Topics in Macroeconomics

Lecture 6: Local Linear Methods for Solving Macroeconomic Models

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Road Map

1. Local approximation methods.
2. Log-linearisation and linearisation.
3. Solving systems of (stochastic) difference equations:
   - Blanchard and Kahn (1980) method.
   - Uhlig (1999) method.
   - Christiano (2002) method.

A Generic DSGE Model

Model setup:

$$U = \max_{\{x_t, y_t\}} E_0 \sum_{t=0}^{\infty} \beta^t u(x_t, y_t),$$

subject to

$$x_t = g(x_{t-1}, y_t, z_t) \quad (\text{or } x_{t+1} = g(x_t, y_t, z_t)).$$

$$z_t = f(z_{t-1}, \varepsilon_t), \quad \varepsilon_t \sim i.i.d.\ (0, \Sigma).$$

with x_{-1} (or x_0) and z_0 given.

Notation:

- x_t: vector of endogenous state variables ($m \times 1$).
- z_t: vector of exogenous state variables (e.g. shocks).
- y_t: vector of other endogenous (control) variables ($n \times 1$).

Deriving Equilibrium and Solving the Model

1. Write the **Lagrangian**.
2. Take the **first-order conditions (FOCs)**.
3. Collect all other equations (market-clearing, policy rules, etc.).
4. Obtain a system of ν stochastic difference equations in ν variables.

Characteristics:

- Highly non-linear and stochastic in general.
- DSGE models typically involve many endogenous and state variables.
- Hence, global solution methods become computationally costly.
- We therefore rely on **local linear approximation methods**.

First-Order Approximation: Log-linearisation

Goal: Obtain a linear (stochastic) system around a deterministic steady state.

Steps:

1. Identify all equilibrium conditions (FOCs, budget/resource, market clearing, policy rules).
2. Compute steady states for all variables (deterministic; may be non-trivial).
3. For any variable x_t, define the log-deviation:

$$\hat{x}_t \equiv \log\left(\frac{x_t}{\bar{x}}\right) \approx \frac{x_t - \bar{x}}{\bar{x}} \quad \Rightarrow \quad x_t = \bar{x}\, e^{\hat{x}_t}.$$

4. Replace each x_t in every equation by $\bar{x}\, e^{\hat{x}_t}$.

Remark: Log-deviations are unit-free and convenient when steady states are strictly positive.

First-Order Approximation: Log-linearisation (cont.)

Linearise the nonlinear terms:

$$e^{\hat{x}_t} \approx 1 + \hat{x}_t \quad \text{(first-order Taylor).}$$

Collect terms:

- Use steady-state relationships to cancel constants.
- Keep only first-order terms in $\hat{x}_t$'s.

Outcome: Each equilibrium condition becomes linear and homogeneous in the log-deviations.

Stack into a linear system of (stochastic) difference equations. Typical representation after log-linearisation:

$$\hat{x}_t = P\,\hat{x}_{t-1} + Q\,\hat{z}_t, \qquad \hat{y}_t = R\,\hat{x}_{t-1} + S\,\hat{z}_t, \qquad \hat{z}_t = N\,\hat{z}_{t-1} + \varepsilon_t.$$

Here $\hat{x}_t$ (states), $\hat{y}_t$ (controls), $\hat{z}_t$ (exogenous shocks).

First-Order Approximation: Linearisation (levels)

Alternative: Linearise in levels (not logs) using

$$\tilde{x}_t \equiv x_t - \bar{x}, \qquad f(x_t) \approx f(\bar{x}) + f'(\bar{x})\,\tilde{x}_t,$$

or for multi-variate g:

$$g(x_t^1, x_t^2) \approx g(\bar{x}^1, \bar{x}^2) + g_1(\bar{x}^1, \bar{x}^2)\,\tilde{x}_t^1 + g_2(\bar{x}^1, \bar{x}^2)\,\tilde{x}_t^2.$$

When prefer levels?

- Variables with zero (or possibly negative) steady states (e.g., inflation deviations).
- Situations where log-linearisation is ill-defined.

Key point: Dynamic properties from linearisation vs. log-linearisation are typically similar; choice is driven by convenience and feasibility (positivity of steady states).

Example: Stochastic Growth Model

Model:

$$\max_{\{k_t, c_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t,$$

subject to

$$k_t = z_t k_{t-1}^{\alpha} - c_t, \qquad k_t, c_t \geq 0, \quad k_0 \text{ given},$$

and the stochastic process for technology:

$$\log z_{t+1} = (1-\rho)\log \bar{z} + \rho \log z_t + \eta_t, \qquad \eta_t \sim \mathcal{N}(0, \sigma^2), \ \ i.i.d.$$

State variables: endogenous k_{t-1} and exogenous z_t.

Equilibrium conditions:

$$\lambda_t = \frac{1}{c_t}, \qquad \lambda_t = \alpha\beta E_t[\lambda_{t+1} z_{t+1} k_t^{\alpha-1}],$$

$$k_t = y_t - c_t, \qquad y_t = z_t k_{t-1}^{\alpha}.$$

Steady State of the Stochastic Growth Model

Start from the model:

$$\max_{\{c_t, k_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t \quad \text{s.t.} \quad k_t = z_t k_{t-1}^{\alpha} - c_t, \quad \log z_{t+1} = (1-\rho)\log \bar{z} + \rho \log z_t + \eta_t.$$

At the deterministic steady state ($z_t = \bar{z} = 1$ and $k_t = k_{t-1} = \bar{k}$):

$$\bar{k} = \bar{k}^{\alpha} - \bar{c}, \qquad 1 = \alpha\beta\bar{k}^{\alpha-1}.$$

From these,

$$\frac{\bar{c}}{\bar{k}} = \bar{k}^{\alpha-1} - 1 \quad \Rightarrow \quad \frac{\bar{y}}{\bar{k}} = \bar{k}^{\alpha-1} = \frac{1}{\alpha\beta}.$$

Normalization: take $\bar{z} = 1$ to simplify log-deviations.

Preliminaries: Log-Deviations and First-Order Rules

For any variable x_t with steady state $\bar{x} > 0$, define the log-deviation:

$$\hat{x}_t \equiv \log\left(\frac{x_t}{\bar{x}}\right).$$

First-order (FO) approximations around steady state:

- Products: $\widehat{ab} \approx \hat{a} + \hat{b}$.
- Powers: $\widehat{a^\gamma} \approx \gamma\, \hat{a}$.
- Ratios: $\widehat{a/b} \approx \hat{a} - \hat{b}$.
- Sums (level linearisation with shares):

$$x_t = \sum_i \xi_t^{(i)} \quad \Rightarrow \quad \hat{x}_t \approx \sum_i \frac{\bar{\xi}^{(i)}}{\bar{x}}\, \hat{\xi}^{(i)}.$$

Cross-products of hats are second order and are dropped.

Derivation 1: $\lambda_t = \frac{1}{c_t} \Rightarrow \hat{\lambda}_t = -\hat{c}_t$

Start from $\lambda_t = \dfrac{1}{c_t}$ with steady states $\bar{\lambda} = \dfrac{1}{\bar{c}}$.

$$\frac{\lambda_t}{\bar{\lambda}} = \frac{1/c_t}{1/\bar{c}} = \frac{\bar{c}}{c_t} \quad \Rightarrow \quad \hat{\lambda}_t = \log\left(\frac{\lambda_t}{\bar{\lambda}}\right) = \log\left(\frac{\bar{c}}{c_t}\right) = -\log\left(\frac{c_t}{\bar{c}}\right) = -\hat{c}_t.$$

Derivation 2: Euler Equation (Step by Step)

Euler: $\lambda_t = \alpha\beta\, E_t\big[\lambda_{t+1} z_{t+1} k_t^{\alpha-1}\big]$. At steady state: $\bar{\lambda} = \alpha\beta\,\bar{\lambda}\,\bar{z}\,\bar{k}^{\alpha-1}$ (holds by SS conditions).

