An Exploration of Distance Labeling and Colorings of Maximally Planar Graphs

By

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Background and Justification for Study

Many developments in graph theory can be traced back to the four color theorem, which states that any planar graph can be colored with 4 colors such that no 2 adjacent vertices share the same color [1, p. 237]. An accepted proof was published by Appel and Haken in 1976 that involved reducing the set of all planar graphs to a finite set of 1936 irreducible configurations, then checking whether each graph in that set was 4 colorable [2]. This was followed by an improvement by Robertson, Sanders and Seymor in 1997 that reduced the set of unavoidable configurations to 633 [3]. Euler's formula, which states that every planar graph must satisfy the equation $v - e + f = 2$, where v refers to the number of vertices, e refers to the number of edges, and f refers to the number of faces, was used in both proofs. Attempts at a proof simple enough to be checked by a human have been attempted, but there is still no known proof that does not require the use of computers. This paper will attempt to explore approaches to the problem that are not reliant on a derivation of Euler's formula or irreducible configurations in the hopes of obtaining a proof based on induction that does not require the use of computers. Although that particular end result is unlikely, it is hoped that this research will lead to further insights into planar graph coloring.

Review of Literature

Augustus De Morgan, credited for popularizing the problem originally posed by Francis Guthrie, believed that the necessity of a fourth color when a cycle of 3 vertices encloses a single vertex was significant to the problem [4, p. 24]. However, he was not able to turn this idea into a proof. Arthur Cayley revived the problem around 1878 [4, p. 62] and introduced the idea of restricting the problem to cubic maps (when countries are represented as regions) or making all regions but the outer region maximally planar (when countries are represented as vertices). He also made the observation that any planar map that can be colored with 4 colors must also be able to be colored such that the vertices on the outer cycle can be colored with 3 colors.

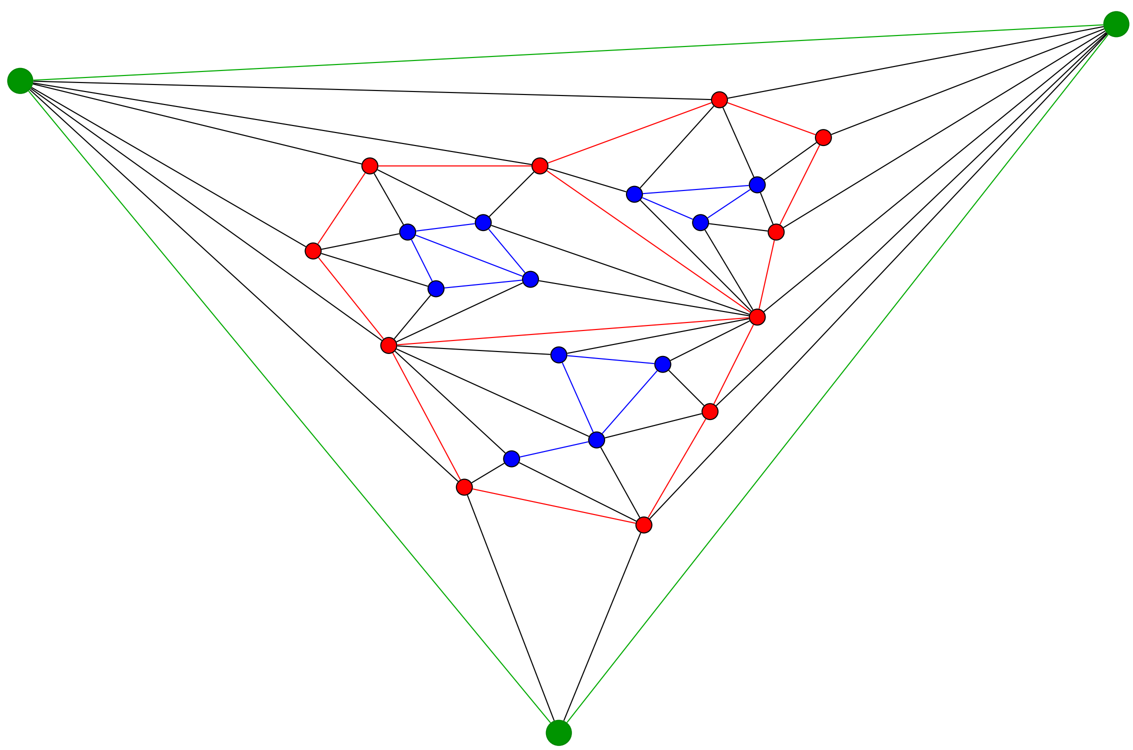
Alfred Kempe made significant advances by introducing the idea of Kempe Chains, or cycles of vertices made up of alternating colors, and by using a derivation of Euler's formula for maps to prove that every planar map must have at least one vertex with degree 5 or less [4, p. 79]. His approach resulted in a proof which was eventually disproved by Heawood, but Heawood was able to use the same approach to prove that all planar graphs could be colored with 5 colors such that no 2 adjacent vertices shared the same color. [4, p. 125]

