

ToFu geometric tools
Intersection of a cone with a plane

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15.10.2019

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Chapter 1

Geometry

1.1 Generic cone and plane

Let's consider a half-cone C_1 (defined only for $z > 0$), with summit on the cartesian frame's origin $(O, \underline{e}_x, \underline{e}_y, \underline{e}_z)$. The cone's axis is the (O, \underline{e}_z) axis. It's angular opening is θ .

Let's consider plane P_1 , of normal \underline{n} , intersection axis (O, \underline{e}_z) at point P of coordinates $(0, 0, Z_P)$. Vector \underline{n} is oriented by angles ϕ and ψ such that one can define the local frame $(P, \underline{e}_1, \underline{e}_2, \underline{n})$:

$$\begin{cases} \underline{e}_1 &= \cos(\phi) \underline{e}_x + \sin(\phi) \underline{e}_y \\ \underline{e}_2 &= (-\sin(\phi) \underline{e}_x + \cos(\phi) \underline{e}_y) \cos(\psi) + \sin(\psi) \underline{e}_z \\ \underline{n} &= \underline{e}_1 \wedge \underline{e}_2 \\ &= (\sin(\phi) \underline{e}_x - \cos(\phi) \underline{e}_y) \sin(\psi) + \cos(\psi) \underline{e}_z \end{cases}$$

We want to find all points M of coordinates (x, y, z) and (x_1, x_2) belonging both to the cone C_1 and the plane P_1 .

$$M \in C_1 \Leftrightarrow \underline{OM} \cdot \underline{e}_z = \cos(\theta) \|\underline{OM}\|$$

$$M \in P_1 \Leftrightarrow \underline{PM} \cdot \underline{n} = 0$$

1.2 Intersection

If M belongs to both P_1 and C_1 , then:

$$(\underline{OM} \cdot \underline{e}_z)^2 = \cos^2(\theta) \|\underline{OM}\|^2$$

Given that:

$$\begin{aligned} \underline{OM} &= \underline{OP} + \underline{PM} \\ &= Z_P \underline{e}_z + x_1 \underline{e}_1 + x_2 \underline{e}_2 \\ &= Z_P \underline{e}_z + x_1 (\cos(\phi) \underline{e}_x + \sin(\phi) \underline{e}_y) + x_2 ((-\sin(\phi) \underline{e}_x + \cos(\phi) \underline{e}_y) \cos(\psi) + \sin(\psi) \underline{e}_z) \\ &= Z_P \underline{e}_z + x_1 \cos(\phi) \underline{e}_x + x_1 \sin(\phi) \underline{e}_y - x_2 \sin(\phi) \cos(\psi) \underline{e}_x + x_2 \cos(\phi) \cos(\psi) \underline{e}_y + x_2 \sin(\psi) \underline{e}_z \\ &= (x_1 \cos(\phi) - x_2 \sin(\phi) \cos(\psi)) \underline{e}_x + (x_1 \sin(\phi) + x_2 \cos(\phi) \cos(\psi)) \underline{e}_y + (Z_P + x_2 \sin(\psi)) \underline{e}_z \end{aligned}$$

We have:

$$\begin{aligned} (\underline{OM} \cdot \underline{e}_z)^2 &= (Z_P + x_2 \sin(\psi))^2 \\ &= Z_P^2 + 2Z_P x_2 \sin(\psi) + x_2^2 \sin^2(\psi) \end{aligned}$$

And:

$$\begin{aligned}
\|\underline{OM}\|^2 &= \|(x_1 \cos(\phi) - x_2 \sin(\phi) \cos(\psi)) \underline{e}_x + (x_1 \sin(\phi) + x_2 \cos(\phi) \cos(\psi)) \underline{e}_y + (Z_P + x_2 \sin(\psi)) \underline{e}_z\|^2 \\
&= (x_1 \cos(\phi) - x_2 \sin(\phi) \cos(\psi))^2 \\
&\quad + (x_1 \sin(\phi) + x_2 \cos(\phi) \cos(\psi))^2 \\
&\quad + (Z_P + x_2 \sin(\psi))^2 \\
&= x_1^2 \cos^2(\phi) - 2x_1 x_2 \cos(\phi) \sin(\phi) \cos(\psi) + x_2^2 \sin^2(\phi) \cos^2(\psi) \\
&\quad + x_1^2 \sin^2(\phi) + 2x_1 x_2 \sin(\phi) \cos(\phi) \cos(\psi) + x_2^2 \cos^2(\phi) \cos^2(\psi) \\
&\quad + Z_P^2 + 2Z_P x_2 \sin(\psi) + x_2^2 \sin^2(\psi) \\
&= x_1^2 + x_2^2 \cos^2(\psi) \\
&\quad + Z_P^2 + 2Z_P x_2 \sin(\psi) + x_2^2 \sin^2(\psi) \\
&= x_1^2 + x_2^2 + 2Z_P x_2 \sin(\psi) + Z_P^2
\end{aligned}$$

Thus:

$$\begin{aligned}
(\underline{OM}, \underline{e}_z)^2 &= \cos(\theta)^2 \|\underline{OM}\|^2 \\
\Leftrightarrow Z_P^2 + 2Z_P x_2 \sin(\psi) + x_2^2 \sin^2(\psi) &= \cos(\theta)^2 (x_1^2 + x_2^2 + 2Z_P x_2 \sin(\psi) + Z_P^2) \\
\Leftrightarrow Z_P^2 (1 - \cos(\theta)^2) + 2Z_P x_2 \sin(\psi) (1 - \cos(\theta)^2) &= x_1^2 \cos(\theta)^2 + x_2^2 (\cos(\theta)^2 - \sin^2(\psi)) \\
\Leftrightarrow Z_P^2 \sin^2(\theta) + 2Z_P x_2 \sin(\psi) \sin(\theta)^2 &= x_1^2 \cos(\theta)^2 + x_2^2 (\cos(\theta)^2 - \sin^2(\psi))
\end{aligned}$$

Considering that by hypothesis $\theta > 0$:

$$\begin{aligned}
(\underline{OM}, \underline{e}_z)^2 &= \cos(\theta)^2 \|\underline{OM}\|^2 \\
\Leftrightarrow x_1^2 \cos(\theta)^2 + x_2^2 (\cos(\theta)^2 - \sin^2(\psi)) - 2Z_P x_2 \sin(\psi) \sin(\theta)^2 - Z_P^2 \sin(\theta)^2 &= 0 \\
\Leftrightarrow x_1^2 \frac{\cos(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} + x_2^2 - 2x_2 Z_P \frac{\sin(\psi) \sin(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} - Z_P^2 \frac{\sin(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} &= 0 \\
\Leftrightarrow x_1^2 \frac{\cos(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} + \left(x_2 - Z_P \frac{\sin(\psi) \sin(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2}\right)^2 - Z_P^2 \frac{\sin(\psi)^2 \sin(\theta)^4}{(\cos(\theta)^2 - \sin(\psi)^2)^2} - Z_P^2 \frac{\sin(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} &= 0 \\
\Leftrightarrow x_1^2 \frac{\cos(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} + \left(x_2 - Z_P \frac{\sin(\psi) \sin(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2}\right)^2 &= Z_P^2 \frac{\sin(\theta)^2}{(\cos(\theta)^2 - \sin(\psi)^2)^2} (\sin(\psi)^2 \sin(\theta)^2 + \cos(\theta)^2 - \sin(\psi)^2) \\
\Leftrightarrow x_1^2 \frac{\cos(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} + \left(x_2 - Z_P \frac{\sin(\psi) \sin(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2}\right)^2 &= Z_P^2 \frac{\sin(\theta)^2}{(\cos(\theta)^2 - \sin(\psi)^2)^2} (-\sin(\psi)^2 \cos(\theta)^2 + \cos(\theta)^2) \\
\Leftrightarrow x_1^2 \frac{\cos(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} + \left(x_2 - Z_P \frac{\sin(\psi) \sin(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2}\right)^2 &= Z_P^2 \frac{\sin(\theta)^2 \cos(\psi)^2 \cos(\theta)^2}{(\cos(\theta)^2 - \sin(\psi)^2)^2} \\
\Leftrightarrow \frac{x_1^2}{Z_P^2 \frac{\sin(\theta)^2 \cos(\psi)^2}{\cos(\theta)^2 - \sin(\psi)^2}} + \frac{\left(x_2 - Z_P \frac{\sin(\psi) \sin(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2}\right)^2}{Z_P^2 \frac{\sin(\theta)^2 \cos(\psi)^2 \cos(\theta)^2}{(\cos(\theta)^2 - \sin(\psi)^2)^2}} &= 1
\end{aligned}$$

