Game Theory: Homework #4

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Collaboration. For this homework we have discussed our answers with Boaz Beukers, Bram Slangen, Marta Freixo Lopes and Ioanna Gogou. Didier and Bram discussed how to display utilities in a tree for exercise 3. Marta and Didier discussed what happens when a game off odds/evens ends after 2 rounds for exercise 3. Ioanna and Didier discussed exercise 1 after both had completed it. Tsatsral and Marta discussed exercise 3 and the equivalence relations.

Exercise 1

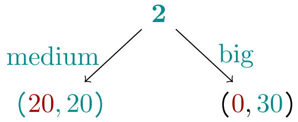
Part (a)

This game can be formally defined as an extensive-form game which is a tuple:
$<N, A, H, Z, \underline{i}, \underline{A}, \sigma, \boldsymbol{u}>$, with the following definitions:

- $N = \{1, 2\}$ is the set of two players;
- $A = \{small, pass, medium, big\}$ is the set of actions;
- $H = \{h_1, h_2\}$ is the set of choice nodes;
- $Z = \{z_1, z_2, z_3\}$ is the set of outcome nodes;
- $\underline{i} : H \to N$ is the turn function that assigns a player who gets to take an action to each non-terminal node, and we have $\underline{i}(h_1) = 1, \underline{i}(h_2) = 2$;
- $\underline{A} : H \to 2^A$ is the action function that assigns to a choice a set of playable actions, and we have $\underline{A}(h_1) = \{small, pass\}, \underline{A}(h_2) = \{medium, big\}$;
- $\sigma : H \times A \to H \cup Z$ with $\sigma(h_1, small) = z_1, \sigma(h_1, pass) = 2, \sigma(h_2, medium) = z_2, \sigma(h_2, big) = z_3$;
- $\boldsymbol{u} = (u_1, u_2)$ where $u_i : Z \to \mathbb{R}$ with $u_1(z_1) = 10, u_1(z_2) = 20, u_1(z_3) = 0$ and $u_2(z_1) = 0, u_2(z_2) = 20, u_2(z_3) = 30$.

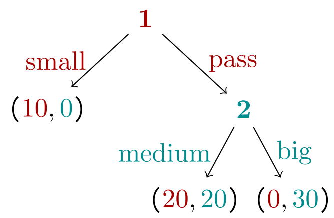
Part (b)

The only (pure) subgame-perfect equilibria is the strategy profile ($small, big$). To derive it, we do backward induction where we first identify the equilibria in the bottom most subtree and work upwards recursively. In this game, the bottom-most subtree is:



Note that this is a one-player subgame with a dominant strategy big for the second player as the payoff of choosing big is higher than that of chosing medium ($30 > 20$).

Moving one level upward, we have the whole tree:



Note that at node **1** and conditioned that the second player will choose big at node **2**, the dominant strategy of the first player is choosing small as its payoff 10 is higher than that of choosing pass (then big will be chosen and the first player will get 0).

Since the root is reached, we have found the one and only (pure) strategy profile which leads to the subgame-perfect equilibria by backward induction, namely, $(small, big)$.

Part (c)

The normal-form game is represented with the following payoff matrix.

	medium	big
small	10, 0	10, 0
pass	20, 20	0, 30

Part (d)

The **pure Nash equilibria** is reached at the strategy $(small, big)$. The column player has a strictly dominant strategy *big* as the payoff of choosing this action is always higher than choosing *medium*. Given that the column player will always play *big*, then the best response of the row player is *small*. Hence, $(small, big)$ is where both players have their best response and have no incentive to deviate, given the choice of the other player. By definition, this is the pure Nash equilibria.

The **mixed Nash equilibria** is reached at the strategy $(1, q)$ with $q \in (0, \frac{1}{2}]$. To derive it, let the strategy of the row player be $(p, 1-p)$ and that of the column player be $(q, 1-q)$, where p and q denotes the probability of choosing the action *small* and *medium*.

