

ToFu tools
Magnetic fields

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Chapter 1

3D field from straight current segments

1.1 Single segment

Let's consider current segment with current I , centered on A , with unit vector \underline{u} and half-length $L_{1/2}$. Considering a point P in this segment, identified by its length to A :

$$\underline{OP} = \underline{OA} + l\underline{u}, \text{ with } l \in [-L_{1/2}; L_{1/2}]$$

Any point M in space can be located by its position with respect to \underline{A}

$$\begin{cases} \underline{OM} = \underline{OA} + l_M \underline{u} + r_M \underline{v} \\ \underline{v} = \underline{AM} - (\underline{AM} \cdot \underline{u}) \underline{u} \end{cases}$$

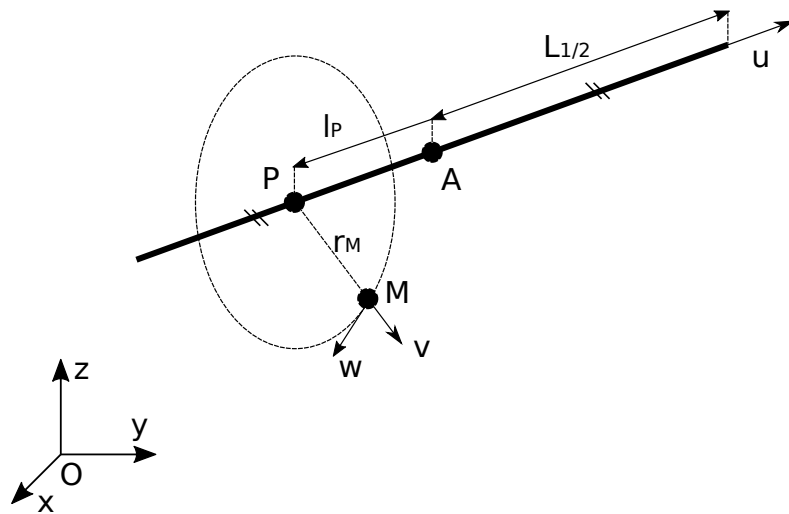


Figure 1.1: Magnetic field at any point M from a straight current segment

The Biot-Savart law stipulates that an elementary length of current creates an elemen-

tary magnetic field at M :

$$\begin{cases} \underline{dI} = I d\underline{l} \\ \underline{PM} = (l_M - l)\underline{u} + r_M \underline{v} \\ \underline{B}_A = \frac{\mu_0}{4\pi} \int_{-L_{1/2}}^{L_{1/2}} \frac{\underline{dI} \wedge \underline{PM}}{\|\underline{PM}\|^3} \end{cases}$$

Introducing $\underline{w} = \underline{u} \wedge \underline{v}$:

$$\underline{dI} \wedge \underline{PM} = I dl r_M \underline{w}$$

and:

$$\|\underline{PM}\| = \sqrt{(l - l_M)^2 + r_M^2}$$

Hence:

$$\underline{B}_A = \frac{\mu_0}{4\pi} I r_M \underline{w} \int_{-L_{1/2}}^{L_{1/2}} \frac{dl}{((l - l_M)^2 + r_M^2)^{3/2}}$$

Introducing $x = l - l_M \Rightarrow dx = dl$:

$$\underline{B}_A = \frac{\mu_0}{4\pi} I r_M \underline{w} \int_{-L_{1/2} - l_M}^{L_{1/2} - l_M} \frac{dx}{(x^2 + r_M^2)^{3/2}}$$

Noticing that:

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + a}} \right) &= \frac{\sqrt{x^2 + a} - x \frac{1}{2} \frac{2x}{\sqrt{x^2 + a}}}{x^2 + a} \\ &= \frac{(x^2 + a) - x^2}{(x^2 + a)^{3/2}} \\ &= \frac{a}{(x^2 + a)^{3/2}} \end{aligned}$$

Hence:

$$\begin{aligned} \underline{B}_A &= \frac{\mu_0}{4\pi} I r_M \underline{w} \left[\frac{1}{r_M^2} \frac{x}{\sqrt{x^2 + r_M^2}} \right]_{-L_{1/2} - l_M}^{L_{1/2} - l_M} \\ &= \frac{\mu_0}{4\pi} \frac{I}{r_M} \underline{w} \left(\frac{L_{1/2} - l_M}{\sqrt{(L_{1/2} - l_M)^2 + r_M^2}} + \frac{L_{1/2} + l_M}{\sqrt{(L_{1/2} + l_M)^2 + r_M^2}} \right) \end{aligned}$$

