Search and Matching

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Abstract

This lecture note introduces some key concepts regarding search and matching theory used in the Master's course in Economics at Sciences Po. First, we introduce the baseline model where we study the firm side, and different from the McCall Model, we endogenize the wage w, which will be obtained through the bargaining process between firms and workers. The matching process will allow us also to have the probability of receiving offers in an endogenous way. After, we allow the matching and the resulting wage to depend on the productivity of the worker. First, we consider the case where the wage does not change after the worker is matched with a firm. Lastly, we consider the case where the productivity of the worker can change once matched with a firm, which means the wage can increase or decrease. To finish this topic we describe an economy where we allow for direct search and posting.

Keywords: Search and Matching

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1 Baseline Model

- Different from the McCall model, the idea now is to look now for the firm side.
- The variables w and α will be endogenized by the market equilibrium, where w is the wage and α is the probability of receiving a a job offer.
- The variable λ denotes the probability of the matching breaks, i.e., it represents the exogenous rate of the worker being fired or ask for resignation.

1. **Flow payoff of the employed worker:**

$$rW(w) = w + \lambda[U - W(w)]. \tag{1}$$

2. **Flow payoff of firms per worker :**

$$rJ(w) = y - w + \lambda[V - J(w)], \tag{2}$$

where we assume $y - w$ is what is left for the firms in real terms per worker (y is the exogenous output per worker assumed to be the same for all firms). The variable V denotes the value of the *vacancy*, i.e., it represents the value for a firm to have an available spot and be able to attract new employees.

The matching process will generate a surplus, which we will denote by S and is given by:

$$S = W(w) - U + J(w) - V,$$

1.1 Bargaining process

In the bargaining process that occurs during the matching, firms and workers will negotiate part of the total resources that each type of agent will receive. Let $\psi \in [0, 1]$ be this fraction.

- Denote by ψ_w the worker's proposal of the fraction of the total surplus he will receive (i.e., the worker proposes the firm will receive a fraction $1 - \psi_w$ of the total surplus S).
- In a similar fashion the firm proposes that the worker receives a fraction ψ_f (i.e., the firm proposes to itself get $1 - \psi_f$ of the total surplus S).
- Suppose there is also a probability θ that the worker does the proposal about the division of the total surplus. This represents his bargaining power, i.e., when $\theta \to 1$ it means that only the worker does the proposal.

In the way the bargaining process was designed, this will be a stationary game since the parameters do not depend on time. Even considering that the process repeats every time, each game does not depend on the realization of the past games. Hence, the game is the same sub-game at any moment in time.

Notice the worker will do the offer until the moment the firm is indifferent between accepting or not accepting the offers, where we will suppose that once the firm is indifferent the firm will hire the worker. Observe that in instances where a firm strictly accepts an offer, the worker still has some margin to increase his gain in the bargaining process. Therefore, the worker will do an offer that will be equal to the discounted value of a new probability that in the future he will do the offer plus the probability that the firm will do the offer:

$$\begin{gathered}(1-\psi_w)\,S=\tfrac{1}{1+r\Delta}\left[\theta\,(1-\psi_w)\,S+(1-\theta)\,(1-\psi_f)\,S\right]\\ \Longrightarrow (1-\psi_w)\,r\Delta=(1-\theta)\,(\psi_w-\psi_f)\,.\end{gathered} \tag{3}$$

On the other hand, the firms also make offers until the moment where the worker is indifferent between accepting it or not:

$$\begin{gathered}\psi_f S=\frac{1}{1+r\Delta}\left[\theta\psi_w S+(1-\theta)\psi_f S\right]\\ \Longrightarrow \psi_f r\Delta=\theta\,(\psi_w-\psi_f)\,.\end{gathered} \tag{4}$$

When $\Delta\rightarrow 0$ we will have the situation where $\psi_f\rightarrow\psi_w\equiv\psi$, i.e., when the interval between the negotiations goes to zero it does not matter who is doing or who is receiving the offer. By dividing the system given by (3) and (4) we have:

$$\frac{1-\psi_w}{\psi_f}=\frac{1-\theta}{\theta}\Longrightarrow\psi=\theta,$$

when the time interval goes to zero. This means the worker will receive a fraction of the total surplus that will be equal to his negotiation power.

