ToFu tools
Magnetic fields

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Contents

Chapter 1

3D field from straight current segments

1.1 Single segment

Let's consider current segment with current I, centered on A, with unit vector $\underline{u}$ and half-length $\mathrm{L}_{1/2}$. Considering a point P in this segment, identified by its length to A:

$$\underline{OP} = \underline{OA} + l\underline{u}, \text{ with } l \in \left[-\mathrm{L}_{1/2}; \mathrm{L}_{1/2}\right]$$

Any point M in space can be located by its position with respect to $\underline{A}$

$$\begin{cases} \underline{OM} = \underline{OA} + l_M\underline{u} + r_M\underline{v} \\ \underline{v} = \underline{AM} - (\underline{AM} \cdot \underline{u})\underline{u} \end{cases}$$

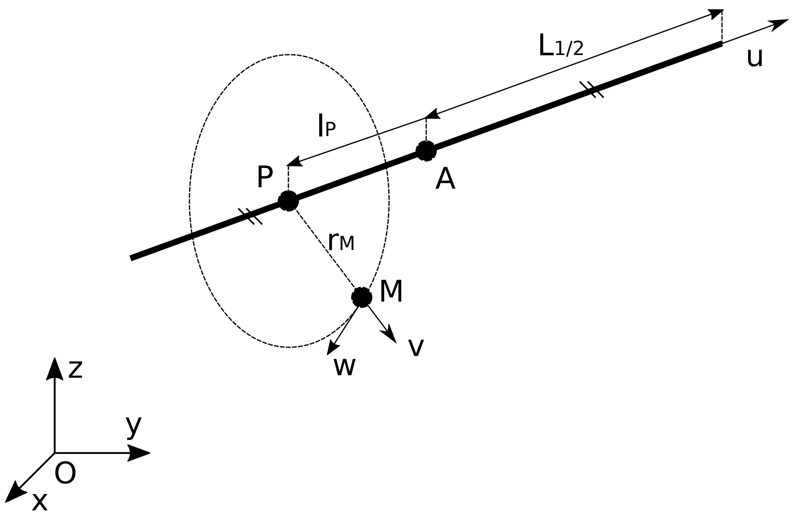

Figure 1.1: MAgnetic field at any point M from a straight current segment

The Biot-Savart law stipulates that an elementary length of current creates an elementary magnetic field at M:

$$\begin{cases} \underline{dI} = Idl\underline{u} \\ \underline{PM} = (l_M - l)\underline{u} + r_M\underline{v} \\ \underline{B_A} = \frac{\mu_0}{4\pi}\int_{-\mathrm{L}_{1/2}}^{\mathrm{L}_{1/2}} \frac{\underline{dI}\wedge\underline{PM}}{\|\underline{PM}\|^3} \end{cases}$$

Introducing $\underline{w} = \underline{u} \wedge \underline{v}$:

$$\underline{dI} \wedge \underline{PM} = Idlr_M\underline{w}$$

and:

$$\|\underline{PM}\| = \sqrt{(l - l_M)^2 + r_M^2}$$

Hence:

$$\underline{B_A} = \frac{\mu_0}{4\pi}Ir_M\underline{w}\int_{-\mathrm{L}_{1/2}}^{\mathrm{L}_{1/2}} \frac{dl}{\left((l - l_M)^2 + r_M^2\right)^{3/2}}$$

Introducing $x = l - l_M \Rightarrow dx = dl$:

$$\underline{B_A} \quad = \frac{\mu_0}{4\pi}Ir_M\underline{w}\int_{-\mathrm{L}_{1/2}-l_M}^{\mathrm{L}_{1/2}-l_M} \frac{dx}{\left(x^2+r_M^2\right)^{3/2}}$$

Noticing that:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+a}}\right) &= \frac{\sqrt{x^2+a}-x\frac{1}{2}\frac{2x}{\sqrt{x^2+a}}}{x^2+a} \\ &= \frac{(x^2+a)-x^2}{(x^2+a)^{3/2}} \\ &= \frac{a}{(x^2+a)^{3/2}} \end{aligned}$$

Hence:

$$\begin{aligned} \underline{B_A} &= \frac{\mu_0}{4\pi}Ir_M\underline{w}\left[\frac{1}{r_M^2}\frac{x}{\sqrt{x^2+r_M^2}}\right]_{-\mathrm{L}_{1/2}-l_M}^{\mathrm{L}_{1/2}-l_M} \\ &= \frac{\mu_0}{4\pi}\frac{I}{r_M}\underline{w}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}} + \frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right) \end{aligned}$$

