Game Theory: Homework #1

Didier Merk (11037172)

Collaboration. For this homework I have discussed my answers with Tsatsral Mendsuren, Boaz Beukers and Marta Freixo Lopes. This was done only after all of us had individually finished the exercises. For Boaz I corrected his possible misunderstanding of exercise 1 and explained something about the intuition behind exercise 3. With Tsatsral I had a longer conversation about the concept of Nash equilibria for question 3 and we discussed our conclusions of the other questions. Marta and I only discussed conclusions of other questions.

Exercise 1

In this exercise I will demonstrate what the pure Nash equilibria are of a normal-form game known as the *Numbers Game*.

Since we are talking about pure Nash equilibria, it is good to re-visit what they are exactly. A Nash equilibrium is a state of a normal-form game where "*no player has an incentive to unilaterally deviate from her assigned strategy*" (slide 14, lecture 1). It basically means that we have a certain action profile, and for all players their utility would remain the same or be lowered if they were to unilaterally (by themselves) pick a different action. The mathematical definition can be found on slide 14 of lecture 1.

Part (a)

The pure Nash equilibrium occurs when all players in the game pick the number 0. To find this pure Nash equilibria of the *Numbers Game*, we need to consider the intuitive definition of a pure Nash equilibrium and what happens in the game. I will do this in four steps:

1. For a pure Nash equilibrium to occur in the *Numbers Game*, all players need to 'win'.
2. For everybody to win the game, everyone needs to have chosen the same number.
3. The average of the Numbers Game needs to be equal to the winning number.
4. The one and only resulting Nash Equalibrium is therefore the action profile where everyone picks the number 0.

Explanation

1. The way the Numbers Game is set up, a player will *always* want to switch when it has not won the game, since it can just switch to the winning number and join the 'winners' of the game. This has the consequence that as long as not all players have won the game, we can not have a pure Nash equilibrium. This is because the action chosen by the one or more losing players was not the *best response* given the action profile; and they will want to 'switch' their action to picking the winning number.
2. This means that a possible pure Nash equilibrium can only occur when *all* players have won the game. It is by definition not possible that everyone wins the game,

when **not** everyone has chosen the same number. This would require everyone to be at an equal distance of the winning number and that winning number being $\frac{2}{3}$ of the average of all chosen numbers. We can therefore conclude that 'everybody winning' can only occur when everyone has chosen the winning number (as opposed to two groups being an equal distance away of the winning number).

3. So far we have been able to conclude that a pure Nash equilibrium can only happen when all players have won the game and that all players need to have guessed the winning number. Since we already concluded that for a pure Nash equilibrium to occur in this game everyone player needs to pick the same number, the average will be equal to the number every player has picked. As long as this average is **not** equal to the winning number, there will be an incentive for a player to switch to the winning number and we do not have a pure Nash equilibrium. Therefore the winning number needs to be equal to the average.

4. The game states that the winning number is equal to $\frac{2}{3}$ times the average number ($avg\_number = \frac{2}{3} * avg\_number$), and we have also concluded that for a pure Nash Equilibrium to occur the winning number needs to be equal to the average number ($win\_number = avg\_number$). A simple calculation ($avg\_number = \frac{2}{3} * avg\_number \Rightarrow avg\_number = 0$), shows that the only possible average for this to happen, is when the average is 0; which only occurs when every player has chosen the number 0.

Conclusion
We can now conclude that the scenario in which every player chooses 0 is the one and only pure Nash equilibrium of the numbers game with infinite action possibilities. This scenario is indeed a pure Nash equilibrium, because every player in this scenario has won, and any player choosing a different action than 0 in this scenario would decrease their utility, since a player can only switch to a non winning number.

Part (b)

We know slightly adjust the Numbers Game, to a scenario in which the players must choose integers. This adds another pure Nash equilibrium to the normal-form game (in addition the one from the previous question), when every player picks the number 1. To find the new pure Nash equilibria of this game, we can follow the same reasoning we gave in question 1a, up until step 3 and 4.

With this new scenario of only being able to pick integer numbers, the winning number does not have to be *exactly* equal to the average number as we stated before in step 3 and 4. Imagine the scenario where every player picks the number 1; the average will therefore be 1 and the winning number will be $\frac{2}{3}$. Since players are only allowed to pick integer numbers, no one wants to switch to a different number, because 1 is the closest possible number to the winning number.

