Practicing Dynare

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Abstract

This paper teaches Dynare by applying it to approximate equilibria and estimate nine dynamic economic models. We hope that it illustrates once again the efficacy of learning by doing.

1. Introduction

This paper describes nine examples that illustrate how Dynare can approximate the solutions of dynamic rational expectations models, simulate them, and estimate them by maximum likelihood and Bayesian methods. Section 2 approximates and estimates a one-sector stochastic growth model. Section 3 approximates and estimates a two-country stochastic growth model. Section 4 follows chapter 11 of Ljungqvist and Sargent (2004) in studying the effects of foreseen fiscal policy in a non-stochastic growth model. Section 5 estimates a rational expectations model of hyperinflation originally formulated by Sargent (1977). Section 6 solves and estimates the permanent income model of Hall (1988). This application is interesting, among other reasons, for the way that it illustrates how Dynare can implement the 'diffuse Kalman filter' needed in situations in which an initial endogenous state variable is unknown and the model has a unit root. Section 7 solves and estimates the Ryoo and Rosen (2004) rational expectations model of a market for engineers. Section 8 estimates a model of consumption growth proposed by Bansal and Yaron (2004). Section 9 solves and estimates the Hansen, Sargent, and Tallarini (1999) model of robust permanent income and pricing. Section 10 solves and estimates the Bansal and Yaron (2004) of asset prices and long-run risk in consumption and dividends. Appendix A tells how the reader can obtain the file examples.zip that contain the *.mod and data files that we used to generate these examples.

Some of our examples bear eloquent witness to the technological improvements brought by Dynare. For example, the maximum likelihood estimates in Sargent (1977) and Hansen, Sargent, and Tallarini (1999) were time-consuming and painful to obtain originally. Dynare has reduced the time and the pain.

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2. The Neoclassical growth model

2.1 The model

We study a widely used stochastic neoclassical growth model with leisure (see, for example, Cooley and Prescott (1995)). A representative household's problem is

$$\max_{\{c_t, l_t\}_{t=0}^{\infty}} E_0 \sum_{t=1}^{\infty} \beta^{t-1} \frac{\left(c_t^{\theta}(1-l_t)^{1-\theta}\right)^{1-\tau}}{1-\tau}$$

subject to the resource constraint

$$c_t + i_t = e^{z_t} k_t^{\alpha} l_t^{1-\alpha} \tag{1}$$

the law of motion for capital

$$k_{t+1} = i_t + (1-\delta)k_t \tag{2}$$

and the stochastic process for productivity

$$z_t = \rho z_{t-1} + s\epsilon_t \tag{3}$$

with $\epsilon_t \sim N(0, \sigma^2)$.

The system of equations characterizing an equilibrium is form by equations (1), (2) and (3), the Euler intertemporal condition

$$\frac{\left(c_t^{\theta}(1-l_t)^{1-\theta}\right)^{1-\tau}}{c_t} = \beta E_t \left[\frac{\left(c_{t+1}^{\theta}(1-l_{t+1})^{1-\theta}\right)^{1-\tau}}{c_{t+1}}(1 + \alpha e^{z_t} k_t^{\alpha-1} l_t^{\alpha} - \delta) \right] \tag{4}$$

and an optimality condition for supply of labor

$$\frac{1-\theta}{\theta} \frac{c_t}{1-l_t} = (1-\alpha)e^{z_t} k_t^{\alpha} l_t^{-\alpha}. \tag{5}$$

Use (1) and (2) to get

$$k_{t+1} = e^{z_t} k_t^{\alpha} l_t^{1-\alpha} - c_t + (1-\delta)k_t. \tag{6}$$

An equilibrium is characterized by a system of four equations (3), (4), (5) and (6).

2.2 Calibration and approximation

2.2.1 Calibration

To estimate the policy function, parameters that enter the model are calibrated as in Table 1. For more on the calibrations of the related parameters, please see, for example, Cooley and Prescott (1995).

2.2.2 Approximation

The following instructions implement this model in Dynare.

```
periods 1000;

var c k lab z;
varexo e;

parameters bet the del alp tau rho s;

bet     = 0.987;
the     = 0.357;
del     = 0.012;
alp     = 0.4;
```

```
tau     = 2;
rho     = 0.95;
s       = 0.007;

model;
    (c^the*(1-lab)^(1-the))^(1-tau)/c=bet*((c(+1)^the*(1-lab(+1))^(1-the))^(1-tau)/c(+1))*
    (1+alp*exp(z(-1))*k(-1)^(alp-1)*lab^(1-alp)-del);
    c=the/(1-the)*(1-alp)*exp(z(-1))*k(-1)^alp*lab^(-alp)*(1-lab);
    k=exp(z(-1))*k(-1)^alp*lab^(1-alp)-c+(1-del)*k(-1);
    z=rho*z(-1)+s*e;
end;

initval;
k   = 1;
c   = 1;
lab = 0.3;
z   = 0;
e   = 0;
end;

shocks;
var e;
stderr 1;
end;

steady;

stoch_simul(dr_algo=0,periods=1000); datasaver('simudata',[]);
```

Parameter	Calibration
β	0.987
θ	0.357
δ	0.012
α	0.4
τ	2
ρ	0.95
s	0.007
σ	1

Table 1: Parameters' calibration

Dynare displays the results reported in Table 2 that provide the coefficients of the approximated policy functions and transition functions.

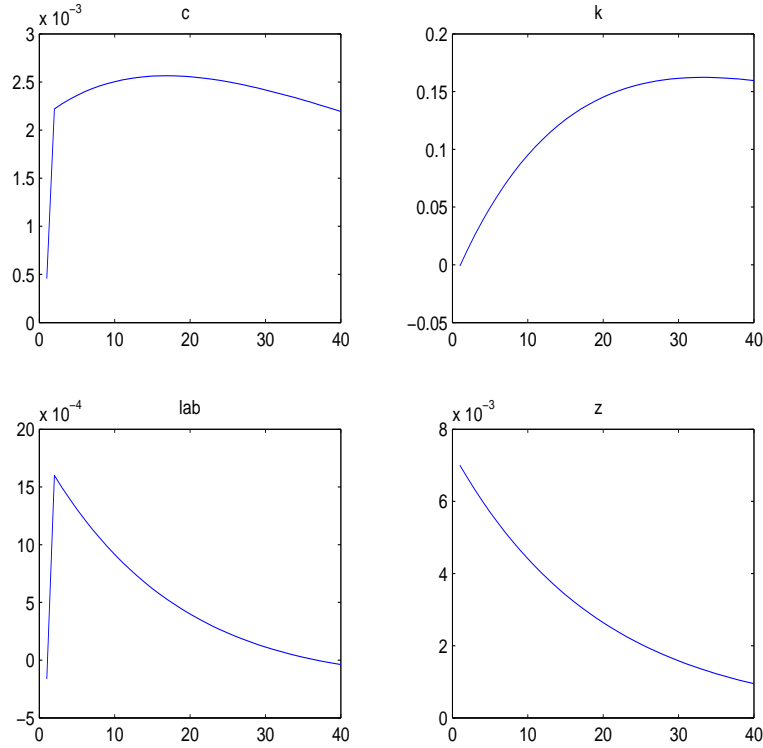|	k	z	c	lab
Constant	54.615371	0	0.960775	0.543716
k (-1)	0.983153	0	0.011727	-0.001668
z (-1)	1.978406	0.950000	0.317480	0.228716
e	-0.000935	0.007000	0.000454	-0.000162
k (-1),k (-1)	-0.000038	0	-0.000030	0.000011
z (-1),k (-1)	0.011128	0	0.002996	0.000762
z (-1),z (-1)	1.253721	0	0.096224	-0.046394
e,e	-0.000003	0	0.000001	0
k (-1),e	-0.000002	0	0.000002	0.000001
z (-1),e	-0.001076	0	0.000430	-0.000056

Table 2: Policy and transition functions.

Figure 1: Impulse response function to a technology shock

2.3 Estimation

The simulated data series for consumption are used to estimate some unknown parameters of the model. Suppose we would like to estimate the preference parameters θ and τ, and the stochastic process for productivity, summarized by two parameters ρ and σ. The following code instructs Dynare to conduct the estimation.

```
periods 1000;

var c k lab z;
varexo e;
```

```
parameters bet del alp rho the tau s;

bet     = 0.987;
the     = 0.357;
del     = 0.012;
alp     = 0.4;
tau     = 2;
rho     = 0.95;
s       = 0.007;

model;
    (c^the*(1-lab)^(1-the))^(1-tau)/c=bet*((c(+1)^the*(1-lab(+1))^(1-the))^(1-tau)/c(+1))
    *(1+alp*exp(z(-1))*k(-1)^(alp-1)*lab^(1-alp)-del);
    c=the/(1-the)*(1-alp)*exp(z(-1))*k(-1)^alp*lab^(-alp)*(1-lab);
    k=exp(z(-1))*k(-1)^alp*lab*(1-alp)-c+(1-del)*k(-1);
    z=rho*z(-1)+s*e;
end;

initval;
k   = 1;
c   = 1;
lab = 0.3;
z   = 0;
e   = 0;
end;

shocks;
var e;
stderr 1;
end;

steady;

check;

estimated_params;
stderr e,inv_gamma_pdf, 0.95,inf;
rho,beta_pdf,0.93,0.02;
the,normal_pdf,0.3,0.05;
tau,normal_pdf,2.1,0.3;
end;

varobs c;

estimation(datafile=simudata,mh_replic=10000);
```

The priors used in estimation are as in Table 3. Table 4 is one of the outputs of Dynare which presents summary statistics of posterior distribution.
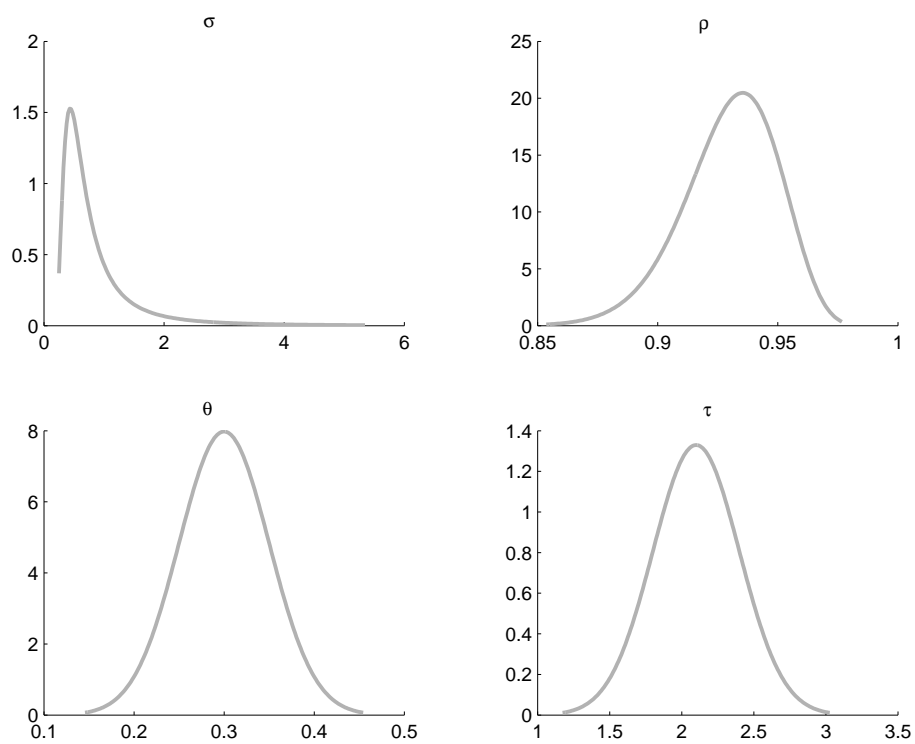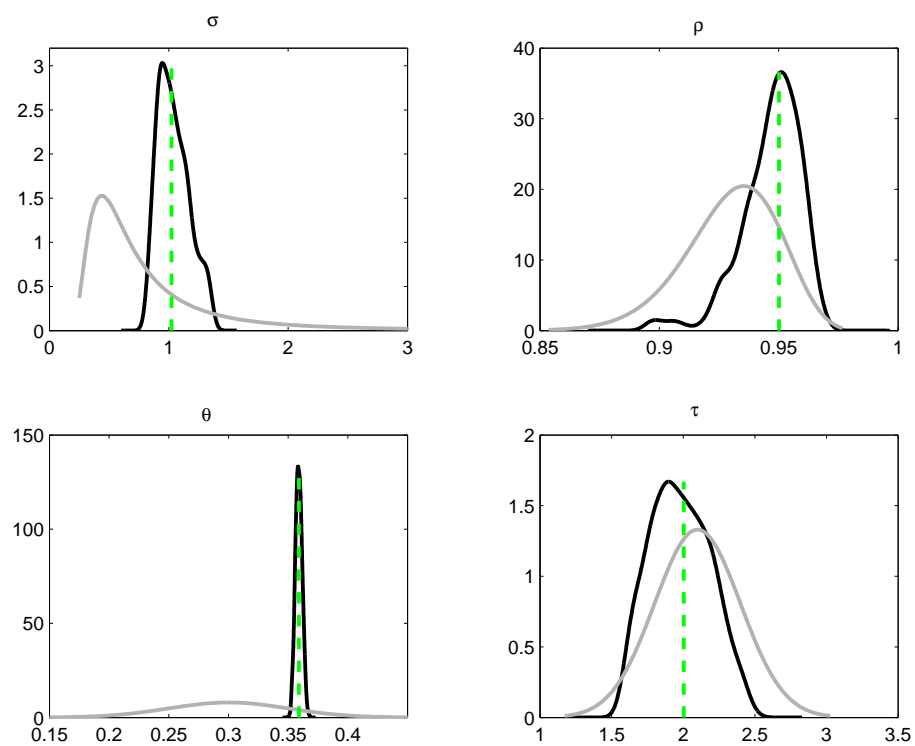
Figure 2: Priors.

Figure 3: Posterior.

Parameter	Distribution	Mean	Std.Dev.
ρ	Beta	0.93	0.02
θ	Normal	0.3	0.05
τ	Normal	2.1	0.3
σ	Inv. Gamma	0.95	inf.

Table 3: Priors

	prior mean	post. mean	conf.	interval	prior dist	prior std
ρ	0.930	0.9480	0.9319	0.9645	beta	0.0200
θ	0.300	0.3589	0.3540	0.3645	norm	0.0500
τ	2.100	2.0046	1.6532	2.3356	norm	0.3000
σ	0.950	1.0227	0.8321	1.2175	invg	Inf

Table 4: Posterior.

3. International Business Cycle Model

We study a simplified version of the two-country production model in Kim and Kim (2003).

3.1 The model

Two countries are identical ex-ante and markets are complete. A solution to a Pareto planner's problem is characterized by the following equations

$$
\begin{aligned}
C_{1,t} &= C_{2,t} \\
C_{1,t}^{-\gamma} &= \beta \mathbb{E}_t C_{1,t+1}^{-\gamma} \left(\alpha A_{1,t+1} K_{1,t+1}^{\alpha-1} + 1 - \delta \right) \\
C_{2,t}^{-\gamma} &= \beta \mathbb{E}_t C_{2,t+1}^{-\gamma} \left(\alpha A_{2,t+1} K_{2,t+1}^{\alpha-1} + 1 - \delta \right) \\
A_{1,t} K_{1,t}^{\alpha} + A_{2,t} K_{2,t}^{\alpha} &= C_{1,t} + C_{2,t} + K_{1,t+1} - (1-\delta) K_{1,t} + K_{2,t+1} - (1-\delta) K_{2,t} \\
\ln A_{1,t+1} &= \rho \ln A_{1,t} + \epsilon_{1,t+1} \\
\ln A_{2,t+1} &= \rho \ln A_{2,t} + \epsilon_{2,t+1}
\end{aligned}
$$

The first three equalities are the first-order conditions with regard to the consumption ($C_{i,t}$), and the fourth one is the world resource constraint. The last two describe the law of motion for the technological shock ($A_{i,t}$). $K_{i,t}$ denotes the capital stock of country i at time t.

