STA2102 Homework 1

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Question 1

a)

$$\begin{aligned}
\frac{f(x)}{g(x)} &= \frac{b}{\sqrt{2\pi}(1-\Phi(b))} \exp\left[\frac{-\left(b+\frac{Y}{b}\right)^2}{2} + b\left(b+\frac{Y}{b}-b\right)\right] \\
&= \frac{b}{\sqrt{2\pi}(1-\Phi(b))} \exp\left[\frac{-b^2-2Y-\frac{Y^2}{b^2}}{2} + Y\right] \\
&= \frac{b}{\sqrt{2\pi}(1-\Phi(b))} \exp\left[\frac{-b^2}{2} - \frac{Y^2}{2b^2}\right] \\
\text{define } M &= \frac{b}{\sqrt{2\pi}(1-\Phi(b))} \exp\left[\frac{-b^2}{2}\right] \\
\text{then } \frac{f(x)}{Mg(x)} &= \exp\frac{-Y^2}{2b^2} \leq 1 \text{ since } e^{-x} \leq \ \forall\, x \geq 0
\end{aligned}$$

Now we accept $X = b + \frac{Y}{b}$ if $U \leq \frac{f(X)}{Mg(X)} = \exp\frac{-Y^2}{2b^2} \implies -2\ln U \geq \frac{Y^2}{b^2}$.

b)

$$\begin{aligned}
\frac{\partial}{\partial x}(\ln f(x) - \ln g(x)) &= \frac{\partial}{\partial x}\left(\frac{-x^2}{2} - \ln(\sqrt{2\pi}(1-\Phi(b))) - \ln(b) + b(x-b)\right) \\
&= -x + b = 0 \\
&\implies x = b \\
&\implies M = \frac{1}{\sqrt{2\pi}(1-\Phi(b))} e^{\frac{-b^2}{2}} \times \frac{1}{b} \\
&\implies P_{accept} = \frac{1}{M} = \frac{b\sqrt{2\pi}(1-\Phi(b))}{e^{\frac{-b^2}{2}}}
\end{aligned}$$

To evaluate the limit,

$$
\begin{aligned}
\lim_{b\to\infty} P_{accept} &= \lim_{b\to\infty} \frac{b\sqrt{2\pi}(1-\Phi(b))}{e^{\frac{-b^2}{2}}} \\
&\overset{H}{=} \lim_{b\to\infty} \frac{\sqrt{2\pi}(1-\Phi(b)) - b\sqrt{x\pi}\phi(b)}{-be^{-\frac{b^2}{2}}} \\
&= \lim_{b\to\infty} \frac{\frac{\sqrt{2\pi}(1-\Phi(b))}{b} - \sqrt{x\pi}\phi(b)}{-e^{-\frac{b^2}{2}}} \\
&\overset{H}{=} \lim_{b\to\infty} \frac{\frac{-\sqrt{2\pi}}{b}(1-\Phi(b)) - \frac{\sqrt{2\pi}\phi(b)}{b} - \sqrt{2\pi}\phi'(b)}{-be^{\frac{2b^2}{2}}} = 0
\end{aligned}
$$

Note that $\phi(b)$ is the density at b.

c)

$$
\begin{aligned}
\frac{\partial}{\partial\lambda}(\ln f(x) - \ln g_\lambda(x)) &= \frac{\partial}{\partial\lambda}\left(\frac{-x^2}{2} - \ln(\sqrt{2\pi}(1-\Phi(b))) + \lambda(x-b) - \ln\lambda\right) = 0 \\
x - b - \frac{1}{\lambda} &= 0 \\
\implies \lambda &= \frac{1}{x-b} \\
\implies \max_{x\geq b}\min_{\lambda>0}(\ln f(x) - \ln g_\lambda(x)) &= \max_{x\geq b}\left(\frac{-x^2}{2} - \ln(\sqrt{2\pi}(1-\Phi(b))) + \ln(x-b) + 1\right) \\
\implies \frac{\partial}{\partial x}&\left(\frac{-x^2}{2} - \ln(\sqrt{2\pi}(1-\Phi(b))) + \ln(x-b) + 1\right) = 0 \\
\implies -x + \frac{1}{x-b} &= 0 \\
\implies x^2 - bx - 1 &= 0 \\
\implies x = \frac{b+\sqrt{b^2+4}}{2} \text{ since } x \leq b &\implies \lambda = \frac{1}{\frac{b+\sqrt{b^2+4}}{2} - b} = \frac{2}{\sqrt{b^2+4} - b}
\end{aligned}
$$

Question 2

a)

