ToFu tools
Magnetic fields

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Chapter 1

3D field from a straight current segment

Let's consider current segment with current I, centered on A, with unit vector $\underline{u}$ and half-length $\mathrm{L}_{1/2}$. Considering a point P in this segment, identified by its length to A:

$$\underline{OP} = \underline{OA} + l\underline{u}, \text{ with } l \in \left[-\mathrm{L}_{1/2}; \mathrm{L}_{1/2}\right]$$

Any point M in space can be located by its position with respect to $\underline{A}$

$$\begin{cases} \underline{OM} = \underline{OA} + l_M \underline{u} + r_M \underline{v} \\ \underline{v} = \underline{AM} - \underline{AM} \cdot \underline{u}\underline{u} \end{cases}$$

The Biot-Savart law stipulates that an elementary length of current creates an elementary magnetic field at M:

$$\begin{cases} \underline{dI} = Idl\underline{u} \\ \underline{PM} = (l_M - l)\underline{u} + r_M \underline{v} \\ \underline{B_A} = \frac{\mu_0}{4\pi} \int_{-\mathrm{L}_{1/2}}^{\mathrm{L}_{1/2}} \frac{\underline{dI} \wedge \underline{PM}}{\|\underline{PM}\|^3} \end{cases}$$

Introducing $\underline{w} = \underline{u} \wedge \underline{v}$:

$$\underline{dI} \wedge \underline{PM} = Idlr_M \underline{w}$$

and:

$$\|\underline{PM}\| = \sqrt{(l - l_M)^2 + r_M^2}$$

Hence:

$$\underline{B_A} = \frac{\mu_0}{4\pi} I r_M \underline{w} \int_{-\mathrm{L}_{1/2}}^{\mathrm{L}_{1/2}} \frac{dl}{\left((l - l_M)^2 + r_M^2\right)^{3/2}}$$

Introducing $x = l - l_M \Rightarrow dx = dl$:

$$\underline{B_A} \quad = \frac{\mu_0}{4\pi} I r_M \underline{w} \int_{-\mathrm{L}_{1/2} - l_M}^{\mathrm{L}_{1/2} - l_M} \frac{dx}{\left(x^2 + r_M^2\right)^{3/2}}$$

Noticing that:

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + a}} \right) &= \frac{\sqrt{x^2+a} - x\frac{1}{2}\frac{2x}{\sqrt{x^2+a}}}{x^2+a} \\ &= \frac{(x^2+a) - x^2}{(x^2+a)^{3/2}} \\ &= \frac{a}{(x^2+a)^{3/2}} \end{aligned}$$

Hence:

$$
\begin{aligned}
\underline{B_A} &= \frac{\mu_0}{4\pi} I r_M \underline{w} \left[\frac{1}{r_M^2} \frac{x}{\sqrt{x^2+r_M^2}} \right]_{-\,\mathrm{L}_{1/2}\,-l_M}^{\mathrm{L}_{1/2}\,-l_M} \\
&= \frac{\mu_0}{4\pi} \frac{I}{r_M} \underline{w} \left(\frac{\mathrm{L}_{1/2}\,-l_M}{\sqrt{(\mathrm{L}_{1/2}\,-l_M)^2+r_M^2}} + \frac{\mathrm{L}_{1/2}\,+l_M}{\sqrt{(\mathrm{L}_{1/2}\,+l_M)^2+r_M^2}} \right)
\end{aligned}
$$

Chapter 2

Circular coil

The magnetic field produced at any point in 3D space by a planar circular coil cannot be derived analytically.

2.1 Circle discretization

Let's consider a circular coil of radius R centered on axis $(A, \underline{n})$.

For a given point M in space, let's make the plane $(O, M, \underline{e}_z)$ a symmetry plane and divide the circle into a N-sided polygon.

2.1.1 N-sided polygon

The polygon is N-sided, with N an even number. The polygon shall have either:

- the same perimeter as the circle: $L = 2\pi R$
- the same area as the circle: $S = \pi R^2$
- the same magnetic field on the center as the circle: $B = \frac{\mu_0 I}{2R}$

For a N-sided regular polygon of height h:

$$\begin{cases} L_i = 2h\tan\left(\frac{\pi}{N}\right) & \text{is the length a single side} \\ S_i = \frac{hL_i}{2} = h^2\tan\left(\frac{\pi}{N}\right) & \text{is the area a single side} \end{cases}$$

Remembering that, at the center, $l_m = 0$ and $r_m = h$ for all, by construction, and that $\mathrm{L}_{1/2} = \frac{L_i}{2} = h\tan\left(\frac{\pi}{N}\right)$, we derive:

$$\begin{aligned} B &= N\frac{\mu_0}{4\pi}\frac{I}{h}\left(\frac{\mathrm{L}_{1/2}}{\sqrt{(\mathrm{L}_{1/2})^2+h^2}} + \frac{\mathrm{L}_{1/2}}{\sqrt{(\mathrm{L}_{1/2})^2+h^2}}\right) \\ &= N\frac{\mu_0}{2\pi}\frac{I}{h}\frac{\mathrm{L}_{1/2}}{\sqrt{(\mathrm{L}_{1/2})^2+h^2}} \\ &= N\frac{\mu_0}{2\pi}\frac{I}{h}\frac{h\tan\left(\frac{\pi}{N}\right)}{\sqrt{h^2\tan^2\left(\frac{\pi}{N}\right)+h^2}} \\ &= N\frac{\mu_0}{2\pi}\frac{I}{h}\frac{\tan\left(\frac{\pi}{N}\right)}{\sqrt{\tan^2\left(\frac{\pi}{N}\right)+1}} \\ &= \frac{\mu_0 I}{2}\frac{1}{h}\frac{N}{\pi}\frac{\tan\left(\frac{\pi}{N}\right)}{\sqrt{\tan^2\left(\frac{\pi}{N}\right)+1}} \end{aligned}$$

Hence,

- same perimeter $\Rightarrow h = R\frac{\frac{\pi}{N}}{\tan\left(\frac{\pi}{N}\right)}$
- same area $\Rightarrow h = R\sqrt{\frac{\frac{\pi}{N}}{\tan\left(\frac{\pi}{N}\right)}}$
- same field $\Rightarrow h = R\frac{\tan\left(\frac{\pi}{N}\right)}{\frac{\pi}{N}}\frac{1}{\sqrt{\tan^2\left(\frac{\pi}{N}\right)+1}}$

Appendix A

Appendices

A.1 Section

A.1.1 Subsection