ToFu geometric tools
Intersection of a cone with a plane

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Contents

Chapter 1

Geometry

1.1 Generic cone and plane

Let's consider a half-cone C_1 (defined only for $z > 0$), with summit on the cartesian frame's origin $(O, \underline{\mathrm{e}}_{\mathrm{x}}, \underline{\mathrm{e}}_{\mathrm{y}}, \underline{\mathrm{e}}_{\mathrm{z}})$. The cone's axis is the $(O, \underline{\mathrm{e}}_{\mathrm{z}})$ axis. It's angular opening is θ.

Let's consider plane P_1, of normal $\underline{\mathrm{n}}$, intersection axis $(O, \underline{\mathrm{e}}_{\mathrm{z}})$ at point P of coordinates $(0, 0, Z_P)$. Vector $\underline{\mathrm{n}}$ is oriented by angles ϕ and ψ such that one can define the local frame $(P, \underline{\mathrm{e}}_1, \underline{\mathrm{e}}_2, \underline{\mathrm{n}})$:

$$\begin{cases} \underline{\mathrm{e}}_1 & = \cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + \sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}} \\ \underline{\mathrm{e}}_2 & = \big(-\sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + \cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}}\big)\cos(\psi) + \sin(\psi)\,\underline{\mathrm{e}}_{\mathrm{z}} \\ \underline{\mathrm{n}} & = \underline{\mathrm{e}}_1 \wedge \underline{\mathrm{e}}_2 \\ & = \big(\sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} - \cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}}\big)\sin(\psi) + \cos(\psi)\,\underline{\mathrm{e}}_{\mathrm{z}} \end{cases}$$

We want to find all points M of coordinates (x, y, z) and (x_1, x_2) belonging both to the cone C_1 and the plane P_1.

$$M \in C_1 \Leftrightarrow \underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}} = \cos(\theta)\|\underline{OM}\|$$

$$M \in P_1 \Leftrightarrow \underline{PM}.\,\underline{\mathrm{n}} = 0$$

1.2 Intersection

If M belongs to both P_1 and C_1, then:

$$(\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}})^2 = \cos(\theta)^2\|\underline{OM}\|^2$$

Given that:

$$\begin{aligned} \underline{OM} & = \underline{OP} + \underline{PM} \\ & = Z_P\,\underline{\mathrm{e}}_{\mathrm{z}} + x_1\,\underline{\mathrm{e}}_1 + x_2\,\underline{\mathrm{e}}_2 \\ & = Z_P\,\underline{\mathrm{e}}_{\mathrm{z}} + x_1\big(\cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + \sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}}\big) + x_2\Big(\big(-\sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + \cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}}\big)\cos(\psi) + \sin(\psi)\,\underline{\mathrm{e}}_{\mathrm{z}}\Big) \\ & = Z_P\,\underline{\mathrm{e}}_{\mathrm{z}} + x_1\cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + x_1\sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}} - x_2\sin(\phi)\cos(\psi)\,\underline{\mathrm{e}}_{\mathrm{x}} + x_2\cos(\phi)\cos(\psi)\,\underline{\mathrm{e}}_{\mathrm{y}} + x_2\sin(\psi)\,\underline{\mathrm{e}}_{\mathrm{z}} \\ & = (x_1\cos(\phi) - x_2\sin(\phi)\cos(\psi))\,\underline{\mathrm{e}}_{\mathrm{x}} + (x_1\sin(\phi) + x_2\cos(\phi)\cos(\psi))\,\underline{\mathrm{e}}_{\mathrm{y}} + (Z_P + x_2\sin(\psi))\,\underline{\mathrm{e}}_{\mathrm{z}} \end{aligned}$$

We have:

$$\begin{aligned} (\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}})^2 & = (Z_P + x_2\sin(\psi))^2 \\ & = Z_P^2 + 2Z_P x_2\sin(\psi) + x_2^2\sin(\psi)^2 \end{aligned}$$