Suppose there exists a unique minimizer for $f_\lambda(x) = \sum_{i=1}^n (y_i - \theta_i)^2 + \lambda\sum_{i=2}^{n-1}(\theta_{i+1} - 2\theta_i + \theta_{i-1]})^2$ and notice that $f_\lambda(x) \geq 0$ almost surely. If $\hat{\theta}_i = y_i$, then $f_\lambda(\hat{\theta}) = 0 + \lambda\sum_{i=2}^{n-1}(a(i+1)+b-2(ai+b)+a(i-1)+b)^2 = 0+0 = 0$. Therefore, $\hat{\theta} = y$ is the minimizer.

b)

$$
Y^* = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
\qquad
X = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & & \ddots & \vdots & \vdots & & & & & & \\
0 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & \sqrt{\lambda} & -2\sqrt{\lambda} & \sqrt{\lambda} & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0 & 0 & \sqrt{\lambda} & -2\sqrt{\lambda} & \sqrt{\lambda} & 0 & \cdots & 0 \\
\vdots & & & & & & & & \ddots & & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \sqrt{\lambda} & -2\sqrt{\lambda} & \sqrt{\lambda}
\end{pmatrix}
$$

c)

Let $\hat{\theta}^{(n)}$ and $\hat{\theta}^{(n+1)}$ denote the iterated values at nth and $(n+1)$th steps, respectively. Then $\hat{\theta}_{\bar{w}}^{(n)} = \hat{\theta}_{\bar{w}}^{(n+1)}$. Therefore,

$$||Y - X_{\bar{w}}\hat{\theta}_{\bar{w}}^{(n+1)} - X\hat{\theta}_{w}^{(n+1)}||^2 = ||Y - X_{\bar{w}}\hat{\theta}_{\bar{w}}^{n} - X\hat{\theta}_{w}^{(n+1)}||^2 \leq ||Y - X_{\bar{w}}\hat{\theta}_{\bar{w}}^{n} - X\hat{\theta}_{w}^{n}||^2$$

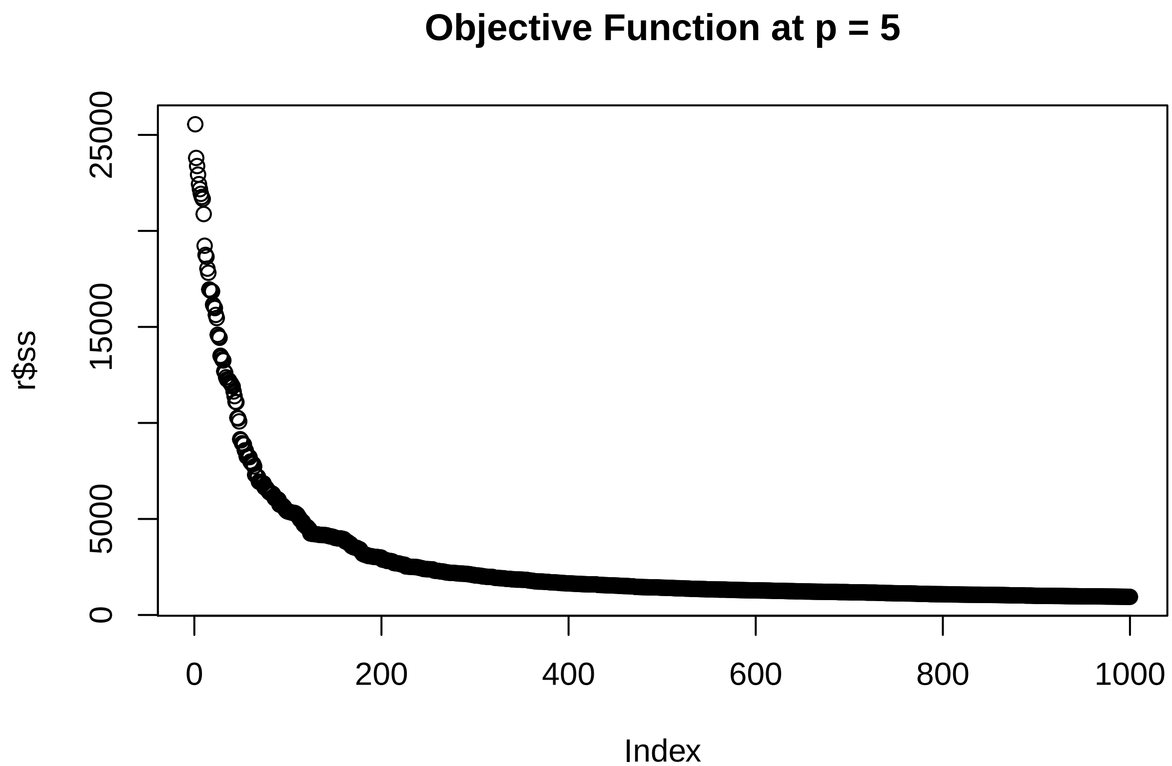
The last inequality holds because of the definition of $\hat{\theta}_{w}^{(n+1)}$ as the minimizer of the function

