Macroeconomics 1

Lecture - Growth facts & The Solow model

Diego de Sousa Rodrigues
diego.desousarodrigues@sciencespo.fr

Sciences Po

2022 Fall

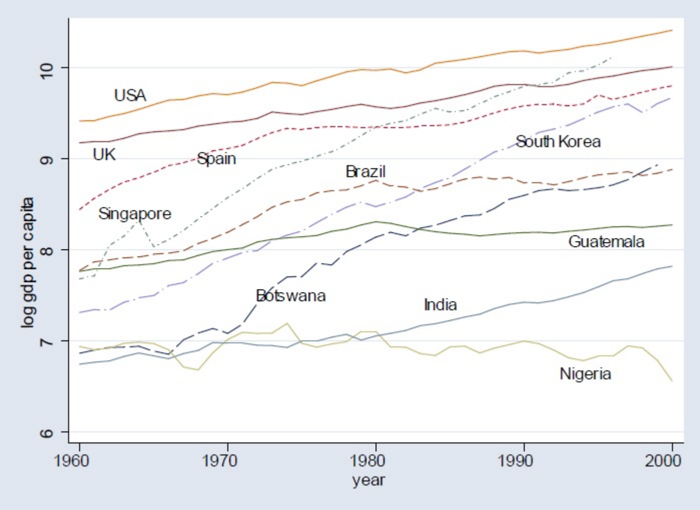
Course plan

- In the first part we will cover **Economic Growth**:
  - **Solow Model.**
  - **Neoclassical growth model.**
  - **Overlapping generations models.**
  - **Endogenous growth models.**
  - **Structural change.**

Course plan

- In the first part we will cover **Economic Growth**:
  - **Solow Model.**
  - **Neoclassical growth model.**
  - **Overlapping generations models.**
  - **Endogenous growth models.**
  - **Structural change.**
- In the second part we will cover **Labor Markets**:
  - **Introduction to Dynamic Programming.**
  - **Introduction to Labor Markets & The Search Models.**
  - **The Search and Matching Model.**
  - **The Search and Matching Model with endogenous job destruction.**

Economic Growth in Selected Countries



Source: Introduction to Modern Economic Growth, Acemoglu.

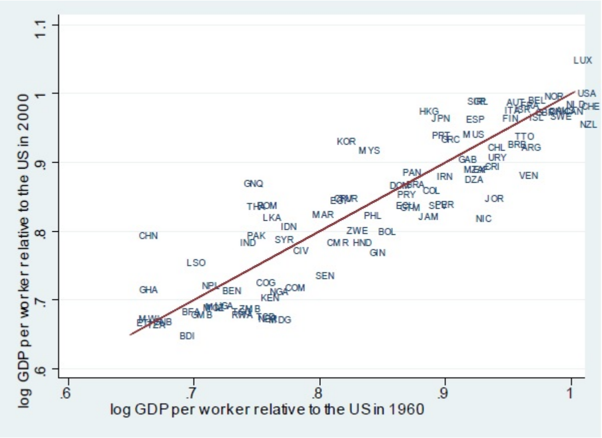
Figure: The evolution of income per capita in selected countries, 1960-2000.

Why should we care about differences in income per capita?

1. **Welfare**.
2. Understand the structure and efficiency of **production and market mechanisms**, i.e., what do income differences reflect?

Origins of Income Differences

- Growth is what can explain cross-country income differences, but **postwar growth alone cannot**:

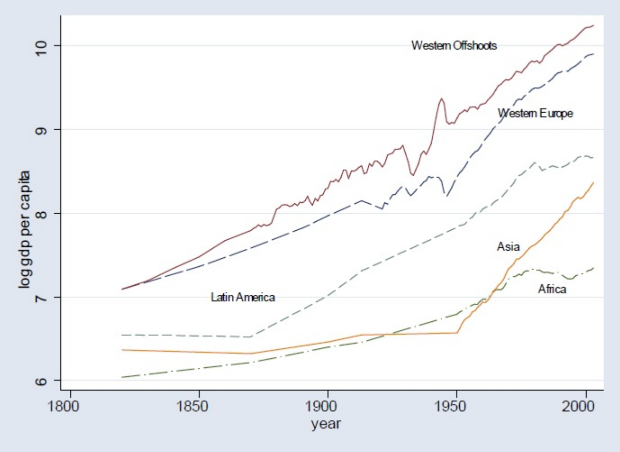


Source: Introduction to Modern Economic Growth, Acemoglu.

Figure: Log GDP per worker in 2000 and log GDP per worker in 1960.

Origins of Income Differences

- Much of the **divergence** took place during the XIXth and early XXth century:

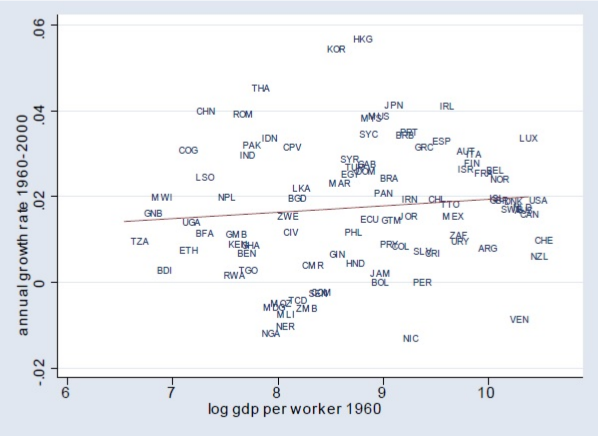


Source: Introduction to Modern Economic Growth, Acemoglu.

Figure: The evolution of average log GDP per capita in Western Offshoots, Western Europe, Latin America, Asia and Africa, 1820-2000.

