Game Theory: Homework #2

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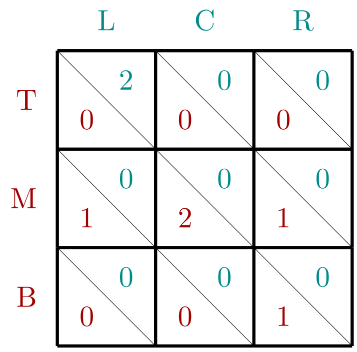
Collaboration. For this homework we have discussed our answers with Boaz Beukers and Marta Freixo Lopes. This was done only after all of us had individually finished the exercises. Boaz and Didier talked about the programming exercise, whether it should be hardcoded or not, and if giving a counter example for question 1 and 3 would be enough. Tsatsral also gave a small tip on how to approach Exercise 3 for Boaz. Marta and Didier had conversations about what to do with infinite cases in the program for question 4.

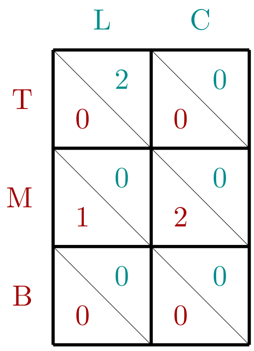
Exercise 1

Iterated elimination of strictly dominated strategies is order-independent, it does not matter in which order you eliminate strategies, you will end up in the same equilibrium. However, for iterated elimination of *weakly* dominated strategies this is not the case. We will demonstrate this by giving an example with the payoff matrix illustrated in Matrix 1. Proposed game, in which different orders of elimination end up in a different equilibrium.

1. Proposed game



1a. Elimination of 'R'



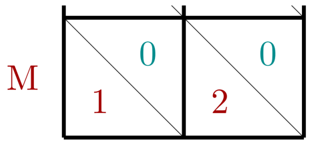
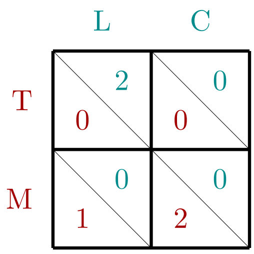


Figure 1: Final resulting cells in scenario 1 after elimination of 'T'



2a. Elimination of 'B'

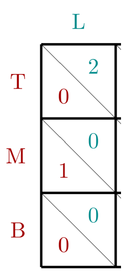


Figure 2: After step 1a, we eliminate 'C'

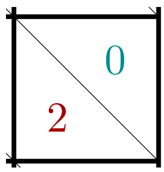


Figure 3: Final resulting cell (MC) in scenario 2, after elimination of T and B

In this game, R is one of the weakly dominated strategies for the column player, as the payoff of choosing R is always smaller than or equal to the payoff of choosing L or C regardless of what the row player selects. Hence, strategy R can be eliminated, resulting the new payoff matrix illustrated in Matrix 1a.

We continue eliminating the dominated strategies in the matrix in Figure 1.1a. For the row player, now there are two strictly dominated strategies available for elimination, namely, T and B. For the column player, there is one weakly dominated strategy to eliminate, namely, C.

If we first eliminate strategy B, we will have the new payoff matrix indicated in Matrix 2a. In Matrix 2a, we are left with a weakly dominated strategy C and a strictly dominated strategy T. If choose to eliminate T with the result indicated in Figure 1, however, no dominated strategies are left. We will have two equilibrias ((0,1,0), (1,0,0)) and ((0,1,0),(0,1,0))

However, in Matrix 1a, if we choose to first eliminate C, we will end up with the new payoff matrix indicated in Figure 2 with two strictly dominated strategies, T and B to eliminate. If we eliminate B and T, we will left with the equilibrium ((0, 1, 0), (1, 0, 0)), illustrated in Figure 3.

Hence, two elimination orders, R-B-T and R-C-B-T indeed give different equilibriums. We have proved with a counter example that the elimination order matters with the presence of weakly dominated strategies.

Exercise 3

The requested normal-form game is proposed as follows. Rowena and Colin are camping together and they need to retrieve the water from a remote spot. If they both don't go, they won't have the water which is essential for their survival. This will result in a negative payoff -3 for both of them. If only one of them goes but they share the retrieved water, the person who doesn't go will have a higher payoff of 3 because he/she gets the water without spending any effort. On the contrary, the person who goes to retrieve the water will get a payoff 0 because the effort and gain cancels out. If they decide to do it together and share the water, each of them will spend less effort and this results in a positive payoff of 2 for both of them. The payoff matrix of this game is therefore defined as follows.