3.2 Calibration and approximation

3.2.1 Calibration

The following parameter values are used.

Parameter	Calibration
β	0.98
δ	0.05
α	0.4
ρ	0.85
σ_{ϵ_1}	0.08
σ_{ϵ_2}	0.08

Table 5: Parameter Values

Dynare computes the policy function as a second-order approximation around the (log) steady-state. The `.mod` Dynare file of the system is as follows.

```
periods 1000;
var c1 c2 k1 k2 a1 a2;
varexo e1 e2;

parameters gamma delta alpha beta rho;

gamma=2;
delta=.05;
alpha=.4;
beta=.98;
rho=.85;

model;
c1=c2;
exp(c1)^(-gamma) =...
```

```
beta*exp(c1(+1))^(-gamma)*(alpha*exp(a1(+1))*exp(k1)^(alpha-1)+1-delta);
exp(c2)^(-gamma) =...
beta*exp(c2(+1))^(-gamma)*(alpha*exp(a2(+1))*exp(k2)^(alpha-1)+1-delta);
exp(c1)+exp(c2)+exp(k1)-exp(k1(-1))*(1-delta)+exp(k2)-exp(k2(-1))*(1-delta)...
= exp(a1)*exp(k1(-1))^alpha+exp(a2)*exp(k2(-1))^alpha;
a1=rho*a1(-1)+e1; a2=rho*a2(-1)+e2; end;

initval;
k1=2.8;
k2=2.8;
c1=.8;
c2=.8;
a1=0;
a2=0;
e1=0;
e2=0;
end;

shocks;
var e1;
stderr .08;
var e2;
stderr .08;
end;

steady;

stoch_simul(dr_algo=0,periods=1000); datatomfile('simudata',[]);
```

The output is given as a matrix of coefficients for the policy function. From this matrix, for example, one can construct the optimal consumption rule for Country 1:

$$\hat{c}_{1,t} = 0.824 + 0.243(\hat{k}_{1,t} + \hat{k}_{2,t}) + 0.140(\hat{a}_{1,t} + \hat{a}_{2,t}) + 0.062(\hat{k}_{1,t-1}^2 + \hat{k}_{2,t-1}^2) - 0.110\hat{k}_{1,t}\hat{k}_{2,t}$$
$$- 0.015(\hat{a}_{1,t}\hat{k}_{1,t} + \hat{a}_{2,t}\hat{k}_{2,t}) - 0.032(\hat{a}_{1,t}\hat{k}_{2,t} + \hat{a}_{2,t}\hat{k}_{1,t}) + 0.044(\hat{a}_{1,t}^2 + \hat{a}_{2,t}^2) - 0.040\hat{a}_{1,t}\hat{a}_{2,t},$$

where $\hat{y}$ denotes the deviation of $\ln Y$ from its log steady-state. Note that Dynare breaks $\hat{a}_{i,t}$ into $\hat{a}_{i,t-1}$ and $\epsilon_{i,t}$, so the output from Dynare looks unnecessarily messy.

Dynare also produces the associated impulse response functions of the system. Figure 4 illustrate the response of the system to a one-standard-deviation shock to ϵ_1.

The last two lines of the code generates sample paths of the variables governed by the approximated policy functions. The simulated process is saved as a `.m` file.

3.3 Estimation

Now assume that we do not know the true parameter values. We can estimate the unknown parameters from the saved data, using the Bayesian econometrics function of Dynare.

We will estimate 4 parameters ρ, α, σ_{ϵ_1}, and σ_{ϵ_2}. Hence, we have to specify the prior distribution of each parameter. In this exercise, ρ and α are given a normal prior, while the priors of σ_{ϵ_1} and σ_{ϵ_2} are inverse gamma distributions. The Dynare code is as follows.

```
periods 1000;
var c1 c2 k1 k2 a1 a2;
varexo e1 e2;

parameters gamma delta alpha beta rho;
```
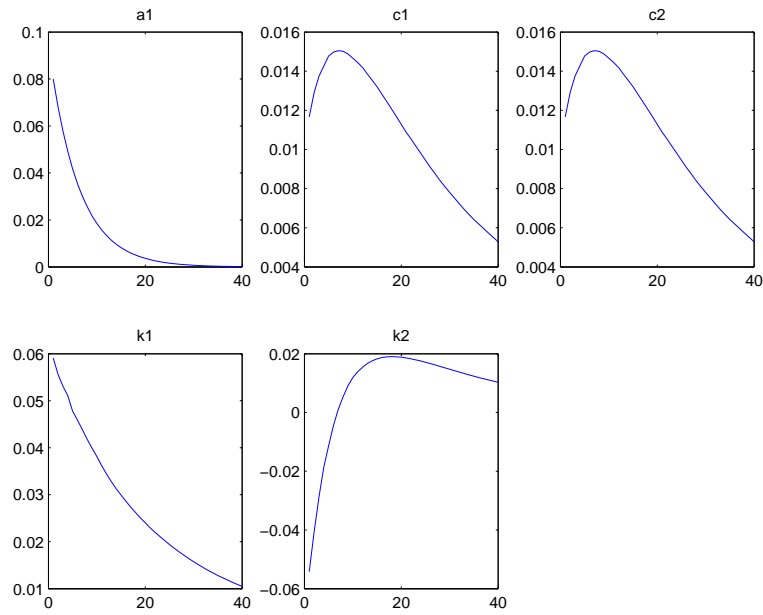
Figure 4: Impulse response function (ϵ_1)

```
gamma=2;
delta=.05;
alpha=.4;
beta=.98;
rho=.85;

model;
c1=c2; exp(c1)^(-gamma) =...
beta*exp(c1(+1))^(-gamma)*(alpha*exp(a1(+1))*exp(k1)^(alpha-1)+1-delta);
exp(c2)^(-gamma) =...
beta*exp(c2(+1))^(-gamma)*(alpha*exp(a2(+1))*exp(k2)^(alpha-1)+1-delta);
exp(c1)+exp(c2)+exp(k1)-exp(k1(-1))*(1-delta)+exp(k2)-exp(k2(-1))*(1-delta)...
= exp(a1)*exp(k1(-1))^alpha+exp(a2)*exp(k2(-1))^alpha;
a1=rho*a1(-1)+e1; a2=rho*a2(-1)+e2; end;

initval;
k1=2.8;
k2=2.8;
c1=.8;
c2=.8;
a1=0;
a2=0;
e1=0;
e2=0;
end;

shocks;
var e1;
stderr .08;
var e2;
```

```
stderr .08;
end;

steady;

estimated_params;
rho, normal_pdf, .84,.05; %
alpha, normal_pdf, .38, .03;
stderr e1, inv_gamma_pdf, .078, inf;
stderr e2,inv_gamma_pdf,.082, inf;
end;

varobs c1 k2;
estimation(datafile=simudata,mh_replic=1000,mh_jscale=.5);
```

Given that there are two exogenous shock variables, we use two observables $\ln C_1$ and $\ln K_2$. The result of the estimation is summarized in Figure 5, where the posterior distribution (in the darker line) of each parameter is contrasted against the given prior distribution. The result can be compared to the true parameter values in Table 5.
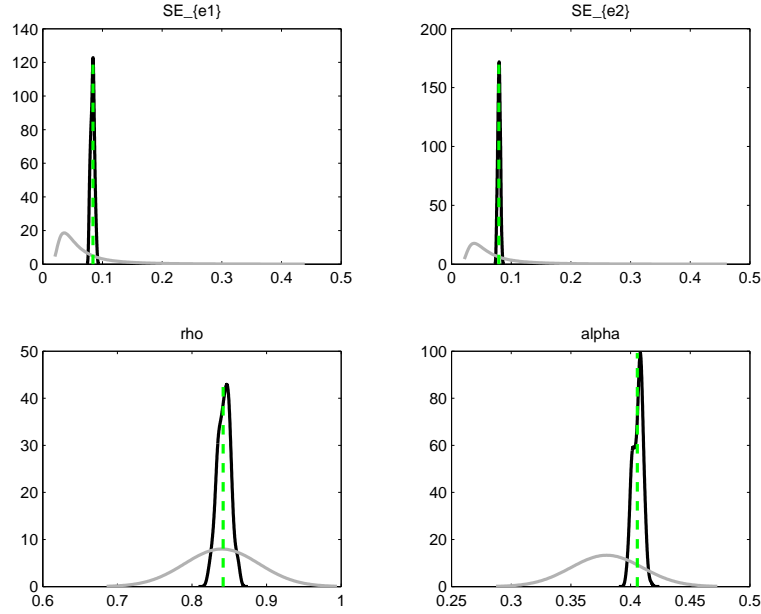

Figure 5: Prior v. Posterior Distribution

4. Fiscal policies in the growth model

This section describes a suite of Dynare programs that replicate the transition experiments performed in chapter 11 of Ljungqvist and Sargent (2004).

4.1 The model

A deterministic growth model has inelastic labor supply, an exogenous stream of government expenditures, and several kinds of distorting taxes. All variables are as described in chapter 11 of Ljungqvist and Sargent (2004). A representative agent maximizes

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}, \quad \beta \in (0,1)$$

11

Feasible allocations satisfy

$$g_t + c_t + k_{t+1} \leq Ak_t^\alpha + (1 - \delta) k_t.$$

The household budget constraint is

$$\sum_{t=0}^{\infty} \{q_t (1 + \tau_{ct}) + (1 - \tau_{it}) q_t [k_{t+1} - (1 - \delta) k_t]\} \leq \sum_{t=0}^{\infty} \{r_t (1 - \tau_{kt}) k_t + w_t\}$$

and the government budget constraint is

$$\sum_{t=0}^{\infty} q_t g_t \leq \sum_{t=0}^{\infty} \{\tau_{ct} q_t c_t - \tau_{it} q_t [k_{t+1} - (1 - \delta) k_t] + r_t \tau_{kt} k_t + w_t\}$$

The following conditions characterize an equilibrium:

$$
\begin{align}
c_t &= Ak_t^\alpha + (1 - \delta) k_t - k_{t+1} - g_t \tag{7}\\
q_t &= \beta^t c_t^{-\gamma} / (1 + \tau_{ct}) \tag{8}\\
r_t/q_t &= A\alpha k_t^{\alpha-1} \tag{9}\\
w_t/q_t &= Ak_t^\alpha - k_t A\alpha k_t^{\alpha-1} \tag{10}\\
R_{t+1} &= \frac{(1 + \tau_{ct})}{(1 + \tau_{ct+1})} \left[\frac{(1 + \tau_{it+1})}{(1 + \tau_{it})} (1 - \delta) + \frac{(1 - \tau_{kt+1})}{(1 - \tau_{it})} A\alpha k_{t+1}^{\alpha-1} \right] \tag{11}\\
s_t/q_t &= (1 - \tau_{kt}) A\alpha k_t^{\alpha-1} + (1 - \delta) \tag{12}\\
c_t^{-\gamma} &= \beta c_{t+1}^{-\gamma} R_{t+1} \tag{13}
\end{align}
$$

Because the other endogenous variables can be expressed as functions of k_t and c_t, in the programs below we use only a subset of the equilibrium conditions to compute equilibrium paths.

4.2 Parameter values

We set the parameters and the baseline values of the exogenous variables at values set in chapter 11 of Ljungqvist and Sargent (2004), as described in the following table are taken from RMT2.

Parameter/exogenous variable	Values
β	0.95
δ	0.2
A	1
α	0.33
γ	2
g	0.2
τ_c	0
τ_i	0
τ_k	0

4.3 Transition experiments

4.3.1 A permanent increase in g (Figure 11.3.1).

The Dynare code below reproduces Figure 11.3.1. which looks at the impact of a permanent increase of 0.2 in g at t=10.

```
// This program replicates figure 11.3.1 from chapter 11 of RMT2 by
// Ljungqvist and Sargent.

// y_ records the simulated endogenous variables in alphabetical order
// ys0_ records the initial steady state
```

```
// ys_ records the terminal steady state
// We check that these line up at the end points
// Note: y_ has ys0_ in the first column, ys_ in the last column, which explains
// why it has 102 elements.
// The sample of size 100 is in between.

// Warning:  All endogenous variables dated time t are treated as jump variables
// in Dynare. So k in the program corresponds to k_{t+1}
// and the same timing holds for the taxes.

//Declares the endogenous variables;
var c k;
//declares the exogenous variables
//investment tax credit, consumption tax, capital tax, government spending
varexo taui tauc tauk g;

parameters bet gam del alpha A;

bet=.95; // discount factor

gam=2; // CRRA parameter

del=.2; // depreciation rate

alpha=.33; // capital's share

A=1; // productivity

// Alignment convention:
// g tauc taui tauk are now columns of ex_. Because of a bad design decision
// the date of ex_(1,:) doesn't necessarily match the date in y_.
// Whether they match depends on the number of lag periods in endogenous versus
// exogenous variables.
// In this example they match because tauc(-1) and taui(-1) enter the model.

// These decisions and the timing conventions mean that y_(:,1) records the
// initial steady state, while y_(:,102) records the terminal steady state
// values. For j > 2, y_(:,j) records [c(j-1) .. k(j-1) .. G(j-1)] where k(j-1)
// means end of period capital in period j-1, which equals k(j) in chapter 11
// notation. Note that the jump in G occurs in y_(;,11), which confirms this
// timing. The jump occurs now in ex_(11,1)

model;
// equation 11.3.8.a
k=A*k(-1)^alpha+(1-del)*k(-1)-c-g;
// equation 11.3.8e + 11.3.8.g
c^(-gam)=
bet*(c(+1)^(-gam))*((1+tauc(-1))/(1+tauc))*((1-taui)*(1-del)/(1-taui(-1))+
 ((1-tauk)/(1-taui(-1)))*alpha*A*k(-1)^(alpha-1));
end;

initval; k=1.5; c=0.6; g = 0.2; tauc = 0; taui = 0; tauk = 0; end;
steady; // put this in if you want to start from the initial
// steady state, comment it out to start from the indicated values
```

```
endval; // The following values determine the new steady state
// after the shocks.
k=1.5; c=0.4; g =.4; tauc =0; taui =0; tauk =0; end;

steady; // We use steady again and the enval provided are initial
// guesses for Dynare to compute the ss.

// The following lines produce a g sequence with a once and for all jump in g shocks
// we use shocks to undo that for the first 9 periods and leave g at
// it's initial value of 0.2

var g; periods 1:9; values 0.2; end;

// now solve the model
simul(periods=100);

// Note: y_ records the simulated endogenous variables in alphabetical order
// ys0_ records the initial steady state
// ys_ records the terminal steady state
// check that these line up at the end points
y_(:,1) -ys0_(:) y_(:,102) - ys_(:)

// Compute the initial steady state for consumption to later do the plots.
co=ys0_(var_index('c')); ko = ys0_(var_index('k'));
// g is in ex_(:,1) since it is stored in alphabetical order
go = ex_(1,1)

// The following equation compute the other endogenous variables use in the
// plots below. Since they are function of capital and consumption, so we can
// compute them from the solved model above.

// These equations were taken from page 333 of RMT2
rbig0=1/bet;
rbig=y_(var_index('c'),2:101).^(-gam)./(bet*y_(var_index('c'),3:102).^(-gam));
rq0=alpha*A*ko^(alpha-1);
rq=alpha*A*y_(var_index('k'),1:100).^(alpha-1);
wq0=A*ko^alpha-ko*alpha*A*ko^(alpha-1);
wq=A*y_(var_index('k'),1:100).^alpha-y_(var_index('k'),1:100).*alpha*A.*...
 y_(var_index('k'),1:100).^(alpha-1);
sq0=(1-ex_(1,4))*A*alpha*ko^(alpha-1)+(1-del);
sq=(1-ex_(1:100,4)')*A*alpha.*y_(var_index('k'),1:100).^(alpha-1)+(1-del);

//Now we plot the responses of the endogenous variables to the shock.

figure subplot(2,3,1) plot([ko*ones(100,1)
y_(var_index('k'),1:100)' ]) // note the timing: we lag capital to
correct for syntax title('k') subplot(2,3,2) plot([co*ones(100,1)
y_(var_index('c'),2:101)' ]) title('c') subplot(2,3,3)
plot([rbig0*ones(100,1) rbig' ]) title('R') subplot(2,3,4)
plot([wq0*ones(100,1) wq' ]) title('w/q') subplot(2,3,5)
plot([sq0*ones(100,1) sq' ]) title('s/q') subplot(2,3,6)
plot([rq0*ones(100,1) rq' ]) title('r/q')
```
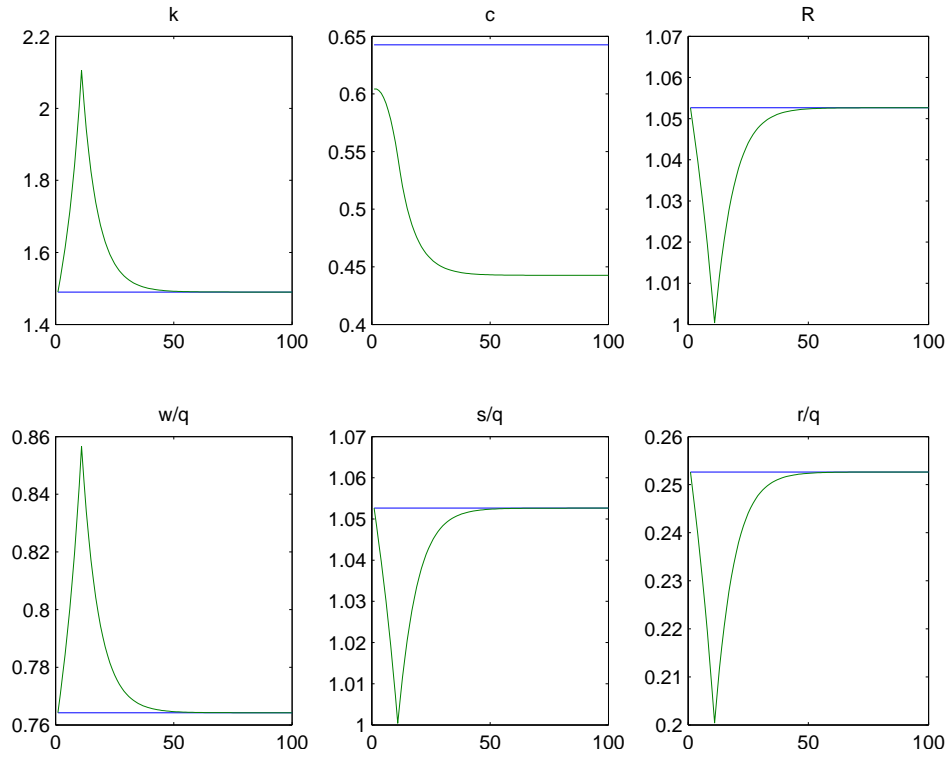
Figure 6: A permanent increase of g (Figure 11.3.1).