No Unconditional Convergence

- In general, **countries with lower GDP/capita do not grow faster**:



Source: Introduction to Modern Economic Growth, Acemoglu

Figure: Annual growth rate of GDP per worker between 1960 and 2000 versus log GDP per worker in 1960 for the entire world.

But Convergence Among Relatively Similar Countries

- See the **convergence among OECD countries**:

Source: Introduction to Modern Economic Growth, Acemoglu.

Figure: Annual growth rate of GDP per worker between 1960 and 2000 versus log GDP per worker in 1960 for core OECD countries.

But Convergence Among Relatively Similar Countries (or regions)

- See the **convergence among US states**:

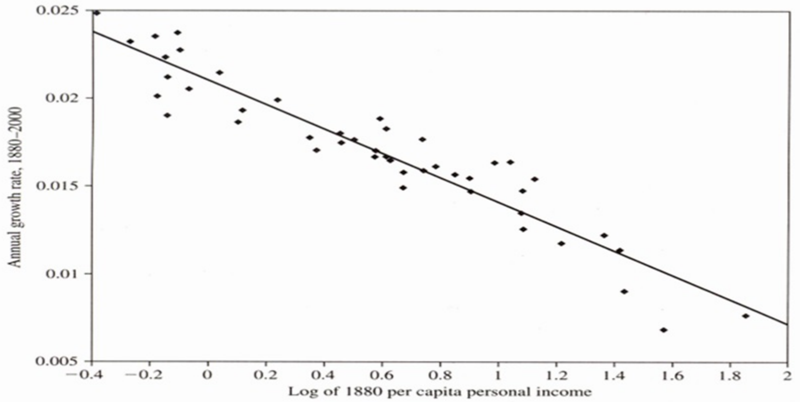


Figure 11.2
Convergence of personal income across U.S. states: 1880 personal income and 1880–2000 income growth. The average growth rate of state per capita income for 1880–2000, shown on the vertical axis, is negatively related to the log of per capita income in 1880, shown on the horizontal axis. Thus, absolute β convergence exists for the U.S. states.

Conditional Convergence

- Barro (1991): look at conditional convergence, i.e. the **income gap between countries that share the same characteristics**.
- To measure conditional convergence: 'Barro growth regression'

$$\underbrace{g_{t,t-1}}_{\text{growth}} = \beta \underbrace{ln(y_{t-1})}_{\substack{\text{initial}\\\text{income}}} + \underbrace{X'_{t-1}}_{\textit{characteristics}} \alpha + \underbrace{\varepsilon_t}_{\substack{\text{unobserved}\\\text{variability}}}$$

Conditional Convergence

- Barro (1991): look at conditional convergence, i.e. the **income gap between countries that share the same characteristics**.
- To measure conditional convergence: 'Barro growth regression'

$$\underbrace{g_{t,t-1}}_{\text{growth}} = \beta \underbrace{ln(y_{t-1})}_{\substack{\text{initial}\\ \text{income}}} + \underbrace{X_{t-1}^{'}}_{\textit{characteristics}} \alpha + \underbrace{\varepsilon_t}_{\substack{\text{unobserved}\\ \text{variability}}}$$

- Useful for **describing the data**, but not for estimating **causal effects** of the variables in X_{t-1}.
- These variables correlate with growth but are not necessarily exogenous (e.g. investment, human capital).

Proximate and Fundamental Drivers of Growth

- Important distinction between the **drivers of growth**:
  - **Proximate reasons**.
  - **Fundamental reasons**.

Proximate and Fundamental Drivers of Growth

- Important distinction between the **drivers of growth**:
  - **Proximate reasons**.
  - **Fundamental reasons**.
- **Proximate reasons** for US being richer than Ghana:

Proximate and Fundamental Drivers of Growth

- Important distinction between the **drivers of growth**:
  - **Proximate reasons**.
  - **Fundamental reasons**.
- **Proximate reasons** for US being richer than Ghana:
  - It has more physical capital, machines, human capital, and better technology.
  - Better infrastructure.
  - Firms in the US invest more, and households save more.
- **Fundamental reasons** - what is the reason for these differences?

Proximate and Fundamental Drivers of Growth

- Important distinction between the **drivers of growth**:
  - **Proximate reasons**.
  - **Fundamental reasons**.
- **Proximate reasons** for US being richer than Ghana:
  - It has more physical capital, machines, human capital, and better technology.
  - Better infrastructure.
  - Firms in the US invest more, and households save more.
- **Fundamental reasons** - what is the reason for these differences?
  - Geography.
  - Institutions.
  - Culture.
  - History.

Proximate and Fundamental Drivers of Growth

- Important distinction between the **drivers of growth**:
  - **Proximate reasons**.
  - **Fundamental reasons**.
- **Proximate reasons** for US being richer than Ghana:
  - It has more physical capital, machines, human capital, and better technology.
  - Better infrastructure.
  - Firms in the US invest more, and households save more.
- **Fundamental reasons** - what is the reason for these differences?
  - Geography.
  - Institutions.
  - Culture.
  - History.
- **Mechanics of growth is about the proximate causes**.
- They can then be **linked to possible fundamental causes of long-run development** (development economics, political economy).

What is the purpose to study Economic Growth? Summing up what we saw so far

- **Explain the relative levels of development** of various countries, as well as their rates of growth, is **challenging**.
  - How did we arrive at the **degree of inequality** we observe today?
- **Until the end of the XVIII, disparities were moderate** among countries since most of them were just above the subsistence level.
- But **during the XIX and early XX, we could see an increase in disparity** among countries.

What is the purpose to study Economics Growth? Summing up what we did so far

- The **natural movement of ideas, factors of production, and people** were expected to bring those countries close to each other.
- But, what we can observe is the existence of **convergence clubs** made up of similar countries when we analyze the growth rate of GDP per capita.

