Game Theory: Homework #6

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Collaboration. Tsatsral Mendsuren and I have discussed whether we had the same results for questions, meaning checking if we had a proof or a counterexample for the same questions. For the left to right proof in question 2 Tsatsral corrected my notation and we discussed the symmetry proof in question 3. Marta Freixo Lopes and I discussed the left to right proof in question 2 as well.

Exercise 1

In this exercise I will shortly discuss some of the basic structural properties often used to describe TU games. We will focus on terms *cohesiveness* and *monotonicity*, either proving or disproving three statements about these basic properties.

Definitions

- **Cohesiveness**: A cohesive TU-game means that when we have $N = C_1 \uplus \cdots \uplus C_K$, it implies $v(N) \geq v(C_1) + \cdots + v(C_K)$
- **Monotonicity**: A monotonic TU-game means that (for all $C, C' \subseteq N$) if we have $C \subseteq C'$ it implies $v(C) \leq v(C')$

Part (a)

In this first part I will **disprove** the statement "*Cohesiveness implies monotonicity*". I will do this by proposing a TU-game that is cohesive, but **not** monotonic.

Consider the following 3-player TU-game $\langle N, v \rangle$, with $N = \{1, 2, 3\}$, in which player 1 is able to generate surplus on their own, but player 2 and 3 are not:

$$\begin{array}{lll} v(\{1\}) = 5 & v(\{1,2\}) = 4 & \\ v(\{2\}) = 0 & v(\{1,3\}) = 6 & v(N) = 10 \\ v(\{3\}) = 0 & v(\{2,3\}) = 5 & \end{array}$$

Using the definitions from above, we can see that this proposed game (where $v(N) = 10$) is indeed cohesive.

$$\begin{array}{lll} v(\{1\}) + v(\{2\}) + v(\{3\}) & = 5 + 0 + 0 & \leq 10 ✓ \\ v(\{1,2\}) + v(\{3\}) & = 4 + 0 & \leq 10 ✓ \\ v(\{1,3\}) + v(\{2\}) & = 6 + 0 & \leq 10 ✓ \\ v(\{2,3\}) + v(\{1\}) & = 5 + 5 & \leq 10 ✓ \\ v(\{1,2,3\}) & = 10 & \leq 10 ✓ \end{array}$$

Assume that $C = \{1\}$ and $C' = \{1, 2\}$, meaning that $C \subseteq C'$. We can also calculate the surplus generated by these two coalitions: $v(C) = v(\{1\}) = 5$ and $v(C') = v(\{1, 2\}) = 4$. This game is not monotonic, since $v(C) \nleq v(C')$ ✗. In conclusion, this game is cohesive yet **not** monotonic, therefore cohesiveness does not imply monotonicity.

Part (b)

In this second part I will **disprove** the statement "*Monotonicity implies cohesiveness*". I will do this by proposing a TU-game that is monotonic, but **not** cohesive.

Consider the following 3-player TU-game $\langle N, v\rangle$, with $N = \{1, 2, 3\}$, in which player 3 is able to generate surplus on their own, but player 1 and 2 are not:

$$\begin{array}{lll} v(\{1\}) = 0 & v(\{1,2\}) = 7 & \\ v(\{2\}) = 0 & v(\{1,3\}) = 6 & v(N) = 10 \\ v(\{3\}) = 4 & v(\{2,3\}) = 5 & \end{array}$$

This game is monotonic since for all $C, C' \subseteq N$, if we have $C \subseteq C'$ we can see that all $v(C) \leq v(C')$ ✓. However, this game is **not** cohesive, since:

$$v(\{3\}) + v(\{1,2\}) = 4 + 7 = 11 > v(N) ✗$$

Part (c)

In this third part I will **disprove** the statement "*All simple games are monotonic*". I will do this by proposing a simple TU-game that is not monotonic. As per the definition, a *simple game* is a TU-game $\langle N, v\rangle$ for which it holds that $v(C) \in \{0, 1\}$ for every possible coalition $C \subseteq N$, and $v(N) = 1$.

