ToFu tools
Magnetic fields

Didier VEZINET

05.11.2020

Contents

Chapter 1

3D field from straight current segments

1.1 Single segment

Let's consider current segment with current I, centered on A, with unit vector $\underline{u}$ and half-length $\mathrm{L}_{1/2}$. Considering a point P in this segment, identified by its length to A:

$$\underline{OP} = \underline{OA} + l\underline{u}, \text{ with } l \in \left[-\mathrm{L}_{1/2}; \mathrm{L}_{1/2}\right]$$

Any point M in space can be located by its position with respect to $\underline{A}$

$$\begin{cases} \underline{OM} = \underline{OA} + l_M\underline{u} + r_M\underline{v} \\ \underline{v} = \underline{AM} - (\underline{AM} \cdot \underline{u})\underline{u} \end{cases}$$

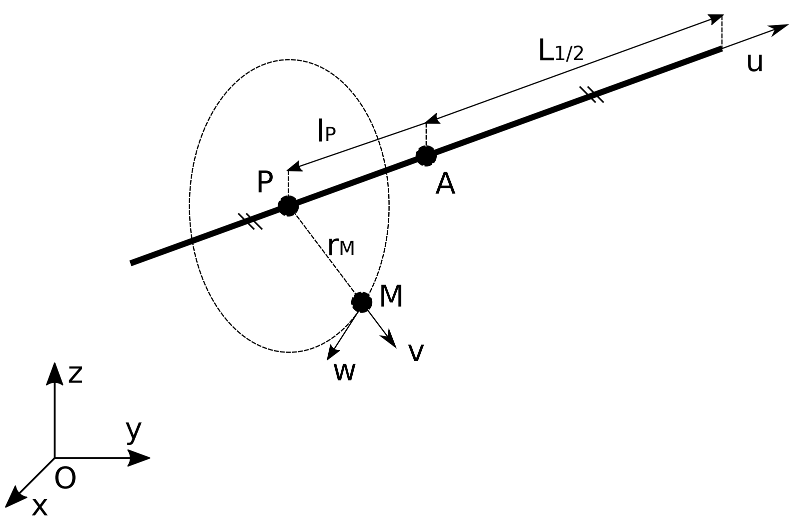

Figure 1.1: MAgnetic field at any point M from a straight current segment

The Biot-Savart law stipulates that an elementary length of current creates an elementary magnetic field at M:

$$\begin{cases} \underline{dI} = Idl\underline{u} \\ \underline{PM} = (l_M - l)\underline{u} + r_M\underline{v} \\ \underline{B_A} = \frac{\mu_0}{4\pi}\int_{-\mathrm{L}_{1/2}}^{\mathrm{L}_{1/2}} \frac{\underline{dI}\wedge\underline{PM}}{\|\underline{PM}\|^3} \end{cases}$$

Introducing $\underline{w} = \underline{u} \wedge \underline{v}$:

$$\underline{dI} \wedge \underline{PM} = Idlr_M\underline{w}$$

and:

$$\|\underline{PM}\| = \sqrt{(l - l_M)^2 + r_M^2}$$

Hence:

$$\underline{B_A} = \frac{\mu_0}{4\pi} I r_M \underline{w} \int_{-\mathrm{L}_{1/2}}^{\mathrm{L}_{1/2}} \frac{dl}{\left((l-l_M)^2 + r_M^2\right)^{3/2}}$$

Introducing $x = l - l_M \Rightarrow dx = dl$:

$$\underline{B_A} = \frac{\mu_0}{4\pi} I r_M \underline{w} \int_{-\mathrm{L}_{1/2}-l_M}^{\mathrm{L}_{1/2}-l_M} \frac{dx}{\left(x^2+r_M^2\right)^{3/2}}$$

Noticing that:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+a}}\right) &= \frac{\sqrt{x^2+a} - x\frac{1}{2}\frac{2x}{\sqrt{x^2+a}}}{x^2+a} \\ &= \frac{(x^2+a)-x^2}{(x^2+a)^{3/2}} \\ &= \frac{a}{(x^2+a)^{3/2}} \end{aligned}$$

Hence:

$$\begin{aligned} \underline{B_A} &= \frac{\mu_0}{4\pi} I r_M \underline{w} \left[\frac{1}{r_M^2}\frac{x}{\sqrt{x^2+r_M^2}}\right]_{-\mathrm{L}_{1/2}-l_M}^{\mathrm{L}_{1/2}-l_M} \\ &= \frac{\mu_0}{4\pi}\frac{I}{r_M}\underline{w}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}} + \frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right) \end{aligned}$$

For numerical evaluation, keep in mind that:

$$\begin{cases} l_M = \underline{u} \cdot \underline{AM} \\ r_M = \|\underline{u} \wedge \underline{AM}\| \\ \underline{w} = \frac{\underline{u}\wedge\underline{AM}}{r_M} \end{cases}$$

1.2 2 mirrored segments

Let's consider 2 current segments $(A, \underline{u})$ and $(A', \underline{u'})$ one being the symmetric of the other via a symmetry plane $(M, \underline{n})$.
Each current segment has its own current I (resp. I').

