Macroeconomics 1

Lecture - Endogenous Growth

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Introduction

Explaining the Great Divergence

- In the Solow and in the Overlapping Generations Model the **rate of growth is exogenous** and does not depend on any decision by firms or the productive sector.
- It was determined by rates of growth of the **population** and **exogenous technical progress**.

New assumptions

1. The rate of growth is sensitive to the rate of **factor accumulation**.
2. **Technical progress is an economic activity** that results from a **rational rational decision by households and firms**.

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- This is important to explain the great divergence without putting the weight on exogenous variables and, thus, unexplained factors.
- The study of endogenous growth models in the way we understand them today began with the work of Romer (1986, 1987, 1990) using **increasing returns** and allowing the model to embed **research**. Lucas (1988) also made contributions using a model with **two accumulated factors** and global constant returns

The AK Model

Introduction

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- This mode is the simplest one, where the production function is **linear** in the capital, which is the only factor of production:

$$Y_t = AK_t.$$

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- This function is **homogeneous of degree one**, a feature that appeared already in other models.

The discrete-time version AK Model

- The **capital evolves** according to:

$$K_{t+1} = (1-\delta)K_t + I_t.$$

- The **budget constraint** is:

$$K_{t+1} + C_t = (1-\delta)K_t + Y_t = (1-\delta)K_t + AK_t.$$

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- The problem of the households is given by:

$$\sum_{t=0}^{\infty} \beta^t \left\{ \log C_t + \lambda_t \left[(1-\delta+A)K_t - C_t - K_{t+1} \right] \right\}$$

- The solution to this problem is ...

The rate of growth of consumption

- The **rate of growth of consumption** will be given by

$$\frac{C_{t+1}}{C_t} = \beta(1 - \delta + A).$$

- It will depend on **preferences** (β), as well as on **technology** (δ and A).

The intertemporal equilibrium

- The economy will settle on a **balanced growth path**, which will result in:

$$\frac{K_{t+1}}{K_t} = \frac{C_{t+1}}{C_t} = \frac{Y_{t+1}}{Y_t}.$$

- This will lead to:

$$\frac{K_{t+1}}{K_t} = 1 - \delta + \frac{I_t}{K_t} = \frac{C_{t+1}}{C_t} = \beta(1 - \delta + A).$$

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- In the end:

$$\frac{I_t}{Y_t} = \frac{\beta A - (1-\beta)(1-\delta)}{A}.$$

Technical Progress and Endogenous Growth

How does the rate of technical progress influence growth?

- If the technology has **constant returns to scale** in the accumulated factors we have **endogenous growth**.
- But the **total returns** to traditional accumulated factors (physical and human capital) seem to be **less than one in reality**.

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- More weight on the role of technical progress and the models now **endogenize the evolution of A_t**.

Rivalry and Excludability

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Returns to scale and Market structure

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- The production function is **not concave** anymore, since for $\lambda \leq 1$:

$$F\left(\lambda A_t, \lambda Z_t\right) \leq F\left(A_t, \lambda Z_t\right) = \lambda F\left(A_t, Z_t\right).$$

- ...

Returns to scale and Market structure

- Notice first that:
$$F\left(A_t, Z_t\right) = Z_t F_Z\left(A_t, Z_t\right).$$
- Therefore, if the nonrival is productive, i.e., $F_A > 0$ we have:
$$A_t F_A\left(A_t, Z_t\right) + Z_t F_Z\left(A_t, Z_t\right) > F\left(A_t, Z_t\right)$$

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- If all **factors were paid for their marginal productivity**, firms would make **losses** and shut down.
- That is why models of **imperfect competition** are the natural structure in these types of models.
- What is the meaning of this regarding the choice of the technology level and the market structure for firms responsible for technology investments?

The Romer Model

The model

- Simple version without capital since capital does not play a role.
- Time is discrete and there will be **three sectors of production**.
- Output is produced with the help of intermediate goods i.
- Each intermediate good i is produced by a **monopolistically competitive firm** i:

$$y_{it} = l_{it}.$$

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$$y_{it} = l_{it}.$$

- The **range of intermediate goods** existing at time t is $[0, N_t]$.
- Those **goods are combined by a competitive firm** to produce output Y_t according to:

$$Y_t = N_t^{1+\nu} \left(\frac{1}{N_t} \int_0^{N_t} y_{it}^{\eta} di \right)^{1/\eta} \quad 0 \leq \eta < 1 \quad \nu \geq 0.$$

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- η is the **elasticity of substitution among the goods**. The more substitutable the goods are, the closer η will be to one, and the more elastic the demand curves are, and so more **competitive is the market**.
- v is the **return to diversification**.

