then by Tits [1976], $(P, B, I)$ is a generalized octagon of order $(q, q^2)$. This generalized octagon and its dual are called the *classical generalized octagons*. These are the only known generalized octagons of order $(s, t)$ with $s > 1$ and $t > 1$. No purely geometrical construction for these octagons is known.

Finally we notice that a similar group-theoretical construction can also be given for the classical generalized quadrangles and hexagons; the groups involved here are classical groups and the groups $\mathsf{G}_2(q)$, ${}^3\mathsf{D}_4(\sqrt[3]{q})$, respectively.

## 4. Nonclassical finite generalized quadrangles

From now on a generalized quadrangle will be briefly denoted by GQ.

### 4.1. *The trivial nonclassical generalized quadrangles, grids and dual grids*

A *grid* is an incidence structure $S = (P, B, I)$ with

$$P = \{x_{ij}\colon\ i = 0, 1, \dots, s_1,\ j = 0, 1, \dots, s_2\}, \quad s_1 > 0,\ s_2 > 0,$$

$$B = \{L_0, \dots, L_{s_1}, M_0, \dots, M_{s_2}\},$$

$x_{ij} I L_k$ if and only if $i = k$, and $x_{ij} I M_k$ if and only if $j = k$. A grid with parameters $s_1$, $s_2$ is a generalized quadrangle if and only if $s_1 = s_2$. Clearly any generalized quadrangle with $t = 1$ is a grid. A *dual grid* is an incidence structure $S = (P, B, I)$ with

$$B = \{L_{ij}\colon\ i = 0, 1, \dots, t_1,\ j = 0, 1, \dots, t_2\}, \quad t_1 > 0,\ t_2 > 0,$$

$$P = \{x_0, \dots, x_{t_1}, y_0, \dots, y_{t_2}\},$$

$L_{ij} I x_k$ if and only if $i = k$, and $L_{ij} I y_k$ if and only if $j = k$. A dual grid with parameters $t_1$, $t_2$ is a generalized quadrangle if and only if $t_1 = t_2$. Clearly any generalized quadrangle with $s = 1$ is a dual grid.

### 4.2. *The nonclassical examples of Tits*

The earliest known nontrivial nonclassical examples of GQ were discovered by Tits and first appeared in Dembowski [1968].

Let $d = 2$ (respectively, $d = 3$) and let $O$ be an oval (respectively, an ovoid) of PG$(d, q)$; for the definitions of oval and ovoid, see Sections 1 and 3 of Chapter 7. Further, let PG$(d, q)$ be embedded as a hyperplane in PG$(d + 1, q)$. Define points as

(i) the points of PG$(d + 1, q) \setminus$ PG$(d, q)$,
(ii) the hyperplanes $X$ of PG$(d + 1, q)$ for which $|X \cap O| = 1$, and
(iii) one new symbol $(\infty)$.

Lines are defined as

(a) the lines of PG$(d + 1, q)$ which are not contained in PG$(d, q)$ and meet $O$ (necessarily in a unique point), and
(b) the points of $O$.

Incidence is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of PG$(d + 1, q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $O$ in it. The point $(\infty)$ is incident with no line of type (a) and all lines of type (b). It is an easy exercise to show that the incidence structure so defined is a GQ with parameters

$$s = t = q,\ v = b = (q + 1)(q^2 + 1), \quad \text{when } d = 2,$$

$$s = q,\ t = q^2,\ v = (q + 1)(q^3 + 1),\ b = (q^2 + 1)(q^3 + 1), \quad \text{when } d = 3.$$

If $d = 2$, the GQ is denoted by $T_2(O)$; if $d = 3$, the GQ is denoted by $T_3(O)$. If no confusion is possible, these quadrangles are also denoted by $T(O)$.

**4.3.** *The examples of Ahrens and Szekeres, Hall, Jr., and Payne*

For each prime power $q$, Ahrens and Szekeres [1969] constructed a GQ with order $(q-1, q+1)$. For $q$ even, these examples were found independently by Hall, Jr., [1971]. Then a construction was found by Payne [1971] which included all these examples and for $q$ even produced some additional ones (see also Payne [1972, 1985a] and Payne and Thas [1984]). These examples yield the only known cases, with $s > 1$ and $t > 1$, in which $s$ and $t$ are not powers of the same prime.

**4.3.1.** *The construction of Ahrens and Szekeres [1969] and Hall, Jr., [1971], for $q$ even*

Let $O$ be a hyperoval (see Section 1 of Chapter 7), i.e. a $(q+2)$-arc, of the projective plane PG$(2,q)$, $q = 2^h$, and let PG$(2,q)$ be embedded as a plane in PG$(3,q)$. Define an incidence structure $T_2^*(O)$ by taking for points just those points of PG$(3,q) \setminus$ PG$(2,q)$ and for lines just those lines of PG$(3,q)$ which are not contained in PG$(2,q)$ and meet $O$ (necessarily in a unique point). The incidence is that inherited from PG$(3,q)$. It is evident that the incidence structure so defined is a GQ with parameters

$$s = q-1,\ t = q+1,\ v = q^3,\ b = q^2(q+2).$$

**4.3.2.** *The construction of Ahrens and Szekeres [1969], for $q$ odd*

Let the elements of $P$ be the points of the affine 3-space AG$(3,q)$ over GF$(q)$, $q$ odd. Elements of $B$ are the following curves of AG$(3,q)$:

(i) $x = \sigma,\ y = a,\ z = b,$

(ii) $x = a,\ y = \sigma,\ z = b,$

(iii) $x = c\sigma^2 - b\sigma + a,\ y = -2c\sigma + b,\ z = \sigma.$

Here the parameter $\sigma$ ranges over GF$(q)$ and $a$, $b$, $c$ are arbitrary elements of GF$(q)$. The incidence $I$ is the natural one. Then $(P, B, I)$ is a GQ of order $(q-1, q+1)$. It will be denoted by AS$(q)$.

**4.3.3.** *The construction of Payne [1971]*

Continuing with the same notation as in 1.3, it is clear that

$$|\{x,y\}^\perp| = t+1 \quad \text{and} \quad |\{x,y\}^{\perp\perp}| \leqslant t+1$$

for each pair of noncollinear points $x$, $y$, and

$$|\{x,y\}^\perp| = |\{x,y\}^{\perp\perp}| = s+1$$

for each pair of distinct collinear points $x$, $y$. The *trace* of a pair $(x,y)$ of distinct points is defined to be the set

$$\mathrm{tr}(x,y) = x^\perp \cap y^\perp = \{x,y\}^\perp;$$

the *span* of the pair $(x, y)$ is the set

$$\mathrm{sp}(x,y) = \{x,y\}^{\perp\perp} = \{u \in P\colon\ u \in z^\perp\ \forall z \in x^\perp \cap y^\perp\}.$$

For $x \not\sim y$, $\mathrm{sp}(x,y)$ is also called the *hyperbolic line* defined by $x$ and $y$. If $x \sim y$, $x \neq y$, or if $x \not\sim y$ and $|\{x,y\}^{\perp\perp}| = t+1$, we say that the pair $(x,y)$ is *regular*. The point $x$ is *regular* provided $(x,y)$ is regular for all $y \in P$, $y \neq x$.

Let $x$ be a regular point of the GQ $S = (P, B, I)$ of order $s$, $s > 1$. Then $P'$ is defined to be the set $P \setminus x^\perp$. The elements of $B'$ are of two types: the elements of type (a) are the lines of $B$ which are not incident with $x$; the elements of type (b) are the hyperbolic lines $\{x,y\}^{\perp\perp}$, $y \not\sim x$. Now let us define the incidence $I'$. If $y \in P'$ and $L \in B'$ is of type (a), then $yI'L$ if and only if $yIL$; if $y \in P'$ and $L \in B'$ is of type (b), then $yI'L$ if and only if $y \in L'$. The incidence structure $P(S,x) = (P', B', I')$ so defined is a GQ of order $(s-1, s+1)$.

A quick look at the examples of order $s$ in 3.2 and 4.2 reveals that regular points and regular lines arise in the following cases (see Payne and Thas [1984]): all lines of $Q(4,q)$ are regular; the points of $Q(4,q)$ are regular if and only if $q$ is even; all points of $W(q)$ are regular; the lines of $W(q)$ are regular if and only if $q$ is even; the unique point $(\infty)$ of type (iii) of $T_2(O)$ is regular if and only if $q$ is even; all lines of type (b) of $T_2(O)$ are regular.

It is easily seen that for $q$ even the GQ $P(T_2(O), (\infty))$ is the GQ $T_2^*(\overline{O})$, with $\overline{O}$ the hyperoval containing the oval $O$. Further, $P(W(q), x)$, with $x$ any point of $\mathrm{PG}(3,q)$, has the following nice description: points are the points of $\mathrm{PG}(3,q)$ not in the plane $\pi_x = x^\zeta$, with $\zeta$ the symplectic polarity defining $W(q)$; lines are the lines of $W(q)$ not in $\pi_x$ together with the lines of $\mathrm{PG}(3,q)$ through $x$ but not in $\pi_x$; incidence is the natural one.

### **4.4.** *Generalized quadrangles as group coset geometries*

The following construction method for GQ was introduced by Kantor [1980]; it was motivated by the parabolic subgroup construction of 3.4.

Let $G$ be a finite group of order $s^2t$, $1 < s$, $1 < t$, together with a family

$$J = \{A_i\colon\ 0 \leqslant i \leqslant t\}$$

of $1+t$ subgroups of $G$, each of order $s$. Assume furthermore that for each $A_i \in J$, there exists a subgroup $A_i^*$ of $G$, of order $st$, containing $A_i$. Put $J^* = \{A_i^*\colon\ 0 \leqslant i \leqslant t\}$ and define as follows a point-line geometry $S = (P, B, I) = S(G, J)$.

Points are of three kinds: (i) the elements of $G$; (ii) the right cosets $A_i^*g$, $A_i^* \in J^*$, $g \in G$; (iii) a symbol $(\infty)$.

Lines are of two kinds: (a) the right cosets $A_ig$, $A_i \in J$, $g \in G$; (b) the symbols $[A_i]$, $A_i \in J$.

A point $g$ of type (i) is incident with each line $A_ig$, $A_i \in J$; a point $A_i^*g$ of type (ii) is incident with $[A_i]$ and with each line $A_ih$ contained in $A_i^*g$; the point $(\infty)$ is incident with each line $[A_i]$ of type (b).

Then Kantor [1980] proved that the following holds: $S(G,J)$ is a GQ of order $(s,t)$ provided

K1: $A_iA_j \cap A_k = \{1\}$, for $i$, $j$, $k$ distinct, and
K2: $A_i^* \cap A_j = \{1\}$, for $i \neq j$.

If the conditions K1 and K2 are satisfied, then one easily sees that

$$A_i^* = \bigcup \{A_i g\colon\ A_i g = A_i \text{ or } A_i g \cap A_j = \varnothing \text{ for all } A_j \in J\},$$

so that $A_i^*$ is uniquely defined by $A_i$.

Suppose K1 and K2 are satisfied. For any $h \in G$ let us define $\theta_h$ by

$$g^{\theta_h} = gh, \quad (A_i g)^{\theta_h} = A_i gh,$$

$$(A_i^* g)^{\theta_h} = A_i^* gh, \quad [A_i]^{\theta_h} = [A_i], \quad (\infty)^{\theta_h} = (\infty),$$

with $g \in G$, $A_i \in J$, $A_i^* \in J^*$. Then $\theta_h$ is an automorphism of $S(G,J)$ which fixes the point $(\infty)$ and all lines of type (b). If $G' = \{\theta_h\colon\ h \in G\}$, then clearly $G' \cong G$ and $G'$ acts regularly on the points of type (i).

If K1 and K2 are satisfied, then $J$ is called a 4-*gonal family for* $G$.

Let $F = \mathrm{GF}(q)$, $q$ any prime power. Let $f\colon\ F^2 \times F^2 \to F$ be a fixed symmetric, nonsingular biadditive map. Put $G = \{(\alpha,c,\beta)\colon\ \alpha,\beta \in F^2,\ c \in F\}$. Define a binary operation on $G$ by:

$$(\alpha,c,\beta)\cdot(\alpha',c',\beta') = \bigl(\alpha+\alpha', c+c'+f(\beta,\alpha'), \beta+\beta'\bigr).$$

This makes $G$ into a group whose centre is $C = \{(0,c,0) \in G\colon\ c \in F\}$. Suppose that for each $u \in F$ there is an additive map $\delta_u\colon\ F^2 \to F^2$ and a map $g_u\colon\ F^2 \to F$ for which

$$g_u(\alpha+\beta) - g_u(\alpha) - g_u(\beta) = f(\alpha^{\delta_u},\beta) = f(\beta^{\delta_u},\alpha),$$

for all $\alpha,\beta \in F^2$, $u \in F$. With such a setup, we can define a family of subgroups of $G$ by:

$$A(u) = \bigl\{\bigl(\alpha, g_u(\alpha), \alpha^{\delta_u}\bigr)\colon\ \alpha \in F^2\bigr\}, \quad u \in F,$$

and

$$A(\infty) = \bigl\{(0,0,\beta) \in G\colon\ \beta \in F^2\bigr\}.$$

Then put $J = \{A(u)\colon\ u \in F \cup \{\infty\}\}$ and $J^* = \{A^*(u)\colon\ u \in F \cup \{\infty\}\}$, with $A^*(u) = A(u)C$. So

$$A^*(u) = \bigl\{\bigl(\alpha, c, \alpha^{\delta_u}\bigr)\colon\ \alpha \in F^2\bigr\}, \quad u \in F,$$

and

$$A^*(\infty) = \bigl\{(0,c,\beta) \in G\colon\ \beta \in F^2\bigr\}.$$

Necessary and sufficient conditions were worked out in Payne [1980] (or see Section 10.4 of Payne and Thas [1984]) for $J$ to be a 4-gonal family.

THEOREM 1. *$J$ is a 4-gonal family for $G$ if and only if*

(i) $\delta(u,r)$: $\alpha \mapsto \alpha^{\delta_u} - \alpha^{\delta_r}$ *is bijective for* $u \neq r$,

(ii) $g_u(\alpha) = g_r(\alpha)$, $u \neq r$, *implies* $\alpha = 0$,

(iii) *if $u$, $r$, and $v$ are distinct, then $\gamma = 0$ is the only solution to*

$$g_u\big(\gamma^{\delta^{-1}(u,v)}\big) - g_v\big(\gamma^{\delta^{-1}(u,v)}\big) + g_v\big(-\gamma^{\delta^{-1}(v,r)}\big) - g_r\big(-\gamma^{\delta^{-1}(v,r)}\big) = 0.$$

Let $\mathcal{C} = \{A_u\colon\ u \in F\}$ be a set of $q$ distinct $2 \times 2$-matrices over $F$. Then $\mathcal{C}$ is called a *$q$-clan* provided $A_u - A_r$ is anisotropic whenever $u \neq r$, i.e. $\alpha(A_u - A_r)\alpha^T = 0$ has only the trivial solution $\alpha = (0,0)$. For $A_u \in \mathcal{C}$, put $K_u = A_u + A_u^T$, and then define $g_u(\alpha) = \alpha A_u \alpha^T$ and $\alpha^{\delta_u} = \alpha K_u$ for $\alpha \in F^2$. Then necessarily $f(\alpha,\beta) = \alpha\beta^T$ for all $\alpha,\beta \in F^2$. With $G$, $A(u)$, $A^*(u)$, $J$ as above, the following theorem is a combination of results of Payne [1980, 1985b] and Kantor [1986].

