The model

This chapter develops the model addressed in the previous chapter, introducing price stickiness and monopolistic competition, which are at the core of New-Keynesian models. The RBC model's general assumptions (2.2.1-2.2.3) remain valid.

Households

As households follow the same optimizing behavior as the RBC model (assumptions 2.2.4-2.2.7 remain valid), there is no need to repeat the problem in this chapter. Thus, the equations that maximize utility will be taken from the previous chapter:

Equation (2.3),

$$K_{j,t+1} = (1-\delta)K_{j,t} + I_{j,t} \tag{3.1}$$

Equation (2.8),

$$C_{j,t}^{\sigma} L_{j,t}^{\varphi} = \frac{W_t}{P_t} \tag{3.2}$$

And equation (2.9),

$$\left(\frac{E_t C_{j,t+1}}{C_{j,t}}\right)^{\sigma} = \beta\left[(1-\delta) + E_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right] \tag{3.3}$$

Firms

As for the problem of the firm, assumption 2.2.9 remains valid. Assumption 2.2.8, on the other hand, ceases to be so in this model. Thus,

Assumption 3.2.1. *It is assumed that the market structure is one of monopolistic competition.*

The economy's producing sector is thus divided into two parts: an intermediate goods sector (wholesale firms) and a final goods sector (retail firms). The intermediate goods sector consists of a large number of companies, each one producing differentiable goods. These companies must decide the quantity of factors of production to be used and the prices of their goods using a production function. In the final goods sector, there is a single firm that, using a specific, pertinent technology, aggregates intermediate goods into one single good that will be consumed by economic agents.

Firms that produce final goods (retail firms)

From an aggregate perspective, monopolistic competition, among other factors, compels us to recognize the fact that consumers buy a large variety of goods, there being a need, therefore, for models that assume that consumers buy only one kind of good (an aggregate bundle with all goods), as in the previous chapter's RBC model.

Assumption 3.2.2. *This aggregate good (bundle of goods) is sold by a retail firm within a structure of perfect competition. That is, it is assumed that a given retail firm is completely identical to any other.*

The theoretical implication of assumption 3.2.2 is that a representative retail firm exists. Because of to this assumption that retailers sell their products in a market that is in perfect competition, there is nothing here that is very different from the idea presented in the RBC model.

With the aim of producing an aggregate good, a retailer must buy a large quantity of goods from the wholesale sector. That is, these are the inputs used in a retail firm's production process. Thus, a retailer buys a large variety of wholesale goods (clothes, electronic products, etc.,) and transforms them into an aggregate good (a bundle of goods) that will be sold to the final agent.

How much is a "large variety of goods"? If on the one hand consumers do not face a literally infinite number of consumption possibilities, on the other, they may buy a large variety of goods that vary in size, color, style etc. For this reason and for mathematical convenience, in this kind of model, "many" is treated as "infinite". It is assumed that there is a continuum of wholesale goods and that each good is indexed within the unit interval [0,1]. Thus, a continuous number of wholesale goods is taken into account rather than a discrete number.

In order to make things clearer, it is assumed that it is possible to represent a particular wholesale good at any point within the unit interval [0,1]. Each of these goods is imperceptible infinitesimally small when compared to the total amount of available goods.

Therefore, it is assumed that each good belonging to the unit interval is produced by a single wholesaler and is imperfectly substitutable by any other good. Thus, these goods are differentiated products, which allows for the possibility of some degree of monopoly power.

To represent the problem that a retail firm faces, its production technology and maximization problem must be described. Since the incorporation of the idea of monopolistic competition in mainstream macroeconomics in the eighties and nineties, the most widely applied functional form for aggregation technology is the Dixit-Stiglitz aggregator (Dixit and Stiglitz, 1977):

$$Y_t = \left(\int_0^1 Y_{j,t}^{\frac{\psi-1}{\psi}} \, dj \right)^{\frac{\psi}{\psi-1}} \tag{3.4}$$

where Y_t is the product of retailers in period t, and $Y_{j,t}$ for $j \in [0,1]$ is wholesale good j, and $\psi > 1$ is the elasticity of substitution between wholesale goods[3].

With P_t as the nominal price of a retail product and $P_{j,t}$ as the nominal price of wholesale good j, the price of each wholesale good is taken as a given by retail firms. Therefore, the problem of the representative retail firm is maximizing its profit function:

$$\max_{Y_{j,t}} P_t Y_t - \int_0^1 P_{j,t} Y_{j,t} dj \tag{3.5}$$

Substituting the aggregator technology in the last expression (Equation (3.4) in equation (3.5)), we get:

$$\max_{Y_{j,t}} P_t \left(\int_0^1 Y_{j,t}^{\frac{\psi-1}{\psi}} \, dj \right)^{\frac{\psi}{\psi-1}} - P_{j,t} \int_0^1 Y_{j,t} dj \tag{3.6}$$

Taking the first-order condition for the above problem,

$$\frac{\psi}{\psi-1} P_t \left(\int_0^1 Y_{j,t}^{\frac{\psi-1}{\psi}} \, dj \right)^{\frac{\psi}{\psi-1}-1} \frac{\psi-1}{\psi} Y_{j,t}^{\frac{\psi-1}{\psi}-1} - P_{j,t} = 0$$

