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PII: S0010-4485(97)00046-8

Computer-Aided Design, Vol. 30, No. 1, pp. 1–9, 1998

0010-4485/98/$19.00+0.00

Research

Cubic parametric curves of given tangent and curvature

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We propose a constructive solution to the problem of finding a cubic parametric curve in a plane if the tangent vectors (derivatives with respect to the parameter) and signed curvatures are given at its end-points but the end-points themselves are unknown. We also show how these curves can be applied to construct blending curves subject to curvature, arc length, inflection and area constraints.

Keywords: Bézier curves, cubics, blending, constraints

INTRODUCTION

Cubic parametric curves, i.e. curves the coordinate functions of which are cubic polynomials of the parameter, play an important role in computer aided geometric design. This is due to the fact that it is the lowest degree curve by means of which one can describe twisted curves—curves whose torsion is not identically zero—and curves with singularities, such as inflection, cusp or loop. There are several papers[1–3] on the properties and representations of parametric cubics.

A parametric cubic can not only be specified by means of quantities appearing in its different representations, like coefficients of its parametric form or control points of its Bézier representation, but by means of different geometric constraints. A trivial geometric constraint is interpolation, i.e. to find a cubic which passes through four given points. In Refs. 4 and 5, the authors solve the problem of how to produce a parametric cubic in a plane if its end-points are given, as well as its tangent directions and curvatures at these points. One can find papers[6–8] where cubic or higher degree parametric curves with fixed endpoints are constructed subject to arc length or different fairing constraints.

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Paper Received: 4 October 1996. Revised: 8 September 1997

Whereas in this previous works the end-points were given, in our paper we assume that the endpoints are unknown but we know the tangent vectors (derivatives with respect to the parameter) and signed curvatures at them. We provide an elementary constructive solution to this problem using the Bézier representation of cubic arcs, then we show how these curves can be applied for the construction of blending curves satisfying conditions in connection with curvature, arc length, inflection and area.

Designers apply blending in order to avoid sharp edges and vertices. Blending in plane means to substitute a vertex with a curve arc or, more generally, to make a smooth transition between two curves. There must be a certain order of continuity at the junction of the blending curve and the curves to be blended. To ensure smoothness this order must be at least one. In CAGD there are two different types of continuity distinguished. One with respect to the parameter, which requires the equality of derivatives at the junction (C^n), and the other called visual or geometric continuity (G^n) which is less strict than the previous one and requires the equality of parameterisation independent geometric quantities like tangent line, curvature or torsion. Originally only circular arcs were used for blending, then to meet special requirements, like that of airfoil design, conic arcs were applied[9]. A rational representation of quadric blends is discussed in Ref. 10.

In computer-aided design surface blending is a major issue. Several methods have been developed for the solution to this problem, especially for parametric surfaces. A comprehensive description of this topic can be found in Ref. 11. In Section 6 (Conclusions) we show how our results could be used in parametric surface blending.

CUBIC PARAMETRIC CURVES OF GIVEN TANGENT AND CURVATURE

The problem is as follows:

Given: tangent vectors (derivatives with respect to the parameter) and signed curvatures at the end-points of a

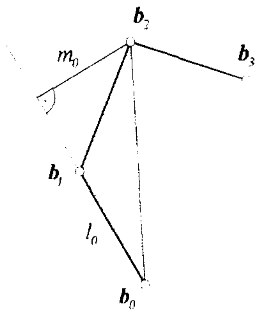

Figure 1 Area proportional to the curvature at $\mathbf{b}_0$

plane cubic parametric curve. (The end-points themselves are unknown.)

Find: the control points of its Bézier representation.

Let us denote the tangent vectors by $\mathbf{e}_0$ and $\mathbf{e}_1$, the curvatures by κ_0 and κ_1 and the control points to be constructed by $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$. We assume that the parameter varies in the range $[0,1]$ which is not a restriction, since a change from a range $[a,b]$ to $[0,1]$ results only in the multiplication of the tangent vectors by $(b-a)$.

If a plane curve $\mathbf{r}(t)$, $t \in [a,b]$ is given we assign to its curvature $\kappa(t) = |\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)|/|\dot{\mathbf{r}}(t)|^3$ the sign of the third component of the vector $\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t)$. Thus the signed curvature of a plane curve $\mathbf{r}(t)$ is defined by the expression

$$\kappa(t) = \frac{\dot{\mathbf{r}}(t) \wedge \ddot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|^3} \tag{1}$$

where $\dot{\mathbf{r}}(t) \wedge \ddot{\mathbf{r}}(t) = \dot{r}_x(t)\ddot{r}_y(t) - \ddot{r}_x(t)\dot{r}_y(t)$. From here on, the operator $\wedge$ will denote this special vector operation.

In order to determine the signed curvature of the plane Bézier curve $\mathbf{b}(t) = \Sigma_{i=0}^{n} \mathbf{b}_i B_i^n(t)$, $t \in [0,1]$ at $t = 0$, we need its first and second derivatives, i.e.

$$\dot{\mathbf{b}}(0) = n(\mathbf{b}_1 - \mathbf{b}_0),$$

$$\ddot{\mathbf{b}}(0) = n(n-1)((\mathbf{b}_2 - \mathbf{b}_1) - (\mathbf{b}_1 - \mathbf{b}_0)).$$