The following figure shows the path of the endogenous variables along the transition to the new steady state.

Next we show the other transition experiments and only the parts of the code that need to be changed for each experiment.

4.3.2 A permanent increase in τ_c (Figure 11.3.2).

A permanent increase of τ_c at t=10 of 20 per cent. (Figure 11.3.2)

```
initval;
k=1.5;
c=0.6;
g = 0.2;
tauc = 0;
taui = 0;
tauk = 0;
end;

steady;

endval;
k=1.5;
c=0.6;
g = 0.2;
tauc =0.2;
taui =0;
tauk =0;
end;
```
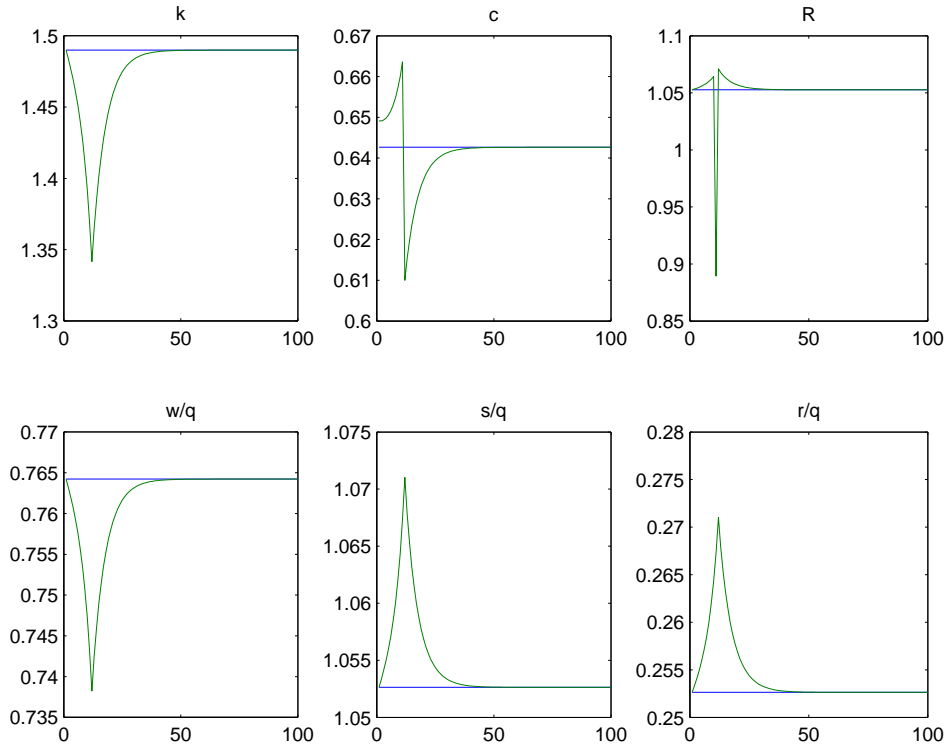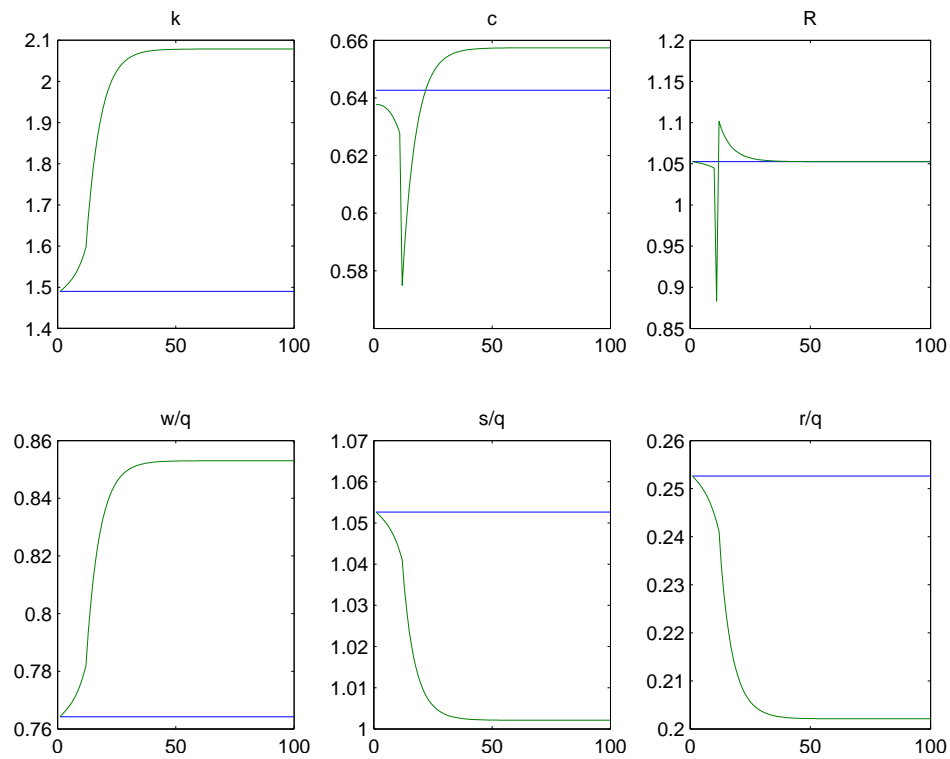
Figure 7: A permanent increase of τ_c (Figure 11.3.2).

```
steady;

shocks;
var tauc;
periods 1:9;
values 0;
end;
```

4.3.3 A permanent increase in τ_i (Figure 11.5.1).

A permanent increase of τ_i at t=10 of 20 percent. (Figure 11.5.1)

```
initval;
k=1.5;
c=0.6;
g = 0.2;
tauc = 0;
taui = 0;
tauk = 0;
end;
steady;

endval;
k=1.5;
c=0.6;
```
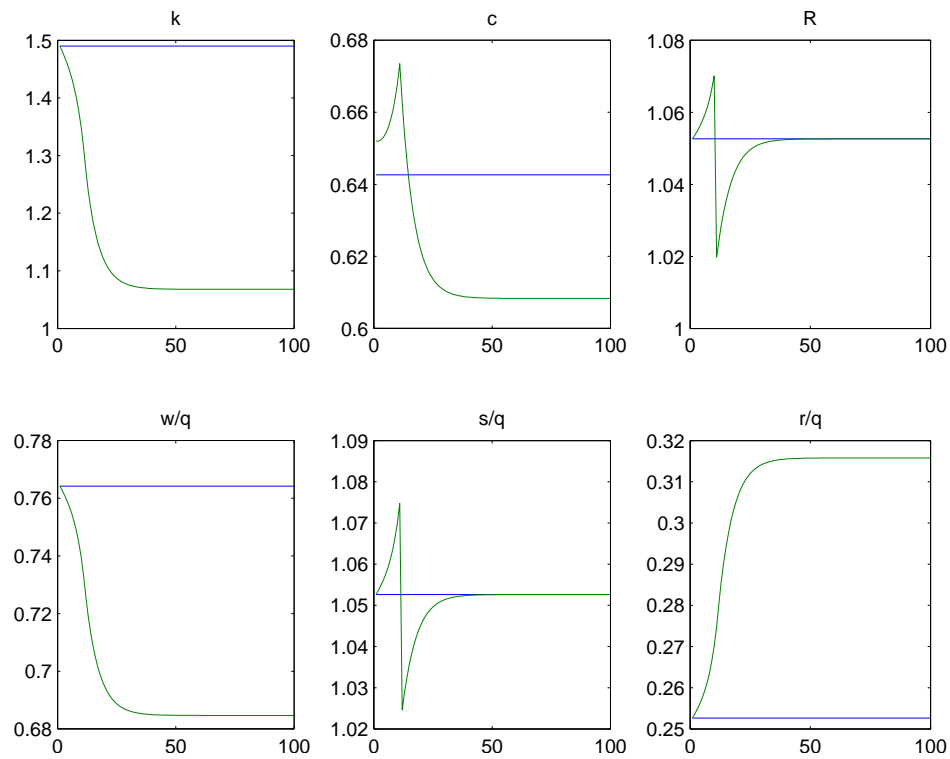
Figure 8: A permanent increase of τ_i (Figure 11.5.1).

```
g =0.2;
tauc =0;
taui =0.20;
tauk =0;
end;
steady;

shocks;
var taui;
periods 1:9;
values 0;
end;
```

4.3.4 A permanent increase in τ_k (Figure 11.5.2).

A permanent increase of τ_k at t=10 of 20 percent. (Figure 11.5.2)

```
initval;
k=1.5;
c=0.6;
g = 0.2;
tauc = 0;
taui = 0;
tauk = 0;
end;
steady;
```

Figure 9: A permanent increase of τ_k (Figure 11.5.2).

```
endval;
k=1.5;
c=0.6;
g =0.2;
tauc =0;
taui =0;
tauk = 0.2;
end;
steady;

shocks;
var tauk;
periods 1:9;
values 0;
end;
```

4.3.5 One time increase in g (Figure 11.7.1).

A one time pulse of g at t=10 of 0.2. (Figure 11.7.1)
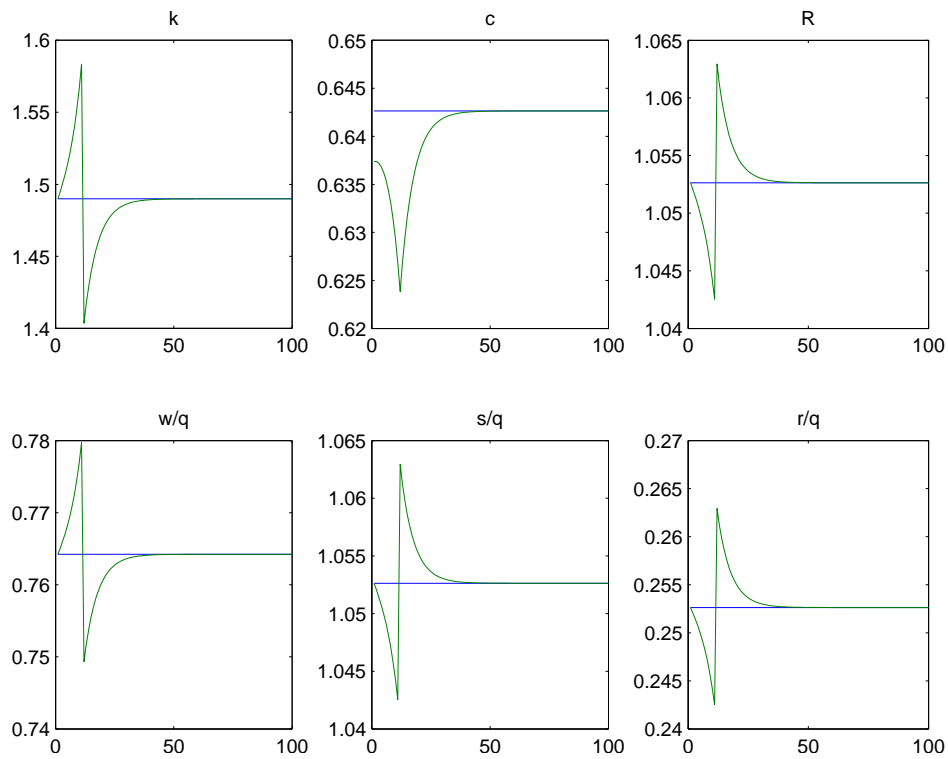
```
initval;
k=1.5;
c=0.6;
g = 0.2;
tauc = 0;
```

Figure 10: One time increase of g (Figure 11.7.1).

```
taui = 0;
tauk = 0;
end;
steady;

endval;
k=1.5;
c=0.6;
g = 0.2;
tauc =0;
taui =0;
tauk =0;
end;
steady;

shocks;
var g;
periods 10;
values 0.4;
end;
```

