Everything Is a Wave

An Introduction to Fourier Approximations

Jai Veilleux

Edmonds College

2023

Fourier Series

Everything Is a Wave

Jai Veilleux

Fourier Series

Breaking it Down

Coefficient Formulas

Square Wave

Crash Course: Complex Numbers

Complex Fourier Series

Revisiting the Square Wave

Applications

Examples

Works Cited

Questions

- John-Baptiste Joseph Fourier - the father of Fourier Analysis (1768 - 1830)
- Fourier Series is a way to approximate any periodic function, signal, curve, etc. using sinusoids
- The general formula for a Fourier Series approximation:

$$F(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{P}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{P}\right)$$

Breaking It Down

$$F(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{P}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n t}{P}\right)$$

- $F(t)$ is the function we are approximating
- P is the period
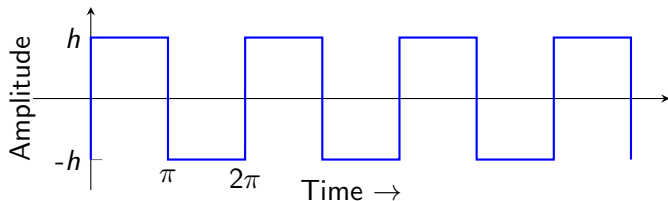- a_0, a_n, and b_n are coefficients we will need to calculate

Coefficient Formulas

- $a_0 = \dfrac{1}{P} \displaystyle\int_0^P [F(t)] \, dt$

- $a_n = \dfrac{2}{P} \displaystyle\int_0^P \left[F(t) \cdot \cos\left(\dfrac{2\pi nt}{P}\right) \right] dt$

- $b_n = \dfrac{2}{P} \displaystyle\int_0^P \left[F(t) \cdot \sin\left(\dfrac{2\pi nt}{P}\right) \right] dt$

Square wave

Fourier Series: Square Wave

1. $a_0 = \int_0^{2\pi} F(t)dt = 0$

2. $a_1 = \dfrac{2}{2\pi} \int_0^{2\pi} \left[F(t) \cdot \cos\left(\dfrac{2\pi \cdot 1 \cdot t}{2\pi} \right) \right]$

3. From 0 to π, $F(t) = h$ and from π to 2π $F(t) = -h$

4. $a_1 = \dfrac{h}{\pi} \int_0^{\pi} \left[\cos(t) \right] dt - \dfrac{h}{\pi} \int_{\pi}^{2\pi} \left[\cos(t) \right] dt = 0$

5. All $a_n = 0$ in our square wave example

6. $b_1 = \dfrac{2}{2\pi} \int_0^{2\pi} \left[F(t) \cdot \sin\left(\dfrac{2\pi \cdot 1 \cdot t}{2\pi} \right) \right] dt$

7. $b_1 = \dfrac{h}{\pi} \int_0^{\pi} \left[\sin(t) \right] dt - \dfrac{h}{\pi} \int_{\pi}^{2\pi} \left[\sin(t) \right] dt = \dfrac{4h}{\pi}$

8. If we continue to work out b_n we find a pattern:

   $b_n = \begin{cases} \frac{4h}{n\pi} & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even} \end{cases}$
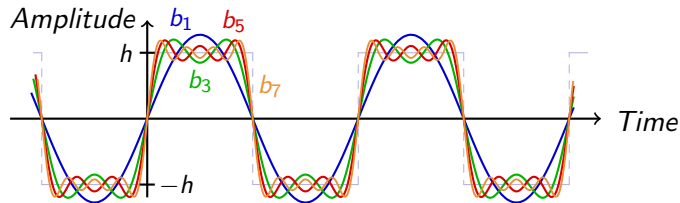
Visualization

- $F(t) = \dfrac{4h}{\pi} \left[b_1 + b_3 + b_5 + b_7 \right]$

- $b_1 = \sin(t)$

- $b_3 = \dfrac{\sin(3t)}{3}$

- $b_5 = \dfrac{\sin(5t)}{5}$

- $b_7 = \dfrac{\sin(7t)}{7}$

Crash Course: Complex Numbers

1. $i = \sqrt{-1}$
2. Standard complex number $z_s = x + iy$
3. Conjugate $z_s^* = x - iy$
4. Magnitude $||z_s|| = \sqrt{z_s z_s^*} = \sqrt{(x + iy)(x - iy)} = \sqrt{x^2 + y^2}$
5. $\theta = \begin{cases} \arctan\left(\frac{y}{x}\right), & x \geq 0 \\ \arctan\left(\frac{y}{x}\right) + 180^\circ, & x < 0 \end{cases}$
6. Polar form $z_p = ||z_s|| e^{i\theta}$
7. Converting back to standard: $\begin{cases} x = ||z_s|| \cos(\theta) \\ y = ||z_s|| \sin(\theta) \end{cases}$