For numerical evaluation, keep in mind that:

$$\begin{cases} l_M = \underline{u}\cdot\underline{AM} \\ r_M = \|\underline{u}\wedge\underline{AM}\| \\ \underline{w} = \frac{\underline{u}\wedge\underline{AM}}{r_M} \end{cases}$$

1.2 2 mirrored segments

Let's consider 2 current segments $(A, \underline{u})$ and $(A', \underline{u'})$ one being the symmetric of the other via a symmetry plane $(M, \underline{n})$.
Each current segment has its own current I (resp. I').

By construction:

$$\begin{cases} l'_M &= l_M \\ r'_M &= r_M \\ L'_{1/2} &= L_{1/2} \\ d_A &= -\underline{AM}\cdot\underline{n} \\ \underline{A'A} &= 2d_A\underline{n} \\ \underline{u'} &= \underline{u} - 2(\underline{u}\cdot\underline{n})\underline{n} \\ \underline{AM} &= l_M\underline{u} + r_M\underline{v} \\ \underline{A'M} &= l_M\underline{u'} + r_M\underline{v'} \end{cases}$$

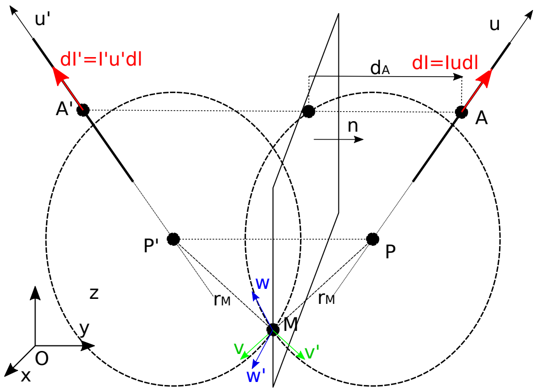

Figure 1.2: Magnetic field at any point M from 2 mirrored straight current segments

The total magnetic field created in M is:

$$
\begin{aligned}
\underline{B} &= \underline{B_A}(I) + \underline{B_{A'}}(I') \\
&= \frac{\mu_0}{4\pi}\frac{I}{r_M}\underline{w}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}}+\frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right)+\frac{\mu_0}{4\pi}\frac{I'}{r_M}\underline{w'}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}}+\frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right) \\
&= \frac{\mu_0}{4\pi}\frac{1}{r_M}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}}+\frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right)(I\underline{w}+I'\underline{w'})
\end{aligned}
$$

Now, considering that:

$$
\begin{aligned}
r_M\underline{v'} &= \underline{A'M} - l_M\underline{u'} \\
&= \underline{A'A} + \underline{AM} - l_M(\underline{u} - 2(\underline{u}\cdot\underline{n})\underline{n}) \\
&= 2d_A\underline{n} + \underline{AM} - l_M\underline{u} + 2l_M(\underline{u}\cdot\underline{n})\underline{n} \\
&= 2d_A\underline{n} + r_M\underline{v} + 2l_M(\underline{u}\cdot\underline{n})\underline{n} \\
\underbrace{\Leftrightarrow}_{r_M\neq 0} \quad \underline{v'} &= \underline{v} + \frac{2}{r_M}\left(l_M(\underline{u}\cdot\underline{n}) + d_A\right)\underline{n}
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\underline{w'} &= \underline{u'}\wedge\underline{v'} \\
&= (\underline{u} - 2(\underline{u}\cdot\underline{n})\underline{n})\wedge\left(\underline{v} + \frac{2}{r_M}\left(l_M(\underline{u}\cdot\underline{n}) + d_A\right)\underline{n}\right) \\
&= \underline{u}\wedge\underline{v} + \frac{2}{r_M}\left(l_M(\underline{u}\cdot\underline{n}) + d_A\right)\underline{u}\wedge\underline{n} - 2(\underline{u}\cdot\underline{n})\underline{n}\wedge\underline{v} \\
&= \underline{w} + 2\left[\frac{(\underline{u}\cdot\underline{n})l_M + d_A}{r_M}\underline{u} + (\underline{u}\cdot\underline{n})\underline{v}\right]\wedge\underline{n} \\
&= \underline{w} + 2\left[\frac{(\underline{u}\cdot\underline{n})(l_M\underline{u} + r_M\underline{v}) + d_A\underline{u}}{r_M}\right]\wedge\underline{n} \\
&= \underline{w} + 2\left[\frac{(\underline{u}\cdot\underline{n})\underline{AM} + d_A\underline{u}}{r_M}\right]\wedge\underline{n}
\end{aligned}
$$

