Department of Economics - ESG-UQAM
Topics in Macroeconomics

Problem Set 4 - Incomplete Markets and Heterogeneous Agents Models

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Instructions:

Answer all questions. Show all steps and include any code or plots you use. You can use any programming language you want (Python, Julia, Matlab, R, etc.). Please submit your answers in a single PDF file. You can use Overleaf or any other LaTeX editor to write your answers.

Question 1 - Labor Supply and Consumption

Assume that an agent i has a utility function that depends positively on consumption, c^i, and negatively on labor supplied to work l^i.

$$u\left(c^i\right) - l^i.$$

Assume that the hourly wage is w, such that the agent's total labor income is wl^i . Moreover assume that this agent has an endowment A^i, such that his budget constraint is:

$$c^i = wl^i + A^i.$$

Solve:

$$\begin{aligned} & \max_{c^i, l^i} u\left(c^i\right) - l^i, \\ \text{s.t.} \quad c^i &= wl^i + A^i. \end{aligned}$$

Show that consumption does not depend on wealth, but that hours worked depend on wealth. Explain this result.

Question 2 - Two Periods Consumption and Saving

Using the same model as above. Now assume the agent lives for two periods, he consumes in the two periods, but can only work when he is young. Young agents can save an amount a_1, remunerated at an interest rate $R_1 < 1/\beta$. With obvious notations, the agent now maximizes:

$$\begin{aligned} & \max_{c_1, c_2, a_1, l_1} u\left(c_1\right) - l_1 + \beta u\left(c_2\right), \\ a_1 + c_1 &= wl_1, \\ c_2 &= R_1 a_1. \end{aligned}$$

a) Show that c_2 is determined by:

$$c_2 = u'^{-1}\left(\frac{1}{\beta R_1 w}\right).$$

How does c_2 evolve when w increases? Explain the intuition.

b) Find the expression for a_1. How does the saving rate a_1 evolve when w increases? Explain.

Question 3 - Two Periods Consumption and Saving with Unemployment Risk

Assume that the agent lives for two periods as before, but can work with a probability α (and is unemployed with a probability $1-\alpha$). When unemployed he earns a income δ.

His budget constraint in period 2 is:

$$\begin{aligned} c_2^{employ} &= wl_2 + R_1 a_1, \\ c_2^{unemploy} &= \delta + R_1 a_1. \end{aligned}$$

The program is now:

$$\begin{aligned} & \max_{c_1, c_2, a_1, l_1, l_2} u(c_1) - l_1 + \beta\left(\alpha\left[u\left(c_2^{employ}\right) - l_2\right] + (1-\alpha)\, u\left(c_2^{unemploy}\right)\right), \\ a_1 + c_1 &= wl_1, \\ c_2^{employ} &= wl_2 + R_1 a_1, \\ c_2^{unemploy} &= \delta + R_1 a_1. \end{aligned}$$

a) Solve this problem. Show that $u'(c_1) = u'\left(c_2^{employ}\right) = 1/w$.

b) Find the Euler equation. Find the value of a_1. Show that:

$$\frac{1}{w}\frac{1}{\beta R} = \frac{\alpha}{w} + (1-\alpha)\, u'(\delta + a_1 R_1).$$

c) Notice α measures the unemployment risk. How does the savings a_1 evolve with α? with R? Explain.

Question 4 - Huggett Economy with Two Income States

We consider an endowment economy. The economy is only composed of a unit mass of agents (no firm).

1. Risk: Each agent has labor income $\varepsilon \in \{\varepsilon_h, \varepsilon_l\}$. $\varepsilon_h > \varepsilon_l$.

$$\Pr[\varepsilon = \varepsilon_{s'} | \varepsilon = \varepsilon_s] = \pi_{ss''}.$$

2. Preferences: There is a continuum of mass 1 of agents. Each agent maximizes:

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}\right].$$

3. Assets and budget constraints: Denote as a the current beginning of period wealth. The wealth at the beginning of next period is denoted a' (think of it as a_{t+1}). The price today of one unit of wealth in the future is q. The budget constraint is:

$$c + qa' = a + \varepsilon.$$

To simplify, it is assumed that agents can not borrow (in the class we have seen: credit limit $-\bar{b}$):

$$a' \geq 0.$$

Denote $V_h(a)$ and $V_l(a)$ the value function for high income (h) and low income (l) agents respectively.

Now answer the following questions:

a) Write the Bellman equation for agents h and l.

b) What is the asset supply? What is the financial market equilibrium? Derive the equilibrium values of $a\prime$ and c_s $(s \in (h, l))$.

c) Denote λ_h and λ_l the Lagrange coefficient on the credit constraint in state h and l. Write the first order and envelop conditions in each case.

d) Write the two Euler equations.

e) Show that:

$$\frac{\lambda_h}{\beta \varepsilon_h^{-\sigma}} < \frac{\lambda_l}{\beta \varepsilon_l^{-\sigma}}.$$

What do you conclude?

f) What is the interest rate? How does it evolve when π_{hh} decreases? Why?