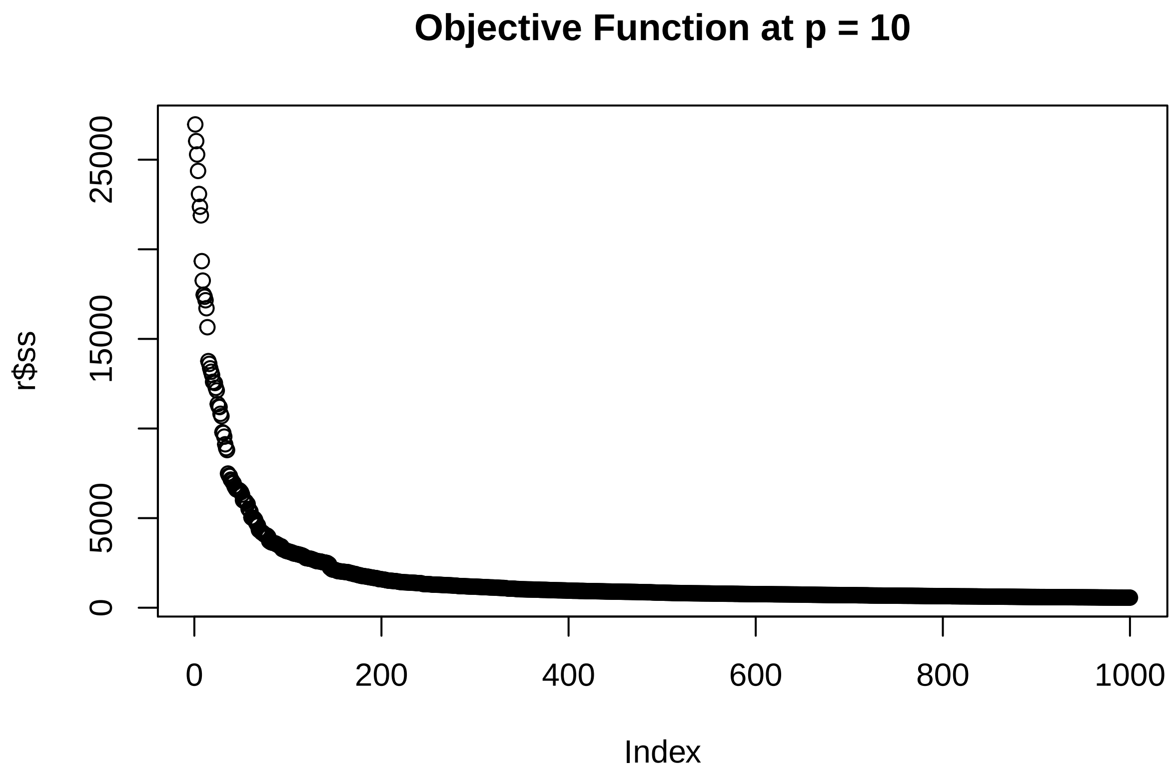
$$f_n(\theta_w) = ||Y - X_{\bar{w}}\hat{\theta}_{\bar{w}}^{n} - X\theta_w||^2$$

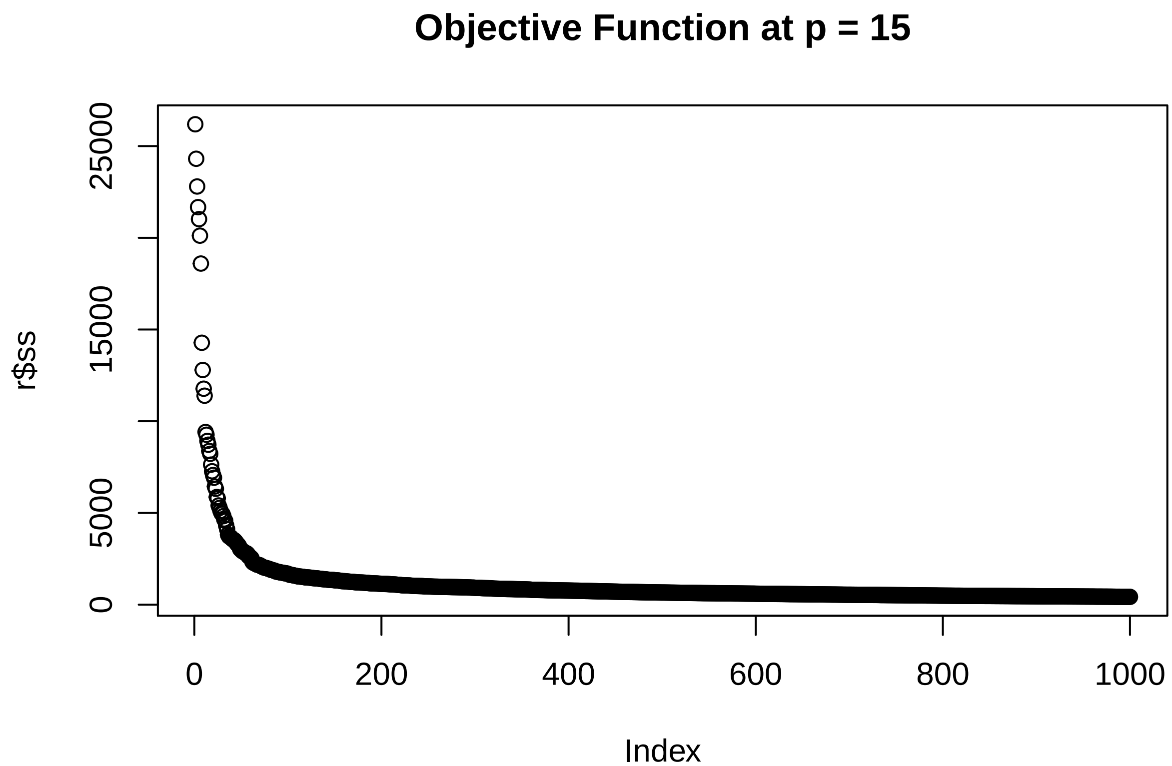
.