THEOREM 2. *The set $J$ is a 4-gonal family for $G$ if and only if $\mathcal{C}$ is a $q$-clan.*

In particular, let $\mathcal{C} = \{A_u\colon\ u \in F\}$ be a set of $q$ upper triangular $2 \times 2$-matrices over $F$, with

$$A_u = \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix}, \quad x_u, y_u, z_u, u \in F.$$

For $q$ odd, $\mathcal{C}$ is a $q$-clan if and only if

$$-\det(K_u - K_r) = (y_u - y_r)^2 - 4(x_u - x_r)(z_u - z_r) \tag{2}$$

is a nonsquare of $F$ whenever $r,u \in F$, $r \neq u$. For $q$ even, $\mathcal{C}$ is a $q$-clan if and only if

$$\operatorname{tr}\big((x_u + x_r)(z_u + z_r)(y_u + y_r)^{-2}\big) = 1 \tag{3}$$

whenever $r,u \in F$, $r \neq u$. (Here $\operatorname{tr}(w)$ is the trace of $w$ over the prime subfield, so that $\operatorname{tr}(w) = 1$ if and only if $X^2 + X + w = 0$ has no solution in $F$.)

In Thas [1987] it was shown that (2) and (3) are exactly the conditions for the planes

$$x_u X_0 + z_u X_1 + y_u X_2 + X_3 = 0$$

of $\mathrm{PG}(3,q)$ to define a flock of the quadratic cone $X_0X_1 = X_2^2$ (for the definition of flock, see Section 10 of Chapter 7).

THEOREM 3. *To any flock of the quadratic cone of* $\mathrm{PG}(3,q)$ *corresponds a* GQ *of order* $(q^2,q)$.

Now we consider all flocks listed in Section 10.5 of Chapter 7; see also the private communication in 10.8(b) of Chapter 7.

(0) To the linear flocks correspond the classical GQ $H(3,q^2)$. For a proof that these are classical we refer to Payne and Thas [1984].

(1) To the flocks *FTW* correspond GQ first discovered by Kantor [1980]. Kantor used the classical generalized hexagons $H(q)$ to construct these GQ.

(2) The flocks $K1$ were derived by Thas [1987] from $q$-clans discovered by Kantor [1986]. The GQ were first mentioned by Kantor [1986].

(3) The flocks $K2$ were derived by Thas [1987] from $q$-clans discovered by Kantor [1986]. The GQ were first mentioned by Kantor [1986].

(4) The flocks $P1$ were derived by Thas [1987] from $q$-clans discovered by Payne [1985b]. The GQ were first mentioned by Payne [1985b].

(5) The flocks $K3$ and the corresponding $q$-clans were derived by Gevaert and Johnson [1988] from Kantor's 'likeable' planes, using the connections between flocks and translation planes discovered by Thas and Walker (see Thas [1987]) and the connection between flocks and $q$-clans discovered by Thas [1987]. The GQ were first mentioned by Gevaert and Johnson [1988].

(6) The flocks $G$ and the corresponding $q$-clans were derived by Gevaert and Johnson [1988] from some semifield planes of Ganley [1981], using the connections between flocks and translation planes discovered by Thas and Walker (see Thas [1987]) and the connection between flocks and $q$-clans discovered by Thas [1987]. The GQ were first mentioned by Gevaert and Johnson [1988].

(7) The $q$-clans and GQ derived from the flocks $Fi$ were first mentioned by Thas [1987].

(8) The $q$-clans, $q \in \{11, 16, 17, 23\}$, and GQ derived from the flocks $C$ were not yet studied in detail.

The following important theorem on derivation of flocks (see Section 10 of Chapter 7) is due to Payne and Rogers [1990].

THEOREM 4. *The process of derivation produces new flocks and new planes, but never new* GQ.

The last examples of 4-gonal families were deduced by Payne [1988, 1989] from the semifield planes of Ganley [1981], but more than just the ideas of Thas [1987] is required. These are the only examples of GQ known to arise from 4-gonal families but not from $q$-clans via the procedure just described. More about these examples will be said in Section 9.

Let $q = 3^h$ and let $n$ be a nonsquare of GF($q$). With the notations of this section, put

$$g_u(\gamma) = \gamma \begin{pmatrix} u & 0 \\ 0 & -nu \end{pmatrix} \gamma^T + \left( \gamma \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \gamma^T \right)^{1/3}$$

$$+ \left( \gamma \begin{pmatrix} 0 & 0 \\ 0 & -n^{-1}u \end{pmatrix} \gamma^T \right)^{1/9},$$

$$\alpha^{\delta_u} = u\alpha,$$

and

$$f(\alpha,\beta) = \alpha \begin{pmatrix} -1 & 0 \\ 0 & -n \end{pmatrix} \beta^T + \left( \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \beta^T \right)^{1/3} + \left( \alpha \begin{pmatrix} 0 & 0 \\ 0 & n^{-1} \end{pmatrix} \beta^T \right)^{1/9}.$$

Then Payne [1988, 1989] shows that the corresponding set $J$ is a 4-gonal family, thus giving a GQ of order $(q^2, q)$. As these results were obtained while he was visiting the University of Rome, Payne calls these GQ the *Roman generalized quadrangles*.

*The construction of Knarr*

Let $F = \{C_1, C_2, \cdots, C_q\}$ be a flock of the quadratic cone $K$ with vertex $x_0$ of PG$(3, q)$, with $q$ odd. The plane of $C_i$ is denoted by $\pi_i, i = 1, 2, \ldots, q$. Let $K$ be embedded in the nonsingular quadric $Q$ of PG$(4, q)$. The polar line of $\pi_i$ with respect to $Q$ is denoted by $L_i$; let

$$L_i \cap Q = \{x_0, x_i\}, \quad i = 1, 2, \ldots, q.$$

Then no point of $Q$ is collinear with all three of $x_0, x_i, x_j$, $1 \leqslant i < j \leqslant q$. In Bader, Lunardon and Thas [1990] it is proved that it is also true that no point of $Q$ is collinear with all three of $x_i, x_j, x_k$, $0 \leqslant i < j < k \leqslant q$ (see also the description of derivation in Section 10.5 of Chapter 7). Such a set $U$ of $q + 1$ points of $Q$ will be called a BLT-*set* in $Q$, following a suggestion of Kantor [1991]. Since the GQ $Q(4, q)$ arising from $Q$ is isomorphic to the dual of the GQ $W(q)$, to a BLT-set in $Q$ corresponds a set $V$ of $q + 1$ lines of $W(q)$ with the property that no line of $W(q)$ is concurrent with three distinct lines of $V$; such a set $V$ will also be called a BLT-*set*.

To $F$ corresponds a GQ $S$ of order $(q^2, q)$. Knarr [1992] proves that $S$ is isomorphic to the following incidence structure.

Start with a symplectic polarity $\zeta$ of PG$(5, q)$. Let $p \in$ PG$(5, q)$ and let PG$(3, q)$ be a 3-dimensional subspace of PG$(5, q)$ for which $p \notin$ PG$(3, q) \subset p^\zeta$. In PG$(3, q)$ $\zeta$ induces a symplectic polarity $\zeta'$, and hence a GQ $W(q)$. Let $V$ be a BLT-set of the GQ $W(q)$ and construct a geometry $S = (P, B, I)$ as follows.

Points: (i) $p$; (ii) lines of PG$(5, q)$ not containing $p$ but contained in one of the planes $\pi_t = pL_t$, with $L_t$ a line of the BLT-set $V$; (iii) points of PG$(5, q)$ not in $p^\zeta$.

Lines: (a) planes $\pi_t = pL_t$, with $L_t \in V$; (b) totally isotropic planes of $\zeta$ not contained in $p^\zeta$ and meeting some $\pi_t$ in a line (not through $p$).

The incidence relation $I$ is just the natural incidence inherited from PG$(5, q)$.

Then Knarr [1992] proves that $S$ is a GQ of order $(q^2, q)$ isomorphic to the GQ arising from the flock $F$.

**4.5.** *Isomorphisms*

For a proof of the following theorem we refer to Payne [1989] and Payne and Thas [1984].

THEOREM 5.

(a) *The* GQ $T_2(O)$ *is isomorphic to the classical* GQ $Q(4,q)$ *if and only if* $O$ *is an irreducible conic*; *it is isomorphic to* $W(q)$ *if and only if* $q$ *is even and* $O$ *is a conic.*

(b) *The* GQ $T_2^*(O)$ *and* AS$(q)$ *are isomorphic to the respective* GQ $P(T_2(O'), (\infty))$, *with* $O' = O \setminus \{x\}$ *and* $x \in O$, *and* $P(W(q), y)$, *with* $y$ *any point of* $W(q)$.

(c) *The* GQ $T_3(O)$ *is isomorphic to* $Q^-(5,q)$ *if and only if* $O$ *is an elliptic quadric of* PG$(3,q)$.

(d) *Apart from a few small values of* $q$, *no* GQ *in any of the nine classes described in* 4.4 (*excluding the classical* GQ *corresponding to the linear flocks*) *is isomorphic to a previously known* GQ.

REMARK. If $q$ is odd, then the oval $O$ is a conic (see Chapter 7), implying $T_2(O) \cong Q(4,q)$. In such a case $T_2(O)$ is not self-dual. If $q$ is even and $O$ is a conic, then $T_2(O)$, which is isomorphic to $Q(4,q)$, is self-dual. The problem of determining all ovals for which $T_2(O)$ is self-dual has been solved (cf. Eich and Payne [1972], and Payne and Thas [1976, 1984]).

If $q$ is odd, then the ovoid $O$ is an elliptic quadric (see Chapter 7), implying $T_3(O) \cong Q^-(5,q)$.

**4.6.** *Open problems*

(a) Are $Q(4,q)$ and $W(q)$ the only GQ of order $q$, $q$ odd and $q \geqslant 3$?

(b) Is AS$(q)$ the only GQ of order $(q-1, q+1)$, $q$ odd?

(c) Is $H(4,q^2)$ the only GQ of order $(q^2, q^3)$, $q \geqslant 3$?

(d) Are the Roman generalized quadrangles the only GQ arising from 4-gonal families but not from $q$-clans?

(e) Are some of the GQ arising from the flocks $C$ sporadic?

(f) Let $F$ be a flock of the quadratic cone $K$ of PG$(3,q)$, with $q$ even. Give a geometrical construction of the GQ of order $(q^2,q)$ arising from $F$.

## 5. Generalized polygons with small parameters

Since generalized digons are trivial incidence structures and since projective planes were considered in great detail in Chapters 4 and 5, we will consider here only generalized $n$-gons with $n \geqslant 4$.

**5.1.** *Generalized quadrangles with small parameters*

Let $S = (P, B, I)$ be a finite GQ of order $(s, t)$, $1 < s \leqslant t$.

**5.1.1.** $s = 2$
By 2.1 $s + t$ divides $st(s + 1)(t + 1)$, and by 2.2 $t \leqslant s^2$. Hence $t \in \{2, 4\}$.

THEOREM 1. *Up to isomorphism there is only one* GQ *of order* 2 *and only one* GQ *of order* $(2, 4)$.

COROLLARY. *The* GQ $W(2)$ *and* $Q(4, 2)$ *are self-dual and mutually isomorphic. The* GQ AS(3) *is isomorphic to the* GQ $Q^-(5, 2)$.

It is easy to show that the GQ of order 2 is unique. The uniqueness of the GQ of order $(2, 4)$ was proved independently at least five times, by Dixmier and Zara [1976], Seidel [1968], Shult [1972], Thas [1974] and Freudenthal [1975].

*Two interesting models of the* GQ *of order* 2
(1) Let $O$ be a hyperoval, i.e. a 6-arc of PG$(2, 4)$. Points of the GQ are the points of PG$(2, 4)$ not on $O$, lines of the GQ are the lines of PG$(2, 4)$ intersecting $O$, incidence is the natural one.

(2) The following construction was apparently first discovered by Sylvester [1904]. A *duad* is an unordered pair $ij = ji$ of distinct integers from among $1, 2, \ldots, 6$. A *syntheme* is a set $\{ij, kl, mn\}$ of three duads for which $i$, $j$, $k$, $l$, $m$, $n$ are distinct. It is routine to verify that Sylvester's syntheme-duad geometry with duads playing the role of points, synthemes playing the role of lines, and containment as the incidence relation, is the unique GQ of order 2.

*Two interesting models of the* GQ *of order* $(2, 4)$
(1) In Payne and Thas [1984] the following construction of the GQ of order $(2, 4)$ is given. In addition to the duads and synthemes given above, let $1, 2, \ldots, 6$ and $1', 2', \ldots, 6'$ denote twelve additional points, and let $\{i, ij, j'\}$, $1 \leqslant i, j \leqslant 6$, $i \neq j$, denote thirty additional lines. It is easy to verify that the 27 points and 45 lines just constructed yield a representation of the unique GQ of order $(2, 4)$.

(2) Points of the GQ are the 27 lines on a general cubic surface $V$ in PG$(3, \boldsymbol{C})$, lines of the GQ are the 45 tritangent planes of $V$, and incidence is inclusion (for the properties of the general cubic surface in PG$(3, \boldsymbol{C})$ we refer, e.g., to Baker [1921–1934]; see also Hirschfeld [1985] where a full chapter is devoted to cubic surfaces).

**5.1.2.** $s = 3$
Again by 2.1 and 2.2 we have $t \in \{3, 5, 6, 9\}$.

THEOREM 2. *Any* GQ *of order* $(3, 5)$ *must be isomorphic to the* GQ $T_2^*(O)$ *arising from a hyperoval in* PG$(2, 4)$, *any* GQ *of order* $(3, 9)$ *must be isomorphic to* $Q^-(5, 3)$, *and a* GQ *of order* 3 *is isomorphic to either* $W(3)$ *or to its dual* $Q(4, 3)$. *Finally, there is no* GQ *of order* $(3, 6)$.

The uniqueness of the GQ of order $(3, 5)$ was proved by Dixmier and Zara [1976], the uniqueness of the GQ of order $(3, 9)$ was proved independently by Dixmier and Zara [1976] and Cameron (see Payne and Thas [1976]), the determination of all GQ of order 3 is due independently to Dixmier and Zara [1976] and Payne [1975]. Dixmier and Zara [1976] proved that there is no GQ of order $(3, 6)$. All these proofs, some of them simplified or streamlined, are also contained in Payne and Thas [1984].