By construction:

$$\begin{cases} l'_M &= l_M \\ r'_M &= r_M \\ L'_{1/2} &= L_{1/2} \\ d_A &= -\underline{AM} \cdot \underline{n} \\ \underline{A'A} &= 2d_A\underline{n} \\ \underline{u'} &= \underline{u} - 2(\underline{u} \cdot \underline{n})\underline{n} \\ \underline{AM} &= l_M\underline{u} + r_M\underline{v} \\ \underline{A'M} &= l_M\underline{u'} + r_M\underline{v'} \end{cases}$$

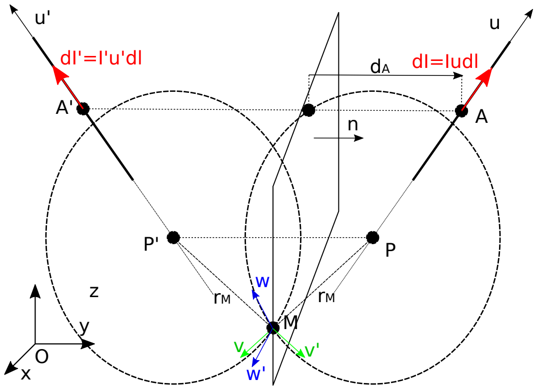

Figure 1.2: Magnetic field at any point M from 2 mirrored straight current segments

The total magnetic field created in M is:

$$\begin{aligned}
\underline{B} &= \underline{B_A}(I) + \underline{B_{A'}}(I') \\
&= \frac{\mu_0}{4\pi}\frac{I}{r_M}\underline{w}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}} + \frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right) + \frac{\mu_0}{4\pi}\frac{I'}{r_M}\underline{w'}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}} + \frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right) \\
&= \frac{\mu_0}{4\pi}\frac{1}{r_M}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}} + \frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right)(I\underline{w} + I'\underline{w'})
\end{aligned}$$

Now, considering that:

$$\begin{aligned}
r_M\underline{v'} &= \underline{A'M} - l_M\underline{u'} \\
&= \underline{A'A} + \underline{AM} - l_M(\underline{u} - 2(\underline{u}\cdot\underline{n})\underline{n}) \\
&= 2d_A\underline{n} + \underline{AM} - l_M\underline{u} + 2l_M(\underline{u}\cdot\underline{n})\underline{n} \\
&= 2d_A\underline{n} + r_M\underline{v} + 2l_M(\underline{u}\cdot\underline{n})\underline{n} \\
\underbrace{\Leftrightarrow}_{r_M\neq 0}\quad \underline{v'} &= \underline{v} + \frac{2}{r_M}\left(l_M(\underline{u}\cdot\underline{n}) + d_A\right)\underline{n}
\end{aligned}$$

Hence:

$$\begin{aligned}
\underline{w'} &= \underline{u'}\wedge\underline{v'} \\
&= (\underline{u} - 2(\underline{u}\cdot\underline{n})\underline{n}) \wedge \left(\underline{v} + \frac{2}{r_M}\left(l_M(\underline{u}\cdot\underline{n}) + d_A\right)\underline{n}\right) \\
&= \underline{u}\wedge\underline{v} + \frac{2}{r_M}\left(l_M(\underline{u}\cdot\underline{n}) + d_A\right)\underline{u}\wedge\underline{n} - 2(\underline{u}\cdot\underline{n})\underline{n}\wedge\underline{v} \\
&= \underline{w} + 2\left[\frac{(\underline{u}\cdot\underline{n})l_M + d_A}{r_M}\underline{u} + (\underline{u}\cdot\underline{n})\underline{v}\right]\wedge\underline{n} \\
&= \underline{w} + 2\left[\frac{(\underline{u}\cdot\underline{n})(l_M\underline{u} + r_M\underline{v}) + d_A\underline{u}}{r_M}\right]\wedge\underline{n} \\
&= \underline{w} + 2\left[\frac{(\underline{u}\cdot\underline{n})\underline{AM} + d_A\underline{u}}{r_M}\right]\wedge\underline{n}
\end{aligned}$$

Remebering that $d_A = -\underline{AM}\cdot\underline{n}$ and that $\underline{a}\wedge(\underline{b}\wedge c) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$:

$$\begin{aligned}
\underline{w'} &= \underline{w} + 2\left[\frac{(\underline{u}\cdot\underline{n})\underline{AM} - (\underline{AM}\cdot\underline{n})\underline{u}}{r_M}\right]\wedge\underline{n} \\
&= \underline{w} + \frac{2}{r_m}\left[\underline{n}\wedge(\underline{AM}\wedge\underline{u})\right]\wedge\underline{n} \\
&= \underline{w} + 2\frac{\sqrt{r_m^2+l_M^2}}{r_m}\left[\underline{n}\wedge(\underline{e}_{AM}\wedge\underline{u})\right]\wedge\underline{n}
\end{aligned}$$