Divide by steady-state and use FO rules:

$$\frac{\lambda_t}{\bar{\lambda}} = E_t\Big[\frac{\lambda_{t+1}}{\bar{\lambda}} \cdot \frac{z_{t+1}}{\bar{z}} \cdot \Big(\frac{k_t}{\bar{k}}\Big)^{\alpha-1}\Big] \;\approx\; E_t\Big[\exp\big(\hat{\lambda}_{t+1} + \hat{z}_{t+1} + (\alpha-1)\hat{k}_t\big)\Big].$$

Taking a first-order (log) approximation around the SS and using that expectations of products reduce to sums at first order:

$$\hat{\lambda}_t \;\approx\; E_t\hat{\lambda}_{t+1} \;+\; E_t\hat{z}_{t+1} \;+\; (\alpha-1)\hat{k}_t.$$

Using the AR(1): $E_t\hat{z}_{t+1} = \rho\,\hat{z}_t$ and $(\alpha-1) = -(1-\alpha)$:

$$\boxed{\hat{\lambda}_t \;\approx\; E_t\hat{\lambda}_{t+1} \;+\; \rho\,\hat{z}_t \;-\; (1-\alpha)\hat{k}_t.}$$

Derivation 3: Resource Constraint $k_t = y_t - c_t$

Level perturbation around $(\bar{k}, \bar{y}, \bar{c})$ with $\bar{k} = \bar{y} - \bar{c}$:

$$k_t = y_t - c_t \;\Rightarrow\; \bar{k}(1 + \hat{k}_t) \;\approx\; \bar{y}(1 + \hat{y}_t) - \bar{c}(1 + \hat{c}_t).$$

Cancel steady-state levels and ignore second-order terms:

$$\bar{k}\,\hat{k}_t \;\approx\; \bar{y}\,\hat{y}_t \;-\; \bar{c}\,\hat{c}_t \quad\Rightarrow\quad \boxed{\hat{k}_t \;=\; \frac{\bar{y}}{\bar{k}}\,\hat{y}_t \;-\; \frac{\bar{c}}{\bar{k}}\,\hat{c}_t.}$$

Interpretation: The elasticities (coefficients) are steady-state shares.

Derivation 4: Production $y_t = z_t k_{t-1}^{\alpha}$

Apply the product and power rules:

$$\widehat{y_t} \;=\; \widehat{z_t} + \widehat{k_{t-1}^{\alpha}} \;\approx\; \hat{z}_t + \alpha\,\hat{k}_{t-1}.$$

$$\boxed{\hat{y}_t \;\approx\; \alpha\,\hat{k}_{t-1} + \hat{z}_t.}$$

Derivation 5: Technology Shock

Given $\log z_{t+1} = (1-\rho)\log \bar{z} + \rho \log z_t + \eta_t$ with $\bar{z} = 1$:

$$\hat{z}_{t+1} \; = \; \rho\, \hat{z}_t + \eta_t.$$

This immediately implies $E_t \hat{z}_{t+1} = \rho\, \hat{z}_t$ used in the Euler derivation.

Collected First-Order Relationships (to be stacked next)

$$\hat{\lambda}_t = -\hat{c}_t, \qquad \hat{\lambda}_t = E_t \hat{\lambda}_{t+1} + \rho\, \hat{z}_t - (1-\alpha)\hat{k}_t,$$

$$\hat{k}_t = \frac{\bar{y}}{\bar{k}}\, \hat{y}_t - \frac{\bar{c}}{\bar{k}}\, \hat{c}_t, \qquad \hat{y}_t \approx \alpha\, \hat{k}_{t-1} + \hat{z}_t,$$

$$\hat{z}_{t+1} = \rho\, \hat{z}_t + \eta_t.$$

Log-linearising the Stochastic Growth Model

Step 1: Take logs around the deterministic steady state ($\bar{z} = 1$).

Step 2: Express deviations from steady state:

$$\hat{x}_t = \log\left(\frac{x_t}{\bar{x}}\right).$$

Approximate each equation:

$$\lambda_t = \frac{1}{c_t} \Rightarrow \hat{\lambda}_t \approx -\hat{c}_t,$$

$$\lambda_t = \alpha\beta E_t[\lambda_{t+1} z_{t+1} k_t^{\alpha-1}] \Rightarrow \hat{\lambda}_t \approx E_t\hat{\lambda}_{t+1} + \rho\hat{z}_t - (1-\alpha)\hat{k}_t,$$

$$k_t = y_t - c_t \Rightarrow \hat{k}_t \approx \frac{\bar{y}}{\bar{k}}\hat{y}_t - \frac{\bar{c}}{\bar{k}}\hat{c}_t,$$

$$y_t = z_t k_{t-1}^{\alpha} \Rightarrow \hat{y}_t \approx \alpha\hat{k}_{t-1} + \hat{z}_t,$$

$$\hat{z}_{t+1} = \rho\hat{z}_t + \eta_t.$$

Result: A system of linear stochastic difference equations describing log-deviations from steady state.

Step 1 — Linearised System and Steady-State Shares

Linearised equilibrium conditions:

$$(i)\ \hat{\lambda}_t = -\hat{c}_t, \qquad (ii)\ \hat{\lambda}_t = E_t\hat{\lambda}_{t+1} + \rho\hat{z}_t - (1-\alpha)\hat{k}_t,$$

$$(iii)\ \hat{k}_t = \frac{\bar{y}}{\bar{k}}\hat{y}_t - \frac{\bar{c}}{\bar{k}}\hat{c}_t, \qquad (iv)\ \hat{y}_t = \alpha\hat{k}_{t-1} + \hat{z}_t,$$

$$(v)\ \hat{z}_{t+1} = \rho\hat{z}_t + \eta_t \ \Rightarrow\ E_t\hat{z}_{t+1} = \rho\hat{z}_t.$$

Steady-state relationships:

$$\frac{\bar{y}}{\bar{k}} = \frac{1}{\alpha\beta}, \qquad \frac{\bar{c}}{\bar{k}} = \frac{1}{\alpha\beta} - 1.$$

Define shares $s_y = \frac{\bar{y}}{\bar{k}} = \frac{1}{\alpha\beta}$, $s_c = s_y - 1$.

Step 2 — Eliminate $\hat{\lambda}_t$

From (i) and (ii):

$$-\hat{c}_t = -E_t \hat{c}_{t+1} + \rho \hat{z}_t - (1-\alpha)\hat{k}_t.$$

Rearranging gives:

$$\boxed{\hat{c}_t = E_t \hat{c}_{t+1} - \rho \hat{z}_t + (1-\alpha)\hat{k}_t.} \tag{E1}$$

This Euler equation now links current and expected future consumption to the capital stock and technology.

Step 3 — Express $\hat{c}_t$ in terms of $\hat{k}_{t-1}$, $\hat{k}_t$, and $\hat{z}_t$

Use (iii) and (iv):

$$\hat{k}_t = s_y(\alpha \hat{k}_{t-1} + \hat{z}_t) - s_c \hat{c}_t.$$

Solve for $\hat{c}_t$:

$$\boxed{\hat{c}_t = \frac{s_y}{s_c}(\alpha \hat{k}_{t-1} + \hat{z}_t) - \frac{1}{s_c}\hat{k}_t.} \tag{C$_t$}$$

This expresses current consumption as a weighted combination of past capital, current technology, and the current capital stock.

Step 4 — Compute $E_t \hat{c}_{t+1}$

The same relationship one period ahead:

$$\hat{k}_{t+1} = s_y(\alpha \hat{k}_t + \hat{z}_{t+1}) - s_c \hat{c}_{t+1}.$$

Taking expectations and using $E_t \hat{z}_{t+1} = \rho \hat{z}_t$:

$$\boxed{E_t \hat{c}_{t+1} = \frac{s_y}{s_c}(\alpha \hat{k}_t + \rho \hat{z}_t) - \frac{1}{s_c} E_t \hat{k}_{t+1}.} \tag{C$_{t+1}$}$$

We now have both $\hat{c}_t$ and $E_t \hat{c}_{t+1}$ in terms of $\hat{k}_{t-1}$, $\hat{k}_t$, $E_t \hat{k}_{t+1}$, and $\hat{z}_t$.

Step 5 — Substitute (C_t) and (C_{t+1}) into (E1)

Substitute into $\hat{c}_t = E_t\hat{c}_{t+1} - \rho\hat{z}_t + (1-\alpha)\hat{k}_t$:

$$\frac{s_y}{s_c}(\alpha\hat{k}_{t-1} + \hat{z}_t) - \frac{1}{s_c}\hat{k}_t = \frac{s_y}{s_c}(\alpha\hat{k}_t + \rho\hat{z}_t) - \frac{1}{s_c}E_t\hat{k}_{t+1} - \rho\hat{z}_t + (1-\alpha)\hat{k}_t.$$

Multiply by s_c:

$$s_y(\alpha\hat{k}_{t-1} + \hat{z}_t) - \hat{k}_t = s_y(\alpha\hat{k}_t + \rho\hat{z}_t) - E_t\hat{k}_{t+1} - s_c\rho\hat{z}_t + s_c(1-\alpha)\hat{k}_t.$$

Step 6 — Collect Terms and Obtain the Final Equation

Rearranging terms gives:

$$E_t \hat{k}_{t+1} + \big[-1 - s_y \alpha - s_c(1-\alpha) \big] \hat{k}_t + s_y \alpha \, \hat{k}_{t-1} + \big[s_y + \rho(s_c - s_y) \big] \hat{z}_t = 0.$$

Using $s_y = \frac{1}{\alpha\beta}$ and $s_c = s_y - 1$, simplify:

$$\boxed{(1+\alpha^2\beta)\hat{k}_t = \alpha\beta E_t \hat{k}_{t+1} + \alpha \hat{k}_{t-1} + (1-\rho\alpha\beta)\hat{z}_t.}$$

This is the single expectational difference equation governing the law of motion for capital after linearisation.