What is the purpose to study Economics Growth? Summing up what we did so far

- The **natural movement of ideas, factors of production, and people** were expected to bring those countries close to each other.
- But, what we can observe is the existence of **convergence clubs** made up of similar countries when we analyze the growth rate of GDP per capita.
- The **exogenous growth** model leads to the result of convergence whereas the **endogenous growth** model naturally generates different growth rates.

THE SOLOW-SWAN MODEL

THE SOLOW-SWAN MODEL

It explains the increase in national production by factor accumulation.

The Solow Growth Model

- Limit of regressions:
  - Only gives **correlations**.
  - We are not sure a given **increase in investment or human capital** will *cause* an increase in growth rate corresponding to the slope.

The Solow Growth Model

- Limit of regressions:
  - Only gives **correlations**.
  - We are not sure a given **increase in investment or human capital** will *cause* an increase in growth rate corresponding to the slope.
- To better understand **proximate causes**, we rely on a *model* of growth.
- Before, the most common approach to economic growth was built on the Harrod-Domar model (assume **labor and capital are perfect substitutes**).

The Solow Model

- **Mechanism**:
  People save a fraction of their income $\rightarrow$ future capital $\rightarrow$ produce output $\rightarrow$ save $\rightarrow$ capital ...

The Solow Model

- **Mechanism**:
  People save a fraction of their income $\rightarrow$ future capital $\rightarrow$ produce output $\rightarrow$ save $\rightarrow$ capital ...

- **Implications**:
  1. Can this mechanism lead to **sustained growth** in the long run, or does it run out of steam eventually?
  2. What is the **relationship between growth and investment/saving**?
  3. Why do countries have **different experiences**?
     - Are countries inherently different?
     - Are they just at different points on the same path?

Set-up

- **Closed economy** with a unique **final good** $\rightarrow$ used for consumption and for investment, i.e., to create new capital.
- Time can be **discrete**, $t = 0, 1, 2, ...$, or **continuous**.

Set-up

- **Closed economy** with a unique **final good** $\rightarrow$ used for consumption and for investment, i.e., to create new capital.
- Time can be **discrete**, $t = 0, 1, 2, ...$, or **continuous**.

- **Two types of agents**:
  - Representative **household** (or large number of identical households) who for now are not optimizing.
  - Representative **firm** (or multiple identical firms) that is optimizing and is in a perfectly competitive environment.
- **Three markets**:
  1. **Labor market**.
  2. **Capital market**.
  3. **Final good market**.

Representative household

- Owns all the **labor** & supplies it **inelastically** (no labor-leisure choice, as in most growth models), labor supply denoted L_t^s.
- Owns all the **capital** & **rent it out** to firms, capital supply denoted K_t^s.
- Owns the **firm** and receives profit Π_t.

Representative household

- Owns all the **labor** & supplies it **inelastically** (no labor-leisure choice, as in most growth models), labor supply denoted L_t^s.
- Owns all the **capital** & **rent it out** to firms, capital supply denoted K_t^s.
- Owns the **firm** and receives profit Π_t.

- Overall receives **income**:

$$Y_t = \underbrace{w_t L_t^s}_{\text{Labor inc.}} + \underbrace{r_t K_t^s}_{\text{Capital inc.}} + \underbrace{\Pi_t.}_{\text{Profit}}$$

- Saves a **constant exogenous fraction $s \in [0,1]$ of his/her disposable income**, taken as given by now.
- **Consumption function**:

Representative household

- Owns all the **labor** & supplies it **inelastically** (no labor-leisure choice, as in most growth models), labor supply denoted L_t^s.
- Owns all the **capital** & **rent it out** to firms, capital supply denoted K_t^s.
- Owns the **firm** and receives profit Π_t.

- Overall receives **income**:

$$Y_t = \underbrace{w_t L_t^s}_{\text{Labor inc.}} + \underbrace{r_t K_t^s}_{\text{Capital inc.}} + \underbrace{\Pi_t.}_{\text{Profit}}$$

- Saves a **constant exogenous fraction $s \in [0,1]$ of his/her disposable income**, taken as given by now.
- **Consumption function**:

$$S_t = sY_t, \tag{1}$$

$$\Leftrightarrow C_t = (1-s)Y_t. \tag{2}$$

- Preferences not specified $\Rightarrow$ we cannot make welfare statements.

Representative firm

- **Aggregate production function** of the firm:

$$Y_t \equiv F(K_t, L_t).$$

- K_t capital,
- L_t Labor,
- $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is the neoclassical production function.

Assumptions

- Production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ is **twice continuously differentiable** in K and L.
- **Marginal products** of K and L are:

- **Positive**:

$$MPK = \frac{\partial F(.)}{\partial K} = F_K > 0. \qquad MPL = \frac{\partial F(.)}{\partial L} = F_L > 0.$$

- **Diminishing**:

$$\frac{\partial^2 F(.)}{\partial K^2} = \frac{\partial F_K}{\partial K} < 0. \qquad \frac{\partial^2 F(.)}{\partial L^2} = \frac{\partial F_L}{\partial L} < 0.$$

Assumptions

- **Constant returns to scale** (CRS) in K and L ($\Rightarrow F$ is homogeneous of degree 1 in K and L): for any $\lambda > 0$:

$$F(\lambda K, \lambda L) = \lambda F(K, L).$$

- **Inada conditions**:

$$\lim_{K\to 0} F_K(.) = \infty. \qquad \lim_{K\to\infty} F_K = 0 \ \ \forall L > 0.$$

$$\lim_{L\to 0} F_L = \infty \qquad \lim_{L\to\infty} F_L = 0. \ \ \forall K > 0.$$

Firm optimization

- Faces a **static problem** i.e. problem can be specified as maximizing profit period by period:

$$\max_{\{K_t^d, L_t^d\}} \underbrace{F(K_t^d, L_t^d)}_{\text{Revenue}} \underbrace{-w_t L_t^d - r_t K_t^d}_{\text{Costs}}.$$

- Price at which the final good is solved is **normalized** to 1.