**5.1.3.** $s = 4$
Using 2.1 and 2.2 it is easy to check that $t \in \{4, 6, 8, 11, 12, 16\}$. Nothing is known about $t = 11$ or $t = 12$. In the other cases unique examples are known, but the uniqueness question is settled only in the case $t = 4$.

THEOREM 3. *A* GQ *of order* 4 *must be isomorphic to* $W(4)$.

The proof of this fact that appears in Payne and Thas [1984] is that of Payne [1977], with a gap filled in by Tits.

**5.2.** *Generalized hexagons with small parameters*

Let $S = (P, B, I)$ be a finite generalized hexagon of order $(s, t)$, $s \leqslant t$. By 1.3 the generalized hexagons of order $(1, t)$, $t > 1$, correspond to the projective planes of order $t$. Now suppose $s = 2$. By 2.1 and 2.2 we have $t \in \{2, 8\}$.

THEOREM 4 (Cohen and Tits [1985]). *Any generalized hexagon of order 2 must be isomorphic to either the generalized hexagon* $H(2)$ *or to its dual. Any generalized hexagon of order* $(2, 8)$ *must be isomorphic to the dual of* $H(8, 2)$.

**5.3.** *Open problems*

(a) Is there a unique GQ of order $(4, t)$, $t \in \{6, 8, 16\}$?
(b) Is there a unique generalized hexagon of order 3?
(c) Does there exist a GQ of order 6?

## 6. Ovoids, spreads, polarities and subpolygons

Again we will consider only finite generalized $n$-gons with $n \geqslant 4$.

**6.1.** *Ovoids and spreads*

An *ovoid* of the finite GQ $S = (P, B, I)$ is a set $O$ of points of $S$ such that each line of $S$ is incident with a unique point of $O$. Dually, a *spread* of $S$ is a set $R$ of lines of $S$ such that each point of $S$ is incident with a unique line of $R$. It is trivial that a GQ with $s = 1$ or $t = 1$ has ovoids and spreads. The following theorem is easy to prove.

THEOREM 1. *If $O$ is an ovoid of the* GQ $S$ *of order* $(s, t)$, *then* $|O| = 1 + st$; *dually, if* $R$ *is a spread of the* GQ $S$ *of order* $(s, t)$, *then* $|R| = 1 + st$.

Let $S = (P, B, I)$ be a finite generalized hexagon, briefly denoted by GH, of order $s$. An *ovoid* of $S$ is a set $O$ of points of $S$, any two at distance 6 in the incidence graph of $S$, such that each point of $P \setminus O$ is collinear with a unique point of $O$. A *spread* of $S$ is defined dually. The following theorem is easy to prove.

THEOREM 2. *If $O$ is an ovoid of the* GH $S$ *of order* $s$, *then* $|O| = s^3 + 1$; *dually, if $R$ is a spread of the* GH $S$ *of order* $s$, *then* $|R| = s^3 + 1$.

**6.2.** *Ovoids and spreads of generalized quadrangles*

THEOREM 3 (Payne and Thas [1984]).

(i) *The* GQ $Q(4, q)$ *always has ovoids. It has spreads if and only if $q$ is even.*

(ii) *The* GQ $Q^-(5, q)$ *has spreads but no ovoids.*

(iii) *The* GQ $H(4, q^2)$ *has no ovoid.*

EXAMPLES.

(a) Consider the GQ $Q(4, q)$ in PG$(4, q)$, and let PG$(3, q)$ be a hyperplane of PG$(4, q)$ which intersects the quadric $Q$ in an elliptic quadric $O$. Then $O$ is an ovoid of $Q(4, q)$.

(b) Since $Q(4, q)$, with $q$ even, is self-dual, any ovoid of $Q(4, q)$ defines a spread of $Q(4, q)$.

(c) Let $U$ be a nonsingular Hermitian curve on the Hermitian variety defining the GQ $H(3, q^2)$. Then $U$ is an ovoid of $H(3, q^2)$. Since $H(3, q^2)$ is isomorphic to the dual of $Q^-(5, q)$, it follows that $Q^-(5, q)$ admits a spread.

Concerning spreads of $H(4, q^2)$ we have just one partial result.

THEOREM 4 (Brouwer [1981]). $H(4, 4)$ *has no spread.*

Using the Klein correspondence (see Section 7 of Chapter 2), it is shown in Thas [1972a] that for $q$ even, to each spread of $W(q)$ (and so, to each ovoid of $Q(4, q)$) corresponds an ovoid of PG$(3, q)$, and conversely. The spread is regular if and only if the ovoid is an elliptic quadric. The spreads corresponding to the Tits ovoids were first discovered by Lüneburg [1965]. In Kantor [1982], for any odd $q$ with $q$ not a prime, a nonregular spread of $W(q)$ is constructed; see also Thas and Payne [to appear] and Thas [to appear].

In Thas [1972a] it is shown that any ovoid of $W(q)$, $q$ even, is an ovoid of PG$(3, q)$. Conversely, any ovoid of PG$(3, q)$, $q$ even, is an ovoid of some $W(q)$ (cf. Section 3.2 of Chapter 7).

Several constructions of spreads of $Q^-(5, q)$ are given in Thas [1983]. It is also shown there that to each spread of $Q^-(5, q)$ corresponds a semipartial geometry (cf. Chapter 10).

Parts (i), (ii), (iii) of the following theorem are due to Payne and Thas [1984]; part (iv) to Thas and Payne [to appear].

THEOREM 5.

(i) *The* GQ $T_2(O)$ *always has an ovoid.*

(ii) *The* GQ $T_3(O)$ *has no ovoid but always has spreads.*

(iii) *The* GQ $P(S,x)$ *always has spreads. It has an ovoid if and only if* $S$ *has an ovoid containing* $x$.

(iv) *Each* GQ *arising from a flock has an ovoid.*

REMARK. Clearly the lines of type (b) of $P(S,x)$, i.e. the pointsets $\{x,y\}^{\perp\perp} \setminus \{x\}$ of $S$, form a spread of $P(S,x)$. Recently Payne and Thas, cf. Payne [1990], discovered a new class of spreads of the GQ $P(T_2(O),(\infty))$ of order $(q-1,q+1)$, with $q$ even and $O$ a conic.

In the construction of Knarr, see Section 4.4, let $\pi$ be a plane in $p^\zeta$ through $p$, but with $\pi \cap \mathrm{PG}(3,q)$ skew to all the lines of the BLT-set $V$. Further, let $\overline{\mathrm{PG}(3,q)}$ be a 3-dimensional subspace of $\mathrm{PG}(5,q)$, with $\pi \subset \overline{\mathrm{PG}(3,q)} \not\subset p^\zeta$. Then $(\overline{\mathrm{PG}(3,q)} \setminus \pi) \cup \{p\}$ is an ovoid of the GQ $S$ arising from the flock $F$ defining $V$.

Finally, Thas and Payne [to appear] show that any Roman GQ has spreads.

### 6.3. *Ovoids and spreads of generalized hexagons*

Consider the classical GH $H(q)$ of order $q$ embedded in the nonsingular quadric $Q$ of $\mathrm{PG}(6,q)$. Let $\mathrm{PG}(5,q)$ be a hyperplane of $\mathrm{PG}(6,q)$ such that $\mathrm{PG}(5,q) \cap Q$ is an elliptic quadric $Q^-$. Then it is shown in Thas [1980a] that the lines of $H(q)$ on $Q^-$ constitute a spread of both the GQ $Q^-(5,q)$ and the GH $H(q)$. Further, it is shown in Thas [1981a] that $O$ is an ovoid of $H(q)$ if and only if $O$ is an ovoid of the polar space $Q(6,q)$. So we have the following theorem (cf. Section 9 of Chapter 7 for results on ovoids of $Q(6,q)$).

THEOREM 6. *The* GH $H(q)$ *always has a spread. It has an ovoid if and only if* $Q(6,q)$ *has an ovoid. In particular* $H(q)$, *with* $q$ *even, has no ovoid, and* $H(q)$, *with* $q = 3^h$, *has an ovoid.*

REMARK. Let $x$ be any point of the GH $H(q)$. Then the $q+1$ lines of $H(q)$ through $x$ all lie in a plane $\pi_x$. Now let $O$ be an ovoid of $H(q)$. It is clear that the $q^3+1$ planes $\pi_x$ corresponding to the points of $O$ are mutually skew. Hence they constitute a spread of the polar space $Q(6,q)$.

### 6.4. *Polarities*

THEOREM 7 (Payne [1968]). *If the* GQ $S = (P,B,I)$ *of order* $s$ *admits a polarity, then either* $s = 1$ *or* $2s$ *is a square. Also, the set of all absolute points of a polarity* $\theta$ *of* $S$ *is an ovoid of* $S$, *and the set of all absolute lines of* $\theta$ *is a spread of* $S$.

Tits [1962] shows that $W(q)$ admits a polarity if and only if $q = 2^{2e+1}$. The corresponding ovoids are the Tits ovoids and the corresponding spreads are the Lüneburg spreads. For more details, we refer to Section 3.2 of Chapter 7 and to Chapter 5.

The first part of the next theorem on polarities of GH is due to Cameron, Thas and Payne [1976], the second part to Ott [1981].

THEOREM 8.

(i) *If $\theta$ is a polarity of the* GH $S$ *of order* $s$*, then the set of absolute points of* $\theta$ *is an ovoid of* $S$*, and the set of absolute lines of* $\theta$ *is a spread of* $S$.

(ii) *If the* GH $S$ *of order* $s$ *admits a polarity, then either* $s = 1$ *or* $3s$ *is a square.*

It is easy to show that the GH of order 1 admits a polarity, and in Section 3.3 it was mentioned that $H(q)$ admits a polarity if and only if $q = 3^{2h+1}$, $h \geqslant 0$.

Finally, Thas [1980a] proves that the spread of $H(q)$ described in the first paragraph of Section 6.3 is never a spread arising from a polarity of $H(q)$.

**6.5.** *Subpolygons*

The generalized $n$-gon $S' = (P', B', I')$ is called a *sub-$n$-gon* of the finite generalized $n$-gon $S = (P, B, I)$ if and only if $P' \subseteq P$, $B' \subseteq B$ and $I' = I \cap ((P' \times B') \cup (B' \times P'))$. If $S' \neq S$, we say that $S'$ is a *proper* sub-$n$-gon of $S$.

THEOREM 9 (Payne and Thas [1984]). *Let* $S' = (P', B', I')$ *be a proper subquadrangle of order* $(s', t')$ *of the finite* GQ $S = (P, B, I)$ *of order* $(s, t)$. *Then either* $s = s'$ *or* $s \geqslant s't'$. *If* $s = s'$ *and if* $x \in P \setminus P'$*, then* $x$ *is collinear with the* $1 + st'$ *points of an ovoid of* $S'$; *if* $s = s't'$ *and if* $x \in P$ *is incident with no line of* $B'$*, then* $x$ *is collinear with exactly* $1 + s'$ *points of* $S'$. *The dual holds, similarly.*

The next results are easy consequences of Theorem 9, although they first appeared in Thas [1972b].

THEOREM 10. *Let* $S' = (P', B', I')$ *be a proper subquadrangle of the* GQ $S = (P, B, I)$, *with* $S$ *having order* $(s, t)$ *and* $S'$ *having order* $(s, t')$*, i.e.* $s = s'$ *and* $t > t'$. *Then we have*:

(i) $t \geqslant s$; *if* $t = s$*, then* $t' = 1$.

(ii) *If* $s > 1$*, then* $t' \leqslant s$; *if* $t' = s \geqslant 2$*, then* $t = s^2$.

(iii) *If* $s = 1$*, then* $1 \leqslant t' < t$ *is the only restriction on* $t'$.

(iv) *If* $s > 1$ *and* $t' > 1$*, then* $\sqrt{s} \leqslant t' \leqslant s$*, and* $s^{3/2} \leqslant t \leqslant s^2$.

(v) *If* $t = s^{3/2} > 1$ *and* $t' > 1$*, then* $t' = \sqrt{s}$.

(vi) *Let* $S'$ *have a proper subquadrangle* $S''$ *of order* $(s, t'')$, $s > 1$. *Then* $t'' = 1$, $t' = s$*, and* $t = s^2$.

EXAMPLES. Here we shall describe some of the known subquadrangles of the finite classical GQ. Examples of subquadrangles of nonclassical GQ are contained in Payne and Thas [1984].

(a) Consider $Q^-(5, q)$ and intersect $Q^-$ with a nontangent hyperplane $\mathrm{PG}(4, q)$. Then the points and lines of $Q' = Q^- \cap \mathrm{PG}(4, q)$ form the GQ $Q'(4, q)$. Here $s^2 = t = q^2$,

$s = s' = t'$, so that $t = s't'$. Since all lines of $Q'(4, q)$ are regular (cf. Section 4.3.3), $Q^-(5, q)$ and $Q'(4, q)$ have subquadrangles with $t'' = 1$ and $s'' = s' = s$.

(b) Similarly, consider $H(4, q^2)$, with $H$ a nonsingular Hermitian variety of PG$(4, q^2)$. Intersect $H$ with a nontangent hyperplane PG$(3, q^2)$. Then the points and lines of $H' = H \cap \mathrm{PG}(3, q^2)$ form the GQ $H'(3, q^2)$. Here $t = s^{3/2} = q^3$, $s = s'$, $t' = \sqrt{s}$, and again $t = s't'$. Since all points of $H'(3, q^2)$ are regular, $H'(3, q^2)$ has subquadrangles with $t'' = t' = \sqrt{s}$ and $s'' = 1$.

(c) Now consider $Q(4, q)$ and extend GF$(q)$ to GF$(q^2)$. Then $Q$ extends to $\overline{Q}$ and $Q(4, q)$ to $\overline{Q}(4, q^2)$. Here $Q(4, q)$ is a subquadrangle of $\overline{Q}(4, q^2)$, and we have $t = s = q^2$ and $t' = s' = q$. Hence $t = s = s't'$.

A theorem which appears to be very useful for several characterization theorems is the following.

THEOREM 11 (Thas [1972b]). *Let $S' = (P', B', I')$ be a substructure of the* GQ $S = (P, B, I)$ *of order* $(s, t)$ *for which the following conditions are satisfied*:

(i) *if $x, y \in P'$, $x \neq y$, and $xILIy$, then $L \in B'$*;

(ii) *each element of $B'$ is incident with $1 + s$ elements of $P'$.*

*Then there are four possibilities*:

(a) *$S'$ is a dual grid* (*and then $s = 1$*);

(b) *the elements of $B'$ are lines which are incident with a distinguished point of $P$, and $P'$ consists of those points of $P$ which are incident with these lines*;

(c) *$B' = \varnothing$ and $P'$ is a set of pairwise noncollinear points of $P$*;

(d) *$S'$ is a subquadrangle of order $(s, t')$.*

The following theorem gives all possibilities for the substructure $S_\theta$ of the fixed elements of an automorphism $\theta$ of the GQ $S$.