For numerical evaluation, keep in mind that:

$$\begin{cases} l_M = \underline{u} \cdot \underline{AM} \\ r_M = \|\underline{u} \wedge \underline{AM}\| \\ \underline{w} = \frac{\underline{u} \wedge \underline{AM}}{r_M} \end{cases}$$

1.2 2 mirrored segments

Let's consider 2 current segments (A, \underline{u}) and (A', \underline{u}') one being the symmetric of the other via a symmetry plane (M, \underline{n}) .

Each current segment has its own current I (resp. I').

By construction:

$$\begin{cases} l'_M &= l_M \\ r'_M &= r_M \\ L'_{1/2} &= L_{1/2} \\ d_A &= -\underline{AM} \cdot \underline{n} \\ \underline{A'A} &= 2d_A \underline{n} \\ \underline{u}' &= \underline{u} - 2(\underline{u} \cdot \underline{n})\underline{n} \\ \underline{AM} &= l_M \underline{u} + r_M \underline{v} \\ \underline{A'M} &= l_M \underline{u}' + r_M \underline{v}' \end{cases}$$

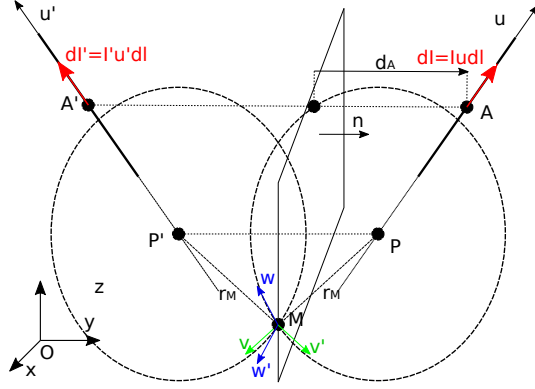


Figure 1.2: Magnetic field at any point M from 2 mirrored straight current segments

The total magnetic field created in M is:

$$\begin{aligned}
 \underline{B} &= \underline{B}_A(I) + \underline{B}_{A'}(I') \\
 &= \frac{\mu_0}{4\pi} \frac{I}{r_M} \underline{w} \left(\frac{L_{1/2} - l_M}{\sqrt{(L_{1/2} - l_M)^2 + r_M^2}} + \frac{L_{1/2} + l_M}{\sqrt{(L_{1/2} + l_M)^2 + r_M^2}} \right) + \frac{\mu_0}{4\pi} \frac{I'}{r_M} \underline{w}' \left(\frac{L_{1/2} - l_M}{\sqrt{(L_{1/2} - l_M)^2 + r_M^2}} + \frac{L_{1/2} + l_M}{\sqrt{(L_{1/2} + l_M)^2 + r_M^2}} \right) \\
 &= \frac{\mu_0}{4\pi} \frac{1}{r_M} \left(\frac{L_{1/2} - l_M}{\sqrt{(L_{1/2} - l_M)^2 + r_M^2}} + \frac{L_{1/2} + l_M}{\sqrt{(L_{1/2} + l_M)^2 + r_M^2}} \right) (I \underline{w} + I' \underline{w}')
 \end{aligned}$$

Now, considering that:

$$\begin{aligned}
 r_M \underline{v}' &= \underline{A'M} - l_M \underline{u}' \\
 &= \underline{A'A} + \underline{AM} - l_M (\underline{u} - 2(\underline{u} \cdot \underline{n}) \underline{n}) \\
 &= 2d_A \underline{n} + \underline{AM} - l_M \underline{u} + 2l_M (\underline{u} \cdot \underline{n}) \underline{n} \\
 &= 2d_A \underline{n} + r_M \underline{v} + 2l_M (\underline{u} \cdot \underline{n}) \underline{n} \\
 \Leftrightarrow \underline{v}' &= \underline{v} + \frac{2}{r_M} (l_M (\underline{u} \cdot \underline{n}) + d_A) \underline{n} \\
 r_M \neq 0
 \end{aligned}$$

