Everything Is a Wave

An Introduction to Fourier Approximations

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Fourier Series

- John-Baptiste Joseph Fourier - the father of Fourier Analysis (1768 - 1830)
- Fourier Series is a way to approximate any periodic function, signal, curve, etc. using sinusoids
- The general formula for a Fourier Series approximation:

$$F(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{P}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{P}\right)$$

Breaking It Down

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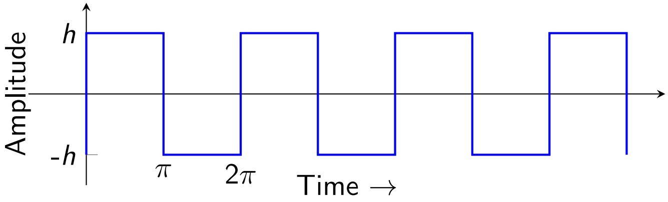
- $F(t)$ is the function we are approximating
- P is the period
- a_0, a_n, and b_n are coefficients we will need to calculate

Coefficient Formulas

- $a_0 = \dfrac{1}{P} \displaystyle\int_0^P [F(t)]\, dt$
- $a_n = \dfrac{2}{P} \displaystyle\int_0^P \left[F(t) \cdot \cos\left(\dfrac{2\pi n t}{P}\right)\right] dt$
- $b_n = \dfrac{2}{P} \displaystyle\int_0^P \left[F(t) \cdot \sin\left(\dfrac{2\pi n t}{P}\right)\right] dt$

Square wave



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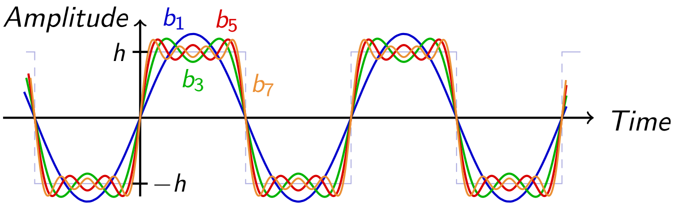
8. If we continue to work out b_n we find a pattern:

$$b_n = \begin{cases} \frac{4h}{n\pi} & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even} \end{cases}$$

Visualization

- $F(t) = \frac{4h}{\pi}\left[b_1 + b_3 + b_5 + b_7\right]$
- $b_1 = \sin(t)$
- $b_3 = \frac{\sin(3t)}{3}$
- $b_5 = \frac{\sin(5t)}{5}$
- $b_7 = \frac{\sin(7t)}{7}$



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6. Polar form $z_p = ||z_s|| e^{i\theta}$
7. Converting back to standard: $\begin{cases} x = ||z_s|| \cos(\theta) \\ y = ||z_s|| \sin(\theta) \end{cases}$

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4. We can prove the complex representation is equal to the general equation using some trig identities:
$$\begin{cases} \cos(t) = \frac{e^{it}+e^{-it}}{2} \\ \sin(t) = \frac{e^{it}-e^{-it}}{2i} \end{cases}$$

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7. As long as $c_n^* = c_{-n}$ the resulting $F(t)$ will be real numbers

