make the same type of decision when deciding the combination of units of labor and capital to be used, analyzing the relative prices of these inputs (W/R).

The model

In this section, the structural model of the economy proposed in this chapter is presented and solved, step by step. This begins with the presentation of the agents (households and firms), following which the equilibrium conditions are shown. Then, the steady state is found and the equations that make up the model's equilibrium are log-linearized.

Assumption 2.2.1. *The economy is closed, with no government or financial sector.*

Assumption 2.2.2. *This economy does not have a currency. That is, it is a barter economy.*

Assumption 2.2.3. *Adjustment costs do not exist.*

Households

Assumption 2.2.4. *The economy in this model is formed by a unitary set of households indexed by $j \in [0,1]$ whose problem is to maximize a particular intertemporal welfare function. To this end, a utility function is used, additively separable into consumption (C) and labor (L).*

It is to be expected that a rise in consumption brings utility (happiness) to households, while a rise in labor hours brings disutility. At this point in the book, this is not surprising, seeing as in the theoretical section, it was mentioned that leisure provides individuals with happiness and that the more time they spend working, the less time they will have for leisure.

Assumption 2.2.5. *Consumption is intertemporally additively separable (no habit formation).*

Assumption 2.2.6. *Population growth is ignored.*

Assumption 2.2.7. *The labor market structure is one of perfect competition (no wage rigidity).*

The representative household optimizes the following welfare function:

$$\max_{C_{j,t},L_{j,t},K_{j,t+1}} E_t \sum_{t=0}^{\infty} \beta^t \left(\frac{C_{j,t}^{1-\sigma}}{1-\sigma} - \frac{L_{j,t}^{1+\varphi}}{1+\varphi} \right) \tag{2.1}$$

where E_t is the expectations operator, β is the intertemporal discount factor, C is the consumption of goods, L is the number of hours worked, σ is the relative risk aversion coefficient, and φ is the marginal disutility in respect of labor supply.

As mentioned in the theoretical section, the utility function[8] must have certain characteristics: $U_C > 0$ and $U_L < 0$, this means that consumption and labor have positive and negative effects, respectively, on the utility of households. On the other hand, $U_{CC} < 0$ and $U_{LL} < 0$ indicate that the utility function is concave[9]. This represents the fact that, as consumption increases, so does utility, albeit at increasingly lower rates.

Households maximize their welfare function, which is subject to their intertemporal budget constraints, which indicates which resources are available and how they are allocated. Thus, it is assumed that households are the owners of the economy's factors of production (capital and labor). Households, providing labor and capital to firms, receive wages and returns on capital, respectively. They also own the firms, and therefore receive dividends. Thus, a household's intertemporal budget constraint can be written in the following way:

$$P_t(C_{j,t} + I_{j,t}) = W_t L_{j,t} + R_t K_{j,t} + \Pi_t \tag{2.2}$$

where P is the general price level, I is level of investment, W is the level of wages, K is the capital stock, R is the return on capital, and Π is the firms' profit (dividends).

[8]The most common utility function used to represent Household choices is the utility function with a constant relative risk aversion (CRRA) (Galí, 2008; Lim and McNelis, 2008; Clarida *et al.*, 2002; Galí and Monacelli, 2005; Christoffel and Kuester, 2008; Christoffel *et al.*, 2009; Ravenna and Walsh, 2006, among others). In the literature, other functions that represent utility do exist, for example: a logarithmic utility function, $U(C_t, L_t) = \ln C_t + \frac{L_t}{L_0} A \ln(1 - L_0)$ (Hansen, 1985); a utility function that is a combination of a logarithmic function and CRRA, $U(C_t, L_t) = \ln(C_t) - \frac{\upsilon}{1+\chi} L_t^{1+\chi}$ (Gertler and Karadi, 2011, among others).

[9]U_C and U_L are the first-order derivatives of the utility function in relation to consumption and labor, respectively, while, U_{CC} and U_{LL} are the second-order derivatives.

An additional equation that shows capital accumulation over time is required.

$$K_{j,t+1} = (1-\delta)K_{j,t} + I_{j,t} \tag{2.3}$$

where δ is the depreciation rate of physical capital.

The problem of the household is solved using the following Lagrangian formed by equations (2.1), (2.2) and (2.3):

$$\mathscr{L} = E_t \sum_{t=0}^{\infty} \beta^t \left\{ \left[\frac{C_{j,t}^{1-\sigma}}{1-\sigma} - \frac{L_{j,t}^{1+\varphi}}{1+\varphi} \right] \right.$$

$$\left. -\lambda_{j,t} \left[P_t C_{j,t} + P_t K_{j,t+1} - P_t(1-\delta)K_{j,t} - W_t L_{j,t} - R_t K_{j,t} - \Pi_t \right] \right\} \tag{2.4}$$

where λ is the Lagrange multiplier.

Solving the previous problem, we arrive at the following first-order conditions:

$$\frac{\partial \mathscr{L}}{\partial C_{j,t}} = C_{j,t}^{-\sigma} - \lambda_{j,t} P_t = 0 \tag{2.5}$$

$$\frac{\partial \mathscr{L}}{\partial L_{j,t}} = -L_{j,t}^{\varphi} + \lambda_{j,t} W_t = 0 \tag{2.6}$$

$$\frac{\partial \mathscr{L}}{\partial K_{j,t+1}} = -\lambda_{j,t} P_t + \beta E_t \lambda_{j,t+1} \left[(1-\delta) E_t P_{t+1} + E_t R_{t+1} \right] = 0 \tag{2.7}$$

Solving for λ_t equations (2.5) and (2.6), we arrive at the household's labor supply equation.

$$C_{j,t}^{\sigma} L_{j,t}^{\varphi} = \frac{W_t}{P_t} \tag{2.8}$$

or,

$$\underbrace{-C_{j,t}^{\sigma} L_{j,t}^{\varphi}}_{\text{Consumption-leisure MRS}} = \underbrace{-\frac{W_t}{P_t}}_{\text{Consumption-leisure relative price}}$$

The labor supply equation states that the consumption-leisure relative price (real wage) must be equal to the leisure-consumption marginal rate of substitution (Theoretical Result 2.1.1). Thus, a rise in consumption, ceteris paribus, is only possible with a rise in the amount of labor hours (less leisure). In other words, there is a trade-off between working less (enjoying less leisure) and consuming more. On the other hand, higher real wages allow consumption to increase without there being a need to give up leisure[10].

