STA2102 Homework 3

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Question 1

a)

$$E((V^\intercal AV)^2) = \sum_{i=1} a_{ii}^2 E(V_i^4) = \sum_{i=1}^n \sum_{j=1}^n a_{ii}a_{jj}E(V_i^2)E(V_j^2) = \sum_{i=1}^n \sum_{j=1}^n a_{ii}a_{jj}$$

Therefore we need to minimize $E(V_i^4)$. Note that $E(V_i^4) = var(V_i^2) + (E(V_i^2))^2 = var(V_i^2) + 1$ is minimized if $var(V_i^2) = 0$ or equivalently V_i^2 is degenerate at some number k. Since $E(V_i^2) = 1$, we have $V_i^2 = 1$ a.s. which implies that $V_i \in \{-1, 1\}$ and $P(-1) = P(1) = 0.5$.

b)

By simple matrix-vector multiplication:

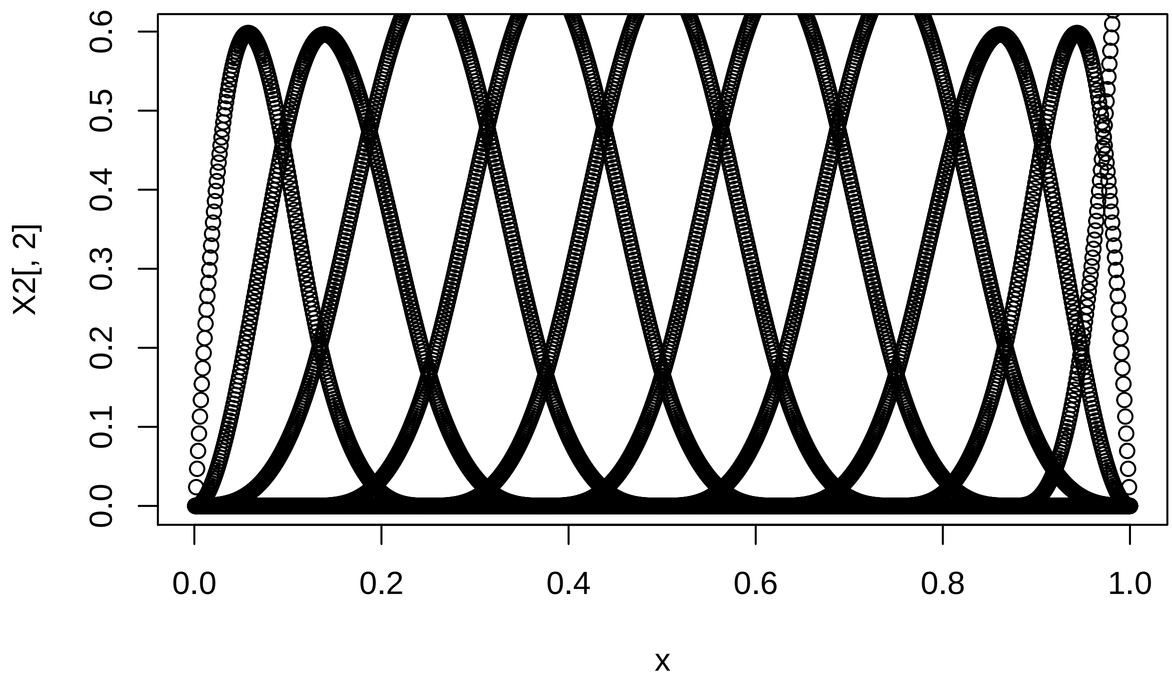
$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} V \\ 0 \end{pmatrix} = \begin{pmatrix} H_{11}V \\ H_{21}V \end{pmatrix}$$