1. Why use complex numbers? It makes the calculations simpler!

2. The complex representation of the Fourier Series can be written as:

$$F(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi nt}{P}}$$

3. Euler's formula relates this back to our sinusoidal representation:

$$e^{it} = \cos(t) + i\sin(t)$$

4. We can prove the complex representation is equal to the general equation using some trig identities:

$$\begin{cases} \cos(t) = \frac{e^{it}+e^{-it}}{2} \\ \sin(t) = \frac{e^{it}-e^{-it}}{2i} \end{cases}$$

Revisiting the Square Wave

1. $c_n = \dfrac{1}{P} \displaystyle\int_0^P \left[F(t) \cdot e^{-i\frac{2\pi n t}{P}} \right] dt$

2. $c_0 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \left[F(t) \cdot e^{-i\frac{2\pi \cdot 0 \cdot t}{2\pi}} \right] dt$

3. $c_0 = \dfrac{h}{2\pi} \left[\displaystyle\int_0^{\pi} [1] \, dt - \int_{\pi}^{2\pi} [1] \, dt \right] = 0$

4. $c_1 = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \left[F(t) \cdot e^{-i\frac{2\pi \cdot 1 \cdot t}{2\pi}} \right] dt$

5. $c_1 = \dfrac{h}{2\pi} \left[\displaystyle\int_0^{\pi} \left[e^{-it} \right] dt - \int_{\pi}^{2\pi} \left[e^{-it} \right] dt \right] = \dfrac{4h}{i2\pi} = \dfrac{2h}{i\pi}$

6. We find a familiar pattern if we continue to work c_n out

   $c_n = \begin{cases} \frac{2h}{in\pi}, & \pm n = \text{odd} \\ 0, & \pm n = \text{even} \end{cases}$

7. As long as $c_n^* = c_{-n}$ the resulting $F(t)$ will be real numbers

Applications

Everything Is a Wave

Jai Veilleux

Fourier Series

Breaking it Down

Coefficient Formulas

Square Wave

Crash Course: Complex Numbers

Complex Fourier Series

Revisiting the Square Wave

Applications

Examples

Works Cited

Questions

1. Fourier approximations are used in many fields including:
   - ▶ Signal processing
   - ▶ Audio
   - ▶ Electrical Engineering
   - ▶ AI
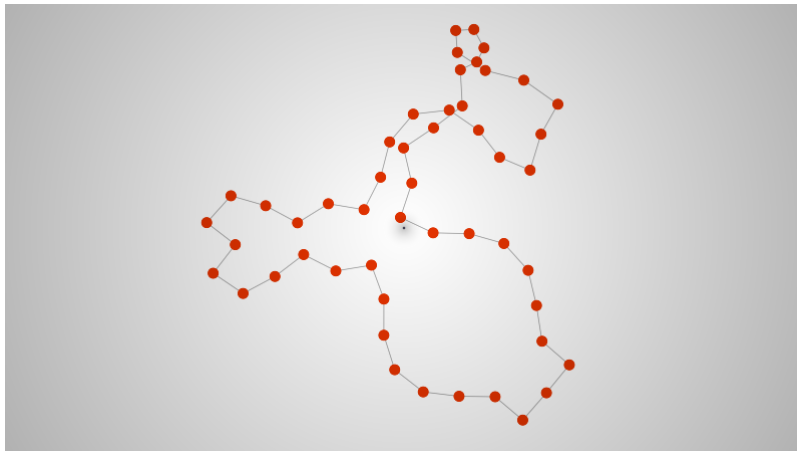   - ▶ Graphical Programming
   - ▶ Computer Vision
   - ▶ And more!
2. It can be taken further to solve more complex problems:
   - ▶ Fourier Transforms are used to approximate non-periodic functions
   - ▶ Fourier Interpolation can approximate a path between points
   - ▶ Fast Fourier Transform is a computational method that takes less steps to solve

