Department of Economics - ESG-UQAM
Topics in Macroeconomics

Problem Set 2 - Dynamic Programming and Value Function Iterations

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Instructions:

Answer all questions. Show all steps and include any code or plots you use. You can use any programming language you want (Python, Julia, Matlab, R, etc.). Please submit your answers in a single PDF file. You can use Overleaf or any other LaTeX editor to write your answers.

Question 1 - Dynamic Programming Discrete Environment

Consider an economy in which the representative consumer lives forever. There is a good in each period that can be consumed or saved as capital as well as labor. The consumer's utility function is

$$\sum_{t=0}^{\infty} \beta^t \left(\log c_t + \gamma \log x_t\right)$$

Here $\beta \in (0,1)$ and $\gamma \in (0,1)$. The consumer is endowed with 1 unit of labor in each period, some of which can be consumed as leisure, x_t, and some of which is supplied as labor, l_t. The consumer is also endowed with $\bar{k}_0$ units of capital in the first period. The feasible allocations satisfy:

$$c_t + k_{t+1} \leq \theta k_t^{\alpha} l_t^{1-\alpha},$$

where $\theta > 0$ and $0 < \alpha < 1$. Notice we also have the following constraints:

$$x_t + l_t \leq 1,$$

$$c_t, x_t, l_t, k_t \geq 0,$$

$$k_0 \leq \bar{k}_0.$$

a) Write down the **Bellman equation** for this problem.

b) **Guessing that the value function** $V(k)$ has the form $a_0 + a_1 \log k$ and that the policy function for labor $l(k)$ is constant, find **analytical solutions** for the value function $V(k)$ and the policy functions $c(k), x(k), l(k), k'(k)$.

c) Define a competitive equilibrium for this economy (either an Arrow-Debreu or a Sequential Market Equilibrium). Explain how would you be able to use the results in (b) to find the competitive equilibrium for this economy.

Question 2 - Dynamic Programming Discrete Environment

Consider a single agent problem where each period, w total output is produced and can be divided into consumption of a perishable good, c_t, and investment in a durable good, d_{xt}. The durable depreciates like a capital good, but is not directly productive. The stock of durables at any date, d_t, produces a flow of services that enters the utility function. Thus, the problem faced by the household with initial stock d_0 is:

$$\begin{gathered}\max_{\{(c_t,d_t,d_{xt})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[u_1(c_t) + u_2(d_t)\right] \\ \text{s.t. } c_t + d_{xt} \leq w \quad \forall t \\ d_{t+1} \leq (1-\delta)d_t + d_{xt} \quad \forall t \\ c_t, d_t, d_{xt} \geq 0 \quad \forall t \\ d_0 \text{ given}\end{gathered}$$

where both u_1 and u_2 are strictly increasing and continuous. Ignore the non-negativity constraints on d_{xt} while solving this problem.

a) State a condition on either u_1 or u_2 (or both) such that you can write an equivalent problem in the following form:

$$\begin{gathered}\max_{\{d_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(d_t, d_{t+1}) \\ \text{s.t. } d_{t+1} \in \Gamma(d_t) \qquad , \\ d_0 \text{ given}\end{gathered}$$

where $\Gamma(d_t) \in \mathbb{R}^+$. What is F? What is the correspondence Γ (i.e., this is the set of possible values where the variable d_{t+1} can be chosen)?

b) Write the **Bellman equation** for this problem.

c) State additional conditions on u_1 and u_2 such that the value function $v(d)$ you found previously is both strictly increasing and strictly concave. Prove or argument with words these two properties.

For the remaining questions, assume that both u_1 and u_2 satisfy the Inada conditions and are continuously differentiable.

d) State the **envelope condition** and the **F.O.C.** for the functional equation problem in (b).

e) Find the **Euler equation** of this problem.

f) Show that **there is a unique steady state** value of the stock, d^*, such that if $d_0 = d^*$, then $d_t = d^* \forall t$. Show that $d^* > 0$.

g) Show that the policy functions for the solution, $c^*(d)$ and $d' = g^*(d)$, are increasing (i.e., you can use the fact that the value function is concave to prove this result).

h) Show or argue that the system is **globally stable**, i.e., for any value $d > d^*$ the value of d decreases until it reaches d^* and for any value $d < d^*$, the value of d increases until it reaches d^*. You have to use the policy functions and assume that they are differentiable.