[3] Smets and Wouters (2007) assume that the elasticity of substitution between intermediate goods is stochastic: $\psi_t = \psi + \nu_t$, with $\nu_t \sim N(0, \sigma_\nu)$.

or,

$$P_t \left(\int_0^1 Y_{j,t}^{\frac{\psi-1}{\psi}} \, dj \right)^{\frac{1}{\psi-1}} Y_{j,t}^{\frac{-1}{\psi}} - P_{j,t} = 0$$

Remember that the aggregator (equation 3.4) may also be written as,

$$Y_t^{\frac{1}{\psi}} = \left(\int_0^1 Y_{j,t}^{\frac{\psi-1}{\psi}} \, dj \right)^{\frac{1}{\psi-1}}$$

The right-hand side of this last equation is the term that should be eliminated from the first-order condition, thus:

$$P_t Y_t^{\frac{1}{\psi}} Y_{j,t}^{\frac{-1}{\psi}} - P_{j,t} = 0$$

Raising the previous expression to the power of $-\psi$ and with some algebraic manipulation, we get:

$$Y_{j,t} = Y_t \left(\frac{P_t}{P_{j,t}} \right)^{\psi} \tag{3.7}$$

This expression is the demand function for wholesale good j, which is directly proportional to aggregate demand (Y_t) and inversely proportional to its relative price level $\left(\frac{1}{\frac{P_{j,t}}{P_t}} \right)$.

Substituting equation (3.7) in equation (3.4),

$$Y_t = \left\{ \int_0^1 \left[Y_t \left(\frac{P_t}{P_{j,t}} \right)^{\psi} \right]^{\frac{\psi-1}{\psi}} dj \right\}^{\frac{\psi}{\psi-1}}$$

$$Y_t = Y_t P_t^{\psi} \left\{ \int_0^1 \left[\left(\frac{1}{P_{j,t}} \right)^{\psi} \right]^{\frac{\psi-1}{\psi}} dj \right\}^{\frac{\psi}{\psi-1}}$$

$$P_t^{\psi} = \left[\int_0^1 \left(P_{j,t}^{\psi} \right)^{\frac{\psi-1}{\psi}} dj \right]^{\frac{\psi}{\psi-1}}$$

$$P_t = \left[\int_0^1 P_{j,t}^{1-\psi} dj \right]^{\frac{1}{1-\psi}} \tag{3.8}$$

Equation (3.8) is the pricing rule for final (retail) goods.

Firms that produce intermediate goods (wholesale firms)

As already described, wholesale firms sell their differentiated products to retail firms.

Assumption 3.2.3. *Owing to the differentiated nature of wholesale goods, wholesale firms have some degree of market power and are thus price setters (market structure of monopolistic competition).*

Assumption 3.2.4. *It is assumed that fixed production costs do not exist. This means that average variable cost is equal to average total cost.*

Assumption 3.2.5. *It is assumed that the per unit production cost of a wholesale product is always the same regardless of the scale of production. This means that it is being assumed that wholesale firms have constant returns to scale, resulting in a marginal production cost, regardless of the quantity produced.*

These two assumptions lead to an opportune mathematical consequence where the marginal cost function coincides with the average total cost function. This, in turn, means that total cost may be expressed simply by multiplying the quantity produced and the marginal cost.

The retail firm solves its problem in two stages. First, the firm takes the prices of the factors of production (return on capital and wages) and determines the amount of capital and labor that it will use to minimize its total production cost:

$$\min_{L_{j,t}, K_{j,t}} W_t L_{j,t} + R_t K_{j,t} \tag{3.9}$$

subject to the following technology,

$$Y_{j,t} = A_t K_{j,t}^{\alpha} L_{j,t}^{1-\alpha} \tag{3.10}$$

with the law of motion of productivity,

$$\log A_t = (1-\rho_A)\log A_{ss} + \rho_A \log A_{t-1} + \varepsilon_t \tag{3.11}$$

where A_{ss} is the value of productivity at the steady state, ρ_A is the autoregressive parameter of productivity, whose absolute value must be less than 1, $|\rho_A| < 1$, to ensure the steadiness of the process, and $\epsilon_t \sim N(0,\sigma_A)$.

Using the Lagrangian to solve the problem of the wholesale firm,

$$\mathscr{L} = W_t L_{j,t} + R_t K_{j,t} + \mu_{j,t}\left(Y_{j,t} - A_t K_{j,t}^{\alpha} L_{j,t}^{1-\alpha}\right) \tag{3.12}$$

The first-order conditions for the previous problem are:

$$\frac{\partial \mathscr{L}}{\partial L_{j,t}} = W_t - (1-\alpha)\mu_{j,t} A_t K_{j,t}^{\alpha} L_{j,t}^{\alpha} = 0 \tag{3.13}$$

$$\frac{\partial \mathscr{L}}{\partial K_{j,t}} = R_t - \alpha\mu_{j,t} A_t K_{j,t}^{\alpha-1} L_{j,t}^{1-\alpha} = 0 \tag{3.14}$$

With $\mu_{j,t} = MC_{j,t}$ (MC - Marginal Cost), equations (3.13) and (3.14), become:

$$L_{j,t} = (1-\alpha) MC_{j,t} \frac{Y_{j,t}}{W_t} \tag{3.15}$$

$$K_{j,t} = \alpha MC_{j,t} \frac{Y_{j,t}}{R_t} \tag{3.16}$$

These two equations represent the demand of a wholesale firm j for labor and capital, respectively.