Remebering that $d_A = -\underline{AM}\cdot\underline{n}$ and that $\underline{a}\wedge(\underline{b}\wedge c) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$:

$$
\begin{aligned}
\underline{w'} &= \underline{w} + 2\left[\frac{(\underline{u}\cdot\underline{n})\underline{AM} - (\underline{AM}\cdot\underline{n})\underline{u}}{r_M}\right]\wedge\underline{n} \\
&= \underline{w} + \frac{2}{r_m}\left[\underline{n}\wedge(\underline{AM}\wedge\underline{u})\right]\wedge\underline{n} \\
&= \underline{w} + 2\frac{\sqrt{r_m^2+l_M^2}}{r_m}\left[\underline{n}\wedge(\underline{e}_{AM}\wedge\underline{u})\right]\wedge\underline{n}
\end{aligned}
$$

Where we have introduced $\underline{AM} = \sqrt{r_M^2 + l_M^2}\underline{e}_{AM}$:

Then, assuming $I' = -I$ we have:

$$
\begin{aligned}
\underline{B} &= \frac{\mu_0}{4\pi}\frac{I}{r_M}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}}+\frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right)(\underline{w}-\underline{w'})\\
&= -\frac{\mu_0}{4\pi}\frac{I}{r_M}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}}+\frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right)2\frac{\sqrt{r_m^2+l_M^2}}{r_m}\left[\underline{n}\wedge(\underline{e}_{AM}\wedge\underline{u})\right]\wedge\underline{n}\\
&= \frac{\mu_0}{2\pi}I\frac{\sqrt{r_m^2+l_M^2}}{r_m^2}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}}+\frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right)\underline{n}\wedge\left[\underline{n}\wedge(\underline{e}_{AM}\wedge\underline{u})\right]
\end{aligned}
$$

For numerical evaluation, keep in mind that:

$$
\begin{cases}
l_M = \underline{u}\cdot\underline{AM}\\
r_M = \|\underline{u}\wedge\underline{AM}\|\\
d_A = -\underline{AM}\cdot\underline{n}
\end{cases}
$$

1.3 4 mirrored segments

Let's consider the 2 previous mirrored current segments and add a pair mirroring them via another plane $(C, \underline{m})$, perpendicular to the first plane $(C, \underline{n}) = (M, \underline{n})$. All segments are lying in the same plane $(C, \underline{a})$
The same current is running through each segment and they all have the same half-length $L_{1/2}$.

The same derivation as previously can be done for pair AA' and pair BB'.

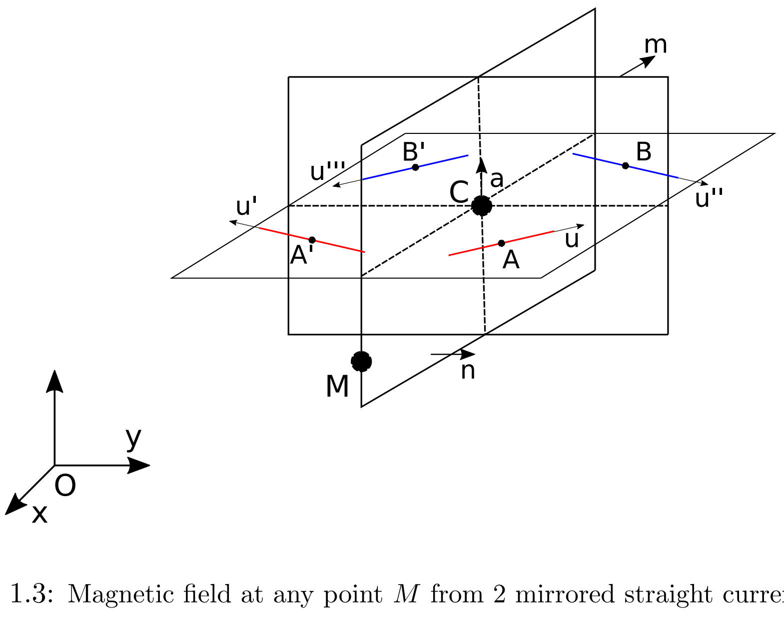

Figure 1.3: Magnetic field at any point M from 2 mirrored straight current segments

