Rules for Sketching Wavefunctions

(Adapted from "Particles Behave Like Waves" by Thomas A. Moore.)

1. Solutions to the Schrödinger equation curve *toward* the x axis in classically allowed regions (where $E - V(x) > 0$) and *away* from the x axis in classically forbidden regions (where $E - V(x) < 0$).

   - Implication: Solutions are *wavelike* in classically allowed regions and *exponential-like* in forbidden regions.
   - Follows from: $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x) \implies \dfrac{d^2\psi}{dx^2} = -\dfrac{2m}{\hbar^2}[E - V(x)]\psi(x)$
   - The classical turning points, where $E - V(x) = 0$, are *inflection points*.

2. The curvature of solutions increases with $|E - V(x)|$. Note that $|E - V(x)|$ corresponds to the distance between the energy line and the potential energy curve on a potential energy graph. Implications:

   - In classically allowed regions, the *local wavelength* of the wavelike part of the solution gets shorter (or local wavenumber k gets larger) as $|E - V(x)|$ increases.
   - Recall that $-\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} \to \dfrac{\hbar^2 k^2}{2m}$ and $k = \dfrac{2\pi}{\lambda}$.
   - In classically forbidden regions, the *length* of any exponential-like tail gets shorter as $|E - V(x)|$ increases.

3. The local amplitude of the wavelike part of a solution *decreases* as the value of $|E - V(x)|$ increases.

   - You are more likely to find the particle in regions where its momentum (therefore velocity) is lower.
   - The following figure illustrates the above points:

1
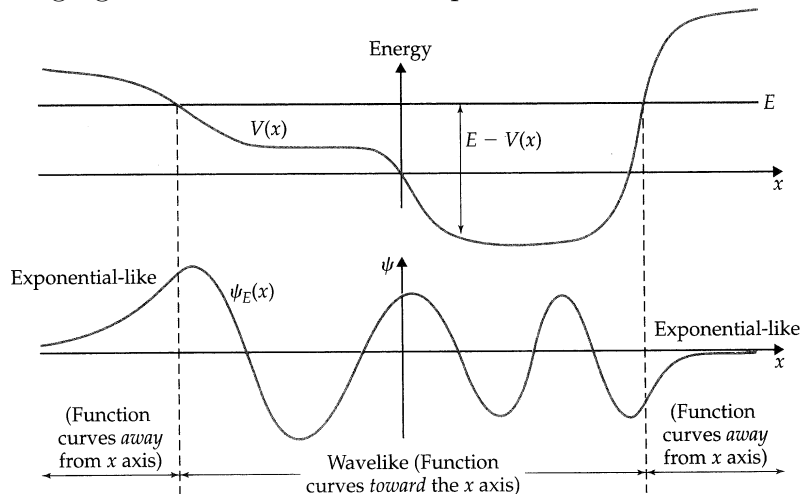
4. Solutions $\psi(x)$ are always *continuous* and *smooth* (no kinks).

- This rule provides *boundary conditions* to match solutions between regions where $V(x)$ changes discontinuously (steps).

- Exception: If $V(x)$ has an *infinite* jump (as for the quanton in an infinite box).

5. Physically reasonable wavefunctions remain finite as $|x| \to \infty$.

- The exponential-like parts of physically reasonable solutions for bound quantons must actually decrease to *zero* as $|x| \to \infty$ in the classically forbidden regions. (Otherwise the total probability would be infinite!)

- Only certain quantized energy values E_n give rise to physically reasonable solutions (that is, energy eigenfunctions) for bound quantons. For other energies there are solutions to the Schrödinger equation but they do not satisfy the rules for physically acceptable solutions.

6. Energy eigenfunctions (stationary-state solutions) have an integer number of bumps in the classically allowed region: the more bumps, the greater the corresponding energy.

