ToFu geometric tools
Intersection of a cone with a plane

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Contents

Chapter 1

Geometry

1.1 Generic cone and plane

Let's consider a half-cone C_1 (defined only for $z > 0$), with summit on the cartesian frame's origin $(O, \underline{\mathrm{e}}_{\mathrm{x}}, \underline{\mathrm{e}}_{\mathrm{y}}, \underline{\mathrm{e}}_{\mathrm{z}})$. The cone's axis is the $(O, \underline{\mathrm{e}}_{\mathrm{z}})$ axis. It's angular opening is θ.

Let's consider plane P_1, of normal $\underline{\mathrm{n}}$, intersection axis $(O, \underline{\mathrm{e}}_{\mathrm{z}})$ at point P of coordinates $(0, 0, Z_P)$. Vector $\underline{\mathrm{n}}$ is oriented by angles ϕ and ψ such that one can define the local frame $(P, \underline{\mathrm{e}}_1, \underline{\mathrm{e}}_2, \underline{\mathrm{n}})$:

$$\begin{cases} \underline{\mathrm{e}}_1 & = \cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + \sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}} \\ \underline{\mathrm{e}}_2 & = \big(-\sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + \cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}}\big)\cos(\psi) + \sin(\psi)\,\underline{\mathrm{e}}_{\mathrm{z}} \\ \underline{\mathrm{n}} & = \underline{\mathrm{e}}_1 \wedge \underline{\mathrm{e}}_2 \\ & = \big(\sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} - \cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}}\big)\sin(\psi) + \cos(\psi)\,\underline{\mathrm{e}}_{\mathrm{z}} \end{cases}$$

We want to find all points M of coordinates (x, y, z) and (x_1, x_2) belonging both to the cone C_1 and the plane P_1.

$$M \in C_1 \Leftrightarrow \underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}} = \cos(\theta)\|\underline{OM}\|$$

$$M \in P_1 \Leftrightarrow \underline{PM}.\,\underline{\mathrm{n}} = 0$$

1.2 Intersection

If M belongs to both P_1 and C_1, then:

$$(\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}})^2 = \cos(\theta)^2\|\underline{OM}\|^2$$

Given that:

$$\begin{aligned} \underline{OM} & = \underline{OP} + \underline{PM} \\ & = Z_P\,\underline{\mathrm{e}}_{\mathrm{z}} + x_1\,\underline{\mathrm{e}}_1 + x_2\,\underline{\mathrm{e}}_2 \\ & = Z_P\,\underline{\mathrm{e}}_{\mathrm{z}} + x_1\big(\cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + \sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}}\big) + x_2\Big(\big(-\sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + \cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}}\big)\cos(\psi) + \sin(\psi)\,\underline{\mathrm{e}}_{\mathrm{z}}\Big) \\ & = Z_P\,\underline{\mathrm{e}}_{\mathrm{z}} + x_1\cos(\phi)\,\underline{\mathrm{e}}_{\mathrm{x}} + x_1\sin(\phi)\,\underline{\mathrm{e}}_{\mathrm{y}} - x_2\sin(\phi)\cos(\psi)\,\underline{\mathrm{e}}_{\mathrm{x}} + x_2\cos(\phi)\cos(\psi)\,\underline{\mathrm{e}}_{\mathrm{y}} + x_2\sin(\psi)\,\underline{\mathrm{e}}_{\mathrm{z}} \\ & = (x_1\cos(\phi) - x_2\sin(\phi)\cos(\psi))\,\underline{\mathrm{e}}_{\mathrm{x}} + (x_1\sin(\phi) + x_2\cos(\phi)\cos(\psi))\,\underline{\mathrm{e}}_{\mathrm{y}} + (Z_P + x_2\sin(\psi))\,\underline{\mathrm{e}}_{\mathrm{z}} \end{aligned}$$