Key elements of the approach by Kempe, namely looking for unavoidable sets and reducible configurations, eventually lead to the existing proof. [4, p. 146]. Birkhoff contributed by further characterizing the unavoidable sets in planar graphs, and by introducing the idea of a chromatic polynomial [4, p. 166], which counts the number of valid colorings for a particular graph with given properties [5]. Wernicke, Franklin and Lebesgue each contributed by attempting to characterize unavoidable sets of configurations [4, p. 169], and Henrich Heesche was responsible for unifying the search for unavoidable sets of configurations and introducing the method of discharging [4, p. 172]. This method of discharging was eventually used to characterize as set of irreducible configurations by Appel and Haken, which then resulted in the currently accepted proof.

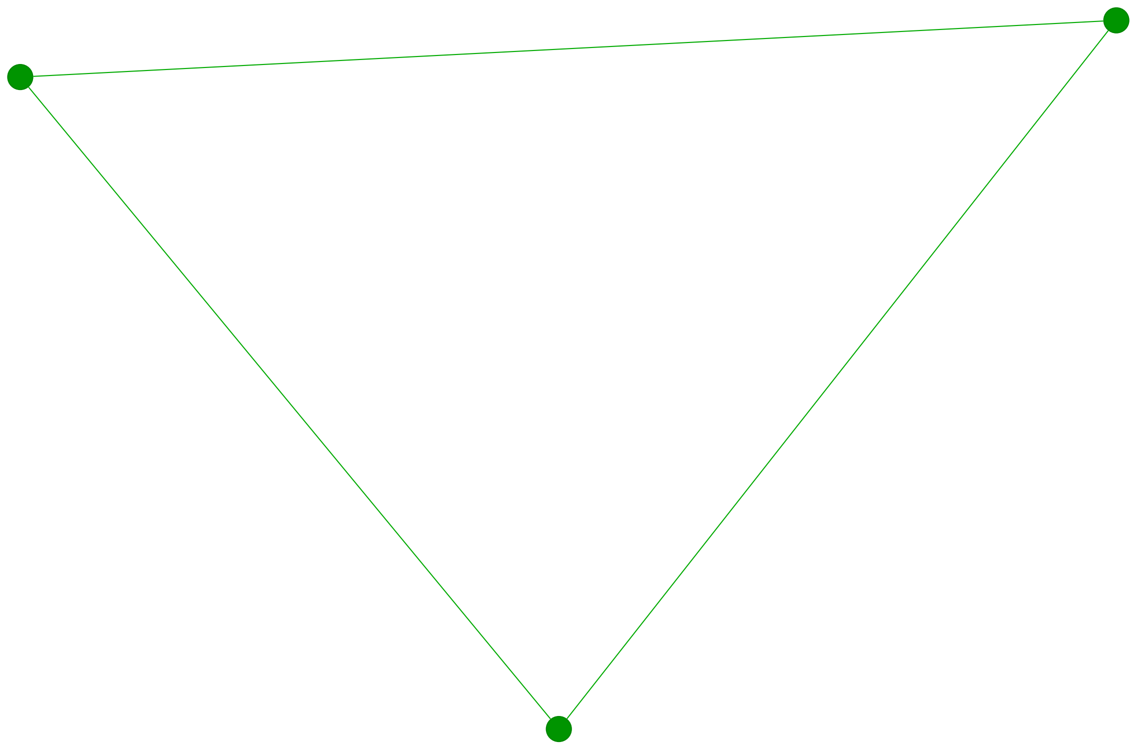
Proofs via induction have not yet been successful. Tait introduced the idea of coloring boundary lines and using those properties to determine the coloring of interior regions, and Weirnecke attempted a proof along the same lines [4, p. 147]. No proof attempts which involve breaking the graph down into subgraphs based on distance labels seem to exist, although this idea does seem similar to De Morgan's original intuition about cycles of vertices which surround another vertex. There is a theorem presented by Müller in 1979 that relates to a method of coloring we will pursue [6], and our method is somewhat related to the colouring number presented by Erdős an Hanjal [7].