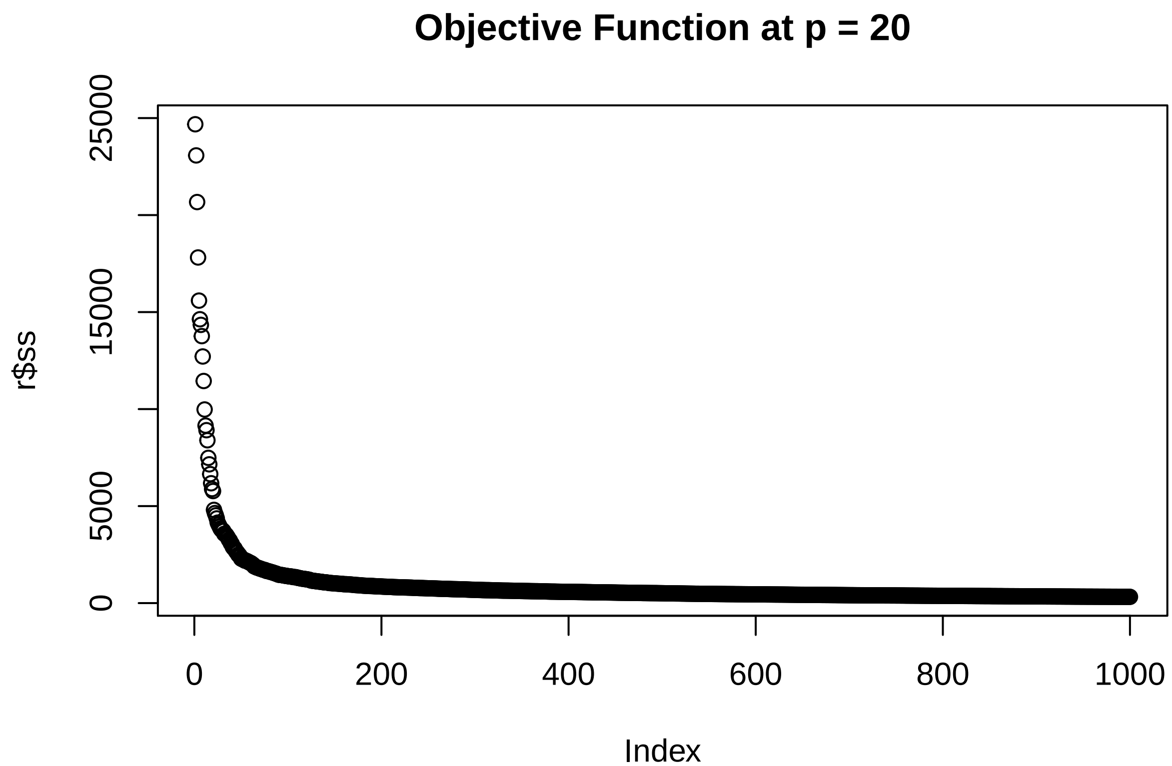
d)

```r
HP <- function(x, lambda, p=20, niter=200){
  n <- length(x)
  a <- c(1, -2, 1)
  aa <- c(a, rep(0, n - 2))
  aaa <- c(rep(aa, n - 3), a)
  mat <- matrix(aaa, ncol=n, byrow=T)
  mat <- rbind(diag(rep(1, n)), sqrt(lambda) * mat)
  xhat <- x
  x <- c(x, rep(0, n - 2))
  sumofsquares <- NULL
  for(i in 1:niter){
    w <- sort(sample(c(1:n), size=p))
    xx <- mat[,w]
    y <- x - mat[,-w] %*% xhat[-w]
    r <- lsfit(xx, y, intercept=F)
    xhat[w] <- r$coef
    sumofsquares <- c(sumofsquares, sum(r$residuals^2))
  }
  r <- list(xhat=xhat, ss=sumofsquares)
  r
}
x <- scan('yield.txt')
p <- seq(5, 50, 5)
for(p_temp in p){
  r <- HP(x,lambda=2000, p=p_temp, niter=1000)
  plot(r$ss, main=sprintf('Objective Function at p = %d', p_temp))
}
```