Firm optimization

- Faces a **static problem** i.e. problem can be specified as maximizing profit period by period:

$$\max_{\{K_t^d, L_t^d\}} \underbrace{F(K_t^d, L_t^d)}_{\text{Revenue}} \underbrace{-w_t L_t^d - r_t K_t^d}_{\text{Costs}}.$$

- Price at which the final good is solved is **normalized** to 1.

- **Optimality** requires:

$$w_t = F_L(K_t^d, L_t^d). \tag{3}$$

$$r_t = F_K(K_t^d, L_t^d). \tag{4}$$

Law of motion of capital & market clearing

- **Law of motion for capital**

$$K_{t+1} = (1 - \delta)K_t + I_t. \tag{5}$$

$\delta \in [0, 1]$: depreciation rate.
K_0: initial capital endowment and is given.

Law of motion of capital & market clearing

- **Law of motion for capital**

$$K_{t+1} = (1-\delta)K_t + I_t. \tag{5}$$

$\delta \in [0,1]$: depreciation rate.
K_0: initial capital endowment and is given.

- **Markets clear**:

1. **Labor market**: There will be a w such that endowment of labor $=$ labor demanded by the firm:

$$L_t^s = L_t^d \equiv L_t. \tag{6}$$

2. **Capital market**: There will be a r such that supply of capital $=$ demand for capital by the firm:

$$K_t^s = K_t^d \equiv K_t. \tag{7}$$

3. **Goods market**:

$$Y_t = C_t + I_t. \tag{8}$$

Equilibrium in the Solow model

An **equilibrium path** in the Solow model is a **series of prices** $\{w_t, r_t\}_{t=0}^{\infty}$ and **quantities** $\{C_t, S_t, I_t, Y_t, K_t, L_t\}_{t=0}^{\infty}$ given $\{L_t^s\}_{t=0}^{\infty}$ and K_0 such that for all $t \geq 0$:

Equilibrium in the Solow model

An **equilibrium path** in the Solow model is a **series of prices** $\{w_t, r_t\}_{t=0}^{\infty}$ and **quantities** $\{C_t, S_t, I_t, Y_t, K_t, L_t\}_{t=0}^{\infty}$ given $\{L_t^s\}_{t=0}^{\infty}$ and K_0 such that for all $t \geq 0$:

- $C_t = (1-s)Y_t$ and $S_t = sY_t$. *((1) and (2) hold)*
- Firms maximize profits given prices. *((3) and (4) hold)*
- Capital evolves according to $K_{t+1} = (1-\delta)K_t + I_t$. *((5) hold)*
- Factor markets clear. *((6), (7) and (8) hold)*

Analysing the Equilibrium and Dynamics

The Production Function

1. Let $Y_t \equiv F(K_t, L_t)$. The production function is assumed to be **homogeneous of degree 1** (CRS). What does that mean?

   ...

The Production Function

1. Let $Y_t \equiv F(K_t, L_t)$. The production function is assumed to be **homogeneous of degree 1** (CRS). What does that mean?

   ...

2. Since the function is homogeneous of degree 1, one can represent it in a **simple manner**. For that we can use the function $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$ and the per capita definitions $y_t \equiv \dfrac{Y_t}{L_t}$ and $k_t \equiv \dfrac{K_t}{L_t}$.

   ...

The Production Function

1. Let $Y_t \equiv F(K_t, L_t)$. The production function is assumed to be **homogeneous of degree 1** (CRS). What does that mean?

   ...

2. Since the function is homogeneous of degree 1, one can represent it in a **simple manner**. For that we can use the function $Y_t = AK_t^{\alpha}L_t^{1-\alpha}$ and the per capita definitions $y_t \equiv \dfrac{Y_t}{L_t}$ and $k_t \equiv \dfrac{K_t}{L_t}$.

   ...

3. Now using the idea above in a more general setting and defining $\lambda \equiv \dfrac{1}{L_t}$ we can rewrite the production function as:

   ...

The Production Function

1. Let $Y_t \equiv F(K_t, L_t)$. The production function is assumed to be **homogeneous of degree 1** (CRS). What does that mean?

   ...

2. Since the function is homogeneous of degree 1, one can represent it in a **simple manner**. For that we can use the function $Y_t = AK_t^{\alpha}L_t^{1-\alpha}$ and the per capita definitions $y_t \equiv \dfrac{Y_t}{L_t}$ and $k_t \equiv \dfrac{K_t}{L_t}$.

   ...

3. Now using the idea above in a more general setting and defining $\lambda \equiv \dfrac{1}{L_t}$ we can rewrite the production function as:

   ...

4. Using the **Assumptions** and the **Inada Conditions**, the graph for the production per capita will be:

   ...