Consider the following 3-player *simple* TU-game $\langle N, v\rangle$, with $N = \{1, 2, 3\}$, in which all players are able to generate surplus on their own:

$$\begin{array}{lll} v(\{1\}) = 1 & v(\{1,2\}) = 0 & \\ v(\{2\}) = 1 & v(\{1,3\}) = 0 & v(N) = 1 \\ v(\{3\}) = 1 & v(\{2,3\}) = 0 & \end{array}$$

This is indeed a simple TU-game ✓, however it is **not** monotonic. Pick for example $C = \{1\}$ and $C' = \{1, 2\}$, meaning that $C \subseteq C'$. When we calculate the surplus generated by these two coalitions we see: $v(C) = v(\{1\}) = 1$ and $v(C') = v(\{1, 2\}) = 0$. This game is therefore not monotonic, since $v(C) \nleq v(C')$ ✗.

Exercise 2

In the lecture it was discussed that a simple game has a nonempty core if and only if it has at least one veto player. Building further on the proof behind this proposition, in this exercise we aim to prove the following representation theorem for the core in simple games:

For a simple game with at least one veto player, an imputation is in the core if and only if it makes a zero payment to every player who is not a veto player.

1. Left to right (⇒)

To prove this theorem I will first start with the left to right (⇒) proof. Here, I will prove that if an imputation is in the core, then all non-veto players will be given a zero payment. I will do this by showing that when a non-zero payment is made to a player, by definition that player has to be a veto-player. This proof is similar to the one shown in lecture 11 (slide 15).

We assume an imputation $\boldsymbol{x}$ and we assume $\boldsymbol{x}$ is in the core. Suppose there are $k \geq 1$ players that receive a non-zero payment, captured in a set called ℓ.

Using the rule of efficiency we know that $\sum_{i \in N} x_i = 1$. Let's now take any $C \subseteq N$, from which we remove any player, or combination of players, that are in ℓ (the set of players that received a non-zero payment). We will then have that $\sum_{i \in C} x_i < 1$. This means that $v(C) = 0$.

What this means intuitively, is that when you remove any player, or combination of players, that have received a non-zero payment from a coalition; that coalition becomes a losing coalition. By definition this means that those players receiving a non-zero payment are actually veto-players. Therefore, when we assume an imputation $\boldsymbol{x}$ and we assume $\boldsymbol{x}$ is in the core, all non-veto players will be given a zero payment.

2. Right to left ($\Leftarrow$)
To prove the right to left part of the theorem I will show that when a zero-payment is made to every player who is not a veto-player, it means that an imputation $\boldsymbol{x}$ is in the core.

Suppose we have ≥ 1 non-veto players and make a zero payment to all of them. We assume an imputation $\boldsymbol{x}$. Then $\boldsymbol{x}$ is in the core for:

- Every *winning* coalition C: Efficiency in general states that $\sum_{i \in N} x_i = v(N)$. In the coalitions we are looking at now we assumed a zero-payment to all *non-veto* players. Also, all *veto*-players have to be in a winning coalition. Therefore $\sum_{i \in C} x_i = 1$, which is equal to $v(C)$ ✓

- Every *losing* coalition C: Similar to the lecture we have that $\sum_{i \in C} x_i \geq 0$ which is $\geq v(C)$ ✓

This means that when we make a zero payment to all non-veto players imputation $\boldsymbol{x}$ is in the core.

Interpretation
This proof gives us a little insight into how powerful veto players are in the division of surplus for solutions that are in the core. Veto players are the only players that will ever receive any payment larger than 0, otherwise the imputation is not in the core. Depending on how you want to interpret 'positive' or 'negative', this result is very interesting and maybe shows that solutions in the core are not always as optimal. It can be seen as a negative result that non-veto players in this scenario always receive zero payment.

Exercise 3

In this exercise we talk about the four axioms characterising the Shapley value for TU-games. These axioms are *efficiency*, *symmetry*, *dummy player axiom* and *additivity*. For each of these axioms I will prove they are also satisfied by the Banzhaf value or give a counterexample to demonstrate they are not satisfied by the Banzhaf value.

