Game Theory: Homework #3

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Collaboration. For this homework we have discussed our answers with Boaz Beukers, Marta Freixo Lopes and Ioanna Gogou. Ioanna and Didier talked more in depth about which questions should be answered for the programming report. Boaz and Didier had discussions about question 1 and 3. Tsatsral and Marta discussed exercise 4, where Marta gave a small hint when we were stuck (which we did not end up using).

Exercise 2

In this exercise we will demonstrate a Python program, that is able to simulate fictitious play for arbitrary 2×2 zero-sum games. The program will be used to give some insight into how to measure the 'distance' between two consecutive profiles of empirical mixed strategies; in addition to giving some insight into how many iterations are required before the fictitious play converges and its *rate of convergence*.

Generating a game

In order to draw some conclusions about the rate of convergence, we will play 500 2×2 zero-sum games for every experiment we conduct. It is very time consuming and inefficient to write out all these games by hand, this is why they will be generated randomly, displayed by the code below.

```python
def generate_game(num_actions_p1=2, num_actions_p2=2):
    # generate random utilities player 1
    u_1 = np.array([random.randrange(-25, 25, 1) for i in
        range(num_actions_p1*num_actions_p2)])

    # player 2 utilities are given (zero-sum game, opposite of u_1)
    u_2 = np.array([0 - i for i in u_1])

    return u_1, u_2;
```

In this two player game, the utilities for the first player are randomly chosen from a uniform distribution, with the minimum value being -25 and the maximum being 24. Since we are playing zero-sum games, meaning the utilities in an action profile add up to zero, the utilities of the second player are now also decided. This gives rise to $50^4 = 6.25$ million different 2×2 zero-sum games. Using the formula below, we see this gives us only a 2% chance of having duplicate games in the 500 games we play!

$$p(dup) = 1 - \frac{\frac{6249999!}{6249499!}}{6250000^{500}} \approx 0.02$$

Playing a game

The `play_game(e)` function simulates fictitious play on one of these random games. It starts by initializing a `actions_played` dictionary containing four keys: `p1`, `p2`, `emp_s1`, and `emp_s2`. Here `p1` and `p2` correspond to the actions played by both players so far; `emp_s1` and `emp_s2` correspond to the empirical mixed strategies of both players so far.

The `mixed_NEs()` function from last week is then used to calculate the mixed Nash equilibria of the generated game. This is useful, since the theory shows us our fictitious play will convergence to a Nash equilibrium, meaning these mixed Nash equilibria gives us the 'real' values a game is supposed to converge to. The `fictitious_play()` function will then proceed to carry out the fictitious play of the generated game, while updating the values in the `actions_played` dictionary.

Convergence

The first question we have to ask ourselves is, when has a game converged? Our variables `emp_s1` and `emp_s2` give us the empirical mixed strategies of both players during the fictitious play. When the difference between the current mixed strategy and previous mixed strategy has become small enough for both players, we say that the game has converged. The question now becomes, ofcourse, what is "*small enough*"?

Convergence margin - *The distance between the current empirical mixed strategy and the previous empirical mixed strategy for a player can be calculated. Once both players' distance falls below a 'convergence margin' we say that a game has converged.*

To test which convergence margin works well, we have tried four different values: 0.1, 0.01, 0.001 and 0.0001. When the fictitious play stops, the value it *has* 'converged to' might not be exactly equal to the value it *should* converge to; which is the real mixed Nash equilibrium we mentioned earlier. We can measure the distance between the converged value and the real value of the mixed Nash equilibrium, as a measure of how well a game has converged.

In the four graphs below we show the results of playing 500 randomly generated games, using these four different convergence margins. On the x-axis we display the distance of a game's converged empirical mixed strategy to the actual mixed Nash equilibrium. On the y-axis we display the number of iterations it took for a game to converge.