Factor Accumulation

- In **discrete time** define the time change of a variable as:

$$\Delta X_t \equiv \dot{X}_t \equiv X_{t+1} - X_t.$$

  - The **growth rate** will be:

$$\frac{\Delta X_t}{X_t} \equiv \frac{\dot{X}_t}{X_t} = \frac{X_{t+1} - X_t}{X_t}.$$

Factor Accumulation

- In **discrete time** define the time change of a variable as:

$$\Delta X_t \equiv \dot{X}_t \equiv X_{t+1} - X_t.$$

  - The **growth rate** will be:

$$\frac{\Delta X_t}{X_t} \equiv \frac{\dot{X}_t}{X_t} = \frac{X_{t+1} - X_t}{X_t}.$$

- In **continuous time** define the time change of a variable as:

$$\dot{X}_t \equiv \frac{\partial X_t}{\partial t}.$$

  - The **growth rate** will be:

$$\frac{\dot{X}_t}{X_t} = \frac{\frac{\partial X_t}{\partial t}}{X_t}.$$

Factor Accumulation

1. The **population grows** at a constant rate n:

$$\frac{\dot{L}_t}{L_t} \equiv n.$$

2. The **capital accumulation** grows as:

$$\dot{K}_t = I_t - \delta K_t,$$

where $I_t = sY_t = sF(K_t, L_t)$.

Short-run Equilibrium and Dynamics

In the **short-run**, the **relative prices of factors adjust** so that **capital and labor** are always fully employed:

$$w_t = \frac{\partial F\left(K_t, L_t\right)}{\partial L_t}.$$

$$r_t = \frac{\partial F\left(K_t, L_t\right)}{\partial K_t}.$$

Short-run Equilibrium and Dynamics

In the **short-run**, the **relative prices of factors adjust** so that **capital and labor** are always fully employed:

$$w_t = \frac{\partial F\left(K_t, L_t\right)}{\partial L_t}.$$

$$r_t = \frac{\partial F\left(K_t, L_t\right)}{\partial K_t}.$$

- Since the function is homogeneous of degree 1 we can use **Euler's Theorem** and show that:

  ...

Short-run Equilibrium and Dynamics

In the **short-run**, the **relative prices of factors adjust** so that **capital and labor** are always fully employed:

$$w_t = \frac{\partial F(K_t, L_t)}{\partial L_t}.$$

$$r_t = \frac{\partial F(K_t, L_t)}{\partial K_t}.$$

- Since the function is homogeneous of degree 1 we can use **Euler's Theorem** and show that:

  ...

$$w_t L_t + r_t K_t = Y_t.$$

- We can also show that:

$$r_t = f'(k_t),$$

$$w_t = f(k_t) - k_t f'(k_t).$$

The Dynamics of the Solow-Swan Model

- So far, we have a **dynamic system** in two variables: **labor and capital**.
- The **homogeneity of the function** F will enable us to reduce the system to k_t **(capital per capita)**.

  ...

The Dynamics of the Solow-Swan Model

- So far, we have a **dynamic system** in two variables: **labor and capital**.
- The **homogeneity of the function** F will enable us to reduce the system to k_t **(capital per capita)**.

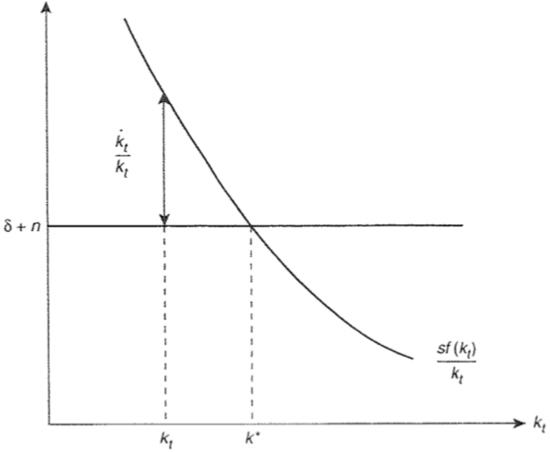
  ...

- The result will be:

$$\frac{\dot{k}_t}{k_t} = \frac{sf(k_t)}{k_t} - (\delta + n)$$

- We can see that is an **increasing function of the saving rates**, and a **decreasing function of the population growth rate and the depreciation rate**.

The Dynamics of the Solow-Swan Model



The Dynamics of the Solow-Swan Model

- The capital per capita will converge towards the **long-run equilibrium** k^*:
  ...

The Dynamics of the Solow-Swan Model

- The capital per capita will converge towards the **long-run equilibrium** k^*:
  ...

- It emphasizes that the **steady-state equilibrium** sets investment $sf(k)$ equal to the amount of capital that needs to be replenished $(\delta + n)k$.

- Now we will analyze the **value of capital per capita** in the Cobb-Douglas case and the **speed of convergence**:
  ...

Does a steady state exist and is it unique?

- The way we drew the curve, yes.
  - It **exists** if the two curves cross.
  - It is **unique** if they only cross once.
- The **Inada conditions** and the **decreasing marginal returns** guarantee that these two hold.

Proof of existence and uniqueness

1. **Existence**:
   - To establish **existence**, note that from Inada conditions (and from L'Hopital's rule), $lim_{k\to 0} f(k)/k = \infty$ and $lim_{k\to\infty} f(k)/k = 0$.
   - Moreover, $f(k)/k$ is continuous (from assumption), so by the **Intermediate Value Theorem** there exists $k^* \in [0, \infty[$ such that $sf(k^*) + (1-\delta-n)k^* = k^* \Leftrightarrow \frac{f(k^*)}{k^*} = \frac{\delta+n}{s}$ is satisfied.

Proof of existence and uniqueness

1. **Existence**:
   - To establish **existence**, note that from Inada conditions (and from L'Hopital's rule), $lim_{k\to 0} f(k)/k = \infty$ and $lim_{k\to\infty} f(k)/k = 0$.
   - Moreover, $f(k)/k$ is continuous (from assumption), so by the **Intermediate Value Theorem** there exists $k^* \in [0, \infty[$ such that $sf(k^*) + (1-\delta-n)k^* = k^* \Leftrightarrow \frac{f(k^*)}{k^*} = \frac{\delta+n}{s}$ is satisfied.