1.2 Benefit of the matching

By definition, for both the worker and the firm the benefit of being in the matching is equal to the fraction of the received surplus:

$$\theta S=W(w)-U\Longrightarrow S=\frac{W(w)-U}{\theta}. \tag{5}$$

$$(1-\theta)S=J(w)-V\Longrightarrow S=\frac{J(w)-V}{(1-\theta)}. \tag{6}$$

Notice that (5) plus (6) is equal to S. The system formed by (5) and (6) allows us to write the following:

$$\theta[J(w)-V]=(1-\theta)[W(w)-U]. \tag{7}$$

The above results are the same as we would obtain by solving the *Generalized Nash Bargaining* problem:

$$\begin{gathered}\max_w[W(w)-U]^{\theta}[J(w)-V]^{(1-\theta)}\\ F.O.C.:\theta[J(w)-V]=(1-\theta)[W(w)-U],\end{gathered}$$

where this problem is designed for cooperative games.

This highlights that the conditions we imposed on our problem, which is a non-cooperative game, led to the same solution as a cooperative game. From now on we can use this result to solve non-cooperative games by using the same F.O.C. of cooperative games.

Notice the F.O.C. of the problem, which is the result highlighted in equation (7) will give rise endogenously to the wage w.

Lastly, notice equations (1) and (2) results in:

$$W(w) = \frac{w + \lambda U}{r + \lambda}, \tag{8}$$

and

$$J(w) = \frac{y - w + \lambda V}{r + \lambda}. \tag{9}$$

2 Finding the probability of receiving offers α

In order to find the origin of this parameter, we will specify the flow payoff for the firms and the workers when the vacancies are not filled:

1. **Flow payoff of the unemployed worker:**

$$rU = b + \alpha_w[W(w) - U]. \tag{10}$$

2. **Flow payoff of the firm with a vacancy:**

$$rV = -k + \alpha_e[J(w) - V], \tag{11}$$

where k represents the fixed cost to keep the vacancy open, α_e is the probability of the firm finding a new employee and α_w is the probability of the worker finding a new firm and, hence, receiving a new offer.
Consider workers in a continuum interval in $[0, 1]$. In this way, we will have a measure u of unemployed workers and v of job postings by the firms. Right now one could think in a function m, similar to the production function $F(K, L)$, that generates matching between u and v:

$$m(u, v),$$

where we assume m has constant returns to scale. Define a new parameter $q = \dfrac{u}{v}$, which can be thought of as the queue for a job. We can write now the probabilities to find the job using this new parameter:

$$\alpha_w = \tfrac{m(u,v)}{u} \implies \alpha_w = m\left(1, \tfrac{v}{u}\right) = m\left(1, \tfrac{1}{\frac{u}{v}}\right) = m\left(1, \tfrac{1}{q}\right) \therefore \alpha_w \equiv \alpha_w(q), \quad {\alpha_w}'(q) < 0.$$
$$\alpha_e = \tfrac{m(u,v)}{v} \implies \alpha_e = m\left(\tfrac{u}{v}, 1\right) = m(q, 1) \therefore \alpha_e \equiv \alpha_e(q), \quad \alpha_e'(q) > 0.$$

2.1 System of equations

The system of equations that characterize the equilibrium is given by:

$$rW(w) = w + \lambda[U - W(w)]. \tag{12}$$

$$rJ(w) = y - w + \lambda[V - J(w)]. \tag{13}$$

$$rU = b + \alpha_w(q)[W(w) - U]. \tag{14}$$

$$rV = -k + \alpha_e(q)[J(w) - V]. \tag{15}$$

$$\theta[J(w) - V] = (1 - \theta)[W(w) - U]. \tag{16}$$

We can derive the unemployment rate by considering the situation where the number of people who were employed and left their job is equal to the number of unemployed people who assumes a job position. This will be the unemployment rate in a steady state:

$$(1 - u)\lambda = u\alpha_w(q) \Longrightarrow u = \frac{\lambda}{\lambda + \alpha_w(q)}. \tag{17}$$

