ToFu geometric tools
Intersection of a LOS with a cone

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Chapter 1

Definitions

1.1 Geometry definition in ToFu

The definition of a fusion device in ToFu is done by defining the edge of a poloidal plane as a set of segments in a 2D plane. The 3D volume is obtained by an extrusion for cylinders or a revolution for tori. We consider an orthonormal direct cylindrical coordinate system $(O, \underline{e}_R, \underline{e}_\theta, \underline{e}_Z)$ associated to the orthonormal direct cartesian coordinate system $(O, \underline{e}_X, \underline{e}_Y, \underline{e}_Z)$. We suppose that all poloidal planes live in (R, Z) and can be obtained after a revolution around the Z axis of the user-defined poloidal plane at $\theta = 0$, $\mathcal{P}_0$. Thus, the torus is axisymmetric around the (O, Z) axis (see Figure 1.1).

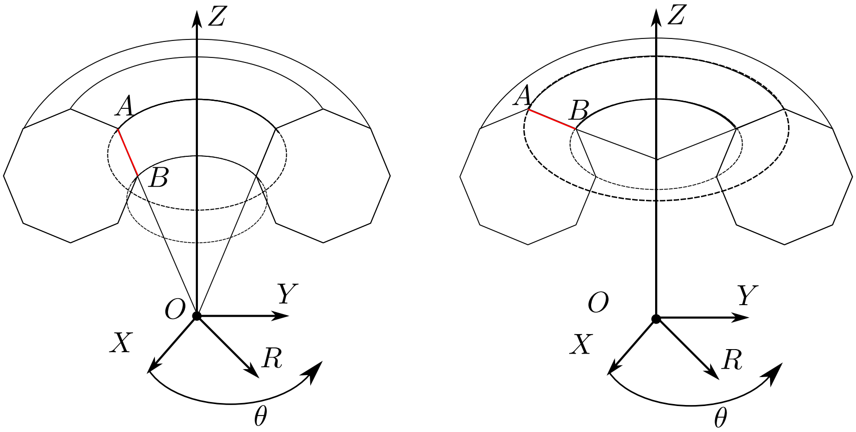

Figure 1.1: Two examples of a circular torus approximated by a revolved octagon. For each segment $\overline{AB}$ of the octagon there is a cone with origin on the (O, Z) axis.

1.2 Notations

In order to simplify the computations, let A and B be the end points of a segment $\mathcal{S}_i$ such that $A \neq B$ and $\mathcal{P} = \cup_{i=1}^{n} \mathcal{S}_i = \cup_{i=1}^{n} \overline{\mathrm{A}_i\mathrm{B}_i}$ with n the number of segments given by the user defining the plane $\mathcal{P}$. We define a right circular cone $\mathcal{C}$ of origin $P = (\mathrm{A}, \mathrm{B}) \cap (\mathrm{O}, \mathrm{Z})$ of generatrix (A, B) and of axis (O, Z) (see Figure 1.1). Thus we can define the edge of the torus as the union of the edges of the frustums $\mathcal{F}_i$ defined by truncating the cones $\mathcal{C}_i$ to the segment $\overline{AB}_i$.

Then, any point M with coordinates (X, Y, Z) or (R, θ, Z) belongs to the frustum $\mathcal{F}$

if and only if

$$\exists q \in [0;1] / \left\{ \begin{array}{l} R - R_A = q(R_B - R_A) \\ Z - Z_A = q(Z_B - Z_A) \end{array} \right.$$

Now let us consider a LOS L (i.e.: a half-infinite line) defined by a point D and a normalized directing vector u, of respective coordinates (X_D, Y_D, Z_D) or (R_D, θ_D, Z_D) and (u_X, u_Y, u_Z). Then, point M belongs to L if and only if:

$$\exists k \in [0; \infty[/ \underline{DM} = k\underline{u}$$

Chapter 2

Computing shortest distance between LOS and Frustum

We want to calculate the shortest distance between a 3D ray $\mathcal{R}$ defined by its origin $\vec{D}$ and its unit directional vector $\vec{u}$ and a frustum $\mathcal{F}$ defined by a segment AB extruded around the axis $\vec{N}$ of coordinates $(0, 0, 1)$. We want to compute the shortest distance between a point P on the ray $\mathcal{R}$ and a point Q on the frustum $\mathcal{F}$.

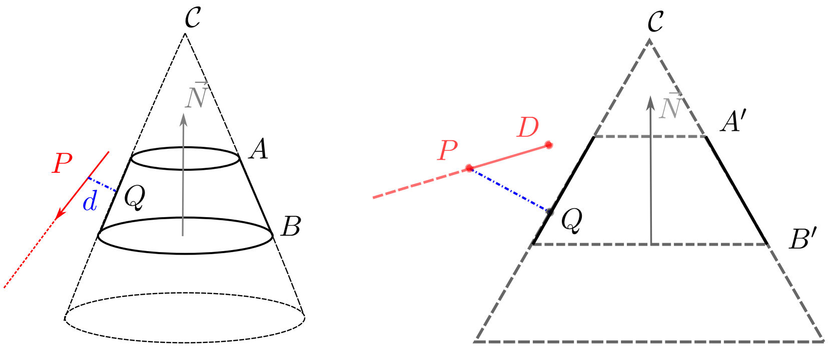

Figure 2.1: Example of closest point between ray and Frustum: 3D space and (R,Q) plane.