Since production technology is the same as in the previous chapter's model, there is no need to work out again the total and marginal cost functions. To this end, it is sufficient to use equation (2.20):

$$MC_{j,t} = \frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha} \tag{3.17}$$

Note that equation (3.17) meets the result of assumptions 3.2.4 and 3.2.5 that $MC_{j,t} = \frac{CT_{j,t}}{Y_{j,t}}$.

Calvo pricing

The second stage of the problem of the wholesale firm is defining the price of its goods. This firm decides how much to produce in each period according to the Calvo rule (Calvo, 1983).

Thus, the wholesale firm has a θ probability of keeping the price of its good fixed in the next period and a $1-\theta$ probability of optimally defining its price. For this second type of firm, having defined the price of its good in t, there is a θ probability that this price remains fixed in t+1, a θ^2 probability that it remains fixed in t+2, and so on. Consequently, the firm must consider these probabilities when defining the price of its good in t.

Definition 3.2.1 (Calvo's rule)**.** *Establishes that in each period t, a fraction $0<\theta<1$ of firms is randomly selected and allowed to define the prices of its goods for the period. The rest of the firms (the θ fraction) keeps the prices of its goods defined by a stickiness rule which, in the literature, may follow one of the three possibilities below:*

1. *Maintain the previous period's price*
$$P_{j,t} = P_{j,t-1}$$
2. *Update the price using the steady state gross inflation rate (π_{ss})*
$$P_{j,t} = \pi_{ss} P_{j,t-1}$$
3. *Update the price using the previous period's gross inflation rate (π_{t-1})*
$$P_{j,t} = \pi_{t-1} P_{j,t-1}$$

Assumption 3.2.6. *In this book, the first rule will be used to determine price stickiness, $P_{j,t} = P_{j,t-1}$.*

Therefore, the problem of the wholesale firm that is capable of readjusting the price of its good is:

$$\max_{P^*_{j,t}} E_t \sum_{i=0}^{\infty} (\beta\theta)^i (P^*_{j,t} Y_{j,t+i} - TC_{j,t+i}) \tag{3.18}$$

Substituting equation (3.7) in equation (3.18),

$$\max_{P^*_{j,t}} E_t \sum_{i=0}^{\infty} (\beta\theta)^i \left[P^*_{j,t} Y_{t+i} \left(\frac{P_{t+i}}{P^*_{j,t}} \right)^{\psi} - Y_{t+i} \left(\frac{P_{t+i}}{P^*_{j,t}} \right)^{\psi} MC_{j,t+i} \right] \tag{3.19}$$

Taking the previous problem's first-order condition,

$$0 = E_t \sum_{i=0}^{\infty} (\beta\theta)^i \left[(1-\psi) Y_{j,t+i} + \psi \frac{Y_{j,t+i}}{P^*_{j,t}} MC_{j,t+i} \right]$$

$$P^*_{j,t} = \left(\frac{\psi}{\psi - 1} \right) E_t \sum_{i=0}^{\infty} (\beta\theta)^i MC_{j,t+i} \tag{3.20}$$

Note that all wholesale firms that fix their prices have the same markup on the same marginal cost. Thus, in all periods, $P^*_{j,t}$ is the same price for all the $1-\theta$ firms that set their prices. Combining equation (3.8)'s pricing rule and the fact that firms within their respective groups firms that define their prices and firms subject to stickiness use the same prices (as they are subject to the same level of technology), the aggregate price level is obtained thus:

$$P_t^{1-\psi} = \int_0^{\theta} P_{t-1}^{1-\psi} dj + \int_{\theta}^{1} P_t^{*\,1-\psi} dj$$

$$P_t^{1-\psi} = \left[j P_{t-1}^{1-\psi} \right]_0^{\theta} + \left[j P_t^{*\,1-\psi} \right]_{\theta}^{1}$$

$$P_t^{1-\psi} = \theta P_{t-1}^{1-\psi} + (1-\theta) P_t^{*\,1-\psi}$$

$$P_t = \left[\theta P_{t-1}^{1-\psi} + (1-\theta) P_t^{*\,1-\psi} \right]^{\frac{1}{1-\psi}} \tag{3.21}$$

It is important to remember that there is a continuum of firms, and the group that can alter its price (and the group that cannot) is chosen randomly, regardless of when each firm last altered its price. This means that the distribution of prices among firms does not change between periods.

The model's equilibrium condition

It is still necessary to establish an equilibrium condition in the goods market.

$$Y_t = C_t + I_t \tag{3.22}$$

In short, this economy's model consists of the following equations in Table 3.1.

Table 3.1: Model structure.