Knowing that from equation (2.5) $\lambda_{j,t} = \frac{C_{j,t}^{-\sigma}}{P_t}$ e $\lambda_{j,t+1} = \frac{C_{j,t+1}^{-\sigma}}{P_{t+1}}$, and using these results in equation (2.7), the Euler equation is found:

$$-C_{j,t}^{-\sigma} + \beta E_t \left\{ \left(\frac{C_{j,t+1}^{-\sigma}}{P_{t+1}} \right) [(1-\delta)P_{t+1} + R_{t+1}] \right\} = 0$$

$$\left(\frac{E_t C_{j,t+1}}{C_{j,t}} \right)^{\sigma} = \beta \left[(1-\delta) + E_t \left(\frac{R_{t+1}}{P_{t+1}} \right) \right] \tag{2.9}$$

The previous equation determines the household's savings decision (in this model, savings is the acquisition of investment goods). Thus, when households decide their level of savings, they compare the utility rendered by consuming an additional amount today with the utility that would be rendered by consuming more in the future. Thus, if interest rate expectations rise, consuming "today" (at t) is more expensive and, ceteris paribus, future consumption (t+1) will rise.

One final remark concerning the Euler equation is worth being made. To simplify it, assume that $\beta = 1$ and $\delta = 1$,

$$\underbrace{-E_t \left[\frac{1}{\pi_{t+1}} \left(\frac{C_{j,t+1}}{C_{j,t}} \right)^{\sigma} \right]}_{\text{TMS } C_t\text{-}C_{t+1}} = \underbrace{-E_t \left(\frac{r_{t+1}}{\pi_{t+1}} \right)}_{\text{relative price } C_t\text{-}C_{t+1}}$$

where $E_t r_{t+1} = E_t \left(\frac{R_{t+1}}{P_{t+1}} \right)$ is the real rate of return on capital.

Thus, this last expression (Theoretical Result 2.1.2) states that the marginal rate of substitution of current consumption for future consumption is equal to the relative price of current consumption in terms of future consumption.

[10]With higher real wages, the consumption of goods will certainly be higher. On the other hand, the same cannot be said for leisure. If the income effect exceeds the substitution effect, leisure will increase, however, in the opposite case, leisure will decrease.

In short, the problem of the household boils down to two choices. The first is an intratemporal choice between acquiring consumption and leisure goods. The other is an intertemporal choice, in which the household must choose between present and future consumption.

Firms

The representative firm is the agent that produces the goods and services that will be either consumed or saved (and then transformed into capital) by households.

Assumption 2.2.8. *There is a continuum of firms indexed by j that maximize profit observing a structure of perfect competition, this means that their profits will be zero ($\Pi_t = 0$, for every t).*

To this end a Cobb-Douglas[11] production function is used:

$$Y_{j,t} = A_t K_{j,t}^{\alpha} L_{j,t}^{1-\alpha} \tag{2.10}$$

where A_t represents productivity, a variable that can be interpreted as the level of general knowledge about the "arts" of production available in an economy, Y_t is the product, and α is the elasticity of the level of production with respect to capital; α can also be thought of as the level of participation of capital in the productive process, whereas $(1-\alpha)$ would be the level of participation of labor. Similarly to the household's utility function, the production function must have certain properties: it must be strictly increasing ($F_K > 0$ and $F_L > 0$), strictly concave ($F_{KK} < 0$ e $F_{LL} < 0$), and twice differentiable. It is also assumed that the production function has constant returns to scale, $F(zK_t, zL_t) = zY_t$. This function must also satisfy the Inada conditions: $\lim_{K\to 0} F_K = \infty$; $\lim_{K\to\infty} F_K = 0$; $\lim_{L\to 0} F_L = \infty$; and $\lim_{L\to\infty} F_L = 0$.

[11] Although many DSGE models use Cobb-Douglas technology, there are alternatives. Another very popular function in the literature is the CES (Constant Elasticity of Substitution) function,

$$F(K_t, L_t) = \left[\alpha K_t^{\rho} + (1-\alpha) L_t^{\rho}\right]^{\frac{1}{\rho}}$$

where $\rho \in (-\infty, 1)$ is a parameter that determines the elasticity of substitution between two inputs.

The problem of the firm is solved by maximizing the Profit function, choosing the amounts of each input (L_t, K_t):

$$\max_{L_{j,t},K_{j,t}} \Pi_{j,t} = A_t K_{j,t}^{\alpha} L_{j,t}^{1-\alpha} P_{j,t} - W_t L_{j,t} - R_t K_{j,t} \tag{2.11}$$

Solving the previous problem, we arrive at the following first-order conditions:

$$\frac{\partial \Pi_{j,t}}{\partial K_{j,t}} = \alpha A_t K_{j,t}^{\alpha-1} L_{j,t}^{1-\alpha} P_{j,t} - R_t = 0 \tag{2.12}$$

$$\frac{\partial \Pi_{j,t}}{\partial L_{j,t}} = (1-\alpha) A_t K_{j,t}^{\alpha} L_{j,t}^{-\alpha} P_{j,t} - W_t = 0 \tag{2.13}$$

From equations (2.12) and (2.13):

$$\underbrace{\frac{R_t}{P_{j,t}}}_{\text{Real MCK}} = \underbrace{\alpha \frac{Y_{j,t}}{K_{j,t}}}_{\text{MPK}} \tag{2.14}$$

$$\underbrace{\frac{W_t}{P_{j,t}}}_{\text{Real MCL}} = \underbrace{(1-\alpha) \frac{Y_{j,t}}{L_{j,t}}}_{\text{MPL}} \tag{2.15}$$

Equations (2.14) and (2.15) represent the demand for capital and labor, respectively (Theoretical Result 2.1.3), in which marginal costs are equal to the marginal products[12].

Note that in equation (2.15) a reduction in real wages means higher demand for labor as, when the real cost of hiring workers reduces, firms increase their demand for labor until the marginal product of labor reduces to the same level as the fall in real wages[13] (Barro, 1997).