We have:

$$\begin{aligned} (\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}})^2 & = (Z_P + x_2\sin(\psi))^2 \\ & = Z_P^2 + 2Z_P x_2\sin(\psi) + x_2^2\sin(\psi)^2 \end{aligned}$$

And:

$$\begin{aligned}
\|\underline{OM}\|^2 &= \|\left(x_1\cos(\phi) - x_2\sin(\phi)\cos(\psi)\right)\underline{\mathrm{e}}_{\mathrm{x}} + \left(x_1\sin(\phi) + x_2\cos(\phi)\cos(\psi)\right)\underline{\mathrm{e}}_{\mathrm{y}} + \left(Z_P + x_2\sin(\psi)\right)\underline{\mathrm{e}}_{\mathrm{z}}\|^2 \\
&= \left(x_1\cos(\phi) - x_2\sin(\phi)\cos(\psi)\right)^2 \\
&\quad + \left(x_1\sin(\phi) + x_2\cos(\phi)\cos(\psi)\right)^2 \\
&\quad + \left(Z_P + x_2\sin(\psi)\right)^2 \\
&= x_1^2\cos(\phi)^2 - 2x_1x_2\cos(\phi)\sin(\phi)\cos(\psi) + x_2^2\sin(\phi)^2\cos(\psi)^2 \\
&\quad + x_1^2\sin(\phi)^2 + 2x_1x_2\sin(\phi)\cos(\phi)\cos(\psi) + x_2^2\cos(\phi)^2\cos(\psi)^2 \\
&\quad + Z_P^2 + 2Z_Px_2\sin(\psi) + x_2^2\sin(\psi)^2 \\
&= x_1^2 + x_2^2\cos(\psi)^2 \\
&\quad + Z_P^2 + 2Z_Px_2\sin(\psi) + x_2^2\sin(\psi)^2 \\
&= x_1^2 + x_2^2 + 2Z_Px_2\sin(\psi) + Z_P^2
\end{aligned}$$

Thus:

$$\begin{aligned}
& (\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}})^2 = \cos(\theta)^2\|\underline{OM}\|^2 \\
\Leftrightarrow\ & Z_P^2 + 2Z_Px_2\sin(\psi) + x_2^2\sin(\psi)^2 = \cos(\theta)^2\left(x_1^2 + x_2^2 + 2Z_Px_2\sin(\psi) + Z_P^2\right) \\
\Leftrightarrow\ & Z_P^2\left(1 - \cos(\theta)^2\right) + 2Z_Px_2\sin(\psi)\left(1 - \cos(\theta)^2\right) = x_1^2\cos(\theta)^2 + x_2^2\left(\cos(\theta)^2 - \sin(\psi)^2\right) \\
\Leftrightarrow\ & Z_P^2\sin(\theta)^2 + 2Z_Px_2\sin(\psi)\sin(\theta)^2 = x_1^2\cos(\theta)^2 + x_2^2\left(\cos(\theta)^2 - \sin(\psi)^2\right)
\end{aligned}$$

Considering that by hypothesis $\theta > 0$:

$$\begin{aligned}
& (\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}})^2 = \cos(\theta)^2\|\underline{OM}\|^2 \\
\Leftrightarrow\ & x_1^2\cos(\theta)^2 + x_2^2\left(\cos(\theta)^2\text{-}\sin(\psi)^2\right) - 2Z_Px_2\sin(\psi)\sin(\theta)^2 - Z_P^2\sin(\theta)^2 = 0 \\
\Leftrightarrow\ & x_1^2\tfrac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + x_2^2 - 2x_2Z_P\tfrac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} - Z_P^2\tfrac{\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} = 0 \\
\Leftrightarrow\ & x_1^2\tfrac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + \left(x_2 - Z_P\tfrac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}\right)^2 - Z_P^2\tfrac{\sin(\psi)^2\sin(\theta)^4}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2} - Z_P^2\tfrac{\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} = 0 \\
\Leftrightarrow\ & x_1^2\tfrac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + \left(x_2 - Z_P\tfrac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}\right)^2 = Z_P^2\tfrac{\sin(\theta)^2}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2}\left(\sin(\psi)^2\sin(\theta)^2 + \cos(\theta)^2\text{-}\sin(\psi)^2\right) \\
\Leftrightarrow\ & x_1^2\tfrac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + \left(x_2 - Z_P\tfrac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}\right)^2 = Z_P^2\tfrac{\sin(\theta)^2}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2}\left(-\sin(\psi)^2\cos(\theta)^2 + \cos(\theta)^2\right) \\
\Leftrightarrow\ & x_1^2\tfrac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} + \left(x_2 - Z_P\tfrac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}\right)^2 = Z_P^2\tfrac{\sin(\theta)^2\cos(\psi)^2\cos(\theta)^2}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2} \\
\Leftrightarrow\ & \frac{x_1^2}{Z_P^2\frac{\sin(\theta)^2\cos(\psi)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}} + \frac{\left(x_2 - Z_P\frac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}\right)^2}{Z_P^2\frac{\sin(\theta)^2\cos(\psi)^2\cos(\theta)^2}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2}} = 1
\end{aligned}$$

Or, in a reduced conic form:

$$\frac{x_1^2}{a^2} + \frac{(x_2 - x_2(C))^2}{b^2} = 1$$

With:

$$\left\{\begin{array}{lll}
x_2(C) & = Z_P\frac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} & x_2 \text{ coordinate of the center} \\
a^2 & = Z_P^2\frac{\sin(\theta)^2\cos(\psi)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} & \text{squared minor radius} \\
b^2 & = Z_P^2\frac{\sin(\theta)^2\cos(\psi)^2\cos(\theta)^2}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2} & \text{squared major radius} \\
b^2 & = a^2\frac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} \Leftrightarrow a^2 = b^2\left(1 - \frac{\sin(\psi)^2}{\cos(\theta)^2}\right) &
\end{array}\right.$$

The distance d_{CF} between the center C and the focal point F can be deduced from:

$$\begin{aligned}
d_{CF}^2 &= b^2 - a^2 \\
&= b^2\frac{\sin(\psi)^2}{\cos(\theta)^2} \\
&= Z_P^2\frac{\sin(\theta)^2\cos(\psi)^2\sin(\psi)^2}{(\cos(\theta)^2\text{-}\sin(\psi)^2)^2}
\end{aligned}$$

Hence, the x_2 coordinate of F is:

$$\begin{aligned} x_2(F) &= x_2(C) \pm d_{CF} \\ &= Z_P \frac{\sin(\psi)\sin(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} \pm Z_P \frac{\sin(\theta)\cos(\psi)\sin(\psi)}{\cos(\theta)^2\text{-}\sin(\psi)^2} \\ &= Z_P \frac{\sin(\psi)\sin(\theta)^2 \pm \sin(\theta)\cos(\psi)\sin(\psi)}{\cos(\theta)^2\text{-}\sin(\psi)^2} \\ &= Z_P \frac{\sin(\psi)\sin(\theta)}{\cos(\theta)^2\text{-}\sin(\psi)^2} \left(\sin(\theta) \pm \cos(\psi)\right) \end{aligned}$$

It is worth noticing that the neither the focal point nor the center correspond to the intersection between the axes and the plane P.

1.3 Parametric equation

In our case, only the axes $(O, \underline{e_z})$, fixed by the crystal's summit and normal, is independent from the cone's angular opening θ. It makes sense to design an ad-hoc coordinate system centered on the ellipse's center C to use its parameterized equation.

Knowing all geometrical parameters, it is possible to compute all points on the ellipse parameterizing them with t:

$$\begin{cases} x_1 = a\cos(t) \\ x_2 = x_2(C) + b\sin(t) \end{cases}$$

BEWARE : parameter t is not the angle of the point with respect to the ellipse's center.