2. **Uniqueness**:
   - To see **uniqueness**, differentiate $f(k)/k$ with respect to k, which gives:

$$\frac{\partial[f(k)/k]}{\partial k} = \frac{f'(k)k - f(k)}{k^2} = -\frac{w}{k^2} < 0,$$

     where the last inequality uses $w_t = F_L(K_t, L_t) = f(k_t) - f'(k_t)k_t$.
   - Since $f(k)/k$ is **everywhere (strictly) decreasing**, there can only exist a unique value k^* that satisfies $\frac{f(k^*)}{k^*} = \frac{\delta+n}{s}$.

Comparative statics

In the steady state

$$k^* = sf(k^*) + (1-\delta-n)k^* \Leftrightarrow \underbrace{(\delta+n)k^*}_{\text{depreciation and pop.growth}} = \underbrace{sf(k^*)}_{\text{investment}}$$

Re-arranging:

$$\frac{f(k^*(s,\delta,n))}{k^*(s,\delta,n)} = \frac{\delta+n}{s}$$

- $\frac{\partial k^*(s,\delta,n)}{\partial \delta} < 0$: **higher depreciation** $\rightarrow$ lower k^*, y^*.
- $\frac{\partial k^*(s,\delta,n)}{\partial s} > 0$: **higher savings rate** $\rightarrow$ higher k^*, y^*.
- $\frac{\partial k^*(s,\delta,n)}{\partial n} < 0$: **higher population growth** $\rightarrow$ lower k^*, y^*.

Note: k^* depends on s, δ, n! Do those comparative statics without considering the Cobb-Douglas function but the general properties of the functions $f(k)$.

Golden rule

The Golden rule

- We took the **saving rate as given**. What would be the **best saving from the point of view of society**?

The Golden rule

- We took the **saving rate as given**. What would be the **best saving from the point of view of society**?
- Define the savings rate as the one that **maximizes the consumption per capita** in the stationary state.
- The problem will thus be:

$$\begin{aligned} &\text{Maximize } (1-s)f(k^*) \\ &\text{subject to} \quad (\delta + n)k^* = \text{sf}(k^*) \end{aligned}$$

...

The Golden rule

- We took the **saving rate as given**. What would be the **best saving from the point of view of society**?

- Define the savings rate as the one that **maximizes the consumption per capita** in the stationary state.

- The problem will thus be:

$$\begin{aligned} &\text{Maximize } (1-s)f(k^*) \\ &\text{subject to} \quad (\delta+n)k^* = sf(k^*) \end{aligned}$$

...

- The **optimal saving rate** is exactly equal to the shares of profits in production, i.e., when $f(k) = Ak^{\alpha}$, we have $s = \alpha$.

The golden rule saving rate

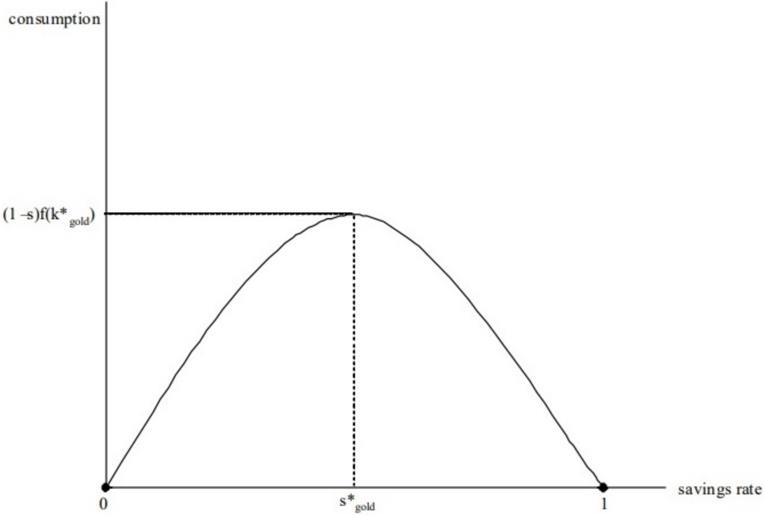


Figure: The 'golden rule' level of savings rate, which maximizes steady-state consumption.

The golden rule saving rate

1. When k^* is below k^*_{gold} , **higher saving will increase consumption**. Why?

The golden rule saving rate

1. When k^* is below k^*_{gold}, **higher saving will increase consumption**. Why?
   - **Capital-labor ratio is too low** so that individuals could increase steady-state consumption *at the cost of reducing consumption during transition* and **increasing saving rate**.

The golden rule saving rate

1. When k^* is below k^*_{gold}, **higher saving will increase consumption**. Why?
   - **Capital-labor ratio is too low** so that individuals could increase steady-state consumption *at the cost of reducing consumption during transition* and **increasing saving rate**.

2. When it is above k^*_{gold}, **steady-state consumption can be increased by saving less**. Why?

The golden rule saving rate

1. When k^* is below k^*_{gold} , **higher saving will increase consumption**. Why?
   - **Capital-labor ratio is too low** so that individuals could increase steady-state consumption *at the cost of reducing consumption during transition* and **increasing saving rate**.

2. When it is above k^*_{gold} , **steady-state consumption can be increased by saving less**. Why?
   - **Capital-labor ratio is too high** because depreciation of capital and population growth are higher than marginal product of capital, so that individuals could both increase steady-state consumption *and consumption during transition* by **decreasing saving rate**.

The golden rule saving rate

1. When k^* is below k^*_{gold} , **higher saving will increase consumption**. Why?
   - **Capital-labor ratio is too low** so that individuals could increase steady-state consumption *at the cost of reducing consumption during transition* and **increasing saving rate**.