Changing the game to have players pick only integer numbers, now means that a Nash equilibrium is also possible when the winning number *rounded to the nearest number* is equal to the average number.

$$
\begin{aligned}
\frac{2}{3} * \text{avg\_number} &> \text{avg\_number} - \frac{1}{2} \\
-\frac{1}{3} * \text{avg\_number} &> -\frac{1}{2} \\
\frac{1}{3} * \text{avg\_number} &< \frac{1}{2} \\
\text{avg\_number} &< \frac{3}{2}
\end{aligned}
$$

The simple calculation above shows that this only happens when the average number is lower than $\frac{3}{2}$. We still have the reasoning from the question 1a which stated that for a Nash equilibrium to occur in this game, all players need to win and everybody needs to have chosen the same number. In addition, we now have found that the average needs to be lower than $\frac{3}{2}$, this means the only pure Nash equilibria in this scenario are when either every player chooses 0 or every player chooses 1, because players can only choose integer numbers.

Part (c)

For this part of the question we can follow the same reasoning we had in question b, however the equation to finding which winning numbers now round up to be equal to the average number slightly changes:

$$
\begin{aligned}
\frac{9}{10} * \text{avg\_number} &> \text{avg\_number} - \frac{1}{2} \\
-\frac{1}{10} * \text{avg\_number} &> -\frac{1}{2} \\
\frac{1}{10} * \text{avg\_number} &< \frac{1}{2} \\
\text{avg\_number} &< 5
\end{aligned}
$$

We see that for the winning number rounded to the nearest integer to be equal to the average, our average needs to be lower than 5. Since the players are only able to pick integers (and the other assumptions for a pure Nash Equilibrium to occur are still the same as in question 1a and 1b), we will now have five pure Nash equilibrium. One where every player picks the number 0; one where every player picks 1; another one where every player picks 2; a fourth one where every players picks 3; and the fifth and final one where every player picks 4.

We can visualise it better by imagining the one hundred players picking the number 5. The average will be 5, and $\frac{9}{10}$ of that will be 4.5. Everybody wins in this scenario, however if any player moves to the number 4, the average will be just below 4.5 and the player that switched positions will now be the sole winner of the game.

Exercise 2

In this exercise I will demonstrate what all (mixed and pure) Nash equilibria are for the following two normal-form games. I will refer to the row-player as Rowena and the column player as Colin.

(a)

	L	R
T	5, 9	6, 3
B	2, 2	9, 5

(b)

	L	R
T	4, 7	6, 3
B	6, 3	1, 3

I also make use of the notation used in the tutorial groups to denote strategy profiles. For example when Rowena plays 'T' with a probability of p and Colin plays 'L' with a probability of q, we have the following strategies: $s_{Rowena} = (p, 1-p)$ and $s_{Colin} = (q, 1-q)$.

Since the second number between brackets (which is the probability of making the other decision) logically follows from the first number, we can also denote this in the more concise form that was used in the tutorials: (p, q). Here p denotes the probability of Rowena playing 'T' and q denotes the probability of Colin playing 'L'. This will be the notation used in the rest of this exercise 2.

Part (a)

In question 1 I already discussed what pure Nash equilibria (NE) are and what they mean. To reiterate, intuitively, a NE occcurs when "*no player has an incentive to unilaterally deviate from her assigned strategy*" (slide 14, lecture 1).

In normal-form game (a) there are two of such pure Nash equilibria. In both these scenarios, neither Rowena and Colin want to *unilaterally* switch their decision, because in both cases their utility would decrease by switching:

(I) Rowena plays 'T' with certainty (probability = 1) and Colin plays 'L' with certainty. In the matrix this is the top-left cell. Using our previously explained notation this Nash equilibrium is denoted as: $(1, 1)$

(II) The second pure Nash equilibrium is when Rowena plays 'B' with certainty and Colin plays 'R' with certainty. In the matrix this is the bottom-right cell. Using the notations as explained above, this Nash equilibrium can be written as: $(0, 0)$.

In mixed strategies (as opposed to pure strategies) we allow player i to play any action A_i with a certain probability, called their strategy. The mixed Nash equilibrium is a situation in which no player has an incentive to unilaterally change her strategy. In normal-form game (a) there is also a mixed Nash equilibrium. To find this mixed Nash equilibrium, we want to find the strategies in which each player is indifferent about their actions. We can calculate this using the formula on slide 11 of lecture 2:

(III) Colin is indifferent: $9p + 2(1-p) = 3p + 5(1-p) \Rightarrow p = \frac{1}{3}$
Rowena is indifferent: $5q + 6(1-q) = 2q + 9(1-q) \Rightarrow q = \frac{1}{2}$
This mixed Nash equilibrium can be denoted as $(\frac{1}{3}, \frac{1}{2})$.