1. Efficiency
The Banzhaf value does **not** satisfy the efficiency axiom. A detailed example of this was given in class (lecture 12, slide 6). The efficiency axioms states that the sum of all payments in the payment vector $\boldsymbol{x}$, should equal the surplus of the grand coalition: $\sum_{i \in N} x_i(N, v) = v(N)$.

Consider the following 3-player TU-game $\langle N, v \rangle$, with $N = \{1, 2, 3\}$, in which no single player can generate any surplus on her own:

$$\begin{array}{lll} v(\{1\}) = 0 & v(\{1,2\}) = 7 & \\ v(\{2\}) = 0 & v(\{1,3\}) = 6 & v(N) = 10 \\ v(\{3\}) = 0 & v(\{2,3\}) = 5 & \end{array}$$

Calculating the Banzhaf values will give us the following payoffs for each player, calculated in the same way as slide 6 of lecture 12:

$$\begin{aligned} \beta_1(N, v) &= \frac{1}{4} \cdot (0 + 7 + 6 + 5) = \frac{18}{4} \\ \beta_2(N, v) &= \frac{1}{4} \cdot (0 + 7 + 5 + 4) = \frac{16}{4} \\ \beta_3(N, v) &= \frac{1}{4} \cdot (0 + 6 + 5 + 3) = \frac{14}{4} \end{aligned}$$

The total amount of payoffs is now $\frac{18}{4} + \frac{16}{4} + \frac{14}{4} = 12$. This amount is higher than $v(N)$, therefore the Banzhaf value does ***not*** satisfy the efficiency axiom.

2. Symmetry
To explain it intuitively, the symmetry axiom states that interchangeable players should get equal payoffs. In this part, I will prove that the Banzhaf value satisfies this axiom. Formally, we can describe the symmetry axiom as: if $v(C \cup \{i\}) = v(C \cup \{j\})$ for all $C \subseteq N \backslash \{i, j\}$, then $x_i(N, v) = x_j(N, v)$. Let's start with the formula to calculate the Banzhaf value:

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \cdot \sum_{C \subseteq N \backslash \{i\}} v(C \cup \{i\}) - v(C)$$

In the symmetry axiom we are talking about $C \subseteq N \backslash \{i, j\}$. We want to rewrite the Banzhaf formula to a similar form, which we can do as follows:

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \cdot \left(\left(\sum_{C \subseteq N \backslash \{i,j\}} v(C \cup \{i\}) - v(C) \right) + \left(\sum_{C \subseteq N \backslash \{i\}, j \in C} v(C \cup \{i\}) - v(C) \right) \right)$$

In this formula we have split the original sum into two sums; the left sum contains all the $C \subseteq N \backslash \{i, j\}$; the right sum completes the set by summing over all $C \subseteq N \backslash \{i\}$ that contain j (denoted by $j \in C$). Together, these two sums form the sum over $C \subseteq N \backslash \{i\}$ that we had in the original Banzhaf formula above.

The right sum can actually be written in a different way. It can be interpreted as the sum over all coalitions in N that do not contain $\{i\}$, but must contain $\{j\}$. In the sum we then take the $v(C \cup \{i\}) - v(C)$. We can rewrite this right sum as follows:

$$\sum_{C \subseteq N \backslash \{i\}, j \in C} v(C \cup \{i\}) - v(C) = \sum_{C \subseteq N, \{i,j\} \in C} v(C) - v(C \backslash \{i\})$$

In this sum we sum over all the coalitions $C \subseteq N$ that contain $\{i, j\}$ and then in the sum we take $v(C) - v(C \backslash \{i\})$. Filled in in the formula this looks as follows:

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \cdot \left(\left(\sum_{C \subseteq N \backslash \{i,j\}} v(C \cup \{i\}) - v(C) \right) + \left(\sum_{C \subseteq N, \{i,j\} \in C} v(C) - v(C \backslash \{i\}) \right) \right)$$