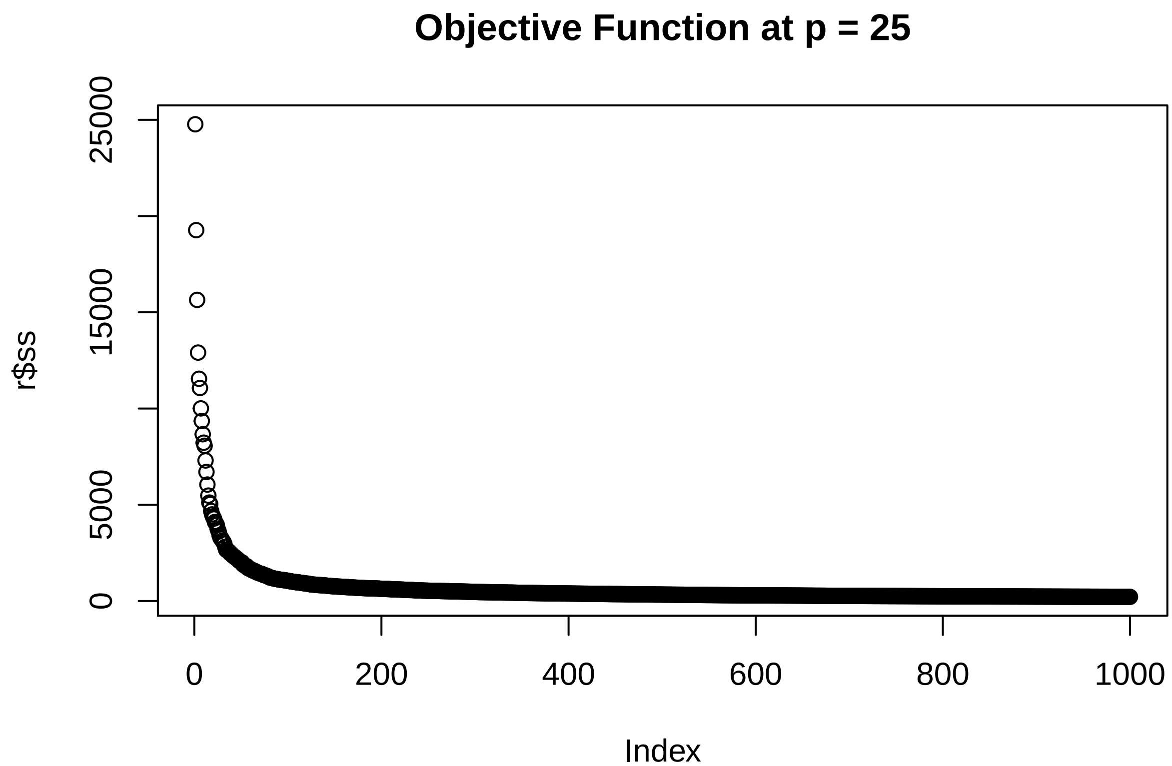

Objective Function at p = 5
r$ss
0
5000
15000
25000
0
200
400
600
800
1000
Index


Objective Function at p = 10
r$ss
0
5000
15000
25000
0
200
400
600
800
1000
Index