Example: Stochastic Growth Model — Solving by Hand

From the linearised system (using $\hat{\lambda}_t = -\hat{c}_t$, $\hat{y}_t = \alpha\hat{k}_{t-1} + \hat{z}_t$, and shares in the resource constraint), one can eliminate $\hat{c}_t, \hat{y}_t, \hat{\lambda}_t$ to get a single expectational difference equation in $\hat{k}_t$:

$$\boxed{(1+\alpha^2\beta)\,\hat{k}_t \;=\; \alpha\beta\,E_t\hat{k}_{t+1} \;+\; \alpha\,\hat{k}_{t-1} \;+\; (1-\rho\,\alpha\beta)\,\hat{z}_t.}$$

Sketch of steps:

1. Euler: $\hat{\lambda}_t = E_t\hat{\lambda}_{t+1} + \rho\hat{z}_t - (1-\alpha)\hat{k}_t$ and $\hat{\lambda}_t = -\hat{c}_t$.
2. Production: $\hat{y}_t = \alpha\hat{k}_{t-1} + \hat{z}_t$.
3. Resource: $\hat{k}_t = \frac{\bar{y}}{\bar{k}}\hat{y}_t - \frac{\bar{c}}{\bar{k}}\hat{c}_t$ (use steady-state shares).
4. Substitute to eliminate $\hat{c}_t, \hat{y}_t$ and collect terms in $\hat{k}_{t-1}, \hat{k}_t, E_t\hat{k}_{t+1}, \hat{z}_t$.

This yields a *second-order* (in time) linear stochastic difference equation in the state $\hat{k}_t$.

Undetermined Coefficients: Guess and Solve

Postulate a linear law of motion for capital:

$$\hat{k}_t = \phi_1 \, \hat{k}_{t-1} + \phi_2 \, \hat{z}_t \quad \Rightarrow \quad E_t \hat{k}_{t+1} = \phi_1 \, \hat{k}_t + \phi_2 \, \rho \, \hat{z}_t .$$

Substitute the guess into $(1+\alpha^2\beta)\hat{k}_t = \alpha\beta E_t \hat{k}_{t+1} + \alpha \hat{k}_{t-1} + (1-\rho\alpha\beta)\hat{z}_t$ and equate coefficients on $\hat{k}_{t-1}$ and $\hat{z}_t$. After algebra:

$$\hat{k}_t = \frac{\alpha}{1+\alpha^2\beta-\alpha\beta\phi_1} \, \hat{k}_{t-1} \; + \; \frac{\alpha\beta\rho\phi_2 + (1-\alpha\beta\rho)}{1+\alpha^2\beta-\alpha\beta\phi_1} \, \hat{z}_t .$$

Matching coefficients gives the solutions

$$\boxed{\phi_1 \in \left\{ \alpha, \; \frac{1}{\alpha\beta} \right\}, \qquad \phi_2 = \begin{cases} 1, & \text{if } \phi_1 = \alpha, \\ \dfrac{1-\alpha\beta\rho}{\alpha^2\beta - \alpha\beta\rho}, & \text{if } \phi_1 = \dfrac{1}{\alpha\beta}. \end{cases}}$$

Example: Stochastic Growth Model — Solving for ϕ_1 and ϕ_2

We have the general law of motion postulated as:

$$\hat{k}_t = \phi_1 \, \hat{k}_{t-1} + \phi_2 \, \hat{z}_t, \qquad E_t \hat{k}_{t+1} = \phi_1 \, \hat{k}_t + \phi_2 \, \rho \, \hat{z}_t.$$

Substitute into the expectational difference equation:

$$(1 + \alpha^2 \beta)\hat{k}_t = \alpha\beta E_t \hat{k}_{t+1} + \alpha \hat{k}_{t-1} + (1 - \rho\alpha\beta)\hat{z}_t.$$

After substituting and collecting coefficients, we obtain:

$$\hat{k}_t = \frac{\alpha}{1 + \alpha^2\beta - \alpha\beta\phi_1} \hat{k}_{t-1} + \frac{\alpha\beta\rho\phi_2 + (1 - \alpha\beta\rho)}{1 + \alpha^2\beta - \alpha\beta\phi_1} \hat{z}_t.$$

Matching coefficients with $\hat{k}_t = \phi_1 \hat{k}_{t-1} + \phi_2 \hat{z}_t$, we find:

$$\boxed{\begin{aligned} \phi_1 &= \frac{\alpha}{1 + \alpha^2\beta - \alpha\beta\phi_1}, \\ \phi_2 &= \frac{\alpha\beta\rho\phi_2 + (1 - \alpha\beta\rho)}{1 + \alpha^2\beta - \alpha\beta\phi_1}. \end{aligned}}$$

Selecting the Stable Solution and Implications

Solving for ϕ_1 and ϕ_2 from the previous system gives two possible roots:

$$\boxed{\phi_1 \in \{\alpha,\, 1/(\alpha\beta)\}, \qquad \phi_2 = \begin{cases} 1, & \text{if } \phi_1 = \alpha, \\ \dfrac{1-\alpha\beta\rho}{\alpha^2\beta - \alpha\beta\rho}, & \text{if } \phi_1 = \dfrac{1}{\alpha\beta}. \end{cases}}$$

Stability condition: since $0 < \alpha, \beta < 1$, the stable (non-explosive) root is

$$\boxed{\phi_1 = \alpha, \quad \phi_2 = 1.}$$

Hence:

$$\hat{k}_t = \alpha\, \hat{k}_{t-1} + \hat{z}_t.$$

Other variables:

$$\hat{y}_t = \alpha \hat{k}_{t-1} + \hat{z}_t, \qquad \hat{c}_t = \alpha \hat{k}_{t-1} + \hat{z}_t.$$

Interpretation: A positive productivity shock $\hat{z}_t$ increases output, consumption, and capital next period — the response is persistent due to α, but the system is stable.

Comments and Transition

Summary of the example:

- The stochastic growth model was log-linearised and solved using the method of undetermined coefficients.
- We obtained a stable solution for capital:

$$\hat{k}_t = \alpha\, \hat{k}_{t-1} + \hat{z}_t,$$

  implying persistence but stationarity of the system.
- The same structure applies to output and consumption:

$$\hat{y}_t = \alpha\, \hat{k}_{t-1} + \hat{z}_t, \qquad \hat{c}_t = \alpha\, \hat{k}_{t-1} + \hat{z}_t.$$

Remarks:

- Local (first-order) methods provide simple linear laws of motion.
- However, many DSGE models are *larger systems* of linear difference equations involving expectations.
- We now study general solution methods for such systems: **Blanchard and Kahn (1980)**, **Uhlig (1999)**, **Christiano (2002)**.

Blanchard–Kahn Method

General setup: Write the system of linear difference equations as:

$$A \begin{bmatrix} \hat{x}_t \\ E_t \hat{y}_{t+1} \end{bmatrix} = B \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} + C \, \hat{z}_t .$$

Assuming A is invertible, pre-multiply by A^{-1}:

$$\begin{bmatrix} \hat{x}_t \\ E_t \hat{y}_{t+1} \end{bmatrix} = F \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} + G \, \hat{z}_t , \qquad F = A^{-1} B, \; G = A^{-1} C.$$

Eigenvalue decomposition:

$$F = HJH^{-1} = \begin{bmatrix} v_1 & \cdots & v_{n+m} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n+m} \end{bmatrix} \begin{bmatrix} v_1 & \cdots & v_{n+m} \end{bmatrix}^{-1} .$$

Eigenvalues ordered so that $|\lambda_1| < |\lambda_2| < \cdots < |\lambda_{n+m}|$, and v_i are the corresponding eigenvectors.

Blanchard–Kahn Method: Existence and Uniqueness

Let h be the number of eigenvalues in J that lie *outside* the unit circle ($|\lambda_i| > 1$).

Proposition — Blanchard and Kahn (1980):

1. If $h = n$, the system of stochastic difference equations has a **unique solution**.
2. If $h > n$, the system has **no solution**.
3. If $h < n$, the system has **infinitely many solutions**.

Interpretation:

- n is the number of non-predetermined (jump) variables — e.g., prices, consumption, or inflation.
- We require one unstable root (eigenvalue >1) for each jump variable.
- When this holds, the model satisfies **saddle-path stability**.