The total magnetic field created in M is:

$$
\begin{aligned}
\underline{B} &= \underline{B_{AA'}}(I)+\underline{B_{BB'}}(I)\\
&= \frac{\mu_0}{2\pi}\frac{I}{r_M}\left(\frac{\mathrm{L}_{1/2}-l_M}{\sqrt{(\mathrm{L}_{1/2}-l_M)^2+r_M^2}}+\frac{\mathrm{L}_{1/2}+l_M}{\sqrt{(\mathrm{L}_{1/2}+l_M)^2+r_M^2}}\right)\left(\underline{n}\wedge\left[\frac{(\underline{u}_A\cdot\underline{n})\underline{AM}+d_A\underline{u}_A}{r_M}\right]\right)\\
&+\frac{\mu_0}{2\pi}\frac{I}{r''_M}\left(\frac{\mathrm{L}_{1/2}-l''_M}{\sqrt{(\mathrm{L}_{1/2}-l''_M)^2+r''^2_M}}+\frac{\mathrm{L}_{1/2}+l''_M}{\sqrt{(\mathrm{L}_{1/2}+l''_M)^2+r''^2_M}}\right)\left(\underline{n}\wedge\left[\frac{(\underline{u}_B\cdot\underline{n})\underline{BM}+d_B\underline{u}_B}{r''_M}\right]\right)
\end{aligned}
$$

By construction, we have: .... cannot get (r''_M, l''_M) from (r_M, l_M).

Chapter 2

Circular coil

The magnetic field produced at any point in 3D space by a planar circular coil cannot be derived analytically.

2.1 Circle discretization

Let's consider a circular coil of radius R centered on axis $(C, \underline{e}_3)$.

For a given point M in space, let's make the plane $(C, M, \underline{e}_3)$ a symmetry plane and divide the circle into a N-sided polygon.

2.1.1 N-sided polygon

The polygon is N-sided, with N an even number. The polygon shall have either:

- the same perimeter as the circle: $L = 2\pi R$
- the same area as the circle: $S = \pi R^2$
- the same magnetic field on the center as the circle: $B(0) = \frac{\mu_0 I}{2R}$

For a N-sided regular polygon of height h:

$$\begin{cases} L_i = 2h \tan\left(\frac{\pi}{N}\right) & \text{is the length of a single side} \\ S_i = \frac{hL_i}{2} = h^2 \tan\left(\frac{\pi}{N}\right) & \text{is the area of a single side} \end{cases}$$

Remembering that, at the center, $l_m = 0$ and $r_m = h$ for all, by construction, and that $\mathrm{L}_{1/2} = \frac{L_i}{2} = h \tan\left(\frac{\pi}{N}\right)$, we derive:

$$\begin{aligned} B(0, N) &= N \frac{\mu_0}{4\pi} \frac{I}{h} \left(\frac{\mathrm{L}_{1/2}}{\sqrt{(\mathrm{L}_{1/2})^2 + h^2}} + \frac{\mathrm{L}_{1/2}}{\sqrt{(\mathrm{L}_{1/2})^2 + h^2}} \right) \\ &= N \frac{\mu_0}{2\pi} \frac{I}{h} \frac{\mathrm{L}_{1/2}}{\sqrt{(\mathrm{L}_{1/2})^2 + h^2}} \\ &= N \frac{\mu_0}{2\pi} \frac{I}{h} \frac{h \tan\left(\frac{\pi}{N}\right)}{\sqrt{h^2 \tan^2\left(\frac{\pi}{N}\right) + h^2}} \\ &= N \frac{\mu_0}{2\pi} \frac{I}{h} \frac{\tan\left(\frac{\pi}{N}\right)}{\sqrt{\tan^2\left(\frac{\pi}{N}\right) + 1}} \\ &= \frac{\mu_0 I}{2} \frac{1}{h} \frac{N}{\pi} \frac{\tan\left(\frac{\pi}{N}\right)}{\sqrt{\tan^2\left(\frac{\pi}{N}\right) + 1}} \end{aligned}$$

Hence,

- same perimeter $\Rightarrow h = R\frac{\frac{\pi}{N}}{\tan(\frac{\pi}{N})}$
- same area $\Rightarrow h = R\sqrt{\frac{\frac{\pi}{N}}{\tan(\frac{\pi}{N})}}$
- same field on axis $\Rightarrow h = R\frac{\tan(\frac{\pi}{N})}{\frac{\pi}{N}}\frac{1}{\sqrt{\tan^2(\frac{\pi}{N})+1}}$

2.1.2 Symetries

Let's consider, for a point M, the 2 symetry planes passing through the center of the circle, containing its axis, and one passing through M, the other one perpendicular to that one.