Now, we would like to parameterize the ellipse not with t but with the angle β with respect to the point P, because it is the physically relevant angle since it is taken with respect to the axis $(O, \underline{e_z})$ and relates to the impact point of the photon beam on the crystal's center. Also, it is the only common element to all ellipses. The angle ϵ taken with respect to the center is not relevant because each ellipse has a different center.

In this perspective:

$$\begin{cases} x_1 = l(\beta)\cos(\beta) \\ x_2 = l(\beta)\sin(\beta) \end{cases}$$

Keeping in mind that the ellipse is defined as:

$$\frac{x_1^2}{a^2} + \frac{(x_2 - x_2(C))^2}{b^2} = 1$$

We can write:

$$\begin{aligned} & l^2 b^2 \cos(\beta)^2 + a^2 \left(l^2 \sin(\beta)^2 - 2l x_2(C)\sin(\beta) + x_2(C)^2\right) = a^2 b^2 \\ \Leftrightarrow\ & l^2 \left(b^2\cos(\beta)^2 + a^2\sin(\beta)^2\right) - 2la^2 x_2(C)\sin(\beta) + a^2 x_2(C)^2 - a^2 b^2 = 0 \end{aligned}$$

Has solutions if:

$$\begin{aligned} & \Delta = 4a^4 x_2(C)^2 \sin(\beta)^2 - 4\left(b^2\cos(\beta)^2 + a^2\sin(\beta)^2\right)\left(a^2 x_2(C)^2 - a^2 b^2\right) \geq 0 \\ \Leftrightarrow\ & \Delta = 4a^2 \left[a^2 x_2(C)^2 \sin(\beta)^2 - \left(b^2\cos(\beta)^2 + a^2\sin(\beta)^2\right)\left(x_2(C)^2 - b^2\right)\right] \geq 0 \\ \Leftrightarrow\ & \Delta = 4a^2 \left[a^2 x_2(C)^2 \sin(\beta)^2 - b^2 x_2(C)^2\cos(\beta)^2 - a^2 x_2(C)^2\sin(\beta)^2 + b^4\cos(\beta)^2 + a^2b^2\sin(\beta)^2\right] \geq 0 \\ \Leftrightarrow\ & \Delta = 4a^2 \left[-b^2 x_2(C)^2\cos(\beta)^2 + b^4\cos(\beta)^2 + a^2b^2\sin(\beta)^2\right] \geq 0 \\ \Leftrightarrow\ & \Delta = 4a^2b^2 \left[-x_2(C)^2\cos(\beta)^2 + b^2\cos(\beta)^2 + a^2\sin(\beta)^2\right] \geq 0 \end{aligned}$$

Which is equivalent to, keeping in mind that $b^2 - a^2 = d_{CF}^2$:

$$\begin{aligned} & \Delta = 4a^2b^2 \left[a^2 + \left(b^2 - a^2 - x_2(C)^2\right)\cos(\beta)^2\right] \geq 0 \\ \Leftrightarrow\ & \left(b^2 - a^2 - x_2(C)^2\right)\cos(\beta)^2 \geq -a^2 \Leftrightarrow \qquad \left(d_{CF}^2 - x_2(C)^2\right)\cos(\beta)^2 \geq -a^2 \end{aligned}$$

If $d_{CF}^2 - x_2(C)^2\ geq0$, this is true for all β values, and this condition is met if:

$$
\begin{aligned}
& b^2 - a^2 - x_2(C)^2 \geq 0 \\
\Leftrightarrow\ & \frac{Z_P^2}{(\cos(\theta)^2 \text{-} \sin(\psi)^2)^2} \left(\sin(\theta)^2 \cos(\psi)^2 \cos(\theta)^2 - \sin(\theta)^2 \cos(\psi)^2 \cos(\theta)^2 + \sin(\theta)^2 \cos(\psi)^2 \sin(\psi)^2 - \sin(\psi)^2 \sin(\right. \\
\Leftrightarrow\ & \sin(\theta)^2 \sin(\psi)^2 \left(\cos(\psi)^2 - \sin(\theta)^2\right) \geq 0
\end{aligned}
$$