	NG	G
NG	-3, -3	3, 0
G	0, 3	2, 2

Until now, the game has two pure Nash equilibrias. Namely, $((1,0),(0,1))$ and $((0,1),(1,0))$ (*the order is first action Go then No-go*) with payoff $(3,0)$ and $(0,3)$. The pure equilibrias are only reached when one of the players goes to retrieve the water and the other stays because it is where the best response of the both players corresponds and, hence, no player has an incentive to switch strategies.

Moreover, the game has a mixed Nash equilibria. Namely, $((\frac{1}{4}, \frac{3}{4}), (\frac{1}{4}, \frac{3}{4}))$ with expected payoff $(\frac{3}{2}, \frac{3}{2})$. To derive it, let the strategy of Rowena be $(p, 1-p)$ and that of Colin be $(q, 1-q)$, where p and q denotes the probability of choosing the action NG and G for Rowena and Colin, respectively.

The mixed equilibria is reached when the expected payoff of the two actions is the same for both players, meaning no particular incentive is there for the players to choose a different strategy. Hence, we have:

$$\begin{aligned} -3q + 3(1-q) &= 0q + 2(1-q); \\ -3p + 3(1-p) &= 0p + 2(1-p). \end{aligned}$$

Solving them leads to $p = 1/4$ and $q = 1/4$ and the mixed strategies are $((\frac{1}{4}, \frac{3}{4}), (\frac{1}{4}, \frac{3}{4}))$. The expected payoff, for example for Rowena, is then $-3 \times \frac{1}{4} \times \frac{1}{4} + 3 \times \frac{1}{4} \times \frac{3}{4} + 0 + 2 \times \frac{3}{4} \times \frac{3}{4} = \frac{3}{2}$. The same goes for Colin, which is also $\frac{3}{2}$

Therefore, if no randomized device is used, the maximal sum of (expected) payoff is **3** for this game.

However, if we add one randomized device to coordinate the players' action, the sum of their expected payoff can be strictly higher than 3 at the acquired correlated equilibrium. The randomized device is defined as follows.

- $D_i = \{red, green\}$ for both players i.
- $\pi(red, green) = \pi(green, red) = \frac{2}{5}, \pi(green, green) = \frac{1}{5}$.
- the two players will use the mapping $\sigma_i : d_i \mapsto \begin{cases} Go & \text{if } d_i = green \\ No\text{-}go & \text{if } d_i = red \end{cases}$

If such randomized device is used by both players, the correlated equilibrium will be reached. The expected utility is then: $\frac{2}{5} \times 0 + \frac{2}{5} \times 3 + \frac{1}{5} \times 2 = \frac{8}{5}$ for Rowena and $\frac{2}{5} \times 3 + \frac{2}{5} \times 0 + \frac{1}{5} \times 2 = \frac{8}{5}$ for Colin. Hence, the sum of their expected utility is $\frac{16}{5} > 3$, with 3 the sum of the payoff of the pure and mixed Nash equilibrias.

Exercise 4

In this exercise we will demonstrate a Python program, that is able to compute all (pure and mixed) Nash equilibria, of a given normal-form game with two players and two actions per player. It works correctly in the cases where the submitted normal-form game has only finitely many equilibria. Below are 3 examples that demonstrate the program works correctly on multiple different scenarios. After this, we explain how we approached this problem.

(a)

	L	R
T	5, 9	6, 3
B	2, 2	9, 5

(b)

	L	R
T	1, 2	2, 1
B	1, 2	2, 1

(c)

	L	R
T	2, 2	4, 4
B	4, 4	6, 6

Input of program using game (a)

```
# Utility matrices for both players
# order = top-left, top-right,
    bottom-left, bottom-right

# player 1 (row player)
u_1 = np.array([[5, 6], [2, 9]])
# player 2 (column player)
u_2 = np.array([[9, 3], [2, 5]])
```

Output of program using game (a)

```
***********************************
The pure Nash equilibria are:
(1, 1)
(0, 0)

The mixed Nash equilibria are:
(1/3, 1/2)
***********************************
```

Output of program using game (b)

```
***********************************
The pure Nash equilibria are:
No pure Nash equilibria found!

The mixed Nash equilibria are:
(1/2, 1/2)
***********************************
```

Output of program using game (c)

```
***********************************
The pure Nash equilibria are:
(0, 0)

The mixed Nash equilibria are:
No mixed Nash equilibria found!
***********************************
```

1. Input

The first thing we needed to do for this program was to decide how to encode the input of the game. We decided to make use of two NumPy arrays u_1 and u_2: here, each matrix contains the utilities of a specific player for all possible action profiles. This is demonstrated in the figure above for the input of game (a).