Intuition Behind the Blanchard–Kahn Conditions

1. **Predetermined vs. Jump variables**
   - *Predetermined (state) variables:* known at time t (e.g., capital stock).
   - *Jump (control) variables:* chosen optimally at t to ensure convergence (e.g., consumption, prices).
2. **Dynamics and stability**
   - Each eigenvalue λ_i describes how a linear combination of variables evolves.
   - If $|\lambda_i| < 1$: stable root : variable converges automatically.
   - If $|\lambda_i| > 1$: unstable root : diverges unless the initial value of a jump variable is chosen to offset it.
3. **Saddle-path logic**
   - The model is *well-behaved* when there are exactly as many unstable roots as jump variables.
   - Jump variables adjust immediately so the economy remains on the unique stable path.
   - Too few unstable roots : indeterminacy (many paths). Too many : no feasible path.

From (A, B, C) to (F, G) and the Eigen-Basis

Starting point:

$$A \begin{bmatrix} \hat{x}_t \\ E_t \hat{y}_{t+1} \end{bmatrix} = B \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} + C\, \hat{z}_t, \qquad A \text{ invertible.}$$

Step 1 — Pre-multiply by A^{-1}:

$$\begin{bmatrix} \hat{x}_t \\ E_t \hat{y}_{t+1} \end{bmatrix} = F \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} + G\, \hat{z}_t, \qquad F := A^{-1} B,\ G := A^{-1} C.$$

Step 2 — Spectral decomposition of F:

$$F = HJH^{-1},$$

where H collects (generalised) eigenvectors of F and J is (block-)diagonal with eigenvalues ordered so that those with $|\lambda| < 1$ come first (stable), and those with $|\lambda| > 1$ last (unstable).

Goal: Change coordinates to the eigen-basis so the dynamics *decouple* along eigen-directions.

Change of Variables and Decoupling

Step 3 — Define transformed variables:

$$\begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix} := H^{-1} \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix}, \qquad \begin{bmatrix} \tilde{x}_t \\ E_t \tilde{y}_{t+1} \end{bmatrix} := H^{-1} \begin{bmatrix} \hat{x}_t \\ E_t \hat{y}_{t+1} \end{bmatrix}.$$

Step 4 — Substitute the definitions into $\begin{bmatrix} \hat{x}_t \\ E_t \hat{y}_{t+1} \end{bmatrix} = F \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} + G \hat{z}_t$:

$$H \begin{bmatrix} \tilde{x}_t \\ E_t \tilde{y}_{t+1} \end{bmatrix} = F\, H \begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix} + G\, \hat{z}_t.$$

Left-multiply by H^{-1} and use $F = HJH^{-1}$:

$$\begin{bmatrix} \tilde{x}_t \\ E_t \tilde{y}_{t+1} \end{bmatrix} = J \begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix} + V\, \hat{z}_t, \qquad V := H^{-1} G.$$

Blanchard–Kahn Method: Change of Variables

When a unique solution exists ($h = n$), we can solve the system:

$$\begin{bmatrix} \hat{x}_t \\ E_t \hat{y}_{t+1} \end{bmatrix} = F \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} + G\, \hat{z}_t.$$

Step 1 — Change of variables:

$$\begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix} = H^{-1} \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix}, \qquad \begin{bmatrix} \tilde{x}_t \\ E_t \tilde{y}_{t+1} \end{bmatrix} = H^{-1} \begin{bmatrix} \hat{x}_t \\ E_t \hat{y}_{t+1} \end{bmatrix}.$$

Step 2 — Substitute into the system:

$$\begin{bmatrix} \tilde{x}_t \\ E_t \tilde{y}_{t+1} \end{bmatrix} = J \begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix} + V\, \hat{z}_t, \qquad \text{where } V = H^{-1} G.$$

Matrix J is diagonal, with eigenvalues ordered so that those inside the unit circle correspond to the stable subsystem.

Blanchard–Kahn Method: Partitioning the System

Partition the diagonal matrix J and vectors as:

$$\begin{bmatrix} \tilde{x}_t \\ E_t \tilde{y}_{t+1} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \hat{z}_t.$$

Interpretation:

- J_1: eigenvalues $|\lambda_i| < 1$ — stable subsystem (predetermined variables).
- J_2: eigenvalues $|\lambda_i| > 1$ — unstable subsystem (forward-looking variables).

Stable subsystem:

$$\tilde{x}_t = J_1 \tilde{x}_{t-1} + V_1 \hat{z}_t.$$

Unstable subsystem:

$$E_t \tilde{y}_{t+1} = J_2 \tilde{y}_t + V_2 \hat{z}_t.$$

These two blocks decouple the dynamics of predetermined and jump variables — only one unique trajectory keeps the unstable subsystem bounded.

Blanchard–Kahn Method: Forward Iteration

Starting from the partitioned system:

$$\tilde{x}_t = J_1 \tilde{x}_{t-1} + V_1 \hat{z}_t,$$

$$E_t \tilde{y}_{t+1} = J_2 \tilde{y}_t + V_2 \hat{z}_t.$$

Step 1 — Forward iteration on the second block:

$$E_t \tilde{y}_{t+1} = J_2 \tilde{y}_t + V_2 \hat{z}_t \Rightarrow \tilde{y}_t = J_2^{-1} E_t \tilde{y}_{t+1} - J_2^{-1} V_2 \hat{z}_t.$$

Step 2 — Iterate forward repeatedly:

$$\tilde{y}_{t+1} = J_2^{-1} E_{t+1} \tilde{y}_{t+2} - J_2^{-1} V_2 \hat{z}_{t+1},$$

and taking expectations gives

$$E_t \tilde{y}_{t+1} = J_2^{-1} E_t \tilde{y}_{t+2} - J_2^{-1} V_2 E_t \hat{z}_{t+1}.$$

Step 3 — Continuing forward iterations: since $|\lambda_i(J_2)| > 1$, the only bounded (non-explosive) solution is

$$\boxed{\tilde{y}_t = -J_2^{-1} V_2 \hat{z}_t.}$$

Blanchard–Kahn Method: Recovering the Original Variables

From the transformed system:

$$\tilde{x}_t = J_1 \tilde{x}_{t-1} + V_1 \hat{z}_t, \qquad \tilde{y}_t = -J_2^{-1} V_2 \hat{z}_t.$$

Return to the original variables:

$$\begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix}.$$

Interpretation:

- The transformation H maps the decoupled (stable/unstable) coordinates back to the model's original variables.
- The system can now be solved using standard methods for uncoupled difference equations.
- Once $\tilde{x}_t$ and $\tilde{y}_t$ are known, we recover $\hat{x}_t$, $\hat{y}_t$ via this inverse mapping.

Conclusion: The Blanchard–Kahn procedure delivers a unique, stable trajectory for the economy if and only if the number of unstable eigenvalues equals the number of jump variables.

Interest Rule in a New Keynesian Model

Log-linearised NK model:

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} \qquad \text{(New Keynesian Phillips Curve)}$$

$$x_t = E_t x_{t+1} - \frac{1}{\gamma}\left(i_t - E_t \pi_{t+1}\right) + \varepsilon_t^x \qquad \text{(IS curve)}$$

$$i_t = \rho_r \, i_{t-1} + \varepsilon_t \qquad \text{(Policy rule)}$$

Matrix form:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{\gamma} & 1 & \frac{1}{\gamma} \\ 0 & 0 & \beta \end{bmatrix}}_{A} \begin{bmatrix} i_t \\ E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \rho_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\kappa & 1 \end{bmatrix}}_{B} \begin{bmatrix} i_{t-1} \\ x_t \\ \pi_t \end{bmatrix} + \underbrace{\begin{bmatrix} \varepsilon_t \\ -\varepsilon_t^x \\ 0 \end{bmatrix}}_{C\varepsilon_t \text{ (stacked shocks)}}$$

(Here we've stacked shocks as ε_t and ε_t^x for compactness.)

Pre-multiplying and (In)determinacy

Pre-multiply by A^{-1}:

$$\begin{bmatrix} i_t \\ E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = W \begin{bmatrix} i_{t-1} \\ x_t \\ \pi_t \end{bmatrix} + A^{-1} \begin{bmatrix} \varepsilon_t \\ -\varepsilon_t^x \\ 0 \end{bmatrix}, \qquad W := A^{-1}B.$$

Closed-form W:

$$W = \begin{bmatrix} \rho_r & 0 & 0 \\ \frac{\rho_r}{\gamma} & 1 + \frac{\kappa}{\beta\gamma} & -\frac{1}{\beta\gamma} \\ 0 & -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$

BK check (from the PDF): This system is *indeterminate*: there is only **one** eigenvalue outside the unit circle but **two** forward-looking (non-predetermined) variables (x_t and π_t).

$$\#\{|\lambda| > 1\} = 1 \ < \ n = 2 \ \Rightarrow \ \text{infinitely many solutions.}$$

Computing Eigenvalues of the System Matrix W

We obtained:

$$W = \begin{bmatrix} \rho_r & 0 & 0 \\ \frac{\rho_r}{\gamma} & 1 + \frac{\kappa}{\beta\gamma} & -\frac{1}{\beta\gamma} \\ 0 & -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}.$$

Goal: Find eigenvalues λ_i solving

$$\det(W - \lambda I_3) = 0.$$

Computing Eigenvalues of the System Matrix W (Cont.)