Where we have introduced $\underline{AM} = \sqrt{r_M^2 + l_M^2}\underline{e}_{AM}$:

Then, assuming $I' = -I$ we have:

$$\begin{aligned}
\underline{B} &= \frac{\mu_0}{4\pi}\frac{I}{r_M}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}}+\frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right)(\underline{w}-\underline{w'}) \\
&= -\frac{\mu_0}{4\pi}\frac{I}{r_M}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}}+\frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right)2\frac{\sqrt{r_m^2+l_M^2}}{r_m}\left[\underline{n}\wedge(\underline{e}_{AM}\wedge\underline{u})\right]\wedge\underline{n} \\
&= \frac{\mu_0}{2\pi}I\frac{\sqrt{r_m^2+l_M^2}}{r_m^2}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}}+\frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right)\underline{n}\wedge\left[\underline{n}\wedge(\underline{e}_{AM}\wedge\underline{u})\right]
\end{aligned}$$

For numerical evaluation, keep in mind that:

$$\begin{cases} l_M = \underline{u}\cdot\underline{AM} \\ r_M = \|\underline{u}\wedge\underline{AM}\| \\ d_A = -\underline{AM}\cdot\underline{n} \end{cases}$$

1.3 4 mirrored segments

Let's consider the 2 previous mirrored current segments and add a pair mirroring them via another plane $(C,\underline{m})$, perpendicular to the first plane $(C,\underline{n}) = (M,\underline{n})$. All segments are lying in the same plane $(C,\underline{a})$
The same current is running through each segment and they all have the same half-length $L_{1/2}$.

The same derivation as previously can be done for pair AA' and pair BB'.

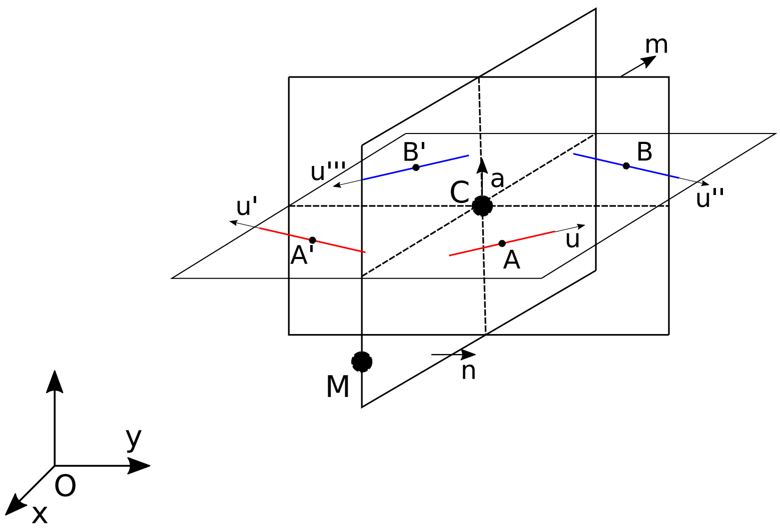

Figure 1.3: Magnetic field at any point M from 2 mirrored straight current segments

The total magnetic field created in M is:

$$\begin{aligned}
\underline{B} &= \underline{B_{AA'}}(I)+\underline{B_{BB'}}(I) \\
&= \frac{\mu_0}{2\pi}\frac{I}{r_M}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}}+\frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right)\left(\underline{n}\wedge\left[\frac{(\underline{u}_A\cdot\underline{n})\underline{AM}+d_A\underline{u}_A}{r_M}\right]\right) \\
&+\frac{\mu_0}{2\pi}\frac{I}{r''_M}\left(\frac{\mathrm{L}_{1/2}-l''_M}{\sqrt{(\mathrm{L}_{1/2}-l''_M)^2+r''^2_M}}+\frac{\mathrm{L}_{1/2}+l''_M}{\sqrt{(\mathrm{L}_{1/2}+l''_M)^2+r''^2_M}}\right)\left(\underline{n}\wedge\left[\frac{(\underline{u}_B\cdot\underline{n})\underline{BM}+d_B\underline{u}_B}{r''_M}\right]\right)
\end{aligned}$$

By construction, we have: .... cannot get (r''_M, l''_M) from (r_M, l_M).

Chapter 2

Circular coil - discretized

The magnetic field produced at any point in 3D space by a planar circular coil cannot be derived analytically.

2.1 Circle discretization

Let's consider a circular coil of radius R centered on axis $(C, \underline{e}_3)$.

For a given point M in space, let's make the plane $(C, M, \underline{e}_3)$ a symmetry plane and divide the circle into a N-sided polygon.