THEOREM 12 (Payne and Thas [1984]). *The substructure $S_\theta = (P_\theta, B_\theta, I_\theta)$ of the fixed elements of an automorphism $\theta$ of the* GQ $S = (P, B, I)$ *of order* $(s, t)$, *is given by at least one of the following*:

(i) *$B_\theta = \varnothing$ and $P_\theta$ is a set of pairwise noncollinear points*;

(i)$'$ *$P_\theta = \varnothing$ and $B_\theta$ is a set of pairwise nonconcurrent lines*;

(ii) *$P_\theta$ contains a point $x$ such that $x \sim y$ for every point $y \in P_\theta$ and each line of $B_\theta$ is incident with $x$*;

(ii)$'$ *$B_\theta$ contains a line $L$ such that $L \sim M$ for every line $M \in B_\theta$ and each point of $P_\theta$ is incident with $L$*;

(iii) *$S_\theta$ is a grid*;

(iii)$'$ *$S_\theta$ is a dual grid*;

(iv) *$S_\theta$ is a subquadrangle of order $(s', t')$, $s' \geqslant 2$ and $t' \geqslant 2$.*

The basic inequality concerning subhexagons is due to Thas.

THEOREM 13 (Thas [1976a]). *If $S' = (P', B', I')$ is a proper subhexagon of order $(s', t')$ of the generalized hexagon $S = (P, B, I)$ of order $(s, t)$, then $st \geqslant s'^2 t'^2$. If $s > s'$ and $st = s'^2 t'^2$, and if $x \in P$ is collinear with no point of $P'$, then there are exactly $1 + t'$ lines of $B'$ at distance 3 (in the incidence graph of $S$) from $x$; the dual holds, similarly.*

Finally, we have the following theorem of Thas on suboctagons.

THEOREM 14 (Thas [1979]). *Let $S' = (P', B', I')$ be a proper suboctagon of order $(s', t')$ of the generalized octagon $S = (P, B, I)$ of order $(s, t)$. Then*:

(i) *either $t = t'$, or*

$$s^3(s+1)t^2 + s\big(s^2 + s'^4 t'^3 - ss'(1+t')\big(1 + s't' + s'^2 t'^2\big)\big)t + (s+1)s'^4 t'^3 \geqslant 0;$$

(ii) *if $s = s'$, then there are the following possibilities*
(a) *$s = 1$ and $t \geqslant t'^2$,*
(b) *$t' = 1$ and $t > s$.*

*The dual holds, similarly.*

REMARK. In Thas [1979] the possibility $s = s'$, $t = s^2$, $s = t'^2$ also is mentioned. But as 2st is a square, see 2.1, this case cannot occur. This was pointed out to us by Van Maldeghem.

**6.6.** *Open problems*

(a) Classify all ovoids of $Q(4, q)$.
(b) Classify all spreads of $Q^-(5, q)$.
(c) Does $H(4, q^2), q > 2$, have a spread?
(d) Does $H(q), q$ odd and $q \neq 3^h$, have an ovoid?
(e) Is any spread of $H(q)$, $q \neq 3^{2h+1}$, a spread of some $Q^-(5, q)$?
(f) Improve (i) of Theorem 14.

## 7. Generalized quadrangles in finite projective and affine spaces

**7.1.** *Generalized quadrangles in finite projective spaces*

A (finite) *projective* GQ $S = (P, B, I)$ is a GQ for which $P$ is a subset of the pointset of some projective space PG$(d, q)$, $B$ is a set of lines of PG$(d, q)$, $P$ is the union of all members of $B$, and the incidence relation $I$ is the one induced by that of PG$(d, q)$. We also say that $S$ is *embedded* in PG$(d, q)$. If PG$(d', q)$ is the subspace of PG$(d, q)$ generated by all points of $P$, then we say that PG$(d', q)$ is the *ambient space* of $S$.

The following fundamental and beautiful theorem is due to Buekenhout and Lefèvre.

THEOREM 1 (Buekenhout and Lefèvre [1974]). *A projective* GQ $S = (P, B, I)$ *with ambient space* $\mathrm{PG}(d, q)$ *must be obtained in one of the following ways*:

(i) *there is a nonsingular quadric* $Q$ *of Witt index* 2 *in* $\mathrm{PG}(d, q)$, $d = 3, 4$ *or* 5, *such that* $P$ *is the set of points of* $Q$ *and* $B$ *is the set of lines on* $Q$;

(ii) *there is a nonsingular Hermitian variety* $H$ *in* $\mathrm{PG}(d, q^2)$, $d = 3$ *or* 4, *such that* $P$ *is the set of points of* $H$ *and* $B$ *is the set of lines on* $H$;

(iii) $d = 3$, $P$ *is the set of all points of* $\mathrm{PG}(3, q)$ *and* $B$ *is the set of all totally isotropic lines with respect to some symplectic polarity of* $\mathrm{PG}(3, q)$.

Hence $S$ must be one of the classical examples described in 3.2.

REMARK. *Weak projective* GQ were considered by Lefèvre-Percsy [1981, 1982]. Here the lines of the GQ are subsets of the lines of $\mathrm{PG}(d, q)$.

**7.2.** *Generalized quadrangles in finite affine spaces*

We say that the GQ $S = (P, B, I)$ is *embedded* in the finite affine space $\mathrm{AG}(d, q)$ if $P$ is a subset of the pointset of $\mathrm{AG}(d, q)$, $B$ is a set of lines of $\mathrm{AG}(d, q)$, $P$ is the union of all members of $B$, and the incidence relation $I$ is the one induced by that of $\mathrm{AG}(d, q)$. If $\mathrm{AG}(d', q)$ is the subspace of $\mathrm{AG}(d, q)$ generated by all points of $P$, then we say that $\mathrm{AG}(d', q)$ is the *ambient space* of $S$. All GQ embedded in $\mathrm{AG}(d, q)$ were determined by Thas [1978a]; the theorem on the embedding in $\mathrm{AG}(3, q)$ was proved independently by Bichara [1978].

We note that in contrast with the projective case, there arise five nontrivial ‘sporadic’ cases in the finite affine case.

THEOREM 2. *If the* GQ $S$ *of order* $(s, t)$ *is embedded in* $\mathrm{AG}(2, s + 1)$, *then the lineset of* $S$ *is the union of two parallel classes of the plane and the pointset of* $S$ *is the pointset of the plane.*

THEOREM 3. *Suppose that the* GQ $S = (P, B, I)$ *of order* $(s, t)$ *is embedded in* $\mathrm{AG}(3, s + 1)$, *and that* $P$ *is not contained in a plane of* $\mathrm{AG}(3, s + 1)$. *Then one of the following cases must occur*:

(i) $s = 1$, $t = 2$ (*trivial case*);

(ii) $t = 1$ *and the elements of* $S$ *are the affine points and affine lines of a hyperbolic quadric of* $\mathrm{PG}(3, s + 1)$, *the projective completion of* $\mathrm{AG}(3, s + 1)$, *which is tangent to the plane at infinity of* $\mathrm{AG}(3, s + 1)$;

(iii) $P$ *is the pointset of* $\mathrm{AG}(3, s + 1)$ *and* $B$ *is the set of all lines of* $\mathrm{AG}(3, s + 1)$ *whose points at infinity are the points of a hyperoval* $O$ *of the plane at infinity of* $\mathrm{AG}(3, s + 1)$, *i.e.* $S = T_2^*(O)$ (*here* $s + 1 = 2^h$ *and* $t = s + 2$);

(iv) $P$ *is the pointset of* $\mathrm{AG}(3, s + 1)$ *and* $B = B_1 \cup B_2$, *where* $B_1$ *is the set of all affine totally isotropic lines with respect to a symplectic polarity* $\theta$ *of the projective completion* $\mathrm{PG}(3, s + 1)$ *of* $\mathrm{AG}(3, s + 1)$ *and where* $B_2$ *is the class of parallel* *lines defined by the pole* $x$ (*the image with respect to* $\theta$) *of the plane at infinity of* AG$(3, s+1)$, *i.e.* $S = P(W(s+1), x)$ (*here* $t = s+2$);

(v) $s = t = 2$, *and up to a collineation of the space* AG$(3,3)$ *there is just one embedding of a* GQ *of order* 2 *in* AG$(3,3)$.

*The embedding of the* GQ *of order* 2 *in* AG$(3,3)$

Let $\omega$ be a plane of AG$(3,3)$ and let $\{L_0, L_1, L_2\}$ and $\{M_x, M_y, M_z\}$ be two classes of parallel lines of $\omega$. Suppose that $\{x_i\} = M_x \cap L_i$, $\{y_i\} = M_y \cap L_i$, and $\{z_i\} = M_z \cap L_i$, $i = 0, 1, 2$. Further, let $N_x$, $N_y$, $N_z$ be three lines containing $x_0$, $y_0$, $z_0$, respectively, such that $N_x \notin \{M_x, L_0\}$, $N_y \notin \{M_y, L_0\}$, $N_z \notin \{M_z, L_0\}$, such that the planes $N_xM_x$, $N_yM_y$, $N_zM_z$ are parallel, and such that the planes $\omega$, $L_0N_x$, $L_0N_y$, $L_0N_z$ are distinct. The points of $N_x$ are $x_0$, $x_3$, $x_4$; the points of $N_y$ are $y_0$, $y_3$, $y_4$; and the points of $N_z$ are $z_0$, $z_3$, $z_4$; where notation is chosen in such a way that $x_3$, $y_3$, $z_3$, and also $x_4$, $y_4$, $z_4$, are collinear. Then the points of the GQ are

$$x_0, \ldots, x_4, y_0, \ldots, y_4, z_0, \ldots, z_4,$$

and the lines are

$$L_0, L_1, L_2, M_x, M_y, M_z, N_x, N_y, N_z, x_3y_4, x_4y_3, x_3z_4, x_4z_3, y_3z_4, y_4z_3.$$

THEOREM 4. *Suppose that the* GQ $S = (P, B, I)$ *of order* $(s, t)$ *is embedded in* AG$(4, s+1)$ *and that* $P$ *is not contained in an* AG$(3, s+1)$. *Then one of the following cases must occur*:

(i) $s = 1$, $t \in \{2, 3, 4, 5, 6, 7\}$ (*trivial case*);

(ii) $s = t = 2$, *and up to a collineation of the space* AG$(4,3)$ *there is just one embedding of the* GQ *with* 15 *points and* 15 *lines in* AG$(4,3)$ (*so that the ambient space is* AG$(4,3)$);

(iii) $s = t = 3$, $S$ *is isomorphic to the* GQ $Q(4,3)$, *and up to a collineation* (*whose companion automorphism is the identity*) *of the space* AG$(4,4)$ *there is just one embedding of a* GQ *of order* 3 *in* AG$(4,4)$;

(iv) $s = 2$, $t = 4$, *and up to a collineation of the space* AG$(4,3)$ *there is just one embedding of the* GQ *with* 27 *points and* 45 *lines in* AG$(4,3)$.

*The embedding of the* GQ *of order* 2 *in* AG$(4,3)$ (*with ambient space* AG$(4,3)$)

Let PG$(3,3)$ be the hyperplane at infinity of AG$(4,3)$; let $\omega_\infty$ be a plane of PG$(3,3)$, and let $l$ be a point of PG$(3,3) \setminus \omega_\infty$. In $\omega_\infty$ choose points $m_{01}$, $m_{02}$, $m_{11}$, $m_{12}$, $m_{21}$, $m_{22}$ in such a way that $m_{01}$, $m_{21}$, $m_{11}$ are collinear, that $m_{11}$, $m_{02}$, $m_{22}$ are collinear, that $m_{21}$, $m_{02}$, $m_{12}$ are collinear, and that $m_{01}$, $m_{22}$, $m_{12}$ are collinear. Let $L$ be an affine line containing $l$, and let the affine points of $L$ be denoted by $p_0$, $p_1$, $p_2$. The points of the GQ are the affine points of the lines

$$p_0m_{01}, p_0m_{02}, p_1m_{11}, p_1m_{12}, p_2m_{21}, p_2m_{22}.$$

The lines of the GQ are the affine lines of the (2-dimensional) hyperbolic quadric containing $p_0m_{01}$, $p_1m_{11}$, $p_2m_{21}$, respectively, $p_0m_{02}$, $p_1m_{11}$, $p_2m_{22}$, respectively, $p_0m_{02}$, $p_1m_{12}$, $p_2m_{21}$, and, respectively, $p_0m_{01}$, $p_1m_{12}$, $p_2m_{22}$.

*The embedding of a* GQ *of order* 3 *in* AG(4, 4)
Let PG(3, 4) be the hyperplane at infinity of AG(4, 4), let $\omega_\infty$ be a plane of PG(3, 4), let $H$ be a Hermitian curve of $\omega_\infty$, and let $l$ be a point of PG(3, 4) $\setminus \omega_\infty$. In $\omega_\infty$ there are exactly four triangles $m_{i1}m_{i2}m_{i3}$, $i = 0, 1, 2, 3$, whose vertices are exterior points of $H$ and whose sides are secants (nontangents) of $H$. Any line $m_{0a}m_{1b}$, with $a, b \in \{1, 2, 3\}$, contains exactly one vertex $m_{2c}$ of $m_{21}m_{22}m_{23}$ and one vertex $m_{3d}$ of $m_{31}m_{32}m_{33}$, and the cross-ratio $\{m_{0a}, m_{1b}; m_{2c}, m_{3d}\}$ is independent of the choice of $a, b \in \{1, 2, 3\}$. Let $L$ be an affine line through $l$, and let $p_0, p_1, p_2, p_3$ be the affine points of $L$, where notation is chosen in such a way that

$$\{p_0, p_1; p_2, p_3\} = \{m_{0a}, m_{1b}; m_{2c},\ m_{3d}\}.$$

The points of the GQ are the 40 affine points of the lines $p_i m_{ij}$, $i = 0, 1, 2, 3$, $j = 1, 2, 3$. The lines of the GQ are the affine lines of the (2-dimensional) hyperbolic quadric containing $p_0m_{0a}$, $p_1m_{1b}$, $p_2m_{2c}$, $p_3m_{3d}$, with $a, b = 1, 2, 3$.

*The embedding of the* GQ *of order* (2, 4) *in* AG(4, 3)
Let PG(3, 3) be the hyperplane at infinity of AG(4, 3), let $\omega_\infty$ be a plane of PG(3, 3), and let $l$ be a point of PG(3, 3)$\setminus\omega_\infty$. In $\omega_\infty$ choose points $m$, $n_x$, $n_y$, $n_z$, $n'_x$, $n'_y$, $n'_z$, $n''_x$, $n''_y$, $n''_z$, in such a way that $m$, $n_x$, $n_y$, $n_z$ are collinear, that $m$, $n'_x$, $n'_y$, $n'_z$ are collinear, that $m$, $n''_x$, $n''_y$, $n''_z$ are collinear, and that $n_a$, $n'_b$, $n''_c$ with $\{a, b, c\} = \{x, y, z\}$ are collinear. Let $L$ be an affine line through $l$, and let $x$, $y$, $z$ be the affine points of $L$. The plane defined by $L$ and $m$ is denoted by $\omega$. The points of the GQ are the 27 affine points of the lines $am$, $an_a$, $an'_a$, $an''_a$, with $a = x, y, z$. The 45 lines of the GQ are the affine lines of $\omega$ having as point at infinity either $l$ or $m$, the affine lines of the (2-dimensional) hyperbolic quadric containing $am$, $bn_b$, $cn_c$, respectively, $am$, $bn'_b$, $cn'_c$, respectively, $am$, $bn''_b$, $cn''_c$, and, respectively, $an_a$, $bn'_b$, $cn''_c$, always with $\{a, b, c\} = \{x, y, z\}$.