This way we can define a 4-symetry for the discretization of the circle, indexed by n.

2.2 Deriving B from discretized 3D cirle

Let's consider a spire as a circle in 3D with center C, axis $(C, \underline{e}_3)$ and radius R. Let's consider a point M in 3D with coordinates (x_M, y_M, z_M).

Local coordinate system $(\underline{e}_1, \underline{e}_2)$ can be defined from:

$$\begin{cases} \underline{CM} = r\underline{e}_1 + z\underline{e}_3 \\ \underline{e}_2 = \underline{e}_3 \wedge \underline{e}_1 \end{cases}$$

Let's discretize the circle into a 4n-sided polygon with 2 symetry planes, including (M, C, e_3).

Depending on the constraint, the height h of each polygon can be derived (cf. 2.1.1). Similarly, the length of the basis of each polygon can also be derived as:

$$\begin{cases} h & \underbrace{=}_{\text{perim.}} R\frac{\frac{\pi}{4n}}{\tan(\frac{\pi}{4n})} \\ & \underbrace{=}_{\text{area}} R\sqrt{\frac{\frac{\pi}{4n}}{\tan(\frac{\pi}{4n})}} \\ & \underbrace{=}_{\text{B(0)}} R\frac{\tan(\frac{\pi}{4n})}{\frac{\pi}{4n}}\frac{1}{\sqrt{\tan^2(\frac{\pi}{4n})+1}} \\ \mathrm{L}_{1/2} & = h\tan(\frac{\pi}{4n}) \end{cases}$$

Knowing h and $\mathrm{L}_{1/2}$ one can write, for $i \in [1; 2n]$ (we only consider one half of the circle due to the symetry):

$$\begin{cases} \theta_i = (i - \frac{1}{2})\frac{\pi}{4n} \\ \underline{CA_i} = h\cos(\theta_i)\underline{e}_1 + h\sin(\theta_i)\underline{e}_2 \\ \underline{u}_i = -\sin(\theta_i)\underline{e}_1 + \cos(\theta_i)\underline{e}_2 \end{cases}$$

Introducing the local coordinates with respect to $(C, \underline{e}_3)$:

$$\underline{CM} = r\underline{e}_1 + z\underline{e}_3 \Rightarrow \begin{cases} z & = \underline{CM} \cdot \underline{e}_3 \\ r & = \|\underline{CM} - z\underline{e}_3\| \\ \underline{e}_1 & = (\underline{CM} - z\underline{e}_3)/r \end{cases}$$

For each side i, we can write:

$$\begin{aligned} \underline{A_iM} &= \underline{CM} - \underline{CA_i} \\ &= r\underline{e}_1 + z\underline{e}_3 - h\cos(\theta_i)\underline{e}_1 - h\sin(\theta_i)\underline{e}_2 \\ &= (r - h\cos(\theta_i))\underline{e}_1 - h\sin(\theta_i)\underline{e}_2 + z\underline{e}_3 \end{aligned}$$

Hence:

$$\begin{array}{rcl}\underline{u}_i \wedge \underline{A_iM} & = & (-\sin(\theta_i))(-h\sin(\theta_i))\underline{e}_3 + (-\sin(\theta_i))(z)(-\underline{e}_2) \\ & & +(\cos(\theta_i))((r-h\cos(\theta_i)))(-\underline{e}_3) + (\cos(\theta_i))(z)(\underline{e}_1) \\ & = & z\cos(\theta_i)\underline{e}_1 + z\sin(\theta_i)\underline{e}_2 + (h\sin^2(\theta_i) + h\cos^2(\theta_i) - r\cos(\theta_i))\underline{e}_3 \\ & = & z\cos(\theta_i)\underline{e}_1 + z\sin(\theta_i)\underline{e}_2 + (h - r\cos(\theta_i))\underline{e}_3\end{array}$$

and:

$$\begin{array}{rcl}\underline{u}_i \cdot \underline{A_iM} & = & (r-h\cos(\theta_i))(-\sin(\theta_i)) + (\cos(\theta_i))(-h\sin(\theta_i)) \\ & = & -r\sin(\theta_i)\end{array}$$