Hence:

$$\begin{aligned}
 \underline{w}' &= \underline{u}' \wedge \underline{v}' \\
 &= (\underline{u} - 2(\underline{u} \cdot \underline{n}) \underline{n}) \wedge \left(\underline{v} + \frac{2}{r_M} (l_M (\underline{u} \cdot \underline{n}) + d_A) \underline{n} \right) \\
 &= \underline{u} \wedge \underline{v} + \frac{2}{r_M} (l_M (\underline{u} \cdot \underline{n}) + d_A) \underline{u} \wedge \underline{n} - 2(\underline{u} \cdot \underline{n}) \underline{n} \wedge \underline{v} \\
 &= \underline{w} + 2 \left[\frac{(\underline{u} \cdot \underline{n}) l_M + d_A}{r_M} \underline{u} + (\underline{u} \cdot \underline{n}) \underline{v} \right] \wedge \underline{n} \\
 &= \underline{w} + 2 \left[\frac{(\underline{u} \cdot \underline{n}) (l_M \underline{u} + r_M \underline{v}) + d_A \underline{u}}{r_M} \right] \wedge \underline{n} \\
 &= \underline{w} + 2 \left[\frac{(\underline{u} \cdot \underline{n}) \underline{AM} + d_A \underline{u}}{r_M} \right] \wedge \underline{n}
 \end{aligned}$$

Remembering that $d_A = -\underline{AM} \cdot \underline{n}$ and that $\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c}) \underline{b} - (\underline{a} \cdot \underline{b}) \underline{c}$:

$$\begin{aligned}
 \underline{w}' &= \underline{w} + 2 \left[\frac{(\underline{u} \cdot \underline{n}) \underline{AM} - (\underline{AM} \cdot \underline{n}) \underline{u}}{r_M} \right] \wedge \underline{n} \\
 &= \underline{w} + \frac{2}{r_m} [\underline{n} \wedge (\underline{AM} \wedge \underline{u})] \wedge \underline{n} \\
 &= \underline{w} + 2 \frac{\sqrt{r_m^2 + l_M^2}}{r_m} [\underline{n} \wedge (\underline{e}_{AM} \wedge \underline{u})] \wedge \underline{n}
 \end{aligned}$$

Where we have introduced $\underline{AM} = \sqrt{r_M^2 + l_M^2} \underline{e}_{AM}$:

Then, assuming $I' = -I$ we have:

$$\begin{aligned}
\underline{B} &= \frac{\mu_0}{4\pi} \frac{I}{r_M} \left(\frac{L_{1/2} - l_M}{\sqrt{(L_{1/2} - l_M)^2 + r_M^2}} + \frac{L_{1/2} + l_M}{\sqrt{(L_{1/2} + l_M)^2 + r_M^2}} \right) (\underline{w} - \underline{w}') \\
&= -\frac{\mu_0}{4\pi} \frac{I}{r_M} \left(\frac{L_{1/2} - l_M}{\sqrt{(L_{1/2} - l_M)^2 + r_M^2}} + \frac{L_{1/2} + l_M}{\sqrt{(L_{1/2} + l_M)^2 + r_M^2}} \right) 2 \frac{\sqrt{r_m^2 + l_M^2}}{r_m} [\underline{n} \wedge (\underline{e}_{AM} \wedge \underline{u})] \wedge \underline{n} \\
&= \frac{\mu_0}{2\pi} I \frac{\sqrt{r_m^2 + l_M^2}}{r_m^2} \left(\frac{L_{1/2} - l_M}{\sqrt{(L_{1/2} - l_M)^2 + r_M^2}} + \frac{L_{1/2} + l_M}{\sqrt{(L_{1/2} + l_M)^2 + r_M^2}} \right) \underline{n} \wedge [\underline{n} \wedge (\underline{e}_{AM} \wedge \underline{u})]
\end{aligned}$$

For numerical evaluation, keep in mind that:

$$\begin{cases} l_M = \underline{u} \cdot \underline{AM} \\ r_M = \|\underline{u} \wedge \underline{AM}\| \\ d_A = -\underline{AM} \cdot \underline{n} \end{cases}$$

1.3 4 mirrored segments

Let's consider the 2 previous mirrored current segments and add a pair mirroring them via another plane (C, \underline{m}) , perpendicular to the first plane $(C, \underline{n}) = (M, \underline{n})$. All segments are lying in the same plane (C, \underline{a})

The same current is running through each segment and they all have the same half-length $L_{1/2}$.