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- The **total labor employed** in the production of intermediate goods:

$$L_t = \int_0^{N_t} \ell_{it} di.$$

The Model

- Consider the **symmetrical situation** where:

$$\ell_{it} = \ell_t = \frac{L_t}{N_t} \quad y_{it} = y_t = \frac{L_t}{N_t}$$

- ...

The Model

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- In the end we have;

$$Y_t = N_t^{\nu} L_t,$$

  where the parameter v measures the return to diversification.

How do the technology and research work in this environment?

- The **number of goods** can be increased by **undertaking research**.
- The reason agents want the good i is because each good i is associated with a **patent**, which gives the owner the **exclusive right to produce and sell it**.
- In the end we have:

$$\frac{N_{t+1} - N_t}{N_t} = aH_t. \tag{1}$$

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- This show how much the number N_t of intermediate goods expands as a function of the quantity of labor H_t devoted to research.

The Model

- The problem of the household will be maximizing the discounted utility:

$$U = \sum_{t=0}^{\infty} \beta^t \log C_t.$$

  subject to:

- $$C_t + v_t \left(N_{t+1} - N_t \right) = \omega_t \Lambda + \pi_t N_t,$$

  where v_t **is the price of a patent**, ω_t **is the real wage**, and π_t **is the flow of real profit that an existing patent yields**.

The Market Equilibrium

- Let p_{it} be the relative price of intermediate good i. The **firms producing final output maximize**:

$$Y_t - \int_0^{N_t} p_{it} y_{it} di = N_t^{1+\nu} \left(\frac{1}{N_t} \int_0^{N_t} y_{it}^{\eta} di \right)^{1/\eta} - \int_0^{N_t} p_{it} y_{it} di.$$

- Now take F.O.C. with respect to y_{it} ...

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- Now take F.O.C. with respect to y_{it} ...

- In the end we have the following **demand curve for a firm** i:

$$y_{it} = \frac{Y_t}{N_t^{1+\nu}} \left(\frac{p_{it}}{N_t^{\nu}} \right)^{-1/(1-\eta)} .$$

The Market Equilibrium

The demand curve for a firm i is given by:

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   $$\varepsilon_d = -\frac{1}{1-\eta}.$$

2. What is the **profit** of the firm i?

   $$\pi = p_{it} y_{it} - \omega_t \ell_{it},$$

   subject to the production function $y_{it} = \ell_{it}$ and the demand curve $y_{it} = \frac{Y_t}{N_t^{1+\nu}} \left(\frac{p_{it}}{N_t^{\nu}} \right)^{-1/(1-\eta)}$.

The Market Equilibrium

3. What will be the **price** the firm will charge for the product?

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$$p_{it} = p_t = \frac{\omega_t}{\eta}. \tag{2}$$

Notice the price will be a **fraction of the monopolistically markup**.

The Market Equilibrium

- Because the final good sector is competitive, the price p_{it} of an intermediate good is equal to its marginal productivity $\dfrac{\partial Y_t}{\partial x_{it}}$:

$$p_{it} = N_t^{\nu} = p_t. \tag{3}$$

- The **research sector** is competitive also, so:

$$v_t a N_t = \omega_t,$$

$$v_t = \frac{\omega_t}{a N_t} \quad \text{if} \quad N_{t+1} > N_t \tag{4}$$

The Market Equilibrium

- Observe that $\ell_t = \dfrac{L_t}{N_t}$, then

$$\pi_t = p_t y_t - \omega_t \ell_t.$$

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- ...
- In the end we have:

$$\pi_t = \frac{(1-\eta)p_t L_t}{N_t}. \tag{5}$$

The Market Equilibrium

- We also have the following conditions:

$$C_t = Y_t = N_t^{\nu} L_t \tag{6}$$

- And also:

$$L_t + H_t = \Lambda. \tag{7}$$

The Market Equilibrium

- The problem of the household is:

$$\sum_{t=0}^{\infty} \beta^t \left\{ \log C_t + \lambda_t \left[\omega_t \Lambda + \pi_t N_t - C_t - v_t \left(N_{t+1} - N_t \right) \right] \right\}$$

- The **F.O.C.s with respect to C_t and N_t** are ...