Or, in a reduced conic form:

$$\frac{x_1^2}{a^2} + \frac{(x_2 - x_2(C))^2}{b^2} = 1$$

With:

$$\begin{cases}
x_2(C) &= Z_P \frac{\sin(\psi) \sin(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} && x_2 \text{ coordinate of the center} \\
a^2 &= Z_P^2 \frac{\sin(\theta)^2 \cos(\psi)^2}{\cos(\theta)^2 - \sin(\psi)^2} && \text{squared minor radius} \\
b^2 &= Z_P^2 \frac{\sin(\theta)^2 \cos(\psi)^2 \cos(\theta)^2}{(\cos(\theta)^2 - \sin(\psi)^2)^2} && \text{squared major radius} \\
b^2 &= a^2 \frac{\cos(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} \Leftrightarrow a^2 = b^2 \left(1 - \frac{\sin(\psi)^2}{\cos(\theta)^2}\right)
\end{cases}$$

The distance d_{CF} between the center C and the focal point F can be deduced from:

$$\begin{aligned}
d_{CF}^2 &= b^2 - a^2 \\
&= b^2 \frac{\sin(\psi)^2}{\cos(\theta)^2} \\
&= Z_P^2 \frac{\sin(\theta)^2 \cos(\psi)^2 \sin(\psi)^2}{(\cos(\theta)^2 - \sin(\psi)^2)^2}
\end{aligned}$$

Hence, the x_2 coordinate of F is:

$$\begin{aligned}
x_2(F) &= x_2(C) \pm d_{CF} \\
&= Z_P \frac{\sin(\psi) \sin(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} \pm Z_P \frac{\sin(\theta) \cos(\psi) \sin(\psi)}{\cos(\theta)^2 - \sin(\psi)^2} \\
&= Z_P \frac{\sin(\psi) \sin(\theta)^2 \pm \sin(\theta) \cos(\psi) \sin(\psi)}{\cos(\theta)^2 - \sin(\psi)^2} \\
&= Z_P \frac{\sin(\psi) \sin(\theta)}{\cos(\theta)^2 - \sin(\psi)^2} (\sin(\theta) \pm \cos(\psi))
\end{aligned}$$

It is worth noticing that the neither the focal point nor the center correspond to the intersection between the axes and the plane P .

1.3 Parametric equation

In our case, only the axes (O, \underline{e}_z) , fixed by the crystal's summit and normal, is independent from the cone's angular opening θ . It makes sense to design an ad-hoc coordinate system centered on the ellipse's center C to use its parameterized equation.

Knowing all geometrical parameters, it is possible to compute all points on the ellipse parameterizing them with angle ϵ :

$$\begin{cases} x_1 = a \cos(\epsilon) \\ x_2 = x_2(C) + b \sin(\epsilon) \end{cases}$$

1.3.1 From bragg angle and to local cartesian coordinates

Keep in mind that the frame $(P, \underline{e}_1, \underline{e}_2)$ is, by definition aligned on the minor and major axes of the ellipse. Hence, for an arbitrary frame $(R, \underline{e}_i, \underline{e}_j)$ on plane P_1 , translated and rotated by α with respect to $(P, \underline{e}_1, \underline{e}_2)$:

$$\begin{cases} \underline{e}_i = \cos(\alpha) \underline{e}_1 + \sin(\alpha) \underline{e}_2 \\ \underline{e}_j = -\sin(\alpha) \underline{e}_1 + \cos(\alpha) \underline{e}_2 \\ \underline{e}_1 = \cos(\alpha) \underline{e}_i - \sin(\alpha) \underline{e}_j \\ \underline{e}_2 = \sin(\alpha) \underline{e}_i + \cos(\alpha) \underline{e}_j \end{cases}$$