The same derivation as previously can be done for pair AA' and pair BB' .

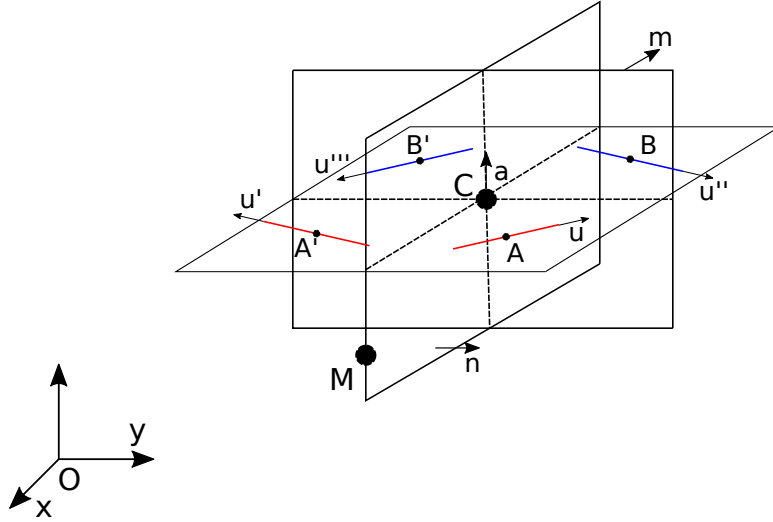


Figure 1.3: Magnetic field at any point M from 2 mirrored straight current segments

The total magnetic field created in M is:

$$\begin{aligned}
\underline{B} &= B_{AA'}(I) + B_{BB'}(I) \\
&= \frac{\mu_0}{2\pi} \frac{I}{r_M} \left(\frac{L_{1/2} - l_M}{\sqrt{(L_{1/2} - l_M)^2 + r_M^2}} + \frac{L_{1/2} + l_M}{\sqrt{(L_{1/2} + l_M)^2 + r_M^2}} \right) \left(\underline{n} \wedge \left[\frac{(\underline{u}_A \cdot \underline{n}) \underline{AM} + d_A \underline{u}_A}{r_M} \right] \right) \\
&\quad + \frac{\mu_0}{2\pi} \frac{I}{r_M} \left(\frac{L_{1/2} - l_M''}{\sqrt{(L_{1/2} - l_M'')^2 + r_M'^2}} + \frac{L_{1/2} + l_M''}{\sqrt{(L_{1/2} + l_M'')^2 + r_M'^2}} \right) \left(\underline{n} \wedge \left[\frac{(\underline{u}_B \cdot \underline{n}) \underline{BM} + d_B \underline{u}_B}{r_M'} \right] \right)
\end{aligned}$$

By construction, we have: cannot get (r''_M, l''_M) from (r_M, l_M) .

Chapter 2

Circular coil

The magnetic field produced at any point in 3D space by a planar circular coil cannot be derived analytically.

2.1 Circle discretization

Let's consider a circular coil of radius R centered on axis (C, \underline{e}_3) .

For a given point M in space, let's make the plane (C, M, \underline{e}_3) a symmetry plane and divide the circle into a N -sided polygon.

2.1.1 N-sided polygon

The polygon is N -sided, with N an even number. The polygon shall have either:

- the same perimeter as the circle: $L = 2\pi R$
- the same area as the circle: $S = \pi R^2$
- the same magnetic field on the center as the circle: $B(0) = \frac{\mu_0 I}{2R}$

For a N -sided regular polygon of height h :

$$\begin{cases} L_i = 2h \tan\left(\frac{\pi}{N}\right) & \text{is the length of a single side} \\ S_i = \frac{hL_i}{2} = h^2 \tan\left(\frac{\pi}{N}\right) & \text{is the area of a single side} \end{cases}$$