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- The **F.O.C.s with respect to C_t and N_t** are ...

$$\lambda_t = \frac{1}{C_t}$$
$$\lambda_t v_t = \beta \lambda_{t+1} (v_{t+1} + \pi_{t+1})$$

The Market Equilibrium

The equations that fully determine the **equilibrium** are: (1), (2), (3), (4), (5), (6), and (7):

$$\frac{N_{t+1} - N_t}{N_t} = aH_t,$$

$$p_{it} = p_t = \frac{\omega_t}{\eta},$$

$$p_{it} = N_t^{\nu} = p_t,$$

$$v_t = \frac{\omega_t}{aN_t},$$

$$\pi_t = \frac{(1-\eta)p_t L_t}{N_t},$$

$$C_t = Y_t = N_t^{\nu} L_t,$$

$$L_t + H_t = \Lambda.$$

The Market Equilibrium

- The **dynamic equation for the evolution of L_t** is:

$$\frac{1}{L_t} = \frac{\beta}{1+a\Lambda}\frac{1}{L_{t+1}} + \frac{\beta a(1-\eta)+\eta a}{\eta(1+a\Lambda)}.$$

- Because the coefficient of $\frac{1}{L_{t+1}}$ is smaller than one we have a **steady state value for L**:

$$L_t = L = \frac{\eta[(1-\beta)+a\Lambda]}{\beta a(1-\eta)+\eta a}.$$

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- The above results in a **constant rate of growth γ_N of the number N_t of goods and patents**:

$$\gamma_N = \frac{N_{t+1}-N_t}{N_t} = a(\Lambda - L) = \frac{\beta a(1-\eta)\Lambda - \eta(1-\beta)}{\beta(1-\eta)+\eta}.$$

The Market Equilibrium

$$\gamma_N = \frac{N_{t+1} - N_t}{N_t} = a(\Lambda - L) = \frac{\beta a(1-\eta)\Lambda - \eta(1-\beta)}{\beta(1-\eta)+\eta}.$$

- Notice the rate of growth of patents is an **increasing function of the monopolistic markup $1/\eta$**. Why?
- We also can see that the **rate of growth depends positively on the size of the working population** Δ. This is called the **scale effect**. In other words, a large economy should grow faster than a smaller one. What do you think about this effect?

The Scale Effect

- Many models predict that the growth rate is increasing in the population and this is tied to the non-rivalry of ideas.
  1. **Demand for innovation** - more people can use any single innovation (what is at play in these models).
  2. **Supply of innovations** - more people have more ideas, and you only need one idea for one innovation
- This was a huge debate during the 90s.

Problematic implications

1. **The scale debate**:
   - **Countries with larger populations do not necessarily grow faster**:
   - But Kremer (1993) shows:
     - World population is positively correlated with world growth over long stretches of time.
     - Among technologically separate societies, those with higher initial populations grew faster.

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     - World population is positively correlated with world growth over long stretches of time.
     - Among technologically separate societies, those with higher initial populations grew faster.
2. **Jone's solution (1995)**:
   - $n > 0$ – population growth gives more incentives to innovate.
   - $\eta = \phi N^{\sigma}$ with $\sigma > 0$ – makes it increasingly costly to innovate.

The Social Optimum

The Social Optimum

- Consider a symmetric situation.
- Then the choice is to choose the quantity devoted to the **production of intermediate goods** and the remaining will be devoted to **research**.
- The program will be:

$$\text{Maximize} \sum_{t=0}^{\infty} \beta^t \log\left(N_t^{\nu} L_t\right) \quad \text{s.t.}$$
$$\log N_{t+1} - \log N_t = \log\left[1 + a\left(\Lambda - L_t\right)\right]$$

The Social Optimum

- The Lagrangian is concave in L_t and in $\log N_t$:

$$\sum_{t=0}^{\infty} \beta^t \left\{ \nu \log N_t + \log L_t + \lambda_t \left[\log \left(1 + a\Lambda - aL_t \right) + \log N_t - \log N_{t+1} \right] \right\}.$$

- The F.O.C.s are ...