It is assumed that productivity shocks follow a first-order autoregressive process, such that:

$$\log A_t = (1-\rho_A) \log A_{ss} + \rho_A \log A_{t-1} + \epsilon_t \tag{2.16}$$

[12] Real MCK is the real marginal cost of capital;Real MCL is the real marginal cost of labor; MPK is the marginal product of capital; and MPL is the marginal product of labor.

[13] The same logic applies to capital (equation 2.14).

where A_{ss} is the value of productivity at the steady state, ρ_A is the autoregressive parameter of productivity, whose absolute value must be less than one ($|\rho_A| < 1$) to ensure the stationary nature of the process and $\epsilon_t \sim N(0, \sigma_A)$.

Assumption 2.2.9. *Productivity growth is ignored in this model.*

As the model follows the RBC approach, the price level must be equal to marginal cost. Thus, to obtain the marginal cost, the input demand equations must first be combined (equations (2.14) and (2.15)):

$$\underbrace{-\frac{W_t}{R_t}}_{\text{ERS}} = \underbrace{-\frac{(1-\alpha)K_{j,t}}{\alpha L_{j,t}}}_{\text{MRTS}}$$

Reminding the reader that this expression represents Theoretical Result 2.1.4. Its right-hand side is the marginal rate of technical substitution, which measures the rate at which labor can be replaced by capital while maintaining a constant level of production. The left-hand side is the economic rate of substitution, which measures the rate at which labor can be replaced by capital while maintaining the same cost.

Rearranging the previous expression,

$$L_{j,t} = \left(\frac{1-\alpha}{\alpha}\right)\frac{R_t}{W_t}K_{j,t} \tag{2.17}$$

and substituting equation (2.17) in the production function (equation (2.10)),

$$Y_{j,t} = A_t K_{j,t}^{\alpha}\left[\left(\frac{1-\alpha}{\alpha}\right)\frac{R_t}{W_t}K_{j,t}\right]^{1-\alpha}$$

$$K_{j,t} = \frac{Y_{j,t}}{A_t}\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{1-\alpha} \tag{2.18}$$

Substituting equation (2.18) in (2.17),

$$L_{j,t} = \frac{Y_{j,t}}{A_t}\left(\frac{1-\alpha}{\alpha}\right)\frac{R_t}{W_t}\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{1-\alpha}$$

$$\left(\frac{1-\alpha}{\alpha}\right)\frac{R_t}{W_t}=\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{-1}$$

$$L_{j,t}=\frac{A_t}{Y_{j,t}}\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{-\alpha} \tag{2.19}$$

total cost (TC) is represented by:

$$TC_{j,t}=W_tL_{j,t}+R_tK_{j,t}$$

substituting equations (2.18) and (2.19) in the total cost function:

$$TC_t=W_t\frac{Y_{j,t}}{A_t}\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{-\alpha}+R_t\frac{Y_{j,t}}{A_t}\left[\left(\frac{\alpha}{1-\alpha}\right)\frac{W_t}{R_t}\right]^{1-\alpha}$$

with a little algebraic massaging, we arrive at:

$$TC_{j,t}=\frac{Y_{j,t}}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha}$$

and the marginal cost is derived from the total cost[14]:

$$MC_{j,t}=\frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha} \tag{2.20}$$

As the marginal cost depends solely on productivity and the prices of the factors of production, it will be the same for all firms ($MC_{j,t}=MC_t$). Knowing that $P_t=MC_t$, we arrive at the general price level,

$$P_t=\frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha} \tag{2.21}$$

The model's equilibrium conditions

Now that each agent's behavior has been described, the interaction between them must be studied in order to determine macroeconomic equilibrium. Households decide how much to consume (C), how much to invest (I) and how much labor to supply (L), with the aim of maximizing utility, taking prices as given. On the other

[14] $MC_{j,t}=\frac{\partial TC_{j,t}}{\partial Y_{j,t}}$.

hand, firms decide how much to produce (Y) using available technology and choosing the factors of production (capital and labor), taking these prices as given.

Therefore, the model's equilibrium consists of the following blocks:

1. a price system, W_t, R_t and P_t;
2. an endowment of values for goods and inputs Y_t, C_t, I_t, L_t and K_t; and
3. a production-possibility frontier described by the following equilibrium condition of the goods market (aggregate supply = aggregate demand).

$$Y_t = C_t + I_t \tag{2.22}$$

Competitive equilibrium consists in finding a sequence of endogenous variables in the model such that the conditions that define equilibrium are satisfied. In short, this economy's model consists of the following equations from Table 2.1[15].

Steady state

After defining the economy's equilibrium, the steady state values must be defined. Indeed, the model presented is steady in the sense that there exists a value for the variables that is maintained over time: an endogenous variable x_t is at the steady state in each t, if $E_t x_{t+1} = x_t = x_{t-1} = x_{ss}$.

Some endogenous variables have their steady state values previously determined (exogenously). This is the case of productivity, which is the source of standard RBC models' shocks - at the steady state $E(\epsilon_t) = 0$. Thus, with equation (2.16) it is not possible to know the value of productivity at the steady state, the literature generally assigning $A_{ss} = 1$. The next step is to remove the variables' time indicators. Therefore, the structural model is:

[15] Because of the symmetry in the preferences of households and in the technology of firms, these two kinds of agents will be represented by representative agents (this eliminates the j subscript).

Table 2.1: Structure of the model.