Research

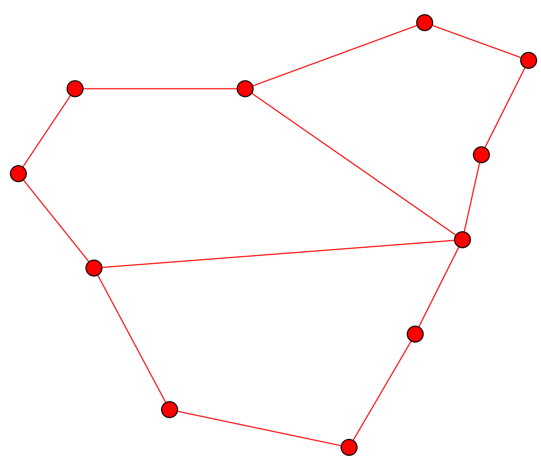
This research will investigate the coloring properties of induced subgraphs of an arbitrary maximally connected planar graph with a fixed embedding in the plane, which we we call G. The set of induced subgraphs of interest is the series of induced subgraphs of G where each element of the series, G_d, is the induced subgraph of G consisting of vertices a minimum distance of d away from some vertex in outermost cycle of G.



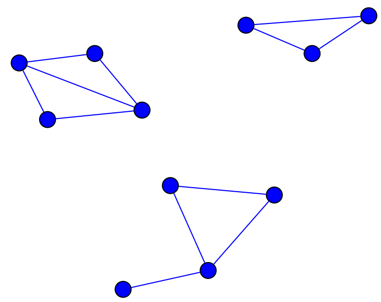
G



G_0



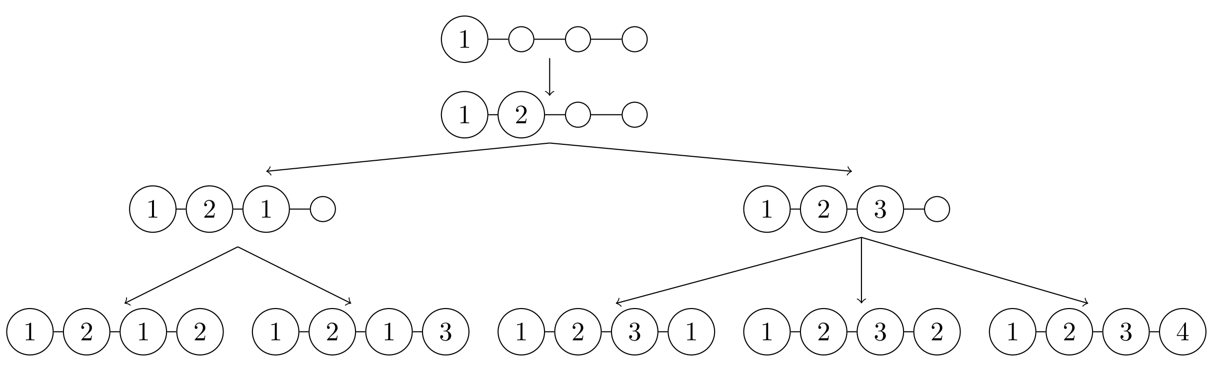
G_1



G_2

We will attempt to prove that each disjoint component of G_{d+1} must sit inside the interior region of a chordless cycle of G_d, and that the inner dual of each G_d is a forest. We will then use this as a framework for investigating the coloring properties of planar graphs.

In addition, we will explore relative chromatic polynomials. A relative chromatic polynomial is a chromatic polynomial that counts the number of unique possible colorings given a fixed coloring of a starting point, where each node in the sequence of nodes being colored can only be colored with a color previously used or a single new color.



Example of all possible relative colorings of a path of length 4 (value of relative coloring polynomial of this path would be 5)

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