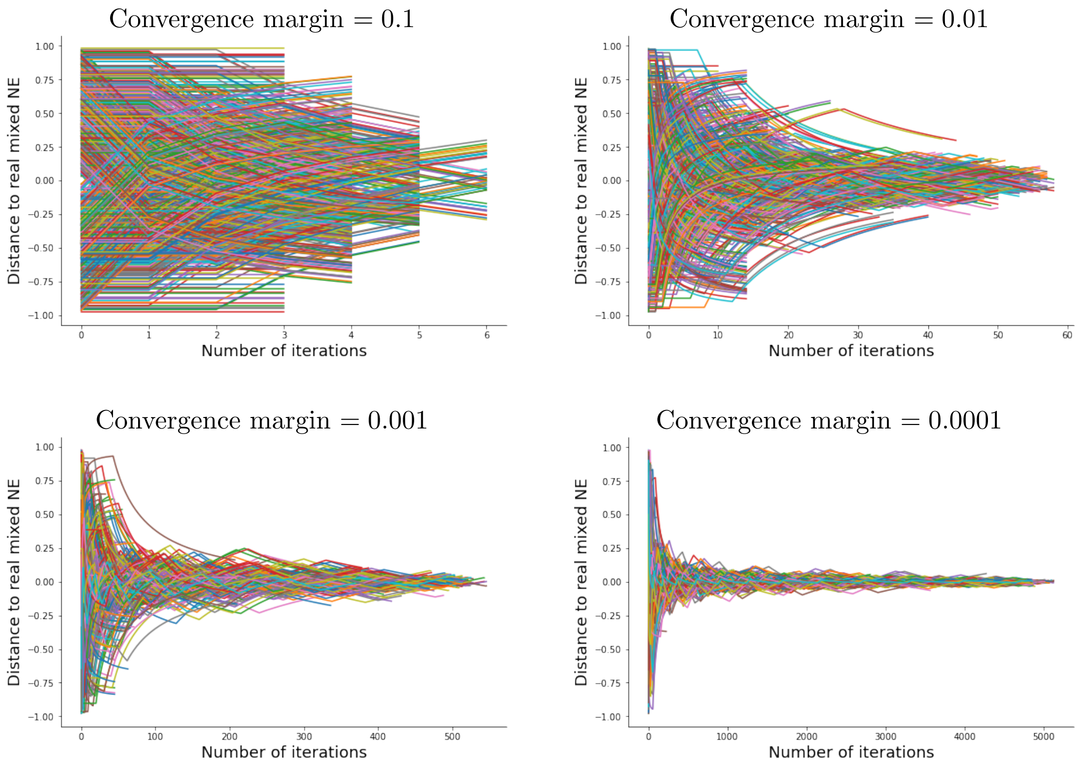

Figure 1: Visualizations of the different convergence margins. On the x-axis the number of iterations are shown and on the y-axis the distance between the converged value and the real mixed Nash equilibrium.

In this figure we can see that with a high convergence margin (0.1) the games converge quickly, with all 500 games converging within 6 iterations. However, the values are still quite far away from the real mixed equilibrium it should converge to. As the convergence margins get smaller, the distance between the value it converged to and the real value get closer to 0 for more and more games. When the convergence margin is 0.0001 for example, 898 out of the 1000 players have converged to a value within 0.02 of the real mixed Nash equilibrium!

Convergence time

We have found that a convergence margin of 0.0001 has a good performance, with almost 90% of games converging very closely to the actual mixed Nash equilibrium. The next question is; how *long* does it take for a game to converge, or how *many iterations*.

In the figure above we already saw that for a convergence margin of 0.0001 there are games that require over five thousand iterations to converge. In the figure below we show for all iterations what percentage of the 500 games has converged. From the figure we see that the order of convergence is slower than linear when the number of iterations increases and the order is approximately logarithmic inferring from the shape of the convergence curve.

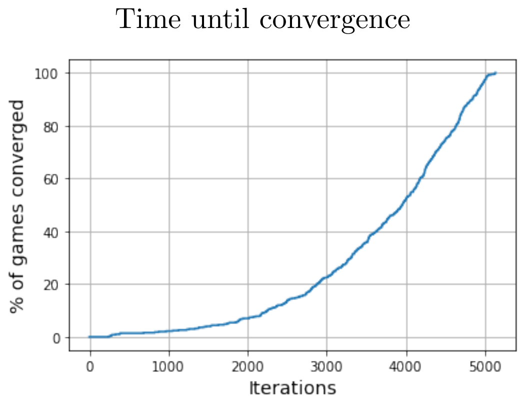

Figure 2: Visualisation of how long it takes for games to converge. On the x-axis the number of iterations is shown; on the y-axis the percentage of games that has converged.

In this figure we can see that the first games start converging after about 400 iterations when the convergence margin is set to 0.0001. Around 4000 iterations half of the games have converged; after a little over 5000 iterations 100% of the games have converged. We can conclude that with a convergence margin of 0.0001 it takes games not more than a little over 5000 iterations to converge to the real mixed Nash equilibrium.

Exercise 3

The given condition where the row player always knows the value of α (which can be either 0 or 2) and the column player is not, can be interpreted as the row player has two types of her own, which is not known to the column player at the time of the game. Specifically, the row player may fall into one of two possible types, namely, 'knowing α is 0' and 'knowing α is 2', and these two types are equally probable. In contrast, the column player has only one type, but she remains unknown the row player's type due to the unknown value of α. Furthermore, the two distinct types of the row player may lead to two different scenarios of the payoff, as illustrated in the following matrices:

	L	R
T	5, 5	0, 10
B	10, 0	1, 1

payoff matrix with if row player has type $\alpha = 0$

	L	R
T	5, 5	2, 10
B	10, 2	1, 1

payoff matrix if row player has type $\alpha = 2$

Hence, the game can be defined as a Bayesian game formally by the definition as follows.