Equation	(Definition)
$C_t^{\sigma} L_t^{\varphi} = \frac{W_t}{P_t}$	(Labor supply)
$\left(\frac{E_t C_{j,t+1}}{C_{j,t}}\right)^{\sigma} = \beta\left[(1-\delta) + E_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right]$	(Euler equation)
$K_{t+1} = (1-\delta)K_t + I_t$	(Law of motion of capital)
$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$	(Production function)
$K_t = \alpha \frac{Y_t}{\frac{R_t}{P_t}}$	(Demand for capital)
$L_t = (1-\alpha)\frac{Y_t}{\frac{W_t}{P_t}}$	(Demand for labor)
$P_t = \frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha}$	(Price level)
$Y_t = C_t + I_t$	(Equilibrium condition)
$\log A_t = (1-\rho_A)\log A_{ss} + \rho_A \log A_{t-1} + \epsilon_t$	(Productivity shock)

Households

$$C_{ss}^{\sigma} L_{ss}^{\varphi} = \frac{W_{ss}}{P_{ss}} \tag{2.23}$$

$$1 = \beta\left(1 - \delta + \frac{R_{ss}}{P_{ss}}\right) \tag{2.24}$$

$$I_{ss} = \delta K_{ss} \tag{2.25}$$

Firms

$$K_{ss} = \alpha \frac{Y_{ss}}{\frac{R_{ss}}{P_{ss}}} \tag{2.26}$$

$$L_{ss} = (1-\alpha)\frac{Y_{ss}}{\frac{W_{ss}}{P_{ss}}} \tag{2.27}$$

$$Y_{ss} = K_{ss}^{\alpha} L_{ss}^{1-\alpha} \tag{2.28}$$

$$P_{ss} = \left(\frac{W_{ss}}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_{ss}}{\alpha}\right)^{\alpha} \tag{2.29}$$

Equilibrium Condition

$$Y_{ss} = C_{ss} + I_{ss} \tag{2.30}$$

The system of equations formed by equations (2.23) to (2.30) will be used to determine the value of eight endogenous variables at the steady state (Y_{ss}, C_{ss}, I_{ss}, K_{ss}, L_{ss}, W_{ss}, R_{ss} and P_{ss}).

The first values that must be determined are the prices (W_{ss}, R_{ss} and P_{ss}). To this end, Walras' law must be taken into consideration.

Proposition 2.2.1 (Walras' Law)**.** *For any price vector* $\boldsymbol{p}$*, has* $\boldsymbol{pz}(\boldsymbol{p}) \equiv 0$*; i.e., the demand excess value is identically zero.*

Proof. In simple terms, the definition of excess demand is written and multiplied by $\mathbf{p}$:

$$\mathbf{pz}(\mathbf{p}) = \mathbf{p}\left[\sum_{i=1}^{n} \mathbf{x}_i(\mathbf{p}, \mathbf{p}\,\mathbf{w}_i) - \sum_{i=1}^{n} \mathbf{w}_i\right] = \sum_{i=1}^{n} \left[\mathbf{p}\,\mathbf{x}_i(\mathbf{p}, \mathbf{p}\,\mathbf{w}_i) - \mathbf{p}\,\mathbf{w}_i\right] = 0$$

since $\mathbf{x}_i(\mathbf{p}, \mathbf{p}\,\mathbf{w}_i)$ satisfies the budget constraint $\mathbf{p}\,\mathbf{x}_i = \mathbf{p}\,\mathbf{w}_i$ for each individual i=1,...,n. □

In other words, Walras' law states that if each individual satisfies his/her budget constraint, the value of his/her excess demand is zero, therefore the sum of excess demand must also be zero. It is important to note that Walras' law states that the value of excess demand is identical to zero - the value of excess demand is zero for all prices (Varian, 1992).

Walras' Law implies the existence of k-1 independent equations in equilibrium with k goods. Thus, if demand is equal to supply in k-1 markets, they will also be equal in the k^{th} market. Consequently, if there are k markets, only k-1 relative prices are required to determine equilibrium.

Provided that the excess aggregate demand function is homogeneous of degree zero, prices can be normalized and demands expressed in terms of relative price: $p_i = \frac{\widehat{p}_i}{\sum_{j=1}^{k} \widehat{p}_j}$. As a consequence, the sum of the normalized prices p_i must always be 1. Thus, attention can be directed to the price vector belonging to the unit simplex of dimension k-1: $S^{k-1} = \left\{\mathbf{p} \in R_+^k : \sum_{i=1}^{k} p_i = 1\right\}$. In short, taking Walras' Law into account, the economy's general price level can be normalized, $P_{ss} = 1$.

To find R_{ss}, equation (2.24) is used,

$$R_{ss} = P_{ss}\left[\left(\frac{1}{\beta}\right) - (1-\delta)\right] \tag{2.31}$$

Note that equation (2.31) shows R_{ss} as a function of only the normalized general price level parameters[16], therefore its value is determined. It simply remains to find the steady state of the wage level (W_{ss}). Thus, from equation (2.29),

$$W_{ss}^{1-\alpha} = P_{ss}(1-\alpha)^{1-\alpha}\left(\frac{\alpha}{R_{ss}}\right)^{\alpha}$$

$$W_{ss} = (1-\alpha)P_{ss}^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{R_{ss}}\right)^{\frac{\alpha}{1-\alpha}} \tag{2.32}$$

The next step is to satisfy the equilibrium condition. To this end, the variables that make up aggregate demand (C_{ss} and I_{ss}) must be determined. The idea underlying the equilibrium condition is formed by the following proposition.

Proposition 2.2.2 (Market adjustment)**.** *Given k markets, if demand is equal to supply in k-1 markets and $p_k > 0$, then demand must equal supply in the k^{th} market.*

Proof. If not, Walras' Law is violated. □

Therefore, to meet the equilibrium condition, the input market conditions must be met. To this end, it is necessary to find the meeting point between the supplies (provided by households) and the demands (provided by firms) of the production inputs (labor and capital) (Figure 2.12).

First, equation (2.27) must be replaced in equation (2.23), solving for C_{ss},

$$C_{ss}^{\sigma}\left[(1-\alpha)\frac{Y_{ss}}{\frac{W_{ss}}{P_{ss}}}\right]^{\varphi} = \frac{W_{ss}}{P_{ss}}$$

[16] In the Dynare simulation, there is no need to substitute R_{ss} in the other equations. It should just be shown before the other steady states.