With the notation $l_0 = |\mathbf{b}_1 - \mathbf{b}_0|$ the curvature is

$$\begin{aligned}\kappa_0 = \kappa(0) &= \frac{\dot{\mathbf{b}}(0) \wedge \ddot{\mathbf{b}}(0)}{|\dot{\mathbf{b}}(0)|^3} \\ &= \left(\frac{n-1}{n}\right) \frac{(\mathbf{b}_1 - \mathbf{b}_0) \wedge (\mathbf{b}_2 - \mathbf{b}_1)}{l_0^3} \\ &= \left(\frac{n-1}{n}\right) \frac{2 \cdot \mathrm{Area}(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2)}{l_0^3},\end{aligned}$$

where $\mathrm{Area}(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2)$ is the signed area of the triangle determined by the control points $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$; see *Figure 1*. Denoting the signed distance of the point $\mathbf{b}_2$ from the line $\mathbf{b}_0, \mathbf{b}_1$ by m_0 we obtain

$$m_0 = \frac{(\mathbf{b}_1 - \mathbf{b}_0)^+}{l_0}(\mathbf{b}_2 - \mathbf{b}_1) \text{ and } \mathrm{Area}(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2) = \frac{l_0 m_0}{2}$$

where $(\mathbf{b}_1 - \mathbf{b}_0)^+$ is the positive normal vector of $(\mathbf{b}_1 - \mathbf{b}_0)$, i.e. $(\mathbf{b}_1 - \mathbf{b}_0)$ is rotated counter clockwise through 90°. Therefore $\kappa_0 = (n-1)m_0/nl_0^2$ and its $n = 3$ special case is

$$\kappa_0 = \frac{2m_0}{3l_0^2}.$$

We can analogously calculate the signed curvature at $t = 1$, for which we obtain

$$\kappa_1 = \kappa(1) = \frac{2}{3}\frac{(\mathbf{b}_2 - \mathbf{b}_1) \wedge (\mathbf{b}_3 - \mathbf{b}_2)}{l_1^3} = \frac{2m_1}{3l_1^2}.$$

where $l_1 = |\mathbf{b}_3 - \mathbf{b}_2|$ and $m_1 = (\mathbf{b}_3 - \mathbf{b}_2)^+(\mathbf{b}_1 - \mathbf{b}_2)/l_1$.

On the basis of the previous derivations we can determine the control points $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ of the Bézier curve we are looking for. Due to the above geometric meaning of the signed curvature

$$m_0 = \frac{3}{2}\kappa_0 l_0^2, \quad m_1 = \frac{3}{2}\kappa_1 l_1^2 \tag{2}$$

with $l_0 = |\mathbf{e}_0|/3$ and $l_1 = |\mathbf{e}_1|/3$. The signed distance of the missing control point $\mathbf{b}_2$ from the tangent line at $\mathbf{b}_0$, which is parallel to $\mathbf{e}_0$, is m_0; and the signed distance of $\mathbf{b}_1$ from the tangent line at $\mathbf{b}_3$, which is parallel to $\mathbf{e}_1$, is m_1.

Further on, we assume that the signed distances m_0 and m_1, gained from the signed curvatures κ_0 and κ_1, satisfy either the inequality $|m_0| > |m_1|$ or the equality $|m_0| = |m_1|$, since in the case of $|m_0| < |m_1|$ we can obtain the case $|m_0| > |m_1|$ with multiplying the given entities $\kappa_0, \kappa_1, \mathbf{e}_0, \mathbf{e}_1$ by -1 and inverting the subscripts. This formal modification does not alter our original problem because the Bézier curve is symmetric in the sense that the two sets of control points $\mathbf{b}_0, \mathbf{b}_1, \cdots, \mathbf{b}_n$ and $\mathbf{b}_n, \mathbf{b}_{n-1}, \cdots, \mathbf{b}_0$ determine the same Bézier curve. Later on we distinguish two cases according to the relative position of the given tangents.

Tangents at the end-points are not parallel

If we do not differentiate congruent solutions we can assume without loss of generality that the vector $\mathbf{e}_0$ is on the x axis and $\frac{\mathbf{e}_0}{3}$ points at the origin (see *Figure 2*) thus

$$\mathbf{b}_0 = \begin{bmatrix} -l_0 \\ 0 \end{bmatrix} \text{ and } \mathbf{b}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Due to the curvature constraints imposed on the end-points the control point $\mathbf{b}_2$ has to be on the line parallel to the x axis at the signed distance m_0; moreover, the line $\mathbf{b}_2, \mathbf{b}_3$ should be parallel to $\mathbf{e}_1$ and be tangential to the circle with radius m_1 and centre at the origin. From the similar triangles of *Figure 2* we obtain

$$\mathbf{b}_2 = \begin{bmatrix} m_0 \dfrac{e_{1x}}{e_{1y}} + m_1 \dfrac{|\mathbf{e}_1|}{e_{1y}} \\ m_0 \end{bmatrix}, \tag{3}$$

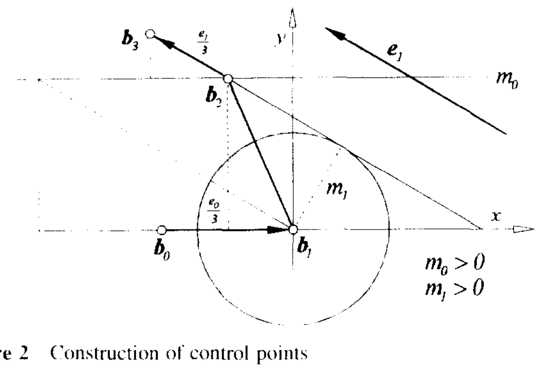

Figure 2 Construction of control points

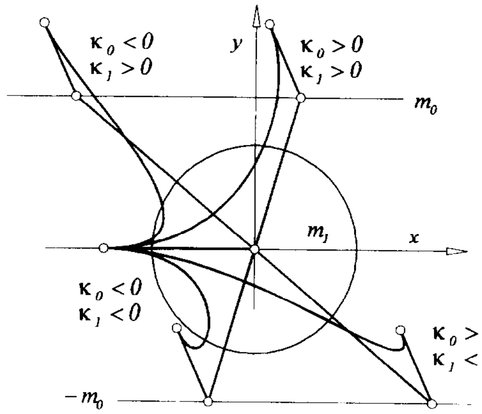

Figure 3 The impact of the curvature's sign on the shape of curve

and from the connection between the control points and derivatives of cubic Bézier curves

$$\mathbf{b}_3 = \mathbf{b}_2 + \frac{1}{3}\mathbf{e}_1 = \begin{bmatrix} m_0\dfrac{e_{1x}}{e_{1y}} + m_1\dfrac{|\mathbf{e}_1|}{e_{1y}} + \dfrac{e_{1x}}{3} \\ m_0 + \dfrac{e_{1y}}{3} \end{bmatrix}.$$