Question 3 - Dynamic Programming Discrete Environment

Assume that agents have the period utility function:

$$u(c_t - hc_{t-1}, l_t), \text{ with } h < 1$$

where l_t is labor and c_t denotes consumption. They are assumed to have a discount factor β and they understand now that consumption affects their habit. Budget constraint is (w is constant to simplify the problem) :

$$c_t + a_{t+1} = (1 + r_t)\, a_t + wl_t$$

a) Write the **Bellman equations** and derive the pricing kernel.

b) What is the **steady state interest rate**?

c) Assume that

$$u(c,l) = \frac{c^{1-\sigma}-1}{1-\sigma} - \frac{l^{1+\varepsilon}}{1+\varepsilon}$$

and that $a_t = 0$. How does the steady-state labor supply change when agents care more about habits?

Question 4 - Dynamic Programming Discrete Environment

Houses are durable goods from which households derive some utility. To model the demand for houses, a simple shortcut consists in introducing houses in the utility function. The goal of this exercise is to be able to use data on house prices and interest rates to derive properties of the demand for houses. Households thus derive utility from consumption and from having houses. The instantaneous utility function is $u(c_t, H_t)$ where H_t is the amount of housing. Households also have access to financial savings denoted b_t at period t, remunerated at a real rate r_t between period t and $t+1$. The program of the households is

$$\max_{\{c_t, H_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t, H_t)$$

$$c_t + b_t + P_t H_t = W_t + (1+r_{t-1}) b_{t-1} + P_t(1-\delta) H_{t-1}$$

where δ is the depreciation rate for houses.

a) Write the **transversality conditions** for this problem for the financial wealth and the stock of housing. Explain the intuition for these conditions.

b) Write the **Bellman equation** for this problem, with the value function denoted $V(b_{t-1}, H_{t-1})$.

c) Compute the **F.O.C.s**

d) Derive the **envelope conditions** and find the **two Euler equations.**

e) We assume that the utility function is

$$u(c_t, H_t) = (c_t^\rho + H_t^\rho)^{\frac{1}{\rho}} \text{ with } \rho < 1$$

Explain what is the economic meaning of the ρ coefficient. Using the two Euler equations, express $\frac{c_t}{H_t}$ as a function of P_t, P_{t+1} and r_t. How can we get ρ from the data?

Question 5 - Programming

For this question you need to use a computer. Feel free to use the language you are more comfortable with. Consider the optimal growth problem:

$$\begin{aligned} &\max \textstyle\sum_{t=0}^{\infty} (0.6)^t \log c_t \\ &\text{s.t. } c_t + k_{t+1} \leq 10 k_t^{0.4} \\ &\quad c_t, k_t \geq 0 \\ &\quad k_0 = \bar{k}_0 \end{aligned}$$

a) Write down the **Euler conditions** and the **transversality condition** for this problem. Calculate the steady state values of c and k.

b) Write down the functional equation that defines the value function for this problem. Guess that the value function has the form $a_0 + a_1 \log k$. Calculate the value function and the policy function. Verify that the policy function generates a path for capital that satisfies the Euler conditions and transversality condition in part (a).

c) Let capital take values for the discrete grid $(2, 4, 6, 8, 10)$. Make the original guess $V_0(k) = 0 \forall k$, and perform the first ten steps of the value function iteration below:

$$V_{i+1}(k) = \max \log \left(10k^{0.4} - k'\right) + 0.6V_i\left(k'\right)$$

Plot your results (value function in terms of the grid on capital and the policy functions for consumption and capital) for each of the steps and comment the pattern you can observe.

d) Using the same grid discrete grid $(2, 4, 6, 8, 10)$ plot the analytical result you obtained in part (b) and compare with the results obtained previously.

e) Now, perform the value function iterations until

$$\max_k |V_{i+1}(k) - V_i(k)| < 10^{-5}$$

Report the value function and the policy function that you obtain. Compare these results with what you obtained in (c) and with the analytical solution in (b).

f) Repeat part (e) for the grid of capital stocks (0.05, 0.10, .., 9.95, 10). What can you observe in this case?