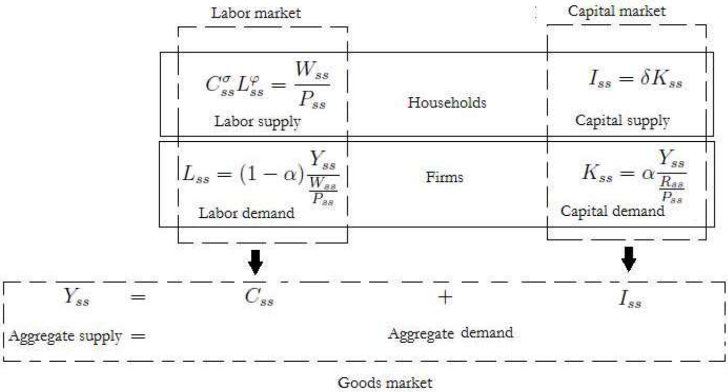

Figure 2.12: Steady state market adjustment structure. The dashed lines represent the labor, capital goods and consumer goods markets.

$$C_{ss}=\frac{1}{Y_{ss}^{\frac{\varphi}{\sigma}}}\left[\frac{W_{ss}}{P_{ss}}\left(\frac{\frac{W_{ss}}{P_{ss}}}{1-\alpha}\right)^{\varphi}\right]^{\frac{1}{\sigma}} \tag{2.33}$$

With the value C_{ss} known, I_{ss} still needs to be found. Consequently, equation (2.26) needs to be replaced in equation (2.25),

$$I_{ss}=\left(\frac{\delta\alpha}{R_{ss}}\right)Y_{ss} \tag{2.34}$$

Finally, Y_{ss} must be found. Substituting equations (2.33) and (2.34) in equation (2.30),

$$Y_{ss}=\frac{1}{Y_{ss}^{\frac{\varphi}{\sigma}}}\left[\frac{W_{ss}}{P_{ss}}\left(\frac{\frac{W_{ss}}{P_{ss}}}{1-\alpha}\right)^{\varphi}\right]^{\frac{1}{\sigma}}+\left(\frac{\delta\alpha}{R_{ss}}\right)Y_{ss}$$

$$\left(1-\frac{\delta\alpha}{R_{ss}}\right)Y_{ss}=\frac{1}{Y_{ss}^{\frac{\varphi}{\sigma}}}\left[\frac{W_{ss}}{P_{ss}}\left(\frac{\frac{W_{ss}}{P_{ss}}}{1-\alpha}\right)^{\varphi}\right]^{\frac{1}{\sigma}}$$

$$Y_{ss}=\left(\frac{R_{ss}}{R_{ss}-\delta\alpha}\right)^{\frac{\sigma}{\sigma+\varphi}}\left[\frac{W_{ss}}{P_{ss}}\left(\frac{\frac{W_{ss}}{P_{ss}}}{1-\alpha}\right)^{\varphi}\right]^{\frac{1}{\sigma+\varphi}} \tag{2.35}$$

The previous procedures are summarized in the presentation of the steady state below:

$$A_{ss}=1$$

$$P_{ss}=1$$

$$R_{ss}=P_{ss}\left[\left(\frac{1}{\beta}\right)-(1-\delta)\right]$$

$$W_{ss}=(1-\alpha)P_{ss}^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{R_{ss}}\right)^{\frac{\alpha}{1-\alpha}}$$

$$Y_{ss}=\left(\frac{R_{ss}}{R_{ss}-\delta\alpha}\right)^{\frac{\sigma}{\sigma+\varphi}}\left[\frac{W_{ss}}{P_{ss}}\left(\frac{\frac{W_{ss}}{P_{ss}}}{1-\alpha}\right)^{\varphi}\right]^{\frac{1}{\sigma+\varphi}}$$

$$I_{ss}=\left(\frac{\delta\alpha}{R_{ss}}\right)Y_{ss}$$

$$C_{ss}=\frac{1}{Y_{ss}^{\frac{\varphi}{\sigma}}}\left[(1-\alpha)^{-\varphi}\left(\frac{W_{ss}}{P_{ss}}\right)^{1+\varphi}\right]^{\frac{1}{\sigma}}$$

$$K_{ss}=\alpha\frac{Y_{ss}}{\frac{R_{ss}}{P_{ss}}}$$

$$L_{ss}=(1-\alpha)\frac{Y_{ss}}{\frac{W_{ss}}{P_{ss}}}$$

Using the previous sequence and the calibrated data shown in Table 2.2, we arrive at the steady state values for the variables (Table 2.3).

Table 2.2: Values of the structural model's parameters.

Parameter	Parameter meaning	Calibrated value
σ	Relative risk aversion coefficient	2
φ	Marginal disutility with regard to supply of labor	1.5
α	Elasticity of level of production in relation to capital	0.35
β	Discount factor	0.985
δ	Depreciation rate	0.025
ρ_A	Autoregressive parameter - productivity	0.95
σ_A	Standard deviation of productivity	0.01

Table 2.3: Values of variables at the steady state.

Variable	Steady state
A	1
R	0.040
W	2.084
Y	2.338
I	0.508
C	1.829
L	0.729
K	20.338

Log-linearization (Uhlig's method)

Handling and solving non-linear models is generally very arduous. When the model is very simple, it is possible to find an approximation of the policy function by recursively solving the value function. On the other hand, linear models are often easier to solve. The problem is converting a non-linear model to a sufficiently adequate linear approximation such that its solution helps in the understanding of the underlying non-linear system's behavior. Thus, a standard procedure is to log-linearize the model around its steady state (some methods use this approach in their solution procedures: Blanchard and Kahn, 1980; Uhlig, 1999; Sims, 2001; and Klein, 2000)[17][18].

Uhlig (1999) recommends a simple method of log-linearization of functions that does not require differentiation: simply replacing a variable X_t with $X_{ss}e^{\widetilde{X}_t}$, where $\widetilde{X}_t = \log X - \log X_{ss}$ represents the log of the variable's deviation in relation to its steady state. Uhlig further proposes the following solution tools:

$$e^{(\widetilde{X}_t + a\widetilde{Y}_t)} \approx 1 + \widetilde{X}_t + a\widetilde{Y}_t \tag{2.36}$$

$$\widetilde{X}_t \widetilde{Y}_t \approx 0 \tag{2.37}$$

$$E_t\left[ae^{\widetilde{X}_{t+1}}\right] \approx a + aE_t\left[\widetilde{X}_{t+1}\right] \tag{2.38}$$

Labor supply

Beginning with the labor supply function,

$$C_t^{\sigma} L_t^{\varphi} = \frac{W_t}{P_t}$$

$X_t = X_{ss}e^{\widetilde{X}_t}$ is replaced in each variable of the previous equation.

$$C_{ss}^{\sigma} L_{ss}^{\varphi} e^{(\sigma\widetilde{C}_t + \varphi\widetilde{L}_t)} = \frac{W_{ss}}{P_{ss}} e^{(\widetilde{W}_t - \widetilde{P}_t)}$$

[17] For further information, see DeJong and Dave (2007) and Canova (2007).