2.2 Solving the system

Notice we have 6 equations, (12) to (17), but 7 unknowns $W(w), J(w), U, V, u, q$ and w. In other words, to solve this system we need another equation. The classical approach to this problem is to assume free entry: $\exists$ free entry in the job vacancies as if there is no limit in the number of job postings that the firms may create or in the number of firms in the market. This implies that that the value of an available vacancy is zero:

$$V = 0. \tag{18}$$

Now we have 7 unknowns $W(w), J(w), U, V, u, q$ and w and 7 equations ((12) to (18)). We wish to explicit all the variables in terms of the total surplus S and for this we will use the most important equations of the system above, e.g., (16) and (18). But before using those equations, first use (18) into (15) and get:

$$0 = -k + \alpha_e(q)J(w) \Longrightarrow k = \alpha_e(q)J(w).$$

Now use (18) into (6) and obtain:

$$(1 - \theta)S = J(w) - 0 \Longrightarrow J(w) = (1 - \theta)S.$$

Using the two last results we obtain:

$$k = \alpha_e(q)(1 - \theta)S, \tag{19}$$

where S and q are the values we need to find. Replace equations (12) to (15) in the surplus equation to get:

$$
\begin{gathered}
S = W(w) - U + J(w) - V \implies rS = rW(w) - rU + rJ(w) - rV \\
\implies rS = w - \lambda[W(w) - U] - b - \alpha_w(q)[W(w) - U] + y - w - \lambda[J(w) - V] - rV \\
\implies rS = y - b - \lambda[W(w) - U] - \alpha_w(q)[W(w) - U] - \lambda[J(w) - V] - rV.
\end{gathered}
$$

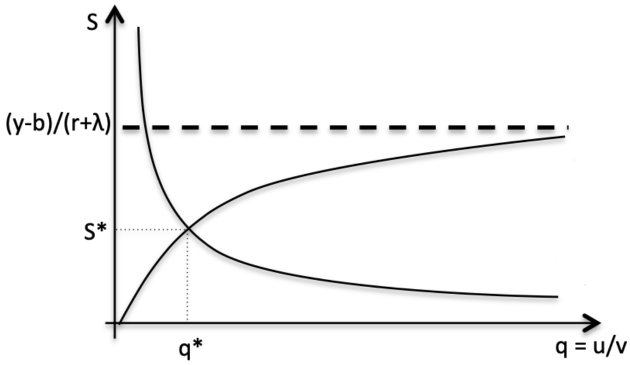
Now using equations (5), (6), and (18) into above:

$$
\begin{gathered}
rS = y - b - \lambda\theta S - \alpha_w(q)\theta S - \lambda(1-\theta)S - rV \\
\implies rS + \lambda S + \alpha_w(q)\theta S = y - b - rV,
\end{gathered}
$$

which will give us:

$$
S = \frac{y-b}{r + \lambda + \alpha_w(q)\theta}. \tag{20}
$$

When $q \to \infty$, $\alpha_w(q)\theta \to 0$ and so, $S \to \dfrac{y-b}{r+\lambda}$. Using (19) we can perceive when $q \to \infty$, $\alpha_e(q) \to \infty$ and, so $(1-\theta)S \to 0$ to keep k fixed. Therefore we can see that equations (19) and (20) give the equilibrium S^* and q^*. This can be also seen in the graph below, where the upward graph is equation (20) and the downward graph is (19):



Once we found S^* we can use equation (13) to find the wage in equilibrium:

$$
r[J(w) - V] = y - w + \lambda[V - J(w)] \implies r(1-\theta)S = y - w - \lambda(1-\theta)S \implies w = y - (r+\lambda)(1-\theta)S.
$$

$$
w^* = y - (r+\lambda)(1-\theta)S^*. \tag{21}
$$

In a similar fashion we can just replace q^* into (17) and obtain:

$$
u^* = \frac{\lambda}{\lambda + \alpha_w(q^*)} \tag{22}
$$

2.3 Static analysis and economic intuition

Suppose a positive shock in b ($b \uparrow$). In other words., we would like to evaluate how the effect of b spreads out in the economy:

1. In the McCall model the increase in b increases the reservation wage $\bar{w}$ and consequently the unemployment rate u. In this scenario when $b \uparrow \Rightarrow w \uparrow$. This occurs because we have

$\theta S = W(w) - U$, so when U increases we need to increase $W(w)$ to keep the fraction of the matching that goes to the worker constant. In other words, wages must increase to improve the outside option of the worker.