The mixed equilibria is reached when the expected payoff of the two actions is the same for both players, meaning no particular incentive is there for the players to choose a different strategy. Hence, we have:

$$10q + 10(1-q) = 20q + 0(1-q);$$
$$0p + 20(1-p) = 0p + 30(1-p).$$

Solving them leads to $p = 1$ and $q = 1/2$.

However, we also notice that the payoff of both actions, 10, is the same for the column player as long as the row player chooses *small*. This is also true intuitively given the rules of the game, because the game will already end as soon as the first player (here represented by the row player) is willing to choose *small*. When this scenario is reached, the row player does not have the incentive to deviate from the strategy as her own strategy is the best response given the strategy of the column player; the column player also does not have the incentive to deviate (if her strategy always makes the row player choose *small*), because her payoff will always be the same regardless how she chooses her action.

By definition, the strategies leading to the above-mentioned scenario are also mixed Nash equilibria. To have it, the payoff of choosing *small* should be higher than choosing *pass*. Thus, the following inequality will then hold:

$$10q + 10(1 - q) > 20q + 0(1 - q).$$

Solving it gives $q < 1/2$.

Combining all we derived, the strategy profiles $(1, q)$ with $q \in (0, \frac{1}{2}]$ are mixed Nash equilibria.

Exercise 3

In this exercise we look at the *Choose Game* done in Seinfeld. It is a two player game, in which one player plays as the 'even player' and the other player plays as the 'odd player'. Both players can choose to put up either one or two fingers. If the total amount of fingers is odd, the odd player wins; if the total amount of fingers is even, the even player wins.

Part (a)

To translate this game to an imperfect imformation game tree, we use some notation. We have an odd player (P_{odd}) and an even player (P_{even}). The game is played in a 'best of three' format, meaning whoever wins two games first is the winner. We denote the winners utility as 1 and the losers' utility as -1. The equivalence classes are shown using a dotted line. The tree of this game as an imperfect information game is shown in Figures 1 - 3.

Because the tree is very large, we have choosen to visualize it in 3 different levels, however it is important to note that this should be seen as one big tree. The three parts of the three each represent the respective rounds of the best of three Choose Game, but should be considered as one single tree and the three levels are actually connected together, shown by the . . . in the visualization.

Specifically, the 4 vertices leading to the "three dots" in Figure 1 should be connected to the roots of the 4 subtrees in Figure 2, from left to right. All the 8 vertices in Figure 2 leading to the "three dots" should be connected to another subtree. Since this specific subtree is identical for all 8 vertices, it is only displayed once in figure 3.

The nodes are the choice node with the annotation illustrating which player's turn it response to; to reiterate, P_{odd} refers to the player who bets odd sum, P_{even} refers to the player who bets even sum; the annotation on the vertices are the playable actions of each choice node (which are "putting out one finger" or "putting out two fingers" for all the choice nodes); the leaves are the outcome leaves, containing the utilities.