The game can be represented as a tuple $(N, \boldsymbol{A}, \boldsymbol{\Theta}, p, \boldsymbol{u})$, where

- $N = \{row, col\}$ is the set of players;
- $\boldsymbol{A} = A_{row} \times A_{col}$ with A_i the set of actions of player i and $A_{row} = \{T, B\}, A_{col} = \{L, R\}$;
- $\boldsymbol{\Theta} = \Theta_{row} \times \Theta_{col}$ with $\Theta_{row} = \{\alpha = 0, \alpha = 2\}, \Theta_{col} = \{\perp\}$ the set of possible types of the row player and the column player;
- $p : \Theta \to [0, 1]$ the common prior over Θ with $p(\alpha = 0, \perp) = \frac{1}{2}$ and $p(\alpha = 2, \perp) = \frac{1}{2}$;
- $\boldsymbol{u} = (u_{row}, u_{col})$ is a profile of utility functions $u_i : \boldsymbol{A} \times \boldsymbol{\Theta} \to \mathbb{R}$.

Pure strategies are of the form $\alpha_i : \Theta_i \to A_i$ for player i. Here they are:

- Row player: *always B, always T, B if type is $\alpha = 0$ else T, T if type is $\alpha = 0$ else B*. To simplify the notation, we use *(B, B), (T, T), (T, B), (B, T)* to represent the above strategies.
- Column player: *L, R* as her own type is certain.

Pure Bayesian-Nash equilibria are reached when the best response of each the player and every type corresponds. To find the best responses, we first need to calculate the ex-ante expected utility of the column player. For a strategy profile $\boldsymbol{s}$ this ex-ante expected utility is:

$$u_{col}(\boldsymbol{s}) = p(\alpha = 0, \perp) \cdot u_{col}(\boldsymbol{s}, \alpha = 0) + p(\alpha = 2, \perp) \cdot u_{col}(\boldsymbol{s}, \alpha = 2)$$

For example, by filling in the corresponding values of p and u_{col} (retrieved from the above-listed payoff matrices) for the strategy profile $s = ((B, B), L)$, the ex-ante expected utility of the column player is

$$u_{col}(((B, B), L)) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 2 = 1.$$

Following this formula, the ex-ante expected utility of the column player for every possible strategy profile with respect to all the types can be computed. The corresponding values are in the following table, with the best response of the column player highlighted in **bold**.

	(B,B)	*(T,T)*	*(B,T)*	*(T,B)*
L	**1**	5	2.5	3.5
R	**1**	**10**	**5.5**	**5.5**

ex-ante expected utility of the column player

Since the type of the column player is certain, the best response of the row player can be directly found in the payoff matrices of the normal-form game w.r.t each type of the row player. Using the above-listed normal-form payoff matrices, when $\alpha = 0$, the best response of the row player is B and B w.r.t L and R; when $\alpha = 2$, the best response of the row player is B and T w.r.t L and R.
As a result, the best response of both players corresponds at the strategy profiles ((B,B), L) and ((B,T), R), which are the two pure Bayesian-Nash equilibria of this game.

Exercise 4

The requested game can be constructed as follows.
Suppose we have the classical normal-form prisoners' dilemma game to play with the following payoff for Rowena and Colin:

	Conf.	Lie
Conf.	-8, -9	0, -10
Lie	-10, 0	-1, -1

payoff matrix of the classical game

Trivially from the matrix, we see that Colin's best response is Confess when Rowena confesses and Confess when Rowena lies, while Rowena's best response is Confess when Colin confesses and Confess when Colin lies. The pure Nash equilibrium is reached at the strategy when the two players both achieve their best response. Based on the analysis above, we have only one pure Nash equilibrium with the strategy (Confess, Confess). Furthermore, there is no mixed Nash equilibrium because both players have their strictly dominant strategy and they will always choose then with full certainty. Hence, if we follow this normal-form game, the highest utility of Rowena is -8 and the highest utility of Colin is -9.

However, we know that 1 in 10 times the court's system will have a *bug* that results in abnormal sentencing judgement for the prisoners, which will lead to a different payoff matrix for this normal-form game. Furthermore, Colin knows a corrupted cop so Colin knows for certain whether the game is abnormal by the time they are interrogated while Rowena only knows the probability of the game having this bug is 0.1. The abnormal game's payoff matrix is illustrated as follows.