4.3.6   One time increase in τ_i (Figure 11.7.2).

A one time pulse of τ_i at t=10 of 20 percent. (Figure 11.7.2)
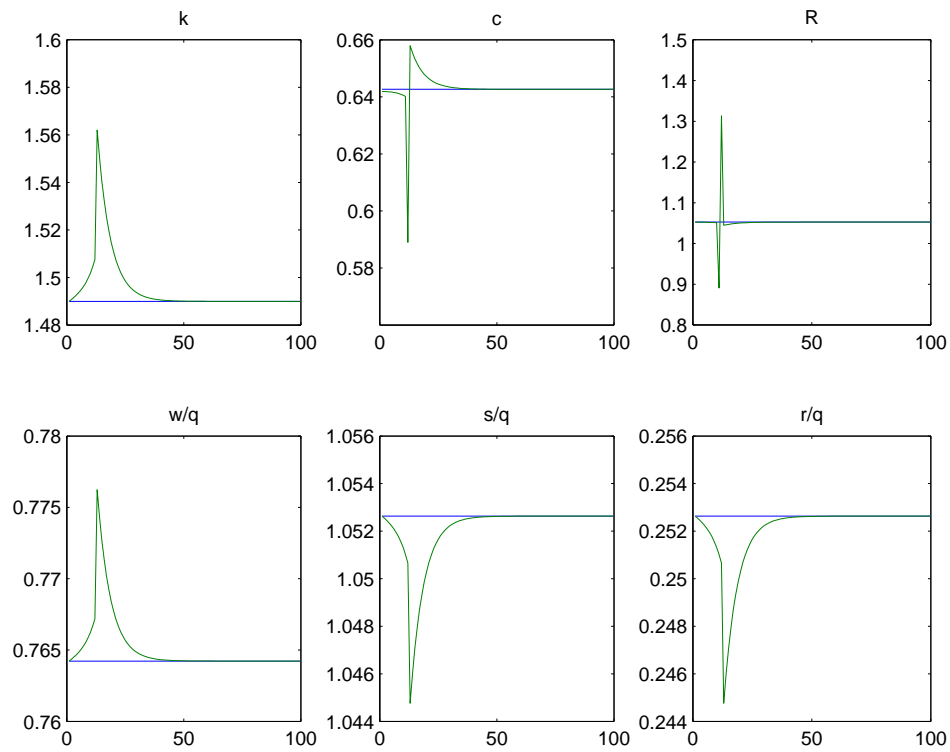
```
initval;
```

Figure 11: One time increase of τ_i (Figure 11.7.2).

```
k=1.5;
c=0.6;
g = 0.2;
tauc = 0;
taui = 0;
tauk = 0;
end;
steady;

endval;
k=1.5;
c=0.6;
g =0.2;
tauc =0;
taui =0;
tauk =0;
end;
steady;

shocks;
var taui;
periods 10;
values 0.2;
end;
```

5. The demand for money during hyperinflations

Sargent (1977) specified and estimated a joint stochastic process for money growth and inflation that was designed to make the adaptive expectations model of Cagan (1956) consistent with rational expectations. The model has the form

$$m_t - p_t = \alpha \pi_t + u_t \tag{14}$$
$$\pi_t = \frac{1-\lambda}{1-\lambda L}(p_t - p_{t-1}) = \frac{1-\lambda}{1-\lambda L} x_t \tag{15}$$
$$\mu_t = m_t - m_{t-1} \tag{16}$$

with $\alpha < 0$, m being the log of money supply, π the expectation of inflation, p the log of the price level and u an erro term whose specification is described below. The hypothesis of rational expectations imposes

$$\pi_t = E_t x_{t+1}. \tag{17}$$

Sargent (1977) reverse engineered a process for x_t such that Cagan's adaptive expectations scheme is equal to rational expectations. The following conditions ensure the rationality of Cagan's expectations scheme

$$u_t = u_{t-1} + \eta_t \tag{18}$$
$$\mu_t = \frac{1-\lambda}{1-\lambda L} x_t + \varepsilon_t \tag{19}$$

η and ε are both serially uncorrelated random error terms with mean zero and variances σ_η^2 and σ_ε^2, respectively. For estimating the model, it is convenient to represent it in the following form

$$\begin{pmatrix} x_t \\ \mu_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1-\lambda & \lambda \end{pmatrix} \begin{pmatrix} x_{t-1} \\ \mu_{t-1} \end{pmatrix} + \begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} - \lambda I \begin{pmatrix} a_{1t-1} \\ a_{2t-1} \end{pmatrix} \tag{20}$$

The a terms are linked to the random terms above via

$$\begin{pmatrix} a_{1t} \\ a_{2t} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1+\alpha(1-\lambda) & -(1-\lambda) \end{pmatrix} (\lambda + (1-\lambda)\alpha)^{-1} \begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \tag{21}$$

The code below estimates representation (20) by maximum likelihood:

```
// this program estimates the model in
// "The Demand for Money during Hyperinflations under Rational Expectations: I"
// by T. Sargent, IER 1977 using maximum likelihood
// variables are defined as follows:
// x=p_t-p_{t-1}, p being the log of the price level
// mu=m_t-m_{t-1}, m being the log of money supply
// note that in contrast to the paper eta and epsilon
// have variance 1 (they are multiplied by the standard deviations)

var x mu a1 a2; varexo epsilon eta; parameters alpha lambda sig_eta
sig_epsilon; lambda=.5921; alpha=-2.344; sig_eta=.001;
sig_epsilon=.001;

model;
x=x(-1)-lambda*a1(-1)+(1/(lambda+alpha*(1-lambda)))*sig_epsilon*epsilon
-(1/(lambda+alpha*(1-lambda)))*sig_eta*eta;
mu=(1-lambda)*x(-1)+lambda*mu(-1)-lambda*a2(-1)+(1+alpha*(1-lambda))/(lambda+alpha*(1-lambda))
*sig_epsilon*epsilon-(1-lambda)/(lambda+alpha*(1-lambda))*sig_eta*eta;
a1=(1/(lambda+alpha*(1-lambda)))*sig_epsilon*epsilon-(1/(lambda+alpha*(1-lambda)))*sig_eta*eta;
a2=(1+alpha*(1-lambda))/(lambda+alpha*(1-lambda))*sig_epsilon*epsilon
-(1-lambda)/(lambda+alpha*(1-lambda))*sig_eta*eta;
```
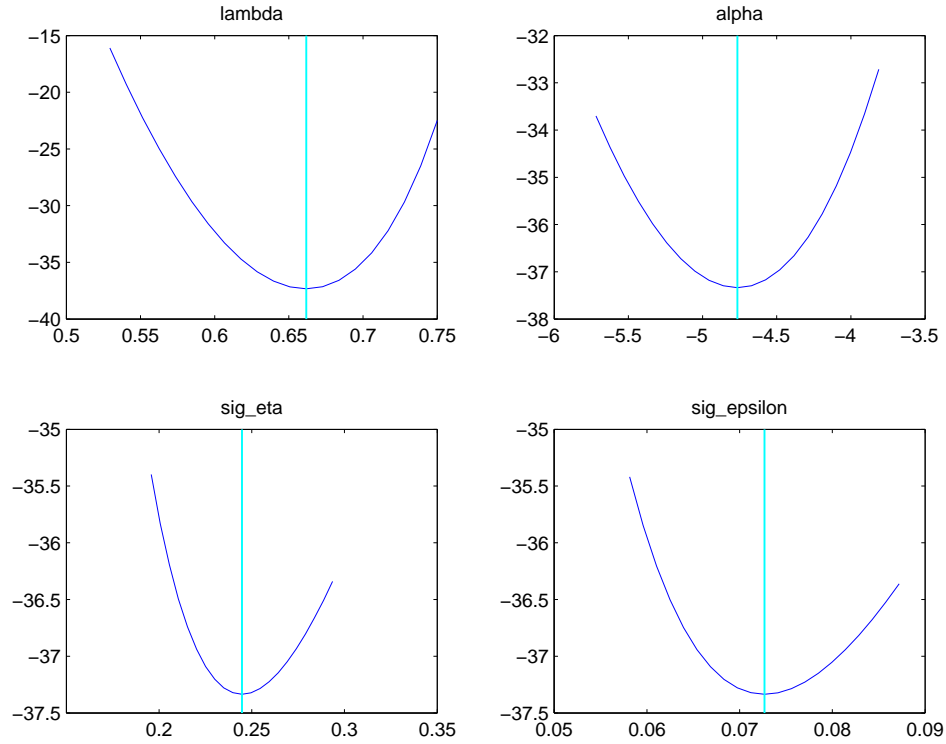
Figure 12: Negative of the Log-Likelihood function for the hyperinflation model

```
end; steady; shocks;
var eta; stderr 1; var epsilon; stderr 1;
end;

estimated_params;
// ML estimation setup
// parameter name, initial value, boundaries_low, ..._up;
lambda, .5, 0.25, 0.75; alpha, -2, -8, -0.1; sig_eta, .0001, 0.0001,
0.3; sig_epsilon, .0001, 0.0001, 0.3; end;

varobs mu x; unit_root_vars x;
estimation(datafile=cagan_data,first_obs=1,nobs=34,mh_replic=0,mode_compute=2,mode_check);
```

The data for the estimation are from Cagan (1956) for Germany[1]. The shape of the negative of the log-likelihood function is plotted in figure 12.

Parameter	Estimate	s.d.	t-stat
λ	0.6617	0.0535	12.3754
α	-4.7654	3.0689	1.5528
σ_η	0.2447	0.2170	1.1275
σ_ε	0.0727	0.0100	7.2946

Table 6: Results from Maximum likelihood

Next we estimate this model using Bayesian techniques. For this we replace the last part of the code above with

---

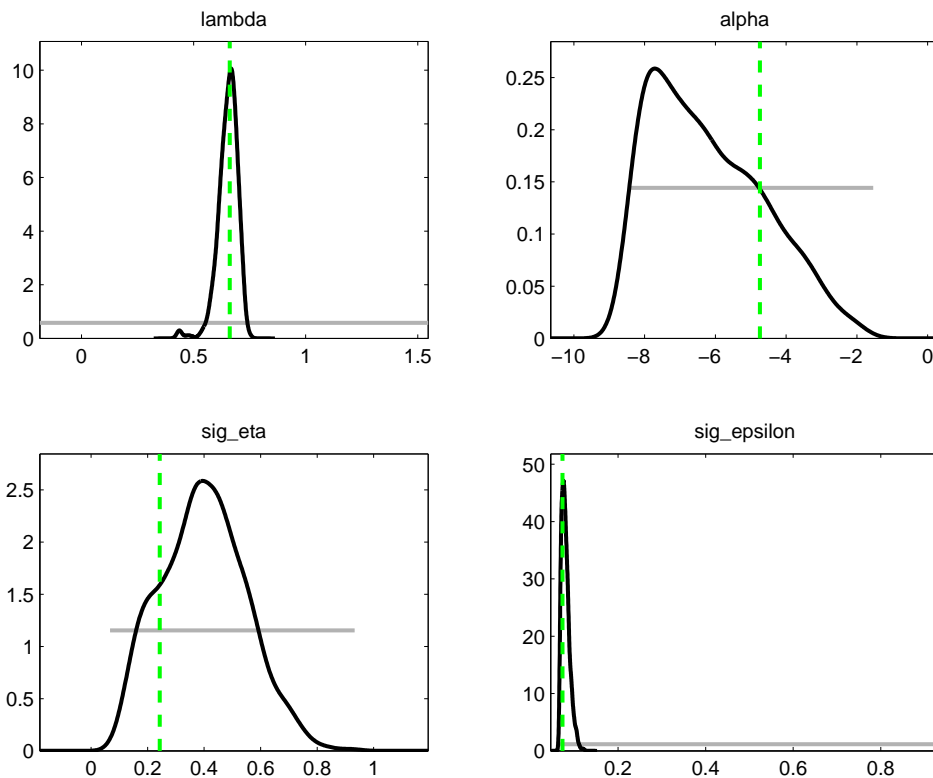[1]The data are demeaned, allowing us to ignore constant terms in the model

Figure 13: Posteriors and priors for the hyperinflation model

```
estimated_params;
// Bayesian setup
lambda, uniform_pdf,    0.68, .5;
alpha, uniform_pdf, -5, 2;
sig_eta,uniform_pdf, .5, 0.25;
sig_epsilon, uniform_pdf, .5, 0.25;
end;


varobs mu x; unit_root_vars x;
estimation(datafile=cagan_data,nobs=34,mh_replic=25000,mh_nblocks=1,mh_jscale=1,mode_compute=4);
```

Note that the priors we have chosen are reasonably loose, but given that there are only 34 observations we have opted to use somewhat informative priors. The posteriors and priors are plotted in figure 13.

6. Solving and estimating Hall's permanent income model

This note describes a suite of Dynare programs to simulate and estimate the permanent income model of Hall(1988).

6.1   The model

A consumer faces the following decision problem:

$$\max_{\{c_t, k_t\}_{t=0}^{\infty}} -\frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left(c_t - b_t\right)^2$$

subject to the following constraints

$$c_t + k_t = Rk_{t-1} + d_t$$
$$d_t = \rho d_{t-1} + (1 - \rho)\, \mu_d + \varepsilon_d$$

$$b_t = \rho_b b_{t-1} + (1 - \rho_b)\, \mu_b + \varepsilon_b$$

We assume that $\beta R = 1$ and that there is no depreciation, which leads to the following law of motion for investment:

$$i_t = k_t - k_{t-1}$$

From the Lagrangian for this problem one obtains the following first order conditions:

$$\mu_{ct} = \mu_{ct+1}$$

$$\mu_{ct} = b_t - c_t$$

where μ_{ct} is the marginal utility of consumption or the Lagrange multiplier.

Note that this problem has a unit root and so there exist no steady state. However, we can supply ONE solution corresponding to an initial k that we give to Dynare.