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} H_{11}^{k-1}V \\ 0 \end{pmatrix} = \begin{pmatrix} H_{11}H_{11}^{k-1}V \\ H_{21}H_{11}^{k-1}V \end{pmatrix} = \begin{pmatrix} H_{11}^{k}V \\ H_{21}H_{11}^{k-1}V \end{pmatrix}$$

c)

```r
leverage <- function(x,w,r=10,m=100){
  set.seed(12345) # modification
  qrx <- qr(x)
  n <- nrow(x)
  lev <- NULL
  for(i in 1:m){
    v <- ifelse(runif(n)>0.5,1,-1)
    v[-w] <- 0
    v0 <- qr.fitted(qrx,v)
    f <- v0
    for(j in 2:r){
      v0[-w] <- 0
      v0 <- qr.fitted(qrx,v0)
      f <- f + v0/j
    }
    lev <- c(lev,sum(v*f))
  }
  std.err <- exp(-mean(lev))*sd(lev)/sqrt(m)
```

```
  lev <- 1 - exp(-mean(lev))
  r <- list(lev=lev,std.err=std.err)
  r
}
x <- c(1:1000)/1000
X1 <- 1
for (k in 1:5) X1 <- cbind(X1,cos(2*k*pi*x),sin(2*k*pi*x))
library(splines) # loads the library of functions to compute B-splines
X2 <- cbind(1,bs(x,df=10))
plot(x,X2[,2])
for (i in 3:11) points(x,X2[,i])
```




```
lev1 <- NULL
err1 <- NULL
lev2 <- NULL
err2 <- NULL
for(k in 1:20){
  w <- (1:1000)[x>(k-1)/20 & x<=k/20]
  r1 <- leverage(X1, w)
  r2 <- leverage(X2, w)
  lev1 <- c(lev1, r1$lev)
  err1 <- c(err1, r1$std.err)
  lev2 <- c(lev2, r2$lev)
  err2 <- c(err2, r2$std.err)
}
data.frame(k=1:20, lev1=lev1, err1=err1, lev2=lev2, err2=err2)
```

```
##     k      lev1       err1      lev2       err2
## 1   1 0.4557024 0.03512688 0.9053758 0.02397850
## 2   2 0.6255833 0.04694883 0.7383969 0.04520032
## 3   3 0.5466491 0.05074565 0.5297106 0.04988424
## 4   4 0.6075882 0.04494969 0.5343172 0.04370109
## 5   5 0.4886954 0.04142627 0.3843179 0.03714324
## 6   6 0.5386923 0.04842888 0.4631653 0.04637430
## 7   7 0.5661153 0.04418983 0.3870280 0.03653007
## 8   8 0.5504269 0.04154915 0.4886912 0.04113533
## 9   9 0.5311669 0.04251794 0.3542561 0.03366749
## 10 10 0.4996194 0.04430449 0.3906994 0.03982617
## 11 11 0.5472358 0.04278452 0.4380903 0.03997192
## 12 12 0.5968979 0.04247119 0.4113550 0.03634399
## 13 13 0.5333564 0.04110098 0.4698133 0.04037578
## 14 14 0.5214683 0.04524945 0.3461899 0.03505435
## 15 15 0.5797373 0.04701736 0.5003101 0.04637395
## 16 16 0.5031542 0.04407215 0.3992349 0.03893502
## 17 17 0.5023650 0.04708470 0.4392638 0.04615961
## 18 18 0.5070157 0.05292635 0.4937906 0.05141883
## 19 19 0.4929380 0.04013808 0.5942051 0.04291982
## 20 20 0.4767470 0.04558771 0.9405694 0.02090240
```

The estimates for the two models are not exactly close to each other. The standard error for model 2 seems to be smaller.