Steps:

1. Construct the characteristic polynomial:

$$\begin{vmatrix} \rho_r - \lambda & 0 & 0 \\ \frac{\rho_r}{\gamma} & 1 + \frac{\kappa}{\beta\gamma} - \lambda & -\frac{1}{\beta\gamma} \\ 0 & -\frac{\kappa}{\beta} & \frac{1}{\beta} - \lambda \end{vmatrix} = 0.$$

2. Expand the determinant (here, the first row simplifies):

$$(\rho_r - \lambda)\left[\left(1 + \frac{\kappa}{\beta\gamma} - \lambda\right)\left(\frac{1}{\beta} - \lambda\right) - \frac{\kappa}{\beta^2\gamma}\right] = 0.$$

3. Hence, one eigenvalue is $\boxed{\lambda_1 = \rho_r}$, and the other two satisfy the quadratic:

$$\left(1 + \frac{\kappa}{\beta\gamma} - \lambda\right)\left(\frac{1}{\beta} - \lambda\right) - \frac{\kappa}{\beta^2\gamma} = 0.$$

Interpreting the Eigenvalues

Step 1 — Solve the quadratic:

$$\lambda_{2,3} = \frac{1}{2}\left[\left(1+\frac{\kappa}{\beta\gamma}+\frac{1}{\beta}\right) \pm \sqrt{\left(1+\frac{\kappa}{\beta\gamma}+\frac{1}{\beta}\right)^2 - 4\left(\frac{1+\frac{\kappa}{\beta\gamma}}{\beta} - \frac{\kappa}{\beta^2\gamma}\right)}\right].$$

Step 2 — Numerical example: For typical NK parameters ($\beta = 0.99$, $\kappa = 0.1$, $\gamma = 1$, $\rho_r = 0.7$):

$$\lambda_1 \approx 0.7, \quad \lambda_2 \approx 0.98, \quad \lambda_3 \approx 1.03.$$

Interpretation:

- One root ($\lambda_1 = 0.7$) corresponds to the policy rule (interest-rate smoothing).
- Two roots relate to the forward-looking IS and Phillips-curve block.
- Only one eigenvalue > 1 (explosive) $\Rightarrow$ one jump variable can stabilize it, but we have *two* jump variables (x_t, π_t).
- Therefore, the system is **indeterminate**: multiple equilibrium paths satisfy expectations consistency.

Economic intuition: Monetary policy is too passive — the Taylor principle is not satisfied strongly enough to pin down a unique equilibrium.

Baseline RBC Model (Intro)

Environment and objects:

- Endogenous: y_t (output), c_t (consumption), k_t (capital), i_t (investment), h_t (hours), r_t (interest/MPK), z_t (TFP).
- Parameters: β (discount), δ (depreciation), θ (capital share), ψ (leisure weight), ρ (TFP persistence), σ (TFP shock s.d.).

Technology and resource constraints:

$$y_t = k_{t-1}^{\theta}\left(e^{z_t} h_t\right)^{1-\theta}, \qquad c_t + i_t = y_t, \qquad i_t = k_t - (1-\delta)\,k_{t-1}.$$

Baseline RBC Model (Cont.)

Optimality conditions:

$$\underbrace{\frac{1}{c_t} \;=\; \beta\, E_t\left[\frac{1}{c_{t+1}}\Big(1 + r_{t+1} - \delta\Big)\right]}_{\text{Euler (intertemporal)}}, \qquad \underbrace{\psi\,\frac{c_t}{1-h_t} \;=\; (1-\theta)\, k_{t-1}^{\theta}\, e^{(1-\theta)z_t}\, h_t^{-\theta}}_{\text{Labor–leisure (intratemporal)}}.$$

$$r_t \;=\; \theta\,\frac{y_t}{k_{t-1}} \;=\; \theta\, k_{t-1}^{\theta-1}\left(e^{z_t} h_t\right)^{1-\theta}.$$

Exogenous shock:

$$z_t \;=\; \rho\, z_{t-1} + e_t, \qquad e_t \sim \mathcal{N}(0,\sigma^2).$$

Interpretation: Households choose $\{c_t, h_t, k_t\}$ to trade off consumption smoothing (Euler) and leisure vs. labor (intratemporal), taking prices from the Cobb–Douglas technology. z_t drives stochastic productivity.

RBC (Chapter 2): Complete Nonlinear Model

Preferences and technology:

$$\max_{\{C_t, L_t, K_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{L_t^{1+\phi}}{1+\phi} \right], \qquad Y_t = K_{t-1}^{\alpha} \left(A_t L_t\right)^{1-\alpha}.$$

Resource constraint and capital accumulation:

$$C_t + I_t = Y_t, \qquad K_t = (1-\delta) K_{t-1} + I_t.$$

Prices (real): factor pricing from Cobb–Douglas:

$$R_t = \alpha \frac{Y_t}{K_{t-1}}, \qquad W_t = (1-\alpha) \frac{Y_t}{L_t}.$$

Optimality conditions:

$$\underbrace{C_t^{-\sigma} = \beta E_t\left[C_{t+1}^{-\sigma}\left(1 + R_{t+1} - \delta\right)\right]}_{\text{Euler (intertemporal)}}, \qquad \underbrace{\frac{L_t^{\phi}}{C_t^{-\sigma}} = W_t}_{\text{Intratemporal labor supply}} .$$

Productivity shock:

$$\log A_t = \rho_A \log A_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}(0, \sigma_\varepsilon^2).$$

RBC (Chapter 2): Setup and Steady State

Endogenous variables:

$$Y_t\ ,\ I_t\ ,\ C_t\ ,\ R_t\ ,\ K_t\ ,\ W_t\ ,\ L_t\ ,\ A_t\ .$$

Exogenous shock: e_t **TFP:** $A_t = \rho_A A_{t-1} + e_t$.

Parameters: σ (risk aversion), ϕ (Frisch inverse), α (capital share), β (discount), δ (depreciation), ρ_A (TFP persistence).

Steady-state objects (from calibration):

$$P^{ss} = 1, \qquad R^{ss} = P^{ss}\left(\tfrac{1}{\beta} - (1-\delta)\right),$$

$$W^{ss} = (1-\alpha)\,(P^{ss})^{\frac{1}{1-\alpha}}\left(\tfrac{\alpha}{R^{ss}}\right)^{\frac{\alpha}{1-\alpha}},$$

$$Y^{ss} = \left(\frac{R^{ss}}{R^{ss}-\delta\alpha}\right)^{\frac{\sigma}{\sigma+\phi}}\left((1-\alpha)^{-\phi}\,(W^{ss}/P^{ss})^{1+\phi}\right)^{\frac{1}{\sigma+\phi}},$$

$$I^{ss} = \tfrac{\delta\alpha}{R^{ss}}\,Y^{ss}, \quad K^{ss} = \alpha\,\tfrac{Y^{ss}}{R^{ss}/P^{ss}}, \quad L^{ss} = (1-\alpha)\,\tfrac{Y^{ss}}{W^{ss}/P^{ss}}, \quad C^{ss} = Y^{ss} - I^{ss}.$$

Convention: The model below is *linearized* around the steady state. Variables like $Y_t, C_t, \ldots$ denote deviations; where levels enter, steady-state scalars (e.g., Y^{ss}) appear as weights.

RBC (Chapter 2): Linearized Equations

Preferences and intratemporal optimality (labor supply):

$$\sigma\, C_t + \phi\, L_t \;=\; W_t.$$

Euler equation (intertemporal):

$$\frac{\sigma}{\beta}\big(C_{t+1} - C_t\big) \;=\; R^{ss}\, R_{t+1}.$$

Capital accumulation:

$$K_t \;=\; (1-\delta)\, K_{t-1} + \delta\, I_t.$$

Production (Cobb–Douglas, log-linearized):

$$Y_t \;=\; A_t + \alpha\, K_{t-1} + (1-\alpha)\, L_t.$$

Factor demands (linearized marginal products):

$$R_t = Y_t - K_{t-1}, \qquad W_t = Y_t - L_t.$$

Resource constraint (linearized with steady-state shares):

$$Y^{ss}\, Y_t \;=\; C^{ss}\, C_t + I^{ss}\, I_t.$$

TFP process:

$$A_t \;=\; \rho_A\, A_{t-1} + e_t.$$

What a Positive Productivity Shock Does (Qualitatively)

- **On impact:** $A_t \uparrow$ raises marginal products $\Rightarrow$ factor demands rise, so $W_t \uparrow$, $R_t \uparrow$.
- **Household response:** Higher income pushes up C_t and I_t; labor L_t rises initially.
- **Over time:** As wages normalize and consumption smoothing bites, L_t gradually falls (more leisure), while K_t keeps rising due to $I_t \uparrow$; effects persist via capital accumulation.
- **Bottom line:** Positive TFP shocks raise $\{C, I, Y, K, L, W, R\}$ on impact; K shows a hump-shaped build-up (often peaking around medium horizons).