2. When it is above k^*_{gold} , **steady-state consumption can be increased by saving less**. Why?
   - **Capital-labor ratio is too high** because depreciation of capital and population growth are higher than marginal product of capital, so that individuals could both increase steady-state consumption *and consumption during transition* by **decreasing saving rate**.

But no utility function, so statements about inefficiency have to be considered with caution.

Transitional Dynamics

Transitional dynamics

- Recall that the **equilibrium path** is the entire path of capital (and all other variables of interest).
- Key questions: **will the economy tend to the steady state? how will the economy behave during the transition?**

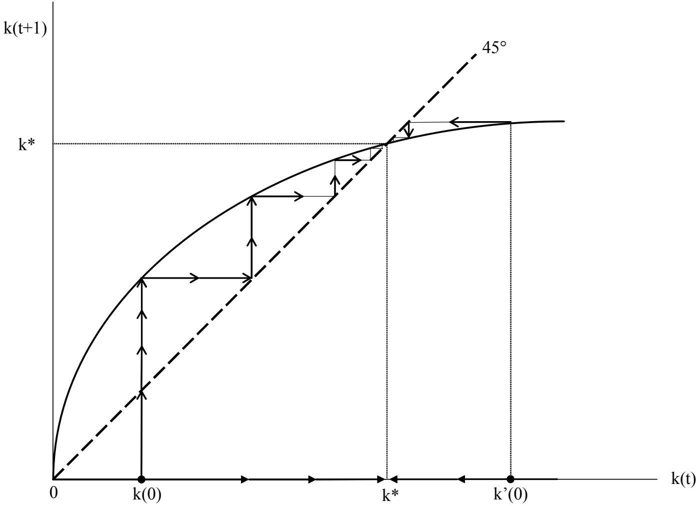
Transitional dynamics

- Recall that the **equilibrium path** is the entire path of capital (and all other variables of interest).

- Key questions: **will the economy tend to the steady state? how will the economy behave during the transition?**

- **Concepts of stability**:
  - **local stability**: the economy converges to the steady state once it is close to it.
  - **global stability**: the economy converges to the steady state regardless of the starting point.

Global stability (for $k_0 > 0$)



Growth in the Solow Model

- The **dynamics of the model** are fully determined by the **evolution of capital**.
- The growth rate of capital is **positive** when $k < k^*$ and **negative** when $k > k^*$:

$$g_t^k \equiv \frac{\dot{k}_t}{k_t} = \frac{sf(k_t) - (\delta + n)k_t}{k_t}$$

$$= s \underbrace{\frac{f(k_t)}{k_t}}_{\substack{\text{decreasing} \\ \text{in k}}} - (\delta + n)$$

Growth in the Solow Model

- The **dynamics of the model** are fully determined by the **evolution of capital**.
- The growth rate of capital is **positive** when $k < k^*$ and **negative** when $k > k^*$:

$$g_t^k \equiv \frac{\dot{k}_t}{k_t} = \frac{sf(k_t) - (\delta + n)k_t}{k_t}$$
$$= s \underbrace{\frac{f(k_t)}{k_t}}_{\substack{\text{decreasing} \\ \text{in k}}} - (\delta + n)$$

- Starting from initial capital stock $k_0 < k^*$, $g_t^k > 0$ and the economy grows towards k^*, **capital accumulation and growth of per capita income**.
- If economy were to start with $k_0 > k^*$, $g_t^k < 0$ and it reaches the steady state by **decumulating capital and contraction of per capita income**.

Growth in the Solow model

- **Implications (country-level)**:
  1. **No growth in the long run**, as $k_{t+1} = k_t = k^*$.
  2. **Growth during the transition**, i.e. as long as $k_t < k^*$.
  3. **Growth rate is declining over time**: g_t^k is monotonously decreasing in k_t.

Growth in the Solow model

- **Implications (country-level)**:
  1. **No growth in the long run**, as $k_{t+1} = k_t = k^*$.
  2. **Growth during the transition**, i.e. as long as $k_t < k^*$.
  3. **Growth rate is declining over time**: g_t^k is monotonously decreasing in k_t.

- **Implication (cross-country)**:
  1. Among similar countries **poor countries should grow faster**.

Technical Progress and Balanced Growth

Technical Progress and Growth

Problems with the previous model:

Technical Progress and Growth

Problems with the previous model:

1. The model predicts a **constant production per capita in the long run**.
2. We observe in most countries, an uninterrupted (although fluctuating) **growth of production per capita**.
3. The **output growth is explained 100 percent by the accumulation of capital**.

Technical Progress and Growth

Problems with the previous model:

1. The model predicts a **constant production per capita in the long run**.
2. We observe in most countries, an uninterrupted (although fluctuating) **growth of production per capita**.
3. The **output growth is explained 100 percent by the accumulation of capital**.

- Taking $f(k) = Ak^{\alpha}$ and log-differentiating it we obtain:

$$\dot{y}_t / y_t = \alpha \dot{k}_t / k_t$$

- What we obtain in fact is something similar to:

$$\dot{y}_t / y_t = \alpha \dot{k}_t / k_t + \dot{z}_t / z_t,$$

where the variable z_t is the Solow residual.

Technical Progress

- What if the **production function $F(K_t, L_t)$ is not stable** in time, but there is a factor that enables more production with the same quantity of factors?
- For instance:

$$Y_t = F(K_t, L_t, t) \quad \frac{\partial F(K_t, L_t, t)}{\partial t} \geq 0$$

Technical Progress

- What if the **production function $F(K_t, L_t)$ is not stable** in time, but there is a factor that enables more production with the same quantity of factors?
- For instance:

$$Y_t = F(K_t, L_t, t) \quad \frac{\partial F(K_t, L_t, t)}{\partial t} \geq 0$$

- One popular way it is to consider a **factor augmenting technical progress**:

$$Y_t = F(B_t K_t, A_t L_t) \quad \dot{A}_t \geq 0 \quad \dot{B}_t \geq 0$$

.