The following Dynare code simulates data from this model:

```
periods 5000;

var c k mu_c b d in;
varexo e_d e_b;

parameters R rho rho_b mu_b mu_d; R=1.05;
rho = 0;
mu_b=30;
mu_d=5;
rho_b = 0;

model(linear);

 c+k = R*k(-1) + d;
 mu_c = b - c;
 mu_c=mu_c(+1);
 d= rho*d(-1)+ mu_d*(1-rho) + e_d;
 b=(1-rho_b)*mu_b+rho_b*b(-1)+e_b;
 in = k - k(-1);
 end;

//With a unit root, there exists no steady state.  Use the following trick.
// Supply ONE solution corresponding to the initial k that you named.

initval;
d=mu_d;
k=100;
c = (R-1)*k +d;
mu_c=mu_b-c;
b=mu_b;
end;
```
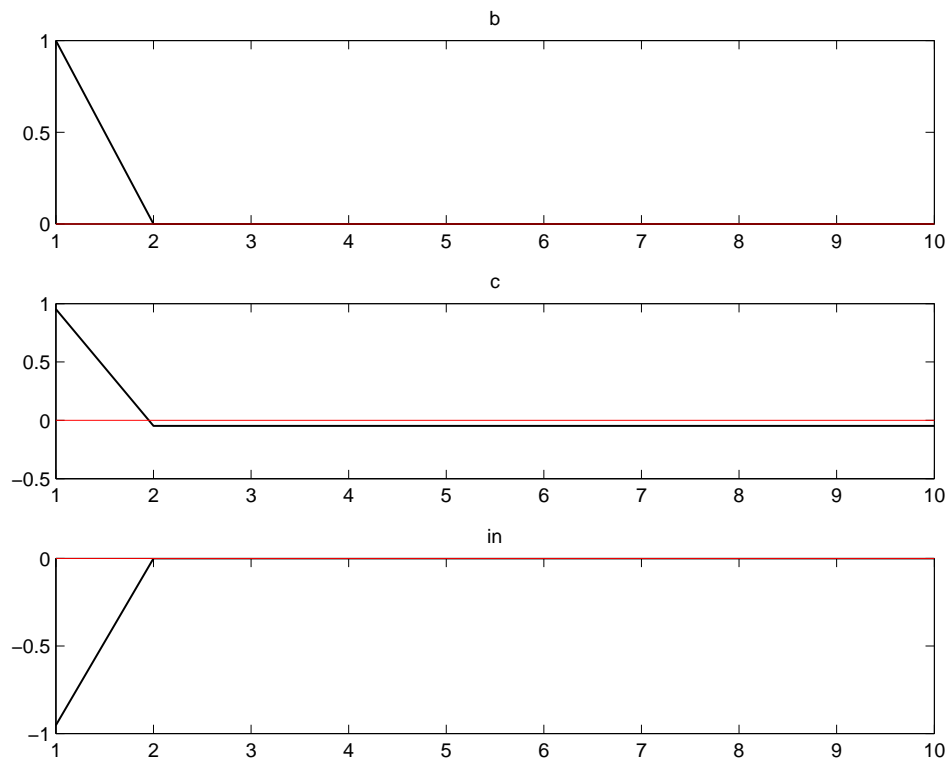
Figure 14: A shock to ε_b

```
shocks;
var e_d;
stderr 1;
var e_b;
stderr 1;
end;

steady;
check;

stoch_simul(dr_algo=1, order=1, periods=500, irf=10);


save data_hall c in;
```

Next, we proceed to estimate a subset of the parameters of the model by ML using simulated data obtained from the previous code. To do so we add the following few lines of code. Notice how we use the command unit_root_vars to initialize the Kalman filter with a diffuse prior. We do this because k is not stationary – Hall's model with $\beta R = 1$ has a solution for k that has a unit root and that is co-integrated with consumption.

```
estimated_params;
// ML estimation setup
// parameter name, initial value, boundaries_low, ..._up;
rho, -0.0159, -0.9, 0.9;
R, 1.0074, 0, 1.5;
end;
```
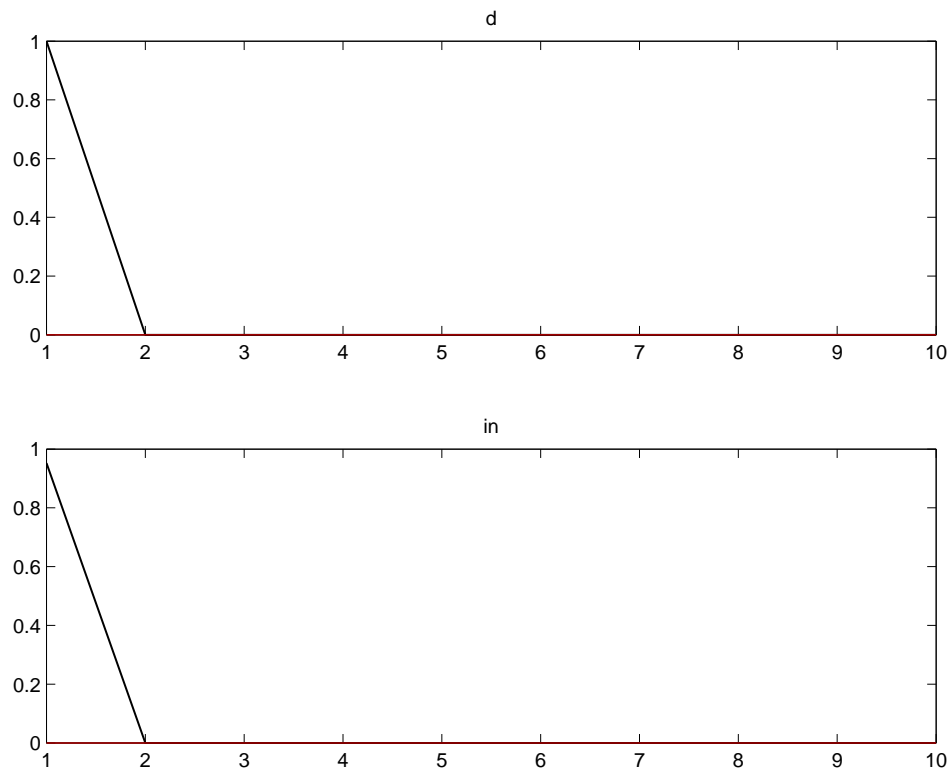
Figure 15: A shock to ε_d

```
varobs c in;
// declare the unit root variables for diffuse filter
unit_root_vars k;
estimation(datafile=data_hall,first_obs=101,nobs=200,mh_replic=0,mode_compute=4,mode_check);
```

We proceed to estimate the same subset of the parameters using Bayesian techniques: This is implemented in Dynare by adding the following few lines of code.

```
estimated_params;
rho, beta_pdf, .1, 0.2;
R, normal_pdf,1.02,0.05;

end;

varobs c in;
// declare the unit root variables for diffuse filter
unit_root_vars k;
estimation(datafile=data_hall,first_obs=101,nobs=200,mh_replic=1000,mh_nblocks=2,mh_jscale=2);
```
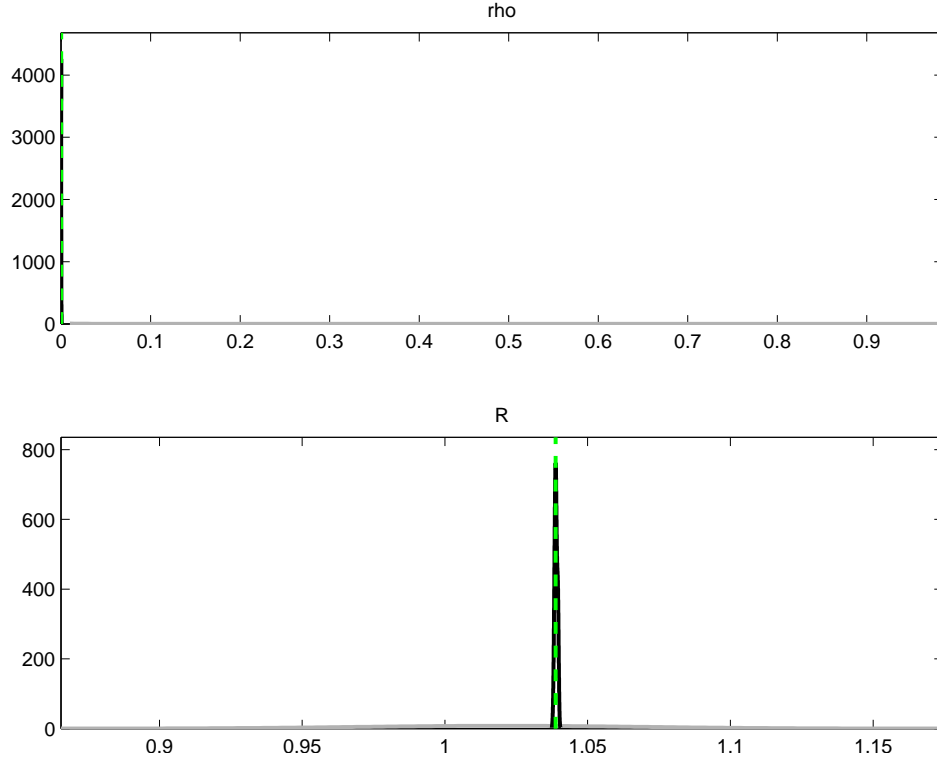
Figure 16: priors and posteriors from the Bayesian estimation of Hall's model

7. Solving and estimating Ryoo and Rosen's (2004) model of the market for engineers

This section describes a suite of Dynare programs to simulate and estimate a model of Aloysius Siow (1984) and Jaewoo Ryoo and Sherwin Rosen (2004) of the labor market for engineers.

7.1 The model

Ryoo and Rosen studied a partial equilibrium model determining a stock of 'engineers' N_t; the number of new entrants into engineering school, s_t; and the wage level w_t of engineers.[2] It takes k periods of schooling to become an engineer. The model consists of the following components:

The present value of wages of an engineer:

$$P_t = (1 - \delta) \beta P_{t+1} + (1 - \delta)^k \beta^k w_{t+k}$$

A supply of new engineering students:

$$s_t = a_0 + a_1 P_t + e_{st}$$

A time-to-build structure of the education process:

$$N_t = (1 - \delta) N_{t-1} + s_{t-k}$$

A demand curve for engineers:

$$N_t = d_0 - d_1 w_t + e_{dt}$$

The following Dynare code simulates data from this model:

---

[2]Sherwin Rosen showed this version of the model to Sargent in Chicago in fall 1995.

```
periods 500;
var s N P W;
varexo e_s e_d;

parameters a0 a1 delta d0 d1 bet k;
a0=10;
a1=1;
d0=1000;
d1=1;
bet=.99;
delta=.02;

model(linear);
s=a0+a1*P+e_s; // flow supply of new entrants
N=(1-delta)*N(-1) + s(-4); // evolution of the stock
N=d0-d1*W+e_d; // stock demand equation
P=bet*(1-delta)*P(+1)+bet^4*(1-delta)^4*W(+4); // present value of
//wages
end;

initval;
s=0;
N=0;
P=0;
W=0;
end;

shocks;
var e_d;
stderr 1;
var e_s;
stderr 1;
end;

steady;
check;

stoch_simul(dr_algo=1, order=1, periods=500, irf=10);

save data_rosen s N P W;
```

Note that in this code we set $k = 4$. Dynare does not let you use a parameter as lag length.

Next, we estimate a subset of the parameters of the model by ML using simulated data obtained from the previous code. To do so we add the following lines of code.

```
estimated_params;
a1, .5, -10, 10;
d1, .5, -20, 40;

stderr e_d, .5, .05, 5;
stderr e_s, .3, .05, 6; end;
// these are the ranges for the parameters

varobs W N;
```
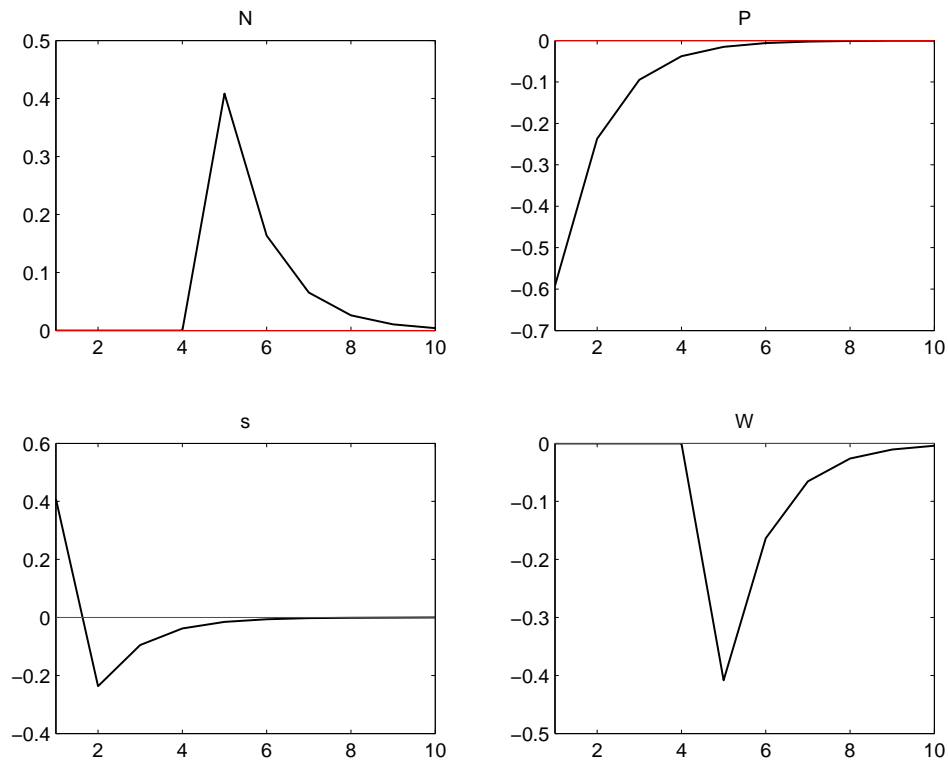
Figure 17: A shock to e_{st}

```
estimation(datafile=data_rosen,first_obs=101,nobs=200,mh_replic=0,mode_compute=4,mode_check);
```

Figure plots the negative of the log-likelihood function for the parameters.

We proceed to estimate the same subset of the parameters using Bayesian techniques. We implemented this in Dynare by adding the following lines of code.

```
estimated_params;
a1, gamma_pdf, .5, .5;
d1, gamma_pdf, 2, .5;

stderr e_d,inv_gamma_pdf,.5, 30;
stderr e_s,inv_gamma_pdf,.2, 30;

// The syntax for to input the priors is the following:
// variable name, prior distribution, parameters of distribution.

end;

varobs W N;
estimation(datafile=data_rosen,first_obs=101,nobs=200,mh_replic=10000,mh_nblocks=2,
  mh_jscale=2,mode_compute=0,mode_file=rosen_estimateML2_mode);
```
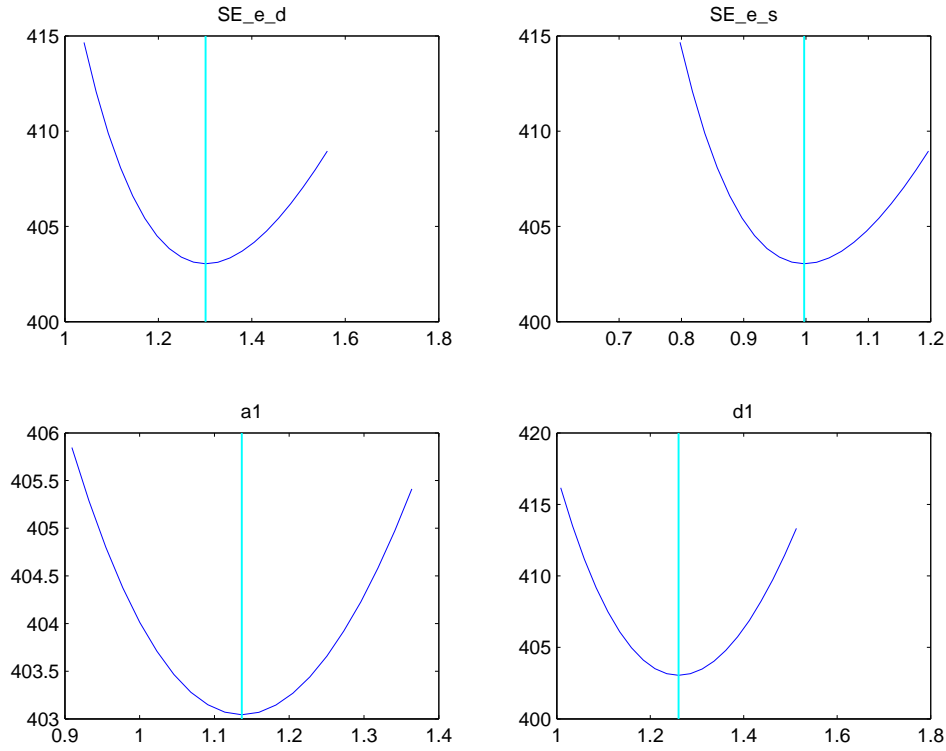
Figure 18: Negative of the Log-Likelihood for Ryoo and Rosen's model