Question 6 - Dynamic Programming Stochastic Environment

Consider the representative consumer has preferences given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right),$$

where $\beta \in (0, 1)$, c_t denotes consumption, $u(.)$ is strictly increasing, strictly concave and twice differentiable, and E_0 is the expectation operator conditional on information at $t = 0$. Note here that, in general, c_t will be random. The representative consumer has 1 unit of labor available in each period, which is supplied inelastically. The production technology is given by:

$$y_t = z_t F\left(k_t, n_t\right),$$

where $F(., .)$ is strictly quasiconcave, homogeneous of degree one, and increasing in both arguments. Here, k_t is the capital input, n_t is the labor input, and z_t is a random technology disturbance. That is, $\{z_t\}_{t=0}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables (each period z_t is an independent draw from a fixed probability distribution $G(z)$). In each period, the current realization, z_t, is learned at the beginning of the period, before decisions are made. The law of motion for the capital stock is:

$$k_{t+1} = i_t + (1 - \delta)k_t,$$

where i_t is investment and δ is the depreciation rate, with $0 < \delta < 1$. The resource constraint for this economy is:

$$c_t + i_t = y_t.$$

a) Explain in words how the **Competitive Equilibrium** is found in this economy, i.e., explain how it works the Competitive Equilibrium in an Arrow-Debreu structure and in a Sequential Market structure. You should be able to say what are the objects that are tradeable in this economy and the moments the negotiation happens.

b) Set up the **Social Planner's Problem**.

c) Write down the **Bellman equation** for this problem.

d) What are the objects you need to determine in the problem you wrote in part (c)?

e) For this part let $F\left(k_t, n_t\right) = k_t^{\alpha} n_t^{1-c}$, with $0 < \alpha < 1$, $u(c_t) = ln(c_t)$, $\delta = 1$, and $E[lnz_t] = \mu$. **Guessing that the value function** $v(k_t, z_t)$ has the form:

$$v(k_t, z_t) = A + Blnk_t + Dlnz_t,$$

find the **analytical solutions** for the value function $v(k_t, z_t)$ and the policy functions $c_t(k_t, z_t)$ and $k_{t+1}(k_t, z_t)$.

f) Does the economy described above converge to a steady state?

g) Explain in words how would you use the policy functions you obtained above to find the competitive equilibrium for this economy.

h) **Programming.** Using the policy functions you obtained in part (e) (i.e., $c_t(k_t, z_t)$ and $k_{t+1}(k_t, z_t)$) and assuming any initial value k_0, determine a sequence $\{z_t\}_{t=0}^{T}$ using a random number generator and fixing T. Use this sequence to find time series for consumption and investment. Observe you can also obtain time series for y_t.

i) What can you say about the $var(lnk_{t+1})$, $var(lnc_t)$, and $var(lny_t)$? Does this model fits the trend we observe in the data for consumption, investment, employment and output? If not, how could you modify this model such that it matches the moments found in the data?

Question 7 - Programming

Consider the following growth problem:

$$\max_{\{c_t, x_t, \ell_t\}} E \sum_{t=0}^{\infty} \beta^t \left\{\log\left(c_t\right) + \psi \log\left(\ell_t\right)\right\}$$

subject to:

$$c_t + x_t = k_t^{\theta} \left(z_t h_t\right)^{1-\theta}$$
$$k_{t+1} = (1-\delta)k_t + x_t$$
$$\log z_t = \rho \log z_{t-1} + \epsilon_t, \quad \epsilon \sim N\left(0, \sigma_\epsilon^2\right)$$
$$h_t + l_t = 1$$
$$c_t, x_t \geq 0 \quad \text{in all states}$$

a) Write the F.O.C. of the problem above with respect to k_{t+1} and h_t (i.e., hours worked).

b) Find the values of the steady state for capital and labor. Notice in this case it is possible to find analytical expressions for k_{ss} and h_{ss}, but this will not always be true. If you want you can use a numerical method to find those values (e.g., Newton method).

c) Use the value for the steady state for capital to build the grid points for the capital (i.e., $[0.25 * k_{ss}, 1.25 * k_{ss}]$, where k_{ss} denotes the capital in the steady state). Notice for the labor you can use a grid point between 0 and 1.

d) Set the parameters for you model, as well as the grid points and do a code where you solve the problem above by **Value Function Iteration.**

   Tip: Notice in this case the planner is choosing the following variables: c_t, k_{t+1} and h_t. It is easy to observe that c_t will be a function of (k_t, k_{t+1}, z_t, h_t). We can then separate the problem in two pieces: a function $log(c_t)$ which will be in the space (k_t, k_{t+1}, z_t, h_t) and a function $\phi log(1-ht)$ which will be in the space h_t. Different from the Value Function Iteration we had seen (i.e., $\psi = 0$), in the first step you can find an auxiliary function in the space (k_t, k_{t+1}, z_t), which maximizes the following $log(c_t(k_t, k_{t+1}, z_t, :)) + \phi log(:)$ (i.e., find where the optimal labor decisions are in the space (k_t, k_{t+1}, z_t)) and after that use this function in the value function iteration and proceed as usual.