Question 2

a)

Let x_m denote the median then

$$\int_{-\infty}^{x_m} \frac{1}{\pi\sigma}\left(\frac{\sigma^2}{(x-\theta)^2+\sigma^2)}\right) dx = \int_{-\infty}^{x_m-\theta} \frac{\sigma}{\pi}\left(\frac{1}{s^2+\sigma^2)}\right) ds = \frac{1}{\pi} arctan \frac{s}{\sigma}\Big|_{-\infty}^{x_m-\theta} = 0.5$$

$$\implies \frac{1}{\pi} arctan \frac{x_m-\theta}{\sigma} + 0.5 = 0.5 \implies x_m = \theta$$

Similarly, let x_q denote the third quartile, then

$$\frac{1}{\pi} arctan \frac{s}{\sigma}\Big|_{0}^{x_r-\theta} = \frac{1}{4} \implies arctan \frac{x_r-\theta}{\sigma} = \frac{\pi}{4} \implies x_r = \theta + \sigma$$

By symmetry around θ, the IQR is 2σ.

b)

$$l(x,\theta,\sigma) = n\log\sigma - n\log\pi - \sum_{i=1}^{n}\log[(x_i-\theta)^2+\sigma^2]$$

$$\frac{\partial l}{\partial \theta} = \sum_{i=1}^{n}\frac{2(x_i-\theta)}{(x_i-\theta)^2+\sigma^2}$$

$$\frac{\partial l}{\partial \sigma} = \frac{n}{\sigma} - \sum_{i=1}^{n}\frac{2\sigma}{(x_i-\theta)^2+\sigma^2)}$$

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \theta^2} &= \sum_{i=1}^{n} \frac{2(x_i-\theta)^2 - 2\sigma^2}{((x_i-\theta)^2+\sigma^2)^2} \\
\frac{\partial^2 l}{\partial \sigma^2} &= \frac{-n}{\sigma^2} - \sum_{i=1}^{n} \frac{2((x_i-\theta)^2+\sigma^2) - 4\sigma^2}{((x_i-\theta)^2+\sigma^2)^2} \\
\frac{\partial^2 l}{\partial \theta \partial \sigma} &= \sum_{i=1}^{n} \frac{4\sigma(x_i-\theta)}{-((x_i-\theta)^2+\sigma^2)^2}
\end{aligned}
$$

```r
library(expm) # required to compute square root of a matrix
```

```
## Loading required package: Matrix
##
## Attaching package: 'expm'
## The following object is masked from 'package:Matrix':
##
##     expm
```

```r
newton_raph <- function(x, n_iter=1000){
  theta <- median(x)
  sigma <- IQR(x) / 2
  res <- c(theta, sigma)
  h1 <- function(x, theta, sigma){
    temp1 <- 0
    for(x_temp in x){
      temp1 <- temp1 + 2*(x_temp-theta)/((x_temp-theta)^2+sigma^2)
    }
    temp2 <- length(x)/sigma
    for(x_temp in x){
      temp2 <- temp2 - 2*sigma/((x_temp-theta)^2+sigma^2)
    }
    return(c(temp1, temp2))
  }
  h2 <- function(x, theta, sigma){
    temp11 <- 0
    for(x_temp in x){
      temp11 <- temp11 + (2*(x_temp-theta)^2-2*sigma^2)/((x_temp-theta)^2+sigma^2)^2
    }
    temp22 <- -length(x)/sigma^2
    for(x_temp in x){
      temp22 <- temp22 - (2*(x_temp-theta)^2-2*sigma^2)/((x_temp-theta)^2+sigma^2)^2
    }
    temp12 <- 0
    for(x_temp in x){
      temp12 <- temp12 - 4*sigma*(x_temp-theta)/((x_temp-theta)^2+sigma^2)^2
    }
    return(matrix(c(temp11, temp12, temp12, temp22), nrow=2))
  }
  for(i in 1:n_iter){
    h1_temp <- h1(x, res[1], res[2])
    h2_temp <- h2(x, res[1], res[2])
    res <- res - solve(h2_temp) %*% h1_temp
  }
  return(list(res=res, err=solve(sqrtm(h2(x, res[1], res[2])))))
```

```r
}
x <- rcauchy(n=1000, location=0,scale=1)
newton_raph(x)
```

```
## $res
##             [,1]
## [1,] -0.03210368
## [2,]  1.00834939
##
## $err
##                 [,1]            [,2]
## [1,] 0-4.588019e-02i 0+1.664797e-05i
## [2,] 0+1.664797e-05i 0-4.434834e-02i
```