Solid angle subtended by a tetrahedron computation

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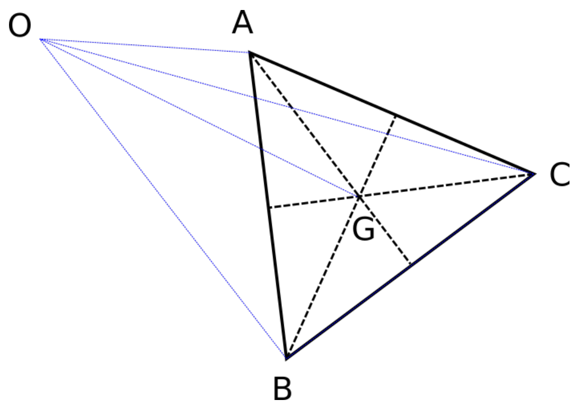

1 Notations

1.1 vectors

- $\underline{\mathrm{A}} = \vec{OA}$
- $\underline{\mathrm{B}} = \vec{OB}$
- $\underline{\mathrm{C}} = \vec{OC}$
- $\underline{\mathrm{G}} = \vec{OG}$
- $\underline{\mathrm{a}} = \vec{GA}$
- $\underline{\mathrm{b}} = \vec{GB}$
- $\underline{\mathrm{c}} = \vec{GC}$

1.2 scalars

- $A = \| \underline{\mathrm{A}} \|$
- $B = \| \underline{\mathrm{B}} \|$
- $C = \| \underline{\mathrm{C}} \|$
- $a = \| \underline{\mathrm{a}} \|$
- $b = \| \underline{\mathrm{b}} \|$
- $c = \| \underline{\mathrm{c}} \|$

2 Computation

The solid angle Ω subtended by the triangular surface ABC is:

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{|\underline{\mathrm{A}}\ \underline{\mathrm{B}}\ \underline{\mathrm{C}}|}{ABC + (\underline{\mathrm{A}} \cdot \underline{\mathrm{B}})\, C + (\underline{\mathrm{A}} \cdot \underline{\mathrm{C}})\, B + (\underline{\mathrm{B}} \cdot \underline{\mathrm{C}})\, A}$$

where $|\underline{\mathrm{A}}\ \underline{\mathrm{B}}\ \underline{\mathrm{C}}| = \underline{\mathrm{A}} \cdot (\underline{\mathrm{B}} \times \underline{\mathrm{C}})$

2.1 Numerator

Given that $\underline{A} = \underline{G} + \underline{a}$ (and resp. with $\underline{B}$ and $\underline{C}$), we get

$$\begin{aligned}
|\underline{A}\ \underline{B}\ \underline{C}| &= \underline{A} \cdot (\underline{B} \times \underline{C}) \\
&= (\underline{G} + \underline{a}) \cdot ((\underline{G} + \underline{b}) \times (\underline{G} + \underline{c})) \\
&= (\underline{G} + \underline{a}) \cdot (\cancel{\underline{G} \times \underline{G}} + \underline{G} \times \underline{c} + \underline{b} \times \underline{G} + \underline{b} \times \underline{c}) \\
&= \cancel{\underline{G} \cdot (\underline{G} \times \underline{c})} + \cancel{\underline{G} \cdot (\underline{b} \times \underline{G})} + \underline{G} \cdot (\underline{b} \times \underline{c}) + \underline{a} \cdot (\underline{G} \times \underline{c}) + \underline{a} \cdot (\underline{b} \times \underline{G}) + \cancel{\underline{a} \cdot (\underline{b} \times \underline{c})} \\
&= \underline{G} \cdot (\underline{b} \times \underline{c}) + \underline{G} \cdot (\underline{c} \times \underline{a}) + \underline{G} \cdot (\underline{a} \times \underline{b}) \\
&= \underline{G} \cdot (\underline{b} \times \underline{c} + \underline{c} \times \underline{a} + \underline{a} \times \underline{b})
\end{aligned}$$

since G is the centroid of ABC,

$$\underline{a} + \underline{b} + \underline{c} = 0 \tag{1}$$

. We obtain

$$\begin{aligned}
|\underline{A}\ \underline{B}\ \underline{C}| &= \underline{G} \cdot (\underline{b} \times \underline{c} + \underline{c} \times \underline{a} + \underline{a} \times \underline{b}) \\
&= \underline{G} \cdot (\underline{b} \times \underline{c} + \underline{c} \times (-\underline{b} - \underline{c}) + (-\underline{b} - \underline{c}) \times \underline{b}) \\
&= G \cdot (\underline{b} \times \underline{c} + \underline{c} \times (-\underline{b}) + \cancel{\underline{c} \times (-\underline{c})} + \cancel{(-b) \times b} + (-c) \times b) \\
&= G \cdot (3b \times c) \\
&= 3G \cdot (b \times c)
\end{aligned} \tag{2}$$

2.2 Denominator

First let's see the term in **C**

$$\begin{aligned}
(\underline{A} \cdot \underline{B})\, C &= (G + a) \cdot (G + b)\, \mathbf{C} \\
&= (\mathbf{G}^2 + G \cdot b + a \cdot G + a \cdot b)\mathbf{C}
\end{aligned}$$

Using (1)

$$\begin{aligned}
(A \cdot B)\, \mathbf{C} &= (\mathbf{G}^2 + G \cdot b + (-b - c) \cdot G + (-b - c) \cdot b)c \\
&= (\mathbf{G}^2 - ||GB||^2 - G \cdot c - c \cdot b)\mathbf{C}
\end{aligned} \tag{3}$$

Now, the term in **B**

$$\begin{aligned}
(A \cdot C)\, \mathbf{B} &= (G + a) \cdot (G + c)\, \mathbf{B} \\
&= (\mathbf{G}^2 + G \cdot c + a \cdot G + a \cdot c)\mathbf{B}
\end{aligned}$$

Using (1)

$$\begin{aligned}
(A \cdot C)\, \mathbf{B} &= (\mathbf{G}^2 + G \cdot c + (-b - c) \cdot G + (-b - c) \cdot c)b \\
&= (\mathbf{G}^2 - ||GC||^2 - G \cdot c - c \cdot b)\mathbf{B}
\end{aligned} \tag{4}$$

For the third term there is no simplification

$$(B \cdot C)\, \mathbf{A} = (G + b) \cdot (G + c)\, \mathbf{A} \tag{5}$$

For the computation of the norms, we can use:

$$\begin{aligned}
\mathbf{A}^2 &= ||OG||^2 + \mathbf{a}^2 - 2G \cdot a \\
&= \mathbf{G}^2 + \mathbf{a}^2 - 2G \cdot a
\end{aligned}$$

and respectively with B and C, we obtain

$$(\mathbf{ABC})^2 = (\mathbf{G}^2 + \mathbf{a} - 2G \cdot a)^2(\mathbf{G}^2 + \mathbf{b} - 2G \cdot b)^2(\mathbf{G}^2 + \mathbf{c} - 2G \cdot c)^2 \tag{6}$$

3 Final formula

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{3G\cdot(b\times c)}{\mathbf{ABC} + (A\cdot B)\,\mathbf{C} + (A\cdot C)\,\mathbf{B} + (B\cdot C)\,\mathbf{A}}$$

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{3G\cdot(b\times c)}{\mathbf{ABC} + (\mathbf{G}^2 - ||GB||^2 - G\cdot c - c\cdot b)\mathbf{C} + (\mathbf{G}^2 - ||GC||^2 - G\cdot c - c\cdot b)\mathbf{B} + (G+b)\cdot(G+c)\,\mathbf{A}}$$