ToFu geometric tools
Intersection of a cone with a plane

Didier VEZINET

15.10.2019

Contents

Chapter 1

Geometry

1.1 Generic cone and plane

Let's consider a half-cone C_1 (defined only for $z > 0$), with summit on the cartesian frame's origin $(O, \underline{\mathrm{e}}_{\mathrm{x}}, \underline{\mathrm{e}}_{\mathrm{y}}, \underline{\mathrm{e}}_{\mathrm{z}})$. The cone's axis is the $(O, \underline{\mathrm{e}}_{\mathrm{z}})$ axis. It's angular opening is θ.

Let's consider plane P_1, of normal $\underline{\mathrm{n}}$, intersection axis $(O, \underline{\mathrm{e}}_{\mathrm{z}})$ at point F of coordinates $(0, 0, Z_F)$. Vector $\underline{\mathrm{n}}$ is oriented by angles ϕ and ψ such that one can define the local frame $(F, \underline{\mathrm{e}}_1, \underline{\mathrm{e}}_2, \underline{\mathrm{n}})$:

$$\begin{cases} \underline{\mathrm{e}}_1 & = \cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + \sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}} \\ \underline{\mathrm{e}}_2 & = \big(-\sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + \cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}}\big)\cos(\psi) + \sin(\psi)\,\underline{\mathrm{e}}_{\mathrm{z}} \\ \underline{\mathrm{n}} & = \underline{\mathrm{e}}_1 \wedge \underline{\mathrm{e}}_2 \\ & = \big(\sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} - \cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}}\big)\sin(\psi) + \cos(\psi)\,\underline{\mathrm{e}}_{\mathrm{z}} \end{cases}$$

We want to find all points M of coordinates (x, y, z) and (x_1, x_2) belonging both to the cone C_1 and the plane P_1.

$$M \in F_1 \Leftrightarrow \underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}} = \cos(\theta)\|\underline{OM}\|$$

$$M \in P_1 \Leftrightarrow \underline{FM}.\,\underline{\mathrm{n}} = 0$$

1.2 Intersection

If M belongs to both P_1 and C_1, then:

$$(\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}})^2 = \cos(\theta)^2\|\underline{OM}\|^2$$

Given that:

$$\begin{aligned} \underline{OM} & = \underline{OF} + \underline{FM} \\ & = Z_F\,\underline{\mathrm{e}}_{\mathrm{z}} + x_1\,\underline{\mathrm{e}}_1 + x_2\,\underline{\mathrm{e}}_2 \\ & = Z_F\,\underline{\mathrm{e}}_{\mathrm{z}} + x_1\big(\cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + \sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}}\big) + x_2\Big(\big(-\sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + \cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}}\big)\cos(\psi) + \sin(\psi)\,\underline{\mathrm{e}}_{\mathrm{z}}\Big) \\ & = Z_F\,\underline{\mathrm{e}}_{\mathrm{z}} + x_1\cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + x_1\sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}} - x_2\sin(\phi)\cos(\psi)\,\underline{\mathrm{e}}_{\mathrm{x}} + x_2\cos(\phi)\cos(\psi)\,\underline{\mathrm{e}}_{\mathrm{y}} + x_2\sin(\psi)\,\underline{\mathrm{e}}_{\mathrm{z}} \\ & = (x_1\cos(\phi) - x_2\sin(\phi)\cos(\psi))\,\underline{\mathrm{e}}_{\mathrm{x}} + (x_1\sin(\phi) + x_2\cos(\phi)\cos(\psi))\,\underline{\mathrm{e}}_{\mathrm{y}} + (Z_F + x_2\sin(\psi))\,\underline{\mathrm{e}}_{\mathrm{z}} \end{aligned}$$

We have:

$$\begin{aligned} (\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}})^2 & = (Z_F + x_2\sin(\psi))^2 \\ & = Z_F^2 + 2Z_F x_2\sin(\psi) + x_2^2\sin(\psi)^2 \end{aligned}$$