2. Pure Nash equilibria

A game can have either pure or mixed Nash equilibria or both. The first thing we did is create a function that calculates whether the inputted two-player normal-form game has any *pure* Nash equilibria. This was done by checking whether players have an incentive to unilaterally deviate from their assigned strategy.

Checking if any players want to deviate from their strategy is done in the function `pure_NEs(u_1, u_2)`. In this function, we examine if the row player wants to unilaterally change their action, by calculating whether the utility of a certain action profile is at least as high or higher than the other utilities in its *column* (since we are 'given' the action of the column player). To see if the column player wants to deviate from their strategy, we can check if the utility of a certain action profile is at least as high or higher than the other utilities in its *row* (since we are given the action of the row player).

3. Mixed Nash equilibria

In addition to checking for pure Nash equilibria, we also need to check the existence of mixed Nash equilibria. In mixed strategies, it is now possible for players to play any action with a certain probability. When no player wants to unilaterally deviate from their mixed strategy, we have found a *mixed* Nash equilibrium.

To find these mixed Nash equilibria we want to find the strategies in which each player is indifferent about their actions. We can calculate this using the formula on slide 11 of lecture 2, where we find the values for p and q that cause the same expected utility between actions for each player. Assume we have a normal-form game as displayed in figure 5:

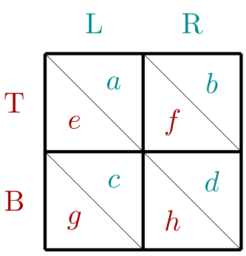


Figure 5: Two-player normal-form game with standardised utilities

For player 2, the column player, to be indifferent, the expected utilities of both actions should be equal. We can find the value of p as follows:

$$\begin{aligned} a \cdot p + c \cdot (1-p) &= b \cdot p + d \cdot (1-p) \\ (a-c) \cdot p + c &= (b-d) \cdot p + d \\ (a-b-c+d) \cdot p &= d - c \\ p &= \frac{d-c}{a-b-c+d} \end{aligned}$$

In a similar way we can find the value of q, for the row player to be indifferent, shown below. Now that we are able to calculate the values for p and q we have found the mixed Nash equilibrium.

$$q = \frac{h-f}{e-f-g+h}$$

4. Special cases

However, there are some special cases. For example when $a-b-c+d=0$ or when $e-f-g+h=0$, because in this scenario our formula is dividing by 0 and will give an undefined answer while there still could be mixed Nash equilibria. We need to intuitively understand what is happening in this scenario.

Take for example a scenario in which $a-b-c+d=0$. From this, it logically follows that $a-c=b-d$. There are now three scenarios:

- $a=b$: When a is equal to b, according to our formula above it means that $a=b=c=d$, meaning we have a utility matrix in which all values are the same for a player. In this scenario the player is always indifferent and the strategy q can take on any value between 0 and 1.

- $a > b$: When a is larger than b, it means that c must be larger than d. In this scenario a and c are both larger than b and d, when we look at the normal-form game in figure 1, this means that there is a *dominant* strategy: the column player will always choose to play 'L', since it always yields higher utility. Therefore q is equal to 1 in this scenario.

- $a < b$: Similar to the statement above, when a is smaller than b, we find that there is a dominant strategy where the column player will always choose to play 'R', and therefore q is equal to 0 in this scenario.

5. General solution

Some of the cases in which there are an infinite amount of equilibria are already covered by our code, as explained in the previous section. However, to turn this into a general solution for two-player normal-form games with two actions for each player, there is another scenario in which an infinite amount of equilibria can occur that needs to be dealt with. Take for example the normal-form game from the first homework below.

Normal-form game (d)

	L	R
T	4, 7	6, 3
B	6, 3	1, 3

Output of program using game (d)

```
***********************************
The pure Nash equilibria are:
(0, 1)

The mixed Nash equilibria are:
(0, 5/7)
***********************************
```

Our program outputs the pure Nash equilibrium (0, 1) and the mixed Nash equilibrium (0, $\frac{5}{7}$). These are correct, however, we saw last week that q can actually take on any value larger or equal to $\frac{5}{7}$. To turn our program into a general solution, it is required to check what the responses of the players are when they play a strategy higher or lower than the found mixed Nash equilibria, and check whether these are *also* Nash equilibria.