Or, in coordinate tranforms:

$$\begin{cases} x_1 = x_1(R) + x_i \cos(\alpha) - x_j \sin(\alpha) \\ x_2 = x_2(R) + x_i \sin(\alpha) + x_j \cos(\alpha) \\ x_i = (x_1 - x_1(R)) \cos(\alpha) + (x_2 - x_2(R)) \sin(\alpha) \\ x_j = -(x_1 - x_1(R)) \sin(\alpha) + (x_2 - x_2(R)) \cos(\alpha) \end{cases}$$

Hence:

$$\begin{cases} x_i = (a \cos(\epsilon) - x_1(R)) \cos(\alpha) + (x_2(C) - x_2(R) + b \sin(\epsilon)) \sin(\alpha) \\ x_j = -(a \cos(\epsilon) - x_1(R)) \sin(\alpha) + (x_2(C) - x_2(R) + b \sin(\epsilon)) \cos(\alpha) \end{cases}$$

1.3.2 From local cartesian coordinates to bragg angle

Knowing (x_i, x_j) and all geometric parameters, we now want to derive (θ, ϵ) .

From the previous equation, we can write:

$$\begin{cases} x_i \cos(\alpha) - x_j \sin(\alpha) = a \cos(\epsilon) - x_1(R) & (1) \\ x_i \sin(\alpha) + x_j \cos(\alpha) = x_2(C) - x_2(R) + b \sin(\epsilon) & (2) \end{cases}$$

The dependency in θ is hidden in the expressions of a , b and $x_2(C)$.

By squaring and summing, it is possible to get rid of the ϵ dependency:

$$\begin{cases} a^2 \cos(\epsilon)^2 = (x_i \cos(\alpha) - x_j \sin(\alpha) + x_1(R))^2 \\ b^2 \sin(\epsilon)^2 = (x_i \sin(\alpha) + x_j \cos(\alpha) - x_2(C) + x_2(R))^2 \end{cases}$$

Hence, keeping in mind that $a^2 = b^2 \frac{\cos(\theta)^2 - \sin(\psi)^2}{\cos(\theta)^2}$ and re-using the definitions of x_1 and x_2 which do not depend on the unknowns (θ, ϵ) :