Not differentiating congruent solutions, there is a unique solution to the problem and this solution is easy to compute, moreover, it can even be constructed by means of a ruler and compasses in the Euclidean sense. *Figure 3* illustrates the impact of the signs of end-curvatures. By means of eqn (3) we can obtain the solutions to those special cases when either or both end-curvatures are zero. Because of their applications later we will discuss these special cases in detail.

Parallel tangents at the end-points

In this case a solution to the problem exists if, and only if, $|m_0| = |m_1|$. From this and from eqn (2) we obtain

$$\frac{|\kappa_0|}{|\kappa_1|} = \frac{l_1^2}{l_0^2},$$

i.e. the equality of the end-curvature's magnitude does not follow from $|m_0| = |m_1|$. A consequence of the equality $|m_0| = |m_1|$ is that either both end-curvatures are zero or neither is. If the curvatures at both end-points are zero, there is an infinite number of degenerate solutions, namely straight-line segments. These solutions are straight-line segments since the control points of the Bézier curve are collinear due to the conditions, and the number of solutions is infinite because the distance between $\mathbf{b}_1$ and $\mathbf{b}_2$ can be arbitrary.

In the case of $\kappa_0 \neq 0$, $\kappa_1 \neq 0$ there are also infinite though not degenerate solutions to the problem. Introducing the notations $m = |m_0| = |m_1|$, $l_0 = \frac{e_{0x}}{3}$ and $l_1 = \frac{e_{1x}}{3}$ the solutions form a one-parameter family of Bézier curves with control points

$$\mathbf{b}_0 = \begin{bmatrix} -l_0 \\ 0 \end{bmatrix}, \ \mathbf{b}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} \nu \\ m \end{bmatrix},$$

$$\mathbf{b}_3 = \begin{bmatrix} \nu + l_1 \\ m \end{bmatrix} \quad \nu \in \Re \tag{4}$$

(see *Figure 4*).

PARALLEL TANGENTS AT THE END-POINTS

Since the $m \neq 0$ case is underdetermined we can specify a further condition for a unique solution. In this section we propose some additional conditions and discuss the application of the resulting curves.

Theorem 1

The members of the family of Bézier curves defined by control points given by eqn (4) can be transformed into each other by an affine elation (shearing) the axis of which is the x axis of the coordinate system.

Proof. In eqn (4) l_0, l_1 and m are fixed and $\nu \in \Re$ is a free parameter. Control polygons determined by them are either of the type *Figure 4a* or of the type *Figure 4b* depending on the signs of l_0 and l_1. The corresponding Bézier curves have zero or one point of inflection, respectively. These properties of the control polygons and the Bézier curves are not influenced by the parameter ν.

The variation of ν results in the translation of the straight line segment $\mathbf{b}_2, \mathbf{b}_3$ along the line $y = m$ while the segment $\mathbf{b}_0, \mathbf{b}_1$ remains unchanged; see *Figure 5*. Consequently, control polygons corresponding to two arbitrarily chosen values ν_1 and ν_2 can be mapped onto each other with an affine elation whose axis is the x axis and the direction of which is parallel to its axis. With other words

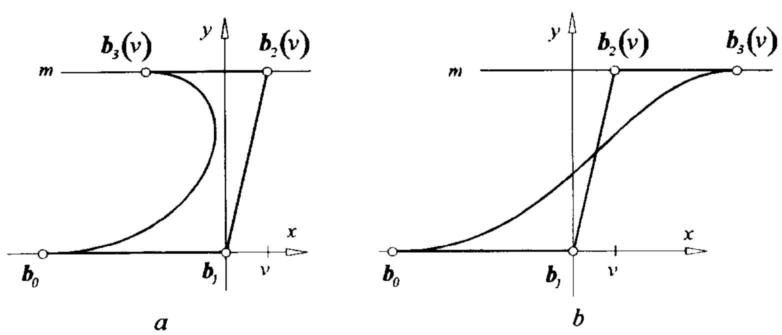

Figure 4 Parallel tangents and non-vanishing curvatures at end-points

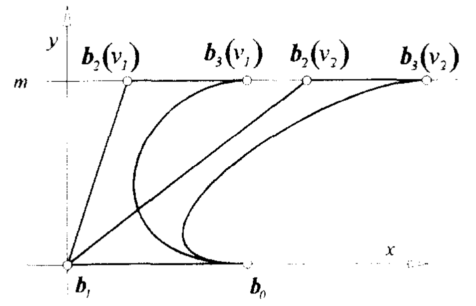

Figure 5 The perspective affinity between Bézier curves of eqn (4)

we can say that the control polygon of ν_1 can be mapped on the control polygon ν_2 with a shearing transformation of the matrix

$$\begin{bmatrix} 1 & \frac{\nu_2-\nu_1}{m} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Taking into account that the Bézier curve is invariant under affine transformation, i.e. the Bézier curve determined by the transformed control points coincide with the point-wise transformation of the curve, the proposition is proven. ♦

Prescribed arc length

The arc length can be a condition for the elimination of the free parameter.