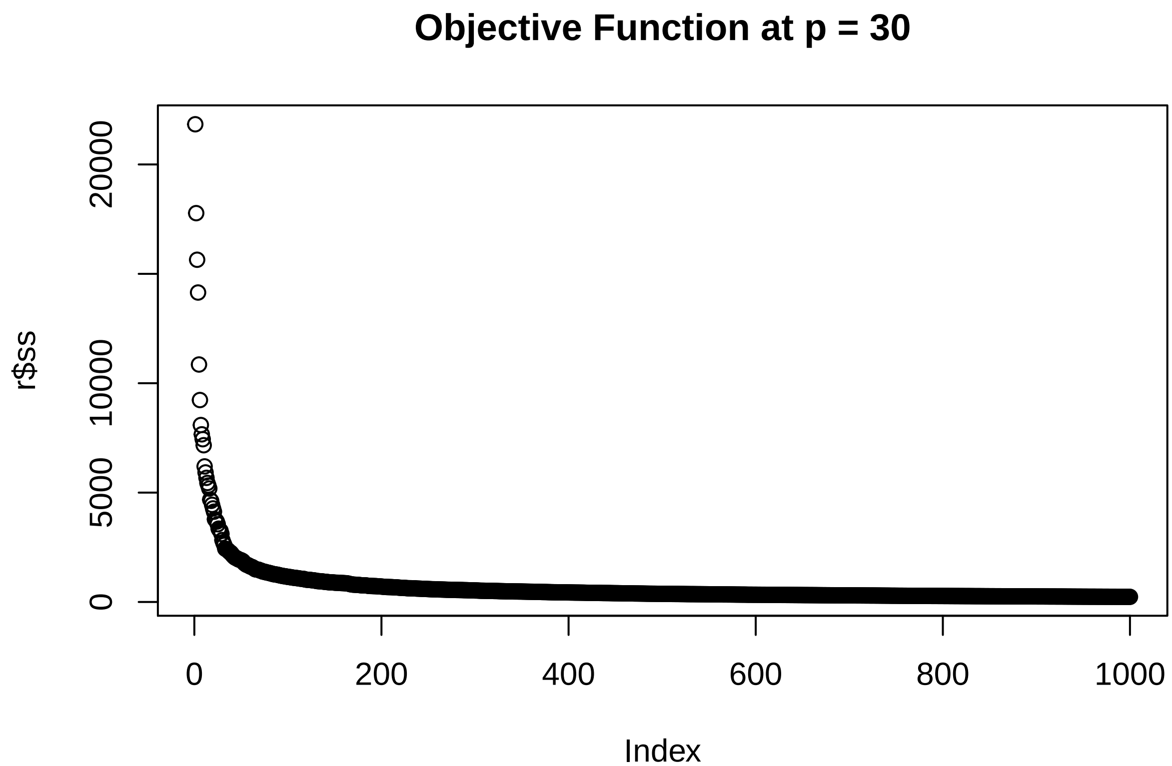

Objective Function at p = 15
r$ss
0
5000
15000
25000
0
200
400
600
800
1000
Index

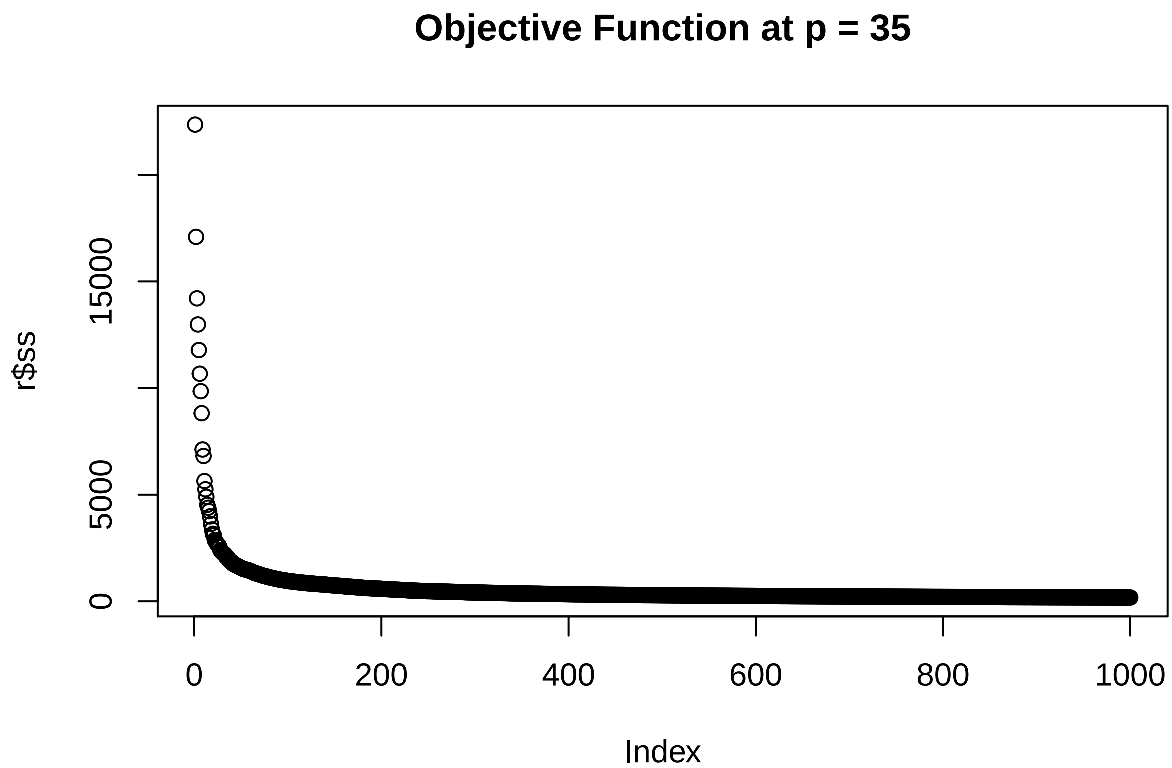


Objective Function at p = 20
r$ss
0
5000
15000
25000
0
200
400
600
800
1000
Index