Equation	(Definition)
$C_t^{\sigma} L_t^{\varphi} = \frac{W_t}{P_t}$	(Labor supply)
$\left(\frac{E_t C_{t+1}}{C_t}\right)^{\sigma} = \beta\left[(1-\delta) + E_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right]$	(Euler Equation)
$K_{t+1} = (1-\delta)K_t + I_t$	(Law of motion of capital)
$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$	(Production function)
$K_t = \alpha MC_t \frac{Y_t}{R_t}$	(Demand for capital)
$L_t = (1-\alpha) MC_t \frac{Y_t}{W_t}$	(Demand for labor)
$MC_t = \frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha}$	(Marginal cost)
$P_t^* = \left(\frac{\psi}{\psi-1}\right) E_t \sum_{i=0}^{\infty} (\beta\theta)^i MC_{t+i}$	(Optimal price level)
$P_t = \left[\theta P_{t-1}^{1-\psi} + (1-\theta) P_t^{*\,1-\psi}\right]^{\frac{1}{1-\psi}}$	(General price level)
$\pi_t = \frac{P_t}{P_{t-1}}$	(Gross inflation rate)
$Y_t = C_t + I_t$	(Equilibrium condition)
$\log A_t = (1-\rho_A)\log A_{ss} + \rho_A \log A_{t-1} + \epsilon_t$	(Productivity shock)

Steady state

As in the previous chapter, armed with the model's solution, the next step is finding the steady state, the point of origin of the simulations that will be performed and the point of reference for log-linearization. The idea that an endogenous variable x_t is at the steady state in each period t, if $E_t x_{t+1} = x_t = x_{t-1} = x_{ss}$, remains valid. The first step is to write the model's steady state.

Households

$$C_{ss}^{\sigma} L_{ss}^{\varphi} = \frac{W_{ss}}{P_{ss}} \tag{3.23}$$

$$1 = \beta \left(1 - \delta + \frac{R_{ss}}{P_{ss}} \right) \tag{3.24}$$

$$\delta K_{ss} = I_{ss} \tag{3.25}$$

Firms

$$L_{ss} = (1-\alpha) MC_{ss} \frac{Y_{ss}}{W_{ss}} \tag{3.26}$$

$$K_{ss} = \alpha MC_{ss} \frac{Y_{ss}}{R_{ss}} \tag{3.27}$$

$$Y_{ss} = K_{ss}^{\alpha} L_{ss}^{1-\alpha} \tag{3.28}$$

$$MC_{ss} = \left(\frac{W_{ss}}{1-\alpha} \right)^{1-\alpha} \left(\frac{R_{ss}}{\alpha} \right)^{\alpha} \tag{3.29}$$

$$P_{ss} = \left(\frac{\psi}{\psi - 1} \right) \left(\frac{1}{1 - \beta\theta} \right) MC_{ss} \tag{3.30}$$

where: $\sum_{i=0}^{\infty} (\beta\theta)^i = \frac{1}{1-\beta\theta}$ [4].

[4]The sum of the infinite terms of a geometric progression is called a geometric series, and the sum is:

$$S_{\infty} = \sum_{n=0}^{\infty} a_1 q^n = \frac{a_1}{1-q}$$

Equilibrium condition

$$Y_{ss} = C_{ss} + I_{ss} \tag{3.31}$$

Initially, the values of prices P_{ss}, R_{ss}, W_{ss} and MC_{ss} must be determined. For the same reasons as described in the previous chapter, the general price level will be normalized ($P_{ss} = 1$), The values of R_{ss} and MC_{ss} are thus also known,

so, from equation (3.24),

$$R_{ss} = P_{ss}\left[\frac{1}{\beta} - (1-\delta)\right] \tag{3.32}$$

and from equation (3.30),

$$MC_{ss} = \left(\frac{\psi - 1}{\psi}\right)(1-\beta\theta)P_{ss} \tag{3.33}$$

With R_{ss} and MC_{ss}, known, the value of W_{ss} is also known, and from equation (3.29),

$$W_{ss}^{1-\alpha} = MC_{ss}(1-\alpha)^{1-\alpha}\left(\frac{\alpha}{R_{ss}}\right)^{\alpha}$$

$$W_{ss} = (1-\alpha)MC_{ss}^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{R_{ss}}\right)^{\frac{\alpha}{1-\alpha}} \tag{3.34}$$

Having determined the values of the prices, the next step is to determine equilibrium in the inputs markets with the aim of obtaining the variables that make up aggregate demand (C_{ss} and I_{ss}). Thus, substituting equation (3.27) in equation (3.25),

$$I_{ss} = \left(\frac{\delta\alpha MC_{ss}}{R_{ss}}\right)Y_{ss} \tag{3.35}$$

and substituting equation (3.26) in equation (3.23),

$$C_{ss}^{\sigma}\left[(1-\alpha)MC_{ss}\frac{Y_{ss}}{W_{ss}}\right]^{\varphi} = \frac{W_{ss}}{P_{ss}}$$

$$C_{ss} = \frac{1}{Y_{ss}^{\frac{\varphi}{\sigma}}}\left\{\frac{W_{ss}}{P_{ss}}\left[\frac{W_{ss}}{(1-\alpha)MC_{ss}}\right]^{\varphi}\right\}^{\frac{1}{\sigma}} \tag{3.36}$$

the expressions for investment and consumption are obtained. Lastly, in order to determine the steady-state output (Y_{ss}), it is necessary to meet the equilibrium condition of the goods market (equation (3.31)) with equations (3.35) and (3.36),

$$Y_{ss} = \left(\frac{\delta\alpha MC_{ss}}{R_{ss}}\right) Y_{ss} + \frac{1}{Y_{ss}^{\frac{\varphi}{\sigma}}} \left\{ \frac{W_{ss}}{P_{ss}} \left[\frac{W_{ss}}{(1-\alpha)MC_{ss}} \right]^{\varphi} \right\}^{\frac{1}{\sigma}}$$