First, let us write some of the equations that Q respects

$$
\begin{aligned}
(Q - C) \cdot N &= \|Q - C\| \cos(\tau_{\mathcal{C}}) \\
R_q - R_A &= q(R_B - R_A) \\
Z_q - Z_A &= q(Z_B - Z_A)
\end{aligned}
$$

where $\tau_{\mathcal{C}}$ is the angle between $\vec{AB}$ and $-\vec{N}$.

$$
\begin{aligned}
-\vec{N} \cdot \vec{AB} &= \|N\| \, \|AB\| \cos(\tau_{\mathcal{C}}) \\
z_A - z_B &= \|AB\| \cos(\tau_{\mathcal{C}}) \\
\tau_{\mathcal{C}} &= \arccos\left(\frac{z_A - z_B}{\|AB\|}\right)
\end{aligned}
$$

We are looking to minimize the distance between P and Q which is equivalent to solve the following system.

$$
\begin{cases}
\dfrac{\partial}{\partial k} \|P-Q\|^2 = 0 & (2.0.1)\\
\dfrac{\partial}{\partial q} \|P-Q\|^2 = 0 & (2.0.2)
\end{cases}
$$

with

$$
\begin{aligned}
\|P-Q\|^2 &= (x_p - x_q)^2 + (y_p - y_q)^2 + (z_p - z_q)^2 \\
&= x_p^2 + y_p^2 + z_p^2 - 2(x_p x_q + y_p y_q + z_p z_q) + x_q^2 + y_q^2 + z_q^2 \\
&= \|P\|^2 - 2 < P, Q > + \|Q\|^2
\end{aligned}
$$

and

$$
\begin{cases}
\dfrac{\partial}{\partial k} < P, Q >= \dfrac{\partial}{\partial k} \left((x_D + k u_x) x_q + (y_D + k u_y) y_q + (z_D + k u_z) z_q\right) & (2.0.3)\\
\dfrac{\partial}{\partial q} < P, Q >= \dfrac{\partial}{\partial q} \left(x_p R_q \cos(\theta_q) + y_p R_q \sin(\theta_q) + z_p(q(z_B - z_A) - z_A)\right) & (2.0.4)
\end{cases}
$$

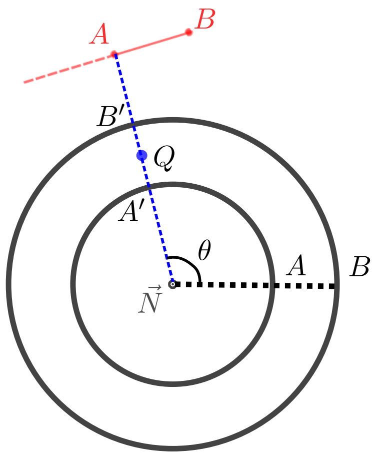

Figure 2.2: Example of closest point between ray and Frustum: (X,Y) plane.

We can see in Figure 2.2, that $\theta_q = \theta_p$. By definition $cos(\theta_p) = x_p/R_p$ and $sin(\theta_p) = y_p/R_p$. Thus $x_p \cos(\theta_q) + y_p \sin(\theta_q) = (x_p^2 + y_p^2)/R_p = R_p$. We introduce this in Equation (2.0.4). The derivation of Equation (2.0.4) is straightforward. We obtain

$$
\begin{cases}
\dfrac{\partial}{\partial k} < P, Q >= u_x x_q + u_y y_q + u_z z_q =< u, Q > & (2.0.5)\\
\dfrac{\partial}{\partial q} < P, Q >= R_p(R_B - R_A) + Z_p(Z_B - Z_A) & (2.0.6)
\end{cases}
$$

Now, let us derivate the remaining terms in $\|P - Q\|^2$

$$
\begin{cases}
\dfrac{\partial}{\partial k}\left\|P\right\|^2 &= \dfrac{\partial}{\partial k}\left((x_D + ku_x)^2 + (y_D + ku_y)^2 + (z_D + ku_z)^2\right) \\
\dfrac{\partial}{\partial q}\left\|Q\right\|^2 &= \dfrac{\partial}{\partial q}\left(R_q^2 + Z_q^2\right) \\
&= \dfrac{\partial}{\partial q}\left((q(R_B - R_A) + R_A)^2 + (q(Z_B - Z_A) + Z_A)^2\right)
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial}{\partial k}\left\|P\right\|^2 &= 2k(u_x + u_y + u_z) + 2(u_x x_D + u_y y_D + u_z z_D) \\
&= 2k\left\|u\right\|^2 + 2 < u, D > \\
\dfrac{\partial}{\partial q}\left\|Q\right\|^2 &= 2q((R_B - R_A)^2 + (Z_B - Z_A)^2) + 2(R_A(R_B - R_A) + Z_A(Z_B - Z_A)) \\
&= 2q\left\|AB\right\|^2 + 2 < OA, AB >
\end{cases}
$$

Thus, Equations (2.0.2)-(2.0.2) become

$$
\begin{cases}
\dfrac{\partial}{\partial k}\left\|P - Q\right\|^2 &= 2k\left\|u\right\|^2 + 2 < u, D > -2 < u, Q >= 0 \\
\dfrac{\partial}{\partial q}\left\|P - Q\right\|^2 &= 2q\left\|AB\right\|^2 + 2 < OA, AB > -2(R_p(R_B - R_A) + Z_p(Z_B - Z_A)) = 0
\end{cases}
$$

$$
\begin{cases}
k &= \dfrac{< u, Q > - < u, D >}{\left\|u\right\|^2} \\
q &= \dfrac{(R_p(R_B - R_A) + Z_p(Z_B - Z_A)) - < OA, AB >}{\left\|AB\right\|^2}
\end{cases}
$$