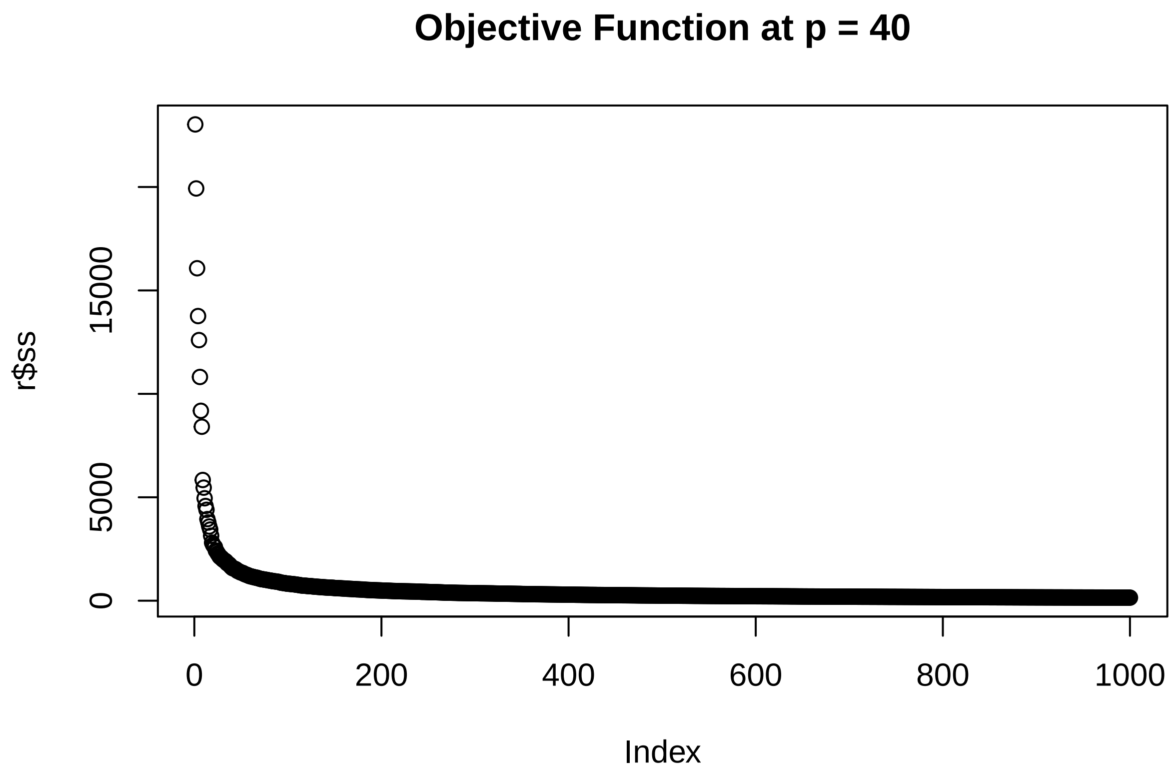


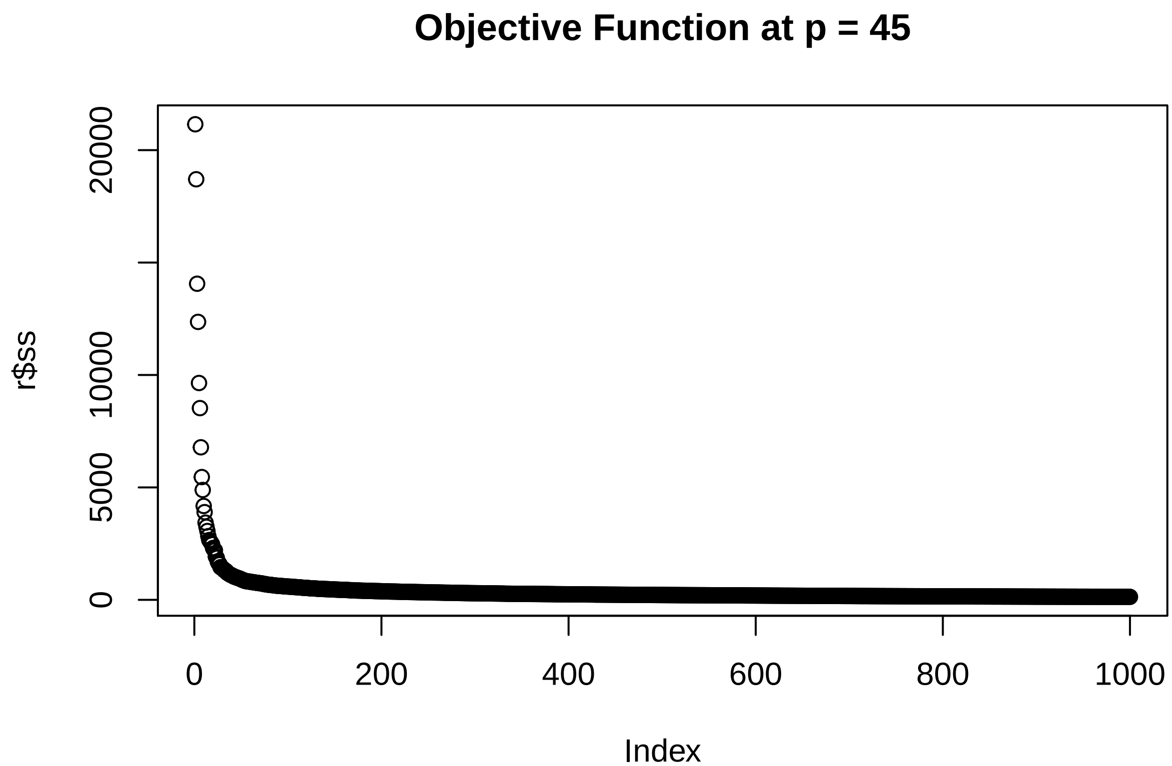
Objective Function at p = 25
r$ss
0
5000
15000
25000
0
200
400
600
800
1000
Index