The arc length of the Bézier curve $\mathbf{b}(t)=\Sigma_{i=0}^{3}\mathbf{b}_iB_i^3(t)$, $t\in[0,1]$ can be expressed in the form

$$\int_0^1|\dot{\mathbf{b}}(t)|\mathrm{d}t=3\int_0^1\sqrt{\mathbf{a}_0^2(1-t)^4+\mathbf{a}_2^2t^4+(4\mathbf{a}_1^2+2\mathbf{a}_0\mathbf{a}_2)t^2(1-t)^2+4\mathbf{a}_0\mathbf{a}_1t(1-t)^3+4\mathbf{a}_1\mathbf{a}_2t^3(1-t)}\mathrm{d}t$$

by means of its derivative $\dot{\mathbf{b}}(t)=\Sigma_{i=0}^{2}\mathbf{a}_iB_i^2(t)$, $(\mathbf{a}_i=\mathbf{b}_{i+1}-\mathbf{b}_i)$.

This is an elliptic integral in general, cf. Ref. 12, accordingly it does not have a closed form.

The arc length of the curves with control points given by eqn (4) is

$$h(\nu)=3\int_0^1\sqrt{f(\nu,t)}\mathrm{d}t \qquad (5)$$

with the notation

$$\mathbf{a}_0=\begin{bmatrix} l_0 \\ 0 \end{bmatrix},\ \mathbf{a}_1=\begin{bmatrix} \nu \\ m \end{bmatrix},\ \mathbf{a}_2=\begin{bmatrix} l_1 \\ 0 \end{bmatrix}$$

and

$$f(\nu,t)=l_0^2(1-t)^4+l_1^2t^4+(4(\nu^2+m^2)+2l_0l_1)t^2(1-t)^2+4l_0\nu t(1-t)^3+4l_1\nu t^3(1-t)\ \nu\in\Re,\ t\in[0,1].$$

Later on we use the following property of Bézier curves, which is proved in Ref.[6].

Theorem 2

Let L denote the length of the control polygon of the Bézier curve determined by the control points $\mathbf{b}_0,\mathbf{b}_1,\cdots,\mathbf{b}_n$ $(L=\Sigma_{i=0}^{n-1}|\mathbf{b}_{i+1}-\mathbf{b}_i|)$, let C denote the length of the chord joining $\mathbf{b}_0$ and $\mathbf{b}_n$ $(C=|\mathbf{b}_n-\mathbf{b}_0|)$, and let h the arc length of the curve. Then the inequality

$$L\geq h\geq C$$

holds. Equality can occur only if the control points are collinear.

Theorem 3

The arc length of the family of Bézier curves defined by control points given by eqn (4) has exactly one minimum.

Proof. First we show that a minimum exists. Using the notation of Theorem 2 $L(\nu)$ and $C(\nu)$ show the following properties:

- $\lim_{\nu\to\pm\infty}L(\nu)=\lim_{\nu\to\pm\infty}C(\nu)=\infty$;
- $L(\nu)$ has a global minimum at $\nu_L=0$ moreover
  - $L(\nu)$ is strictly increasing if $\nu\in[\nu_L,\infty)$ and
  - $L(\nu)$ is strictly decreasing if $\nu\in(-\infty\nu_L]$;
- $C(\nu)$ has a global minimum at $\nu_C=-l_0-l_1$ moreover
  - $C(\nu)$ is strictly increasing if $\nu\in[\nu_C,\infty)$ and
  - $C(\nu)$ is strictly decreasing if $\nu\in(-\infty,\nu_C]$.

According to Theorem 2 the inequality

$$L(\nu)\geq h(\nu)\geq C(\nu)$$

holds. Combining this with the above properties of $L(\nu)$ and $C(\nu)$ it follows that $h(\nu)$, i.e. the arc length, has a minimum.

Now we prove that there can only be one minimum of $h(\nu)$. In order to show this we examine the function $S(\nu)=h(\nu)/3$.

The functions $\sqrt{f(\nu,t)}$ and $(\partial/\partial\nu)\sqrt{f(\nu,t)}$ are continuous on the rectangular domain $t\in[0,1]$, $\nu\in[-\nu_0,\nu_0]$, $\forall\nu_0\in\Re$, thus $s(\nu)$ is differentiable with respect to ν and

$$\dot{s}(\nu)=\frac{\mathrm{d}}{\mathrm{d}\nu}s(\nu)=\int_0^1\frac{\partial}{\partial\nu}\sqrt{f(\nu,t)}\mathrm{d}t.$$

To prove our theorem it is enough to show that $\dot{s}(\nu)$ can have at most one root, for which it is enough to prove that

$$\dot{s}(\nu)=\int_0^1\frac{\partial}{\partial\nu}\sqrt{f(\nu,t)}\mathrm{d}t=\int_0^1\frac{\frac{\partial}{\partial\nu}f(\nu,t)}{2\sqrt{f(\nu,t)}}\mathrm{d}t \qquad (6)$$

is a strictly monotonic function.

If the integrand in eqn (6) is strictly monotonic in ν then $\dot{s}(\nu)$ is strictly monotonic. The integrand in eqn (6) is strictly

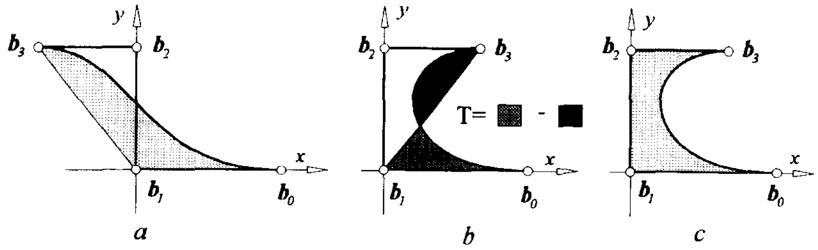

Figure 6 Area of plane figures determined by the origin and Bézier curves of eqn (4)

monotonic in ν if its derivative with respect to ν is either positive or negative $\forall\nu$ except a finite set of t values, i.e. with the notation

$$s_1(\nu)=\frac{\partial}{\partial\nu}\frac{\frac{\partial}{\partial\nu}f(\nu,t)}{2\sqrt{f(\nu,t)}}=\frac{2f(\nu,t)\frac{\partial^2}{\partial\nu^2}f(\nu,t)-\left(\frac{\partial}{\partial\nu}f(\nu,t)\right)^2}{4(f(\nu,t))^{3/2}}$$

either $s_1(\nu)>0$ or $s_1(\nu)<0$, $\forall\nu$.