[18] It is important to point out that the models can be solved directly using Dynare without the need of linearization

Then, the equation's Uhlig rule is used (2.36),

$$C_{ss}^{\sigma} L_{ss}^{\varphi}(1+\sigma \widetilde{C}_t + \varphi \widetilde{L}_t) = \frac{W_{ss}}{P_{ss}}(1+\widetilde{W}_t - \widetilde{P}_t)$$

given that at the steady state, $C_{ss}^{\sigma} L_{ss}^{\varphi} = \frac{W_{ss}}{P_{ss}}$, we arrive at:

$$\sigma \widetilde{C}_t + \varphi \widetilde{L}_t = \widetilde{W}_t - \widetilde{P}_t \tag{2.39}$$

Euler equation for consumption

The same procedure will be used for the Euler equation.

$$\frac{1}{\beta} E_t \left(\frac{C_{t+1}}{C_t} \right)^{\sigma} = (1-\delta) + E_t \left(\frac{R_{t+1}}{P_{t+1}} \right)$$

$X_t = X_{ss} e^{\widetilde{X}_t}$ is replaced in each variable of the previous equation.

$$\frac{1}{\beta} \left(\frac{C_{ss}^{\sigma}}{C_{ss}^{\sigma}} \right) e^{(\sigma E_t \widetilde{C}_{t+1} - \sigma \widetilde{C}_t)} = (1-\delta) + \frac{R_{ss}}{P_{ss}} e^{\left[E_t(\widetilde{R}_{t+1} - \widetilde{P}_{t+1})\right]}$$

Then, the equation's Uhlig rule is used (2.36),

$$\frac{1}{\beta}\left[1+\sigma(E_t \widetilde{C}_{t+1} - \widetilde{C}_t)\right] = (1-\delta) + \frac{R_{ss}}{P_{ss}}\left[1+E_t(\widetilde{R}_{t+1} - \widetilde{P}_{t+1})\right]$$

given that at the steady state, $\frac{1}{\beta} = \frac{R_{ss}}{P_{ss}} + (1-\delta)$, we arrive at:

$$\frac{\sigma}{\beta}(E_t \widetilde{C}_{t+1} - \widetilde{C}_t) = \frac{R_{ss}}{P_{ss}} E_t(\widetilde{R}_{t+1} - \widetilde{P}_{t+1}) \tag{2.40}$$

Return on capital

Rewriting demand for capital,

$$R_t = \alpha \frac{Y_t}{K_t}$$

substituting $X_t = X_{ss} e^{\widetilde{X}_t}$ in each variable of the previous equation.

$$\frac{R_{ss}}{P_{ss}} e^{(\widetilde{R}_t - \widetilde{P}_t)} = \alpha \frac{Y_{ss}}{K_{ss}} e^{(\widetilde{Y}_t - \widetilde{K}_t)}$$

Now, the equation's Uhlig rule is used (2.36),

$$\frac{R_{ss}}{P_{ss}}(1+\widetilde{R}_t-\widetilde{P}_t)=\alpha\frac{Y_{ss}}{K_{ss}}(1+\widetilde{Y}_t-\widetilde{K}_t)$$

knowing that at the steady state, $\frac{R_{ss}}{P_{ss}}=\alpha\frac{Y_{ss}}{K_{ss}}$, we arrive at:

$$\widetilde{R}_t-\widetilde{P}_t=\widetilde{Y}_t-\widetilde{K}_t \tag{2.41}$$

Wage levels

Demand for labor is:

$$\frac{W_t}{P_t}=(1-\alpha)\frac{Y_t}{L_t}$$

Substituting $X_t=X_{ss}e^{\widetilde{X}_t}$ in each variable of the previous equation:

$$\frac{W_{ss}}{P_{ss}}e^{(\widetilde{W}_t-\widetilde{P}_t)}=(1-\alpha)\frac{Y_{ss}}{L_{ss}}e^{(\widetilde{Y}_t-\widetilde{L}_t)}$$

Now, the equation's Uhlig rule is used (2.36),

$$\frac{W_{ss}}{P_{ss}}(1+\widetilde{W}_t-\widetilde{P}_t)=(1-\alpha)\frac{Y_{ss}}{L_{ss}}(1+\widetilde{Y}_t-\widetilde{L}_t)$$

Knowing that at the steady state, $\frac{W_{ss}}{P_{ss}}=(1-\alpha)\frac{Y_{ss}}{L_{ss}}$, we get:

$$\widetilde{W}_t-\widetilde{P}_t=\widetilde{Y}_t-\widetilde{L}_t \tag{2.42}$$

Production function

Using the same procedure as before for the production function:

$$Y_t=A_tK_t^{\alpha}L_t^{1-\alpha}$$

$$Y_{ss}e^{\widetilde{Y}_t}=A_{ss}K_{ss}^{\alpha}L_{ss}^{1-\alpha}e^{(\widetilde{A}_t+\alpha\widetilde{K}_t+(1-\alpha)\widetilde{L}_t)}$$

$$Y_{ss}(1+\widetilde{Y}_t)=A_{ss}K_{ss}^{\alpha}L_{ss}^{1-\alpha}(1+\widetilde{A}_t+\alpha\widetilde{K}_t+(1-\alpha)\widetilde{L}_t)$$

Knowing that at the steady state, $Y_{ss} = A_{ss}K_{ss}^{\alpha}L_{ss}^{1-\alpha}$:

$$\widetilde{Y}_t = \widetilde{A}_t + \alpha\widetilde{K}_t + (1-\alpha)\widetilde{L}_t \tag{2.43}$$

