

Topics in Macroeconomics

Lecture 9: Economies with Aggregate and Idiosyncratic Shocks

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Roadmap

1. Aiyagari model with aggregate productivity shocks.
2. Krussel and Smith algorithm.
3. Results of the regression.

1. Aiyagari model with aggregate productivity shocks.

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Environment

- ▶ **Aggregate shock:** $z \in \{z_b, z_g\} = \{0.99, 1.01\}$.
- ▶ **Idiosyncratic shock:** $e \in \{0, 1\}$ (unemployed / employed).
- ▶ **Joint evolution:** Markov chain,

$$\pi(z', e' \mid z, e) = \Pr(z_{t+1} = z', e_{t+1} = e' \mid z_t = z, e_t = e).$$

- ▶ **Production:**

$$Y_t = Z_t K_t^\alpha H_t^{1-\alpha}, \quad w(K, H, Z) = (1 - \alpha) Z K^\alpha H^{-\alpha},$$
$$R(K, H, Z) = 1 + \alpha Z K^{\alpha-1} H^{1-\alpha} - \delta.$$

Households

Continuum of ex-ante identical agents (unit mass). Idiosyncratic employment $e \in \{0, 1\}$.
Markets are incomplete; savings via assets a (physical capital). Labor supply is endogenous.

Recursive problem:

$$V(a, e, K, H, Z) = \max_{c, a', n} \frac{(c^\eta (1 - n)^{1-\eta})^{1-\mu}}{1 - \mu} + \beta \mathbb{E}[V(a', e', K', H', Z') \mid e, K, H, Z]$$

s.t.

$$c \leq R(K, H, Z) a - a' + w(K, H, Z) e n, \quad a' \geq 0.$$

Key implication: With endogenous H , prices (R, w) are *not* pinned down solely by K ; **agents must forecast both.**

Perceived law of motion (PLM)

Agents forecast next period aggregates via state-contingent linear rules:

$$\log K' = b_z^0 + b_z^1 \log K, \quad \log H' = d_z^0 + d_z^1 \log K, \quad z \in \{z_b, z_g\}.$$

Motivation: If labor were exogenous, K alone would forecast K/H and thus (R, w) . With endogenous H , we need separate PLMs for K' and H' .

FOCs and endogenous grid method (EGM)

First-order conditions:

$$\frac{u_n}{u_c} = w, \quad u_c(c, n) = \beta \mathbb{E}[R(K', H', Z') u_c(c', n')].$$

Labor policy:

$$n^*(c, e, K, H, Z) = \begin{cases} 1 - \frac{1-\eta}{\eta} \frac{c}{w(K, H, Z)}, & e = 1, \\ 0, & e = 0. \end{cases}$$

Closed-form for c given constant RHS target C from Euler:

$$c = \begin{cases} \left(\frac{C}{\eta} \left(\frac{1-\eta}{\eta w(K, H, Z)} \right)^{-(1-\mu)(1-\eta)} \right)^{-1/\mu}, & e = 1, \\ \left(\frac{C}{\eta} \right)^{-1/\mu}, & e = 0. \end{cases}$$

Algorithm: Define grid in a' (endogenous grid), back out (c, n) from FOCs, and then implied a via budget.

Calibration and shock process

- ▶ **Preferences:** $\beta = 0.99$, $\mu = 1$ (CRRA=1), utility over c and leisure.
- ▶ **Technology:** $\delta = 0.0025$, Cobb–Douglas with α .
- ▶ **Shocks (Z, e) calibrated to match:**
 - ▶ Unemployment rate: 4% in expansions, 10% in recessions.
 - ▶ Expected duration: expansions ≈ 8 quarters, recessions ≈ 8 quarters.
 - ▶ Unemployment duration: 1.5 (expansions), 2.5 (recessions).
- ▶ **Solver:** Carroll's EGM to handle high-dimensional state \Rightarrow substantial speed-up.

2. Krussel & Smith algorithm.

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Krusell & Smith (1998) algorithm: idea

Goal: Find a *perceived* law of motion (PLM) for (K', H') that is consistent with the *actual* law of motion (ALM) induced by optimal policies.