$$\left(1 - \frac{\delta\alpha MC_{ss}}{R_{ss}}\right) Y_{ss} = \frac{1}{Y_{ss}^{\frac{\varphi}{\sigma}}} \left\{ \frac{W_{ss}}{P_{ss}} \left[\frac{W_{ss}}{(1-\alpha)MC_{ss}} \right]^{\varphi} \right\}^{\frac{1}{\sigma}}$$

$$Y_{ss}^{1+\frac{\varphi}{\sigma}} = \left(\frac{R_{ss}}{R_{ss} - \delta\alpha MC_{ss}}\right) \left\{ \frac{W_{ss}}{P_{ss}} \left[\frac{W_{ss}}{(1-\alpha)MC_{ss}} \right]^{\varphi} \right\}^{\frac{1}{\sigma}}$$

$$Y_{ss} = \left(\frac{R_{ss}}{R_{ss} - \delta\alpha MC_{ss}}\right)^{\frac{\sigma}{\sigma+\varphi}} \left\{ \frac{W_{ss}}{P_{ss}} \left[\frac{W_{ss}}{(1-\alpha)MC_{ss}} \right]^{\varphi} \right\}^{\frac{1}{\sigma+\varphi}} \tag{3.37}$$

The previous procedures are summed up in the presentation of the steady state below:

$$A_{ss} = 1$$

$$P_{ss} = 1$$

$$R_{ss} = P_{ss} \left[\frac{1}{\beta} - (1-\delta) \right]$$

$$MC_{ss} = \left(\frac{\psi - 1}{\psi}\right) (1 - \beta\theta) P_{ss}$$

$$W_{ss} = (1-\alpha) MC_{ss}^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{R_{ss}}\right)^{\frac{\alpha}{1-\alpha}}$$

$$Y_{ss} = \left(\frac{R_{ss}}{R_{ss} - \delta\alpha MC_{ss}}\right)^{\frac{\sigma}{\sigma+\varphi}} \left\{ \frac{W_{ss}}{P_{ss}} \left[\frac{W_{ss}}{(1-\alpha)MC_{ss}} \right]^{\varphi} \right\}^{\frac{1}{\sigma+\varphi}}$$

$$I_{ss} = \left(\frac{\delta \alpha MC_{ss}}{R_{ss}}\right) Y_{ss}$$

$$C_{ss} = \frac{1}{Y_{ss}^{\frac{\varphi}{\sigma}}} \left\{ \frac{W_{ss}}{P_{ss}} \left[\frac{W_{ss}}{(1-\alpha) MC_{ss}} \right]^{\varphi} \right\}^{\frac{1}{\sigma}}$$

$$L_{ss} = (1-\alpha) MC_{ss} \frac{Y_{ss}}{W_{ss}}$$

$$K_{ss} = \alpha MC_{ss} \frac{Y_{ss}}{R_{ss}}$$

Table 3.2 shows the calibrated values that will be used in the NK model's simulation. Table 3.3 shows the steady state numerically.

Table 3.2: Values of the structural model's parameters.

Parameter	Meaning of the parameter	Calibrated
σ	Relative risk aversion coefficient	2
φ	Marginal disutility with respect to labor supply	1.5
α	Elasticity of output with respect to capital	0.35
β	Discount factor	0.985
δ	Depreciation rate	0.025
ρ_A	Autoregressive parameter of productivity	0.95
σ_A	Standard deviation of productivity	0.01
θ	Price stickiness parameter	0.75
ψ	Elasticity of substitution between intermediate goods	8

Table 3.3: Values of variables at the steady state.

Variable	Steady state
A	1
R	0.040
MC	0.2286
W	0.2152
Y	0.778
I	0.039
C	0.739
L	0.537
K	1.547

Log-linearization (Uhlig's method)

In this section, the "trick" developed by Uhlig (1999) will continue to be used in the log-linearization procedure. Some of the NK model's equations have already been log-linearized in the previous chapter. To avoid unnecessary superposition, these equations will simply be reproduced. Efforts will be focused on obtaining the "New-Keynesian Phillips Equation", which will be developed step by step.

Thus, using the previous chapter's equations:

$$\sigma \widetilde{C}_t + \varphi \widetilde{L}_t = \widetilde{W}_t - \widetilde{P}_t \tag{3.38}$$

$$\frac{\sigma}{\beta}(E_t \widetilde{C}_{t+1} - \widetilde{C}_t) = \frac{R_{ss}}{P_{ss}} E_t(\widetilde{R}_{t+1} - \widetilde{P}_{t+1}) \tag{3.39}$$

$$\widetilde{R}_t = \widetilde{MC}_t + \widetilde{Y}_t - \widetilde{K}_t \tag{3.40}$$

$$\widetilde{W}_t = \widetilde{MC}_t + \widetilde{Y}_t - \widetilde{L}_t \tag{3.41}$$

$$\widetilde{Y}_t = \widetilde{A}_t + \alpha \widetilde{K}_t + (1-\alpha)\widetilde{L}_t \tag{3.42}$$

$$\widetilde{K}_{t+1} = (1-\delta)\widetilde{K}_t + \delta \widetilde{I}_t \tag{3.43}$$

$$Y_{ss}\widetilde{Y}_t = C_{ss}\widetilde{C}_t + I_{ss}\widetilde{I}_t \tag{3.44}$$

$$\widetilde{A}_t = \rho_A \widetilde{A}_{t-1} + \epsilon_t \tag{3.45}$$

Marginal cost

The marginal cost equation is a new element in this chapter, but its log-linearization is similar to that demonstrated in the previous chapter:

From equation (3.17),

$$MC_{ss}(1+\widetilde{MC}_t) = \left(\frac{W_{ss}}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_{ss}}{\alpha}\right)^{\alpha}[1-\widetilde{A}_t + (1-\alpha)\widetilde{W}_t + \alpha\widetilde{R}_t]$$

with,

$$MC_{ss} = \left(\frac{W_{ss}}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_{ss}}{\alpha}\right)^{\alpha}$$

therefore,

$$\widetilde{MC}_t = (1-\alpha)\widetilde{W}_t + \alpha\widetilde{R}_t - \widetilde{A}_t \tag{3.46}$$

Determining the New-Keynesian Phillips curve

Beginning with the log-linearization of the equation that defines the optimal price level, equation (3.20),

$$P_{ss}(1+\widetilde{P}_t^*) = \left(\frac{\psi}{\psi-1}\right)\left(\frac{1-\beta\theta}{1-\beta\theta}\right)MC_{ss}E_t\sum_{i=0}^{\infty}(\beta\theta)^i(1+\widetilde{MC}_{t+i})$$

with,

$$P_{ss} = \left(\frac{\psi}{\psi-1}\right)\left(\frac{1}{1-\beta\theta}\right)MC_{ss}$$

we arrive at,

$$1+\widetilde{P}_t^* = 1+(1-\beta\theta)E_t\sum_{i=0}^{\infty}(\beta\theta)^i\widetilde{MC}_{t+i}$$

$$\widetilde{P}_t^* = (1-\beta\theta)E_t\sum_{i=0}^{\infty}(\beta\theta)^i\widetilde{MC}_{t+i} \tag{3.47}$$

The next step is log-linearizing the final goods' markup rule. From equation (3.21),

$$P_{ss}^{1-\psi}[1+(1-\psi)\widetilde{P}_t] = \theta P_{ss}^{1-\psi}[1+(1-\psi)\widetilde{P}_{t-1}]+(1-\theta)P_{ss}^{1-\psi}[1+(1-\psi)\widetilde{P}_t^*]$$

$$1+\widetilde{P}_t = \theta + \theta\widetilde{P}_{t-1} + 1 - \theta + (1-\theta)\widetilde{P}_t^*$$

$$\widetilde{P}_t = \theta\widetilde{P}_{t-1} + (1-\theta)\widetilde{P}_t^* \tag{3.48}$$

Equation (3.47) must then be substituted in equation (3.48),

$$\widetilde{P}_t = \theta\widetilde{P}_{t-1} + (1-\theta)(1-\beta\theta)E_t\sum_{i=0}^{\infty}(\beta\theta)^i\widetilde{MC}_{t+i} \tag{3.49}$$

The second element of the right-hand side of equation (3.49) possesses an infinite sum of the future nominal marginal cost. Therefore, it is necessary to find a way to remove this term. To this end, a technique known as "quasi-differencing" can be used. Both sides of equation (3.49) must be multiplied by the lag operator $(1-\beta\theta L^{-1})$. This lag operator, applied to a variable X_t, will result in $L^{-1}X_t = X_{t+1}$.

Therefore, multiplying equation (3.49) by $(1-\beta\theta L^{-1})$:

$$\widetilde{P}_t - \beta\theta E_t\widetilde{P}_{t+1} = \theta\widetilde{P}_{t-1} + (1-\theta)(1-\beta\theta)E_t\sum_{i=0}^{\infty}(\beta\theta)^i\widetilde{MC}_{t+i} - \beta\theta\theta\widetilde{P}_t$$
$$-\beta\theta(1-\theta)(1-\beta\theta)E_t\sum_{i=0}^{\infty}(\beta\theta)^i\widetilde{MC}_{t+1+i}$$

As quasi-differencing is applied in order to cancel out the t+1 terms, the previous equation results in:

$$\widetilde{P}_t - \beta\theta E_t\widetilde{P}_{t+1} = \theta\widetilde{P}_{t-1} - \beta\theta\theta\widetilde{P}_t + (1-\theta)(1-\beta\theta)\widetilde{MC}_t$$

Then, P_t must be deducted from the nominal marginal cost,

$$\widetilde{P}_t-\beta\theta E_t\widetilde{P}_{t+1} = \theta\widetilde{P}_{t-1}-\beta\theta\theta\widetilde{P}_t+(1-\theta)(1-\beta\theta)\widetilde{P}_t+(1-\theta)(1-\beta\theta)(\widetilde{MC}_t-\widetilde{P}_t)$$

$$\widetilde{P}_t - \beta\theta E_t\widetilde{P}_{t+1} = \theta\widetilde{P}_{t-1} - \beta\theta\theta\widetilde{P}_t + \widetilde{P}_t - \beta\theta\widetilde{P}_t - \theta\widetilde{P}_t + \beta\theta\theta\widetilde{P}_t$$
$$+(1-\theta)(1-\beta\theta)(\widetilde{MC}_t - \widetilde{P}_t)$$

$$\theta(\widetilde{P}_t - \widetilde{P}_{t-1}) = \beta\theta(E_t\widetilde{P}_{t+1} - \widetilde{P}_t) + (1-\theta)(1-\beta\theta)(\widetilde{MC}_t - \widetilde{P}_t)$$