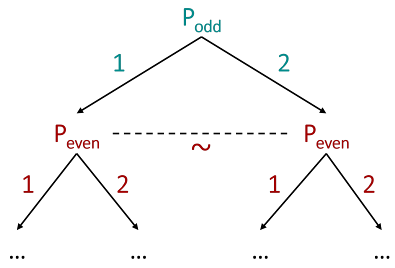


Figure 1: Round 1

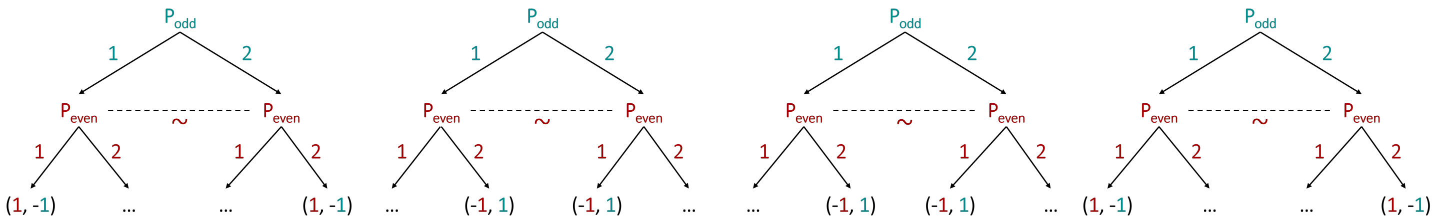


Figure 2: Round 2

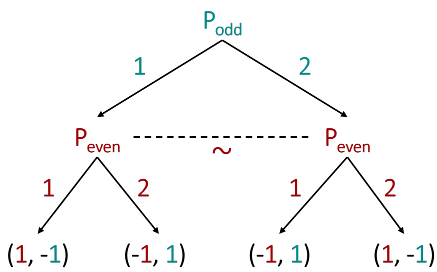


Figure 3: Round 3

Since the two players simultaneously make a choice in each round, no player knows which state she is at. To show this in a tree structure, the choice node of the second player after the choice node of the first player should be *indistinguishable*. However, since the players *know* the outcome of the finished rounds and might condition their further actions based on the existing outcome, the choice nodes across different rounds are *distinguishable*. Hence, no dotted lines for equivalence are drawn across rounds.

Moreover, the game ends as soon as one of the players has won two rounds. Outcome leaves are attached to the paths of this scenario in Figure 2. If no player wins after two rounds, this means each of them has own one round and the result of the entire game is solely dependent on the outcome of the third round. The game can be interpreted as a one round game from now. Thus, the subtree representing the third round is the same in all eight cases if the third round is necessary.

The statistics of the game are the following:

- the number of outcome nodes: $|Z| = 40$;
- the number of choice nodes for the first player (P_{odd}): $|H_1| = 13$;
- the number of equivalence classes of choice nodes distinguishable by the first player (P_{odd}): $|H_1/\sim 1| = 0$;
- the number of choice nodes for the second player (P_{even}): $|H_2| = 26$;
- the number of equivalence classes of choice nodes distinguishable by the second player (P_{even}): $|H_2/\sim 2| = 13$;

Part (b)

In this question we will discuss how many pure strategies there are for each player in this game.

The pure strategy of P_{odd} has the structure such as $1; 1-2-1-1; 1-1-2-1-2-2-1-1$. 1 means the action she can choose in the first round, $1-2-1-1$ means the action she can (potentially) choose in the 4 distinguishable choice nodes on the same level of the tree representing the second round, $1-1-2-1-2-2-1-1$ means the action she can (potentially) choose in the 8 distinguishable choice nodes on the same level of the tree representing the third round. There are two playable actions at each choice node, so the total number of pure strategies of P_{odd} is $2 \cdot 2^4 \cdot 2^8 = 8192$.

Similarly, the pure strategy of P_{even} also has the structure such as $1; 1-2-1-1; 1-2-1-1-2-1-1-2$. There are more choice nodes for P_{even} but the structure of the pure strategies is the same as the odd player. This is because the choice nodes within the same round of the game are considered equivalent for P_{even} and the pure strategies are the same for these equivalent choice nodes. Because there are two choices of actions to play at each pair of indistinguishable choice nodes, and there are 1, 4 and 8 pairs of equivalent choice nodes in each of the three rounds, the total number of pure stratgies is $2 \cdot 2^4 \cdot 2^8 = 8192$, the same as the number of pure strategies of P_{odd}.

Part (c)

The normal-form game corresponding to this imperfect-information game can be represented as the two-dimensional matrix in Figure 4. The annotation outside the matrix boundaries represents the number of fingers the players may put out in each round of this game. For example, 111 means the player who chooses to bet odd puts out one finger in the first round, one finger in the second round and one finger in the third round. Similarly, 212 means the player who chooses to bet even, places 2 fingers, 1 finger and 2 fingers in the first, second and third round respectively.