According to the symmetry axiom we have $v(C \cup \{i\}) = v(C \cup \{j\})$ for all $C \subseteq N \backslash \{i, j\}$, meaning we can rewrite this to:

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \cdot \left(\left(\sum_{C \subseteq N \backslash \{i,j\}} v(C \cup \{j\}) - v(C) \right) + \left(\sum_{C \subseteq N, \{i,j\} \in C} v(C) - v(C \backslash \{j\}) \right) \right)$$

Using the same rewriting we have just done to arrive at this equation, but now backwards, we can arrive at the conclusion that:

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \cdot \sum_{C \subseteq N \backslash \{j\}} v(C \cup \{j\}) - v(C) = \beta_j(N, v)$$

This means that when we have $v(C \cup \{i\}) = v(C \cup \{j\})$ for all $C \subseteq N \backslash \{i, j\}$ it means that $\beta_i(N, v) = \beta_j(N, v)$. Therefore, the Banzhaf value **satisfies** the symmetry axiom.

3. Dummy player axiom

The Dummy player axiom states that if we have a 'dummy player', which can be explained as a player i whose marginal contribution is always the same, player i's payment should be equalling the surplus of the coalition of just player i. Again, we can describe this formally as: if $v(C \cup i) - v(C) = v(\{i\})$ for all coalitions $C \subseteq N \backslash i$, then we should have $x_i(N, v) = v(\{i\})$

The Banzhaf value satisfies this axiom, which I will prove using the formula to calculate the Banzhaf value:

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \cdot \sum_{C \subseteq N \backslash \{i\}} v(C \cup \{i\}) - v(C)$$

When player i is a 'dummy player', using the formal definition as explained above, we can rewrite this formula as follows:

$$\beta_i(N, v) = \frac{1}{2^{n-1}} \cdot \left(\sum_{C \subseteq N \backslash \{i\}} 1 \right) \cdot v(\{i\})$$

Here, the $\left(\sum_{C\subseteq N\backslash\{i\}} 1\right)$ indicates that we have to multiply $v(\{i\})$ with the amount of coalitions in $N\backslash\{i\}$. This is equal to 2^{n-1}, meaning we can therefore rewrite the above equation to:

$$\beta_i(N,v) = \frac{1}{2^{n-1}} \cdot 2^{n-1} \cdot v(\{i\})$$

It is obvious to see this is equal to:

$$\beta_i(N,v) = 1 \cdot v(\{i\} = v(\{i\})$$

Meaning we have satisfied the dummy player axiom for the Banzhaf value.

4. Additivity

The additivity axiom states that we should have $x_i(N, v_1 + v_2) = x_i(N, v_1) + x_1(N, v_2)$ for the characteristic function $[v_1 + v_2] : C \mapsto v_1(C) + v_2(C)$. For the Banzhaf value this is quite straightforward to prove as follows:

$$\beta_i(N, v_1 + v_2) = \frac{1}{2^{n-1}} \cdot \sum_{C\subseteq N\backslash\{i\}} v_1(C \cup \{i\}) - v_1(C) + v_2(C \cup \{i\}) - v_2(C)$$

We can now group split the sum, while keeping the factors containing v_1 together and keeping the factors containing v_2 together:

$$\beta_i(N, v_1{+}v_2) = \frac{1}{2^{n-1}} \cdot \left(\sum_{C\subseteq N\backslash\{i\}} v_1(C \cup \{i\}) - v_1(C) \right) + \frac{1}{2^{n-1}} \cdot \left(\sum_{C\subseteq N\backslash\{i\}} v_2(C \cup \{i\}) - v_2(C) \right)$$

We can recognize these values on the left and right side of the + sign as being $\beta_i(N, v_1)$ and $\beta_i(N, v_2)$ respectively. Therefore:

$$\beta_i(N, v_1 + v_2) = \beta_i(N, v_1) + \beta_i(N, v_2)$$

This demonstrates that the Banzhaf value satisfies the additivity axiom.