$$\begin{aligned} b^2 &= \frac{\cos(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} x_1^2 + (x_2 - x_2(C))^2 \\ \Leftrightarrow b^2 (\cos(\theta)^2 - \sin(\psi)^2) &= \cos(\theta)^2 x_1^2 + (\cos(\theta)^2 - \sin(\psi)^2) (x_2^2 - 2x_2 x_2(C) + x_2(C)^2) \\ \Leftrightarrow Z_P^2 \frac{\sin(\theta)^2 \cos(\psi)^2 \cos(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} &= \cos(\theta)^2 x_1^2 + (\cos(\theta)^2 - \sin(\psi)^2) (x_2^2 - 2x_2 x_2(C) + x_2(C)^2) \\ \Leftrightarrow Z_P^2 \frac{\sin(\theta)^2 \cos(\psi)^2 \cos(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} &= \cos(\theta)^2 (x_1^2 + x_2^2) - \sin(\psi)^2 x_2^2 - (\cos(\theta)^2 - \sin(\psi)^2) (2x_2 x_2(C) - x_2(C)^2) \\ \Leftrightarrow Z_P^2 \sin(\theta)^2 \cos(\psi)^2 \cos(\theta)^2 &= \cos(\theta)^2 (\cos(\theta)^2 - \sin(\psi)^2) (x_1^2 + x_2^2) - \sin(\psi)^2 (\cos(\theta)^2 - \sin(\psi)^2) x_2^2 \\ &\quad - (\cos(\theta)^2 - \sin(\psi)^2)^2 (2x_2 x_2(C) - x_2(C)^2) \\ \Leftrightarrow Z_P^2 \sin(\theta)^2 \cos(\psi)^2 \cos(\theta)^2 &= \cos(\theta)^4 (x_1^2 + x_2^2) - \cos(\theta)^2 \sin(\psi)^2 (x_1^2 + 2x_2^2) + \sin(\psi)^4 x_2^2 \\ &\quad - (\cos(\theta)^2 - \sin(\psi)^2)^2 (2x_2 x_2(C) - x_2(C)^2) \\ \Leftrightarrow Z_P^2 \sin(\theta)^2 \cos(\psi)^2 \cos(\theta)^2 &= \cos(\theta)^4 (x_1^2 + x_2^2) - \cos(\theta)^2 \sin(\psi)^2 (x_1^2 + 2x_2^2) + \sin(\psi)^4 x_2^2 \\ &\quad - [2x_2 Z_P \sin(\psi) \sin(\theta)^2 (\cos(\theta)^2 - \sin(\psi)^2) - Z_P^2 \sin(\psi)^2 \sin(\theta)^4] \\ \Leftrightarrow Z_P^2 \cos(\psi)^2 (\cos(\theta)^2 - \cos(\theta)^4) &= \cos(\theta)^4 (x_1^2 + x_2^2) - \cos(\theta)^2 \sin(\psi)^2 (x_1^2 + 2x_2^2) + \sin(\psi)^4 x_2^2 \\ &\quad - 2x_2 Z_P \sin(\psi) \sin(\theta)^2 (\cos(\theta)^2 - \sin(\psi)^2) \\ &\quad + Z_P^2 \sin(\psi)^2 (1 - 2\cos(\theta)^2 + \cos(\theta)^4) \\ \Leftrightarrow Z_P^2 \cos(\psi)^2 (\cos(\theta)^2 - \cos(\theta)^4) &= \cos(\theta)^4 (x_1^2 + x_2^2) - \cos(\theta)^2 \sin(\psi)^2 (x_1^2 + 2x_2^2) + \sin(\psi)^4 x_2^2 \\ &\quad - 2x_2 Z_P \sin(\psi) (\cos(\theta)^2 - \cos(\theta)^4 - \sin(\psi)^2 (1 - \cos(\theta)^2)) \\ &\quad + Z_P^2 \sin(\psi)^2 (1 - 2\cos(\theta)^2 + \cos(\theta)^4) \end{aligned}$$

Then introducing $X = \cos(\theta)^2$:

$$\begin{aligned} b^2 &= \frac{\cos(\theta)^2}{\cos(\theta)^2 - \sin(\psi)^2} x_1^2 + (x_2 - x_2(C))^2 \\ \Leftrightarrow Z_P^2 \cos(\psi)^2 (X - X^2) &= X^2 (x_1^2 + x_2^2) - X \sin(\psi)^2 (x_1^2 + 2x_2^2) + \sin(\psi)^4 x_2^2 \\ &\quad - 2x_2 Z_P \sin(\psi) (X - X^2 - \sin(\psi)^2 + X \sin(\psi)^2) \\ &\quad + Z_P^2 \sin(\psi)^2 (1 - 2X + X^2) \end{aligned}$$

Which boils down to:

$$\begin{aligned}
0 &= X^2 [x_1^2 + x_2^2 + 2x_2 Z_P \sin(\psi) + Z_P^2 \sin(\psi)^2 + Z_P^2 \cos(\psi)^2] \\
&\quad + X [-\sin(\psi)^2 (x_1^2 + 2x_2^2) - (1 + \sin(\psi)^2) 2x_2 Z_P \sin(\psi) - 2Z_P^2 \sin(\psi)^2 - Z_P^2 \cos(\psi)^2] \\
&\quad + \sin(\psi)^4 x_2^2 + 2x_2 Z_P \sin(\psi)^3 + Z_P^2 \sin(\psi)^2 \\
\Leftrightarrow 0 &= X^2 [x_1^2 + (x_2 + Z_P \sin(\psi))^2 + Z_P^2 \cos(\psi)^2] \\
&\quad + X [-\sin(\psi)^2 (x_1^2 + 2x_2^2) - (1 + \sin(\psi)^2) 2x_2 Z_P \sin(\psi) - 2Z_P^2 \sin(\psi)^2 - Z_P^2 \cos(\psi)^2] \\
&\quad + \sin(\psi)^2 (\sin(\psi)^2 x_2^2 + 2x_2 Z_P \sin(\psi) + Z_P^2) \\
\Leftrightarrow 0 &= X^2 [x_1^2 + (x_2 + Z_P \sin(\psi))^2 + Z_P^2 \cos(\psi)^2] \\
&\quad - X [\sin(\psi)^2 (x_1^2 + 2x_2^2) + (1 + \sin(\psi)^2) 2x_2 Z_P \sin(\psi) + 2Z_P^2 \sin(\psi)^2 + Z_P^2 \cos(\psi)^2] \\
&\quad + \sin(\psi)^2 (x_2 \sin(\psi) + Z_P) \\
\Leftrightarrow 0 &= X^2 [x_1^2 + (x_2 + Z_P \sin(\psi))^2 + Z_P^2 \cos(\psi)^2] \\
&\quad - X [\sin(\psi)^2 (x_1^2 + x_2^2) + x_2^2 \sin(\psi)^2 + 2x_2 Z_P \sin(\psi) + 2x_2 Z_P \sin(\psi)^3 + Z_P^2 + Z_P^2 \sin(\psi)^2] \\
&\quad + \sin(\psi)^2 (x_2 \sin(\psi) + Z_P) \\
\Leftrightarrow 0 &= X^2 [x_1^2 + (x_2 + Z_P \sin(\psi))^2 + Z_P^2 - Z_P^2 \sin(\psi)^2] \\
&\quad - X [\sin(\psi)^2 (x_1^2 + x_2^2) + (x_2 \sin(\psi) + Z_P)^2 + Z_P \sin(\psi)^2 (2x_2 \sin(\psi) + Z_P)] \\
&\quad + \sin(\psi)^2 (x_2 \sin(\psi) + Z_P) \\
\Leftrightarrow 0 &= X^2 [x_1^2 + x_2^2 + (x_2 \sin(\psi) + Z_P)^2 - x_2^2 \sin(\psi)^2] \\
&\quad - X [\sin(\psi)^2 (x_1^2 + x_2^2) + (x_2 \sin(\psi) + Z_P)^2 + Z_P \sin(\psi)^2 (2x_2 \sin(\psi) + Z_P)] \\
&\quad + \sin(\psi)^2 (x_2 \sin(\psi) + Z_P) \\
\Leftrightarrow 0 &= AX^2 - BX + C
\end{aligned}$$

With:

$$\begin{cases}
A = x_1^2 + x_2^2 + (x_2 \sin(\psi) + Z_P)^2 - x_2^2 \sin(\psi)^2 \\
B = \sin(\psi)^2 (x_1^2 + x_2^2) + (x_2 \sin(\psi) + Z_P)^2 + Z_P \sin(\psi)^2 (2x_2 \sin(\psi) + Z_P) \\
C = \sin(\psi)^2 (x_2 \sin(\psi) + Z_P)
\end{cases}$$

Solutions exist if:

$$\begin{aligned}
\Delta &\geq 0 \\
\Leftrightarrow B^2 - 4AC &\geq 0
\end{aligned}$$

In which case, only solutions in $[0, 1]$ are acceptable:

$$\cos(\theta)^2 = \frac{B \pm \sqrt{(\Delta)}}{2A} \in [0, 1]$$

And by definition, $\theta \in [0, \frac{\pi}{2}]$, hence $\cos(\theta) \geq 0$ and:

$$\theta = \arccos \left(\sqrt{\frac{B \pm \sqrt{(\Delta)}}{2A}} \right)$$

Appendix A

Appendices

A.1 Section

A.1.1 Subsection