Example 1: Fourier Approximation

Video (external viewer)

Example 2: Fourier Interpolation
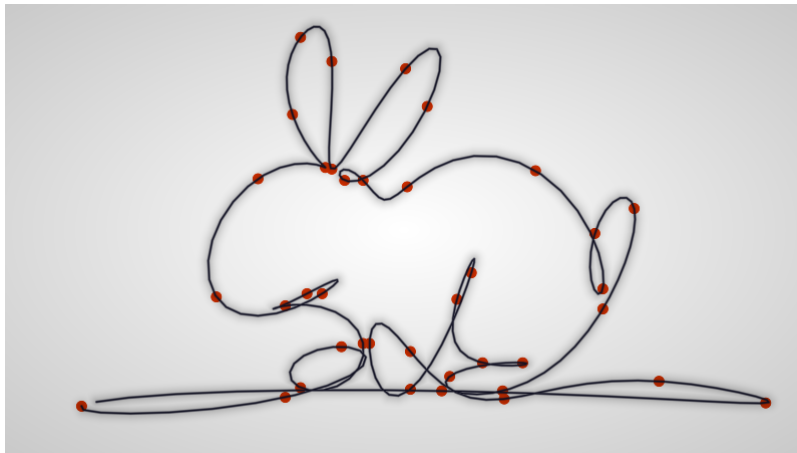
Video (external viewer)

Example 3: Image Files

▶ Our waves can be represented as color data, which is used for image synthesis and JPG compression
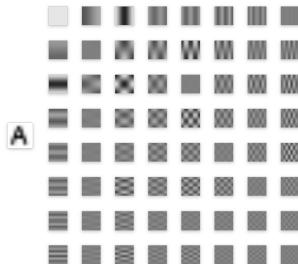
Works Cited Pt. 1

- Inigo Quilez: Fourier Series https://iquilezles.org/articles/fourier/
- Inigo Quilez: Visual for Example 1 & 2
- Better Explained: An Interactive Guide to the Fourier Transform https://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/
- Better Explained: Intuitive Understanding of Euler's Formula https://betterexplained.com/articles/intuitive-understanding-of-eulers-formula/
- Jez Swanson: An Interactive Introduction to Fourier Transforms https://www.jezzamon.com/fourier/
- Jez Swanson: Visuals for Example 3

Works Cited Pt. 2

▶ Joao Neto: Fourier Transform A R Tutorial
  http://www.di.fc.ul.pt/∼jpn/r/fourier/fourier.html

▶ Diego Unzueta: Fourier Transforms Animated Visualization
  https://towardsdatascience.com/fourier-transforms-animated-visualization-5bdb43b4b3d2

▶ Math is Fun: Fourier Series
  https://www.mathsisfun.com/calculus/fourier-series.html

▶ Darryl Morrell: Fourier Series Example
  https://www.youtube.com/watch?v=ci1gnhE8Kv8

▶ thefouriertransform.com:
  https://www.thefouriertransform.com/series/fourier.php

Works Cited Pt. 3

▶ Izaak Neutelings: Code for visual on "Visualization" slide
  https://tikz.net/fourier_series/

▶ Torbjorn T: Code for visual on "Square Wave" slide
  https://tex.stackexchange.com/a/113050

▶ Keith Lantz: FFT Rendering of an ocean, visuals on Questions slide
  https://www.keithlantz.net/2011/11/ocean-simulation-part-two-using-the-fast-fourier-transform/

Questions?
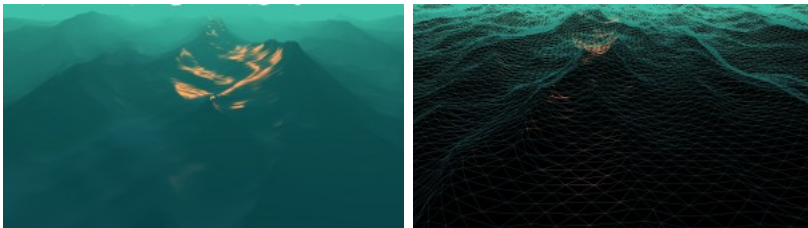
Waves rendered with Fast Fourier Transform!