Law of motion of capital

The law of motion of capital is:

$$K_{t+1} = (1-\delta)K_t + I_t$$

$$K_{ss}e^{\widetilde{K}_{t+1}} = (1-\delta)K_{ss}e^{\widetilde{K}_t} + I_{ss}e^{\widetilde{I}_t}$$

$$K_{ss}(1+\widetilde{K}_{t+1}) = (1-\delta)K_{ss}(1+\widetilde{K}_t) + I_{ss}(1+\widetilde{I}_t)$$

Dividing both sides of the previous equation by K_{ss},

$$(1+\widetilde{K}_{t+1}) = (1-\delta) + (1-\delta)\widetilde{K}_t + \frac{I_{ss}}{K_{ss}} + \frac{I_{ss}}{K_{ss}}\widetilde{I}_t$$

Knowing that at the steady state, $I_{ss} = \delta K_{ss}$:

$$\widetilde{K}_{t+1} = (1-\delta)\widetilde{K}_t + \delta\widetilde{I}_t \tag{2.44}$$

Equilibrium condition

It simply remains to find the equilibrium condition's log-linear equation.

$$Y_t = C_t + I_t$$

$$Y_{ss}e^{\widetilde{Y}_t} = C_{ss}e^{\widetilde{C}_t} + I_{ss}e^{\widetilde{I}_t}$$

$$Y_{ss}(1+\widetilde{Y}_t) = C_{ss}(1+\widetilde{C}_t) + I_{ss}(1+\widetilde{I}_t)$$

Knowing that at the steady state, $Y_{ss} = C_{ss} + I_{ss}$:

$$Y_{ss}\widetilde{Y}_t = C_{ss}\widetilde{C}_t + I_{ss}\widetilde{I}_t \tag{2.45}$$

Technological shock

The process of motion of productivity is:

$$\log A_t = (1-\rho_A)\log A_{ss} + \rho_A \log A_{t-1} + \epsilon_t$$

Using a little algebra, we arrive at:

$$\widetilde{A}_t = \rho_A \widetilde{A}_{t-1} + \epsilon_t \tag{2.46}$$

Table 2.4 summarizes the log-linear model.

Table 2.4: Structure of the log-linear model.

Equation	(Definition)
$\sigma\widetilde{C}_t + \varphi\widetilde{L}_t = \widetilde{w}_t$	(Labor supply)
$\frac{\sigma}{\beta}(E_t\widetilde{C}_{t+1} - \widetilde{C}_t) = \frac{R_{ss}}{P_{ss}}E_t(\widetilde{R}_{t+1} - \widetilde{P}_{t+1})$	(Euler equation)
$\widetilde{K}_{t+1} = (1-\delta)\widetilde{K}_t + \delta\widetilde{I}_t$	(Law of motion capital)
$\widetilde{Y}_t = \widetilde{A}_t + \alpha\widetilde{K}_t + (1-\alpha)\widetilde{L}_t$	(Production function)
$\widetilde{K}_t = \widetilde{Y}_t - \widetilde{r}_t$	(Demand for capital)
$\widetilde{L}_t = \widetilde{Y}_t - \widetilde{w}_t$	(Demand for labor)
$Y_{ss}\widetilde{Y}_t = C_{ss}\widetilde{C}_t + I_{ss}\widetilde{I}_t$	(Equilibrium condition)
$\widetilde{A}_t = \rho_A\widetilde{A}_{t-1} + \epsilon_t$	(Productivity shock)

Here, we have $\widetilde{w}_t = \widetilde{W}_t - \widetilde{P}_t$ and $\widetilde{r}_t = \widetilde{R}_t - \widetilde{P}_t$, which represent wages and the real interest rate, respectively.

Productivity shock

In this section, the results of a productivity shock on the RBC economy in this chapter, will be analyzed. Firstly, the result of the productivity shock on the variables will be checked, following which we will seek to identify the theoretical standards presented at the start of the chapter in the simulation.

Figure 2.13 shows the effects of a positive disruption on the total productivity of the factors of production. The first evidence of this shock is the rise in the marginal productivities of labor and capital (equations (2.41) and (2.42)). Firms thus increase their demand for these inputs. This then affects the prices of the factors of production – wages (W) and return on capital (R) – increasing household income. With higher income, this agent responds by acquiring more consumer and investment goods (equation 2.45)). With regard to inputs, initially both labor and capital increase, but as wages decrease with time, households seek more leisure (labor supply reduces). On the other hand, with the rise in investments, the stock of capital shows a growing tendency until period 20 (equation 2.44)), forming a bell curve. In short, positive productivity shocks increase the consumption variables (C and I), demand for inputs (L and K) and the prices of these factors of production (W and R).

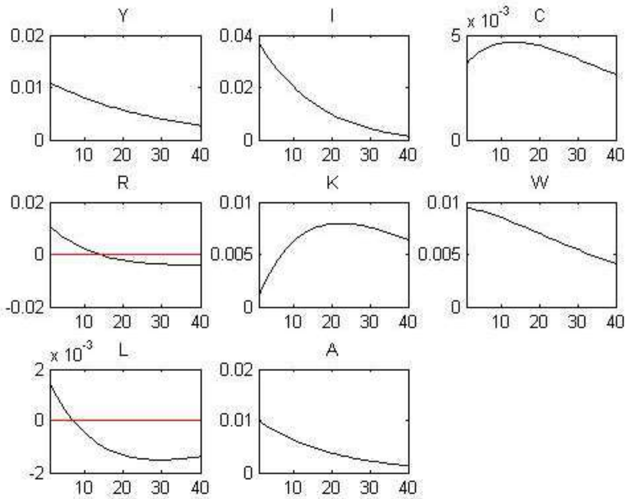

Figure 2.13: Effects of a productivity shock. Results of a Dynare simulation (impulse-response functions).

The aim of the group of figures in 2.14 to 2.17 is to understand how households “manage” their welfare given a positive productivity shock. Figure 2.14 shows the results of this shock on the leisure-wages locus. It can be seen that the inflection point for the wage level is 57% higher than its steady state level. Thus, increases that are lower than this value cause households to seek more leisure (income effect). On the other hand, wage increases higher than the inflection point cause the substitution effect to dominate – leisure becomes more expensive. Figure 2.15 seeks to present a relationship with figure 2.7, its theoretical approach. We find that when leisure is at relatively low levels (bottom left corner of Figure 2.15), the preference of households is to increase leisure. This occurs until the variable is 23% higher than the steady state (inflection point I). From this point onwards, the substitution effect is dominant.