Which is true if we have an ellipse, which is the only case of interest. Hence $\Delta = 4a^2b^2\left[a^2 + \left(b^2 - a^2 - x_2(C)^2\right)\cos(\beta)^2\right] \geq 0$, so:

$$
\begin{aligned}
& l_{1,2} = \tfrac{2a^2 x_2(C)\sin(\beta) \pm \sqrt{\Delta}}{2(b^2\cos(\beta)^2 + a^2\sin(\beta)^2)} \\
\Leftrightarrow\ & l_{1,2} = \tfrac{a^2 x_2(C)\sin(\beta) \pm ab\sqrt{a^2 + (b^2 - a^2 - x_2(C)^2)\cos(\beta)^2}}{b^2\cos(\beta)^2 + a^2\sin(\beta)^2}
\end{aligned}
$$

And we only want the positive solution:

$$
l = \frac{a^2 x_2(C)\sin(\beta) + ab\sqrt{a^2 + (b^2 - a^2 - x_2(C)^2)\cos(\beta)^2}}{b^2\cos(\beta)^2 + a^2\sin(\beta)^2}
$$

1.3.1 From bragg angle and parameter to local cartesian coordinates

Keep in mind that the frame $(P, \underline{\mathrm{e}}_1, \underline{\mathrm{e}}_2)$ is, by definition aligned on the minor and major axes of the ellipse. Hence, for an arbitrary frame $(R, \underline{e}_i, \underline{e}_j)$ on plane P_1, translated and rotated by α with respect to $(P, \underline{\mathrm{e}}_1, \underline{\mathrm{e}}_2)$:

$$
\begin{cases}
\underline{e}_i = \cos(\alpha)\,\underline{\mathrm{e}}_1 + \sin(\alpha)\,\underline{\mathrm{e}}_2 \\
\underline{e}_j = -\sin(\alpha)\,\underline{\mathrm{e}}_1 + \cos(\alpha)\,\underline{\mathrm{e}}_2 \\
\underline{\mathrm{e}}_1 = \cos(\alpha)\underline{e_i} - \sin(\alpha)\underline{e_j}\ \underline{\mathrm{e}}_2 = \sin(\alpha)\underline{e_i} + \cos(\alpha)\underline{e_j}
\end{cases}
$$

Or, in coordinate tranforms:

$$
\begin{cases}
x_1 = x_1(R) + x_i\cos(\alpha) - x_j\sin(\alpha) \\
x_2 = x_2(R) + x_i\sin(\alpha) + x_j\cos(\alpha) \\
x_i = (x_1 - x_1(R))\cos(\alpha) + (x_2 - x_2(R))\sin(\alpha) \\
x_j = -(x_1 - x_1(R))\sin(\alpha) + (x_2 - x_2(R))\cos(\alpha)
\end{cases}
$$

Hence:

$$
\begin{cases}
x_i = (a\cos(\epsilon) - x_1(R))\cos(\alpha) + (x_2(C) - x_2(R) + b\sin(\epsilon))\sin(\alpha) \\
x_j = -\,(a\cos(\epsilon) - x_1(R))\sin(\alpha) + (x_2(C) - x_2(R) + b\sin(\epsilon))\cos(\alpha)
\end{cases}
$$

But

$$
\begin{cases}
\|\underline{PM}\|^2 = x_1^2 + x_2^2 = \\
x_1 = \|\underline{PM}\|\cos(\beta) \\
x_2 = \|\underline{PM}\|\sin(\beta)
\end{cases}
$$

1.3.2 From local cartesian coordinates to bragg angle

Knowing (x_i, x_j) and all geometric parameters, we now want to derive (θ, ϵ).

