Lecture Notes

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Subject	Topics in Macroeconomics
Lecture 6	Local Linear Methods for Solving Macroeconomic Models

Contents

1 A Basic RBC Model

We are going to start by the social planner problem, which is given by:

$$\max \mathbb{E}_t \{ \sum_{t=j}^{\infty} \beta^{t-j} \ln(c_j) \} \tag{1}$$

subject to:

$$c_j + k_j = z_j k_{j-1}^{\alpha} \quad k_j \geq 0, \quad c_j \geq 0 \tag{2}$$

$$\ln(z_{j+1}) = \rho \ln(z_j) + \eta_j, \quad \eta_j \sim N(0, \sigma_\eta^2) \tag{3}$$

In this model we have that the technology process $\{z_t\}$ is defined as an $AR(1)$ process. Observe that the Lagrange of this problem is given by:

$$\mathbb{L} = \mathbb{E}_t \{ \beta^{t-j} (\ln(c_j) + \lambda_j [z_j k_{j-1}^{\alpha} - k_j - c_j]) \} \tag{4}$$

We have that the first order conditions of the problem are given by:

$$\frac{\partial \mathbb{L}}{\partial c_t} = \frac{1}{c_t} - \lambda_t = 0 \implies \frac{1}{c_t} = \lambda_t \tag{5}$$

$$\frac{\partial \mathbb{L}}{\partial k_t} = -\lambda_t + \beta \mathbb{E}_t [\alpha \lambda_{t+1} z_{t+1} k_t^{\alpha-1}] = 0 \tag{6}$$

The transversality condition is given by the following equation:

$$\lim_{T \to \infty} \frac{\beta^{T-j}}{c_t} k_T = 0 \tag{7}$$

Now the next step is to parametrize the model and solve the model numerically. In order to do so this method assigns parameter values based on either long-run steady state relationships of moments or moments from other studies such as microeconomic evidence:

Step 1 - Steady State.

Now we solve the model for the non stochastic steady state in which $z_{t+1} = z_t = 1$. In other words $z_t = 1 \quad \forall t$. By the Euler equation (6) and the resource constraint represented by equation (2) we have the following equations, where the variables with overline are the variables in the steady state:

$$1 = \beta \alpha \overline{k}^{\alpha-1} \implies \overline{k} = (\beta\alpha)^{\frac{1}{1-\alpha}} \tag{8}$$

$$\overline{c} = (\beta\alpha)^{\frac{\alpha}{1-\alpha}} - (\beta\alpha)^{\frac{1}{1-\alpha}} \implies \overline{c} = (\beta\alpha)^{\frac{\alpha}{1-\alpha}}(1-\beta\alpha) \tag{9}$$

Step 2 - Log-linearisation. Now we are going to define the following variables. First note that for any variable x we have that the following condition holds:

$$\hat{x} := \frac{(x-\overline{x})}{\overline{x}} \equiv \frac{\partial x}{\overline{x}} \approx \ln(\frac{x}{\overline{x}}) \tag{10}$$

The expression above is the percentage deviation of x from the point $\overline{x}$. By using this kind of evaluation with the variables that we have we obtain the following conditions:

$$\hat{c}_t := \ln(c_t) - \ln(\overline{c}) = \ln(\frac{c_t}{\overline{c}}) \implies c_t = \overline{c}\mathrm{e}^{\hat{c}_t} \tag{11}$$

$$\hat{k}_t := \ln(k_t) - \ln(\overline{k}) = \ln(\frac{k_t}{\overline{k}}) \implies k_t = \overline{k}\mathrm{e}^{\hat{k}_t} \tag{12}$$

$$\hat{\lambda}_t := \ln(\lambda_t) - \ln(\overline{\lambda}) = \ln(\frac{\lambda_t}{\overline{\lambda}}) \implies \lambda_t = \overline{\lambda}\mathrm{e}^{\hat{\lambda}_t} \tag{13}$$

$$\hat{z}_t := \ln(z_t) - \ln(\overline{z}) = \ln(\frac{z_t}{\overline{z}}) \implies z_t = \mathrm{e}^{\hat{z}_t} \tag{14}$$

