Department of Economics - ESG-UQAM
Topics in Macroeconomics

Problem Set 1 - Root-Finding and Function Approximation, Neoclassical Growth Model, Complete Markets Model

Professor: Diego de Sousa Rodrigues
de_sousa_rodrigues.diego@uqam.ca

Fall 2025

Instructions:

Answer all questions. Show all steps and include any code or plots you use. You can use any programming language you want (Python, Julia, Matlab, R, etc.). Please submit your answers in a single PDF file. You can use Overleaf or any other LaTeX editor to write your answers.

Question 1 - Bisection Method

Consider the function:

$$b(x) \;=\; e^{x} - e^{2.2087}.$$

Starting from the interval $x \in [0,4]$, find x^* such that $b(x^*) = 0$ using the bisection method.

Question 2 - Newton's Method

Consider the function:

$$d(x) \;=\; x^{-5} - x^{-3} - c.$$

a) Set $c = 1$ and plot d on $x \in [0.6, 10]$. Find x^* such that $d(x^*) = 0$ using Newton's method.

b) Construct an *equidistant* grid for c containing 10 nodes between 1 and 8, and for each value of c on the grid find x^*.

c) Construct a new equidistant grid for c containing 1000 nodes between 1 and 10, and *plot your solution for each value on this grid using a spline.*

d) Relabel your solution x^* from part (b) as $x(c)$. Find the inverse function $c(x)$, and plot it on an equidistant grid of 1000 nodes on $x \in [0.6, 10]$ using a spline approximation.

e) Now find the solution to:

$$0 \;=\; c(x) + x.$$

Question 3 - Approximation Methods

Consider the function $h(x)$ on the domain $x \in [-2, 2]$:

$$h(x) \;=\; \begin{cases} (x-0.5)^2, & 0 \le x \le 2, \\ (x+0.5)^2, & -2 \le x < 0. \end{cases}$$

a) Approximate $h(x)$ with a cubic spline, using $n = 5$ equally spaced nodes. Plot the function along with your approximation and calculate the *root mean squared error* (RMSE) of your approximation over a fine grid with step size 0.0001.

b) Approximate $h(x)$ with a cubic spline, using $n = 10$ equally spaced nodes. Plot this new approximation along with both the actual function and your approximation from (a), and compute the RMSE over the same fine grid. *Explain the differences* in your answers.

Question 4 - Neoclassical Growth Model with taxes

There is a continuum of identical households (measure one). Each household has N_t members, which grows at rate η. Each member has one unit of time; let h_t and l_t be per-member hours worked and leisure, respectively, with $h_t + l_t = 1$. Preferences:

$$U \;=\; \sum_{t=0}^{\infty} \beta^t N_t \big[\ln(c_t) + \theta \ln(l_t)\big].$$

Productivity follows $A_{t+1} = (1+\gamma)A_t$. Let K_t be capital and H_t total hours in production. Technology:

$$Y_t \;=\; K_t^{\alpha} (A_t H_t)^{1-\alpha}, \qquad K_{t+1} \;=\; (1-\delta)K_t + I_t,$$

where I_t is investment att time t and K_{t+1} governs how the capital evolves. Government finances G_t via taxes on consumption (τ^c), labor income (τ^h), and capital income (τ^k), with a balanced budget each period:

$$G_t \;=\; \tau^c N_t c_t + \tau^h w_t N_t h_t + \tau^k r_t K_t.$$

Resource constraint:

$$Y_t \;=\; N_t c_t + I_t + G_t.$$

a) Define a competitive equilibrium and write down the equations that describe the equilibrium of the system.

b) What are the variables' growth rates along a balanced-growth-path equilibrium? (Hint: assume that taxes are constant and that G grows at the same rate of Y so that G/Y is constant.)

c) Write down the equivalent stationary system.

d) Calibrate $\beta = 0.96$, $\delta = 0.10$, $\gamma = 0.02$, $\alpha = 0.4$, $\theta = 1$, $\tau^c = 0.17$, $\tau^h = 0.27$, $\tau^k = 0.15$. Write a program to solve the *transitional dynamics* using Newton, Secant, or `fsolve`. Assume $K_0 = 0.8\, K_{SS}$.

e) Plot the dynamics of per-capita capital, consumption, investment, labor, and output.

f) Now our goal is to study a tax reform. Suppose that the government wants to reduce the tax on capital income by increasing the tax on labor income and keeping the same level of spending. (In the long run, the tax reform is revenue neutral). Write a program to implement this tax reform. Assume that the initial capital stock is the steady-state capital stock prior to the reform. Show the dynamics and analyze this reform.