And since here $\underline{n} = \underline{e}_2$:

$$\underline{A_iM} \cdot \underline{n} \quad = \quad -h\sin(\theta_i)$$

Hence:

$$\left\{\begin{array}{ll} r_{Mi} & = \sqrt{z^2 + (h - r\cos(\theta_i))^2} \\ l_{Mi} & = -r\sin(\theta_i) \\ d_{Ai} & = h\sin(\theta_i)\end{array}\right.$$

Thus, according to 1.2 the magnetic field produced at M by the two segments A_i and A_{4n+1-i} mirrored throught $(C, \underline{e}_2)$ is:

$$\underline{B}_i \;=\frac{\mu_0}{2\pi}\frac{I}{r_{Mi}^2}\left(\frac{\mathrm{L}_{1/2}-l_{Mi}}{\sqrt{(\mathrm{L}_{1/2}-l_{Mi})^2+r_{Mi}^2}} + \frac{\mathrm{L}_{1/2}+l_{Mi}}{\sqrt{(\mathrm{L}_{1/2}+l_{Mi})^2+r_{Mi}^2}}\right)\underline{e}_2 \wedge \left[\underline{e}_2 \wedge (\underline{A_iM} \wedge \underline{u_i})\right]$$

Where:

$$\begin{array}{rrcl} & \underline{e}_2 \wedge (\underline{A_iM} \wedge \underline{u_i}) & = & z\cos(\theta_i)\underline{e}_3 - (h - r\cos(\theta_i))\underline{e}_1 \\ \Rightarrow & \underline{e}_2 \wedge \left[\underline{e}_2 \wedge (\underline{A_iM} \wedge \underline{u_i})\right] & = & z\cos(\theta_i)\underline{e}_1 + (h - r\cos(\theta_i))\underline{e}_3\end{array}$$

And:

$$\left\{\begin{array}{rcl}\frac{\mathrm{L}_{1/2}-l_{Mi}}{\sqrt{(\mathrm{L}_{1/2}-l_{Mi})^2+r_{Mi}^2}} & = & \frac{\mathrm{L}_{1/2}+r\sin(\theta_i)}{\sqrt{(\mathrm{L}_{1/2}+r\sin(\theta_i))^2+z^2+(h-r\cos(\theta_i))^2}} \\ & = & \frac{\mathrm{L}_{1/2}+r\sin(\theta_i)}{\sqrt{\mathrm{L}_{1/2}{}^2+2\,\mathrm{L}_{1/2}\,r\sin(\theta_i)+r^2+z^2+h^2-2hr\cos(\theta_i)}} \\ \frac{\mathrm{L}_{1/2}+l_{Mi}}{\sqrt{(\mathrm{L}_{1/2}+l_{Mi})^2+r_{Mi}^2}} & = & \frac{\mathrm{L}_{1/2}-r\sin(\theta_i)}{\sqrt{\mathrm{L}_{1/2}{}^2-2\,\mathrm{L}_{1/2}\,r\sin(\theta_i)+r^2+z^2+h^2-2hr\cos(\theta_i)}}\end{array}\right.$$

Hence, introducing $\alpha_i = \mathrm{L}_{1/2}{}^2 + r^2 + z^2 + h^2 - 2hr\cos(\theta_i)$:

$$\underline{B}_i \;= \frac{\mu_0}{2\pi}\frac{I}{z^2+(h-r\cos(\theta_i))^2}\left(\frac{\mathrm{L}_{1/2}+r\sin(\theta_i)}{\sqrt{\alpha_i+2\,\mathrm{L}_{1/2}\,r\sin(\theta_i)}} + \frac{\mathrm{L}_{1/2}-r\sin(\theta_i)}{\sqrt{\alpha_i-2\,\mathrm{L}_{1/2}\,r\sin(\theta_i)}}\right)(z\cos(\theta_i)\underline{e}_1 + (h - r\cos(\theta_i))\underline{e}_3)$$

Hence, numerically:

1. Get all θ_i, $\cos(\theta_i)$ and $\sin(\theta_i)$ from n.
2. Get h, $\mathrm{L}_{1/2}$ for each spire, defined by $(C, R, \underline{e}_3)$
3. Get r, z and $\underline{e}_1$ for each pair $(M, spire)$
4. Sum all terms

Appendix A

Appendices

A.1 Section

A.1.1 Subsection