THEOREM 5. *Suppose that the* GQ $S = (P, B, I)$ *of order* $(s, t)$ *is embedded in* AG($d$, $s + 1$), $d \geqslant 5$, *and that* $P$ *is not contained in any hyperplane* AG$(d - 1, s + 1)$. *Then one of the following cases must occur*:

(i) $s = 1$ *and* $t \in \{[d/2], \ldots, 2^{d-1} - 1\}$, *with* $[d/2]$ *the greatest integer less than or equal to* $d/2$ (*trivial case*);

(ii) $d = 5$, $s = 2$, $t = 4$, *and up to a collineation of the space* AG(5, 3) *there is just one embedding of the* GQ *with* 27 *points and* 45 *lines in* AG(5, 3) (*so that the ambient space is* AG(5, 3)).

*The embedding of the* GQ *of order* (2, 4) *in* AG(5, 3) (*with ambient space* AG(5, 3))
Let PG(4, 3) be the hyperplane at infinity of AG(5, 3), let $H_\infty$ be a hyperplane of PG(4, 3) and let $l$ be a point of PG(4, 3) $\setminus H_\infty$. In $H_\infty$ choose points $m_x$, $m_y$, $m_z$, $n_x$, $n_y$, $n_z$, $n'_x$, $n'_y$, $n'_z$, $n''_x$, $n''_y$, $n''_z$ in such a way that $m_x$, $m_y$, $m_z$ are collinear, that $m_x$, $m_y$, $m_z$, $n_x$, $n_y$, $n_z$ are in a plane $\omega_\infty$, that $m_x$, $m_y$, $m_z$, $n'_x$, $n'_y$, $n'_z$ are in a plane $\omega'_\infty$, that $m_x$, $m_y$, $m_z$, $n''_x$, $n''_y$, $n''_z$ are in a plane $\omega''_\infty$, that $m_a$, $n_b$, $n_c$ are collinear, that $m_a$, $n'_b$, $n'_c$ are collinear, that $m_a$, $n''_b$, $n''_c$ are collinear, and that $n_a$, $n'_b$, $n''_c$ are collinear, always with $\{a, b, c\} = \{x, y, z\}$. Let $L$ be an affine line through $l$, and let $x$, $y$, $z$ be

the affine points of $L$. The points of the GQ are the 27 affine points of the lines $am_a$, $an_a$, $an'_a$, $an''_a$, with $a = x, y, z$. The 45 lines of the GQ are the affine lines of the (2-dimensional) hyperbolic quadric containing $xm_x$, $ym_y$, $zm_z$, respectively, $am_a$, $bn_b$, $cn_c$, respectively, $am_a$, $bn'_b$, $cn'_c$, respectively, $am_a$, $bn''_b$, $cn''_c$, and, respectively, $an_a$, $bn'_b$, $cn''_c$, always with $\{a, b, c\} = \{x, y, z\}$.

## 8. Combinatorial characterizations of the finite classical generalized quadrangles and hexagons

*Introduction*

In this section we review the most important combinatorial characterizations of the finite classical generalized polygons. Several of these theorems appeared to be very useful and were important tools in the proofs of certain results concerning strongly regular graphs, coding theory, the classification of collineation groups in projective spaces, etc.

In the first part characterizations of the classical GQ $W(q)$ and $Q(4, q)$ are given. The second part will contain characterizations of $Q^-(5, q)$ and $H(3, q^2)$. Next characterizations of $H(4, q^2)$ are given. Then there is a section with conditions characterizing several classical GQ at the same time. Next we have two characterizations of all classical GQ and their duals, and the section on GQ ends with references on combinatorial characterizations of nonclassical GQ. Detailed proofs of Theorems 1 to 23, and of Theorem 25, of this section can be found in Payne and Thas [1984].

Finally, important combinatorial characterizations of the finite classical generalized hexagons are given.

### 8.1. *Characterizations of $W(q)$ and $Q(4, q)$*

Let $S = (P, B, I)$ be a finite GQ of order $(s, t)$. If $x \sim y$, $x \neq y$, or if $x \not\sim y$ and $|\{x, y\}^{\perp\perp}| = t + 1$, we say the pair $(x, y)$ is *regular*. The point $x$ is *regular* provided $(x, y)$ is regular for all $y \in P \setminus \{x\}$. A point $x$ is *coregular* provided each line incident with $x$ is regular. The pair $(x, y)$, $x \not\sim y$, is *antiregular* provided

$$\left|z \cap \{x, y\}^{\perp}\right| \leqslant 2 \quad \text{for all} \quad z \in P \setminus \{x, y\}.$$

A point $x$ is *antiregular* provided $(x, y)$ is antiregular for all $y \in P \setminus x^{\perp}$.

A *triad* (of points) is a triple of pairwise noncollinear points. Given a triad $T$, a *centre* of $T$ is just a point of $T^{\perp}$.

The *closure* of the pair $(x, y)$ is $\mathrm{cl}(x, y) = \{z \in P\colon\ z^{\perp} \cap \{x, y\}^{\perp\perp} \neq \varnothing\}$.

THEOREM 1. *Let $S = (P, B, I)$ be a* GQ *of order $s > 1$.*

(a) *For a regular point $x$, the incidence structure $\pi_x$ with pointset $x^{\perp}$, with lineset the set of spans $\{y, z\}^{\perp\perp}$, where $y, z \in x^{\perp}$ with $y \neq z$, and with the natural incidence, is a projective plane of order $s$.*

(b) *For an antiregular point $x$ and a point $y$ in $x^\perp \setminus \{x\}$, the incidence structure $\pi(x,y)$ with pointset $x^\perp \setminus \{x,y\}^\perp$, with lines the sets $\{x,z\}^{\perp\perp} \setminus \{x\}$ with $x \sim z \not\sim y$ and the sets $\{x,u\}^\perp \setminus \{y\}$ with $y \sim u \not\sim x$, and with the natural incidence, is an affine plane of order $s$.*

In 4.3.3 it was observed that all points of $W(q)$ are regular. Dually, all lines of $Q(4,q)$ are regular. For $q$ even $W(q)$ is self-dual, and so for $q$ even all lines of $W(q)$ are regular. Dually, all points of $Q(4,q)$, $q$ even, are regular. Further, each point of $Q(4,q)$, $q$ odd, is antiregular, and, dually, each line of $W(q)$, $q$ odd, is antiregular. Finally, each point of the GQ $H(3,q^2)$ is regular and, dually, all lines of $Q^-(5,q)$ are regular.

Historically, the next result is probably the oldest combinatorial characterization of a class of GQ. A proof is essentially contained in a paper by Singleton [1966] (although he erroneously thought he had proved a stronger result), but the first satisfactory treatment may have been given by Benson [1970]. No doubt it was discovered independently by several authors, e.g., Tallini [1971].

THEOREM 2. *A* GQ *$S$ of order $s$, $s \neq 1$, is isomorphic to $W(s)$ if and only if all its points are regular.*

The next result is a slight generalization of the preceding theorem.

THEOREM 3 (Thas [1977]). *A* GQ *$S$ of order $(s,t)$, $s \neq 1$, is isomorphic to $W(s)$ if and only if each hyperbolic line has at least $s+1$ points.*

THEOREM 4 (Thas [1973]). *A* GQ *$S$ of order $s$, $s \neq 1$, is isomorphic to $W(2^h)$ if and only if it has an ovoid $O$, each triad of which has at least one centre.*

THEOREM 5 (Thas [1973]). *A* GQ *$S$ of order $s$, $s \neq 1$, is isomorphic to $W(2^h)$ if and only if it has an ovoid $O$, each point of which is regular.*

THEOREM 6 (Payne and Thas [1976]). *A* GQ *$S$ of order $s$, $s \neq 1$, is isomorphic to $W(2^h)$ if and only if it has a regular pair $(L_1, L_2)$ of nonconcurrent lines with the property that any triad of points lying on lines of $\{L_1, L_2\}^\perp$ has at least one centre.*

THEOREM 7 (Mazzocca [1973], Payne and Thas [1976]). *Let $S$ be a* GQ *of order $s$, $s \neq 1$, having an antiregular point $x$. Then $S$ is isomorphic to $Q(4,s)$ if and only if there is a point $y$, $y \in x^\perp \setminus \{x\}$, for which the associated affine plane $\pi(x,y)$ is Desarguesian.*

There is an easy corollary.

COROLLARY. *Let $S$ be a* GQ *of order $s$, $s \neq 1$, having an antiregular point $x$. If $s \leqslant 8$, i.e. if $s \in \{3,5,7\}$, then $S$ is isomorphic to $Q(4,s)$.*

**8.2.** *Characterizations of $Q^-(5,q)$ and $H(3,q^2)$*

Let $S=(P,B,I)$ be a GQ of order $(s,t)$, with $s^2=t>1$. By Theorem 5 of Section 2 for any triad $\{x,y,z\}$ we have $|\{x,y,z\}^\perp|=s+1$. Clearly $|\{x,y,z\}^{\perp\perp}|\leqslant s+1$. We say $\{x,y,z\}$ is 3-*regular* provided $|\{x,y,z\}^{\perp\perp}|=s+1$. The point $x$ is called 3-*regular* if and only if each triad containing $x$ is 3-regular.

Consider the classical GQ $Q^-(5,q)$ and the corresponding orthogonal polarity $\zeta$. If $T=\{x,y,z\}$ is a triad of $Q^-(5,q)$, then $T^\perp$ is the conic $Q^-\cap\pi$, where $\pi$ is the polar plane of the plane $xyz$, and $T^{\perp\perp}$ is the conic $Q^-\cap xyz$. So $|T^{\perp\perp}|=q+1$, and consequently each point of $Q^-(5,q)$ is 3-regular. Dually, each line of $H(3,q^2)$ is 3-regular.

The following characterization theorem is very important, not only for the theory of GQ, but also for other areas in combinatorics.

THEOREM 8 (Thas [1978b]). *Let $S$ be a* GQ *of order $(s,s^2)$, $s\neq 1$.*

(i) *$S\cong Q^-(5,s)$ if and only if all points of $S$ are 3-regular.*

(ii) *When $s$ is odd, then $S\cong Q^-(5,s)$ if and only if it has a 3-regular point.*

(iii) *When $s$ is even, then $S\cong Q^-(5,s)$ if and only if it has at least one 3-regular point not incident with some regular line.*

REMARK. Independently Mazzocca [1974] proved (i) for $s$ odd.

Next we consider the role of subquadrangles in characterizing $Q^-(5,q)$.

THEOREM 9 (Thas [1978b]).

(i) *A* GQ *$S$ of order $(s,t)$, $s>1$, is isomorphic to $Q^-(5,s)$ if and only if every triad of lines with at least one centre is contained in a proper subquadrangle of order $(s,t')$.*

(ii) *A* GQ *$S$ of order $(s,t)$, $s>1$ and $t>1$, is isomorphic to $Q^-(5,s)$ if and only if for each triad $\{u,u',u''\}$ with distinct centers $x$, $x'$ the five points $u,u',u'',x,x'$ are contained in a proper subquadrangle of order $(s,t')$.*

Let $S$ be a GQ of order $(s,t)$, and let $\{L_1,L_2,L_3\}$ and $\{M_1,M_2,M_3\}$ be two triads of lines for which $L_i\not\sim M_j$ if and only if $\{i,j\}=\{1,2\}$. Let $x_i$ be the point defined by $L_iIx_iIM_i$, $i=1,2$. This configuration $\Gamma$ of seven distinct points and six distinct lines is called a *broken grid* with *carriers* $x_1$ and $x_2$. We say $\Gamma$ satisfies *axiom* (D) *with respect to the pair* $(L_1,L_2)$ provided the following holds: if $L_4\in\{M_1,M_2\}^\perp$ with $L_4\not\sim L_i$, $i=1,2,3$, then $\{L_1,L_2,L_4\}$ has at least one centre. Interchanging $L_i$ and $M_i$ gives the definition of axiom (D) for $\Gamma$ with respect to the pair $(M_1,M_2)$. Further, $\Gamma$ is said to satisfy *axiom* (D) provided it satisfies axiom (D) with respect to both pairs $(L_1,L_2)$ and $(M_1,M_2)$.

Let $x$ be any point of $S$. Then $S$ is said to satisfy *axiom (D)$'_x$* if the broken grid $\Gamma$ satisfies axiom (D) with respect to $(L_1,L_2)$ whenever $xIL_1$; it satisfies *axiom* (D)$''_x$ if $\Gamma$ satisfies axiom (D) with respect to $(M_1,M_2)$ whenever $xIL_1$.

THEOREM 10 (Thas [1978b]). *Let $S$ be a* GQ *of order* $(s,t)$, *with* $s \neq t$, $s > 1$, $t > 1$.

(i) *If $s$ is odd, then $S \cong Q^-(5,s)$ if and only if $S$ contains a coregular point $x$ for which* $(\mathrm{D})'_x$ *or* $(\mathrm{D})''_x$ *is satisfied.*

(ii) *If $s$ is even, then $S \cong Q^-(5,s)$ if and only if all lines of $S$ are regular and $S$ contains a point $x$ for which* $(\mathrm{D})'_x$ *or* $(\mathrm{D})''_x$ *is satisfied.*

In order to conclude this section dealing with characterizations of $Q^-(5,s)$, we introduce one more basic concept. Let $S = (P,B,I)$ be a GQ of order $(s,t)$. If $B^{\perp\perp}$ is the set of all spans $\{x,y\}^{\perp\perp}$ with $x \not\sim y$, then let $S^{\perp\perp} = (P, B^{\perp\perp}, \in)$. For $x \in P$, say that $S$ satisfies *property* $(\mathrm{A})_x$ if for any $M = \{y,z\}^{\perp\perp} \in B^{\perp\perp}$ with $x \in \{y,z\}^\perp$, and any $u \in \mathrm{cl}(y,z) \cap (x^\perp \setminus \{x\})$ with $u \notin M$, the substructure of $S^{\perp\perp}$ generated by $M$ and $u$ is a dual affine plane. The GQ $S$ is said to satisfy *property* (A) if it satisfies $(\mathrm{A})_x$ for all $x \in P$. So $S$ satisfies (A) if for any $M = \{y,z\}^{\perp\perp} \in B^{\perp\perp}$ and any $u \in \mathrm{cl}(y,z) \setminus (\{y,z\}^\perp \cup \{y,z\}^{\perp\perp})$, the substructure of $S^{\perp\perp}$ generated by $M$ and $u$ is a dual affine plane. The duals of $(\mathrm{A})_x$ and (A) are denoted by $(\widehat{\mathrm{A}})_L$ and $(\widehat{\mathrm{A}})$, respectively.