RBC Lab: Three Quick Modifications

Goal: See how small code changes shift IRFs and moments.

1. **Shock persistence** — make A_t more/less persistent: change ρ_A.
2. **Labor elasticity** — make labor more/less elastic: change ϕ.
3. **Add fiscal shock** — introduce government spending G_t with AR(1), and include it in the resource constraint.

What to look for:

- Higher ρ_A: longer-lived responses; capital builds up more.
- Lower ϕ: larger labor response on impact; wage adjusts less.
- G shock: output rises; consumption typically crowds out (falls) when G uses resources.

Two One-Line Tweaks

1) Shock persistence (TFP): change ρ_A in the parameter block.

$$\rho_A \in \{0.50,\ 0.95\ \text{(baseline)},\ 0.99\}$$

Higher $\rho_A \Rightarrow$ longer IRFs; capital hump increases.

2) Labor supply elasticity: change ϕ (Frisch inverse).

$$\phi \in \{0.5,\ 1.5\ \text{(baseline)},\ 5\}$$

Smaller $\phi \Rightarrow$ labor responds more; wage adjusts less.

Add a Simple Fiscal Shock

New variables: G_t (gov. spending), shock e_t^G.
New equation (AR(1)):

$$G_t \;=\; \rho_G \, G_{t-1} + e_t^G .$$

Resource constraint (linearized with shares):

$$\bar{Y}\, Y_t \;=\; \bar{C}\, C_t + \bar{I}\, I_t + \bar{G}\, G_t ,$$

with $\bar{G} \equiv G^{ss}$ (e.g., 20% of output).
Predictions after G shock:

- Y rises on impact.
- C often falls (crowding out); I may fall or move modestly depending on parameters.
- K adjusts sluggishly; effects depend on ρ_G.

How to Run and What to Read

Run:

1. Change one parameter (ρ_A or ϕ) or toggle the G shock size.
2. `dynare RBC_G.mod`
3. Inspect IRFs for $\{Y, C, I, K, L, W, R\}$.

Quick checks:

- Does higher ρ_A lengthen the half-life of IRFs?
- Does smaller ϕ amplify L's impact response?
- After a G shock, does C fall and Y rise?

Tip: Keep two MATLAB figures open (baseline vs. change) to compare curves visually.

New Keynesian Model (Linearized) — Setup

Endogenous: Y_t , I_t , C_t , K_t , L_t , W_t , R_t , MC_t , P_t , Π_t , A_t.
Exogenous shock: e_t (TFP innovation), $A_t = \rho_A A_{t-1} + e_t$.
Parameters: σ (risk aversion), ϕ (Frisch inverse), α (capital share), β (discount), δ (depreciation), ρ_A (TFP persistence), ψ (desired markup: $\mu = \psi/(\psi - 1)$), θ (Calvo stickiness).

Steady-state auxiliaries (used as constants in the linearization):

$$P^{ss} = 1, \quad R^{ss} = \tfrac{1}{\beta} - (1 - \delta), \quad MC^{ss} = \tfrac{\psi-1}{\psi}(1 - \beta\theta),$$

$$W^{ss} = (1-\alpha)\left(MC^{ss}\right)^{\frac{1}{1-\alpha}} \left(\tfrac{\alpha}{R^{ss}}\right)^{\frac{\alpha}{1-\alpha}}, \quad Y^{ss}, I^{ss}, C^{ss}, L^{ss}, K^{ss} \text{ as in the Dynare code.}$$

Households, Technology, and Factor Pricing (Linearized)

Intratemporal (labor supply): $\sigma\, C_t + \phi\, L_t \;=\; W_t - P_t.$

Euler (intertemporal): $\dfrac{\sigma}{\beta}\,(C_{t+1} - C_t) \;=\; R^{ss}\,(R_{t+1} - P_{t+1}).$

Capital accumulation: $K_t \;=\; (1-\delta)\,K_{t-1} + \delta\, I_t.$

Production (Cobb–Douglas): $Y_t \;=\; A_t + \alpha\, K_{t-1} + (1-\alpha)\, L_t.$

Factor demands (linearized MPN/MPK): $K_{t-1} = Y_t - R_t, \qquad L_t = Y_t - W_t.$

Marginal cost: $MC_t \;=\; (1-\alpha)\, W_t + \alpha\, R_t - A_t.$

Price Setting and Inflation Dynamics

New Keynesian Phillips Curve (Calvo):

$$\Pi_t \;=\; \beta\,\Pi_{t+1} \;+\; \underbrace{\frac{(1-\theta)(1-\beta\theta)}{\theta}}_{\kappa(\theta,\beta)}\,(MC_t - P_t).$$

Inflation definition: $\Pi_t \;=\; P_t - P_{t-1}.$

Resource constraint (linearized with shares):

$$Y^{ss}\,Y_t \;=\; C^{ss}\,C_t + I^{ss}\,I_t.$$

TFP process: $A_t \;=\; \rho_A A_{t-1} + e_t.$

Notes:

- $\theta\uparrow$ (more stickiness) $\Rightarrow \kappa\downarrow \Rightarrow$ inflation responds more sluggishly to MC_t.
- $\psi\uparrow$ (higher desired markup) raises MC^{ss} and affects steady-state factor prices used in the linearization.

TFP Shock: NK vs RBC - What to Expect

Impact mechanism (both models): $A_t \uparrow$ raises marginal products $\Rightarrow W_t \uparrow$, $R_t \uparrow$, and firms demand more inputs.
Key NK differences (sticky prices):

- **Aggregate demand channel:** With sticky prices (θ high), consumption C_t typically rises more than in RBC.
- **Inflation dynamics:** Π_t responds via the NKPC; higher stickiness (κ small) $\Rightarrow$ muted Π_t, real wages can be elevated for longer.
- **Wage and labor:** Income effect can dominate, so L_t may fall sooner/stronger than in RBC despite $Y_t \uparrow$.

Medium-run: As K_t builds, R_t declines toward steady state; W_t normalizes slowly when θ is large.
Bottom line: In NK, TFP shocks transmit strongly via *demand* (consumption) and sticky prices; in RBC, transmission is mainly via *supply*.

Uhlig's Method: Overview

Idea: Solve a system of linear stochastic difference equations using the *method of undetermined coefficients* in matrix form.

Step 1 — (Log-)Linearise all relevant equations.
Let:

- x_t: $m \times 1$ vector of **endogenous state variables** (predetermined)
- y_t: $n \times 1$ vector of **control (jump) variables**
- z_t: $k \times 1$ vector of **exogenous state variables (shocks)**

Step 2 — Write the system in compact matrix form:

$$
\begin{aligned}
0 &= E_t\left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right], \\
z_{t+1} &= Nz_t + \varepsilon_{t+1}, \\
0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t.
\end{aligned}
$$

with N containing only stable eigenvalues ($|\lambda_i(N)| < 1$).

Uhlig's Method: Dimensions and Solution Form

Matrix dimensions:

- C is $l \times n$ with $\text{rank}(C) = n$ (and $l \geq n$)
- F is $(m + n - l) \times m$ — at most as many expectational equations as state variables
- Other matrices conform in dimensions so that the total number of equations equals the total number of unknowns

Looking for a solution of the form:

$$\boxed{\begin{aligned} x_t &= P x_{t-1} + Q z_t, \\ y_t &= R x_{t-1} + S z_t. \end{aligned}}$$

Intuition:

- P: transition matrix for endogenous states.
- Q: response of states to exogenous shocks.
- R, S: control (jump) responses to states and shocks.

Goal: Determine (P, Q, R, S) such that the system above is satisfied for all (x_{t-1}, z_t).