8.   Estimation of Bansal and Yaron(2004) consumption growth process

Let consumption growth have the state space representation

$$
\begin{aligned}
x_t &= \rho x_{t-1} + \sigma_x \varepsilon_{x,t} \\
\Delta c_t &= \mu_c + x_{t-1} + \sigma_c \varepsilon_{c,t}
\end{aligned}
$$

where the shocks $\varepsilon_{c,t}$ and $\varepsilon_{x,t}$ are i.i.d. standard normal. We use quarterly growth rate of per capita U.S. consumption over the period 1947.2-2003.3 to estimate the four unknown parameters. The following Dynare code does this by maximum likelihood.

```
var x y;
varexo e_x e_u;

parameters rho sig_x sig_u mu_y;

rho = .98;
mu_y=.015;
sig_x=0.00025;
sig_u=.0078;

model(linear);
x=rho*x(-1) + sig_x*e_x;
y=mu_y + x(-1) +sig_u*e_u;
end;

initval;
```
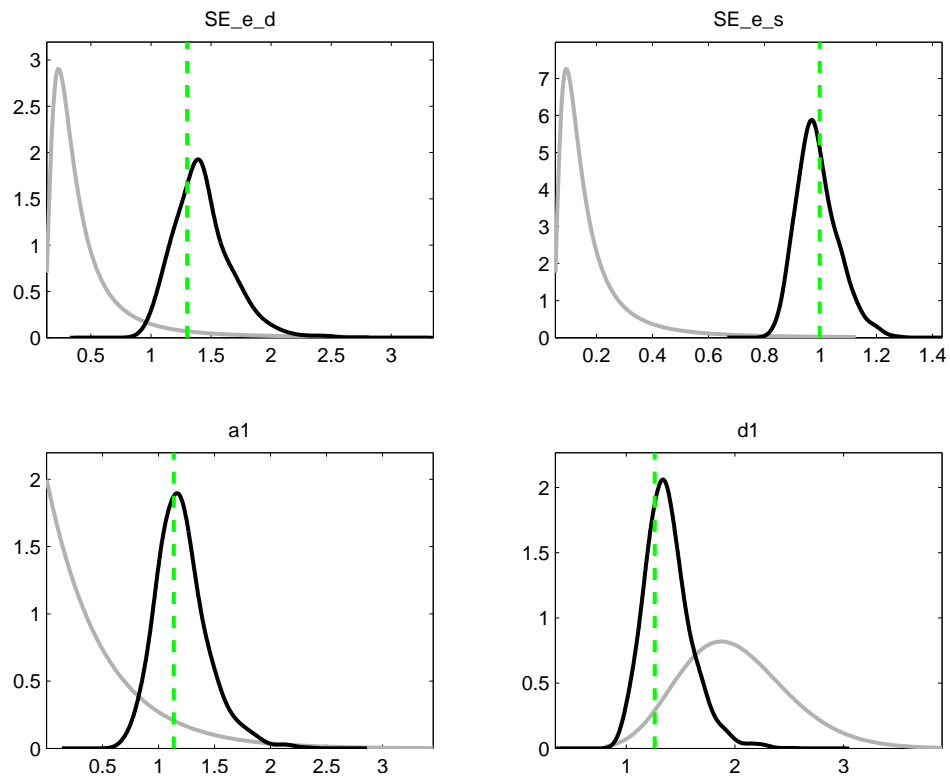
Figure 19: priors and posteriors from the Bayesian estimation of Ryoo and Rosen's model

```
x=0;
y=mu_y;
end;

steady;

shocks;
var e_x;
stderr 1;
var e_u;
stderr 1;
end;

estimated_params;
// ML estimation setup
// parameter name, initial value, boundaries_low, ..._up;
rho, 0, -0.999, 0.999;  // use this for unconstrained max
likelihood
// rho, .980, .975, .999 ;  // use this for long run risk model
// sig_x, .0004,.0001,.05 ; // use this for the long run risk model
sig_x, .0005, .00000000001, .01;  // use this for unconstrained
max likelihood sig_u, .007,.001, .1; mu_y, .014, .0001, .04;

end;

varobs y;
estimation(datafile=data_consRicardoypg,first_obs=1,nobs=227,mh_replic=0,mode_compute=4,mode_check);
//  mode_compute=4 uses Chris Sim's csminwel algorithm to maximize the likelihood function.
//  Using mode_compute=1 which uses the Matlab function fmincon or using mode_compute=2 which uses
//  Lester Ingber's Adaptive Simulated Annealing algorithm produce almost identical results. However,
//  model_compute=3 which uses fminunc gives the wrong answer.
```

Table 7 reports the estimated parameter from Dynare.

Parameter	Estimate	s.d.	t-stat
rho	0.5623	0.1927	2.9179
sigx	0.0030	0.0011	2.6921
sigu	0.0040	0.0007	5.4519
muy	0.0056	0.0005	10.7613

Table 7: Results from Maximum likelihood

Figure 20 also plots the negative of the log-likelihood function along several dimensions.

Next, we estimate the same model using Bayesian techniques and two different sets of priors. The first set of priors are tight around the calibrated values used in Bansal and Yaron (2004). To accomplish this, we can simply change the following lines of code from our MLE program above.

```
estimated_params;

rho, beta_pdf, .98, .01;
mu_y, uniform_pdf, .005, .0025;

sig_u, inv_gamma_pdf, .003, inf;
sig_x, inv_gamma_pdf, .003, inf;
```
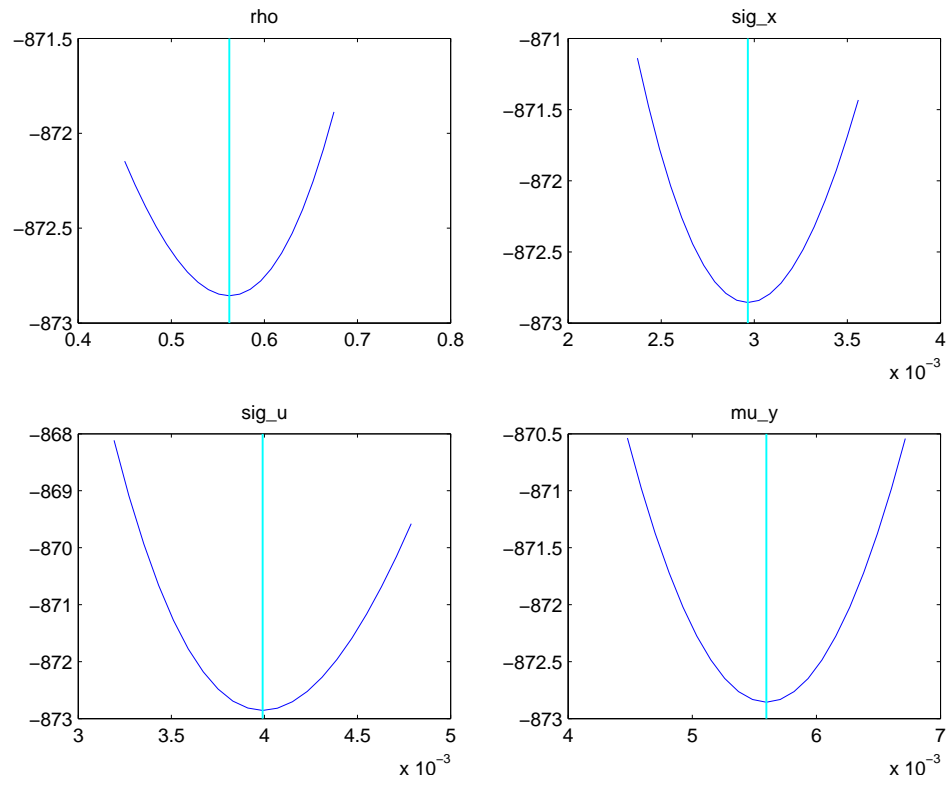
Figure 20: Negative of the Log-Likelihood function for the consumption growth model
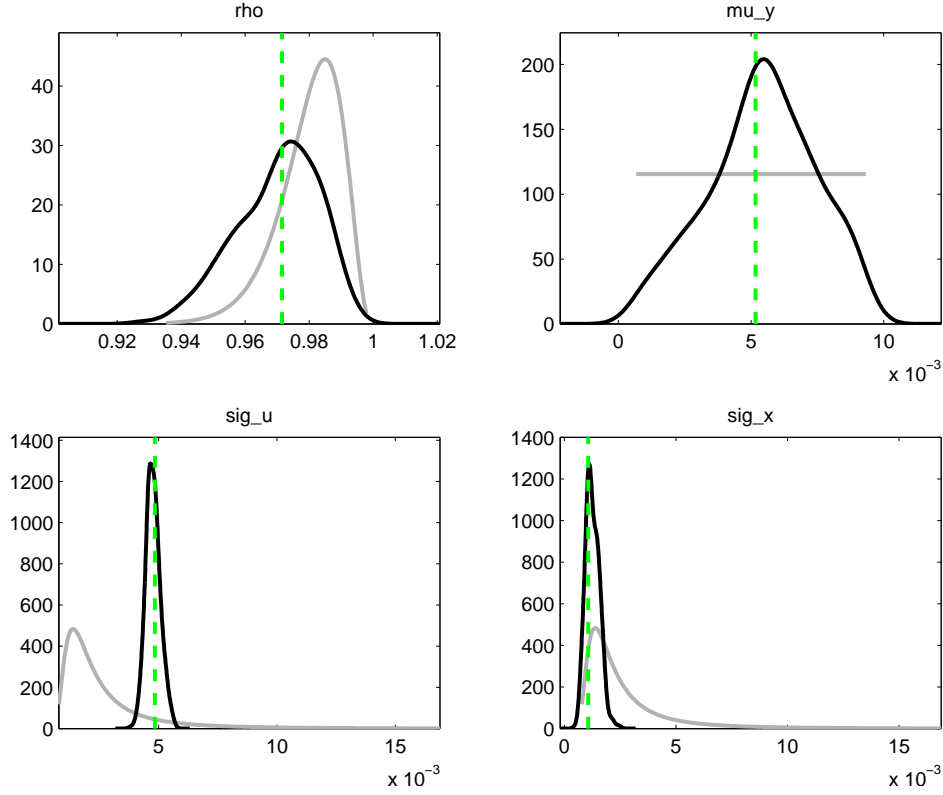
Figure 21: Prior and Posteriors for the Bansal and Yaron (2004) endownment process

```
// The syntax for to input the priors is the following:
// variable name, prior distribution, parameters of distribution.

end;

varobs y;
estimation(datafile=data_consRicardoypg,first_obs=1,nobs=227,mh_replic=5000,mh_nblocks=1,mh_jscale=1)
```

Figure 21 plots the prior and the posterior densities for the estimated parameters. The posteriors are plotted in black and priors in gray. A vertical green line denotes a posterior mean.

Next, we set vague priors for ρ and for μ_y.

```
rho, beta_pdf, .5, .2;
mu_y, uniform_pdf, .005, .0025;

sig_u, inv_gamma_pdf, .003, inf;
sig_x, inv_gamma_pdf, .003, inf;
```

Figure 22 plots the prior and the posterior densities for the estimated parameters. Notice that with a more diffuse prior on ρ, the posteriors for μ_y are tighter.

9. Robust permanent income and pricing

Hansen, Sargent, and Tallarini (1999) study consumption, saving, and security market prices in a permanent income model in which the agents mistrust their specification of the economic environment. The preferences of a
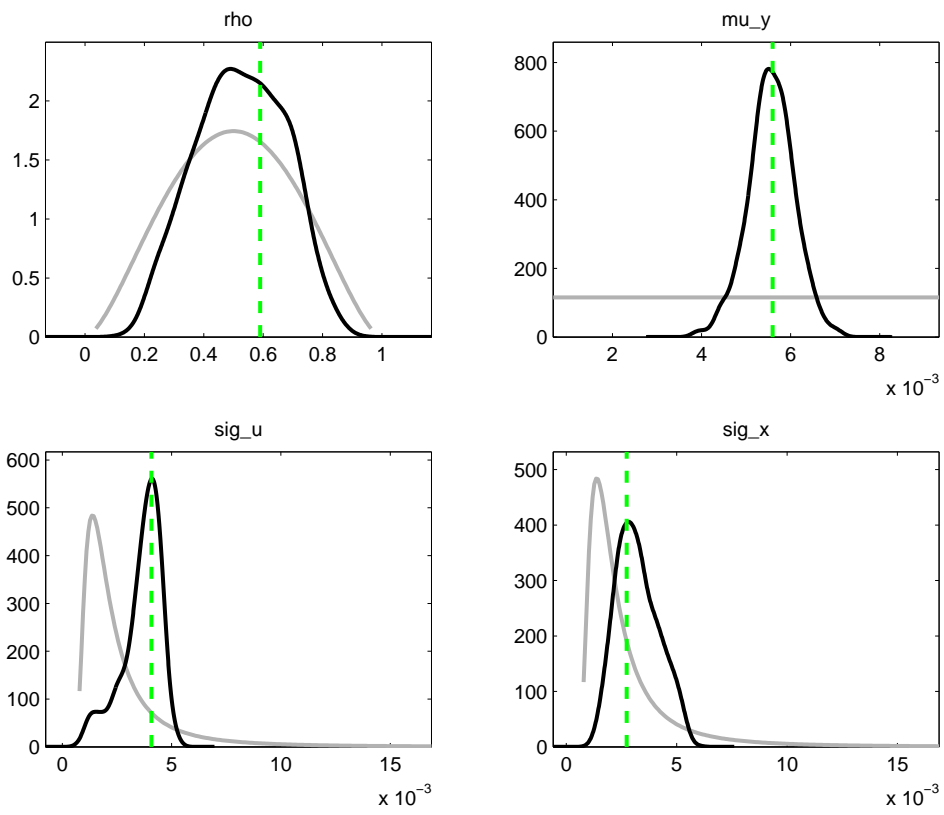
Figure 22: Prior and Posteriors for the Bansal and Yaron (2004) endownment process

representative consumer are defined in terms of the following risk-sensitivity recursion:

$$U_t = u_t + \beta R_t(U_{t+1}) \tag{22}$$

$$R_t(U_{t+1}) \equiv \frac{2}{\sigma} \log E_t[\exp(\frac{\sigma U_{t+1}}{2})] \tag{23}$$

with u_t being the per-period utility function. Note that $\sigma = 0$ corresponds to Von Neumann-Morgenstern preferences. The authors use a linear quadratic framework described below. Furthermore, they show that for a given (β, σ) pair there exists another pair $(\tilde{\beta}, 0)$ that gives the same equilibrium quantities. This observation simplifies solution and estimation of the model significantly. For the $\sigma = 0$ case and HST's linear-quadratic habit persistence model, preferences become

$$U_0 = E_0 \sum_{t=0}^{\infty} \beta^t(-(s_t - b)^2) \tag{24}$$

with b being a subsistence point and s defined as

$$s_t = (1 + \lambda)c_t - \lambda h_{t-1} \tag{25}$$
$$h_t = \delta_h h_{t-1} + (1 - \delta_h)c_t \tag{26}$$

where c_t is current period consumption and h_{t-1} the habit stock. There is a linear production technology:

$$c_t + i_t = \gamma k_{t-1} + d_t \tag{27}$$
$$k_t = \delta_k k_{t-1} + i_t \tag{28}$$
$$d_t = \mu_d + \hat{d}_t + \bar{d}_t \tag{29}$$
$$\bar{d}_t = (\phi_1 + \phi_2)\bar{d}_{t-1} - \phi_1\phi_2\bar{d}_{t-2} + c_{\bar{d}} * e_{1t} \tag{30}$$
$$\hat{d}_t = (\alpha_1 + \alpha_2)\hat{d}_{t-1} - \alpha_1\alpha_2\hat{d}_{t-2} + c_{\hat{d}} * e_{2t} \tag{31}$$

e_{1t} and e_{1t} are standard normal iid shocks. Finally, we define the gross return on capital as

$$R \equiv \delta_k + \gamma \tag{32}$$

and impose

$$R\beta = 1. \tag{33}$$

To solve the planning problem, we formulate a Lagrangian with multipliers $2\beta^t\mu_{st}$ on (25), $2\beta^t\mu_{ht}$ on (26) and $2\beta^t\mu_{ct}$ on (27) (after substituting for investment i_t). The first order necessary conditions are:

$$\mu_{st} = b - s_t \tag{34}$$
$$\mu_{ct} = (1 + \lambda)\mu_{st} + (1 - \delta_h)\mu_{ht} \tag{35}$$
$$\mu_{ht} = \beta E_t[\delta_h\mu_{ht+1} - \lambda\mu_{st+1}] \tag{36}$$
$$\mu_{ct} = \mu_{ct+1} \tag{37}$$

plus (25)-(28). The following code estimates the model using maximum likelihood:

```
// Estimates the Hansen Sargent and Tallarini model by maximum likelihood.

var s c h k i d dhat dbar mus muc muh gamma R; varexo e_dhat e_dbar;

parameters lambda deltah deltak mud b bet phi1 phi2 cdbar alpha1
alpha2 cdhat;
bet=0.9971;
deltah=0.682;
lambda=2.443;
alpha1=0.813;
alpha2=0.189;
```

```
phi1=0.998;
phi2=0.704;
mud=13.710;
cdhat=0.155;
cdbar=0.108;
b=32;
deltak=0.975;

model(linear);
R=deltak+gamma;
R*bet=1;
s=(1+lambda)*c-lambda*h(-1);
h=deltah*h(-1)+(1-deltah)*c;
k=deltak*k(-1)+i;
c+i=gamma*k(-1)+d;
mus=b-s;
muc=(1+lambda)*mus+(1-deltah)*muh;
muh=bet*(deltah*muh(+1)-lambda*mus(+1));
muc=bet*R*muc(+1);
d=mud+dbar+dhat;
dbar=(phi1+phi2)*dbar(-1) - phi1*phi2*dbar(-2) +
cdbar*e_dbar;
dhat=(alpha1+alpha2)*dhat(-1) -
alpha1*alpha2*dhat(-2)+ cdhat*e_dhat;
end;

shocks;
var e_dhat;
stderr 1;
var e_dbar;
stderr 1;
end;

estimated_params;
bet, .91, .9, .99999;
deltah, 0.4, 0.1, 0.8;
lambda, 2, 0.1, 50;
alpha1, 0.8, 0.6, 0.99999;
alpha2, 0.2,0.01,0.5;
phi1, 0.8, 0.6, 0.99999;
phi2, 0.5, 0.3, 0.9;
mud, 10, 1,50;

cdhat, 0.1, 0.05, 0.2;
cdbar, 0.1, 0.05, 0.2;

end;

varobs c i;
estimation(datafile=dataHST,first_obs=1,nobs=500,mode_compute=4,mode_check);
```

