ToFu geometric tools
Intersection of a cone with a circle (magnetic axis)

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Contents

Chapter 1

Geometry

1.1 Generic cone and plane

Let's consider a cartesian frame $(O, \underline{\mathrm{e}}_{\mathrm{x}}, \underline{\mathrm{e}}_{\mathrm{y}}, \underline{\mathrm{e}}_{\mathrm{z}})$. Let's consider a half-cone defined by its axis $(S, \underline{\mathrm{n}})$ and half-opening $\alpha = \pi/2 - \theta_{\mathrm{bragg}}$. The coordinates of S are (x_S, y_S, z_S). The coordinates of $\underline{\mathrm{n}}$ are (n_x, n_y, n_z). Let's consider a circle of axis $(O, \underline{\mathrm{e}}_{\mathrm{z}})$, of radius R, centered on C or coordinates $(0, 0, \mathrm{Z_C})$.

Let's consider point M of coordinates (x, y, z) and (R, θ, z) belonging to both the cone and the circle.

$$\begin{cases} M \in \mathit{circle} & \Leftrightarrow \underline{OM} = \mathrm{Z_C}\, \underline{\mathrm{e}}_{\mathrm{z}} + R(\cos(\theta)\, \underline{\mathrm{e}}_{\mathrm{x}} + \sin(\theta)\, \underline{\mathrm{e}}_{\mathrm{y}}) \\ M \in \mathit{cone} & \Leftrightarrow \underline{SM}.\, \underline{\mathrm{n}} = \cos(\alpha) \|\underline{SM}\| \end{cases}$$

1.2 Intersection

If M belongs to both the circle and the cone, then:

$$\begin{aligned} & \left[(\underline{SO} + \underline{OM})\,.\,\underline{\mathrm{n}}\right]^2 = \cos^2(\alpha)\|\underline{SO} + \underline{OM}\|^2 \\ \Leftrightarrow\ & (\underline{SO}.\,\underline{\mathrm{n}})^2 + (\underline{OM}.\,\underline{\mathrm{n}})^2 + 2\,(\underline{SO}.\,\underline{\mathrm{n}})\,(\underline{OM}.\,\underline{\mathrm{n}}) = \cos^2(\alpha)\left[\|\underline{SO}\|^2 + \|\underline{OM}\|^2 + 2\underline{SO}.\underline{OM}\right] \\ \Leftrightarrow\ & (\underline{OM}.\,\underline{\mathrm{n}})^2 + 2\,(\underline{SO}.\,\underline{\mathrm{n}})\,(\underline{OM}.\,\underline{\mathrm{n}}) - \cos^2(\alpha)\|\underline{OM}\|^2 - 2\cos^2(\alpha)\underline{SO}.\underline{OM} + A = 0 \end{aligned}$$

Where we have introduced $A = (\underline{SO}.\,\underline{\mathrm{n}})^2 - \cos^2(\alpha)\|\underline{SO}\|^2$

Now, we can write:

$$\begin{cases} \|\underline{OM}\|^2 & = & \mathrm{Z_C}^2 + R^2 \\ \underline{OM}.\,\underline{\mathrm{n}} & = & \mathrm{Z_C}\, n_z + R\cos(\theta) n_x + R\sin(\theta) n_y \\ (\underline{OM}.\,\underline{\mathrm{n}})^2 & = & (\mathrm{Z_C}\, n_z)^2 + (R\cos(\theta) n_x)^2 + (R\sin(\theta) n_y)^2 \\ & & +2\,\mathrm{Z_C}\, R\cos(\theta) n_x n_Z + 2\,\mathrm{Z_C}\, R\sin(\theta) n_y n_z + 2R^2\cos(\theta)\sin(\theta) n_x n_y \\ \underline{SO}.\underline{OM} & = & -\,\mathrm{Z_C}\, z_S - R x_S\cos(\theta) - R y_S\sin(\theta) \end{cases}$$

Hence:

$$\begin{aligned} & (\underline{OM}.\,\underline{\mathrm{n}})^2 + 2\,(\underline{SO}.\,\underline{\mathrm{n}})\,(\underline{OM}.\,\underline{\mathrm{n}}) - \cos^2(\alpha)\|\underline{OM}\|^2 - 2\cos^2(\alpha)\underline{SO}.\underline{OM} + A \\ = \ & (\mathrm{Z_C}\, n_z)^2 + (R\cos(\theta) n_x)^2 + (R\sin(\theta) n_y)^2 \\ & +2\,\mathrm{Z_C}\, R\cos(\theta) n_x n_Z + 2\,\mathrm{Z_C}\, R\sin(\theta) n_y n_z + 2R^2\cos(\theta)\sin(\theta) n_x n_y \\ & +2\,(\underline{SO}.\,\underline{\mathrm{n}})\,(\mathrm{Z_C}\, n_z + R\cos(\theta) n_x + R\sin(\theta) n_y) \\ & -\cos^2(\alpha)\,\mathrm{Z_C}^2 - \cos^2(\alpha) R^2 \\ & +2\cos^2(\alpha)\,(\mathrm{Z_C}\, z_S + R x_S\cos(\theta) + R y_S\sin(\theta)) + A \end{aligned}$$

1.3 Parametric equation

1.3.1 From bragg angle and parameter to local cartesian coordinates

Appendix A

Appendices

A.1 Section

A.1.1 Subsection