Uhlig's Method: Deriving the System of Equations

Substitute the conjectured solutions

$$x_t = Px_{t-1} + Qz_t, \qquad y_t = Rx_{t-1} + Sz_t$$

into the system:

$$\begin{aligned} 0 &= E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t], \\ 0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\ z_{t+1} &= Nz_t + \varepsilon_{t+1}. \end{aligned}$$

Using $E_t(\varepsilon_{t+1}) = 0$ and substituting forward expectations:

$$\begin{aligned} 0 &= (AP + B + CR)\, x_{t-1} + (AQ + CS + D)\, z_t, \\ 0 &= \left(FP^2 + GP + H + JRP + KR\right) x_{t-1} \\ &\quad + \left(FPQ + FQN + GQ + JRQ + JSN + KS + LN + M\right) z_t. \end{aligned}$$

Uhlig's Method: System of Matrix Equations

To hold for all (x_{t-1}, z_t), the following must hold:

$$AP + B + CR = 0, \tag{1}$$

$$AQ + CS + D = 0, \tag{2}$$

$$FP^2 + GP + H + JRP + KR = 0, \tag{3}$$

$$FPQ + FQN + GQ + JRQ + JSN + KS + LN + M = 0. \tag{4}$$

Uhlig's Method: Solving the Quadratic Matrix Equation

Special case: $n = l$, i.e. C is square and invertible.

From (1) and (3), eliminate R:

$$AP + B + CR = 0 \quad \Rightarrow \quad R = -C^{-1}(AP + B).$$

Substitute into (3):

$$FP^2 + GP + H + JRP + KR = 0.$$

After simplification:

$$\boxed{\Psi P^2 = \Gamma P + \Theta,}$$

where

$$\Psi = F - JC^{-1}A, \qquad \Gamma = JC^{-1}B - G + KC^{-1}A, \qquad \Theta = KC^{-1}B - H.$$

This is a **quadratic matrix equation** in P. It can be solved via the **generalised eigenvalue problem** (QZ or generalized Schur decomposition).

Uhlig's Method: The Generalised Eigenvalue Problem

The quadratic matrix equation

$$\Psi P^2 = \Gamma P + \Theta$$

is non-trivial to solve directly, but can be rewritten as a **generalised eigenvalue problem**.

Step 1 — Rewrite as a second-order system:

$$\Psi x_{t+1} = \Gamma x_t + \Theta x_{t-1}.$$

Step 2 — Stack variables:

$$w_t = \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Psi & 0 \\ 0 & I_m \end{bmatrix}, \quad \Xi = \begin{bmatrix} \Gamma & \Theta \\ I_m & 0 \end{bmatrix}.$$

Then:

$$\Delta w_{t+1} = \Xi w_t.$$

The goal is to find (λ_i, v_i) such that

$$\Xi v_i = \lambda_i \Delta v_i,$$

which delivers eigenvalues λ_i and eigenvectors v_i (via the QZ decomposition).

Uhlig's Method: Stability and Selection of Eigenvalues

Generalised eigenvalue problem:

$$\Xi v_i = \lambda_i \Delta v_i \quad \Rightarrow \quad \Psi V \Lambda^2 = \Gamma V \Lambda + \Theta V.$$

Matrix solution:

$$P = V \Lambda V^{-1},$$

where V contains the eigenvectors and Λ is diagonal with eigenvalues λ_i.

Which eigenvalues to select?

- Choose the m eigenvalues that have m linearly independent eigenvectors (so V^{-1} exists).
- Among these, retain those **inside the unit circle** ($|\lambda_i| < 1$) to ensure stability.

Proposition (Uhlig): If all eigenvalues of P satisfy $|\lambda_i| < 1$, then the solution

$$x_t = P x_{t-1} + Q z_t$$

is **stable**, and the associated (R, S) matrices can be recovered easily from (6)–(9).

Implementation: MATLAB's `qz` or `eig` functions (or Uhlig's toolkit) can compute this directly.

Uhlig's Method: Case with Non-Square C ($n < l$)

When C is not square ($n < l$), equation (6)

$$AP + B + CR = 0$$

cannot be inverted directly. We then use the **Moore–Penrose pseudo-inverse** of C, denoted C^+.

Idea:

- Replace C^{-1} with C^+ in all algebraic expressions.
- C^+ generalises the matrix inverse to rectangular or rank-deficient matrices:

$$C^+ = (C'C)^{-1}C' \quad \text{when } C \text{ has full column rank.}$$

- In MATLAB, compute with: `pinv(C)`.

Everything else unchanged:

$$\Psi = F - JC^+A, \quad \Gamma = JC^+B - G + KC^+A, \quad \Theta = KC^+B - H.$$

Then solve again:

$$\Psi P^2 = \Gamma P + \Theta$$

by QZ decomposition as before.

Uhlig's Toolkit and Practical Implementation

Practical implementation:

- Uhlig's MATLAB toolkit automates all steps:
  1. Linearises the model equations.
  2. Builds matrices $A, B, C, D, F, G, H, J, K, L, M, N$.
  3. Solves for P, Q, R, S using the generalised eigenvalue problem.
- Uses built-in functions: `[AA,BB,Q,Z] = qz(Xi,Delta)` or `eig(Xi,Delta)`.
- Once P is found, the remaining matrices follow from eqs. (6)–(9).

Stability check: All eigenvalues of P must satisfy $|\lambda_i| < 1$.

Toolkit download:

`Uhlig MATLAB Toolkit (version 41)`

Reference:

Uhlig (1999). A Toolkit for Analysing Non-Linear Economic Models Easily, in Marimon and Scott.

Example: Stochastic Growth Model with Uhlig's Method

Ordering of variables:

- Endogenous state: k_t
- Non-state (jump) variables: λ_t, c_t, y_t
- Exogenous state: z_t

Linearised equilibrium equations:

$$0 = E_t\big[\, 0\hat{k}_{t+1} - (1-\alpha)\hat{k}_t + 0\hat{k}_{t-1} + \hat{\lambda}_{t+1} + 0\hat{c}_{t+1} + 0\hat{y}_{t+1} - \hat{\lambda}_t + 0\hat{c}_t + 0\hat{y}_t + \hat{z}_{t+1} \,\big],$$

$$0 = 0\hat{k}_t + 0\hat{k}_{t-1} + \hat{\lambda}_t + \hat{c}_t + 0\hat{y}_t + 0\hat{z}_t,$$

$$0 = \hat{k}_t + 0\hat{k}_{t-1} + 0\hat{\lambda}_t + \frac{1-\alpha\beta}{\alpha\beta}\hat{c}_t - \frac{1}{\alpha\beta}\hat{y}_t + 0\hat{z}_t,$$

$$0 = 0\hat{k}_t - \alpha\hat{k}_{t-1} + 0\hat{\lambda}_t + 0\hat{c}_t + \hat{y}_t - \hat{z}_t.$$

Technology process: $\hat{z}_{t+1} = \rho\hat{z}_t + \eta_t$.

Example: Stochastic Growth Model with Uhlig's Method (continued)

From the system, matrices are identified as:

$$F = [0], \quad G = [-(1-\alpha)], \quad H = [0],$$
$$J = [1\ 0\ 0], \quad K = [-1\ 0\ 0],$$
$$L = [1], \quad M = [0].$$

$$A = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ -\alpha \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \frac{1-\alpha\beta}{\alpha\beta} & -\frac{1}{\alpha\beta} \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Next: Use these matrices within Uhlig's algorithm to recover P, Q, R, S via the generalised eigenvalue problem.

Christiano's Method: Overview

Idea: A generalisation of the undetermined-coefficients approach — allows for:

- multiple leads and lags,
- heterogeneous information sets (E_t, $E_{t-1}, \ldots$).

General system:

$$E_t\left[\sum_{i=0}^{r} \alpha_i z_{t+r-1-i} + \sum_{i=0}^{r-1} \beta_i s_{t+r-1-i}\right] = 0,$$

where:

- z_t: all endogenous variables (states + controls),
- s_t: exogenous shocks.

Shock process: Any ARMA(p, q) shock can be rewritten as a VAR(1):

$$\theta_t = \rho\theta_{t-1} + \eta_t, \qquad \eta_t \sim (0, \Sigma_\eta).$$

If all information sets coincide: $s_t = \theta_t$, $P = \rho$, $\varepsilon_t = \eta_t$.

Christiano's Method: Structure of Variables

Notation and dimensions:

- z_{1t}: $n_1 \times 1$ vector of endogenous variables determined within period t.
- z_{2t}: $qn_1 \times 1$ vector of lagged elements of z_{1t} (if $q > 0$).
- $z_t = [z'_{1t}, z'_{2t}]'$ of dimension $n_1(1+q) \times 1$.
- s_t: $m \times 1$ vector of exogenous shocks, $s_t = Ps_{t-1} + \varepsilon_t$.

Coefficient matrices:

- α_i: $n_1 \times n_1(1+q)$ — coefficients on endogenous variables.
- β_i: $n_1 \times m$ — coefficients on shocks.
- τ: indicator for information sets ($\tau_{ij} = 1$ if shock i known in eq. j, 0 otherwise).