By this fact and the equations that we already derived we obtain the following conditions that characterize the equilibrium conditions in this system:

Equilibrium conditions:

$$\lambda_t = \frac{1}{c_t} \tag{15}$$

$$\lambda_t = \beta\mathbb{E}_t[\alpha\lambda_{t+1}z_{t+1}k_t^{\alpha-1}] \tag{16}$$

$$c_t + k_t = z_t k_{t-1}^{\alpha} \tag{17}$$

$$\ln z_{t+1} = \rho \ln z_t + \eta_t \tag{18}$$

$$\lim_{T\to\infty} \frac{\beta^{T-j}}{c_t} k_T = 0 \tag{19}$$

Now one can observe that we have the following equation by using the kind of derivation we obtained by equation (10).

$$\overline{\lambda}\mathrm{e}^{\hat{\lambda}_t} = \frac{1}{\overline{c}\mathrm{e}^{\hat{c}_t}} \tag{20}$$

Now observe that a first order Taylor Expansion is given by:

$$f(x)|x^* \approx f(x^*) + f'(x^*)(x-x^*) \tag{21}$$

By doing a first order Taylor approximation around the value of λ in the steady state we obtain the following equation:

$$\overline{\lambda} + \overline{\lambda}(\hat{\lambda}_t - \overline{\hat{\lambda}}) = \frac{1}{\overline{c}} - \frac{1}{\overline{c}}(\hat{c}_t - \overline{\hat{c}}) \tag{22}$$

Observe that the values of $\overline{\hat{\lambda}}$ and $\overline{\hat{\lambda}}$ are zero and that $\overline{\lambda} = \frac{1}{\overline{c}}$, then we get that:

$$(\hat{\lambda}_t - \overline{\hat{\lambda}}) = (\hat{c}_t - \overline{\hat{c}}) \tag{23}$$

The equation above lead us to the following equation:

$$\hat{\lambda}_t = -\hat{c}_t \tag{24}$$

By using equation (16) and by replacing conditions (11) to (14) in the equation (16) lead us to the following result:

$$\lambda_t = \beta \mathbb{E}_t[\alpha \lambda_{t+1} z_{t+1} k_t^{\alpha-1}] \qquad \Longrightarrow \tag{25}$$

$$\lambda_t = \beta \mathbb{E}_t[\alpha(\overline{\lambda} \mathrm{e}^{\hat{\lambda}_{t+1}}) \mathrm{e}^{\hat{z_{t+1}}} (\overline{k} \mathrm{e}^{\hat{k}_t})^{\alpha-1}] \tag{26}$$

Now do a first order Taylor expansion around the stationary state. By using this first order Taylor expansion make us to achieve the equation below:

$$(\overline{\lambda} + \overline{\lambda}\hat{\lambda}_t) = \beta\alpha\overline{k}^{\alpha-1}\overline{\lambda} + \mathbb{E}_t\beta\alpha\overline{k}^{\alpha-1}\overline{\lambda}(\hat{z}_{t+1}) - \mathbb{E}_t\beta\alpha(1-\alpha)\overline{k}^{\alpha-1}\overline{\lambda}\hat{k}_t + \mathbb{E}_t\beta\alpha\overline{k}^{\alpha-1}\overline{\lambda}\hat{\lambda}_{t+1} \tag{27}$$

Observe that:

$$\overline{\lambda}\hat{\lambda}_t = \beta\alpha\overline{k}^{\alpha-1}\overline{\lambda}\mathbb{E}_t\hat{z}_{t+1} - (1-\alpha)\beta\alpha\overline{k}^{\alpha-1}\overline{\lambda}\hat{k}_t + \beta\alpha\overline{k}^{\alpha-1}\overline{\lambda}\mathbb{E}_t\hat{\lambda}_{t+1} \tag{28}$$

The equation above yield us to the following result:

$$\hat{\lambda}_t = \mathbb{E}_t\hat{z}_{t+1} - (1-\alpha)\hat{k}_t + \mathbb{E}_t\hat{\lambda}_{t+1} \tag{29}$$