Question 5 - Hugget Economy

Consider the Huggett (1993) economy we have seen in class. In this model, the environment is characterized by an infinitely-lived measure of households with preferences given by:

$$\mathrm{E}_0 \sum_{t=0} \beta^t u(c_t),$$

where $u(c_t) = \dfrac{c_t^{1-\sigma}}{1-\sigma}$.

We will assume that agents face an idiosyncratic shock each period given by $e_t \in \mathcal{E}$. There is a probability to change the idiosyncratic shock in each period given by $\pi(e'|e)$. For simplicity we will assume that in each period t, $e_t \in \{e_h, e_l\}$. As in class, also assume that there is no aggregate uncertainty.

Households are faced with the constraints:

$$a' + c = e + (1+r)a,$$
$$a' \geq -b.$$

Notice that in this case the productivity idiosyncratic shock is going to determine the income of the agents in this model.

Now use the following parameter values: period length is two months, $\beta = 0.96$ (annual basis), coefficient of relative risk-aversion $\sigma = 1.5$, $e_h = 1, e_l = 0, 1$, $\pi(e_h|e_h) = 0.925$, and $\pi(e_h|e_l) = 0.5$. Consider a stationary equilibrium. Answer the following questions:

a) In this equilibrium what is average income, and what is the average duration of an unemployment spell? (You will need to calculate the unconditional stationary distribution of the associated Markov Process).

   Set the borrowing constraint equal to one year's average income. Construct an equally-spaced grid on asset holdings, with a maximum value equal to $3X$ average income. Let the number of grid points $N = 20$. Suppose the interest rate r $= 3.4\%$ (annual basis).

b) Solve for optimal decision rules across the grid.

c) Imagine one agent, who starts out with zero assets and the high endowment. Simulate the evolution of the agent's wealth, income and consumption for 10,000 periods, each period drawing an endowment according to the Markov process described above. Plot a histogram for asset holdings over this simulation and the income distribution of agents on this economy.

d) Suppose the net supply of assets is zero. What is the market clearing interest rate?

e) Suppose we were to increase the value for the risk-aversion coefficient, σ, from 1.5 to 3. What would happen to the equilibrium interest rate? Can you provide some intuition for this result?

Question 6 - Aiyagari Model with Labor Supply and Income Taxes

This problem set asks to compute an Aiyagari-style incomplete markets model as the one we have seen in class but now we have endogenous labor and income taxes. Here, small letters denote individuals choice and capital letters denote the aggregate ones. The setup of the model is as follows:

- There is a continuum of ex-ante identical agents who have preferences over consumption and leisure given by the following utility function:

$$\mathbb{E}_0 \sum \beta^t \left\{ \frac{\left(c_{i,t}^{\eta} l_{i,t}^{1-\eta} \right)^{1-\mu}}{1-\mu} \right\}.$$

  Households supply labor, save in a risk-free bonds subject to a debt limit. They also pay an income tax τ_y and receive transfers T_t. Let w_t and r_t denote the pre-tax wage rate and return on savings. The budget constraint of a typical households will be:

$$c_{i,t} + a_{i,t+1} \leq (1-\tau_y)\, e_{i,t} w_t \,(1-l_{i,t}) + T_t + (1 + (1-\tau_y)\, r_t)\, a_{i,t},$$

  where $a_{i,t+1} \geq -\underline{a}$, $l_{i,t} \leq 1$ and $c_{i,t} \geq 0$, the labor supply is $N_t = \int e_{i,t}(1-l_{i,t})$.

- The government budget constraint is:

$$G_t + T_t + (1-\tau_y)\, r_t B_t = B_{t+1} + \tau_y\,(w_t N_t + r_t A_t)\,.$$

  where G_t is government spending and B_t is debt.

- Technology: There is a representative firm that uses capital, K_t, and labor, N_t, to operate a CRS technology that produces output according to:

$$Y_t = F(K_t, N_t) = A K_t^{\theta} N_t^{1-\theta}.$$

  The above will pin down the rental rate $r_t = F_K - \delta$ and wage rate $w_t = F_N$.

- Asset markets: Agents can trade claims to one-period risk-free bonds, capital and government bonds.

$$A_t = K_t + B_t.$$

For your baseline calculations set $\beta = 0.98$, $\mu = 1.5$, and $\eta = 0.3$, $\tau_y = 0.4$, $\rho = 0.6$, $\sigma = 0.3$, $\delta = 0.075$, $\theta = 0.3$. It would be useful to set A such that the steady state $Y = 1$. You want to set transfers T_t suhc that aggregate transfers to output is roughly 10 % and G_t such that government expenses are 20 % of output. You can discretize the AR(1) process for skills using 5 states. Now answer the following questions:

a) Compute the stationary equilibrium for this economy.

b) Plot the consumption and savings functions as a function of individual assets for low, medium, and high skill agent.

c) Plot the supply and demand curves as a function of the interest rate and explain how to pin down the market clearing prices.

d) Plot the distribution of agents across states. Compute the mean, st, Gini coefficients and the Lorenz curve for wealth. In what respects do you think the model misses the wealth distribution in the data? Why?

e) Now lower the tax rate to 20 %. Make a table that describes how the aggregates and distributional moments change when you do this tax reform.