There are 8 rows and 8 columns in the matrix. Because the players can only choose from two playable actions in the three rounds, the total number of unique action profiles of each player across three rounds is $2^3 = 8$.

The numbers in each cell means the payoff of its corresponding player (distinguished by color). The value of them is determined by the outcome of the corresponding actions chosen by the players represented in each cell. For example, in the cell of 111 and 221, the player who bets odd will already win after two rounds, so the payoff of the row/even player who bets even is -1 and the payoff of the column/odd player who bets odd is 1 (Since the actions they choose during the third round do not influence the outcome of the entire game, the payoff for choosing (111, 222), choosing (112, 221) and choosing (112, 222) are (-1, 1) as well).

	111	121	122	112	211	221	222	212
111	1, -1	1, -1	-1, 1	1, -1	1, -1	-1, 1	-1, 1	-1, 1
121	1, -1	1, -1	1, -1	-1, 1	-1, 1	1, -1	-1, 1	-1, 1
122	-1, 1	1, -1	1, -1	1, -1	-1, 1	-1, 1	1, -1	-1, 1
112	1, -1	-1, 1	1, -1	1, -1	-1, 1	-1, 1	-1, 1	1, -1
211	1, -1	-1, 1	-1, 1	-1, 1	1, -1	1, -1	-1, 1	1, -1
221	-1, 1	1, -1	-1, 1	-1, 1	1, -1	1, -1	1, -1	-1, 1
222	-1, 1	-1, 1	1, -1	-1, 1	-1, 1	1, -1	1, -1	1, -1
212	-1, 1	-1, 1	-1, 1	1, -1	1, -1	-1, 1	1, -1	1, -1

Figure 4: The game of Exercise 3 represented as a normal-form game

Exercise 4

In this exercise we write a Python program that studies how well people can play a game called Simplified Poker and how well certain strategies fare against the equilibrium strategies that were discussed in class.

Simplified Poker
In Simplified Poker two players agree to play a simplified version of Poker, where both players bet 1\$. There are three cards in this game: the Jack (lowest), the Queen and the King (highest). Both players get dealt one of these cards at random and player 1 is then asked whether he wants to *bet* 1 extra dollar or *fold*. If player 1 chooses to *bet*, player 2 is asked the same question.

If a player chooses to *fold*, the other player gets the pot (and therefore wins 1\$); if both players choose to *bet*, we get to a showdown in which the player with the highest card takes the whole put (and therefore winning 2\$).

Obvious strategies
To evaluate how well people can play Simplified Poker, first there are some 'obvious strategies' that need to be discussed. To do this we will use the terms p_J, p_Q, p_K and q_J, q_Q, q_K to express the probabilities of players *betting* on certain cards. Here p and q denote player 1 and 2 respectively and $_J$, $_Q$ and $_K$ denote the Jack, Queen and King respectively. This means that p_J denotes the probability of player 1 betting when receiving a Jack.

As discussed in class there are some strategies that should be very obvious to each player. For example, when a player receives a King, she is guarenteed to win and will therefore always bet when receiving a King ($p_K = q_K = 1$). Similarly we can deduce that Player 2 should never bet when receiving a Jack ($q_J = 0$) and player 1 should always bet when receiving a Queen ($p_Q = 1$). These strategies, displayed in table 1, are therefore assumed by us to be 'fixed'.

Card	*Player 1*	*Player 2*
J♥	...	0.0
Q♥	1.0	...
K♥	1.0	1.0

Table 1: 'Obvious' betting strategies for both players

Remaining strategies

There are two strategies left of which it is not immediately clear to see what the best betting tacting would be: p_J and q_Q. To simulate what are good strategies, we wrote a python program that can play Simplified Poker using different strategies.

To see which strategies maximize your payoff, we simulated 101 different strategies for p_J: 0.00, 0.01, . . . , 1.00; and tested how well they did against 101 different strategies for q_Q (also $0.00, 0.01, \ldots 1.00$). The rest of the betting strategies are the 'obvious' numbers as they are displayed in table 1.