	Conf.	Lie
Conf.	-9, -8	-7, -5
Lie	-10, -9	3, -10

payoff matrix of the abnormal game

This abnormal game does not have a pure Nash equilibrium as there is no strategy where both of the players have their best response reached. However, there is one mixed Nash equilibrium with strategy $((\frac{1}{4}, \frac{3}{4}), (\frac{10}{11}, \frac{1}{11}))$ and expected utility (-8.82, -8.75).

To derive it, let the strategy of Rowena be $(p, 1-p)$ and that of Colin be $(q, 1-q)$, where p and q denotes the probability of choosing the action Confess and Lie.

The mixed equilibria is reached when the expected payoff of the two actions is the same for both players, meaning no particular incentive is there for the players to choose a different strategy. Hence, we have:

$$-9q - 7(1-q) = -10q + 3(1-q);$$
$$-8p - 9(1-p) = -5p - 10(1-p).$$

Solving them leads to $p = 1/3$ and $q = 11/12$ and the mixed strategies are $((\frac{1}{4}, \frac{3}{4}), (\frac{10}{11}, \frac{1}{11}))$. For example, the expected utility for Rowena is:

$$\frac{11}{12} \cdot \frac{1}{3} \cdot (-9) + \frac{11}{12} \cdot \frac{1}{3} \cdot (-10) + \frac{1}{12} \cdot \frac{2}{3} \cdot (-7) + \frac{11}{12} \cdot \frac{2}{3} \cdot 3 = -8.82$$

The same computation applied for Colin, which leads to an expected utility -8.75. With the classical game being played with the probability of 0.9 and the abnormal game being played with the probability of 0.1, we say that the game becomes a Bayesian game with two types which can be interpreted as the types the row player (knowing the game is classical or abnormal).

So we have $A_{row} = A_{col} = \{Confess, Lie\}$, $\Theta_{row} = \{\perp\}$ and $\Theta_{col} = \{classical, abnormal\}$ with $p(\perp, classical) = 0.9, p(\perp, abnormal) = 0.1$. Pure strategies are:

- Colin: *always confess, always lie, confess-if-classical-game, confess-if-abnormal-game*. To simplify the notation, we use *(C, C), (L, L), (C, L), (L, C)* to represent the above strategies.
- Rowena: *confess, lie* as her own type is certain. We use C and L to denote them.

Pure Bayesian-Nash equilibria are reached when the best response of each the player and every type corresponds. To find the best responses, we first need to calculate the ex-ante expected utility of Rowena. For a strategy profile $\boldsymbol{s}$ this ex-ante expected utility is:

$$u_{row}(\boldsymbol{s}) = p(\perp, classical) \cdot u_{row}(\boldsymbol{s}, classical) + p(\perp, abnormal) \cdot u_{row}(\boldsymbol{s}, abnormal)$$

For example, by filling in the corresponding values of p and u_{col} (retrieved from the payoff matrices of the two underlying normal-form games) for the strategy profile $s = (C, (C, L))$, the ex-ante expected utility of the column player is

$$u_{col}((C, (C, L))) = \frac{1}{10} \cdot (-7) + \frac{9}{10} \cdot (-8) = -7.9.$$

Following this formula, the ex-ante expected utility of the column player for every possible strategy profile with respect to all the types can be computed. The corresponding values are in the following table, with the best response of the Rowena highlighted in **bold**.

	(C,C)	*(L,L)*	*(C,L)*	*(L,C)*
C	**-8.1**	**-0.7**	**-7.9**	**-0.9**
L	-10	-1.2	-9.3	-1.9

ex-ante expected utility of Rowena

Since the type of Rowena is certain, the best response of Colin can be directly found in the payoff matrices of the normal-form game w.r.t each type of the column player. Using the above-listed normal-form payoff matrices, when the game is classical, the best response of the Colin is always C w.r.t C and L of Rowena, since C is the strictly dominant strategy ; similarly, when the game is abnormal, the best response of Colin is L and C w.r.t to C

and L

As a result, the best response of both players correspond at the strategy profile ((C, (C, L)), which is the only pure Bayesian-Nash equilibria of this game. At this Bayesian-Nash equilibria, the expected utility of Rowena is -7.9 and the expected utility of Colin is $0.9 \cdot (-9) + 0.1 \cdot (-5) = -8.6$. Recall that the highest utility of the underlying normal-form games is $-8(< -7.9)$ and $-8.75(< -8.6)$, we conclude that the pure Bayesian-Nash equilibrium of this constructed game provides strictly higher expected utility to all users than any of the Nash equilibria of any of the underlying normal-form games would.