To estimate the model using Bayesian techniques simply replace the last part of the code with
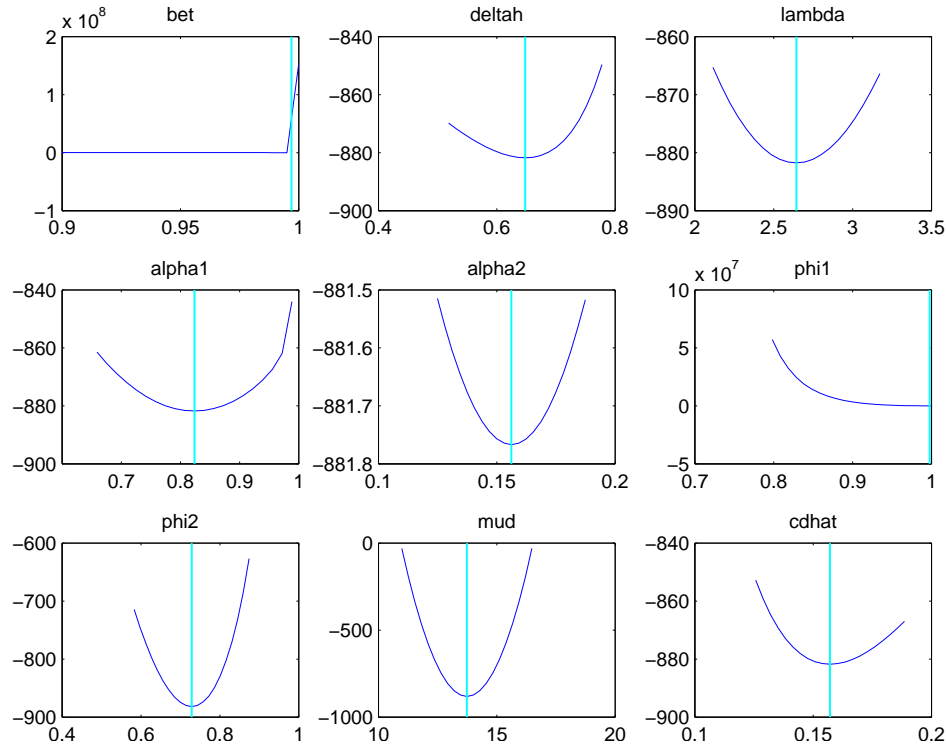
Figure 23: negative of the log-likelihood for the robust permanent income model

```
estimated_params;
bet,uniform_pdf, .9499999999, 0.0288675134306;
deltah,uniform_pdf, 0.45, 0.202072594216;
lambda,uniform_pdf,25.05,14.4048892163;

alpha1,uniform_pdf, 0.8, 0.115470053809;
alpha2,uniform_pdf, 0.25, 0.144337567297;
phi1,uniform_pdf,0.8,0.115470053809;

phi2,uniform_pdf, 0.5, 0.288675134595;
mud,uniform_pdf, 24.5, 14.1450815951;
cdhat,uniform_pdf,0.175,0.0721687836487;

cdbar,uniform_pdf, 0.175, 0.0721687836487;

end;

varobs c i;
estimation(datafile=dataHST,nobs=500,mode_check,mh_replic=10000,mh_nblocks=1,mh_jscale=0.2);
```

The sliced negative of the log-likelihood function is given by pictures (23) and (24). The maximization routine used here (mode compute=4) is Chris Sims' csminwel. Note that this procedure uses a random number generator, so different runs of this code can lead to different estimates (the routine might get stuck in a local extremum). Thus, it is important to run the code several times.
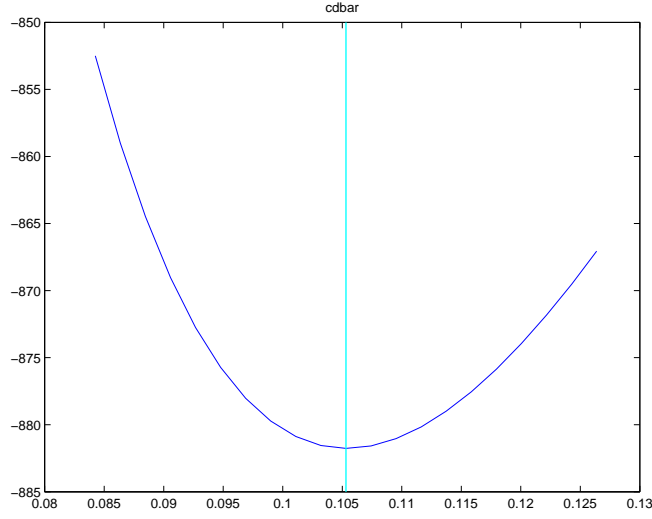
Figure 24: negative of the log-likelihood for the robust permanent income model

10. Asset pricing with risks for the long run

10.1 Introduction

Following Bansal and Yaron (2004), we specify dividend and consumption growth as containing a small but highly persistent predictable component. These dynamics, when combined with Epstein and Zin (1989) preferences, help explain financial phenomena including the equity premium puzzle. What allows this specification to be successful is the ability of Epstein and Zin preferences to separate risk aversion from intertemporal elasticity of substitution, possibly allowing both to be larger than one. Dynare enables us to compute policy functions for the response of price-dividend ratios to an increase in the long-run predictable component of dividend and consumption growth. Using the specification just described, this impact turns out to be positive. In economic terms, the intertemporal substitution effect dominates the wealth effect. In response to higher expected growth rates, agents buy more assets and this increases the wealth to consumption ratio. In the standard power utility (constant relative risk aversion) model, the wealth effect dominates the substitution effect. Dynare shows that the policy functions for the latter specification display the opposite sign for the feedback of the predictable component of growth on price-dividend ratios. Hence the former specification helps explain high equity premia, because when the IES is larger than one, agents fear that a reduction in economic growth prospects will lower asset prices.

10.2 Setup of the model

Specify preferences as follows:

$$U_t = \left[(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta \left(E_t \left[U_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}} \tag{38}$$

where

$$\theta = \frac{1-\gamma}{1-1/\Psi} \tag{39}$$

Here γ reflects the degree of risk aversion, while Ψ take into account intertemporal elasticity of substitution. When $\theta = 1$, preferences collapse to the standard CRRA case. Hence this model can be regarded as a general one.

There are N assets in this economy, whose one period ahead return is denoted as $R_{j,t+1}$, $\forall j = 1, ..., N$. The share of investment in asset j is denoted as $\omega_{j,t}$. Hence there are $N-1$ linearly independent elements in $\omega_t =$

$\begin{bmatrix} \omega_{1,t} & \omega_{2,t} & ... & \omega_{N,t} \end{bmatrix}'$, since portfolio shares must satisfy:

$$\sum_{j=1}^{N} \omega_{j,t} = 1, \forall t \tag{40}$$

The representative agent is endowed with an initial stock of the consumption asset A_0 that can be either consumed or invested in the N assets in the competitive market. Therefore, her wealth evolves according to:

$$A_{t+1} = (A_t - C_t)\omega_t' R_t \tag{41}$$

Following Epstein and Zin (1991), first order conditions to price an asset that entitles to a claim on a stream of consumption good imply:

$$v_{c,t} = E_t \left[\delta^\theta e^{\left[-\frac{\theta}{\Psi}\Delta c_{t+1} + (\theta-1)r_{c,t+1}\right]}(1 + v_{c,t+1})e^{\Delta c_{t+1}} \right] \tag{42}$$

where

$$
\begin{aligned}
\Delta c_t &= \log(C_t) - \log(C_{t-1}) \\
r_{c,t+1} &= \log R_{c,t+1} \\
R_{c,t+1} &= \frac{(1 + v_{c,t+1})e^{\Delta c_{t+1}}}{v_{c,t}} \\
v_{c,t} &= \frac{P_{c,t}}{C_t}
\end{aligned}
$$

Similarly, first order condition to price an asset that entitles to a dividend D_t in each period imply:

$$v_{d,t} = E_t \left[\delta^\theta e^{\left[-\frac{\theta}{\Psi}\Delta c_{t+1} + (\theta-1)r_{c,t+1}\right]}(1 + v_{d,t+1})e^{\Delta d_{t+1}} \right] \tag{43}$$

where

$$
\begin{aligned}
\Delta d_t &= \log(D_t - D_{t-1}) \\
v_{d,t} &= \frac{P_{d,t}}{D_t}
\end{aligned}
$$

We close the model, by specifying exogenous processes for consumption and dividend growth:

$$
\begin{aligned}
\Delta c_t &= \mu_c + x_{t-1} + \varepsilon_{c,t} & (44) \\
\Delta d_t &= \mu_d + \lambda_{dx}x_{t-1} + \varepsilon_{d,t} & (45) \\
x_t &= \rho_x x_{t-1} + \varepsilon_{x,t} & (46)
\end{aligned}
$$

where the shocks $\varepsilon_{c,t}$, $\varepsilon_{d,t}$ and $\varepsilon_{s,t}$ are i.i.d. normal with mean zero and standard deviations σ_c, σ_d and σ_x, respectively. For stationarity $|\rho_x| < 1$. The economy is completely described by equations (42)-(46).

10.3   Calibration and simulation

We calibrate the parameters of the model in two different ways. Our guidelines in doing this are:

- use a reasonable risk aversion

- model the process of x as highly persistent

One important feature that, following the lines of Bansal and Yaron (2004), we would like to display is the different way in which x_t affects price-dividend ratios in the presence of Epstein and Zin utility functions. As we stressed before the channel through which a highly persistent and predictable component of consumption and dividend growth combined with Epstein-Zin preferences helps explaining the equity premium puzzle is the possibility of having intertemporal elasticity of substitution and risk aversion greater than one at the same time. Hence we want to experiment both a calibration where this condition is met and a standard constant relative risk aversion preferences environment, where risk aversion greater than one implies IES smaller than unity. The latter specification is a particular case in our model, as it only requires to put $\theta = 1$.

10.3.1 CRRA specification

We calibrate parameters as in Table 8. The numbers are the same as in Bansal and Yaron (2004) except for risk IES, that is now the inverse of risk aversion.

Parameter	Calibration
Ψ	0.133
θ	1
γ	7.5
δ	0.99
μ_c	0.0015
σ_c	0.0078
μ_d	0.0015
λ_{dx}	3
σ_d	0.0351
ρ_x	0.979
σ_x	0.0003432

Table 8: Parameters' calibration. The value of θ is calibrated to satisfy equation (39).

The Dynare code reported below implements a first order approximation of the model.

```
periods 500;
var  dc, dd, v_c, v_d, x; varexo e_c, e_x, e_d;

parameters DELTA THETA PSI MU_C MU_D RHO_X LAMBDA_DX;

DELTA=.99;
PSI=1.5;
THETA=(1-7.5)/(1-1/PSI);
MU_C=0.0015;
MU_D=0.0015;
RHO_X=.979;
LAMBDA_DX=3;

model;
v_c = DELTA^THETA *exp((-THETA/PSI)*dc(+1)+(THETA-1)*log((1+v_c(+1))*exp(dc(+1))/v_c))* (1+v_c(+1))*exp(dc(+1)); %
v_d = DELTA^THETA * exp((-THETA/PSI)*dc(+1) + (THETA-1)*log((1+v_c(+1))*exp(dc(+1))/v_c) ) * (1+v_d(+1))*exp(dd(+1)); %
dc  = MU_C  + x(-1) + e_c;
dd  = MU_D + LAMBDA_DX*x(-1) + e_d; %
x   = RHO_X * x(-1) + e_x;
end;

initval;
v_c=15;
v_d=15;
dc=MU_C;
dd=MU_D;
x=0;
e_c=0;
e_x=0;
e_d=0;
end;

shocks;
var e_c;
stderr .0078;
var e_x;
stderr .0078*.044;%
var e_d;
stderr .0078*4.5;
end;

steady(solve_algo=0); check;

stoch_simul(dr_algo=1, order=1, periods=1000, irf=30);
datasaver('simudata',[]);
```

Dynare computes policy and transition functions as reported in Table 9. Since these equations are expressed in

terms of x_{t-1}, it is possible to express the system in terms of time t state variable using equation (46):

$$
\begin{aligned}
v_{c,t} &= 50.005 - 8086.753(x_t - \overline{x}) \\
v_{d,t} &= 50.005 - 5598.521(x_t - \overline{x})
\end{aligned}
$$

Equations (44)-(46) complete the description of the solution.

	x	Δc	Δd	v_c	v_d
Constant	0	0.0015	0.0015	50.005844	50.005844
x (-1)	0.979	1	3	-7916.931499	-5480.952577
e_c	0	1	0	0	0
e_d	0	0	1	0	0
e_x	1	0	0	-8086.753319	-5598.521529

Table 9: Policy and transition functions, CRRA calibration (Matlab output).

