

**An Exporation of Distance Labellings and
Colorings of Maximally Planar Graphs**

By

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Background and Justification for Study

The ways in which planar graphs can be colored is a historically important area of graph theory. The four color theorem, which states that any planar graph can be colored such that no 2 adjacent vertices share the same color using a set of only 4 colors, helped motivate a great deal of development in the field. The first proof of the four color theorem was discovered by Appel and Haken in 1976 and involved reducing the set of all planar graphs to a finite set of unavoidable configurations that were then checked for contradictions via a computer. In 1994, Robertson et al presented a great improvement to the proof that reduced the set of unavoidable configurations to 633, but the proof uses the same general approach and is still reliant on computers. The existing proof is reliant on Euler's formula, which can be used to determine whether an assertion about the number of vertices, edges, and faces of a planar graph contradicts the formula $v - e + f = 2$. Instead of utilizing an extension of Euler's formula, this paper will explore whether distance labellings might be used to characterize planar graphs in a way that lends itself to inductive claims about planar graph colorings.

Review of Literature

A review of relevant literature was obtained primarily through two sources: "Four Colors Suffice" by Robin Wilson and "Handbook of Combinatorics, Volume I" by Graham, Grotsche, and Lovasz. "Four Colors Suffice" presented a narrative overview of the history of the four color theorem and detailed the evolution of the existing proof. In particular, attempts at an inductive proof related to characterizing the graph in terms of some sort of distance labelling were looked for. The book stated that Tait attempted a proof by induction, but it did not involve distance labelling and was concerned with coloring the edges of a cycle that would go through every vertex in the graph once (102-105). Weirnecke attempted a proof along the same lines (147). No explicit mention of approaches that used distance labels was found. The section named "Coloring, Stable Sets and Perfect Graphs" of "Handbook of Combinatorics" was surveyed for any theorems related to distance labelling. Theorem 2.4 relates to the distance of circuits and a means of describing "unique" colorings that is relevant to the approach this paper will pursue, but does not appear to be directly relevant. A number of theorems can be found that state alternative ways of stating the 4 color theorems that might be useful when combined with the method to be explored.

Research

This research will investigate the coloring properties of induced subgraphs of an arbitrary maximally connected planar graph with a fixed embedding in the plane, which we call G . The set of induced subgraphs of interest is the series of induced subgraphs of G where each element of the series, G_d , is the induced subgraph of G consisting of vertices a minimum distance of d away from some vertex in outermost cycle of G , where d is equal to the i th element in the series. We will prove that each disjoint component of G_{d+1} must sit inside the interior region of a chordless cycle of G_d . We will then attempt to come up with a constructive argument for counting the minimum number of possible k colorings of each disjoint component $g \in G_d$, ie $P(g, k) | g \in G_d$ when each chordless cycle $s \in g$ is only allowed to have the minimum number of k colorings permitted by being arbitrarily maximally planarly connected to $g_x \in G_{d+1}$ based on the minimum value for $P(g_x, k)$, where g_k is the disjoint component of G_{d+1} that sits inside the interior region of s .

We will also briefly discuss the idea of a relative coloring. A relative coloring is a coloring formed by sequentially coloring vertices on a path starting at a fixed point, where colors are the natural numbers and the fixed point is given the color 1, such that any uncolored vertex must either be colored with a color that's been used, ie $c \in C$, or $\max(C) + 1$. This is similar to the coloring number used by Erdos.

Theorem 0.1. *Assume a maximally planar graph G with a fixed embedding in the plane, let $\lambda : V(G) \mapsto N$ be the minimum distance between $v \in V(G)$ and some point on the outer cycle of G , and let G_d be the induced graph of G_d such that $\{v \in V(G), \lambda(v) = d\} = V(G_d)$. Then the inner dual of G_d will be a forest, G_d will be within the interior region of a chordless cycle in G_{d-1} , and any two chordless cycles within G_d share at most 1 edge and 2 vertices.*

Definition 0.2. Let the relative coloring polynomial $P_{relative}(G, k)$ be number of relative colorings of G starting at a fixed point using at most k colors.

Theorem 0.3. *If L_n is a path of length n , then*

$$P_{relative}(L_n, 4) = \begin{cases} 1 & n < 3 \\ \sum_{i=0}^{n-3} 3^i + 1, & n \geq 3 \end{cases}$$

(1)

Theorem 0.4. *If S_n is a chordless cycle of length $n \geq 3$, then*

$$P_{relative}(S_n, 4) = \sum_{j=0}^{n-2} \left(\sum_{i=0}^{j+1} 3^{n-i-1} (-1)^i \right) + (1 - (-1)^{n-1})/2 \quad (2)$$