g) Calculate the long run welfare implications of this tax reform in terms of consumption:

$$\big[\ln\big((1+\omega)c^{\text{new}}\big) + \gamma \ln(l^{\text{new}})\big] - \big[\ln(c^{\text{old}}) + \gamma \ln(l^{\text{old}})\big],$$

where ω is the compensating variation in consumption, i,e., the percentage of consumption that the household must be compensated (or pay) to accept this tax reform

Question 5 - Infinite-Horizon Savings Problem

Consider the following savings problem. An infinitely-lived household has preferences over consumption at each date given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t).$$

The household has wealth given by W_0 in period 0. The budget constraint in each period is:

$$c_t + W_{t+1} \leq RW_t.$$

Also assume $c_t, W_{t+1} \geq 0 \quad \forall t \geq 0$.

a) Suppose $\beta R = 1$. Set up and characterize the solution to the household's problem.

b) Suppose $\beta R \neq 1$. Characterize the solution to the household's problem (i.e., characterize the solution for the case in which we have $\beta R < 1$ and for the case in which we have $\beta R > 1$).

Question 6- Complete Markets

A pure endowment economy consists of two types of infinitely lived consumers, each of whom has the same utility function:

$$u\left(c_0^i, c_1^i, \ldots\right) = \sum_{t=0}^{\infty} \beta^t \log c_t^i,$$

where $0 < \beta < 1$ is a common discount factor. Suppose that consumer 1 has the endowments $\left(e_0^1, e_1^1, e_2^1, e_3^1, \ldots\right) = (5, 3, 5, 3, \ldots)$ and that consumer 2 has the endowments $\left(e_0^2, e_1^2, e_2^2, e_3^2, \ldots\right) = (3, 5, 3, 5, \ldots)$.

a) Describe an Arrow-Debreu structure for this economy, explaining when markets are open, who trades with whom, and so on. Define an Arrow-Debreu Equilibrium. Be careful to include definitions of all the objects of which the competitive equilibrium is defined.

b) Compute the Arrow-Debreu Equilibrium of this economy.

c) Describe a Sequential Market structure for this economy, explaining when markets are open, who trades with whom, and so on. Define a Sequential Market Equilibrium. Be careful to include definitions of all the objects of which the competitive equilibrium is defined.

d) Compute the Sequential Market Equilibrium for this economy (including the one-period gross interest rates).

e) Redo the calculation in b) by considering the following stream of endowments (in this case there will be a price for when the aggregate endowment is high and a different one for when the aggregate endowment is low):

$$\left(e_0^1, e_1^1, e_2^1, e_3^1, \ldots\right) = (5, 3, 5, 3, \ldots)$$
$$\left(e_0^2, e_1^2, e_2^2, e_3^2, \ldots\right) = (4, 4, 4, 4, \ldots)$$

Question 7 - Complete Markets

A pure endowment economy consists of two types of consumers. Consumers of type 1 order consumption streams of the good according to the utility function:

$$\sum_{t=0}^{\infty} \beta^t c_t^1,$$

and consumer of type 2 order consumption streams according to:

$$\sum_{t=0}^{\infty} \beta^t \ln\left(c_t^2\right),$$

where $c_t^i \geq 0$ is the consumption of a type i consumer and $0 < \beta < 1$ is a common discount factor. The consumption good is tradable but non-storable. There are equal numbers of the two types of consumers. The consumer of type 1 is endowed with the consumption sequence:

$$e_t^1 = \mu > 0 \quad \forall t \geq 0,$$

The consumer of type 2 is endowed with the consumption sequence:

$$\begin{aligned}&\text{If } t \text{ is even (pair), } e_t^2 = 0,\\ &\text{If } t \text{ is odd (impair), } e_t^2 = \alpha,\end{aligned}$$

where $\alpha = \mu \left(1 + \beta^{-1}\right)$.

a) Define an Arrow-Debreu Equilibrium. Be careful to include definitions of all the objects of which the competitive equilibrium is defined.

b) Compute the Arrow-Debreu Equilibrium of this economy.

c) Compute the time 0 wealths of the two types of consumers using the equilibrium prices found in the previous item.

d) Define a Sequential Market Equilibrium. Be careful to include definitions of all the objects of which the competitive equilibrium is defined.

e) Compute the Sequential Market Equilibrium for this economy (including the one-period gross interest rates).