By taking expectation of the expression in (18) and inserting it into the above equation yield the following result:

$$\hat{\lambda}_t = \rho\hat{z}_t - (1-\alpha)\hat{k}_t + \mathbb{E}_t\hat{\lambda}_{t+1} \tag{30}$$

Now we are going to do the same for the equation of consumption:

$$c_t + k_t = z_t k_{t-1}^{\alpha} \tag{31}$$

$$(\overline{c}\mathrm{e}^{\hat{c}_t}) + (\overline{k}\mathrm{e}^{\hat{k}_t}) = \mathrm{e}^{\hat{z}_t}(\overline{k}\mathrm{e}^{\hat{k}_{t-1}})^{\alpha} \tag{32}$$

$$y_t = z_t k_{t-1}^{\alpha} \quad \text{which implies that} \tag{33}$$

$$(\overline{c}\mathrm{e}^{\hat{c}_t}) + (\overline{k}\mathrm{e}^{\hat{k}_t}) = (\overline{y}\mathrm{e}^{\hat{y}_t}) \tag{34}$$

Observe that we can write equation (34) as the following one:

$$\overline{c} + \overline{c}\hat{c}_t + \overline{k} + \overline{k}\hat{k}_t = \overline{y} + \overline{y}\hat{y}_t \tag{35}$$

The equation above implies that:

$$\overline{c}\hat{c}_t + \overline{k}\hat{k}_t = \overline{y}\hat{y}_t \qquad \Longrightarrow \tag{36}$$

$$\hat{k}_t = \frac{\overline{y}}{\overline{k}}\hat{y}_t - \frac{\overline{c}}{\overline{k}}\hat{c}_t \tag{37}$$

Observe that we have that $y_t = z_t k_{t-1}^{\alpha}$ and by using the conditions that characterize the equations in the steady state we obtain that:

$$(\overline{y}\mathrm{e}^{\hat{y}_t}) = \mathrm{e}^{\hat{z}_t}(\overline{k}\mathrm{e}^{\hat{k}_{t-1}})^{\alpha} \;\Longrightarrow \tag{38}$$

$$\overline{y} + \overline{y}\hat{y}_t = \overline{k}^{\alpha} + \overline{k}^{\alpha}\hat{z}_t + \alpha\overline{k}\alpha\hat{k}_{t-1} \tag{39}$$

The equation that we have just written can be summarized in the equations below that characterize the equilibrium of the system:

Equilibrium conditions of the system:

$$\hat{\lambda}_t = -\hat{c}_t \tag{40}$$

$$\hat{\lambda}_t = \rho\hat{z}_t - (1-\alpha)\hat{k}_t + \mathbb{E}_t\hat{\lambda}_{t+1} \tag{41}$$

$$\hat{k}_t = \frac{\overline{y}}{\overline{k}}\hat{y}_t - \frac{\overline{c}}{\overline{k}}\hat{c}_t \tag{42}$$

$$\hat{y}_t = \hat{z}_t + \alpha\hat{k}_{t-1} \tag{43}$$

$$\hat{z}_{t+1} = \rho\hat{z}_t + \eta_t \tag{44}$$

Observe that above we have 5 unknowns and 5 non linear equations, however we have a problem in solving the equation above because we still have the expectations. We have to solve this through a guess of a value for k_t. We are going to guess the value o k_t by the following expression:

$$\hat{k}_t = \phi_1\hat{k}_{t-1} + \phi_2\hat{z}_t \tag{45}$$

By using the five equations above we obtain that:

$$(1+\alpha^2\beta)\hat{k}_t = \alpha\beta\mathbb{E}_t\hat{k}_{t+1} + \alpha\hat{k}_{t-1} - (1-\rho-\alpha\beta)\hat{z}_t \tag{46}$$

By taking expectations of equation (45) one can get the following result:

$$\mathbb{E}[\hat{k}_{t+1}] = \phi_1 \hat{k}_t + \phi_2 \rho \hat{z}_t \tag{47}$$