$$\frac{\partial}{\partial\nu}f(\nu,t)=8\nu t^2(1-t)^2+4(l_1t^3(1-t)+l_0t(1-t)^3),$$

$$\frac{\partial^2}{\partial\nu^2}f(\nu,t)=8t^2(1-t)^2$$

after substitution and simplification the numerator of $s_1(\nu)$ becomes

$$64m^2t^4(1-t)^4$$

which is positive except for $t=0$ and $t=1$. The same property holds for the denominator of $\dot{s}(\nu)$, therefore $s(\nu)$ is strictly increasing, thus $s(\nu)$ has exactly one minimum. The same holds for $h(\nu)$ defined by eqn (5), i.e. for the arc length. ♦

The member of the family of curves given by eqn (4) of minimum arc length can usually only be determined by means of numerical procedures. $l_0=l_1$ is an exception since in this case $\nu_L=\nu_C=0$ thus $h(\nu)$ has a minimum at $\nu=0$.

For the numerical root finding the values ν_L and ν_C of Theorem 3 provide suitable initial values. For any given arc length $s>h_{\min}$ there are two computable ν_s values for which $h(\nu_s)=s$. In other terms, $\forall s>h_{\min}$ there are two cubic Bézier curves with the control points given by eqn (4) the arc length of which is s.

Prescribed area

By means of the free parameter of eqn (4) we can produce a cubic Bézier curve which encloses, with possibly other lines, a plane figure of given area. For this purpose we need an explicit formula relating the area to Bézier curves. The basic idea of the following theorem can be found in Ref. 13, so we omit its proof.

Theorem 4

The signed area of the sector bounded by the plane Bézier curve $\mathbf{b}(t)=\Sigma_{i=0}^{n}\mathbf{b}_iB_i^n(t)$, $t\in[0,1]$ and the straight line segments joining the end-points $\mathbf{b}_0$ and $\mathbf{b}_n$ of the curve with the origin is

$$T=\frac{n}{2(2n-1)}\sum_{i=0}^{n-1}\sum_{j=i+1}^{n}\alpha_{ij}^n(\mathbf{b}_i\wedge\mathbf{b}_j),$$

where

$$\alpha_{ij}^n=\frac{\binom{n-1}{i}\binom{n-1}{j-1}-\binom{n-1}{j}\binom{n-1}{i-1}}{\binom{2(n-1)}{i+j-1}}.$$

Further on we need the case $n=3$ when

$$\alpha_{01}^3=1,\ \alpha_{02}^3=\tfrac{1}{2},\ \alpha_{03}^3=\tfrac{1}{6},\ \alpha_{12}^3=\tfrac{1}{2},\ \alpha_{13}^3=\tfrac{1}{2}\text{ and }\alpha_{23}^3=1.$$

Theorem 5

The signed area of the sector bounded by the Bézier curve given by the control points in eqn (4) and the straight line segments joining its end-points with the origin is

$$T=\frac{m}{10}(-2l_0-3l_1)\ \forall\nu\in\Re. \tag{7}$$

Proof. First we calculate the signed area of the case $\nu=0$ using Theorem 4. This area is

$$T=\frac{3}{10}\sum_{i=0}^{2}\sum_{j=i+1}^{3}\alpha_{ij}^3(\mathbf{b}_i\wedge\mathbf{b}_j)=\frac{m}{10}(-2l_0-3l_1).$$

This area is illustrated in *Figure 6a* and *Figure 6b*.

As we have shown in Theorem 1 the $\nu\neq0$ general case can be obtained from the $\nu=0$ special case by means of a shearing transformation. Taking into account the area preserving property of shearing transformation our statement is proven. ♦

In applications, in general, we need the area of a closed region in plane bounded by a Bézier curve (or curves) and some other lines in the plane. These areas can be determined as a sum of certain plane figures' area and area(s) of the type Theorem 4. The area of plane figure in *Figure 6c* is $m(l_1+l_0)/5$ which is the sum of the signed area of the Bézier curve in the sense of Theorem 5 and the area of the triangle $\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3$.

Prescribed tangent direction at the inflection point

Bézier curves with the control points given in eqn (4) have exactly one inflection point if $l_0l_1>0$. Our objective is to choose the value of the free parameter ν in (4) to ensure the parallelism of the tangent line at the inflection point to a given direction.

For this purpose we first determine the tangent line at the inflection point in case of $\nu=0$ then utilising the inflection point preserving nature of affinity we determine the ν value which corresponds to the prescribed tangent direction by means of the affinity of Theorem 1.

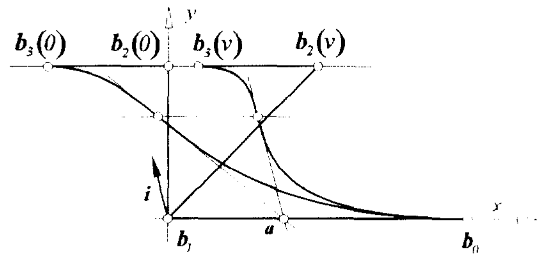

Figure 7 Prescribed tangent direction at the inflection point

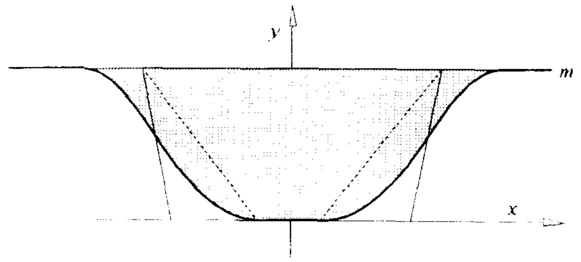

Figure 8 A bend of prescribed area

A necessary condition for the existence of the inflection point is the vanishing of the numerator of eqn (1), i.e. $\dot{\mathbf{b}}(t) \wedge \ddot{\mathbf{b}}(t) = 0$ for some $t \in (0, 1)$. This condition leads to the second order equation