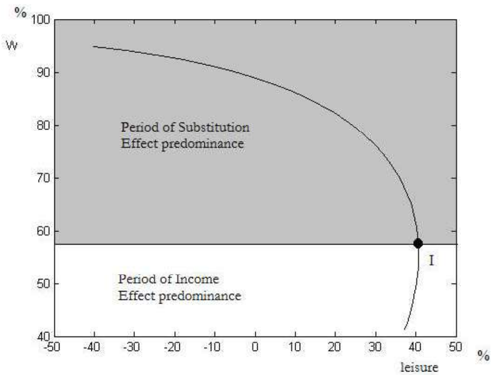

Figure 2.14: Leisure-wage locus. The x and y axes measure deviation in relation to the variable's steady state in percentage terms. Point I is the inflection point between where the substitution effect and income effect dominate.

When the concern is the intertemporal tradeoff, figures 2.16 and 2.17 are used. In this first figure (present consumption versus future consumption), the inflection point is practically at the steady state level of returns on capital. When the shock raises the value of this variable, today's consumption becomes more expensive in relation to future consumption. Households thus reduce their acquisition of consumption goods in the current period (substitution effect). If, on the other hand, the return on capital reduces in relation to its steady state, households are relatively poorer, causing them to reduce present consumption (income effect). When the tradeoff is intertemporal leisure (Figure 2.17), only the substitution effect is

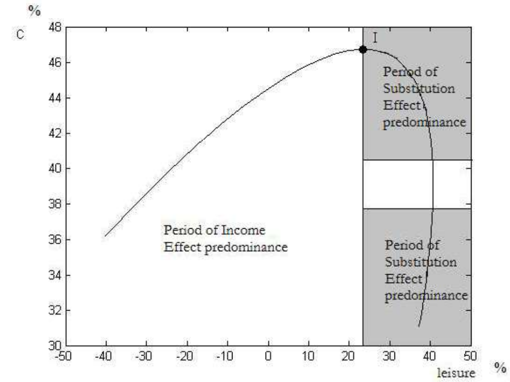

Figure 2.15: Leisure-consumption locus. The x and y axes measure deviations in relation to the variable's steady state in percentage terms. Point I is the inflection point between where the substitution effect and income effect dominate.

present. When R increases, future leisure becomes cheaper in relation to the current period. Therefore, a rise in the return on capital motivates households to work more today and less in the future.

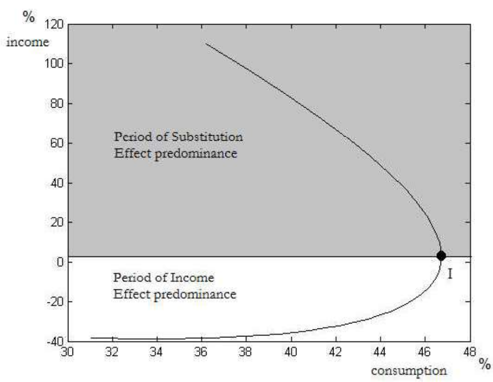

Figure 2.16: Consumption-income locus. The unit on the x and y axes measures percentage deviations in relation to the variable's steady state. Point I is the inflection point between which the substitution effect and income effect dominate.

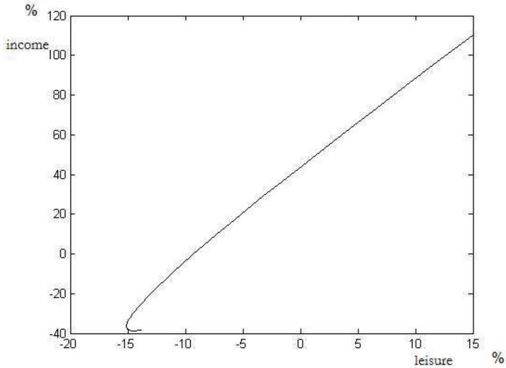

Figure 2.17: Leisure-income locus. The unit on the x and y axes measure, in percentage terms, deviations in relation to the variable's steady state. Point I is the inflection point between which the substitution effect and income effect dominate.

BOX 2.1 - Basic log-linear RBC model on Dynare

```
//RBC model - Chapter 2 (UNDERSTANDING DSGE MODELS)
//note: W and R are real in the simulation

var Y I C R K W L A ;
varexo e;
parameters sigma phi alpha beta delta rhoa;

sigma = 2;
phi = 1.5;
alpha = 0.35;
beta = 0.985;
delta = 0.025;
rhoa = 0.95;

model(linear);
#Pss = 1;
#Rss = Pss*((1/beta)-(1-delta));
#Wss = (1-alpha)*(Pss^(1/(1-alpha)))*((alpha/Rss)^(alpha/(1-alpha)));
#Yss = ((Rss/(Rss-delta*alpha))^(sigma/(sigma+phi)))
*(((1-alpha)^(-phi))*((Wss/Pss)^(1+phi)))^(1/(sigma+phi));
#Kss = alpha*(Yss/Rss/Pss);
#Iss = delta*Kss;
#Css = Yss - Iss;
#Lss = (1-alpha)*(Yss/Wss/Pss);
//1-Labor supply
sigma*C + phi*L = W;
//2-Euler equation
(sigma/beta)*(C(+1)-C)=Rss*R(+1);
//3-Law of motion of capital
K = (1-delta)*K(-1)+delta*I;
//4-Production function
Y = A + alpha*K(-1) + (1-alpha)*L;
//5-Demand for capital
R = Y - K(-1);
//6-Demand for labor
W = Y - L;
//7-Equilibrium condition
Yss*Y = Css*C + Iss*I;
//8-Productivity shock
A = rhoa*A(-1) + e;
end;

steady;
check;
model_diagnostics;
model_info;

shocks;
var e;
stderr 0.01;
end;

stoch_simul;
```