By replacing this value in the expression (46) yield us the following result:

$$(1+\alpha^2\beta)\hat{k}_t = \alpha\beta\phi_1\hat{k}_t + \alpha\beta\phi_2\rho\hat{z}_t + \alpha\hat{k}_{t-1} - (1-\rho-\alpha\beta)\hat{z}_t \tag{48}$$

By isolating the value of $\hat{k}_t$ we achieve the following equation:

$$\hat{k}_t = \frac{\alpha}{1+\alpha^2\beta-\alpha\beta\phi_1} + \frac{(1+\alpha\beta\phi_2\rho-\rho\alpha\beta)}{1+\alpha^2\beta-\alpha\beta\phi_1}\hat{z}_t \tag{49}$$

Observe that the vale of ϕ_1 is given by the following expression:

$$\phi_1 = \frac{\alpha}{1+\alpha^2\beta-\alpha\beta\phi_1} \tag{50}$$

By isolating the value of ϕ_1 we get that we can have the both values listed below:

$$\phi_1 = \frac{1}{\alpha\beta} \quad \text{or} \quad \phi_1 = \alpha \tag{51}$$

By transversality condition we eliminate the solution $\phi_1 = \frac{1}{\alpha\beta}$, since in this case the process would be explosive. Therefore given the value of $\phi_1 = \alpha$ then we have the value of $\phi_2 = 1$ and equation (45) becomes:

$$\hat{k}_t = \alpha\hat{k}_{t-1} + \hat{z}_t \tag{52}$$

Observe also that we can obtain the following equations:

$$\hat{y}_t = \alpha\hat{k}_{t-1} + \hat{z}_t \tag{53}$$

$$\hat{c}_t = \alpha\hat{k}_{t-1} + \hat{z}_t \tag{54}$$

2 A generic DSGE model

We first begin by defining the model of the problem:

$$U = \max_{x_t, y_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(x_t, y_t) \tag{55}$$

subject to

$$x_t = g(x_{t-1}, y_t, z_t) \quad (\text{or} \quad x_{t+1} = g(x_t, y_t, z_t)) \tag{56}$$

$$z_t = f(z_{t-1}, \epsilon_t), \quad \epsilon_t \sim iid(0, \Sigma) \tag{57}$$

$$x_-1 \quad (\text{or} \quad x_0) \quad \text{and} \quad z_0 \quad \text{given} \tag{58}$$

Observe that:

1. The vector of endogenous state variables is given by x_t, which is a m x 1 vector;

2. The vector of exogeneous state variables (stochastic shocks) is given by: z_t;

3. The vector of all other endogenous variables (including control variables) is given by y_t, which is a n x 1 vector. In order to solve the model we have to do the following steps:

1. Write the Lagrangian;

2. Take first order conditions;

3. Collect all other equations that come from market clearing, policy rules etc;

4. Equilibrium conditions: a system of v (first-order) difference equations o v variables.

In general we have that DSGE models have large number of variables and large numbers of state variables $\implies$ very complicated to solve with Global Methods.

3 Blanchard-Kahn Method

Write the system of linear difference equations as

$$A \begin{bmatrix} \hat{x}_t \\ \mathbb{E}_t \hat{y}_{t+1} \end{bmatrix} = B \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} + C\hat{z}_t \tag{59}$$

In order to solve the system above we assume that z_t is a stable stochastic process. Moreover two issues are important for solving equation above:

(a) The matrix A may not be singular;

(b) Some components of x_t are predetermined and others are non-predetermined.

The method of Blanchard and Kahn (1980) assume that A is non singular. As a result, we define $F = A^{-1}B$ and apply the Jordan decomposition to F. Therefore by assuming that A is invertible, we transform to:

$$\begin{bmatrix} \hat{x}_t \\ \mathbb{E}_t \hat{y}_{t+1} \end{bmatrix} = F \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} + G\hat{z}_t \tag{60}$$

where we have that $F = A^{-1}B$ is a (n+m) x (n+m) and $G = A^{-1}C$.