We simulated 10,000 games of Simplified Poker for every strategy combination, for a total of $101 \times 101 \times 10,000 = 102,010,000$ Simplified Poker games. The average return of player 1 was then calculated for each of the strategy combinations p_J and q_Q. The results of this empirical experiment are displayed in figure 5.

(a) Empirical values (b) Gaussian filter ($\sigma = 5$)

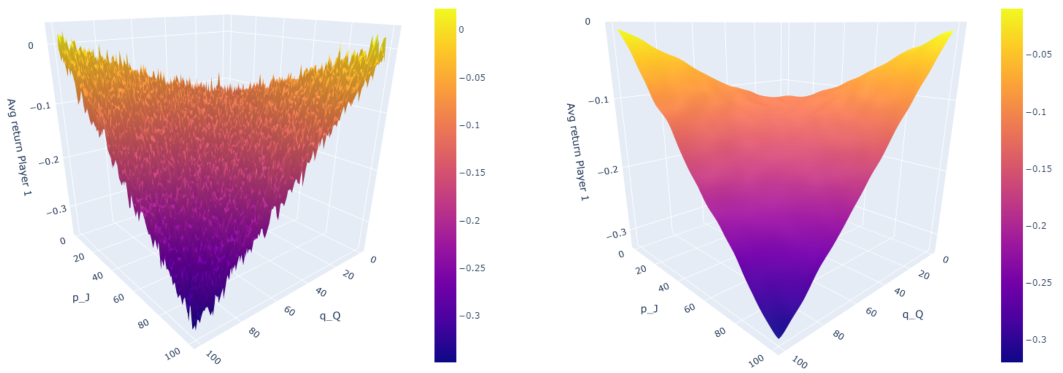


Figure 5: Here, figure 5a displays the empirical results of the 10,000 games using different p_J and q_Q and 5b visualizes the same results with a Gaussian filter to give an idea of the general trend. The q_Q axis indicates q_Q in percentages and p_J axis indicates p_J in percentages. On the Z-axis the average return per game of player 1 is shown. Since it is a zero-sum game the return of player 2 can be deduced from this number as 0 minus the return of player 1.

In this figure we can easily see what would be the best response for a player when the other player plays with a certain strategy. For example, we can clearly see that when $q_Q = 100\%$, the best response for player 1 is to play $p_J = 0\%$, since then his returns seem to be the highest (around 0). For every value of p_J and q_Q we can use this figure to see what we measured to be the best response.

Equilibrium strategies

In figure 5 there seems to be a 'flat' point in the results, which is often described as a saddle point. This point is roughly located at $p_J = q_Q = 33$ and shown in figure 6, which shows the results from a different perspective.

This point in the results special, because it shows that at this point neither player wants to unilaterally deviate from their chosen strategy. It means that if they were to unilaterally switch their strategy, the best they can do is equal their achieved score. What we have found here, is empirical evidence of the mixed equilibrium discussed in class, which stated that $p_J = q_Q = \frac{1}{3}$. For the full 3D results we would like to refer the reader to the submitted notebook.

(a) Empirical values (b) Gaussian filter ($\sigma = 5$)