Remembering that, at the center, $l_m = 0$ and $r_m = h$ for all, by construction, and that $L_{1/2} = \frac{L_i}{2} = h \tan\left(\frac{\pi}{N}\right)$, we derive:

$$\begin{aligned} B(0, N) &= N \frac{\mu_0}{4\pi} \frac{I}{h} \left(\frac{L_{1/2}}{\sqrt{(L_{1/2})^2 + h^2}} + \frac{L_{1/2}}{\sqrt{(L_{1/2})^2 + h^2}} \right) \\ &= N \frac{\mu_0}{2\pi} \frac{I}{h} \frac{L_{1/2}}{\sqrt{(L_{1/2})^2 + h^2}} \\ &= N \frac{\mu_0}{2\pi} \frac{I}{h} \frac{h \tan\left(\frac{\pi}{N}\right)}{\sqrt{h^2 \tan^2\left(\frac{\pi}{N}\right) + h^2}} \\ &= N \frac{\mu_0}{2\pi} \frac{I}{h} \frac{\tan\left(\frac{\pi}{N}\right)}{\sqrt{\tan^2\left(\frac{\pi}{N}\right) + 1}} \\ &= \frac{\mu_0 I}{2} \frac{1}{h} \frac{N}{\pi} \frac{\tan\left(\frac{\pi}{N}\right)}{\sqrt{\tan^2\left(\frac{\pi}{N}\right) + 1}} \end{aligned}$$

Hence,

- same perimeter $\Rightarrow h = R \frac{\frac{\pi}{N}}{\tan(\frac{\pi}{N})}$
- same area $\Rightarrow h = R \sqrt{\frac{\frac{\pi}{N}}{\tan(\frac{\pi}{N})}}$
- same field on axis $\Rightarrow h = R \frac{\tan(\frac{\pi}{N})}{\frac{\pi}{N}} \frac{1}{\sqrt{\tan^2(\frac{\pi}{N})+1}}$

2.1.2 Symetries

Let's consider, for a point M , the 2 symetry planes passing through the center of the circle, containing its axis, and one passing through M , the other one perpendicular to that one.

This way we can define a 4-symetry for the discretization of the circle, indexed by n .

2.2 Deriving B from discretized 3D cirle

Let's consider a spire as a circle in 3D with center C , axis (C, \underline{e}_3) and radius R . Let's consider a point M in 3D with coordinates (x_M, y_M, z_M) .

Local coordinate system $(\underline{e}_1, \underline{e}_2)$ can be defined from:

$$\begin{cases} \underline{CM} = r\underline{e}_1 + z\underline{e}_3 \\ \underline{e}_2 = \underline{e}_3 \wedge \underline{e}_1 \end{cases}$$

Let's discretize the circle into a $4n$ -sided polygon with 2 symetry planes, including (M, C, \underline{e}_3) .

Depending on the constraint, the height h of each polygon can be derived (cf. 2.1.1). Similarly, the length of the basis of each polygon can also be derived as:

$$\left\{ \begin{array}{l} h \\ \text{perim.} \\ \text{area} \\ \text{B(0)} \\ L_{1/2} \end{array} \right. = \begin{cases} \underbrace{R \frac{\frac{\pi}{4n}}{\tan(\frac{\pi}{4n})}} \\ \underbrace{R \sqrt{\frac{\frac{\pi}{4n}}{\tan(\frac{\pi}{4n})}}} \\ \underbrace{R \frac{\tan(\frac{\pi}{4n})}{\frac{\pi}{4n}} \frac{1}{\sqrt{\tan^2(\frac{\pi}{4n})+1}}} \\ h \tan(\frac{\pi}{4n}) \end{cases}$$

Knowing h and $L_{1/2}$ one can write, for $i \in [1; 2n]$ (we only consider one half of the circle due to the symetry):

$$\begin{cases} \theta_i = (i - \frac{1}{2}) \frac{\pi}{4n} \\ \underline{CA_i} = h \cos(\theta_i) \underline{e}_1 + h \sin(\theta_i) \underline{e}_2 \\ \underline{u_i} = -\sin(\theta_i) \underline{e}_1 + \cos(\theta_i) \underline{e}_2 \end{cases}$$

Introducing the local coordinates with respect to (C, \underline{e}_3) :

$$\underline{CM} = r\underline{e}_1 + z\underline{e}_3 \Rightarrow \begin{cases} z &= \underline{CM} \cdot \underline{e}_3 \\ r &= \|\underline{CM} - z\underline{e}_3\| \\ \underline{e}_1 &= (\underline{CM} - z\underline{e}_3)/r \end{cases}$$