From the previous equation, we can write:

$$
\begin{cases}
x_i\cos(\alpha) - x_j\sin(\alpha) = a\cos(\epsilon) - x_1(R) & (1) \\
x_i\sin(\alpha) + x_j\cos(\alpha) = x_2(C) - x_2(R) + b\sin(\epsilon) & (2)
\end{cases}
$$

The dependency in θ is hidden in the expressions of a, b and $x_2(C)$.

By squaring and summing, it is possible to get rid of the ϵ dependency:

$$\left\{\begin{array}{l} a^2\cos(\epsilon)^2 = \left(x_i\cos(\alpha) - x_j\sin(\alpha) + x_1(R)\right)^2 \\ b^2\sin(\epsilon)^2 = \left(x_i\sin(\alpha) + x_j\cos(\alpha) - x_2(C) + x_2(R)\right)^2 \end{array}\right.$$

Hence, keeping in mind that $a^2 = b^2\frac{\cos(\theta)^2\text{-}\sin(\psi)^2}{\cos(\theta)^2}$ and re-using the definitions of x_1 and x_2 which do not depend on the unknowns (θ, ϵ):

$$\begin{array}{lll}
 & b^2 & = \frac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}x_1^2 + \left(x_2 - x_2(C)\right)^2 \\
\Leftrightarrow & b^2\left(\cos(\theta)^2\text{-}\sin(\psi)^2\right) & = \cos(\theta)^2x_1^2 + \left(\cos(\theta)^2\text{-}\sin(\psi)^2\right)\left(x_2^2 - 2x_2x_2(C) + x_2(C)^2\right) \\
\Leftrightarrow & Z_P^2\frac{\sin(\theta)^2\cos(\psi)^2\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} & = \cos(\theta)^2x_1^2 + \left(\cos(\theta)^2\text{-}\sin(\psi)^2\right)\left(x_2^2 - 2x_2x_2(C) + x_2(C)^2\right) \\
\Leftrightarrow & Z_P^2\frac{\sin(\theta)^2\cos(\psi)^2\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2} & = \cos(\theta)^2\left(x_1^2 + x_2^2\right) - \sin(\psi)^2x_2^2 - \left(\cos(\theta)^2\text{-}\sin(\psi)^2\right)\left(2x_2x_2(C) - x_2(\right. \\
\Leftrightarrow & Z_P^2\sin(\theta)^2\cos(\psi)^2\cos(\theta)^2 & = \cos(\theta)^2\left(\cos(\theta)^2\text{-}\sin(\psi)^2\right)\left(x_1^2 + x_2^2\right) - \sin(\psi)^2\left(\cos(\theta)^2\text{-}\sin(\psi)^2\right)x_2^2 \\
\Leftrightarrow & & - \left(\cos(\theta)^2\text{-}\sin(\psi)^2\right)^2\left(2x_2x_2(C) - x_2(C)^2\right) \\
\Leftrightarrow & Z_P^2\sin(\theta)^2\cos(\psi)^2\cos(\theta)^2 & = \cos(\theta)^4\left(x_1^2 + x_2^2\right) - \cos(\theta)^2\sin(\psi)^2\left(x_1^2 + 2x_2^2\right) + \sin(\psi)^4x_2^2 \\
 & & - \left(\cos(\theta)^2\text{-}\sin(\psi)^2\right)^2\left(2x_2x_2(C) - x_2(C)^2\right) \\
\Leftrightarrow & Z_P^2\sin(\theta)^2\cos(\psi)^2\cos(\theta)^2 & = \cos(\theta)^4\left(x_1^2 + x_2^2\right) - \cos(\theta)^2\sin(\psi)^2\left(x_1^2 + 2x_2^2\right) + \sin(\psi)^4x_2^2 \\
 & & - \left[2x_2Z_P\sin(\psi)\sin(\theta)^2\left(\cos(\theta)^2\text{-}\sin(\psi)^2\right) - Z_P^2\sin(\psi)^2\sin(\theta)^4\right] \\
\Leftrightarrow & Z_P^2\cos(\psi)^2\left(\cos(\theta)^2 - \cos(\theta)^4\right) & = \cos(\theta)^4\left(x_1^2 + x_2^2\right) - \cos(\theta)^2\sin(\psi)^2\left(x_1^2 + 2x_2^2\right) + \sin(\psi)^4x_2^2 \\
 & & -2x_2Z_P\sin(\psi)\sin(\theta)^2\left(\cos(\theta)^2\text{-}\sin(\psi)^2\right) \\
 & & +Z_P^2\sin(\psi)^2\left(1 - 2\cos(\theta)^2 + \cos(\theta)^4\right) \\
\Leftrightarrow & Z_P^2\cos(\psi)^2\left(\cos(\theta)^2 - \cos(\theta)^4\right) & = \cos(\theta)^4\left(x_1^2 + x_2^2\right) - \cos(\theta)^2\sin(\psi)^2\left(x_1^2 + 2x_2^2\right) + \sin(\psi)^4x_2^2 \\
 & & -2x_2Z_P\sin(\psi)\left(\cos(\theta)^2 - \cos(\theta)^4 - \sin(\psi)^2\left(1 - \cos(\theta)^2\right)\right) \\
 & & +Z_P^2\sin(\psi)^2\left(1 - 2\cos(\theta)^2 + \cos(\theta)^4\right)
\end{array}$$