$$(l_0 - l_1)t^2 - 2l_0 t + l_0 = 0. \tag{8}$$

(i) If $l_0 = l_1$ eqn (8) becomes linear, the solution is $t = 0.5$ and the inflection point is $\mathbf{b}(1/2) = 1/2[0\,m]^{\mathrm{T}}$. The derivative at the inflection point is $\dot{\mathbf{b}}(1/2) = 3/2[l\,m]^{\mathrm{T}}$ with the notations $l_0 = l_1 = l$, and the tangent line intersects the x axis, i.e. the axis of affinity of Theorem 1, at the point $\mathbf{a} = 1/2[-l\,0]^{\mathrm{T}}$.

If $\nu \neq 0$ the point of the curve at $t = 0.5$ is $\mathbf{b}(1/2) = 1/2[\nu\,m]^{\mathrm{T}}$ and the direction vector of the line joining this point with $\mathbf{a}$ is $\mathbf{b}(1/2) - \mathbf{a} = 1/2[\nu + l\,m]^{\mathrm{T}}$.

If an arbitrarily chosen direction vector $\mathbf{i} = [i_x\,i_y]^{\mathrm{T}}, i_y \neq 0$ is given the corresponding ν value $\nu = m i_x/i_y - l$.

(ii) If $l_0 \neq l_1$ the discriminant of eqn (8) is $4l_0l_1$ which is always positive because of the condition $l_0l_1 > 0$, thus there are two distinct roots of the equation. From these we need the one in the range (0,1) which always exists due to the shape of the control polygon.

The roots of the equation are $t_{1,2} = (l_0 \pm \sqrt{l_0l_1})/(l_0 - l_1)$. The cotangent of the angle between the x axis and the tangent line at any point of the curve is $\dot{b}_x(t)/\dot{b}_y(t) = \nu/m + (l_1 t/(1-t) + l_0(1-t)/t)2m$.

Denoting the root of eqn (8) which is in (0,1) by t_0, for an arbitrarily chosen direction $\mathbf{i} = [i_x\,i_y]^{\mathrm{T}}, i_y \neq 0$ the corresponding ν value is

$$\nu = \frac{i_x}{i_y}m - \frac{1}{2}\left(l_1\frac{t_0}{1-t_0} + l_0\frac{1-t_0}{t_0}\right).$$

This general case is illustrated in *Figure 7*.

Applications

Based on our previous results we can produce such cubic transition curves to two parallel lines that satisfy the condition $|\kappa_0|/|\kappa_1| = l_1^2/l_0^2$ and a further requirement in connection with its arc length, area or tangent line at the inflection point. Here we list a few examples of such cubic blends.

(1) Find a C^1 cubic blend of prescribed arc length to two parallel lines at the distance m. (Given: m, l_0, l_1 and the arc length.)
(2) Find a G^1 cubic blend of prescribed arc length and non-zero curvatures at the end-points to two parallel lines at the distance m. (Given: $m, \kappa_0 \neq 0$, $\kappa_1 \neq 0$ and the arc length.)
(3) Find a C^1 cubic blend with prescribed tangent direction at its inflection point to two parallel lines at the distance m. (Given: m, l_0, l_1 $(l_0l_1 > 0)$ and the angle.)
(4) Find a G^1 cubic blend of prescribed non-zero curvatures at its end-points and tangent direction at its inflection point to two parallel lines at the distance m. (Given: m, κ_0, κ_1 $(\kappa_0\kappa_1 < 0)$ and the angle.)
(5) Bend a wire or a sheet to the depth m to get a symmetric plane figure of a prescribed area; see *Figure 8*. (Given: m, l_0, l_1 $(l_0l_1 > 0)$ and the area; or m, κ_0, κ_1 $(\kappa_0\kappa_1 < 0)$ and the area.)

VANISHING CURVATURE AT ONE OF THE END-POINTS

Since the role of the two end-points is interchangeable, we can assume without loss of generality that $\kappa_1 = 0$, thus $m_1 = 0$. Therefore the control points $\mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b}_3$ are collinear and

$$\mathbf{b}_0 = \begin{bmatrix} -l_0 \\ 0 \end{bmatrix},\ \mathbf{b}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \mathbf{b}_2 = \begin{bmatrix} m_0\dfrac{e_{1x}}{e_{1y}} \\ m_0 \end{bmatrix},$$

$$\mathbf{b}_3 = \begin{bmatrix} m_0\dfrac{e_{1x}}{e_{1y}} + \dfrac{e_{1x}}{3} \\ m_0 + \dfrac{e_{1y}}{3} \end{bmatrix}.$$

These control points and the corresponding Bézier curves are illustrated in *Figure 9*.

This special case of our original problem can also be used to solve blending problems. The curvature at $\mathbf{b}_3$ is independent of l_1, i.e. there is a G^2 continuity between the line $\mathbf{b}_3, \mathbf{b}_2$ and the Bézier curve $\forall l_1 \neq 0$. Therefore, if we require only G^2 continuity at $\mathbf{b}_3$, l_1 becomes a free parameter which can be used to fulfil additional conditions. For our further studies it is practical to describe control points by means of the angle between the positive x axis and the tangent $\mathbf{e}_1$; see *Figure 9*. Thus we obtain the representation

$$\mathbf{b}_0 = \begin{bmatrix} -l_0 \\ 0 \end{bmatrix},\ \mathbf{b}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \mathbf{b}_2 = \begin{bmatrix} m_0\cot\alpha \\ m_0 \end{bmatrix},$$

$$\mathbf{b}_3 = \begin{bmatrix} m_0\cot\alpha + l_1\cos\alpha \\ m_0 + l_1\sin\alpha \end{bmatrix} \tag{9}$$

Arc length condition

The free parameter l_1 can be used to produce a cubic curve of prescribed arc length.