And:

$$\begin{aligned}
\|\underline{OM}\|^2 &= \|\left(x_1\cos(\phi) - x_2\sin(\phi)\cos(\psi)\right)\underline{\mathrm{e}}_{\mathrm{x}} + \left(x_1\sin(\phi) + x_2\cos(\phi)\cos(\psi)\right)\underline{\mathrm{e}}_{\mathrm{y}} + \left(Z_P + x_2\sin(\psi)\right)\underline{\mathrm{e}}_{\mathrm{z}}\|^2 \\
&= \left(x_1\cos(\phi) - x_2\sin(\phi)\cos(\psi)\right)^2 \\
&\quad + \left(x_1\sin(\phi) + x_2\cos(\phi)\cos(\psi)\right)^2 \\
&\quad + \left(Z_P + x_2\sin(\psi)\right)^2 \\
&= x_1^2\cos(\phi)^2 - 2x_1x_2\cos(\phi)\sin(\phi)\cos(\psi) + x_2^2\sin(\phi)^2\cos(\psi)^2 \\
&\quad + x_1^2\sin(\phi)^2 + 2x_1x_2\sin(\phi)\cos(\phi)\cos(\psi) + x_2^2\cos(\phi)^2\cos(\psi)^2 \\
&\quad + Z_P^2 + 2Z_Px_2\sin(\psi) + x_2^2\sin(\psi)^2 \\
&= x_1^2 + x_2^2\cos(\psi)^2 \\
&\quad + Z_P^2 + 2Z_Px_2\sin(\psi) + x_2^2\sin(\psi)^2 \\
&= x_1^2 + x_2^2 + 2Z_Px_2\sin(\psi) + Z_P^2
\end{aligned}$$

Thus:

$$\begin{aligned}
& (\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}})^2 = \cos(\theta)^2\|\underline{OM}\|^2 \\
\Leftrightarrow\ & Z_P^2 + 2Z_Px_2\sin(\psi) + x_2^2\sin(\psi)^2 = \cos(\theta)^2\left(x_1^2 + x_2^2 + 2Z_Px_2\sin(\psi) + Z_P^2\right) \\
\Leftrightarrow\ & Z_P^2\left(1-\cos(\theta)^2\right) + 2Z_Px_2\sin(\psi)\left(1-\cos(\theta)^2\right) = x_1^2\cos(\theta)^2 + x_2^2\left(\cos(\theta)^2 - \sin(\psi)^2\right) \\
\Leftrightarrow\ & Z_P^2\sin(\theta)^2 + 2Z_Px_2\sin(\psi)\sin(\theta)^2 = x_1^2\cos(\theta)^2 + x_2^2\left(\cos(\theta)^2 - \sin(\psi)^2\right)
\end{aligned}$$

Considering that by hypothesis $\theta > 0$:

$$\begin{aligned}
& (\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}})^2 = \cos(\theta)^2\|\underline{OM}\|^2 \\
\Leftrightarrow\ & x_1^2\cos(\theta)^2 + x_2^2\left(\cos(\theta)^2\text{-}\sin(\psi)^2\right) - 2Z_Px_2\sin(\psi)\sin(\theta)^2 - Z_P^2\sin(\theta)^2 = 0 \\
\Leftrightarrow\ & x_1^2\tfrac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + x_2^2 - 2x_2Z_P\tfrac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} - Z_P^2\tfrac{\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} = 0 \\
\Leftrightarrow\ & x_1^2\tfrac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + \left(x_2 - Z_P\tfrac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}\right)^2 - Z_P^2\tfrac{\sin(\psi)^2\sin(\theta)^4}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2} - Z_P^2\tfrac{\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} = 0 \\
\Leftrightarrow\ & x_1^2\tfrac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + \left(x_2 - Z_P\tfrac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}\right)^2 = Z_P^2\tfrac{\sin(\theta)^2}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2}\left(\sin(\psi)^2\sin(\theta)^2 + \cos(\theta)^2\text{-}\sin(\psi)^2\right) \\
\Leftrightarrow\ & x_1^2\tfrac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + \left(x_2 - Z_P\tfrac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}\right)^2 = Z_P^2\tfrac{\sin(\theta)^2}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2}\left(-\sin(\psi)^2\cos(\theta)^2 + \cos(\theta)^2\right) \\
\Leftrightarrow\ & x_1^2\tfrac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + \left(x_2 - Z_P\tfrac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}\right)^2 = Z_P^2\tfrac{\sin(\theta)^2\cos(\psi)^2\cos(\theta)^2}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2} \\
\Leftrightarrow\ & \frac{x_1^2}{Z_P^2\frac{\sin(\theta)^2\cos(\psi)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}} + \frac{\left(x_2 - Z_P\frac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}\right)^2}{Z_P^2\frac{\sin(\theta)^2\cos(\psi)^2\cos(\theta)^2}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2}} = 1
\end{aligned}$$