2.1.1 N-sided polygon

The polygon is N-sided, with N an even number. The polygon shall have either:

- the same perimeter as the circle: $L = 2\pi R$
- the same area as the circle: $S = \pi R^2$
- the same magnetic field on the center as the circle: $B(0) = \frac{\mu_0 I}{2R}$

For a N-sided regular polygon of height h:

$$\begin{cases} L_i = 2h \tan\left(\frac{\pi}{N}\right) & \text{is the length of a single side} \\ S_i = \frac{hL_i}{2} = h^2 \tan\left(\frac{\pi}{N}\right) & \text{is the area of a single side} \end{cases}$$

Remembering that, at the center, $l_m = 0$ and $r_m = h$ for all, by construction, and that $\mathrm{L}_{1/2} = \frac{L_i}{2} = h \tan\left(\frac{\pi}{N}\right)$, we derive:

$$\begin{aligned} B(0, N) &= N \frac{\mu_0}{4\pi} \frac{I}{h} \left(\frac{\mathrm{L}_{1/2}}{\sqrt{(\mathrm{L}_{1/2})^2 + h^2}} + \frac{\mathrm{L}_{1/2}}{\sqrt{(\mathrm{L}_{1/2})^2 + h^2}} \right) \\ &= N \frac{\mu_0}{2\pi} \frac{I}{h} \frac{\mathrm{L}_{1/2}}{\sqrt{(\mathrm{L}_{1/2})^2 + h^2}} \\ &= N \frac{\mu_0}{2\pi} \frac{I}{h} \frac{h \tan\left(\frac{\pi}{N}\right)}{\sqrt{h^2 \tan^2\left(\frac{\pi}{N}\right) + h^2}} \\ &= N \frac{\mu_0}{2\pi} \frac{I}{h} \frac{\tan\left(\frac{\pi}{N}\right)}{\sqrt{\tan^2\left(\frac{\pi}{N}\right) + 1}} \\ &= \frac{\mu_0 I}{2} \frac{1}{h} \frac{N}{\pi} \frac{\tan\left(\frac{\pi}{N}\right)}{\sqrt{\tan^2\left(\frac{\pi}{N}\right) + 1}} \end{aligned}$$

Hence,

- same perimeter $\Rightarrow h = R\frac{\frac{\pi}{N}}{\tan(\frac{\pi}{N})}$
- same area $\Rightarrow h = R\sqrt{\frac{\frac{\pi}{N}}{\tan(\frac{\pi}{N})}}$
- same field on axis $\Rightarrow h = R\frac{\tan(\frac{\pi}{N})}{\frac{\pi}{N}}\frac{1}{\sqrt{\tan^2(\frac{\pi}{N})+1}}$

2.1.2 Symetries

Let's consider, for a point M, the 2 symetry planes passing through the center of the circle, containing its axis, and one passing through M, the other one perpendicular to that one.

This way we can define a 4-symetry for the discretization of the circle, indexed by n.

2.2 Deriving B from discretized 3D cirle

Let's consider a spire as a circle in 3D with center C, axis $(C, \underline{e}_3)$ and radius R. Let's consider a point M in 3D with coordinates (x_M, y_M, z_M).

Local coordinate system $(\underline{e}_1, \underline{e}_2)$ can be defined from:

$$\begin{cases} \underline{CM} = r\underline{e}_1 + z\underline{e}_3 \\ \underline{e}_2 = \underline{e}_3 \wedge \underline{e}_1 \end{cases}$$

Let's discretize the circle into a 4n-sided polygon with 2 symetry planes, including (M, C, e_3).

Depending on the constraint, the height h of each polygon can be derived (cf. 2.1.1). Similarly, the length of the basis of each polygon can also be derived as:

$$\begin{cases} h & \underbrace{=}_{\text{perim.}} R\frac{\frac{\pi}{4n}}{\tan(\frac{\pi}{4n})} \\ & \underbrace{=}_{\text{area}} R\sqrt{\frac{\frac{\pi}{4n}}{\tan(\frac{\pi}{4n})}} \\ & \underbrace{=}_{\text{B(0)}} R\frac{\tan(\frac{\pi}{4n})}{\frac{\pi}{4n}}\frac{1}{\sqrt{\tan^2(\frac{\pi}{4n})+1}} \\ \mathrm{L}_{1/2} & = h\tan(\frac{\pi}{4n}) \end{cases}$$

Knowing h and $\mathrm{L}_{1/2}$ one can write, for $i \in [1; 2n]$ (we only consider one half of the circle due to the symetry):

$$\begin{cases} \theta_i = (i - \frac{1}{2})\frac{\pi}{4n} \\ \underline{CA_i} = h\cos(\theta_i)\underline{e}_1 + h\sin(\theta_i)\underline{e}_2 \\ \underline{u}_i = -\sin(\theta_i)\underline{e}_1 + \cos(\theta_i)\underline{e}_2 \end{cases}$$

Introducing the local coordinates with respect to $(C, \underline{e}_3)$:

$$\underline{CM} = r\underline{e}_1 + z\underline{e}_3 \Rightarrow \begin{cases} z & = \underline{CM} \cdot \underline{e}_3 \\ r & = \|\underline{CM} - z\underline{e}_3\| \\ \underline{e}_1 & = (\underline{CM} - z\underline{e}_3)/r \end{cases}$$