Steps (high level):

1. Choose PLMs: $\log K' = b_z^0 + b_z^1 \log K$, $\log H' = d_z^0 + d_z^1 \log K$.
2. Guess coefficients (b, d) for each $z \in \{z_b, z_g\}$.
3. Solve household problem given PLMs, obtain policies $(c, a', n)(a, e, K, H, Z)$.

Krusell & Smith (1998) algorithm: simulation and updating

1. Simulate economy with N agents for T periods:

$$\hat{K}_t = \frac{1}{N} \sum_{i=1}^N a_{it}, \quad \hat{H}_t = \frac{1}{N} \sum_{i=1}^N e_{it} n_{it}.$$

Labor market must clear each period: adjust w_t until \hat{H}_t matches the implied aggregate H_t .

2. Discard burn-in S periods. For each regime $z \in \{z_b, z_g\}$, run regressions:

$$\log \hat{K}_{t+1} = B_z^0 + B_z^1 \log \hat{K}_t, \quad \log \hat{H}_{t+1} = D_z^0 + D_z^1 \log \hat{H}_t.$$

3. Convergence: if $\max\{|b_z^i - B_z^i|, |d_z^i - D_z^i|\} < \varepsilon$ for all z and $i \in \{0, 1\}$, stop; else set $(b, d) \leftarrow (B, D)$ and repeat.

Accuracy & fit

- ▶ Check R^2 and out-of-sample fit of the PLMs.
- ▶ If fit is poor, enrich the PLM (additional moments, nonlinearity, or adding distributional states).
- ▶ This procedure approximates the full rational expectations equilibrium in an incomplete-markets setting.

3. Results of the regression.

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Capital regression: our estimates vs. KS

Our estimates:

$$\text{Recession: } \ln K_{t+1} = 0.11495 + 0.95422 \ln K_t, \quad R^2 = 0.999,$$

$$\text{Expansion: } \ln K_{t+1} = 0.101475 + 0.957641 \ln K_t, \quad R^2 = 0.999.$$

Krusell & Smith (1998):

$$\text{Recession: } \ln K_{t+1} = 0.114 + 0.953 \ln K_t,$$

$$\text{Expansion: } \ln K_{t+1} = 0.123 + 0.951 \ln K_t.$$

Labor regression: our estimates vs. KS

Our estimates:

$$\text{Recession: } \ln H_t = -0.68278 - 0.217856 \ln K_t, \quad R^2 = 0.995,$$

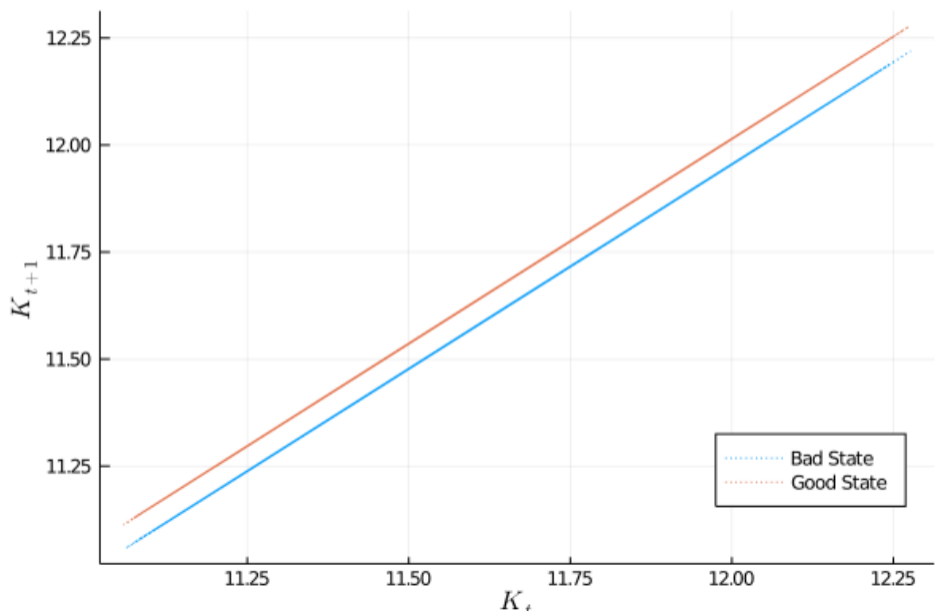
$$\text{Expansion: } \ln H_t = -0.605365 - 0.226267 \ln K_t, \quad R^2 = 0.997.$$