It is also interesting to look at the impulse response functions to a shock to x_t. Figure 25 reports the response of the variables of the system to a one standard deviation shock to x. According to the rationale discussed above, both price-dividend and price-consumption ratios fall.
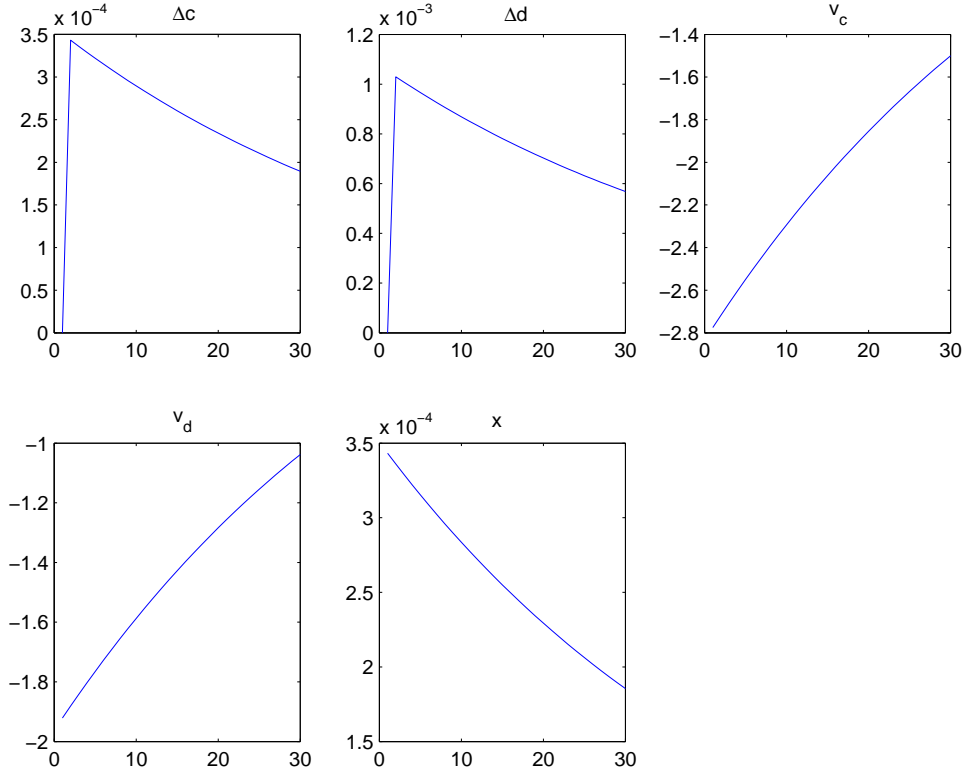

Figure 25: Impulse response function to a shock to x_t (CRRA preferences).

10.3.2 Epstein-Zin specification

This section focuses on the more general Epstein-Zin setup. In particular, we allow both risk aversion and IES to be larger than one. Ψ and, via equation (39), θ are the only two parameters to differ between Table 8 and Table 10.

Parameter	Calibration
Ψ	1.5
θ	-19.5
γ	7.5
δ	0.99
μ_c	0.0015
σ_c	0.0078
μ_d	0.0015
λ_{dx}	3
σ_d	0.0351
ρ_x	0.979
σ_x	0.0003432

Table 10: Parameters' calibration. The value of θ is calibrated to satisfy equation (39).

By running the Dynare program with the change in the calibration of Ψ and θ, the policy functions for v_c and v_d change to:

$$v_{c,t} = 104.209 + 1146.215(x_t - \overline{x})$$
$$v_{d,t} = 104.209 + 8023.510(x_t - \overline{x})$$

Therefore, a shock to x_t will now have a positive effect on price-dividend ratios. Separating risk aversion from intertemporal substitution and modelling both as greater than unity produces the effect of changing the sign of the impact of the forecastable part of consumption and dividend growth on price-dividend ratios. The IRF reported in Figure 26 stress this finding.
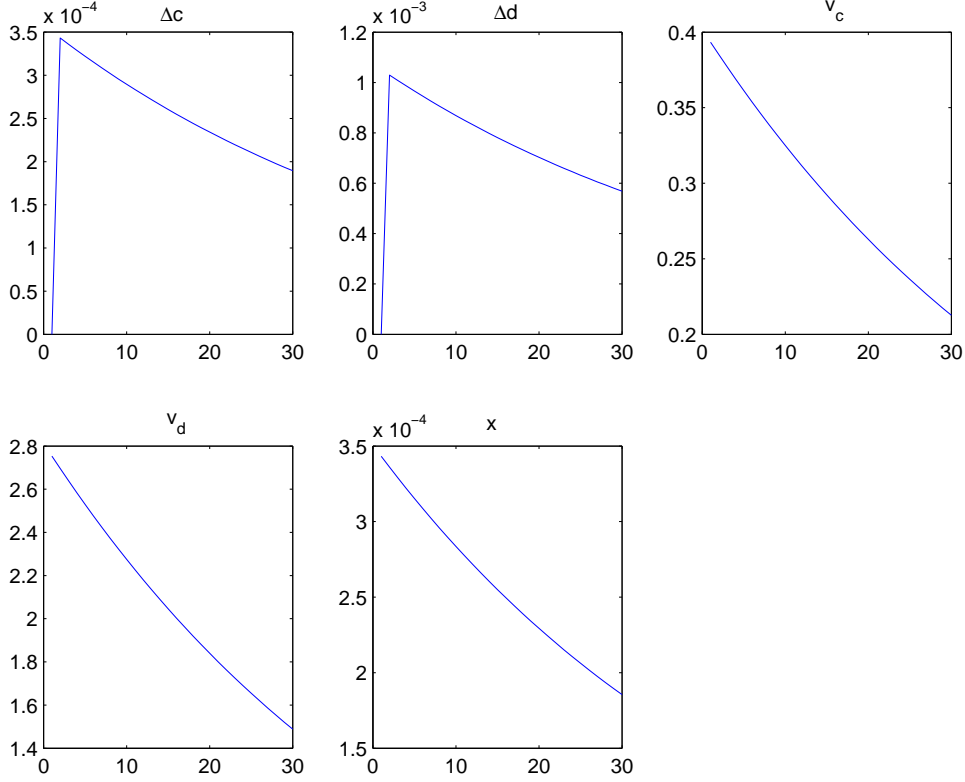

Figure 26: Impulse response function to a shock to x_t (CRRA preferences).

10.4 Bayesian estimation

We perform a Bayesian estimation of the general model, using simulated series. The model has three shocks ε_c, ε_x and ε_d and three observables v_d, Δc and Δd. The priors that we use are reported in Table 11. A slight modification to the program reported in the previous pages enables us to estimate the parameters of the model in Dynare. The following code was been used:

```
var  dc, dd, v_c, v_d, x;
varexo e_c, e_x, e_d;

parameters DELTA THETA PSI MU_C MU_D RHO_X LAMBDA_DX;

DELTA=.99;
PSI=1.5;
THETA=(1-7.5)/(1-1/PSI);
MU_C=0.0015;
MU_D=0.0015;
RHO_X=.979;
LAMBDA_DX=3;

model;

v_c = DELTA^THETA * exp((-THETA/PSI)*dc(+1) + (THETA-1)*log((1+v_c(+1))*exp(dc(+1))/v_c) ) * (1+v_c(+1))*exp(dc(+1)); %
v_d = DELTA^THETA * exp((-THETA/PSI)*dc(+1) + (THETA-1)*log((1+v_c(+1))*exp(dc(+1))/v_c) ) * (1+v_d(+1))*exp(dd(+1)); %
dc  = MU_C  + x(-1) + e_c; %
dd  = MU_D + LAMBDA_DX*x(-1) + e_d; %
x = RHO_X * x(-1) + e_x;
end;

initval;
v_c=15;
v_d=15;
dc=MU_C;
dd=MU_D;
x=0;
e_c=0;
e_x=0;
e_d=0;
end;

shocks;
var e_d;
stderr .001;
var e_c;
stderr .001;
var e_x;
stderr .001;
end;

steady;

estimated_params;
DELTA, beta_pdf, 0.99,.001;
THETA,normal_pdf,-7.5, 0.005;
PSI,normal_pdf,1.5, 0.005;
MU_C,normal_pdf,0.001, 0.005;
MU_D,normal_pdf,0.001, 0.005;
RHO_X,normal_pdf,.98, 0.05;
LAMBDA_DX,normal_pdf,3, 0.05;%
stderr e_d,inv_gamma_pdf,.01, inf; %
stderr e_x,inv_gamma_pdf,.001, inf;
stderr e_c,inv_gamma_pdf,.0001, inf;
end;

varobs v_d dd dc;

estimation(datafile=simudata,mh_replic=5000);
```

In terms of Dynare's output Figure 27 displays priors' distributions, while Figure 28 reports a comparison between prior's and posterior's distributions. Since the data generating process is exactly the model that we are estimating, it is not very surprising that the distribution of the posterior has very little variance around the true parameters.

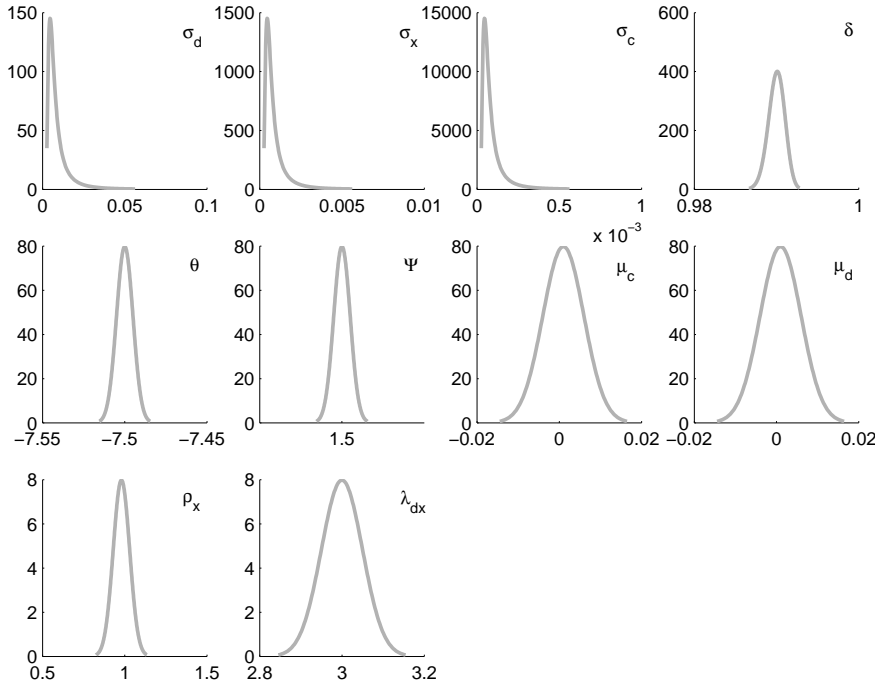| Parameter	Prior Distribution	Mean	Std. Dev.
Ψ	Normal	1.5	.005
θ	Normal	-19.5	.005
δ	Beta	0.99	.001
μ_c	Normal	0.0015	.005
σ_c	Inv. Gamma	0.0078	inf
μ_d	Normal	0.0015	.005
λ_{dx}	Normal	3	.05
σ_d	Inv. Gamma	0.0351	inf
ρ_x	Normal	0.979	.05
σ_x	Inv. Gamma	0.0003432	inf

Table 11: Priors.

Figure 27: Priors.

Dynare also produces a table of summary statistics of the estimation. As already seen from the Figures, the mean of the posterior distributions is almost coincidental with the correct values and the estimated confidence interval is very tight around these values. Table 12 reports these numbers.

|	Prior Mean	Post. Mean	Confidence	Interval	Prior dbn	Post std dev
δ	0.99	0.9896	0.988	0.991	Beta	0.001
θ	-7.5	-7.501	-7.5102	- 7.4934	Normal	0.005
Ψ	1.5	1.4981	1.4898	1.5049	Normal	0.005
μ_c	0.001	0.0012	0.0001	0.0021	Normal	0.005
μ_d	0.001	0.0017	-0.0004	0.0044	Normal	0.005
ρ_x	0.98	0.9786	0.9753	0.9813	Normal	0.05
λ_{dx}	3	2.9963	2.9205	3.0812	Normal	0.05
σ_d	0.01	0.0368	0.0355	0.038	Inv. Gamma	Inf
σ_x	0.001	0.0003	0.0003	0.0003	Inv. Gamma	Inf
σ_c	0	0.0076	0.0074	0.0079	Inv. Gamma	Inf

Table 12: Posterior.
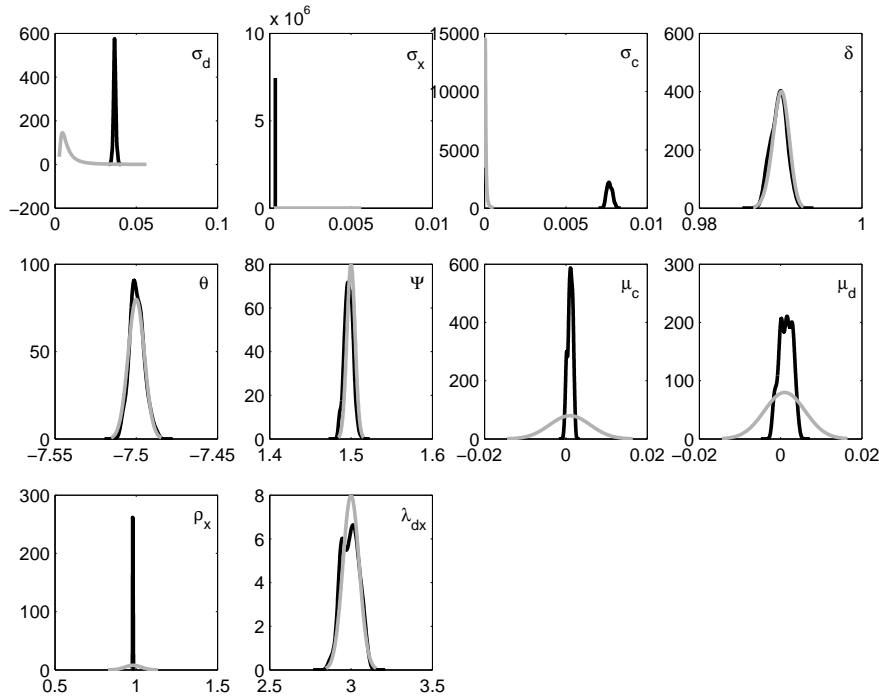
Figure 28: Posterior.

A. List of Dynare Files

This appendix lists the dynare files used in this paper. All files and the data used are contained in the file examples.zip. In case simulated data was used in the estimation one first needs to run the relevant dynare code to simulate the data.

- GrowthApproximate.mod - solves the neoclassical growth model and simulates data

- GrowthEstimate.mod - estimates the neoclassical growth model

- TwocountryApprox.mod - solves the two country production economy and simulates data

- TwocountryEstim.mod - estimates the two country production economy

- Fig11XX.mod - solves a deterministic growth model and reproduces the graph XX in chapter 11 of Ljungqvist and Sargent (2004)

- sargent77.mod - solves the model in Sargent (1977)

- sargent77ML.mod, sargent77Bayes.mod - estimate the model in Sargent (1977) using maximum likelihood and Bayesian MCMC methods, respectively

- hall1.mod - solves the model in Hall (1988) and simulates data

- hall1estimateML.mod, hall1estimateBayes.mod - estimate the model in Hall (1988) using maximum likelihood and Bayesian MCMC methods, respectively

- rosen.mod - solves the model in Ryoo and Rosen (2004) and simulates data

- rosenestimateML.mod, rosenestimateBayes.mod - estimate the model in Ryoo and Rosen (2004) using maximum likelihood and Bayesian MCMC methods, respectively

- BansalYaronML.mod, BansalYaronBayes.mod - estimate the consumption-growth process of Bansal and Yaron (2004) using maximum likelihood and Bayesian MCMC methods, respectively

- HSTML.mod, HSTBayes.mod - estimate the model in Hansen, Sargent, and Tallarini (1999) using maximum likelihood and Bayesian MCMC methods, respectively

- AssetPricingApproximation.mod - solves the model of Bansal and Yaron (2004) and simulates data

- AssetPricingEstimate.mod - estimates the model of Bansal and Yaron (2004)

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