THEOREM 11 (Thas [1981b]). *Let $S$ be a* GQ *of order* $(s,t)$, $s \neq t$, $t > 1$.

(i) *If $s > 1$, $s$ odd, then $S$ is isomorphic to $Q^-(5,s)$ if and only if* $(\widehat{\mathrm{A}})_L$ *is satisfied for all lines $L$ incident with some coregular point $x$.*

(ii) *If $s$ is even, then $S$ is isomorphic to $Q^-(5,s)$ if and only if all lines of $S$ are regular and* $(\widehat{\mathrm{A}})_L$ *is satisfied for all lines $L$ incident with some point $x$.*

Let $S = (P,B,I)$ be a GQ of order $(s,t)$ and let

$$B^* = \{\{x,y\}^{\perp\perp}\colon\ x,y \in P,\ x \neq y\}.$$

Then $S^* = (P, B^*, \in)$ is a linear space (cf. Chapter 6). So as to have no confusion between collinearity in $S$ and collinearity in $S^*$, points $x_1, x_2, \ldots$ of $P$ which are on a line of $S^*$ will be called *$S^*$-collinear*. A *linear variety* of $S^*$ is a subset $P' \subseteq P$ such that $x,y \in P'$, $x \neq y$, implies $\{x,y\}^{\perp\perp} \subseteq P'$. If $P \neq P'$ and $|P'| > 1$, the linear variety is *proper*; if $P'$ is generated by three points which are not $S^*$-collinear, $P'$ is said to be a *plane* of $S^*$.

Now we state a fundamental characterization of the GQ $H(3,s)$.

THEOREM 12 (Tallini [1971]). *Let $S = (P,B,I)$ be a* GQ *of order* $(s,t)$, *with* $s \neq t$, $s > 1$ *and* $t > 1$. *Then $S$ is isomorphic to $H(3,s)$ if and only if*

(i) *all points of $S$ are regular, and*

(ii) *if the lines $L$ and $L'$ of $B^*$ are contained in a proper linear variety of $S^*$, then also the lines $L^\perp$ and $L'^\perp$ of $B^*$ are contained in a proper linear variety of $S^*$.*

**8.3.** *Characterizations of* $H(4,q^2)$

An elegant characterization of $H(4,q^2)$ is the following theorem.

THEOREM 13 (Thas [1976b]). *A* GQ $S$ *of order* $(s,t)$, $s^3 = t^2$ *and* $s \neq 1$, *is isomorphic to the classical* GQ $H(4,s)$ *if and only if every hyperbolic line has at least* $\sqrt{s}+1$ *points.*

Relying on Theorem 13 one obtains the following characterization.

THEOREM 14 (Payne and Thas [1976]). *Let* $S$ *have order* $(s,t)$ *with* $1 < s^3 \leqslant t^2$. *Then* $S$ *is isomorphic to* $H(4,s)$ *if and only if each trace* $\{x,y\}^\perp$, *with* $x \sim y$, *is a plane of* $S^*$ *which is generated by any three non-*$S^*$*-collinear points in it.*

**8.4.** *Theorems simultaneously characterizing several classical generalized quadrangles*

First two new definitions are required. A point $u$ of $S$ is called *semiregular* provided that $z \in \mathrm{cl}(x,y)$ whenever $u$ is the unique centre of the triad $\{x,y,z\}$. And a point $u$ has *property* (H) provided $z \in \mathrm{cl}(x,y)$ if and only if $x \in \mathrm{cl}(y,z)$, whenever $\{x,y,z\}$ is a triad consisting of points in $u^\perp$. It follows easily that any semiregular point has property (H).

We give some examples.

In $W(q)$, $Q(4,q)$, $Q^-(5,q)$ and $H(3,q^2)$ all points and lines are semiregular and have property (H). In $H(4,q^2)$ all points are semiregular and have property (H); all lines have property (H). No line of $H(4,q^2)$ is semiregular. So property (H) does not imply semiregularity.

THEOREM 15 (Thas [1977]). *Let* $S$ *have order* $(s,t)$ *with* $s \neq 1$. *Then* $|\{x,y\}^{\perp\perp}| \geqslant s^2/t+1$ *for all* $x,y$, *with* $x \not\sim y$, *if and only if one of the following occurs*:

(i) $t = s^2$;

(ii) $S \cong W(s)$;

(iii) $S \cong H(4,s)$.

THEOREM 16 (Thas [1977], Thas and Payne [1976]). *In the* GQ $S$ *of order* $(s,t)$ *each point has property* $(H)$ *if and only if one of the following holds*:

(i) *each point is regular*;

(ii) *each hyperbolic line has exactly two points*;

(iii) $S \cong H(4,s)$.

THEOREM 17 (Thas [1977], Thas and Payne [1976]). *Let* $S$ *be a* GQ *of order* $(s,t)$. *Then each point is semiregular if and only if one of the following occurs*:

(i) $s > t$ *and each point is regular*;

(ii) $s = t$ *and* $S \cong W(s)$;

(iii) $s = t$ *and each point is antiregular*;

(iv) $s < t$, *each hyperbolic line has exactly two points, and no triad of points has a unique centre*;

(v) $S \cong H(4,s)$.

THEOREM 18 (Thas [1977]). *In a* GQ $S$ *of order* $(s,t)$ *all triads* $\{x,y,z\}$ *with* $z \notin \mathrm{cl}(x,y)$ *have a constant number of centers if and only if one of the following occurs*:

(i) *all points are regular*;

(ii) $s^2 = t$;

(iii) $S \cong H(4,s)$.

THEOREM 19 (Thas [1977]). *The* GQ $S$ *of order* $(s,t)$, $s > 1$, *is isomorphic to one of* $W(s)$, $Q^-(5,s)$ *or* $H(4,s)$ *if and only if for each triad* $\{x,y,z\}$ *with* $x \notin \mathrm{cl}(y,z)$ *the set* $\{x\} \cup \{y,z\}^\perp$ *is contained in a proper subquadrangle of order* $(s,t')$.

THEOREM 20 (Thas [1977]). *Let* $S$ *be a* GQ *of order* $(s,t)$ *for which not all points are regular. Then* $S$ *is isomorphic to* $Q(4,s)$, *with* $s$ *odd, to* $Q^-(5,s)$ *or to* $H(4,s)$ *if and only if each set* $\{x\} \cup \{y,z\}^\perp$, *where* $\{x,y,z\}$ *is a triad with at least one centre and* $x \notin \mathrm{cl}(y,z)$, *is contained in a proper subquadrangle of order* $(s,t')$.

Next, we give a characterization in terms of matroids.

A finite *matroid* (which, in Chapter 6, is called a dimensional linear space) is a pair $(P,M)$ where $P$ is a finite set of elements called *points* and $M$ is a closure operator which associates to each subset $X$ of $P$ a subset $\overline{X}$ (the *closure* of $X$) of $P$, such that the following conditions are satisfied:

(i) $\overline{\varnothing} = \varnothing$, and $\overline{\{x\}} = \{x\}$ for all $x \in P$;

(ii) $X \subseteq \overline{X}$ for all $X \subseteq P$;

(iii) $X \subseteq \overline{Y} \Rightarrow \overline{X} \subseteq \overline{Y}$ for all $X, Y \subseteq P$;

(iv) $y \in \overline{X \cup \{x\}}$, $y \notin \overline{X} \Rightarrow x \in \overline{X \cup \{y\}}$ for all $x,y \in P$ and $X \subseteq P$.

The sets $\overline{X}$ are called the *closed sets* of the matroid $(P,M)$. It is easy to prove that the intersection of closed sets is always closed. A closed set $C$ has *dimension* $h$ if $h+1$ is the minimum number of points of any subset of $C$ whose closure coincides with $C$. The closed sets of dimension one are the *lines* of the matroid.

THEOREM 21 (Mazzocca and Olanda [1979]). *Suppose that* $S = (P,B,I)$ *is a* GQ *of order* $(s,t)$, $s > 1$ *and* $t > 1$. *Then* $P$ *is the pointset and*

$$B^* = \big\{\{x,y\}^{\perp\perp} \colon\ x,y \in P \ \text{ and } \ x \neq y\big\}$$

*is the lineset of some matroid* $(P,M)$ *having all sets* $x^\perp$, $x \in P$, *as closed sets, if and only if one of the following occurs*:

(i) $S \cong W(s)$;

(ii) $S \cong Q(4,s)$;

(iii) $S \cong H(4,s)$;

(iv) $S \cong Q^-(5,s)$;

(v) *all points of* $S$ *are regular,* $s = t^2$, *and every three non-*$S^*$*-collinear points are contained in a proper linear variety of the linear space* $S^* = (P,B^*,\in)$.

REMARK. The original proof of Theorem 21 is contained in Mazzocca and Olanda [1979], but a very short proof is given by Payne and Thas [1984].

Let $S$ be a thick GQ of order $(s,t)$. A *quadrilateral* of $S$ is just a subquadrangle of order $(1,1)$. A quadrilateral $S'$ is said to be *opposite a line* $L$ if the lines of $S'$ are not concurrent with $L$. If $S'$ is opposite $L$, the four lines incident with the points of $S'$ and concurrent with $L$ are called the *lines of perspectivity of* $S'$ *from* $L$. Two quadrilaterals $S_1$ and $S_2$ are in *perspective from* $L$ if either

(a) $S_1 = S_2$ and $S_1$ is opposite $L$; or

(b) (i) $S_1 \neq S_2$, (ii) $S_1$ and $S_2$ are both opposite $L$, (iii) the lines of perspectivity of $S_1$, and of $S_2$, from $L$ are the same.

THEOREM 22 (Ronan [1980a]). *The* GQ $S = (P, B, I)$ *of order* $(s,t)$, $s > 1$ *and* $t > 2$, *is isomorphic to* $Q(4,s)$ *or* $Q^-(5,s)$ *if and only if given a quadrilateral* $S_1$ *opposite a line* $L$ *and a point* $x'$, *not incident with* $L$ *but incident with a line of perspectivity of* $S_1$ *from* $L$, *there is a quadrilateral* $S_2$ *containing* $x'$ *and in perspective with* $S_1$ *from* $L$.

REMARK. If $t = 2$ and $s > 1$, then by Section 5 $S \cong Q(4,2)$ or $S \cong H(3,4)$. One can check that in these two cases the quadrilateral condition of the preceding theorem is satisfied.

## 8.5. *Characterizations of all thick classical and dual classical generalized quadrangles*

The reader is reminded of properties (A) and $(\widehat{\mathrm{A}})$ introduced in 8.2. Let $B^{\perp\perp}$ be the set of all hyperbolic lines of the GQ $S = (P, B, I)$, and let $S^{\perp\perp} = (P, B^{\perp\perp}, \in)$. We say that $S$ satisfies *property* (A) if for any $M = \{y,z\}^{\perp\perp} \in B^{\perp\perp}$ and any $u \in \mathrm{cl}(y,z) \setminus (\{y,z\}^{\perp} \cup \{y,z\}^{\perp\perp})$ the substructure of $S^{\perp\perp}$ generated by $M$ and $u$ is a dual affine plane. The dual of (A) is denoted by $(\widehat{\mathrm{A}})$.

THEOREM 23 (Thas [1981b]). *Let* $S = (P, B, I)$ *be a thick* GQ *of order* $(s,t)$. *Then* $S$ *is a classical or a dual classical* GQ *if and only if it satisfies either condition* (A) *or* $(\widehat{\mathrm{A}})$.

Let $S = (P, B, I)$ be a GQ of order $(s,t)$ and let $x, y$ be distinct points of the line $L$. Further, let $S_i$ be a quadrilateral of $S$ with points $z_1^i, \dots, z_4^i$, lines $M_1^i, \dots, M_4^i$, and assume

$$z_1^i I M_1^i I z_2^i I M_2^i I z_3^i I M_3^i I z_4^i I M_4^i I z_1^i, \quad \text{with } i = 1, 2.$$

Then we call $S_1$ and $S_2$ in *perspective from* $\{x, L, y\}$ if for every $i \in \{1,2,3,4\}$, $z_i^1$ and $z_i^2$ are collinear with a same point on $L$, and $M_i^1$ and $M_i^2$ are concurrent with a same line through $x$, respectively $y$. A quadrilateral $S'$ is called a $\{x, L, y\}$*-quadrilateral* if at least one of its points, say $z'$, is collinear with $x$ or $y$, say $x$, and if neither $x$ nor $y$ is incident with any of the lines of $S'$; in such a case the line $z'x$ is called a *base-line of the pair* $(\{x, L, y\}, S')$. We call $S$ $\{x, L, y\}$*-Desarguesian* if for every $\{x, L, y\}$-quadrilateral $S_1$ and every point $z_1^2$, with $z_1^2 \neq x, y$, on any base-line, there exists a $\{x, L, y\}$-quadrilateral $S_2$ containing $z_1^2$ which is in perspective with $S_1$ from $\{x, L, y\}$.

THEOREM 24 (Thas and Van Maldeghem [1990]). *A finite thick* GQ *is classical or dual classical if and only if it is* $\{x, L, y\}$*-Desarguesian for all sets* $\{x, L, y\}$ *with* $x, y$ *distinct points on the line* $L$.

REMARK. Theorem 22 by Ronan is an easy corollary of Theorem 24.

Let $\{p, L\}$ be a *flag* of the GQ $S = (P, B, I)$, i.e. let $pIL$. The quadrilateral $S'$ is said to be *opposite the flag* $\{p, L\}$ if the lines of $S'$ are not concurrent with $L$ and if the points of $S'$ are not collinear with $p$. Let $S_i$ be a quadrilateral of $S$ with points $z_1^i, \dots, z_4^i$, lines $M_1^i, \dots, M_4^i$, and assume

$$z_1^i I M_1^i I z_2^i I M_2^i I z_3^i I M_3^i I z_4^i I M_4^i I z_1^i, \quad \text{with } i = 1, 2.$$

If $S_1$ and $S_2$ are opposite a flag $\{p, L\}$, then we say that $S_1$ and $S_2$ are in *perspective from* $\{p, L\}$ provided $z_i^1$ and $z_i^2$ are collinear with a same point on $L$, and $M_i^1$ and $M_i^2$ are concurrent with a common line through $p$, $i = 1, 2, 3, 4$. Note that this definition is self-dual. The GQ $S$, with flag $\{p, L\}$ is said to be $\{p, L\}$*-Desarguesian* provided the following condition holds: for any quadrilateral $S_1$ opposite $\{p, L\}$ and containing a flag $\{z_1^1, M_1^1\}$ and any flag $\{z_1^2, M_1^2\}$ satisfying

(i) $z_1^2$ is not incident with $L$ and $p$ is not incident with $M_1^2$,
(ii) $z_1^1 \sim r_1 \sim z_1^2$ for some $r_1$ incident with $L$,
(iii) $M_1^1 \sim R_1 \sim M_1^2$ for some $R_1$ incident with $p$,

there is a quadrilateral $S_2$ opposite $\{p, L\}$, containing the flag $\{z_1^2, M_1^2\}$ and in perspective with $S_1$ from $\{p, L\}$.