For each side i , we can write:

$$\begin{aligned} \underline{A_i M} &= \underline{CM} - \underline{CA_i} \\ &= r\underline{e}_1 + z\underline{e}_3 - h \cos(\theta_i) \underline{e}_1 - h \sin(\theta_i) \underline{e}_2 \\ &= (r - h \cos(\theta_i)) \underline{e}_1 - h \sin(\theta_i) \underline{e}_2 + z\underline{e}_3 \end{aligned}$$

Hence:

$$\begin{aligned}
\underline{u}_i \wedge \underline{A_i M} &= (-\sin(\theta_i))(-h \sin(\theta_i))\underline{e}_3 + (-\sin(\theta_i))(z)(-\underline{e}_2) \\
&\quad + (\cos(\theta_i))((r - h \cos(\theta_i)))(-\underline{e}_3) + (\cos(\theta_i))(z)(\underline{e}_1) \\
&= z \cos(\theta_i)\underline{e}_1 + z \sin(\theta_i)\underline{e}_2 + (h \sin^2(\theta_i) + h \cos^2(\theta_i) - r \cos(\theta_i))\underline{e}_3 \\
&= z \cos(\theta_i)\underline{e}_1 + z \sin(\theta_i)\underline{e}_2 + (h - r \cos(\theta_i))\underline{e}_3
\end{aligned}$$

and:

$$\begin{aligned}
\underline{u}_i \cdot \underline{A_i M} &= (r - h \cos(\theta_i))(-\sin(\theta_i)) + (\cos(\theta_i))(-h \sin(\theta_i)) \\
&= -r \sin(\theta_i)
\end{aligned}$$

And since here $\underline{n} = \underline{e}_2$:

$$\underline{A_i M} \cdot \underline{n} = -h \sin(\theta_i)$$

Hence:

$$\begin{cases} r_{Mi} &= \sqrt{z^2 + (h - r \cos(\theta_i))^2} \\ l_{Mi} &= -r \sin(\theta_i) \\ d_{Ai} &= h \sin(\theta_i) \end{cases}$$

Thus, according to 1.2 the magnetic field produced at M by the two segments A_i and A_{4n+1-i} mirrored throught (C, \underline{e}_2) is:

$$\underline{B}_i = \frac{\mu_0}{2\pi} \frac{I}{r_{Mi}^2} \left(\frac{L_{1/2} - l_{Mi}}{\sqrt{(L_{1/2} - l_{Mi})^2 + r_{Mi}^2}} + \frac{L_{1/2} + l_{Mi}}{\sqrt{(L_{1/2} + l_{Mi})^2 + r_{Mi}^2}} \right) \underline{e}_2 \wedge [\underline{e}_2 \wedge (\underline{A_i M} \wedge \underline{u}_i)]$$

Where:

$$\begin{aligned}
&\underline{e}_2 \wedge (\underline{A_i M} \wedge \underline{u}_i) &= z \cos(\theta_i)\underline{e}_3 - (h - r \cos(\theta_i))\underline{e}_1 \\
\Rightarrow \underline{e}_2 \wedge [\underline{e}_2 \wedge (\underline{A_i M} \wedge \underline{u}_i)] &= z \cos(\theta_i)\underline{e}_1 + (h - r \cos(\theta_i))\underline{e}_3
\end{aligned}$$

And:

$$\begin{cases} \frac{L_{1/2} - l_{Mi}}{\sqrt{(L_{1/2} - l_{Mi})^2 + r_{Mi}^2}} &= \frac{L_{1/2} + r \sin(\theta_i)}{\sqrt{(L_{1/2} + r \sin(\theta_i))^2 + z^2 + (h - r \cos(\theta_i))^2}} \\ &= \frac{L_{1/2} + r \sin(\theta_i)}{\sqrt{L_{1/2}^2 + 2 L_{1/2} r \sin(\theta_i) + r^2 + z^2 + h^2 - 2hr \cos(\theta_i)}} \\ \frac{L_{1/2} + l_{Mi}}{\sqrt{(L_{1/2} + l_{Mi})^2 + r_{Mi}^2}} &= \frac{L_{1/2} - r \sin(\theta_i)}{\sqrt{L_{1/2}^2 - 2 L_{1/2} r \sin(\theta_i) + r^2 + z^2 + h^2 - 2hr \cos(\theta_i)}} \end{cases}$$