2. Before we had that the increase in b increases u because of the increase in the reservation wage. Now the increase in u occurs because you decrease the profit per worker by the firm $y - w$, which increases q. So, in the end, we have $q \uparrow \Longrightarrow \alpha_e(q) \uparrow \wedge \alpha_w(q) \downarrow \Longrightarrow u \uparrow$.

2.4 Extension

Solve the problem assuming the hypothesis of free entry is not valid and instead of it, assume the number of firms in the economy is fixed and equal to M. In other words, assume that each firm offers only one job posting and equalize the number of employed workers to the number of job postings filled to obtain the last equation:

$$(1-u) = M - v \Longrightarrow (1-u) = M - \frac{u}{q}.$$

Now use this equation to solve the model and analyze the effects of a positive shock in the unemployment insurance b.

3 Math-specific productivity

In the previous model, workers and firms take time until matching, but once they matched, the wage w was the same for all possible matchings. Now, we will assume that the firm and the worker have access to the productivity of the worker, which will influence the wage received, $w(y)$. Assume that this productivity is exogenous and comes from a random distribution $F(y)$. Notice that the observed y is the same for the firms and the workers. We also assume that once observed, the productivity does not change after the matching occurs. As before we will write the flow payoffs for the firms and the workers:

$$rW(w(y)) = w(y) + \lambda[U - W(w(y))], \tag{23}$$

$$rJ(w(y)) = y - w(y) + \lambda[V - J(w(y))], \tag{24}$$

$$rU = b + \alpha_w(q) \int_{\bar{y}}^{\infty} [W(w(y)) - U] dF(y), \tag{25}$$

$$rV = -k + \alpha_e(q) \int_{\bar{u}}^{\infty} [J(w(y)) - V] dF(y), \tag{26}$$

where we can observe that there will be a threshold productivity $\bar{y}$ from which the workers will accept the offers. This puts additional constraint in the equation of the division of the surplus S, e.g., which will be valid only for the instances where $y > \bar{y}$. The surplus that before was a function of the wage w now will be a function of the productivity y, i.e., $S \equiv S(y)$. In this case, there will be a no-surplus condition:

$$S(\bar{y}) = 0. \tag{27}$$

As in the previous model, we still have the variables λ and θ with the same meaning. The surplus-sharing process will occur in the same way as the bargaining process. The matching function will be the same and the queue equation continues to be given by $q = \frac{u}{v}$. We will still assume the hypothesis of free entry and the same probabilities $\alpha_e(q)$ and $\alpha_w(q)$. Therefore, our system of equations will be completed by:

$$V = 0, \tag{28}$$
$$\theta[J(w) - V] = (1-\theta)[W(w) - U], \quad \forall y \geq \bar{y}, \tag{29}$$

and with the equation for the unemployment rate in the steady state:

$$(1-u)\lambda = u\alpha_w(q)[1 - F(\bar{y})] \Longrightarrow u = \frac{\lambda}{\lambda + u\alpha_w(q)[1 - F(\bar{y})]}, \tag{30}$$

where we can notice we have 8 equations (equations (23) to (30)) and 8 unknown variables $W(w(y)), J(w(y)), U, V, w(y), \bar{y}, q$ and u. The last four unknown variables are the ones we are more interested in. By working only with equations (28) and (29) we can obtain a graph and the $\bar{y}$ and q in the equilibrium, e.g., in a similar fashion as we had done previously. Use equation (28) into (26):

$$\begin{aligned} V = 0 &\Longrightarrow k = \alpha_e(q) \int_{\bar{y}}^{\infty} [J(w(y)) - V] dF(y) \\ &\Longrightarrow k = \alpha_e(q) \int_{\bar{y}}^{\infty} [(1-\theta)S(y)] dF(y) \end{aligned} \tag{31}$$

Notice we used (29) above in a similar fashion as we had done in the previous section. Using the surplus equation we obtain:

$$S(y) = W(w(y)) - U + J(w(y)) - V \Longrightarrow rS(y) = rW(w(y)) - rU + rJ(w(y)) - rV.$$