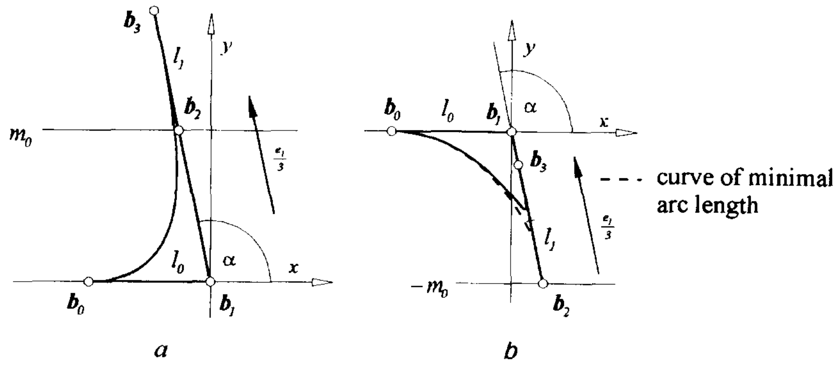

Figure 9 Zero curvature at $\mathbf{b}_3$

Theorem 6

The arc length of the family of cubics of control points (9) and parameter $0 \le l_1 \in \Re$ has exactly one minimum.

Proof. The proof is analogous to that of Theorem 3. By means of the control polygon and the de Casteljau algorithm it can be proved that the arc length has a minimum. For the arc length calculation we need the function

$$\begin{aligned} f(l_1,t) &= l_0^2(1-t)^2 + l_1^2 t^4 \\ &+ \left(4\frac{m_0^2}{\sin^2\alpha} + 2l_0 l_1\cos\alpha\right)t^2(1-t)^2 \\ &+ 4l_0 m_0\frac{\cos\alpha}{\sin\alpha}t(1-t)^3 \\ &+ 4\frac{m_0 l_1}{\sin\alpha}t^3(1-t),\ 0 < l_1 \in \Re,\ t \in [0,1] \end{aligned}$$

from which we can derive the function

$$\begin{aligned} s_1(l_1) &= \frac{\partial}{\partial l_1}\frac{\frac{\partial}{\partial l_1}f(l_1,t)}{2\sqrt{f(l_1,t)}} \\ &= \frac{2f(l_1,t)\frac{\partial^2}{\partial l_1^2}f(l_1,t) - \left(\frac{\partial}{\partial l_1}f(l_1,t)\right)^2}{4(f(l_1,t))^{3/2}} \end{aligned}$$

which is a positive function since

$$\frac{\partial}{\partial l_1}f(l_1,t) = 2l_1 t^4 + 2l_0\cos\alpha t^2(1-t)^2 + \frac{4m_0}{\sin\alpha}t^3(1-t),$$

$$\frac{\partial^2}{\partial l_1^2}f(l_1,t) = 2t^4$$

thus

$$s_1(l_1) = \frac{l_0^2\sin^2\alpha t^4(1-t^4)}{(f(l_1,t))^{3/2}}$$

which is positive except when $t=0$ and $t=1$.♦

There is no exact form for the calculation of this minimum but a consequence of this theorem is that for any arc length which is greater than the minimum there are at most two elements of the family with the specified arc length. In the case of *Figure 9a* the arc length of the curves $\mathbf{b}(l_1,t)$ is strictly increasing and their minimum is at the minimum of l_1, whereas in the case of *Figure 9b* the minimum of the arc length is not at the minimum of l_1. In the latter case there can be two members of the family with the specified arc length, and the corresponding l_1 values can be numerically computed.

Area condition

The signed area of the sector bounded by the Bézier curves of the control points given in eqn (9) and the straight line segments joining the end-points of the curve with the origin is

$$T(l_1) = \frac{-l_0}{10}\left(2m_0 + \frac{l_1}{2}\sin\alpha\right)$$

cf. Theorem 4. Utilising this, one can determine the value of the parameter l_1 which corresponds to an arbitrarily given area.

BOTH END-CURVATURES VANISH

If $\kappa_0 = \kappa_1 = 0$ then $m_0 = m_1 = 0$, i.e. the control points $\mathbf{b}_1$ and $\mathbf{b}_2$ of the Bézier curve coincide. Thus the coordinates of

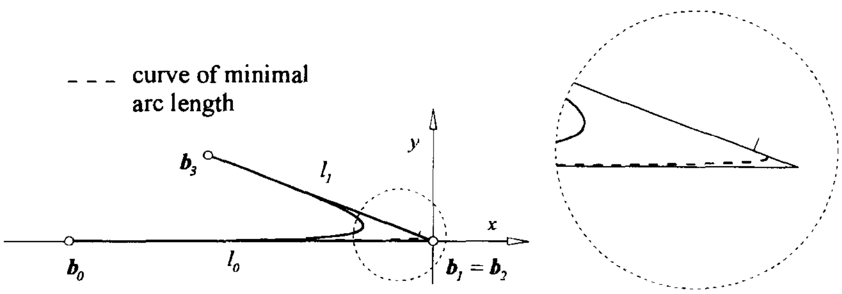

Figure 10 Zero curvature at both end-points

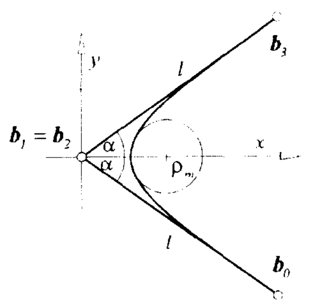

Figure 11 Zero curvature at both end-points, symmetric case

the control points are

$$\mathbf{b}_0 = \begin{bmatrix} -l_0 \\ 0 \end{bmatrix}, \mathbf{b}_1 = \mathbf{b}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} \frac{e_{1x}}{3} \\ \frac{e_{1y}}{3} \end{bmatrix}.$$

This type of Bézier curve enables us to construct curvature continuous blending curves to two intersecting straight lines. Since curvatures at the end-points are independent of the length $l_0 \neq 0$ and $l_1 \neq 0$ we have two free parameters to fulfil further requirements if G^2 continuity is satisfactory at the junctions.