Lastly, both sides of the previous equation must be divided by θ, With the gross inflation rates in t and t+1 being : $\widetilde{\pi}_t = \widetilde{P}_t - \widetilde{P}_{t-1}$; and $E_t\widetilde{\pi}_{t+1} = E_t\widetilde{P}_{t+1} - \widetilde{P}_t$, we arrive at the New-Keynesian Phillips equation:

$$\widetilde{\pi}_t = \beta E_t\widetilde{\pi}_{t+1} + \left[\frac{(1-\theta)(1-\beta\theta)}{\theta}\right](\widetilde{MC}_t - \widetilde{P}_t) \tag{3.50}$$

Table 3.4 sums up the log-linear model.

Table 3.4: Structure of the log-linear model.

Equation	(Definition)
$\sigma\widetilde{C}_t + \varphi\widetilde{L}_t = \widetilde{W}_t - \widetilde{P}_t$	(Labor supply)
$\frac{\sigma}{\beta}(E_t\widetilde{C}_{t+1} - \widetilde{C}_t) = \frac{R_{ss}}{P_{ss}}E_t(\widetilde{R}_{t+1} - \widetilde{P}_{t+1})$	(Euler Equation)
$\widetilde{K}_{t+1} = (1-\delta)\widetilde{K}_t + \delta\widetilde{I}_t$	(Law of motion of capital)
$\widetilde{Y}_t = \widetilde{A}_t + \alpha\widetilde{K}_t + (1-\alpha)\widetilde{L}_t$	(Production function)
$\widetilde{K}_t = \widetilde{MC}_t + \widetilde{Y}_t - \widetilde{R}_t$	(Demand for labor)
$\widetilde{L}_t = \widetilde{MC}_t + \widetilde{Y}_t - \widetilde{W}_t$	(Demanda por trabalho)
$\widetilde{MC}_t = [(1-\alpha)\widetilde{W}_t + \alpha\widetilde{R}_t - \widetilde{A}_t]$	(Marginal cost)
$\widetilde{\pi}_t = \beta E_t\widetilde{\pi}_{t+1} + \left[\frac{(1-\theta)(1-\beta\theta)}{\theta}\right](\widetilde{MC}_t - \widetilde{P}_t)$	(Phillips equation)
$\widetilde{\pi}_t = \widetilde{P}_t - \widetilde{P}_{t-1}$	(Gross inflation rate)
$Y_{ss}\widetilde{Y}_t = C_{ss}\widetilde{C}_t + I_{ss}\widetilde{I}_t$	(Equilibrium condition)
$\widetilde{A}_t = \rho_A\widetilde{A}_{t-1} + \epsilon_t$	(Productivity shock)

Productivity shock

In this section, the results of a positive productivity shock on the model's variables will be discussed. First, the NK model's impulse-response functions will be analyzed (figure 3.3), following which the differences between the two models' results (RBC vs NK) will be discussed (figures 3.4 and 3.5).

The productivity shock in question caused the values of the marginal products of labor and capital to rise. Consequently, firms increased their demand for inputs (labor and capital). The prices of these inputs thus responded positively to this greater level of demand. Bearing in mind that higher wages increase the income of households, if, on the one hand, this higher level of income increases the acquisition of goods (I and C), on the other, it increases the demand for another "good", leisure (income effect). This fall in labor supply explains the higher resistance of wages returning to the steady state, while returns on capital fell below even their initial level (steady state) in period 10. With the growth in aggregate supply, the elements that make up aggregate demand increase, most notably investments, whose result is 10 times greater than that of consumption goods. This higher capital supply (strong growth in investments) explains the returns on capital's swifter return to the steady state.

In short, higher productivity increased the spending variables (consumption and investment) and input prices, with wages showing greater persistence when returning to the steady state. With regard to the factors of production, capital widened, exhibiting a bell-shaped curve with an inflection point in period 20. However, labor supply decreased because of to a strong predominance of the income effect.

After the NK model is analyzed individually, it becomes important to understand the effects of price stickiness on macroeconomic variables. Figure 3.4 shows the results of the IRFs for the RBC and NK models. It can be said that the effects on product, investments, capital stock and real prices of inputs ($W/P, R/P$) were not significantly different. On the other hand, the effects on the acquisition of consumer goods (an element of demand) and on labor supply (an element of supply) were essentially different. In the NK model, the productivity shock led to a significant rise in consumption compared to the results of the RBC model. In the latter, the effects on labor supply were greater compared to the former. In short, the productivity shock affected product via aggregate demand in the NK model, while in the RBC model product was affected via aggregate supply.

This difference in the behavior of households regarding the acquisition of "goods" is explained in figure 3.5. Greater price stickiness causes real wages, at the inflection point between the periods in which the income and substitution effects dominate, to practically double in value. In the RBC model, this point indicated a value of 57% in relation to the steady state, while in the NK model it reached 98%. Thus, with price stickiness (price levels adjusting more slowly to the equilibrium level), productivity had less effect on the components of household income (W/P, R/P).