THEOREM 25 (Van Maldeghem, Payne and Thas [1994]). *A finite thick* GQ $S$ *is classical or dual classical if and only if* $S$ *is* $\{p, L\}$*-Desarguesian for all flags* $\{p, L\}$.

## 8.6. *Characterizations of nonclassical generalized quadrangles*

There are many interesting characterizations of nonclassical GQ. As an illustration we just mention one of these theorems.

THEOREM 26 (Thas [1978b]). *A* GQ *of order* $(s, s^2)$, $s > 1$, *is isomorphic to* $T_3(O)$ *if and only if it has a* 3*-regular point.*

For literature on the subject we refer to De Finis [1985], De Soete [1987a,b], De Soete and Thas [1984, 1986a,b, 1987], Payne [1985a], and Thas [1974, 1975, 1978b, 1981b].

## 8.7. *Characterizations of the finite classical generalized hexagons*

Let $S = (P, B, I)$ be a (finite) generalized hexagon of order $(s, t)$, and let $d(\cdot, \cdot)$ denote distance in the incidence graph of $S$. So if $x$ and $y$ are distinct points, then $d(x, y) = 2, 4,$ or 6. For $x \in P$, let

$$x^{\perp^*} = \{y \in P\colon\ d(x, y) \leqslant 4\},$$

and for distinct points $x, y$ let

$$\{x, y\}^{\perp^*} = x^{\perp^*} \cap y^{\perp^*} \quad \text{and} \quad \{x, y\}^{\perp^*\perp^*} = \bigcap \bigl\{z^{\perp^*}\colon\ z \in x^{\perp^*} \cap y^{\perp^*}\bigr\}.$$

For $x \in P$, let

$$x^{\tilde{\perp}} = \bigl\{L \in B\colon\ d(x, L) \leqslant 3\bigr\},$$

and for distinct points $x, y$ let

$$\{x, y\}^{\tilde{\perp}} = x^{\tilde{\perp}} \cap y^{\tilde{\perp}} \quad \text{and} \quad \{x, y\}^{\tilde{\perp}\tilde{\perp}} = \bigcap \bigl\{M^{\tilde{\perp}}\colon\ M \in x^{\tilde{\perp}} \cap y^{\tilde{\perp}}\bigr\}.$$

THEOREM 27 (Ronan [1980b]). *A thick finite generalized hexagon $S = (P, B, I)$ of order $(s, t)$ is isomorphic to $H(s, t)$, with $s = t^3$, or $H(s)$ if and only if $|\{x, y\}^{\perp^*\perp^*}| = 1 + t$ for all pairs $(x, y)$, $x, y \in P$, with $d(x, y) = 4$.*

REMARK. Yanushka [1976] proved the following weaker theorem: a thick finite generalized hexagon $S = (P, B, I)$ of order $s$ is isomorphic to $H(s)$ if and only if $|\{x, y\}^{\perp^*\perp^*}| = 1 + s$ for all pairs $(x, y)$, $x, y \in P$, with $d(x, y) = 4$.

Yanushka's theorem motivated Kantor to give Ronan Theorem 27 as a thesis subject.

The following theorem is an interesting corollary of the theorem of Yanushka and Ronan.

THEOREM 28 (Thas [1980b]). *Let $S = (P, B, I)$ be a generalized hexagon of order $(s, t)$, with $2 \leqslant t \leqslant s$. Then $S$ is isomorphic to the classical generalized hexagon $H(s)$ if and only if for any three points $x, y, z$ we have*

$$\bigl\{u \in P\colon\ d(u, x) \leqslant 2,\ d(u, y) \leqslant 4 \ \ \textit{and} \ \ d(u, z) \leqslant 4\bigr\} \neq \varnothing.$$

In order to conclude this section dealing with characterizations of the known generalized hexagons, we introduce two more basic concepts. Let $S = (P, B, I)$ be a generalized hexagon of order $(s, t)$. If for any two points $x, y$ with $d(x, y) = 6$ we have $|\{x, y\}^{\tilde{\perp}\tilde{\perp}}| = 1 + s$, then $S$ is said to satisfy the *regulus condition*. For $z, u \in P$ with $d(z, u) = 6$,

$$z^u = \bigl\{v \in P\colon\ d(v, z) = 2\bigr\} \cap \bigl\{w \in P\colon\ d(w, u) = 4\bigr\}.$$

Now let $x, y, z$ be three points such that $d(x, y) = 4$, $d(x, z) = d(y, z) = 6$ and $d(z, x * y) = 4$ with $x * y$ the unique point for which $d(x, x * y) = d(y, x * y) = 2$. Then we call $z^x \cap z^y$ an *intersection set* if $z^x \neq z^y$.

THEOREM 29 (Ronan [1981]). *If $S$ is a finite thick generalized hexagon of order $(s, t)$ with the regulus condition, then all intersection sets have one point if and only if $S \cong H(s)$ or $S \cong H(s, \sqrt[3]{s})$, two points if and only if $S$ is isomorphic to the dual of $H(s)$, and $s^2 + 1$ points if and only if $S$ is isomorphic to the dual of $H(s^3, s)$.*

THEOREM 30 (Ronan [1980c]).

(a) *A finite thick generalized hexagon $S$ of order $(s,t)$, with $s = t^3$, satisfies the regulus condition if and only if it is isomorphic to the classical generalized hexagon $H(s,t)$.*

(b) *A finite thick generalized hexagon $S$ of order $(s,t)$, with $t = s^3$, satisfies the regulus condition if and only if it is isomorphic to the dual of the classical generalized hexagon $H(t,s)$.*

REMARK. In Cameron and Kantor [1979] the classical generalized hexagon $H(s)$ is characterized in terms of metrically regular graphs embedded in PG$(n,s)$.

### **8.8.** *A combinatorial characterization of all finite thick classical generalized $n$-gons, with $n \geqslant 4$, and their duals*

In Van Maldeghem [1990] a common combinatorial characterization of all finite thick classical generalized $n$-gons, $n \geqslant 4$, and their duals is given; it is a generalization of Theorem 24.

### **8.9.** *Open problems*

(a) Is every GQ of order $s$, $s$ odd and $s > 1$, for which each point is antiregular, isomorphic to $Q(4,s)$?

(b) Are the planes in Theorem 1 always Desarguesian?

(c) Is every GQ of order $(s,t)$, $1 < s < t$, with all lines regular, isomorphic to $Q^-(5,s)$?

(d) Does there exist a GQ of order $(s,t)$, with $s < t < s^2$, for which all lines are regular?

## **9. Automorphisms of generalized polygons**

### **9.1.** *Elation generalized quadrangles and translation generalized quadrangles*

Let $S = (P,B,I)$ be a GQ of order $(s,t)$, $s \neq 1$, $t \neq 1$. A collineation $\theta$ of $S$ is a *whorl* about the point $p$ provided $\theta$ fixes each line incident with $p$. Let $\theta$ be a whorl about $p$. If $\theta = \mathrm{id}$ or if $\theta$ fixes no point of $P \setminus p^\perp$, then $\theta$ is an *elation* about $p$. If $\theta$ fixes each point of $p^\perp$, then $\theta$ is a *symmetry* about $p$. Any symmetry about $p$ is automatically an elation about $p$ (cf. Payne and Thas [1984]). Let $p,p' \in P$, $p \not\sim p'$. A *generalized homology* with *centers* $p,p'$ is an automorphism $\theta$ of $S$ which is a whorl about both $p$ and $p'$. The group of all generalized homologies with centers $p,p'$ is denoted $H(p,p')$.

If there is a group $G$ of elations about $p$ acting regularly on $P \setminus p^\perp$, we say that $S$ is an *elation generalized quadrangle* (EGQ) with *elation group* $G$ and *base point* $p$. Briefly, we say that $(S^{(p)},G)$ or $S^{(p)}$ is an EGQ. Most known examples of GQ or their duals are EGQ, the notable exceptions being those of order $(s-1,s+1)$ and their duals. If the group $G$ is Abelian, then we say that the EGQ $(S^{(p)},G)$ is a *translation* *generalized quadrangle* (TGQ) with *translation group* $G$ and *base point* $p$. For any TGQ the parameters $s$ and $t$ satisfy $s \leqslant t$ (cf. Payne and Thas [1984]).

Let $(S^{(p)}, G)$ be an EGQ of order $(s, t)$, and let $y$ be a given point of $P \setminus p^{\perp}$. Let $L_0, L_1, \ldots, L_t$ be the lines incident with $p$, and define $z_i$ and $M_i$ by $L_i I z_i I M_i I y$, $0 \leqslant i \leqslant t$. Put

$$A_i = \{\theta \in G\colon\ M_i^{\theta} = M_i\}, \quad A_i^* = \{\theta \in G\colon\ z_i^{\theta} = z_i\}, \quad 0 \leqslant i \leqslant t.$$

Then conditions K1 and K2 of 4.4 are satisfied. Conversely, if conditions K1 and K2 are satisfied then the corresponding GQ $S(G, J)$ is an EGQ with base point $(\infty)$. Moreover, it follows rather easily that $G$ acts by right multiplication as a (maximal) group of elations about $(\infty)$.

Let $(S^{(p)}, G)$ be a TGQ with $A_i$ and $A_i^*$ as above. The *kernel* $K$ of $S^{(p)}$ or $(S^{(p)}, G)$ or of the 4-gonal family $J = \{A_i\colon\ 0 \leqslant i \leqslant t\}$, is the set of all endomorphisms $\alpha$ of $G$ for which $A_i^{\alpha} \subset A_i$, $0 \leqslant i \leqslant t$. The following basic results are due to Payne and Thas [1984].

THEOREM 1. *With the usual addition and multiplication of endomorphisms the kernel $K$ is a field, so that $A_i^{\alpha} = A_i$, $(A_i^*)^{\alpha} = A_i^*$ for all $i = 0, 1, \ldots, t$ and all $\alpha \in K \setminus \{0\}$.*

THEOREM 2. *The group $G$ is elementary Abelian, and $s$ and $t$ must be powers of the same prime. If $s < t$, then there is a prime power $q$ and an odd integer $a$ for which $s = q^a$ and $t = q^{a+1}$. If $s$ (or $t$) is even then either $s = t$ or $s^2 = t$.*

THEOREM 3. *If $(S^{(p)}, G)$ is a* TGQ *of order $(s, t)$, then $G$ is the complete set of all elations about $p$.*

THEOREM 4. *The multiplicative group of the kernel of the* TGQ *$S^{(p)}$ is isomorphic to the group $H(p, p')$ of generalized homologies about $p$ and $p'$, with $p \not\sim p'$.*

**9.2.** *The sets $O(n, m, q)$ and the generalized quadrangles $T(n, m, q)$*

In PG$(2n + m - 1, q)$ consider a set $O(n, m, q)$ of $q^m + 1$ $(n - 1)$-dimensional subspaces

$$\mathrm{PG}^{(0)}(n-1, q),\ \ldots,\ \mathrm{PG}^{(q^m)}(n-1, q),$$

every three of which generate a PG$(3n - 1, q)$, and such that each element $\mathrm{PG}^{(i)}(n-1, q)$ of $O(n, m, q)$ is contained in a $\mathrm{PG}^{(i)}(n + m - 1, q)$ having no point in common with any $\mathrm{PG}^{(j)}(n-1, q)$ for $j \neq i$. It is easy to check that $\mathrm{PG}^{(i)}(n + m - 1, q)$ is uniquely determined, $i = 0, \ldots, q^m$. The space $\mathrm{PG}^{(i)}(n + m - 1, q)$ is called the *tangent space* of $O(n, m, q)$ at $\mathrm{PG}^{(i)}(n - 1, q)$. Embed PG$(2n + m - 1, q)$ in a PG$(2n + m, q)$, and construct a point-line geometry $T(n, m, q)$ as follows.

Points are of three types:

(i) the points of PG$(2n + m, q) \setminus$ PG$(2n + m - 1, q)$;

(ii) the $(n + m)$-dimensional subspaces of PG$(2n + m, q)$ which intersect PG$(2n + m - 1, q)$ in one of the $\mathrm{PG}^{(i)}(n + m - 1, q)$;

(iii) the symbol $(\infty)$.

Lines are of two types:

(a) the $n$-dimensional subspaces of PG$(2n+m,q)$ which intersect PG$(2n+m-1,q)$ in a PG$^{(i)}(n-1,q)$;

(b) the elements of $O(n,m,q)$.

Incidence in $T(n,m,q)$ is defined as follows. A point of type (i) is incident only with lines of type (a); here the incidence is that of PG$(2n+m,q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of $O(n,m,q)$ contained in it. The point $(\infty)$ is incident with no line of type (a) and with all lines of type (b).

The following theorems are due to Payne and Thas [1984].

THEOREM 5. *$T(n,m,q)$ is a* TGQ *of order $(q^n,q^m)$ with base point $(\infty)$ for which* GF$(q)$ *is a subfield of the kernel. Moreover, the translations of $T(n,m,q)$ induce the translations of the affine space* AG$(2n+m,q)=$ PG$(2n+m,q)\setminus$ PG$(2n+m-1,q)$. *Conversely, every* TGQ *for which* GF$(q)$ *is a subfield of the kernel is isomorphic to a $T(n,m,q)$. It follows that the theory of* TGQ *is equivalent to the theory of the sets $O(n,m,q)$.*

REMARK. For $n=m=1$, $O(n,m,q)=O(1,1,q)=O$ is an oval of PG$(2,q)$ and $T(1,1,q)$ is the GQ $T_2(O)$ of Tits associated with $O$; for $n=1$ and $m=2$ $O(n,m,q)=O(1,2,q)=O$ is an ovoid of PG$(3,q)$ and $T(1,2,q)$ is the GQ $T_3(O)$ of Tits associated with $O$.