For each side i, we can write:

$$\begin{aligned} \underline{A_iM} & = \underline{CM} - \underline{CA_i} \\ & = r\underline{e}_1 + z\underline{e}_3 - h\cos(\theta_i)\underline{e}_1 - h\sin(\theta_i)\underline{e}_2 \\ & = (r - h\cos(\theta_i))\underline{e}_1 - h\sin(\theta_i)\underline{e}_2 + z\underline{e}_3 \end{aligned}$$

Hence:

$$\begin{aligned}
\underline{u_i} \wedge \underline{A_i M} &= (-\sin(\theta_i))(-h\sin(\theta_i))\underline{e}_3 + (-\sin(\theta_i))(z)(-\underline{e}_2)\\
&\quad +(\cos(\theta_i))((r - h\cos(\theta_i)))(-\underline{e}_3) + (\cos(\theta_i))(z)(\underline{e}_1)\\
&= z\cos(\theta_i)\underline{e}_1 + z\sin(\theta_i)\underline{e}_2 + (h\sin^2(\theta_i) + h\cos^2(\theta_i) - r\cos(\theta_i))\underline{e}_3\\
&= z\cos(\theta_i)\underline{e}_1 + z\sin(\theta_i)\underline{e}_2 + (h - r\cos(\theta_i))\underline{e}_3
\end{aligned}$$

and:

$$\begin{aligned}
\underline{u}_i \cdot \underline{A_i M} &= (r - h\cos(\theta_i))(-\sin(\theta_i)) + (\cos(\theta_i))(-h\sin(\theta_i))\\
&= -r\sin(\theta_i)
\end{aligned}$$

And since here $\underline{n} = \underline{e}_2$:

$$\underline{A_i M} \cdot \underline{n} = -h\sin(\theta_i)$$

Hence:

$$\begin{cases} r_{Mi} &= \sqrt{z^2 + (h - r\cos(\theta_i))^2}\\ l_{Mi} &= -r\sin(\theta_i)\\ d_{Ai} &= h\sin(\theta_i)\end{cases}$$

Thus, according to 1.2 the magnetic field produced at M by the two segments A_i and A_{4n+1-i} mirrored throught $(C, \underline{e}_2)$ is:

$$\underline{B}_i = \frac{\mu_0}{2\pi}\frac{I}{r_{Mi}^2}\left(\frac{\mathrm{L}_{1/2} - l_{Mi}}{\sqrt{(\mathrm{L}_{1/2} - l_{Mi})^2 + r_{Mi}^2}} + \frac{\mathrm{L}_{1/2} + l_{Mi}}{\sqrt{(\mathrm{L}_{1/2} + l_{Mi})^2 + r_{Mi}^2}}\right)\underline{e}_2 \wedge \left[\underline{e}_2 \wedge (\underline{A_i M} \wedge \underline{u_i})\right]$$

Where:

$$\begin{aligned}
&\underline{e}_2 \wedge (\underline{A_i M} \wedge \underline{u_i}) &&= z\cos(\theta_i)\underline{e}_3 - (h - r\cos(\theta_i))\underline{e}_1\\
\Rightarrow\ &\underline{e}_2 \wedge \left[\underline{e}_2 \wedge (\underline{A_i M} \wedge \underline{u_i})\right] &&= z\cos(\theta_i)\underline{e}_1 + (h - r\cos(\theta_i))\underline{e}_3
\end{aligned}$$

And:

$$\begin{cases}
\frac{\mathrm{L}_{1/2} - l_{Mi}}{\sqrt{(\mathrm{L}_{1/2} - l_{Mi})^2 + r_{Mi}^2}} &= \frac{\mathrm{L}_{1/2} + r\sin(\theta_i)}{\sqrt{(\mathrm{L}_{1/2} + r\sin(\theta_i))^2 + z^2 + (h - r\cos(\theta_i))^2}}\\
&= \frac{\mathrm{L}_{1/2} + r\sin(\theta_i)}{\sqrt{\mathrm{L}_{1/2}{}^2 + 2\,\mathrm{L}_{1/2}\, r\sin(\theta_i) + r^2 + z^2 + h^2 - 2hr\cos(\theta_i)}}\\
\frac{\mathrm{L}_{1/2} + l_{Mi}}{\sqrt{(\mathrm{L}_{1/2} + l_{Mi})^2 + r_{Mi}^2}} &= \frac{\mathrm{L}_{1/2} - r\sin(\theta_i)}{\sqrt{\mathrm{L}_{1/2}{}^2 - 2\,\mathrm{L}_{1/2}\, r\sin(\theta_i) + r^2 + z^2 + h^2 - 2hr\cos(\theta_i)}}
\end{cases}$$