Then introducing $X = \cos(\theta)^2$:

$$\begin{array}{lll}
 & b^2 & = \frac{\cos(\theta)^2}{\cos(\theta)^2\text{-}\sin(\psi)^2}x_1^2 + \left(x_2 - x_2(C)\right)^2 \\
\Leftrightarrow & Z_P^2\cos(\psi)^2\left(X - X^2\right) & = X^2\left(x_1^2 + x_2^2\right) - X\sin(\psi)^2\left(x_1^2 + 2x_2^2\right) + \sin(\psi)^4x_2^2 \\
 & & -2x_2Z_P\sin(\psi)\left(X - X^2 - \sin(\psi)^2 + X\sin(\psi)^2\right) \\
 & & +Z_P^2\sin(\psi)^2\left(1 - 2X + X^2\right)
\end{array}$$

Which boils down to:

$$\begin{array}{rcl}
0 & = & X^2\left[x_1^2+x_2^2+2x_2Z_P\sin(\psi)+Z_P^2\sin(\psi)^2+Z_P^2\cos(\psi)^2\right] \\
& & +X\left[-\sin(\psi)^2\left(x_1^2+2x_2^2\right)-\left(1+\sin(\psi)^2\right)2x_2Z_P\sin(\psi)-2Z_P^2\sin(\psi)^2-Z_P^2\cos(\psi)^2\right] \\
& & +\sin(\psi)^4x_2^2+2x_2Z_P\sin(\psi)^3+Z_P^2\sin(\psi)^2 \\
\Leftrightarrow\ 0 & = & X^2\left[x_1^2+\left(x_2+Z_P\sin(\psi)\right)^2+Z_P^2\cos(\psi)^2\right] \\
& & +X\left[-\sin(\psi)^2\left(x_1^2+2x_2^2\right)-\left(1+\sin(\psi)^2\right)2x_2Z_P\sin(\psi)-2Z_P^2\sin(\psi)^2-Z_P^2\cos(\psi)^2\right] \\
& & +\sin(\psi)^2\left(\sin(\psi)^2x_2^2+2x_2Z_P\sin(\psi)+Z_P^2\right) \\
\Leftrightarrow\ 0 & = & X^2\left[x_1^2+\left(x_2+Z_P\sin(\psi)\right)^2+Z_P^2\cos(\psi)^2\right] \\
& & -X\left[\sin(\psi)^2\left(x_1^2+2x_2^2\right)+\left(1+\sin(\psi)^2\right)2x_2Z_P\sin(\psi)+2Z_P^2\sin(\psi)^2+Z_P^2\cos(\psi)^2\right] \\
& & +\sin(\psi)^2\left(x_2\sin(\psi)+Z_P\right) \\
\Leftrightarrow\ 0 & = & X^2\left[x_1^2+\left(x_2+Z_P\sin(\psi)\right)^2+Z_P^2\cos(\psi)^2\right] \\
& & -X\left[\sin(\psi)^2\left(x_1^2+x_2^2\right)+x_2^2\sin(\psi)^2+2x_2Z_P\sin(\psi)+2x_2Z_P\sin(\psi)^3+Z_P^2+Z_P^2\sin(\psi)^2\right] \\
& & +\sin(\psi)^2\left(x_2\sin(\psi)+Z_P\right) \\
\Leftrightarrow\ 0 & = & X^2\left[x_1^2+\left(x_2+Z_P\sin(\psi)\right)^2+Z_P^2-Z_P^2\sin(\psi)^2\right] \\
& & -X\left[\sin(\psi)^2\left(x_1^2+x_2^2\right)+\left(x_2\sin(\psi)+Z_P\right)^2+Z_P\sin(\psi)^2\left(2x_2\sin(\psi)+Z_P\right)\right] \\
& & +\sin(\psi)^2\left(x_2\sin(\psi)+Z_P\right) \\
\Leftrightarrow\ 0 & = & X^2\left[x_1^2+x_2^2+\left(x_2\sin(\psi)+Z_P\right)^2-x_2^2\sin(\psi)^2\right] \\
& & -X\left[\sin(\psi)^2\left(x_1^2+x_2^2\right)+\left(x_2\sin(\psi)+Z_P\right)^2+Z_P\sin(\psi)^2\left(2x_2\sin(\psi)+Z_P\right)\right] \\
& & +\sin(\psi)^2\left(x_2\sin(\psi)+Z_P\right) \\
\Leftrightarrow\ 0 & = & AX^2-BX+C
\end{array}$$