For the sake of simplicity it is practical to describe the control points by means of the angle between the positive x axis and the tangent $\mathbf{e}_1$; see *Figure 10*.

$$\mathbf{b}_0 = \begin{bmatrix} -l_0 \\ 0 \end{bmatrix}, \mathbf{b}_1 = \mathbf{b}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} l_1\cos\alpha \\ l_1\sin\alpha \end{bmatrix}. \qquad (10)$$

Arc length condition

Fixing one of the free parameters, arc length conditions can also be satisfied since control points given by eqn (10) are the $m_0 = 0$ special case of control points given by eqn (9). The element with minimum arc length of the family of curves can only be determined numerically. However its existence enables us to find such a value of the parameter l_1 for which the arc length of the corresponding curve is a prescribed value which is greater or equal to the minimum arc length. There are at most two of such l_1 values. The role of l_0 and l_1 is interchangeable.

Area condition

The signed area of the sector bounded by the Bézier curves given by eqn (10) and the straight line segments joining its end-points with the origin is

$$T(l_0, l_1) = \frac{-l_0 l_1 \sin\alpha}{20}$$

cf. Theorem 4. This area is a function in l_0 and l_1. These parameters can be utilised to find such elements of the family which meet prescribed area requirements.

Symmetric case ($l_0 = l_1$)
The $l_0 = l_1$ special case is worth dealing with not only because of the simpler formulas but for their frequent applications in blending.

For the sake of simplification we rotate the control polygon according to *Figure 11* and we apply the notation $l = l_0 = l_1$. The coordinates of control points become

$$\mathbf{b}_0 = \begin{bmatrix} l\cos\alpha \\ -l\sin\alpha \end{bmatrix}, \mathbf{b}_1 = \mathbf{b}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} l\cos\alpha \\ l\sin\alpha \end{bmatrix}. \qquad (11)$$

These control polygons and the corresponding Bézier curves can be mapped onto each other by means of similarity transformations, the centre of similitude of which is the origin. The arc length of these curves is

$$h(l) = 3l \int_0^1 \sqrt{(1-t)^4 + t^4 - 2\cos 2\alpha t^2 (1-t)^2}\, \mathrm{d}t$$

and the area in the sense of Theorem 4 is

$$T(l) = \frac{l^2 \sin 2\alpha}{20}.$$

By means of the free parameter l we can prescribe the maximum of the Bézier curve's curvature. The coordinate functions and derivatives of the Bézier curves given by eqn (11) are

$$b_x(t) = l\cos\alpha((1-t)^3 + t^3), \dot{b}_x(t) = 3l\cos\alpha(2t-1),$$

$$\ddot{b}_x(t) = 6l\cos\alpha,$$

$$b_y(t) = l\sin\alpha(t^3 - (1-t)^3), \dot{b}_y(t) = 3l\sin\alpha(t^2 + (1-t)^2),$$

$$\ddot{b}_y(t) = 6l\sin\alpha(2t-1).$$

Thus the curvature is $\kappa(t) = |\dot{\mathbf{b}}(t) \times \ddot{\mathbf{b}}(t)|/|\dot{\mathbf{b}}(t)|^3 = (2\sin 2\alpha/3l)t(1-t)/(n(t))^{3/2}$, where $n(t) = \cos^2\alpha(2t-1)^2 + \sin^2\alpha(t^2 + (1-t)^2)^2$. Owing to the symmetry, its maximum is at $t = \frac{1}{2}$, i.e. the maximum is $\kappa(1/2) = 8\cos\alpha/3l\sin^2\alpha$.

If the required minimum of radius of curvature ρ_m is given the corresponding unique l can be determined from the formula

$$l = \frac{8\rho_m \cos\alpha}{3\sin^2\alpha}$$

see *Figure 11*.

RESULTS AND CONCLUSIONS

We have presented a constructive solution to the problem of finding the cubic curve in plane specified by tangent vectors and signed curvatures at its end-points (the end-points are unknown). We have applied the Bézier representation of parametric cubics and we have differentiated two cases according to the relative position of the given tangent vectors.

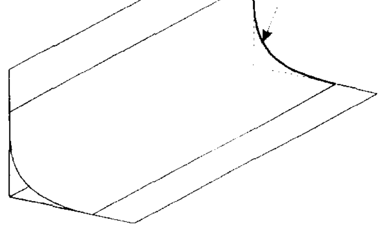

Figure 12 Blending surface defined by a profile curve

If the tangent vectors are parallel we gain a one-parameter family of cubics. This free parameter can be used to fulfil further conditions, just like arc length, area and tangent direction at inflection point. We have pointed out that these curves can be used for constraint-based curve blending with continuity C^1 or G^1.

If the tangent vectors at the end-points are not parallel, we have examined those special cases when either or both end-curvatures are zero. In these special cases we gain a family of curves if we ignore the length of those tangent vectors the curvature at which is zero. By means of this free parameter we can construct cubics that satisfy additional arc length, area or curvature conditions. These curves can be used for constraint-based curve blending of continuity G^2.

The resulted curves can not only be used for curve blending but also for edge blends [11] of parametric surfaces. By means of our results one can construct various profile curves (see *Figure 12*) subject to constraints, for parametric surface blending, especially for the blending of polyhedra.[14]

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