Hence, introducing $\alpha_i = \mathrm{L}_{1/2}{}^2 + r^2 + z^2 + h^2 - 2hr\cos(\theta_i)$:

$$\underline{B}_i = \frac{\mu_0}{2\pi}\frac{I}{z^2 + (h - r\cos(\theta_i))^2}\left(\frac{\mathrm{L}_{1/2} + r\sin(\theta_i)}{\sqrt{\alpha_i + 2\,\mathrm{L}_{1/2}\, r\sin(\theta_i)}} + \frac{\mathrm{L}_{1/2} - r\sin(\theta_i)}{\sqrt{\alpha_i - 2\,\mathrm{L}_{1/2}\, r\sin(\theta_i)}}\right)(z\cos(\theta_i)\underline{e}_1 + (h - r\cos(\theta_i))\underline{e}_3)$$

Hence, numerically:

1. Get all θ_i, $\cos(\theta_i)$ and $\sin(\theta_i)$ from n.
2. Get h, $\mathrm{L}_{1/2}$ for each spire, defined by $(C, R, \underline{e}_3)$
3. Get r, z and $\underline{e}_1$ for each pair $(M, spire)$
4. Sum all terms

2.3 Numerical Applications

2.3.1 Convergence study

Increasing n, we see that:

- Convergence is quite fast
- The inner side of the coil converges faster
- The remaining error is found in close vicinity to the coil itself
- A remaining error of $\approx 0.1\%$ can be obtained with $n = 10$ within $\approx 10\%$ of the coil radius around the coil itself

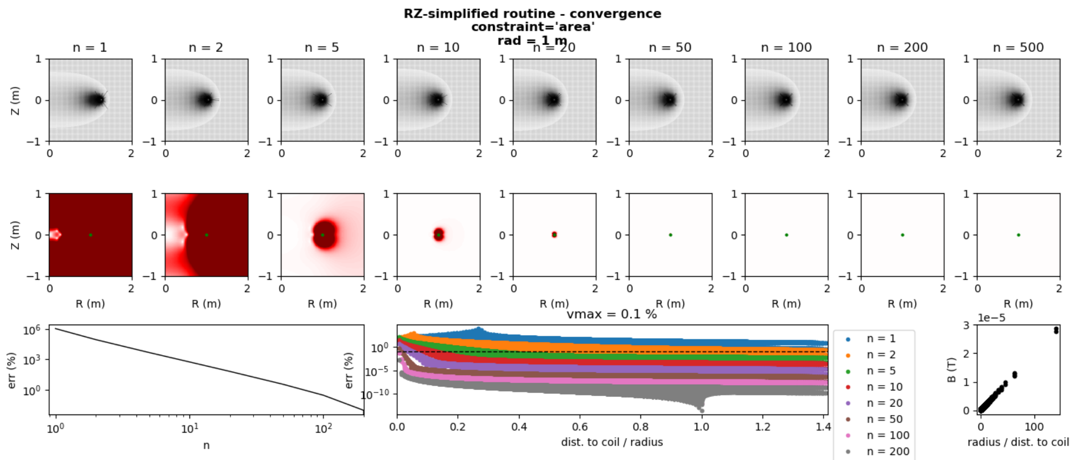

Figure 2.1: Poloidal magnetic field from a single circular coil discretized as a 4n-sided polygon, convergence study

Chapter 3

Circular coil - exact

The magnetic field produced at any point in 3D space by a planar circular coil can be writen (but not solved) analytically:

Let's consider a point M (x, y, z) in space and a point P on the spire.

The elementary magnetic field produced at M by current element $\underline{dj} = IRd\theta\underline{e}_\theta$ is:

$$d\underline{B}(\underline{M}, \underline{A}) = \frac{\mu_0}{4\pi} \frac{IRd\theta\underline{e}_\theta \wedge \underline{AM}}{\|\underline{AM}\|^3}$$

The problem is anti-symmetric regarding the magnetic field around the $(C, M, \underline{e}_z)$ plane, hence the total magnetic field at M will be contained inside the symmetry plane.

Considering $\theta' = \theta - \theta_M$ as the angular reference, we can define:

$$\underline{CM} = R_M\underline{e}_r(\theta' = 0) + (Z_M - Z_C)\underline{e}_z$$

And:

$$\begin{aligned} \underline{CA} &= R\underline{e}_r(\theta') \\ &= R(\cos(\theta')\underline{e}_r(0) + \sin(\theta')\underline{e}_{\theta'}(0)) \end{aligned}$$

In this context:

$$\begin{aligned} \underline{AM} &= \underline{AC} + \underline{CM} \\ &= -R\underline{e}_r(\theta') + \underline{CM} \end{aligned}$$

Hence the total magnetic field:

$$\underline{B} \;= \tfrac{\mu_0}{4\pi} IR \int_{-\pi}^{\pi} \tfrac{\underline{e}_\theta(\theta') \wedge \underline{AM}}{\|\underline{AM}\|^3} d\theta'$$