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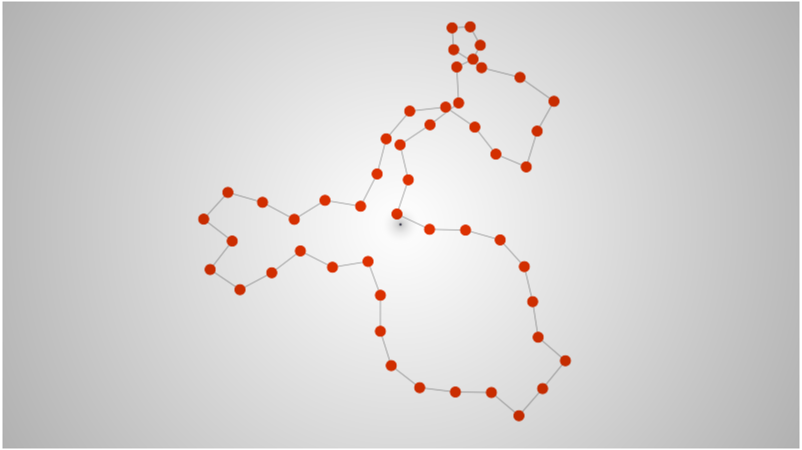
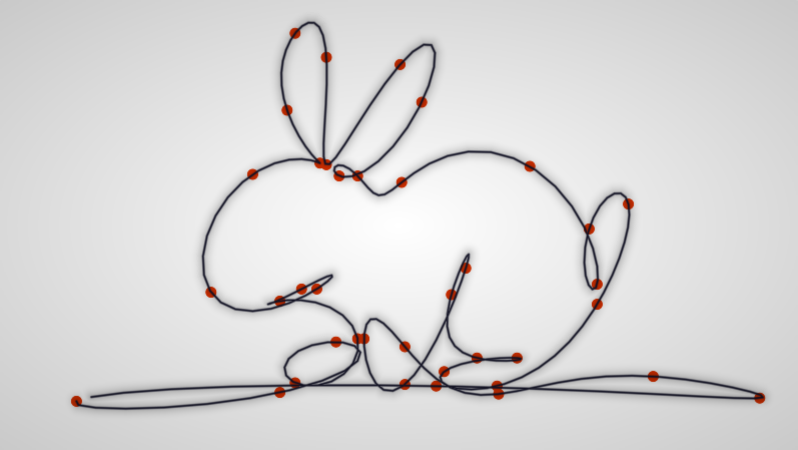
   - Fourier Interpolation can approximate a path between points
   - Fast Fourier Transform is a computational method that takes less steps to solve

Example 1: Fourier Interpolation



Video (external viewer)

Example 2: Fourier Transform



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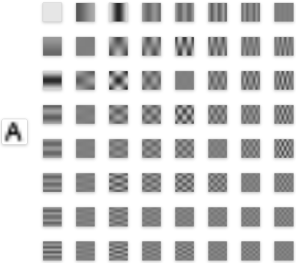
Example 3: Image Files

- Our waves can be represented as color data, which is used for image synthesis and JPG compression



Works Cited Pt. 1

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- Inigo Quilez: Visual for Example 1 & 2
- Better Explained: An Interactive Guide to the Fourier Transform https://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/
- Better Explained: Intuitive Understanding of Euler's Formula https://betterexplained.com/articles/intuitive-understanding-of-eulers-formula/
- Jez Swanson: An Interactive Introduction to Fourier Transforms https://www.jezzamon.com/fourier/
- Jez Swanson: Visuals for Example 3

Works Cited Pt. 2

- Joao Neto: Fourier Transform A R Tutorial
  http://www.di.fc.ul.pt/∼jpn/r/fourier/fourier.html
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  https://towardsdatascience.com/fourier-transforms-animated-visualization-5bdb43b4b3d2
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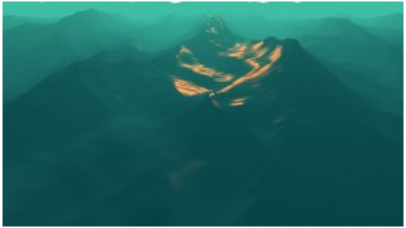
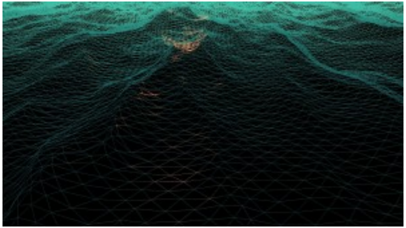
  https://www.mathsisfun.com/calculus/fourier-series.html
- Darryl Morrell: Fourier Series Example
  https://www.youtube.com/watch?v=ci1gnhE8Kv8
- thefouriertransform.com:
  https://www.thefouriertransform.com/series/fourier.php

Works Cited Pt. 3

- Izaak Neutelings: Code for visual on "Visualization" slide https://tikz.net/fourier_series/
- Torbjorn T: Code for visual on "Square Wave" slide https://tex.stackexchange.com/a/113050
- Keith Lantz: FFT Rendering of an ocean, visuals on Questions slide https://www.keithlantz.net/2011/11/ocean-simulation-part-two-using-the-fast-fourier-transform/

Questions?

Waves rendered with Fast Fourier Transform!