With:

$$\left\{\begin{array}{l}
A=x_1^2+x_2^2+\left(x_2\sin(\psi)+Z_P\right)^2-x_2^2\sin(\psi)^2 \\
B=\sin(\psi)^2\left(x_1^2+x_2^2\right)+\left(x_2\sin(\psi)+Z_P\right)^2+Z_P\sin(\psi)^2\left(2x_2\sin(\psi)+Z_P\right) \\
C=\sin(\psi)^2\left(x_2\sin(\psi)+Z_P\right)
\end{array}\right.$$

Solutions exist if:

$$\begin{array}{rcl}
& \Delta & \geq\ 0 \\
\Leftrightarrow & B^2-4AC & \geq\ 0
\end{array}$$

In which case, only solutions in $[0,1]$ are acceptable:

$$\cos(\theta)^2=\frac{B\pm\sqrt(\Delta)}{2A}\in[0,1]$$

And by definition, $\theta\in\left[0,\frac{\pi}{2}\right]$, hence $\cos(\theta)\geq 0$ and:

$$\theta=\arccos\left(\sqrt{\frac{B\pm\sqrt(\Delta)}{2A}}\right)$$

Alternative method for θ

By definition:

$$\underline{OM}.\,\underline{\mathrm{e}}_{\mathrm{z}}=\cos(\theta)\|OM\|$$

And:

$$\begin{array}{rcl}
\underline{OM} & = & \underline{OP}+\underline{PR}+\underline{RM} \\
& = &
\end{array}$$

1.4 Generalization

This time, the crystal of curvature radius R has center C of coordinates (x_C, y_C, z_C) in the tokamak's frame $(O, \underline{\mathrm{e}}_\mathrm{x}, \underline{\mathrm{e}}_\mathrm{y}, \underline{\mathrm{e}}_\mathrm{z})$.
From C, a spherical coordinates system is defined with angles (ϕ, θ) such that $(C, , ,)$ is a direct orthonormal frame, and that the crystal's summit S lies at coordinates $(R, 0, \pi/2)$.

Appendix A

Appendices

A.1 Section

A.1.1 Subsection