Noticing that:

$$\begin{aligned} \underline{e}_\theta(\theta') \wedge \underline{AM} &= R\underline{e}_z + \underline{e}_\theta(\theta') \wedge \underline{CM} \\ &= R\underline{e}_z + R_M\left(-\sin(\theta')\underline{e}_r(0) + \cos(\theta')\underline{e}_\theta(0)\right) \wedge \underline{e}_r(0) + (Z_M - Z_C)\,\underline{e}_r(\theta') \\ &= R\underline{e}_z - R_M\cos(\theta')\underline{e}_z + (Z_M - Z_C)\,\underline{e}_r(\theta') \\ &= (R - R_M\cos(\theta'))\underline{e}_z + (Z_M - Z_C)\left(\cos(\theta')\underline{e}_r(0) + \sin(\theta')\underline{e}_\theta(0)\right) \\ &= (R - R_M\cos(\theta'))\underline{e}_z + \Delta Z\cos(\theta')\underline{e}_r(0) + \Delta Z\sin(\theta')\underline{e}_\theta(0) \end{aligned}$$

Introducing $\theta'' = -\theta'$ and noticing that $\cos(\theta') = \cos(\theta'')$ and $\sin(\theta') = -\sin(\theta'')$ we can write:

$$\begin{aligned} \underline{e}_\theta(\theta') \wedge \underline{AM} &= (R - R_M\cos(\theta'))\underline{e}_z + \Delta Z\cos(\theta')\underline{e}_r(0) + \Delta Z\sin(\theta')\underline{e}_\theta(0) \\ &= (R - R_M\cos(\theta''))\underline{e}_z + \Delta Z\cos(\theta'')\underline{e}_r(0) - \Delta Z\sin(\theta'')\underline{e}_\theta(0) \\ &= \underline{e}_\theta(\theta'') \wedge \underline{AM} - 2\Delta Z\sin(\theta'')\underline{e}_\theta(0) \end{aligned}$$

Hence, splitting the integral in halves:

$$\begin{aligned}
\underline{B} &= \frac{\mu_0}{4\pi} IR \left(\int_{-\pi}^{0} \frac{\underline{e}_\theta(\theta')\wedge\underline{AM}}{\|\underline{AM}\|^3} d\theta' + \int_0^{\pi} \frac{\underline{e}_\theta(\theta')\wedge\underline{AM}}{\|\underline{AM}\|^3} d\theta' \right)\\
&= \frac{\mu_0}{4\pi} IR \left(-\int_0^{-\pi} \frac{\underline{e}_\theta(\theta')\wedge\underline{AM}}{\|\underline{AM}\|^3} d\theta' + \int_0^{\pi} \frac{\underline{e}_\theta(\theta')\wedge\underline{AM}}{\|\underline{AM}\|^3} d\theta' \right)\\
&= \frac{\mu_0}{4\pi} IR \left(\int_0^{\pi} \frac{\underline{e}_\theta(\theta'')\wedge\underline{AM}}{\|\underline{AM}\|^3} d\theta'' - 2\Delta Z \int_0^{\pi} \frac{\sin(\theta'')}{\|\underline{AM}\|^3} d\theta'' \underline{e}_\theta(0) + \int_0^{\pi} \frac{\underline{e}_\theta(\theta')\wedge\underline{AM}}{\|\underline{AM}\|^3} d\theta' \right)\\
&= 2\frac{\mu_0}{4\pi} IR \left(\int_0^{\pi} \frac{\underline{e}_\theta(\theta')\wedge\underline{AM}}{\|\underline{AM}\|^3} d\theta' - \Delta Z \int_0^{\pi} \frac{\sin(\theta')}{\|\underline{AM}\|^3} d\theta' \underline{e}_\theta(0) \right)\\
&= 2\frac{\mu_0}{4\pi} IR \left(\int_0^{\pi} \frac{(R - R_M\cos(\theta'))\underline{e}_z + \Delta Z\cos(\theta')\underline{e}_r(0)}{\|\underline{AM}\|^3} d\theta' + (1-1)\Delta Z \int_0^{\pi} \frac{\sin(\theta')}{\|\underline{AM}\|^3} d\theta'' \underline{e}_\theta(0) \right)\\
&= 2\frac{\mu_0}{4\pi} IR \left(R\int_0^{\pi} \frac{1 - r\cos(\theta')}{\|\underline{AM}\|^3} d\theta' \underline{e}_z + \Delta Z \int_0^{\pi} \frac{\cos(\theta')}{\|\underline{AM}\|^3} d\theta' \underline{e}_r(0) \right)
\end{aligned}$$

Introducing:

$$\begin{cases} r = \frac{R_M}{R} \\ z = \frac{Z}{R} \end{cases}$$

Now, writing:

$$\begin{aligned}
\|\underline{AM}\|^2 &= (-R\underline{e}_r(\theta) + \underline{CM})^2\\
&= R^2 - 2R\underline{e}_r(\theta)\cdot\underline{CM} + \|\underline{CM}\|^2\\
&= R^2 - 2RR_M\cos(\theta) + R_M^2 + \Delta Z^2\\
&= R^2\left(1 + r^2 + z^2 - 2r\cos(\theta)\right)
\end{aligned}$$