Krusell & Smith (1998):

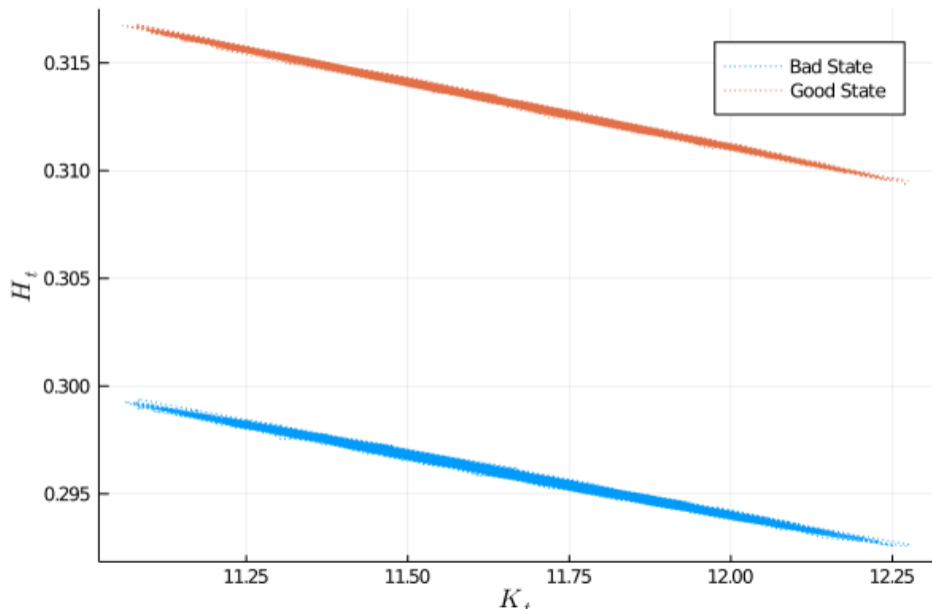
$$\text{Recession: } \ln H_t = -0.592 - 0.255 \ln K_t,$$