THEOREM 6. *The following hold for any $O(n,m,q)$:*

(i) *$n=m$ or $n(a+1)=ma$ with $a$ odd;*

(ii) *if $q$ is even, then $n=m$ or $m=2n$;*

(iii) *if $n\neq m$, then each point of* PG$(2n+m-1,q)$ *which is not contained in an element of $O(n,m,q)$ belongs to 0 or $1+q^{m-n}$ tangent spaces of $O(n,m,q)$;*

(iv) *if $m=2n$, then each point of* PG$(4n-1,q)$ *which is not contained in an element of $O(n,2n,q)$ belongs to exactly $1+q^n$ tangent spaces of $O(n,2n,q)$;*

(v) *if $m=n$ and $q$ is odd, then each point of* PG$(3n-1,q)$ *which is not contained in an element of $O(n,n,q)$ belongs to* 0 *or* 2 *tangent spaces of $O(n,n,q)$; if $m=n$ and $q$ is even, then each point of* PG$(3n-1,q)$ *which is not contained in an element of $O(n,n,q)$ belongs to* 1 *or $q^n+1$ tangent spaces of $O(n,n,q)$;*

(vi) *if $n\neq m$, then each hyperplane of* PG$(2n+m-1,q)$ *which does not contain a tangent space of $O(n,m,q)$ contains* 0 *or $q^{m-n}+1$ elements of $O(n,m,q)$;*

(vii) *if $m=2n$, then each hyperplane of* PG$(4n-1,q)$ *which does not contain a tangent space of $O(n,2n,q)$ contains exactly $1+q^n$ elements of $O(n,2n,q)$;*

(viii) *if $n\neq m$, then the $q^m+1$ tangent spaces of $O(n,m,q)$ form an $O^*(n,m,q)$ in the dual space of* PG$(2n+m-1,q)$ *(the tangent spaces of $O^*(n,m,q)$ are the elements of $O(n,m,q)$); so in addition to $T(n,m,q)$ there arises a* TGQ *$T^*(n,m,q)$, called the translation dual of $T(n,m,q)$.*

THEOREM 7. *Let* $(S^{(p)}, G)$ *be a* TGQ *arising from the set* $O(n, 2n, q)$. *Then* $S \cong T_3(O')$ *for some ovoid* $O'$ *if and only if one of the following holds*:

(i) $|H(p, p')| = s - 1 = q^n - 1$ *for any point* $p'$, *with* $p' \not\sim p$;

(ii) *for each point* $z$ *not contained in an element of* $O(n, 2n, q)$, *the* $q^n + 1$ *tangent spaces containing* $z$ *have exactly* $(q^n - 1)/(q - 1)$ *points in common*;

(iii) *each* $\mathrm{PG}(3n - 1, q)$ *containing at least three elements of* $O(n, 2n, q)$ *contains exactly* $q^n + 1$ *elements of* $O(n, 2n, q)$.

**9.3.** *The known translation generalized quadrangles*

As already mentioned, the GQ $T_2(O)$ and $T_3(O)$ of Tits (of respective orders $(q, q)$ and $(q, q^2)$) are TGQ. Here the kernel is isomorphic to $\mathrm{GF}(q)$, hence has maximal size.

The GQ arising from the flocks K1 (see 4.4 and Chapter 7, 10.5) are TGQ. The kernel is the subfield of $\mathrm{GF}(q)$ consisting of the elements fixed by the automorphism $\sigma$ of $\mathrm{GF}(q)$. Any such TGQ is isomorphic to its translation dual.

The duals of the GQ arising from the flocks $G$ (see 4.4 and Chapter 7, 10.5) are TGQ. The kernel of such a TGQ is always GF(3). The translation duals of these TGQ are the Roman GQ of Payne (see 4.4). Finally, it can be shown that for $q > 9$ the original TGQ is not isomorphic to its translation dual.

REMARK. For $q$ even no GQ arising from a nonlinear flock is a TGQ.

For proofs and many other results on the subject we refer to Johnson [1987], Payne [1985c, 1988, 1989] and Rogers [1990].

**9.4.** *Moufang conditions for finite generalized quadrangles*

In this section we always assume that $S = (P, B, I)$ is a finite thick GQ of order $(s, t)$.

For any chosen point $p$, let us define the following condition:

$(\mathrm{M})_p$: for any two lines $A$ and $A'$ of $S$ incident with $p$, the group of collineations of $S$ fixing $A$ and $A'$ pointwise and $p$ linewise is transitive on the lines ($\neq A$) incident with a given point $x$ on $A$ ($x \neq p$).

The GQ $S$ is said to satisfy *condition* (M) provided it satisfies $(\mathrm{M})_p$ for all points $p \in P$. For a fixed line $L \in B$ let $(\widehat{\mathrm{M}})_L$ be the condition that is the dual of $(\mathrm{M})_p$ and let *condition* $(\widehat{\mathrm{M}})$ be the dual of (M). If $S$ satisfies both (M) and $(\widehat{\mathrm{M}})$, then it is said to be a *Moufang* GQ. It is, e.g., easy to show that any TGQ $S^{(p)}$ satisfies $(\mathrm{M})_p$.

Tits [1976a] shows that from a celebrated theorem of Fong and Seitz [1973, 1974] we have

THEOREM 8. *The* GQ $S$ *is Moufang if and only if it is classical or dual classical.*

In a paper by Thas, Payne and Van Maldeghem [1991] the following result is proved.

THEOREM 9. *The* GQ $S$ *satisfies* (M) *if and only if it satisfies* $(\widehat{\mathrm{M}})$.

As a corollary we have a considerable improvement of Theorem 8.

THEOREM 10. *The* GQ $S$ *satisfies* (M) (*respectively,* ($\widehat{\mathrm{M}}$)) *if and only if it is classical or dual classical.*

**9.5.** *Other characterizations of finite generalized quadrangles using automorphisms*

In this section we state four theorems characterizing finite GQ by automorphisms.

Assume that $S = (P, B, I)$ is a finite thick GQ of order $(s, t)$.

THEOREM 11 (Ealy, Jr., [1977]). *Let the group of symmetries about each point of $S$ have even order. Then $s$ is a power of* 2 *and one of the following must hold*: (i) $S \cong W(s)$, (ii) $S \cong H(3, s)$, (iii) $S \cong H(4, s)$.

THEOREM 12 (Walker [1977]). *Let $G$ be a group of automorphisms of $S$ leaving no point or line of $S$ fixed. Suppose that $S$ has a point $p$ and a line $L$ for which the group of symmetries about $p$, and the group of symmetries about $L$, has order at least* 3 *and is a subgroup of $G$. Then $S$ contains a $G$-invariant subquadrangle $S' \cong W(2^n)$* (*for some integer $n \geqslant 2$*) *such that the restriction of $G$ to this subquadrangle contains* $\mathrm{PSp}(4, 2^n)$.

THEOREM 13 (Thas [1985, 1986]). *The* GQ $S$ *is classical if and only if* $|H(p, p')| = s - 1$ *for all $p, p' \in P$ with $p \not\sim p'$.*

Given a flag $\{p, L\}$ of $S$, a $\{p, L\}$-collineation is a collineation of $S$ which fixes each point on $L$ and each line through $p$. For any line $N$ incident with $p$, $N \neq L$, and any point $u$ incident with $L$, $u \neq p$, the group $G(p, L)$ of all $\{p, L\}$-collineations acts semiregularly on the lines $M$ concurrent with $N$, $p$ not incident with $M$, and on the points $w$ collinear with $u, w$ not incident with $L$. If the group $G(p, L)$ is transitive on the lines $M$, or equivalently, on the points $w$, then we say that $S$ is *$\{p, L\}$-transitive*.

THEOREM 14 (Van Maldeghem, Payne and Thas [1992]). *The* GQ $S$ *is classical or dual classical if and only if $S$ is $\{p, L\}$-transitive for all flags $\{p, L\}$.*

REMARK. In Van Maldeghem, Payne and Thas [1994] it is also shown that for a given flag $\{p, L\}$ the GQ $S$ is $\{p, L\}$-transitive if and only if it is $\{p, L\}$-Desarguesian (cf. Theorem 25 of Section 8).

**9.6.** *Moufang generalized $n$-gons with $n > 4$*

Let $S = (P, B, I)$ be a finite thick generalized hexagon of order $(s, t)$. The generalized hexagon is said to satisfy *condition* (M) if for all distinct $A, A', A'' \in B$ and all distinct $p, p' \in P$, with $AIpIA'Ip'IA''$, the group of collineations of $S$ fixing $A, A', A''$ pointwise and $p, p'$ linewise is transitive on the lines ($\neq A$) incident with a given point $x$ on $A$ ($x \neq p$). Further, let *condition* ($\widehat{\mathrm{M}}$) be the dual of (M). If $S$ satisfies both (M) and ($\widehat{\mathrm{M}}$), then it is said to be a *Moufang generalized hexagon*.

Let $S = (P, B, I)$ be a finite thick generalized octagon of order $(s, t)$. The generalized octagon is said to satisfy *condition* (M) if for all distinct $A, A', A'', A''' \in B$ and all distinct $p, p', p'' \in P$, with $AIpIA'Ip'IA''Ip''IA'''$, the group of collineations of $S$ fixing $A, A', A'', A'''$ pointwise and $p, p', p''$ linewise is transitive on the lines ($\neq A$) incident with a given point $x$ on $A$ ($x \neq p$). Further, let *condition* ($\widehat{\mathrm{M}}$) be the dual of (M). If $S$ satisfies both (M) and ($\widehat{\mathrm{M}}$), then it is said to be a *Moufang generalized octagon*.

Tits [1976a] shows that from Fong and Seitz [1973, 1974] we get:

THEOREM 15. *The finite thick generalized $n$-gon $S$, $n \in \{6, 8\}$, is Moufang if and only if it is classical or dual classical.*

Let $S = (P, B, I)$ be a finite thick generalized $n$-gon, $n \in \{6, 8\}$, of order $(s, t)$, and let $d(\cdot, \cdot)$ denote distance in the incidence graph of $S$. If $p$ and $p'$ are at distance $n$, then a collineation $\theta$ of $S$ fixing all lines incident with $p$ and $p'$ is called a *generalized homology* with *centers* $p$ and $p'$. The group of all generalized homologies with centers $p$ and $p'$ is denoted by $H(p, p')$.

Let $d(p, p') = n$ and let $u$ be a point for which $d(u, p) = 2$ and $d(u, p') = n - 2$. Then $S$ is called $\{p, p'\}$*-transitive* if $H(p, p')$ is transitive on the set of points, distinct from $u$ and $p$, which are incident with the line $pu$. The generalized $n$-gon $S$ is called $\{p, p'\}$*-quasi-transitive* if the group $H(p, p')$ is transitive on the set of lines, distinct from $up$ and from the line incident with $u$ and at distance $n - 3$ from $p'$, which are incident with the point $u$.

THEOREM 16 (Van Maldeghem [1991a,b]). *Let $S$ be a finite thick generalized $n$-gon, $n \in \{6, 8\}$, of order $(s, t)$.*

(i) *If $n = 6$, then $S$ is classical or dual classical if and only if $S$ is $\{p, p'\}$-transitive for all points $p, p'$ with $d(p, p') = 6$ and $\{L, L'\}$-transitive for all lines $L, L'$ with $d(L, L') = 6$.*

(ii) *If $n = 8$ and $s \neq 2 \neq t$, then $S$ is classical if and only if either $S$ or its dual is $\{p, p'\}$-transitive for all points $p, p'$ with $d(p, p') = 8$ and $\{L, L'\}$-quasi-transitive for all lines $L, L'$ with $d(L, L') = 8$.*

REMARK. Other characterizations of classical generalized $n$-gons, $n \in \{6, 8\}$, are given by Cameron and Kantor [1979], Walker [1980, 1982a,b], Van Maldeghem [1990] and Van Maldeghem and Weiss [1992].

## 9.7. *Epimorphisms*

It is well known that an epimorphism between two finite thick generalized 3-gons, i.e. finite projective planes, must be an isomorphism (see Hughes and Piper [1973]). The following theorem generalizes this result to generalized $n$-gons, $n > 2$.

THEOREM 17 (Pasini [1984]). *Every epimorphism between two finite thick generalized $n$-gons, $n > 2$, is an isomorphism.*

**9.8.** *Open problems*

(a) Does there exist a TGQ of order $(q^a, q^{a+1})$, $q$ odd and $a > 1$?
(b) Is every TGQ of order $(q, q^2)$ with all lines regular isomorphic to $Q^-(5, q)$?
(c) Is every TGQ of order $q$ isomorphic to a GQ $T_2(O)$ of Tits ?
(d) Does there exist a TGQ of order $(q, q^2)$, $q$ even, not isomorphic to a GQ $T_3(O)$ of Tits ?
(e) For $q$ odd, classify the TGQ arising from flocks.
(f) For generalized hexagons, does $\{p, p'\}$-transitivity, with $p, p'$ points for which $d(p, p') = 6$, imply $\{L, L'\}$-transitivity, with $L, L'$ lines for which $d(L, L') = 6$?
(g) For generalized octagons, does $\{p, p'\}$-transitivity, with $p, p'$ points for which $d(p, p') = 8$, imply $\{L, L'\}$-quasi-transitivity, with $L, L'$ lines for which $d(L, L') = 8$?
(h) Does Theorem 16 hold for $n = 8$ and $s$ and/or $t$ equal to 2?
(i) Does the Moufang condition (M) imply the Moufang condition ($\widehat{\mathrm{M}}$) for finite thick generalized $n$-gons, $n > 4$?

## 10. Infinite generalized polygons

In this section we define generalized polygons, without the assumption of finiteness.

A *generalized $n$-gon*, $n \geqslant 2$, is an incidence structure $S = (P, B, I)$ of points and lines satisfying:

(i) $P \neq \varnothing$, $B \neq \varnothing$, each point is incident with at least two lines, each line is incident with at least two points;
(ii) the incidence graph of $S$ has girth $2n$ and diameter $n$.

With such a definition grids and dual grids are GQ. To avoid this, one can consider, e.g., only thick generalized polygons.

For infinite generalized polygons there is no Feit–Higman theorem.

It is clear how to generalize conditions (M) and ($\widehat{\mathrm{M}}$) to generalized $n$-gons, $n \geqslant 4$. A generalized $n$-gon, $n \geqslant 4$, satisfying both (M) and ($\widehat{\mathrm{M}}$) is said to be a *Moufang generalized polygon*. In such a case, by Tits [1976b, 1979] and Weiss [1979], we have $n \in \{4, 6, 8\}$, and, by Tits [1976a, 1977a, 1983, 1990], all Moufang generalized $n$-gons, $n \geqslant 4$, are known.

Further, Theorem 22 of 8.4 and Theorems 27 and 29 of 8.7 are particular cases of theorems by Ronan on generalized quadrangles, respectively hexagons, without the finiteness condition. Also the main results of Thas and Van Maldeghem [1990] and Van Maldeghem [1990] hold in the infinite case.

The GQ embedded in an infinite projective space were completely determined by Dienst [1980a,b].

Work on topological GQ was done by Forst [1979] and Schroth [1988]. See also Section 6 of Chapter 23.

An interesting problem on generalized polygons is the following one: does there exist a generalized $n$-gon, $n \geqslant 4$, with a finite number of points on any line and an infinite number of lines through any point? For thick GQ it is not difficult to show that for $s = 2$ there is no such GQ. By a group theoretical argument Kantor [1984] proved that for $s = 3$ there is no such GQ either; Kantor described his proof as complicated and asked for a simpler combinatorial argument. Shortly afterwards Brouwer found indeed an ingenious combinatorial proof; see Brouwer [1991]. To my knowledge no other partial solutions to the problem are known.

Finally, an important reference on infinite polygons is Tits [1977b].

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