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- The F.O.C.s are ...

$$\frac{1}{L_t} = \frac{a\lambda_t}{1 + a\Lambda - aL_t}$$
$$\lambda_{t-1} = \beta\left(\lambda_t + \nu\right)$$

The Social Optimum

- Notice the coefficient of λ_t is smaller than 1, then we can set λ_t to its value in the steady state:

$$\lambda_t = \frac{\beta\nu}{1-\beta}$$

- Then we find that:

$$L_t = L = \frac{(1-\beta)(1+a\Lambda)}{a(1-\beta+\beta\nu)}.$$

- And the **optimal rate of growth** of patents γ_N^* will be:

$$\gamma_N^* = \frac{\beta a\nu\Lambda-(1-\beta)}{1-\beta+\beta\nu}.$$

Comparing Market Outcome and Social Optimum

- The market equilibrium rate of growth γ_N is:

$$\gamma_N = \frac{N_{t+1} - N_t}{N_t} = a(\Lambda - L) = \frac{\beta a(1-\eta)\Lambda - \eta(1-\beta)}{\beta(1-\eta)+\eta}.$$

- Whereas the optimal one is γ_N^*:

$$\gamma_N^* = \frac{\beta a\nu\Lambda - (1-\beta)}{1-\beta+\beta\nu}.$$

- In principle **there is no a priori ranking between the two**, and the market growth rate can be too high or too low.
- In the early literature is found that the **amount of research was always found to be too low**.

Comparing Market Outcome and Social Optimum

- Instead of using the formula:

$$Y_t = N_t^{1+\nu} \left(\frac{1}{N_t} \int_0^{N_t} y_{it}^{\eta} di \right)^{1/\eta} \quad 0 \leq \eta < 1 \quad \nu \geq 0$$

- Use the formula:

$$Y_t = \left(\int_0^{N_t} y_{it}^{\eta} di \right)^{1/\eta} \quad 0 \leq \eta < 1$$

- This is the same as choosing $\nu = 1/\eta - 1$.
- Insert this value in the γ_N^* and by comparing the results we would have"

Comparing Market Outcome and Social Optimum

$$\gamma_N = \frac{\beta a(1-\eta)\Lambda - \eta(1-\beta)}{\beta(1-\eta)+\eta}$$
$$\gamma_N^* = \frac{\beta a(1-\eta)\Lambda - \eta(1-\beta)}{\beta(1-\eta)+\eta(1-\beta)}$$

- Under this specification we can see that the **market growth of patents γ_N is systematically lower than the socially optimum value γ_N^***.
- But this is valid only for a particular choice value for the returns to specialization.

Endogenous Productivity Increases

The Model

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- Households have a total quantity of labor Δ and an intertemporal utility:

$$U = \sum_{t=0}^{\infty} \beta^t \log C_t$$

- Consumption goods are produced with the function:

$$Y_t = N \left(\frac{1}{N} \int_0^N y_{it}^{\eta} di \right)^{1/\eta}.$$

- Intermediate goods are produced with:

$$y_{it} = q_{it} \ell_{it},$$

where **q_{it} is the productivity of sector i**.

The Model

- The Productivity may differ across firms during a period, but at the end, all firms have costlessly access to the best technology, which will be $\bar{q}_t$:

$$\bar{q}_t = \max_i q_{it}.$$

- Firms can **raise the productivity** from $\bar{q}_{t-1}$ to q_{it} at a **labor cost** h_{it}:

$$h_{it} = \Psi\left(\frac{q_{it}}{\bar{q}_{t-1}}\right) \quad \Psi' > 0 \quad \Psi'' > 0$$

The Short-run Equilibrium

- The firms producing the final good maximize profits:

$$Y_t - \int_0^N p_{it} y_{it} di = N \left(\frac{1}{N} \int_0^N y_{it}^{\eta} di \right)^{1/\eta} - \int_0^N p_{it} y_{it} di.$$

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- ...
- Maximizing in y_{it} yields the **demand for the intermediate i**:

$$y_{it} = \frac{Y_t}{N} p_{it}^{-1/(1-\eta)}.$$

The Short-run Equilibrium

- The firms will maximize:

$$p_{it}y_{it} - \omega_t \ell_{it} - \omega_t h_{it}.$$

subject to: $y_{it} = q_{it}\ell_{it}$, $h_{it} = \Psi\left(\frac{q_{it}}{\bar{q}_{t-1}}\right)$, and $y_{it} = \frac{Y_t}{N} p_{it}^{-1/(1-\eta)}$.