$$\text{Expansion: } \ln H_t = -0.544 - 0.252 \ln K_t.$$

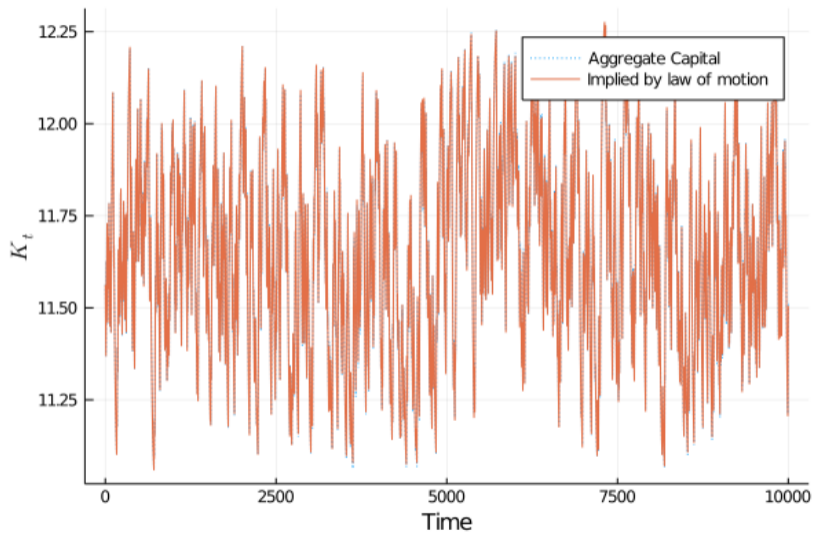
Aggregate capital: K_{t+1} vs K_t



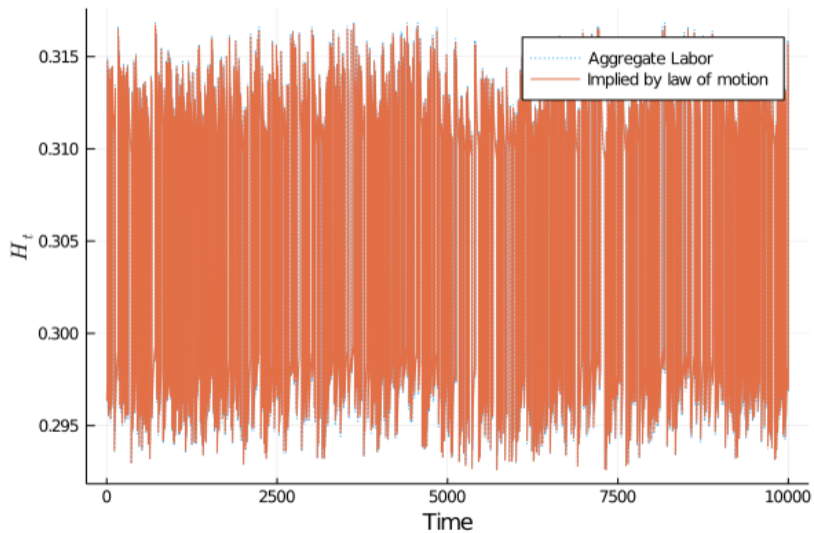
Aggregate labor vs capital



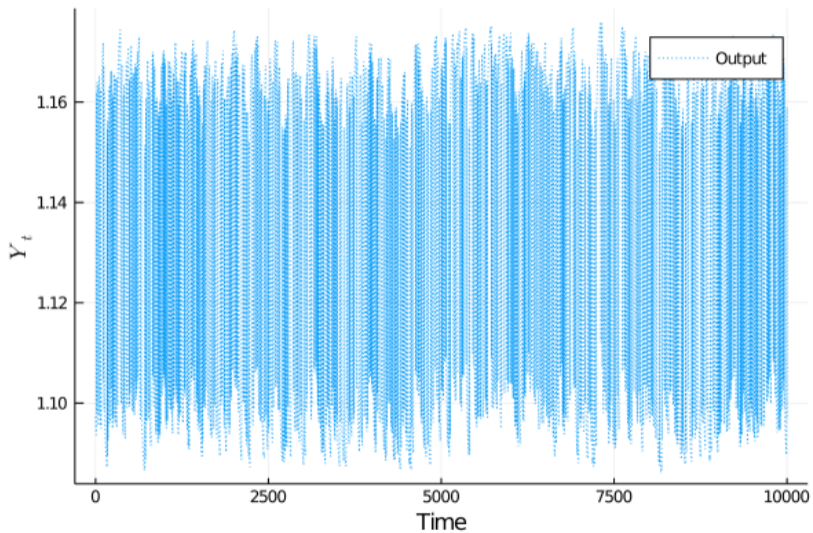
Actual vs implied paths: Capital



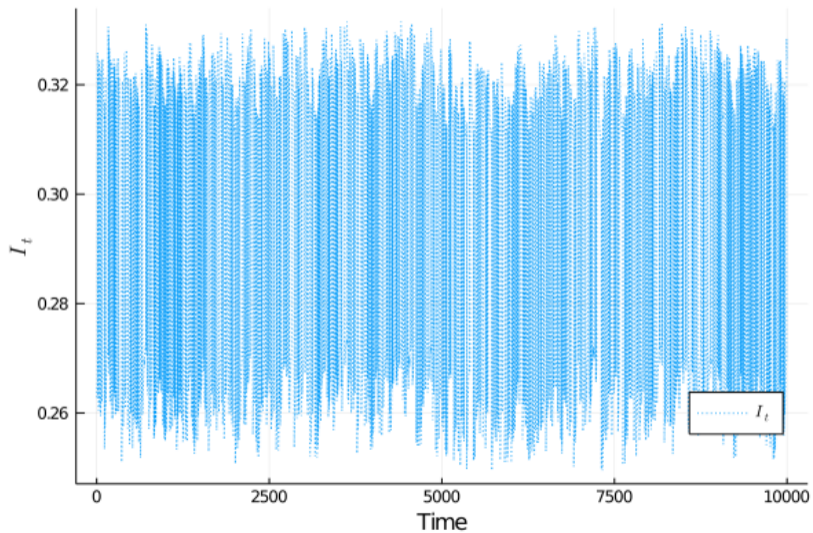
Actual vs implied paths: Labor



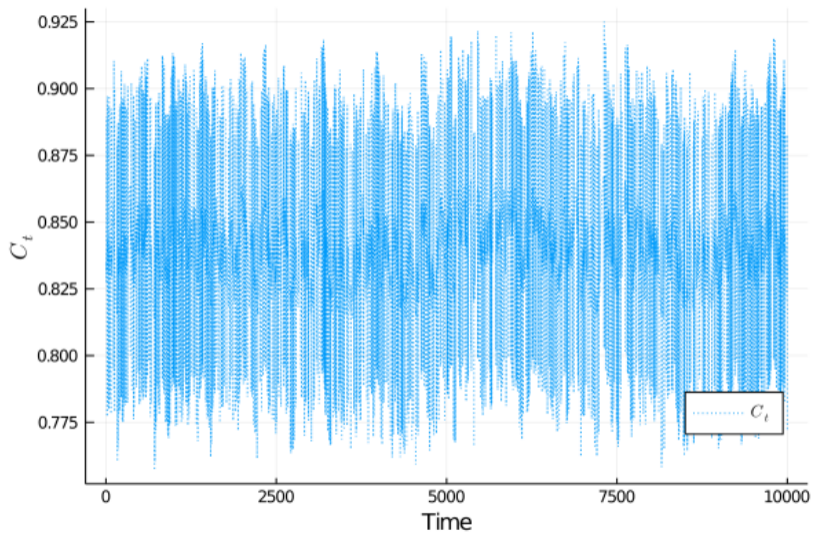
Aggregate output



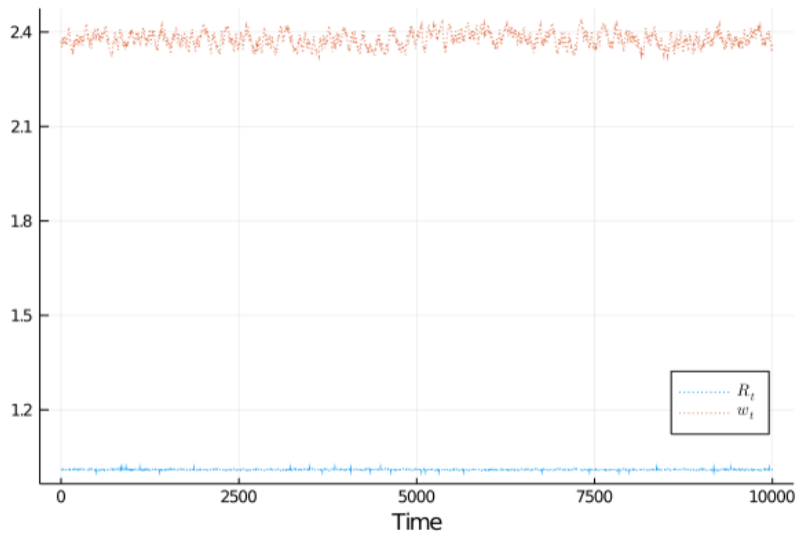
Aggregate investment



Aggregate consumption



Prices: return on capital and wages



Moments: Correlation matrix (Y , C , I)

	Y_t	C_t	I_t
Y_t	1.000	0.570	0.744
C_t	0.570	1.000	-0.124
I_t	0.744	-0.124	1.000

Moments: Means

	Y_t	C_t	I_t
Mean	1.13097	0.839699	0.29127

Moments: Standard deviations

	Y_t	C_t	I_t
Std. dev.	0.03413	0.02298	0.02826

Takeaways

- ▶ **KS algorithm finds aggregation rules (K', H')** that summarize a high-dimensional distribution.
- ▶ **Endogenous labor requires forecasting both K' and H'** (prices depend on K/H).
- ▶ **EGM is effective to solve household problems with many states.**
- ▶ Our estimated PLMs closely match Krusell & Smith (1998) benchmarks.