Or, in a reduced conic form:

$$\frac{x_1^2}{a^2} + \frac{(x_2 - x_2(C))^2}{b^2} = 1$$

With:

$$\left\{\begin{array}{lll}
x_2(C) & = Z_P\frac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} & x_2 \text{ coordinate of the center} \\
a^2 & = Z_P^2\frac{\sin(\theta)^2\cos(\psi)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} & \text{squared minor radius} \\
b^2 & = Z_P^2\frac{\sin(\theta)^2\cos(\psi)^2\cos(\theta)^2}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2} & \text{squared major radius} \\
b^2 & = a^2\frac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} \Leftrightarrow a^2 = b^2\left(1 - \frac{\sin(\psi)^2}{\cos(\theta)^2}\right) &
\end{array}\right.$$

The distance d_{CF} between the center C and the focal point F can be deduced from:

$$\begin{aligned}
d_{CF}^2 &= b^2 - a^2 \\
&= b^2\tfrac{\sin(\psi)^2}{\cos(\theta)^2} \\
&= Z_P^2\tfrac{\sin(\theta)^2\cos(\psi)^2\sin(\psi)^2}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2}
\end{aligned}$$

Hence, the x_2 coordinate of F is:

$$\begin{aligned} x_2(F) &= x_2(C) \pm d_{CF} \\ &= Z_P \frac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} \pm Z_P \frac{\sin(\theta)\cos(\psi)\sin(\psi)}{\cos(\theta)^2\text{-}\sin(\psi)^2} \\ &= Z_P \frac{\sin(\psi)\sin(\theta)^2 \pm \sin(\theta)\cos(\psi)\sin(\psi)}{\cos(\theta)^2\text{-}\sin(\psi)^2} \\ &= Z_P \frac{\sin(\psi)\sin(\theta)}{\cos(\theta)^2\text{-}\sin(\psi)^2} \left(\sin(\theta) \pm \cos(\psi)\right) \end{aligned}$$

It is worth noticing that the neither the focal point nor the center correspond to the intersection between the axes and the plane P.

1.3 Parametric equation

In our case, only the axes $(O, \underline{\mathrm{e}}_\mathrm{z})$, fixed by the crystal's summit and normal, is independent from the cone's angular opening θ. It makes sense to design an ad-hoc coordinate system centered on the ellipse's center C to use its parameterized equation.

Knowing all geometrical parameters, it is possible to compute all points on the ellipse parameterizing them with angle ϵ:

$$\left\{ \begin{array}{l} x_1 = a\cos(\epsilon) \\ x_2 = x_2(C) + b\sin(\epsilon) \end{array} \right.$$

Keep in mind that the frame $(P, \underline{\mathrm{e}}_1, \underline{\mathrm{e}}_2)$ is, by definition ligned on the minor and major axes of the ellipse. Hence, for an arbitrary frame $(R, \underline{e}_i, \underline{e}_j)$ on plane P_1, translated and rotated by α with respect to $(P, \underline{\mathrm{e}}_1, \underline{\mathrm{e}}_2)$:

$$\left\{ \begin{array}{l} \underline{e}_i = \cos(\alpha)\,\underline{\mathrm{e}}_1 + \sin(\alpha)\,\underline{\mathrm{e}}_2 \\ \underline{e}_j = -\sin(\alpha)\,\underline{\mathrm{e}}_1 + \cos(\alpha)\,\underline{\mathrm{e}}_2 \\ \underline{\mathrm{e}}_1 = \cos(\alpha)\underline{e_i} - \sin(\alpha)\underline{e}_j\,\underline{\mathrm{e}}_2 = \sin(\alpha)\underline{e_i} + \cos(\alpha)\underline{e}_j \end{array} \right.$$

Or, in coordinate tranforms:

$$\left\{ \begin{array}{l} x_1 = x_1(R) + x_i\cos(\alpha) - x_j\sin(\alpha) \\ x_2 = x_2(R) + x_i\sin(\alpha) + x_j\cos(\alpha) \\ x_i = (x_1 - x_1(R))\cos(\alpha) + (x_2 - x_2(R))\sin(\alpha) \\ x_j = -(x_1 - x_1(R))\sin(\alpha) + (x_2 - x_2(R))\cos(\alpha) \end{array} \right.$$

Hence:

$$\left\{ \begin{array}{l} x_i = (a\cos(\epsilon) - x_1(R))\cos(\alpha) + (x_2(C) - x_2(R) + b\sin(\epsilon))\sin(\alpha) \\ x_j = -\,(a\cos(\epsilon) - x_1(R))\sin(\alpha) + (x_2(C) - x_2(R) + b\sin(\epsilon))\cos(\alpha) \end{array} \right.$$

Appendix A

Acceleration radiation from a unique point-like charge

A.1 Retarded time and potential

A.1.1 Retarded time

Hence $\frac{dR(t_r)}{c} + dt_r = dt$