# STA2102 Midterm

Kevin Zhang 1002225264

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## Question 1

```
x <- scan("EV.txt", what=list(0,0))
a <- x[[1]]
theta <- x[[2]]
l <- max(a)
n <- length(a)
t <- numeric(0)
p <- numeric(n*l + 1)
phi_value <- numeric(n*l + 1)
# compute phi(t0) to phi(t_nl)
for(j in 0:(n*l)){
  phi_value_temp <- 1
  for(i in 1:n){
    temp <- 1-theta[i]+theta[i]*exp(-2*pi*complex(imaginary=1)*j*a[i]/(n*l+1))
    phi_value_temp <- phi_value_temp * temp
  }
  phi_value[j+1] <- phi_value_temp
}
# compute p(s=o) to p(s=nl)
for(s in 0:(n*l)){
  p_temp <- numeric(n*l + 1)
  for(j in 0:(n*l)){
    p_temp[j+1] <- phi_value[j+1]*exp(2*pi*complex(imaginary=1)*j*s/(n*l+1))
  }
  p[s+1] <- round(Re(sum(p_temp)/(n*l+1)), 5)
}
p1 <- sum(p[(270+1):(300+1)])
p2 <- p[269+1]
print(sprintf('The answer to part a is %f', p1))
```

```
## [1] "The answer to part a is 0.172250"
```

```
print(sprintf('The answer to part b is %f', p2))
```

```
## [1] "The answer to part b is 0.002800"
```

## Question 2

### a)

Case 1: $y < 0$

$$P(U_1 - U_2 \leq y) = \int_{-y}^{1} \int_{0}^{U_2+y} dU_1 dU_2 = \frac{1}{2}(1-y^2) + y(1-y)$$
$$\implies \frac{\partial P(Y \leq y)}{y} = -y + 1 + 2y = 1 + y = 1 - |y|$$

Case 2: $y \geq 0$

$$P(U_1 - U_2 \leq y) = \int_{0}^{1-y} \int_{0}^{U_2+y} dU_1 dU_2 + \int_{1-y}^{1} dU_2 = \frac{1}{2}(1-y)^2 + y(1-y) + y$$
$$\implies \frac{\partial P(Y \leq y)}{y} = y - 1 + 1 - 2y + 1 = 1 - y = 1 - |y|$$

Therefore, $g(y) = 1 - |y|$ for $y \in [-1, 1]$.

### b)

The problem is to find $M$ equal to the max of $\frac{f(x)}{g(x)}$ which is symmetric around 0, so we only look at the interval $[0, 1]$. At $x = 0$, this evaluate to $\frac{3}{4}$. At $x = 1$, by L'Hospital's rule, this evaluates to $\lim_{x \to 1} \frac{\frac{3x}{2}}{x} = \frac{3}{2}$. At $x \in (0, 1)$, we have

$$\frac{\partial}{\partial x}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{-3}{2}x(1-x) + \frac{3}{4}(1-x^2)}{(1-x)^2} = 0$$

This implies that $\frac{-3}{2}x(1-x) + \frac{3}{4}(1-x^2) = 0 \implies x^2 - 2x + 1 = 0 \implies x \in \{-1, 1\}$. This contradicts with $x \in (0, 1)$. Therefore, the max value is equal to $\frac{3}{2}$ at $x \in \{-1, 1\}$. Therefore, the acception probability is $\frac{1}{M} = \frac{2}{3}$.

## Question 3

### a)

First, by iterative method, $x_k = \sum_{i=0}^{k-1} B^i b + B^k x_0$. Notice that $B^2 = uv^\intercal uv^\intercal = qB$ where $q = v^\intercal u$. Therefore, $B^k x_0 = q^{k-1} x_0 \to 0$ as $k \to \infty$ since $|q| < 1$. Now

$$(I - B)\sum_{i=0}^{k-1} B^i = \sum_{i=0}^{k-1} B^i - \sum_{i=1}^{k} B^i = I - B^k$$
$$\implies \sum_{i=0}^{k-1} B^i = (I - B)^{-1}(I - B^k)$$
$$\sum_{i=0}^{k-1} B^i \to (I - B)^{-1} \text{ since } B^k \to 0 \text{ or } I - B^k \to I$$

Therefore, $x_k = \sum_{i=0}^{k-1} B^i b + B^k x_0 \to (I - B)^{-1} b = x^*$.

### b)

```
B <- 0.1 * matrix(c(-1, -1,  1, -2, -3,
                    -1,  1, -1,  3,  2,
                    -1,  3,  2,  1, -1,
```

```r
                      1,  2,  1, -1, -3,
                      1, -1,  3, -1,  2), nrow=5, byrow=TRUE)
eg <- eigen(B)
eg$values
```

```
## [1]  0.2820183+0.0983104i  0.2820183-0.0983104i -0.0568430+0.2437277i
## [4] -0.0568430-0.2437277i -0.1503507+0.0000000i
```

```r
max(Mod(eg$values))
```

```
## [1] 0.2986625
```

There are 5 different eigenvalues therefore the eigenspace is full rank (rank is 5). Therefore, $||Bx||_2 \leq |\lambda_{max}|$. Note that in this case $|\lambda_{max}| \approx 0.299 < 1$. Therefore, $||B||_2 < 1$ and subsequently $\sum_{i=0}^{k} B^i \to (I - B)^{-1}$ and $B^k \to 0$ which implies that the algorithm converges.