- The state with the lowest possible energy will have one bump (or zero nodes). The next highest will have two bumps (one node), and so on:
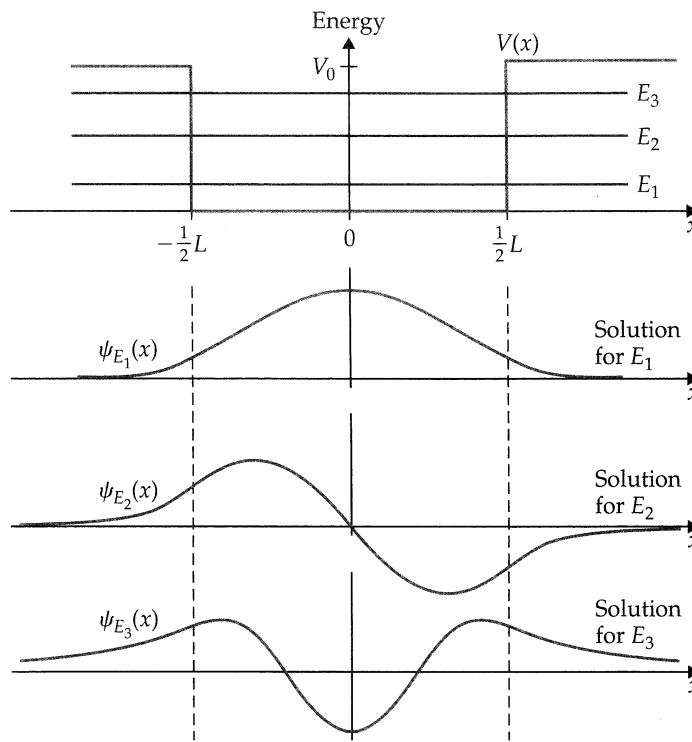
Here is an example that illustrates all of the rules. This is the wavefunction for the *seventh* energy level. Thus, there are seven bumps (six nodes).

(a)

(b)
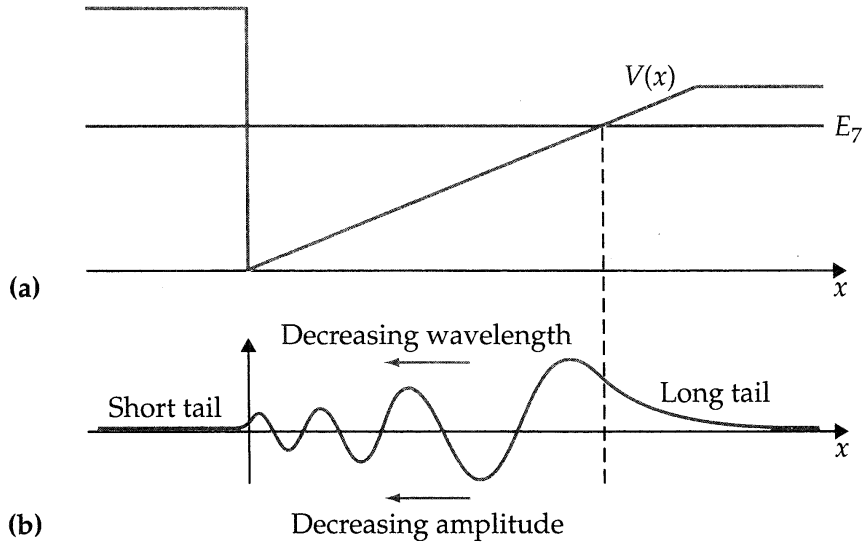
Here is an example of "tunneling" (see section Q11.3):
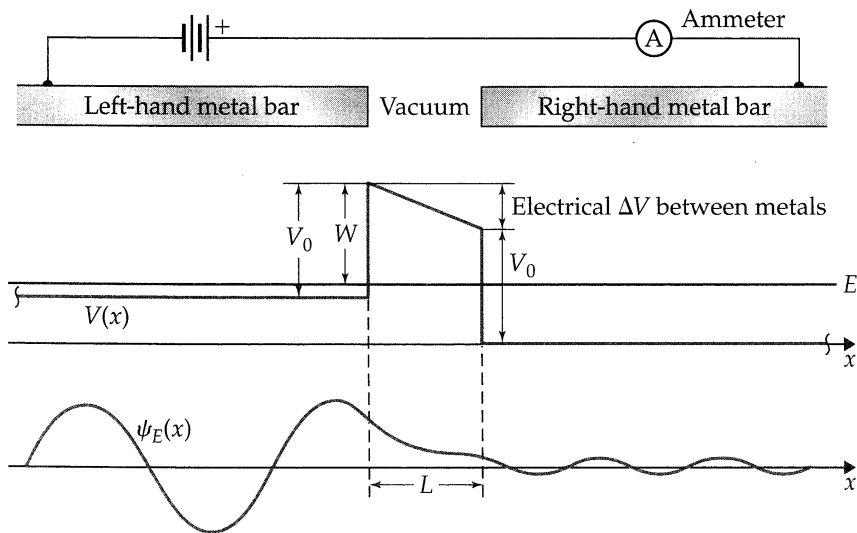
Note the classically forbidden region in the middle, where two exponentials combine, as in this figure:

(a) $f(x) = Ae^{+|a|x}$

(b) $f(x) = Ae^{-|a|x}$

(c) $f(x) = Ae^{+|a|x} + Ae^{-|a|x}$