Hence:

$$\begin{aligned}
\underline{B} &= 2\frac{\mu_0}{4\pi} IR \left(R\int_0^{\pi} \frac{1 - r\cos(\theta)}{\|\underline{AM}\|^3} d\theta \underline{e}_z + \Delta Z \int_0^{\pi} \frac{\cos(\theta)}{\|\underline{AM}\|^3} d\theta \underline{e}_r(0) \right)\\
&= 2\frac{\mu_0}{4\pi}\frac{I}{R} \left(\int_0^{\pi} \underbrace{\frac{1 - r\cos(\theta)}{\left(1 + r^2 + z^2 - 2r\cos(\theta)\right)^{3/2}}}_{f_z(\theta,r,z)} d\theta \underline{e}_z + z\int_0^{\pi} \underbrace{\frac{\cos(\theta)}{\left(1 + r^2 + z^2 - 2r\cos(\theta)\right)^{3/2}}}_{f_r(\theta,r,z)} d\theta \underline{e}_r(0) \right)
\end{aligned}$$

3.1 Analyzing components

3.1.1 Sign of the integrands

Both r and z components are integrals of functions f_r and f_z, each with the same denominator.

This denominator is always positive, indeed:

$$\begin{aligned}
&1 + r^2 + z^2 - 2r\cos(\theta) \geq 0\\
&(1-r)^2 + 2r(1 - \cos(\theta)) + z^2 \geq 0
\end{aligned}$$

Which is always true because all terms are positive.

Hence:

$$\begin{cases} f_r(\theta, r, z) \geq 0 & \Leftrightarrow \cos(\theta) \geq 0 & \Leftrightarrow \theta \leq \frac{\pi}{2} \\ f_z(\theta, r, z) \geq 0 & \Leftrightarrow r\cos(\theta) \leq 1 & \Leftrightarrow \theta \geq \arccos\left(\frac{1}{r}\right) \end{cases}$$

It can be noticed that if $r \leq 1$ (i.e.: $R_M \leq R$), the z component of the magnetic field is always negative (or positive, depending on the sign of the current I).

However, if $r > 1$, there is a value of z (for fixed r) for which the component changes sign.

3.1.2 Zero vertical component

The vertical component is always non-zero if $r < 1$. So here we assume $r \geq 1$.

In the context, let's investigate the monotonicity of the integrands, keeping in mind that $1 + r^2 + z^2 - 2r\cos(\theta) > 0$ and $\sin(\theta) \geq 0$ in $[0; \pi]$.

$$
\begin{array}{ll}
 & \partial_\theta f_z(\theta, r, z) \geq 0 \\
\Leftrightarrow & r\sin(\theta)(1 + r^2 + z^2 - 2r\cos(\theta))^{3/2} - (1 - r\cos(\theta)\frac{3}{2}\sqrt{1 + r^2 + z^2 - 2r\cos(\theta)}(2r\sin(\theta))) \geq 0 \\
\Leftrightarrow & r\sin(\theta)(1 + r^2 + z^2 - 2r\cos(\theta)) - 3(1 - r\cos(\theta))r\sin(\theta))) \geq 0 \\
\Leftrightarrow & \sin(\theta)(1 + r^2 + z^2 - 2r\cos(\theta)) - 3(1 - r\cos(\theta))\sin(\theta))) \geq 0 \\
\Leftrightarrow & 1 + r^2 + z^2 - 2r\cos(\theta) - 3(1 - r\cos(\theta)) \geq 0 \\
\Leftrightarrow & r^2 + z^2 + r\cos(\theta) \geq 2 \\
\Leftrightarrow & \cos(\theta) \geq \frac{2-r^2-z^2}{r} \\
\Leftrightarrow & \theta \leq \arccos\left(\frac{2-r^2-z^2}{r}\right)
\end{array}
$$

We have 2 thresholds, and:

$$
\begin{array}{ll}
 & \frac{2-r^2-z^2}{r} \leq \frac{1}{r} \\
\Leftrightarrow & 2 - r^2 - z^2 \leq 1 \\
\Leftrightarrow & 1 \leq r^2 + z^2
\end{array}
$$

Which is always true because we assumed $r \geq 1$.
Hence, θ lives in:

$$
\begin{cases}
\theta \in [0; \arccos(\frac{1}{r})] & \Rightarrow f_z(\theta, r, z) \leq 0 \\
\theta \in [\arccos(\frac{1}{r}); \arccos(\frac{2-r^2-z^2}{r})] & \Rightarrow \partial_\theta f_z(\theta, r, z) \geq 0
\end{cases}
$$

Hence, a summary of the behaviour of f_z can be written:

	0	$\arccos(\frac{1}{r})$	$\arccos(\frac{2-r^2-z^2}{r})$	π
$\partial_\theta f_z$	≥ 0	≥ 0	≤ 0	≤ 0
f_z	≤ 0	≥ 0	≥ 0	≥ 0

3.2 Re-writing as elliptic integrals

Both components can be written as elliptic integrals.
The result be found in:

[1] J. E. Lane, Simple Analytic Expressions for the Magnetic Field of a Circular Current Loop, NASA/TM-2013-217919, 2001.

Appendix A

Appendices

A.1 Section

A.1.1 Subsection