There are no other Nash equilibria, since if Rowena changes her strategy to $p = \frac{1}{3} - \epsilon$, Colin's best response is $p = 0$ to which Rowena's best response is $q = 0$. Which is the Nash equilibrium we already had at (II): $(0, 0)$. Similarly if Rowena changes her strategy to $p = \frac{1}{3} + \epsilon$ we arive at the $(1, 1)$ pure NE we already had at (I). The same thing would happen if Colin would change his strategy to either $q = \frac{1}{2} - \epsilon$ or $q = \frac{1}{2} + \epsilon$.

In conclusion, we have two pure NE: $(1, 1)$ and $(0, 0)$; and one mixed NE: $((\frac{1}{3}, \frac{1}{2}))$

Part (b)

For this question we follow the similar structure as in part (a), and we find one pure Nash equilibrium. This cell is the only cell in the normal-form game matrix in which no player would unilaterally decide to switch because both Rowena's and Colin's utility would go down by switching:

(I) Rowena plays 'B' with certainty and Colin plays 'L' with certainty. In the matrix this is the bottom-left cell. Using the notation this Nash equilibrium is denoted as: $(0, 1)$

We can also find the mixed Nash equilibria as before:

(II) Colin is indifferent: $7p + 3(1 - p) = 3p + 3(1 - p) \Rightarrow p = 0$
Rowena is indifferent: $4q + 6(1 - q) = 6q + 1(1 - q) \Rightarrow q = \frac{5}{7}$
This mixed Nash equilibrium can be denoted as $(0, \frac{5}{7})$.

However, this is not the only mixed Nash equilibrium. When Colin changes his strategy to $q = \frac{5}{7} + \epsilon$, Rowena's best response is still $q = 0$ (meaning playing 'B' with certainty). This means that there are many (infinitely) more mixed Nash Equilibria: $(0, q)$ with q being the set of numbers larger or equal to $\frac{5}{7}$ and smaller than 1: $\{q \mid \frac{5}{7} \leq q < 1\}$.

This is not the case when Rowena changes her strategy to $p = 0 + \epsilon$, because Colin will have an incentive to always play 'L'. It is also not the case when Colin would change his strategy to $q = \frac{5}{7} - \epsilon$, since Rowena's best response would be to always play 'T'.

Therefore we have the Nash equilibria $(0, q)$ with $\{q \mid \frac{5}{7} \leq q \leq 1\}$. Here, $(0, 1)$ is the pure NE and all the others are mixed NE.

Exercise 3

In this exercise I will a counterexample to **disprove** the claim that every normal-form game $\langle N, \boldsymbol{A}, \boldsymbol{u} \rangle$ has a Nash equilibrium, even when $A_1 \cup \cdots \cup A_n$ might be infinite.

The game I propose is a two player normal-form game where every player submits a positive integer, which can be any number from 1 to infinity. Both players submit at the same time, and whichever player submits the highest number wins the game. The utility of the winning player is 1, the utility of the losing player is -1 and when it is a draw both utilities are 0.

The first thing to demonstrate is that there is no pure Nash equilibrium in this game. This is because whenever the two players have submitted two different numbers, the losing player will always have an incentive to unilaterally switch to an action that consists of submitting a number that was higher than the other player and therefore winning the game instead of losing.

To explain it in terms closer to the mathmatical definition of a pure Nash equilibrium (as given on slide 14 of lecture 1); when there is a losing player, it means there will always be a player (the losing player), whose action a_i was **not** the best response given the winning player's action, since there is an infinite amount of options.

A similar thing occurs when players submit an equal number, both players will have an incentive to unilaterally switch to a higher number, since this would guarentee a higher utility given the other player's choice. Therefore there are no pure Nash equilibria in this game.

Similarly we can see that there are also no mixed Nash equilibria for this game. The losing player will always have an incentive to switch his strategy to something that ensures winning. There is never a strategy that is a best response, since their are infinitely many options.

For example if both players have a uniform distribution over all numbers as their strategy (a probability of $\frac{1}{\inf}$ for every number being picked), the player who loses will always have an incentive to change his strategy to something that would increase his expected utility given the winning player's strategy. This will result in an endless loop of there never being a best response for all players at the same time.

We can conclude using this counterexample that **not** every normal-form game $\langle N, \boldsymbol{A}, \boldsymbol{u} \rangle$ has a Nash equilibrium, when $A_1 \cup \cdots \cup A_n$ is infinite.