Replacing (23), (24), (25) and (28) into above we obtain:

$$\begin{aligned} rS(y) = w(y) - \lambda[W(w(y)) - U] - b - \alpha_w(q) \textstyle\int_{\bar{y}}^{\infty} [W(w(y)) - U] dF(y) + y - w(y) - \lambda[J(w(y)) - V] - 0 \\ \Longrightarrow rS(y) = w(y) - \lambda\theta S(y) - b - \alpha_w(q) \textstyle\int_{\bar{y}}^{\infty} [\theta S(y')] dF(y') + y - w(y) - \lambda(1-\theta)S(y) - 0 \\ \Longrightarrow (r+\lambda)S(y) = y - b - \alpha_w(q) \textstyle\int_{\bar{y}}^{\infty} [\theta S(y')] dF(y') \\ \therefore S(y) = \frac{y - b - \alpha_w(q) \int_{\bar{y}}^{\infty} [\theta S(y')] dF(y')}{r + \lambda}, \quad \forall y \geq \bar{y} \end{aligned} \tag{32}$$

where one can notice that the productivity y' inside the integral is the outside option, i.e., different from the productivity y in the matching that comes from the equation $J(w(y))$. The term $\alpha_w(q) \int_{\bar{y}}^{\infty} [\theta S(y')] dF(y')$ can be simplified when we evaluate it in $\bar{y}$ and since (27) guarantee us that $S(\bar{y}) = 0$ we would have:

$$\begin{aligned} S(\bar{y}) = \frac{\bar{y} - b - \alpha_w(q) \int_{\bar{y}}^{\infty} [\theta S(y')] dF(y')}{r + \lambda} = 0 \Longrightarrow \bar{y} = b + \alpha_w(q) \textstyle\int_{\bar{y}}^{\infty} [\theta S(y')] dF(y') \\ \Longrightarrow \alpha_w(q) \textstyle\int_{\bar{y}}^{\infty} [\theta S(y')] dF(y') = \bar{y} - b \Longrightarrow S(y) = \frac{y - b - (\bar{y} - b)}{r + \lambda} \Longrightarrow S(y) = \frac{y - \bar{y}}{r + \lambda}. \end{aligned}$$

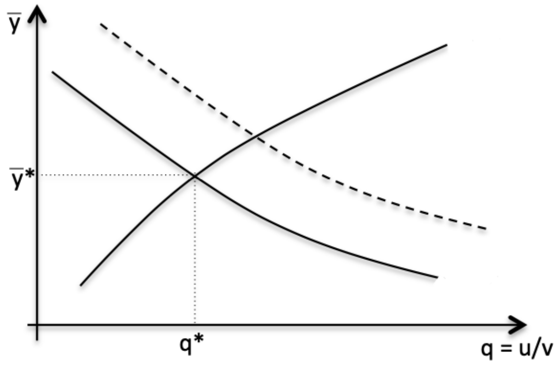
Finally, we have:

$$\bar{y} = b + \frac{\alpha_w(q)\theta}{r+\lambda} \int_{\bar{y}}^{\infty} [y - \bar{y}] dF(y), \tag{33}$$

and using the same result we can rewrite (31) as:

$$k = \frac{\alpha_e(q)(1-\theta)}{r+\lambda} \int_{\bar{y}}^{\infty} [y - \bar{y}] dF(y), \tag{34}$$

where one can perceive equations (33) and (34) represent a system of two equations and two unknowns. We can use the same graph approach. In (34) observe that an increase in q increase $\alpha_e(q)$ and since k is constant, the value of the integral must decrease and the only way for this to happen is through an increase in $\bar{y}$. On the other hand, in (33), the same increase in q diminishes the value $\alpha_w(q)$ and, again, since b is a constant, the value $\bar{y}$ must decrease to the equality continue to be valid. The graph below allow us to find the optimal values $\bar{y}^*$ and q^*:



3.1 Static analysis and economic intuition

We now analyze the impact that a positive shock in the unemployment insurance b generates in the variables q and $\bar{y}$.

1. $b \uparrow \rightarrow \bar{y} \uparrow$, since with the increase in the outside option the worker will only accept a proposal in case the productivity and, hence, the wage is higher. A direct consequence of this is that there will be a lower number of matchings.

2. $b \uparrow \rightarrow w(y) \uparrow$, since the worker is more selective in the bargaining process.

3. By joining both results, we will have a lower number of matching ($u \uparrow$), because the matches are less profitable to the firms given the increase in the wage ($v \downarrow$), i.e., $b \uparrow \rightarrow w(y) \uparrow \rightarrow u \uparrow v \downarrow \rightarrow q \uparrow$.

