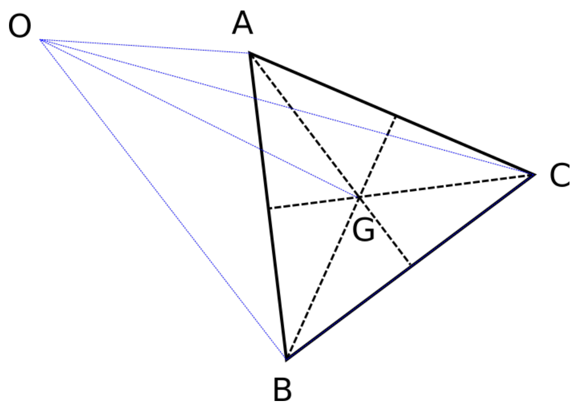
Solid angle subtended by a tetrahedron computation

May 21, 2021



1 Notations

Let

- A be the vector $\vec{OA}$
- B be the vector $\vec{OB}$
- C be the vector $\vec{OC}$
- a be the vector $\vec{GA}$
- b be the vector $\vec{GB}$
- c be the vector c
- $\mathbf{A}$ be the magnitude of the vector $\vec{OA}$
- $\mathbf{B}$ be the magnitude of the vector $\vec{OB}$
- $\mathbf{C}$ be the magnitude of the vector $\vec{OC}$

2 Computation

The solid angle Ω subtended by the triangular surface ABC is:

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{|A\ B\ C|}{\mathbf{ABC} + (A \cdot B)\,\mathbf{C} + (A \cdot C)\,\mathbf{B} + (B \cdot C)\,\mathbf{A}}$$

where $|A\ B\ C| = A \cdot (B \times C)$

2.1 Numerator

Given that $A = \vec{OA} = G + a$ (and resp. with B and C), we get

$$\begin{aligned}
|A\ B\ C| &= A \cdot (B \times C) \\
&= A \cdot ((G + b) \times (G + c)) \\
&= (G + a) \cdot (\cancel{G \times G} + G \times c + b \times G + b \times c) \\
&= \cancel{G \cdot (G \times c)} + \cancel{G \cdot (b \times G)} \\
&\quad + G \cdot (b \times c) + a \cdot (G \times c) + a \cdot (b \times G) + \cancel{a \cdot (b \times c)} \\
&= G \cdot (b \times c) + G \cdot (c \times a) + G \cdot (a \times b) \\
&= G \cdot (b \times c + c \times a + a \times b)
\end{aligned}$$

since G is the centroid of ABC,

$$a + b + c = 0 \tag{1}$$

. We obtain

$$\begin{aligned}
|A\ B\ C| &= G \cdot (b \times c + c \times a + a \times b) \\
&= G \cdot (b \times c + c \times (-b - c) + (-b - c) \times b) \\
&= G \cdot (b \times c + c \times (-b) + \cancel{c \times (-c)} + \cancel{(-b) \times b} + (-c) \times b) \\
&= G \cdot (3b \times c) \\
&= 3G \cdot (b \times c)
\end{aligned} \tag{2}$$

2.2 Denominator

First let's see the term in $\mathbf{C}$

$$\begin{aligned}
(A \cdot B)\,\mathbf{C} &= (G + a) \cdot (G + b)\,\mathbf{C} \\
&= (\mathbf{G}^2 + G \cdot b + a \cdot G + a \cdot b)\mathbf{C}
\end{aligned}$$

Using (1)

$$\begin{aligned}
(A \cdot B)\,\mathbf{C} &= (\mathbf{G}^2 + G \cdot b + (-b - c) \cdot G + (-b - c) \cdot b)c \\
&= (\mathbf{G}^2 - ||GB||^2 - G \cdot c - c \cdot b)\mathbf{C}
\end{aligned} \tag{3}$$

Now, the term in $\mathbf{B}$

$$\begin{aligned}
(A \cdot C)\,\mathbf{B} &= (G + a) \cdot (G + c)\,\mathbf{B} \\
&= (\mathbf{G}^2 + G \cdot c + a \cdot G + a \cdot c)\mathbf{B}
\end{aligned}$$

Using (1)

$$\begin{aligned}
(A \cdot C)\,\mathbf{B} &= (\mathbf{G}^2 + G \cdot c + (-b - c) \cdot G + (-b - c) \cdot c)b \\
&= (\mathbf{G}^2 - ||GC||^2 - G \cdot c - c \cdot b)\mathbf{B}
\end{aligned} \tag{4}$$

For the third term there is no simplification

$$(B \cdot C)\,\mathbf{A} = (G + b) \cdot (G + c)\,\mathbf{A} \tag{5}$$

For the computation of the norms, we can use:

$$\begin{aligned}
\mathbf{A}^2 &= ||OG||^2 + \mathbf{a}^2 - 2G \cdot a \\
&= \mathbf{G}^2 + \mathbf{a}^2 - 2G \cdot a
\end{aligned}$$

and respectively with B and C, we obtain

$$(\mathbf{ABC})^2 = (\mathbf{G}^2 + \mathbf{a} - 2G \cdot a)^2(\mathbf{G}^2 + \mathbf{b} - 2G \cdot b)^2(\mathbf{G}^2 + \mathbf{c} - 2G \cdot c)^2 \tag{6}$$

3 Final formula

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{3G \cdot (b \times c)}{\mathbf{ABC} + (A \cdot B)\,\mathbf{C} + (A \cdot C)\,\mathbf{B} + (B \cdot C)\,\mathbf{A}}$$

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{3G \cdot (b \times c)}{\mathbf{ABC} + (\mathbf{G}^2 - ||GB||^2 - G \cdot c - c \cdot b)\mathbf{C} + (\mathbf{G}^2 - ||GC||^2 - G \cdot c - c \cdot b)\mathbf{B} + (G + b) \cdot (G + c)\,\mathbf{A}}$$