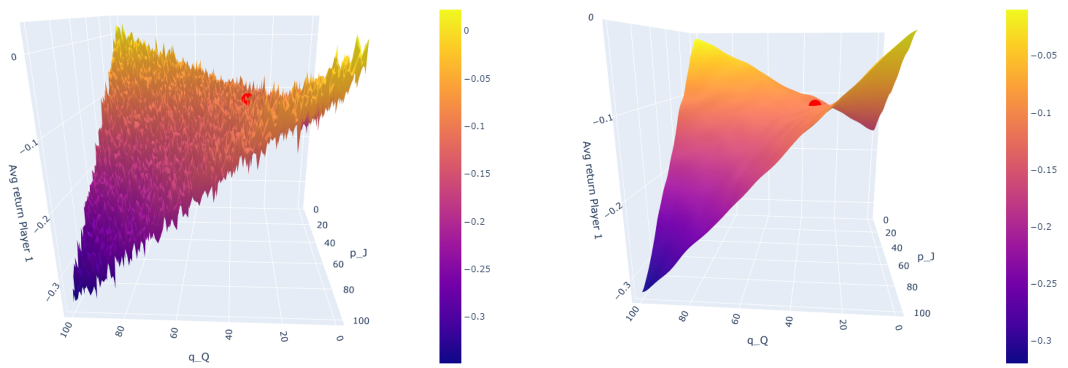


Figure 6: Visualization of the empirical results of the Simplified Poker simulations. The same visualization as figure 5, however from a different perspective and the presumed 'saddle point' highlighted in red.

Response versus equilibrium

To get a more in-depth analysis of how well players can respond when they know their opponent plays the equilibrium strategy, we have simulated two more settings of Simplified Poker, displayed in table 2 and 3.

Card	*Player 1*	*Player 2*
J♥	...	0.0
Q♥	1.0	$\frac{1}{3}$
K♥	1.0	1.0

(a) Table 2: fixed q_Q

Card	*Player 1*	*Player 2*
J♥	$\frac{1}{3}$	0.0
Q♥	1.0	...
K♥	1.0	1.0

(b) Table 3: fixed p_J

In one scenario we fixed q_Q to be $\frac{1}{3}$ and let p_J vary from 0.00 to 1.00; in the other scenario we fixed p_J and let q_Q vary from 0.00 to 1.00. Every strategy configuration played 100,000 games against the equilibrium opponent for a total of $101 \times 100,000 \times 2 = 20,200,000$

games. For both scenarios the average return of the player *not* playing the fixed strategy was calculated, which is displayed in figure 8.

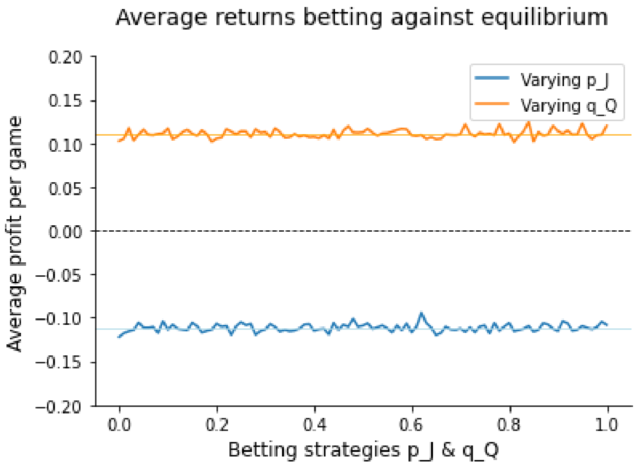


Figure 8: The average returns of a player playing against an equilibrium strategy. The blue line shows the average profit per game for player 1 under different p_J, with the light blue line displaying the average (=-0.111). The orange line shows the average profit per game for player 2 under different q_Q, with the light orange line displaying the average (=0.111).

The result in this figure show that the average profit per game seems to be constant. This means that when a player is playing against someone who is betting according to the equilibrium strategy, it does not matter which value for p_J or q_Q you choose, your return will always be similar.

The average return measured for player 1 when player 2 plays an equilibrium strategy is -0.111 (the light blue line in figure 8; and the average return measured for player 2 when player 1 plays an equilibrium strategy is 0.111 (the light orange line). This corresponds with the theory which states that when you play an equilibrium strategy as player 1 or player 2 your average return is $-\frac{1}{9}$ and $\frac{1}{9}$ respectively.

Conclusion

In this report we have found empirical evidence for the existence of a mixed Nash equilibrium. In addition we generated a graph that displays what the best strategy is under different opposing strategies. Finally, we showed that when your opponent plays an equilibrium strategy, your expected profit under varying p_J and q_Q is constant.