And:

$$
\begin{aligned}
\|\underline{OM}\|^2 &= \|\left(x_1\cos(\phi) - x_2\sin(\phi)\cos(\psi)\right)\underline{\mathrm{e}}_{\mathrm{x}} + \left(x_1\sin(\phi) + x_2\cos(\phi)\cos(\psi)\right)\underline{\mathrm{e}}_{\mathrm{y}} + \left(Z_F + x_2\sin(\psi)\right)\underline{\mathrm{e}}_{\mathrm{z}}\|^2 \\
&= \left(x_1\cos(\phi) - x_2\sin(\phi)\cos(\psi)\right)^2 \\
&\quad + \left(x_1\sin(\phi) + x_2\cos(\phi)\cos(\psi)\right)^2 \\
&\quad + \left(Z_F + x_2\sin(\psi)\right)^2 \\
&= x_1^2\cos(\phi)^2 - 2x_1x_2\cos(\phi)\sin(\phi)\cos(\psi) + x_2^2\sin(\phi)^2\cos(\psi)^2 \\
&\quad + x_1^2\sin(\phi)^2 + 2x_1x_2\sin(\phi)\cos(\phi)\cos(\psi) + x_2^2\cos(\phi)^2\cos(\psi)^2 \\
&\quad + Z_F^2 + 2Z_Fx_2\sin(\psi) + x_2^2\sin(\psi)^2 \\
&= x_1^2 + x_2^2\cos(\psi)^2 \\
&\quad + Z_F^2 + 2Z_Fx_2\sin(\psi) + x_2^2\sin(\psi)^2 \\
&= x_1^2 + x_2^2 + 2Z_Fx_2\sin(\psi) + Z_F^2
\end{aligned}
$$

Thus:

$$
\begin{aligned}
& (\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}})^2 = \cos(\theta)^2\|\underline{OM}\|^2 \\
\Leftrightarrow\ & Z_F^2 + 2Z_Fx_2\sin(\psi) + x_2^2\sin(\psi)^2 = \cos(\theta)^2\left(x_1^2 + x_2^2 + 2Z_Fx_2\sin(\psi) + Z_F^2\right) \\
\Leftrightarrow\ & Z_F^2\left(1 - \cos(\theta)^2\right) + 2Z_Fx_2\sin(\psi)\left(1 - \cos(\theta)^2\right) = x_1^2\cos(\theta)^2 + x_2^2\left(\cos(\theta)^2 - \sin(\psi)^2\right) \\
\Leftrightarrow\ & Z_F^2\sin(\theta)^2 + 2Z_Fx_2\sin(\psi)\sin(\theta)^2 = x_1^2\cos(\theta)^2 + x_2^2\left(\cos(\theta)^2 - \sin(\psi)^2\right)
\end{aligned}
$$

Considering that by hypothesis $\theta > 0$:

$$
\begin{aligned}
& (\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}})^2 = \cos(\theta)^2\|\underline{OM}\|^2 \\
\Leftrightarrow\ & x_1^2\cos(\theta)^2 + x_2^2\left(\cos(\theta)^2\text{-}\sin(\psi)^2\right) - 2Z_Fx_2\sin(\psi)\sin(\theta)^2 - Z_F^2\sin(\theta)^2 = 0
\end{aligned}
$$

This looks like an almost reduced conic equation, except that x_2 needs to be rescaled, because F, the intersection between the axis $(O, \underline{\mathrm{e}}_{\mathrm{z}})$ and plane P_1 is in fact not the center of the conic but one of its focal points.
Computing the x_2 coordinate of the center is easy by testing the particular case $x_1 = 0$, finding 2 solutions (the two extrema of the conic), and taking the middle.

$$
\begin{aligned}
& x_1 = 0 \\
\Rightarrow\ & x_2^2\left(\cos(\theta)^2\text{-}\sin(\psi)^2\right) - 2Z_Fx_2\sin(\psi)\sin(\theta)^2 - Z_F^2\sin(\theta)^2 = 0
\end{aligned}
$$

Introducing

$$
\begin{aligned}
\Delta &= 4Z_F^2\sin(\psi)^2\sin(\theta)^4 + 4Z_F^2\sin(\theta)^2\left(\cos(\theta)^2\text{-}\sin(\psi)^2\right) \\
&= 4Z_F^2\sin(\theta)^2\left(\sin(\psi)^2\sin(\theta)^2 + \cos(\theta)^2\text{-}\sin(\psi)^2\right) \\
&= 4Z_F^2\sin(\theta)^2\left(-\sin(\psi)^2\cos(\theta)^2 + \cos(\theta)^2\right) \\
&= 4Z_F^2\sin(\theta)^2\cos(\theta)^2\cos(\psi)^2
\end{aligned}
$$

Hence, solutions always exist. The solutions are:

$$
\begin{aligned}
x_2^{1,2} &= \frac{2Z_F\sin(\psi)\sin(\theta)^2 \pm 2Z_F\sin(\theta)\cos(\theta)\cos(\psi)}{2(\cos(\theta)^2\text{-}\sin(\psi)^2)} \\
&= Z_F\sin(\theta)\frac{\sin(\psi)\sin(\theta) \pm \cos(\theta)\cos(\psi)}{\cos(\theta)^2\text{-}\sin(\psi)^2}
\end{aligned}
$$