Leads/lags:

$$r > q, \quad r = k + 1 \text{ if } t + k \text{ is the largest lead.}$$

This notation nests most linearised DSGE systems.

Christiano's Method: Compact Solution Representation

We seek a linear solution of the form

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}}_{A} \begin{bmatrix} z_{1,t-1} \\ z_{2,t-1} \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{B} \begin{bmatrix} \theta_t \\ \theta_{t-1} \end{bmatrix},$$

with initial condition z_{-1}.

Steps in Christiano's algorithm:

1. Drop expectations to obtain a deterministic first-order system $aw_{t+1} + bw_t = 0$, where w_t stacks all relevant z's.
2. Use QZ decomposition (Sims, 2000) to solve for eigenvalues and identify stable blocks : yields A.
3. Given A, recover B easily (similar to the undetermined-coefficients step in Uhlig's method).

Functions: `solvea.m` and `solveb.m` implement the algorithm.
Available at:
`http://faculty.wcas.northwestern.edu/~lchrist/research/Solve/main.htm`

Example: Growth Model with Christiano's Method

Model equations (linearised):

$$
\begin{aligned}
0 &= E_t\big[-(1-\alpha)\hat{k}_t + \hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{\omega}_{t+1}\big], \\
0 &= \hat{\lambda}_t + \hat{c}_t, \\
0 &= \hat{k}_t + \frac{1-\alpha\beta}{\alpha\beta}\hat{c}_t - \frac{1}{\alpha\beta}\hat{y}_t, \\
0 &= -\alpha\hat{k}_{t-1} + \hat{y}_t - \hat{\omega}_t.
\end{aligned}
$$

Variable ordering:

$$
z_{1t} = \begin{bmatrix} \hat{k}_t \\ \hat{\lambda}_t \\ \hat{c}_t \\ \hat{y}_t \end{bmatrix}, \qquad q = 0, \qquad s_t = \hat{\omega}_t = \psi\,\hat{\omega}_{t-1} + \varepsilon_t.
$$

Expectations: $E_t[\cdot]$ over next-period values. Largest lead is one ($r = 2$), and all shocks are contemporaneously observable.

Example: Growth Model with Christiano's Method (continued)

System representation:

$$E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0.$$

Matrices:

$$\alpha_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} -(1-\alpha) & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & \frac{1-\alpha\beta}{\alpha\beta} & -\frac{1}{\alpha\beta} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \end{bmatrix}.$$

$$\beta_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \beta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

These matrices are used as inputs to `solva.m` and `solvb.m` to compute the policy matrices A and B.

Example 2: New Keynesian model with lagged inflation

New Keynesian model with lagged inflation:

$$y_t = E_t(y_{t+1}) - \frac{1}{\sigma}(i_t - E_t(p_{t+1} - p_t))$$

$$m_t = \sigma y_t - \beta i_t + p_t$$

$$p_t - p_{t-1} = \beta E_t(p_{t+1} - p_t) - \beta\gamma(p_t - p_{t-1}) + \gamma(p_{t-1} - p_{t-2}) + \kappa y_t$$

Variables:

$$z_{1t} = \begin{bmatrix} y_t \\ i_t \\ p_t \end{bmatrix}, \qquad z_{2t} = \begin{bmatrix} y_{t-1} \\ i_{t-1} \\ p_{t-1} \end{bmatrix} \Rightarrow z_t = \begin{bmatrix} y_t \\ i_t \\ p_t \\ y_{t-1} \\ i_{t-1} \\ p_{t-1} \end{bmatrix}.$$

Example 2: New Keynesian model with lagged inflation (continued)

and

$$0 = E_t\left[y_t - y_{t+1} + \frac{1}{\sigma} i_t - \frac{1}{\sigma} p_{t+1} + \frac{1}{\sigma} p_t\right],$$

$$0 = E_t[m_t - \sigma y_t + \beta i_t - p_t],$$

$$0 = E_t[-\beta p_{t+1} + (1 + \beta\gamma + \beta)p_t - (1 + \gamma + \beta\gamma)p_{t-1} - \kappa y_t + \gamma p_{t-2}].$$

Shock: $s_t = m_t = \psi m_{t-1} + \eta_t$
Parameters: $r = 2,\ q = 1,\ \tau = [1\ 1\ 1]$
Representation:

$$E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0.$$

Example 2: New Keynesian model with lagged inflation (continued)

Matrices:

$$\alpha_0 = \begin{bmatrix} -\frac{1}{\sigma} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} 1 & \frac{1}{\sigma} & \frac{1}{\sigma} & 0 & 0 & 0 \\ 0 & \beta & -1 & 0 & 0 & 0 \\ 0 & 0 & 1+\beta\gamma+\beta & 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_2 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(1+\gamma+\beta\gamma) & 0 & 0 & \gamma \end{bmatrix}.$$

$$\beta_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \beta_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Example 2: New Keynesian model with lagged inflation (continued)

Parameterisation for simulation:

$$\beta = 0.99, \qquad \sigma = 1, \qquad \kappa = 0.1, \qquad \gamma = 0.5, \qquad \psi = 0.7.$$

Information structure:

$$r = 2, \qquad q = 1, \qquad \tau = [\,1\ 1\ 1\,].$$

Shock process:

$$s_t = m_t = \psi\, m_{t-1} + \eta_t, \qquad \eta_t \sim (0, \sigma_\eta^2).$$

Representation:

$$E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0.$$

Next: Feed $(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1)$ and parameter values into `solva.m` / `solvb.m` to obtain the reduced-form policy matrices (A, B).

Example 2: New Keynesian model with lagged inflation (continued)

Try these numbers to practice:

$$\sigma = 1, \qquad \beta = 0.99, \qquad \omega = \tfrac{3}{4}, \qquad \kappa = \frac{(1-\omega)(1-\omega\beta)}{\omega}, \qquad \psi = 0.7, \qquad \gamma = 0.66.$$

We have that:

$$z_t = A z_{t-1} + B\theta_t, \qquad z_t = \begin{bmatrix} y_t \\ i_t \\ p_t \\ y_{t-1} \\ i_{t-1} \\ p_{t-1} \end{bmatrix}.$$

Hence:

$$y_t = -1.1443\, p_{t-1} + 0.3923\, p_{t-2} + 0.6372\, m_t,$$

$$i_t = -0.2313\, m_t,$$

$$p_t = 1.1443\, p_{t-1} - 0.3923\, p_{t-2} + 0.1338\, m_t.$$

Generating Impulse Response Functions

Procedure:

- Generate a sequence of exogenous θ_t shocks by setting

$$\eta_t = \begin{cases} \sigma_\eta, & t = q+2, \\ 0, & \text{otherwise.} \end{cases}$$

- Alternatively, set $\eta_{q+2} = 1$ if absolute magnitudes are unimportant.
- Construct s_t as

$$s_t = \begin{cases} \theta_t, & \text{if all information sets coincide,} \\ (\theta_t, \theta_{t-1})', & \text{otherwise.} \end{cases}$$

- Set $z_1 = 0$.
- For $t = 1, \dots, T$, iterate:

$$z_t = A z_{t-1} + B s_t.$$

- Extract z_{1t} (the endogenous variables) and plot the series.

Impulse Response Functions

Impulse responses to a positive money shock in the New Keynesian model with lagged inflation:

- y_t (output) decreases on impact due to tighter monetary policy.
- i_t (nominal rate) rises following the money shock.
- p_t (inflation) gradually declines — inflation inertia smooths the response.

Simulation output:

Comparisons and Comments

Uhlig's approach:

- Very user-friendly and practical (especially via the MATLAB toolkit).
- Requires identification of state / predetermined variables.
- Handles only one lead and one lag.

Christiano's method:

- More general — allows:
  - Unlimited leads and lags.
  - Different information sets (expectations dated at various times).
- Slightly more complex to implement but highly flexible.

Other toolboxes:

- **DYNARE:** pre-processor and MATLAB/GAUSS routines for solving non-linear DSGE models with forward-looking variables.
- Convenient, widely used, but largely a "black-box" solver.

Readings and References

Key References:

- Blanchard and Kahn (1980): *The Solution of Linear Difference Models under Rational Expectations.* Econometrica.
- Christiano (2002): *Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients.* Computational Economics. Website: `http://faculty.wcas.northwestern.edu/~lchrist/research/Solve/main.htm`
- Heer and Maussner (2005): Section 2.3.
- Miao (2014): Chapter 2.
- Sims (2000): *Solving Linear Rational Expectations Models.* Working Paper.
- Uhlig (1999): *A Toolkit for Analysing Non-Linear Economic Models Easily.* In Marimon and Scott. Website: `http://www2.wiwi.hu-berlin.de/institute/wpol/html/toolkit.htm`