# Solid angle subtended by a tetrahedron computation

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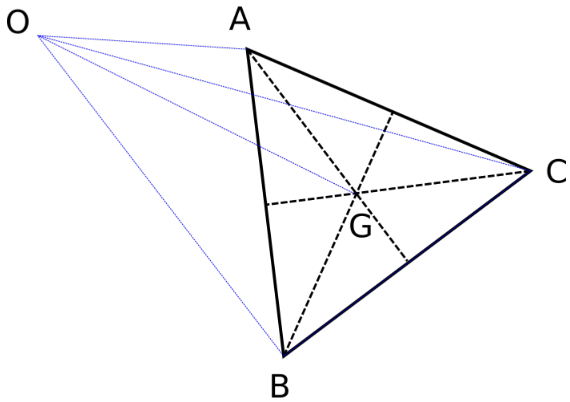


## 1 Notations

Let

- $\vec{a}$ be the vector $\vec{OA}$
- $\vec{b}$ be the vector $\vec{OB}$
- $\vec{c}$ be the vector $\vec{OC}$
- $a$ be the magnitude of the vector $\vec{OA}$
- $b$ be the magnitude of the vector $\vec{OB}$
- $c$ be the magnitude of the vector $\vec{OC}$

## 2 Computation

The solid angle $\Omega$ subtended by the triangular surface $ABC$ is:

$$\tan\left(\frac{1}{2}\Omega\right) = \frac{\left|\vec{a}\ \vec{b}\ \vec{c}\right|}{abc + \left(\vec{a}\cdot\vec{b}\right)c + \left(\vec{a}\cdot\vec{c}\right)b + \left(\vec{b}\cdot\vec{c}\right)a}$$

where $\left|\vec{a}\ \vec{b}\ \vec{c}\right| = \vec{a}\cdot(\vec{b}\times\vec{c})$

## 2.1 Numerator

Given that $\vec{a} = \vec{OA} = \vec{OG} + \vec{GA}$ (and resp. with $B$ and $C$), we get

$$\begin{aligned}
\left|\vec{a}\ \vec{b}\ \vec{c}\right| &= \vec{a} \cdot (\vec{b} \times \vec{c}) \\
&= \vec{a} \cdot ((\vec{OG} + \vec{GB}) \times (\vec{OG} + \vec{GC})) \\
&= (\vec{OG} + \vec{GA}) \cdot (\cancel{\vec{OG} \times \vec{OG}} + \vec{OG} \times \vec{GC} + \vec{GB} \times \vec{OG} + \vec{GB} \times \vec{GC}) \\
&= \cancel{\vec{OG} \cdot (\vec{OG} \times \vec{GC})} + \cancel{\vec{OG} \cdot (\vec{GB} \times \vec{OG})} \\
&+ \vec{OG} \cdot (\vec{GB} \times \vec{GC}) + \vec{GA} \cdot (\vec{OG} \times \vec{GC}) + \vec{GA} \cdot (\vec{GB} \times \vec{OG}) + \cancel{\vec{GA} \cdot (\vec{GB} \times \vec{GC})} \\
&= \vec{OG} \cdot (\vec{GB} \times \vec{GC}) + \vec{OG} \cdot (\vec{GC} \times \vec{GA}) + \vec{OG} \cdot (\vec{GA} \times \vec{GB}) \\
&= \vec{OG} \cdot (\vec{GB} \times \vec{GC}) + \vec{OG} \cdot (\vec{GC} \times \vec{GA}) + \vec{OG} \cdot (\vec{GA} \times \vec{GB}) \\
&= \vec{OG} \cdot \left(\vec{GB} \times \vec{GC} + \vec{GC} \times \vec{GA} + \vec{GA} \times \vec{GB}\right)
\end{aligned}$$

since $G$ is the centroid of $ABC$,

$$\vec{GA} + \vec{GB} + \vec{GC} = 0 \tag{1}$$

. We obtain

$$\begin{aligned}
\left|\vec{a}\ \vec{b}\ \vec{c}\right| &= \vec{OG} \cdot \left(\vec{GB} \times \vec{GC} + \vec{GC} \times \vec{GA} + \vec{GA} \times \vec{GB}\right) \\
&= \vec{OG} \cdot \left(\vec{GB} \times \vec{GC} + \vec{GC} \times (-\vec{GB} - \vec{GC}) + (-\vec{GB} - \vec{GC}) \times \vec{GB}\right) \\
&= \vec{OG} \cdot \left(\vec{GB} \times \vec{GC} + \vec{GC} \times (-\vec{GB}) + \cancel{\vec{GC} \times (-\vec{GC})} + \cancel{(-\vec{GB}) \times \vec{GB}} + (-\vec{GC}) \times \vec{GB}\right) \\
&= \vec{OG} \cdot \left(3\vec{GB} \times \vec{GC}\right) \\
&= 3\vec{OG} \cdot \left(\vec{GB} \times \vec{GC}\right) \qquad (2)
\end{aligned}$$

## 2.2 Denominator

First let's see the term in $c$

$$\begin{aligned}
\left(\vec{a} \cdot \vec{b}\right) c = &\left(\vec{OG} + \vec{GA}\right) \cdot \left(\vec{OG} + \vec{GB}\right) c \\
= &(g^2 + \vec{OG} \cdot \vec{GB} + \vec{GA} \cdot \vec{OG} + \vec{GA} \cdot \vec{GB})c
\end{aligned}$$

Using (1)

$$\begin{aligned}
\left(\vec{a} \cdot \vec{b}\right) c = &(g^2 + \vec{OG} \cdot \vec{GB} + (-\vec{GB} - \vec{GC}) \cdot \vec{OG} + (-\vec{GB} - \vec{GC}) \cdot \vec{GB})c \\
= &(g^2 - ||GB||^2 - \vec{OG} \cdot \vec{GC} - \vec{GC} \cdot \vec{GB})c \qquad (3)
\end{aligned}$$

Now, the term in $b$

$$\begin{aligned}
(\vec{a} \cdot \vec{c})\, c = &\left(\vec{OG} + \vec{GA}\right) \cdot \left(\vec{OG} + \vec{GC}\right) b \\
= &(g^2 + \vec{OG} \cdot \vec{GC} + \vec{GA} \cdot \vec{OG} + \vec{GA} \cdot \vec{GC})b
\end{aligned}$$

Using (1)

$$
\begin{aligned}
(\vec{a}\cdot\vec{c})\, b =& (g^2 + \vec{OG}\cdot\vec{GC} + (-\vec{GB}-\vec{GC})\cdot\vec{OG} + (-\vec{GB}-\vec{GC})\cdot\vec{GC})b \\
=& (g^2 - ||GC||^2 - \vec{OG}\cdot\vec{GC} - \vec{GC}\cdot\vec{GB})b \qquad (4)
\end{aligned}
$$

For the third term there is no simplification

$$
\left(\vec{b}\cdot\vec{c}\right) a = \left(\vec{OG}+\vec{GB}\right)\cdot\left(\vec{OG}+\vec{GC}\right) a \qquad (5)
$$

For the computation of the norms, we can use:

$$
\begin{aligned}
a^2 &= ||OG||^2 + ||GA||^2 \\
&= g^2 + ||GA||^2
\end{aligned}
$$

thus

$$
\begin{aligned}
g^2 &= a^2 - ||GA||^2 \\
&= b^2 - ||GB||^2 \\
&= c^2 - ||GC||^2 \qquad (6)
\end{aligned}
$$

and

$$
(abc)^2 = (g^2 + ||GA||)^2 (g^2 + ||GB||)^2 (g^2 + ||GC||)^2 \qquad (7)
$$