Thus, the average is:

$$
x_2(C) = < x_2^{1,2} > \quad = \frac{1}{2}2Z_F\sin(\theta)\frac{\sin(\psi)\sin(\theta)}{\cos(\theta)^2\text{-}\sin(\psi)^2} \quad = Z_F\sin(\theta)^2\frac{\sin(\psi)}{\cos(\theta)^2\text{-}\sin(\psi)^2}
$$

Hence, let's introduce the new coordinate:

$$
X_2 = x_2 - x_2(C) = x_2 - Z_F\sin(\theta)^2\frac{\sin(\psi)}{\cos(\theta)^2\text{-}\sin(\psi)^2}
$$

Then:

$$\begin{aligned}
& X_2^2 && = x_2^2 - 2Z_F x_2 \sin(\theta)^2 \tfrac{\sin(\psi)}{\cos(\theta)^2\text{-}\sin(\psi)^2} + \sin(\theta)^4 \left[\tfrac{\sin(\psi)}{\cos(\theta)^2\text{-}\sin(\psi)^2}\right]^2 \\
\Leftrightarrow\ & X_2^2 \left(\cos(\theta)^2\text{-}\sin(\psi)^2\right) && = x_2^2 \left(\cos(\theta)^2\text{-}\sin(\psi)^2\right) - 2Z_F x_2 \sin(\theta)^2 \sin(\psi) + \sin(\theta)^4 \tfrac{\sin(\psi)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}
\end{aligned}$$

Hence:

$$\begin{aligned}
& (\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}})^2 = \cos(\theta)^2 \|\underline{OM}\|^2 \\
\Leftrightarrow\ & x_1^2 \cos(\theta)^2 + X_2^2 \left(\cos(\theta)^2\text{-}\sin(\psi)^2\right) - \tfrac{\sin(\psi)^2 \sin(\theta)^4}{\cos(\theta)^2\text{-}\sin(\psi)^2} - Z_F^2 \sin(\theta)^2 = 0 \\
\Leftrightarrow\ & x_1^2 \cos(\theta)^2 + X_2^2 \left(\cos(\theta)^2\text{-}\sin(\psi)^2\right) = \left[\tfrac{\sin(\psi)^2 \sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + Z_F^2\right] \sin(\theta)^2
\end{aligned}$$

Of the form $\frac{x_1^2}{a^2} + \frac{X_2^2}{b^2} = 1$:

$$\frac{x_1^2}{\left[\frac{\sin(\psi)^2 \sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + Z_F^2\right] \tan(\theta)^2} + \frac{X_2^2}{\left[\frac{\sin(\psi)^2 \sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + Z_F^2\right] \frac{\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}} = 1$$

With $b^2 = \left(1 - \frac{\sin(\psi)^2}{\cos(\theta)^2}\right) a^2$

1.3 Focal-centered polar coordinates

In our case, only the axes $(O, \underline{\mathrm{e}}_{\mathrm{z}})$, fixed by the crystal's summit and normal, is independent from the cone's angular opening θ. It makes sense to design an ad-hoc coordinate system centered not on the ellipse's center C, but on the focal point F. The ellipse will be parameterized by an angle ϵ around F, which is in fact the angle around the crystal's summit O at which the ray beam is coming / leaving. The other coordinate will thus be the radius r, in the plane, from F.

We have seen that the ellipse can be defined by:

$$\frac{x_1^2}{\left[\frac{\sin(\psi)^2 \sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + Z_F^2\right] \tan(\theta)^2} + \frac{X_2^2}{\left[\frac{\sin(\psi)^2 \sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + Z_F^2\right] \frac{\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}} = 1$$

By definition, in the local frame defined from the plane's orientation:

$$\underline{FM} = r\cos(\epsilon)\,\underline{\mathrm{e}}_1 + r\sin(\epsilon)\,\underline{\mathrm{e}}_2$$

Hence:

$$\begin{cases} r\cos(\epsilon) = x_1 \\ r\sin(\epsilon) = x_2 = X_2 + Z_F \sin(\theta)^2 \frac{\sin(\psi)}{\cos(\theta)^2\text{-}\sin(\psi)^2} \end{cases} \quad ll$$

Necesarrily:

$$r^2 = x_1^2 + \left(X_2 + Z_F \sin(\theta)^2 \frac{\sin(\psi)}{\cos(\theta)^2\text{-}\sin(\psi)^2}\right)^2$$

Appendix A

Acceleration radiation from a unique point-like charge

A.1 Retarded time and potential

A.1.1 Retarded time

Hence $\frac{dR(t_r)}{c} + dt_r = dt$