Hence, introducing $\alpha_i = L_{1/2}^2 + r^2 + z^2 + h^2 - 2hr \cos(\theta_i)$:

$$\underline{B}_i = \frac{\mu_0}{2\pi} \frac{I}{z^2 + (h - r \cos(\theta_i))^2} \left(\frac{L_{1/2} + r \sin(\theta_i)}{\sqrt{\alpha_i + 2 L_{1/2} r \sin(\theta_i)}} + \frac{L_{1/2} - r \sin(\theta_i)}{\sqrt{\alpha_i - 2 L_{1/2} r \sin(\theta_i)}} \right) (z \cos(\theta_i)\underline{e}_1 + (h - r \cos(\theta_i))\underline{e}_3)$$

Hence, numerically:

1. Get all θ_i , $\cos(\theta_i)$ and $\sin(\theta_i)$ from n .
2. Get h , $L_{1/2}$ for each spire, defined by (C, R, \underline{e}_3)
3. Get r , z and \underline{e}_1 for each pair (M, spire)
4. Sum all terms

2.3 Numerical Applications

2.3.1 Convergence study

Increasing n , we see that:

- Convergence is quite fast
- The inner side of the coil converges faster
- The remaining error is found in close vicinity to the coil itself
- A remaining error of $\approx 0.1\%$ can be obtained with $n = 10$ within $\approx 10\%$ of the coil radius around the coil itself

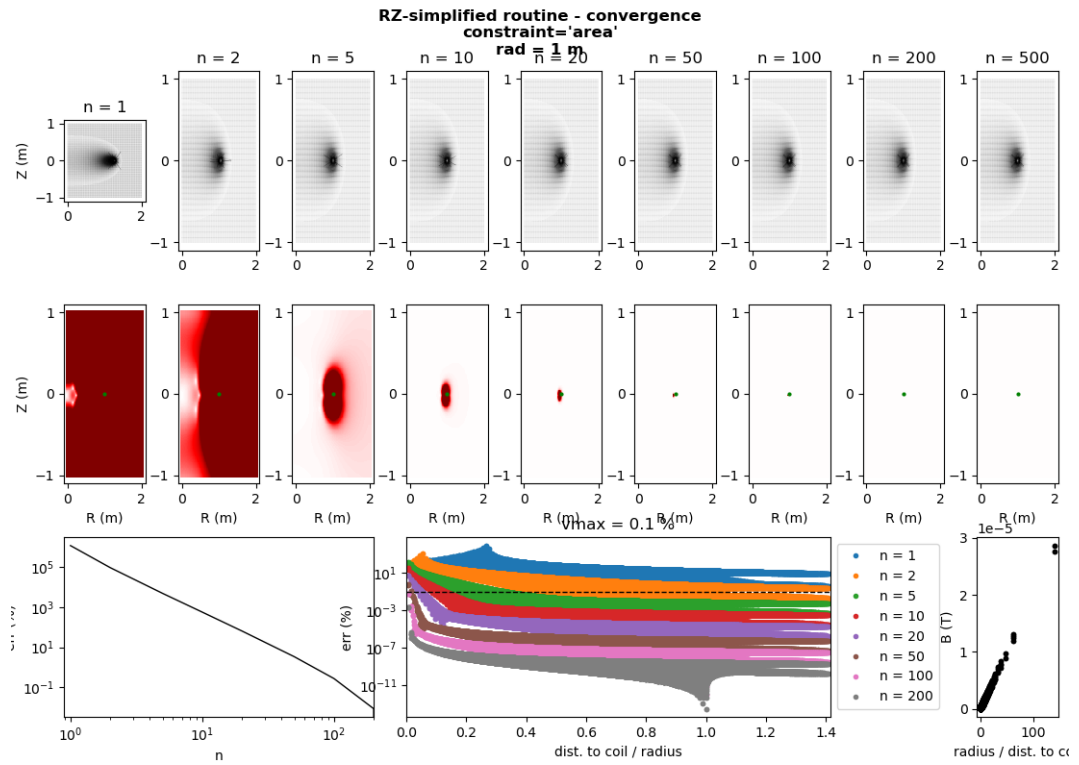


Figure 2.1: Poloidal magnetic field from a single circular coil discretized as a 4n-sided polygon, convergence study

Appendix A

Appendices

A.1 Section

A.1.1 Subsection