4. Lastly notice $b \uparrow \rightarrow q \uparrow \bar{y} \uparrow$ decreases both $\alpha_w(q)$ as well as $[1 - F(\bar{y})]$ and both effects increase the unemployment rate in the steady state, u.

5. Notice that a positive shock in b dislocates the curve given by (33), which right now is represented by the dashed curve. This movement increases $\bar{y}^*$ and q^*.

6. As an exercise recovers the other variables of the system of equations as in the model of the previous section.

4 Match-specific productivity with variations in the productivity

Now we will assume the productivity can change after the worker accepts the job offer. The probability of the productivity change will be exogenous and represented by the variable λ. The flow payoff of the employed worker it will be given by his wage plus the positive variation in his earnings when the productivity is above $\bar{y}$ weighted by the probability λ and with the same probability the worker can enter in an unemployment situation when the productivity goes below $\bar{y}$:

$$rW(w(y)) = w(y) + \lambda \int_{\bar{y}}^{\infty} \left[W\left(w\left(y'\right)\right) - W(w(y))\right] dF\left(y'\right) + \lambda F(\bar{y})[U - W(w(y))]. \tag{35}$$

The flow payoff of the unemployed worker will be:

$$rU = b + \alpha_w(q)[W(w(y)) - U]. \tag{36}$$

The remaining equations and the method to solve the system is the same as before.

As an exercise elucidate the other equations of the model and analyze a positive impact in k, i.e., in the cost that the firm faces keeping a job position opened.

5 Direct Search and Posting

In this section, we work with a model different from the ones in previous sections. First, we suppose only one period, and the events will happen in the following order:

1. Firms choose and announce the wages.
2. Workers observe the wages.
3. Workers apply for job postings.

Notice there is a trade-off when the firm announces a job with a high wage w. On one hand, the firm will receive more candidates. On the other hand, the worker will feel discouraged to apply for the job because more people will apply for this job.

For the worker to apply for this job posting the following must be true:

$$\left[\alpha_w(q)\right] w + \left[1 - \alpha_w(q)\right] b > U.$$

For the firm to keep the vacancy open we must have:

$$V = -k + \alpha_e(q)[y - w].$$

Therefore, the firms will choose a wage w that maximizes this value, i.e., firms will solve the problem:

$$\max_{w} V = -k + \alpha_e(q)[y - w] \quad s.a \quad [\alpha_w(q)]\, w + [1 - \alpha_w(q)]\, b = U. \tag{37}$$

Isolating w in the restriction and replacing in the objective function V:

$$\max_{q} \alpha_e(q) \left[y - \left(\frac{U - b}{\alpha_w(q)} + b \right) \right],$$

where k was taken out of the objective function because it does not change the result. Now the firm will maximize in q:

$$\alpha_e(q) = \frac{m(u,v)}{v} = \frac{u}{v} \frac{m(u,v)}{u} = q\alpha_w(q) \Longrightarrow \alpha_w(q) = \frac{\alpha_e(q)}{q}.$$

Using this result and doing the simplifications we obtain:

$$\max_{q} \alpha_e(q)(y - b) - (U - b)q \Longrightarrow (F.O.C.): \quad \alpha_e'(q)(y - b) = U - b. \tag{38}$$

From (38) we can obtain q. Suppose that each firm offers only one job, which implies $q = \frac{u}{1} = u$, i.e., the firm chooses u by knowing it is affected by the wage w.

Now we will open the expression for U and replace it in the result given by the F.O.C. to find the wage w that will be announced by the firm:

$U = [\alpha_w(q)]\, w + [1 - \alpha_w(q)]\, b = \frac{\alpha_e(q)}{q} w + b - \frac{\alpha_e(q)}{q} b \Longrightarrow \alpha_e'(q)(y - b) = \frac{\alpha_e(q)}{q} w + b - \frac{\alpha_e(q)}{q} b - b$
$\therefore w = b + \frac{q\alpha_e'(q)}{\alpha_e(q)}(y - b)$.

In the previous models, the wage was expressed as $w = b + \epsilon(y - b)$, where ϵ was a totally exogenous parameter. In the model above we could find an endogenous mechanism that can replace ϵ. For instance, if the matching function has a Cobb-Douglas format $m(u,v) = u^{\gamma} v^{1-\gamma}$ we can show that $w = b + \gamma(y - b)$. Besides γ being exogenous, the economic intuition will be better.