By applying the Jordan decomposition to F lead us to the following:

$$F = HJH^{-1} = \begin{bmatrix} v_1 & \dots & v_{n+m} \end{bmatrix} \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_{n+m} \end{bmatrix} \begin{bmatrix} v_1 & \dots & v_{n+m} \end{bmatrix}^{-1} \tag{61}$$

where the eigenvalues are ordered such that $|\lambda_1| < |\lambda_2| < \ldots < |\lambda_{n+m}|$ and v_i are their corresponding eigenvectors. Let h be the number of eigenvalues of J that are outside the unit circle, $|\lambda_i| > 1$.

Proposition Blanchard-Kahn, 1980

(a) If $h = n$, the system of stochastic difference equations has a unique solution.

(b) If $h > n$, the system of linear stochastic difference equations has no solutions.

(c) If $h < n$, the system of linear stochastic difference equations has infinite solutions.

Uniqueness: For every free variable, you need one eigenvalue that is larger than one (saddle path stability).

Multiplicity: Not enough eigenvalues larger than one (indeterminacy).

Now suppose that the solution is unique.

In order to solve the system of difference equations:

$$\begin{bmatrix} \hat{x}_t \\ \mathbb{E}_t\hat{y}_{t+1} \end{bmatrix} = F \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} + G\hat{z}_t \tag{62}$$

we do a change of variables according to:

$$\begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix} = H^{-1} \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{x}_t \\ \mathbb{E}_t\tilde{y}_{t-1} \end{bmatrix} = H^{-1} \begin{bmatrix} \hat{x}_t \\ \mathbb{E}_t\hat{y}_{t+1} \end{bmatrix} \tag{63}$$

which implies that:

$$\begin{bmatrix} \tilde{x}_t \\ \mathbb{E}_t\tilde{y}_{t+1} \end{bmatrix} = J \begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix} + V\hat{z}_t \tag{64}$$

where $V = H^{-1}G$.

The matrix J is diagonal, with the eigenvalues in the diagonal ordered from smallest to largest. By partitioning the matrix we can obtain that:

$$\begin{bmatrix} \tilde{x}_t \\ \mathbb{E}_t\tilde{y}_{t+1} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \hat{z}_t \tag{65}$$

Therefore we have the following equation:

$$\tilde{x}_t = J_1\tilde{x}_{t-1} + V_1\hat{z}_t \tag{66}$$

The system above is a system of uncoupled difference equations which is stable, since all eigenvalues in J_1 are inside the unit circle. Moreover we also have that:

$$\mathbb{E}_t\tilde{y}_{t+1} = J_2\tilde{y}_t + V_2\hat{z}_t \implies \tilde{y}_t = J_2^{-1}\mathbb{E}_t\tilde{y}_{t+1} - J_2^{-1}V_2\hat{z}_t \tag{67}$$

By doing the iteration of the expression (67) and taking expectations we the achieve that:

$$\tilde{y}_{t+1} = J_2^{-1}\mathbb{E}_{t+1}\tilde{y}_{t+2} - J_2^{-1}V_2\hat{z}_{t+1} \tag{68}$$

$$\mathbb{E}_t\tilde{y}_{t+1} = J_2^{-1}\mathbb{E}_t\mathbb{E}_{t+1}\tilde{y}_{t+2} - J_2^{-1}V_2\mathbb{E}_t\hat{z}_{t+1} = J_2^{-1}\mathbb{E}_t\tilde{y}_{t+2} \tag{69}$$

After many iterations we the achieve that:

$$\tilde{y}_t = -J_2^{-1}V_2\hat{z}_t \tag{70}$$

Then we have that:

$$\tilde{x}_t = J_1 \tilde{x}_{t-1} + V_1 \hat{z}_t \tag{71}$$

$$\tilde{y}_t = -J_2^{-1} V_2 \hat{z}_t \tag{72}$$

This system can be solved with known methods for uncoupled difference equations, and we recover the original variables from using the above solutions and the change of variables

$$\begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix} \tag{73}$$