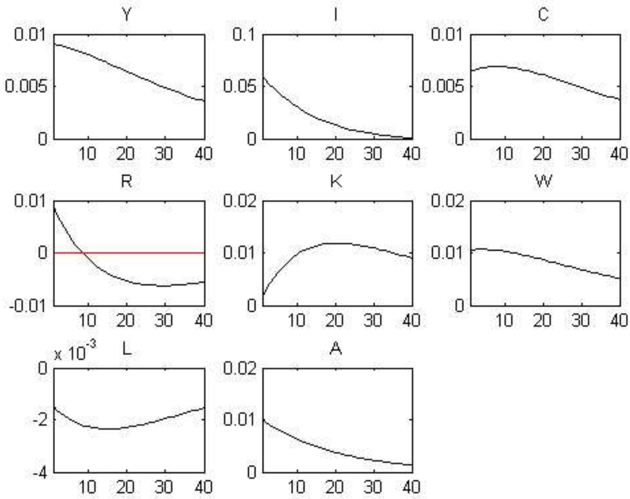

Figure 3.3: Effects of a productivity shock. Dynare simulation results (Impulse-response functions).

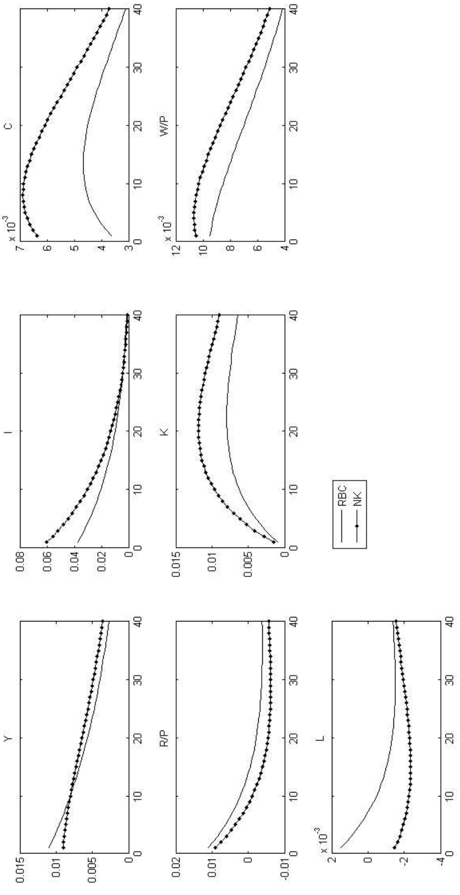

Figure 3.4: Comparison between impulse-response functions in the RBC and NK models.

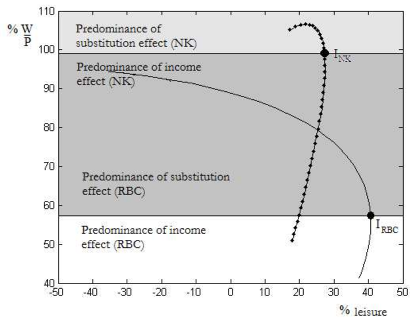

Figure 3.5: Leisure-wages locus. The x and y axes measure the variable's deviation in relation to the steady state in percentage terms. Point I is the inflection point between which the substitution effect and income effect dominate.

BOX 3.1 - Basic log-linear NK model on Dynare.

```
//NK model - Chapter 3 (UNDERSTANDING DSGE MODELS)
var Y I C R K W L MC P PI A;
varexo e;
parameters sigma phi alpha beta delta rhoa psi theta;

sigma = 2;
phi = 1.5;
alpha = 0.35;
beta = 0.985;
delta = 0.025;
rhoa = 0.95;
psi = 8;
theta = 0.75;

model(linear);
#Pss = 1;
#Rss = Pss*((1/beta)-(1-delta));
#MCss = ((psi-1)/psi)*(1-beta*theta)*Pss;
#Wss = (1-alpha)*(MCss^(1/(1-alpha)))*((alpha/Rss)^(alpha/(1-alpha)));
#Yss = ((Rss/(Rss-delta*alpha*MCss))^(sigma/(sigma+phi)))
*((Wss/Pss)*(Wss/((1-alpha)*MCss))^phi)^(1/(sigma+phi));
#Kss = alpha*MCss*(Yss/Rss);
#Iss = delta*Kss;
#Css = Yss - Iss;
#Lss = (1-alpha)*MCss*(Yss/Wss);
//1-Labor supply
sigma*C + phi*L = W - P;
//2-Euler equation
(sigma/beta)*(C(+1)-C)=(Rss/Pss)*(R(+1)-P(+1));
//3-Law of motion of capital
K = (1-delta)*K(-1) + delta*I;
//4-Production function
Y = A + alpha*K(-1) + (1-alpha)*L;
//5-Demand for capital
K(-1) = Y - R;
//6-Demand for labor
L = Y - W;
//7-Marginal cost
MC = (1-alpha)*W + alpha*R - A;
//8-Phillips equation
PI = beta*PI(+1)+((1-theta)*(1-beta*theta)/theta)*(MC-P);
//9-Gross inflation rate
PI = P - P(-1);
//10-Goods market equilibrium condition
Yss*Y = Css*C + Iss*I;
//11-Productivity shock
A = rhoa*A(-1) + e;
end;

steady;
check(qz_zero_threshold=1e-20);

shocks;
var e;
stderr 0.01;
```