- **Maximizing with respect to p_{it}** yields:

$$p_{it} = \frac{\omega_t}{\eta q_{it}}.$$

- In the end the **profit** will become:

$$(1-\eta)\frac{Y_t}{N}\left(\frac{\eta q_{it}}{\omega_t}\right)^{\eta/(1-\eta)} - \omega_t \Psi\left(\frac{q_{it}}{\bar{q}_{t-1}}\right).$$

The Short-run Equilibrium

- Taking F.O.C. with respect to q_{it} will lead to:

$$\eta \frac{Y_t}{N} \left(\frac{\omega_t}{\eta} \right)^{-\eta/(1-\eta)} q_{it}^{\eta/(1-\eta)-1} = \frac{\omega_t}{\bar{q}_{t-1}} \Psi' \left(\frac{q_{it}}{\bar{q}_{t-1}} \right)$$

- All sectors will have the same level of productivity, so that:

$$q_{it} = q_t = \bar{q}_t.$$

- The real wage will be:

$$\omega_t = \eta q_t.$$

The Short-run Equilibrium

- The equation will simplify to:

$$\frac{Y_t}{q_t N_t} = \frac{q_t}{q_{t-1}} \Psi' \left(\frac{q_t}{q_{t-1}} \right).$$

- Denote the gross rate of growth of productivity as:

$$\mathcal{G}_t = \frac{q_t}{q_{t-1}}.$$

- Then:

$$\frac{Y_t}{q_t N} = \mathcal{G}_t \Psi' \left(\mathcal{G}_t \right).$$

The Short-run Equilibrium

- The condition for the **equilibrium in labor market** is given by:

$$\Psi\left(\mathcal{G}_t\right) + \frac{Y_t}{q_t N} = \frac{\Lambda}{N}.$$

- Notice we have:

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- Taking derivative we have:

$$\left[2\Psi'\left(\mathcal{G}_t\right) + \mathcal{G}_t \Psi''\left(\mathcal{G}_t\right)\right] d\mathcal{G}_t = d\left(\frac{\Lambda}{N}\right).$$

- So that:

$$\frac{\partial \mathcal{G}_t}{\partial(\Lambda/N)} > 0$$

A Model without the Scale Effect

A Model without Scale Effects

- In the two models we have see the rate of growth is an increasing function of the size of the population Λ.
- Consider instead of having a fixed number N that there will be **free entry until the point the profits go to zero**.
- This will lead to:

$$(1-\eta)\frac{Y_t}{N_t}\left(\frac{\eta q_{it}}{\omega_t}\right)^{\eta/(1-\eta)} = \omega_t \Psi\left(\frac{q_{it}}{\bar{q}_{t-1}}\right)$$

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- Since $\omega_t = \eta q_t$ we can find:

$$\frac{1-\eta}{\eta}\frac{Y_t}{q_t N_t} = \Psi\left(\frac{q_t}{q_{t-1}}\right).$$

- And the condition for the equilibrium rate of growth is:

$$(1-\eta)\mathcal{G}_t \Psi'(\mathcal{G}_t) = \eta \Psi(\mathcal{G}_t).$$

- Notice the rate of growth no longer depends on the size of the economy. it depends, however, on the shape of the technical progress Ψ and on η.

Conclusion

Some conclusions

- Key element 1: **non-rivalry of ideas**.
- Key element 2: **externalities or monopolistic competition**.
- In models of purposeful innovations, the pace of growth is determined by **incentives market structure, competition policy, taxes, patents, and property rights**.
- **Equilibrium is typically not Pareto optimal** due to externality and monopolistic competition in practice: barriers to R&D may be more important than monopoly distortions.