Objective Function at p = 30
r$ss
0
5000
10000
20000
0
200
400
600
800
1000
Index

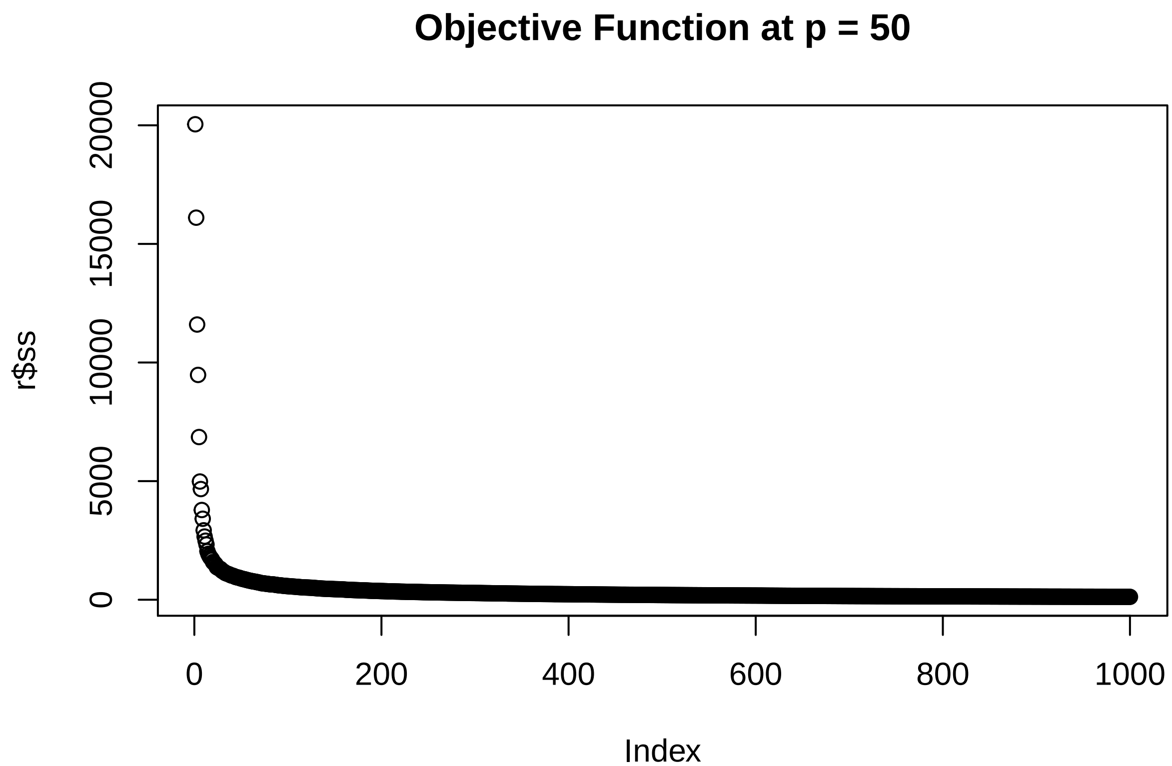


Objective Function at p = 35
r$ss
0
5000
15000
0
200
400
600
800
1000
Index


Objective Function at p = 40
r$ss
0
5000
15000
0
200
400
600
800
1000
Index



Objective Function at p = 45
r$ss
0
5000
10000
20000
0
200
400
600
800
1000
Index



As p increases, the rate of convergence gets faster.