- The idea is that production is a function of **effective labor** and **effective capital**.

Balanced growth

- An equilibrium path consistent with some of the **Kaldor facts**:

Balanced growth

- An equilibrium path consistent with some of the **Kaldor facts**:

  1. The **shares of national income received by labor and capital** are roughly **constant** over long periods of time.

Balanced growth

- An equilibrium path consistent with some of the **Kaldor facts**:

  1. The **shares of national income received by labor and capital** are roughly **constant** over long periods of time.

  2. The **rate of growth of the capital stock per worker** is roughly **constant** over long periods of time.

Balanced growth

- An equilibrium path consistent with some of the **Kaldor facts**:

  1. The **shares of national income received by labor and capital** are roughly **constant** over long periods of time.

  2. The **rate of growth of the capital stock per worker** is roughly **constant** over long periods of time.

  3. The **rate of growth of output per worker** is roughly **constant** over long periods of time.

Balanced growth

- An equilibrium path consistent with some of the **Kaldor facts**:

  1. The **shares of national income received by labor and capital** are roughly **constant** over long periods of time.
  2. The **rate of growth of the capital stock per worker** is roughly **constant** over long periods of time.
  3. The **rate of growth of output per worker** is roughly **constant** over long periods of time.
  4. The **capital/output ratio** is roughly **constant** over long periods of time.

Balanced growth

- An equilibrium path consistent with some of the **Kaldor facts**:

  1. The **shares of national income received by labor and capital** are roughly **constant** over long periods of time.
  2. The **rate of growth of the capital stock per worker** is roughly **constant** over long periods of time.
  3. The **rate of growth of output per worker** is roughly **constant** over long periods of time.
  4. The **capital/output ratio** is roughly **constant** over long periods of time.
  5. The **rate of return on investment** is roughly **constant** over long periods of time.

Balanced growth

- An equilibrium path consistent with some of the **Kaldor facts**:

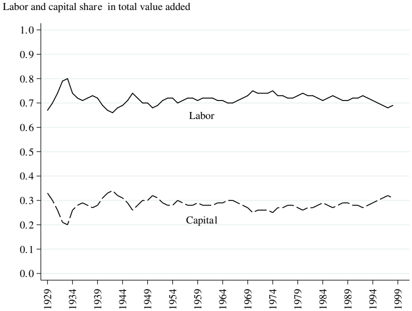
  1. The **shares of national income received by labor and capital** are roughly **constant** over long periods of time.
  2. The **rate of growth of the capital stock per worker** is roughly **constant** over long periods of time.
  3. The **rate of growth of output per worker** is roughly **constant** over long periods of time.
  4. The **capital/output ratio** is roughly **constant** over long periods of time.
  5. The **rate of return on investment** is roughly **constant** over long periods of time.
  6. There are **appreciable variations** (2 to 5 percent) in the **rate of growth of labor productivity and output** among countries.

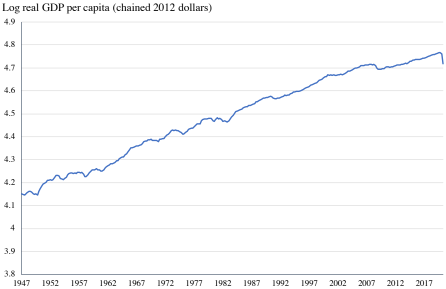
Balanced growth

Some critiques:

- We know that the **capital share of national income is not really constant**, and has been **increasing over the last 30 years or so** (President Biden's Tax Plan argues that corporate taxes should be raised to address a declining share of national income accruing to labor `https://fred.stlouisfed.org/series/LABSHPUSA156NRUG`).
- Nevertheless, its **relative constancy** for **almost a century** might be an argument in favor of Kaldor facts.
- More importantly, **balanced growth is a simple set of properties of the equilibrium path**.

Balanced growth in the US





Balanced growth in the US

- In the US:
  - $\approx$ **constant factor shares**.
  - $\approx$ **constant growth of real GDP per capita**.
- Attributes of balanced growth.

The augmented Solow model to generate a balanced growth path

- Positive, constant, **exogenous population growth and technological progress**.
- To have a model that matches the Kaldor facts, we need **labor-augmenting** (also know as Harrod-neutral) technology

$$F(K_t, L_t, A_t) = F(K_t, A_t L_t)$$

The augmented Solow model to generate a balanced growth path

- Positive, constant, **exogenous population growth and technological progress**.
- To have a model that matches the Kaldor facts, we need **labor-augmenting** (also know as Harrod-neutral) technology

$$F(K_t, L_t, A_t) = F(K_t, A_t L_t)$$

- This is because this **form of technology gives a constant capital-output ratio**.
- Model in continuous time:

$$\frac{\dot{L}}{L} = n.$$
$$\frac{\dot{A}}{A} = \gamma_a.$$

From discrete to continuous time

	discrete	continuous
rate of change:	$\Delta X_t = X_{t+1} - X_t$	$\dot{X}(t) = \frac{dX(t)}{dt}$
growth rate:	$\frac{\Delta X_t}{X_t}$	$\frac{\dot{X}(t)}{X(t)}$

From discrete to continuous time

	discrete	continuous
rate of change:	$\Delta X_t = X_{t+1} - X_t$	$\dot{X}(t) = \frac{dX(t)}{dt}$
growth rate:	$\frac{\Delta X_t}{X_t}$	$\frac{\dot{X}(t)}{X(t)}$

- Discrete $\rightarrow$ continuous growth:

$$\begin{aligned}\lim_{\Delta t \to 0}\left(\frac{X_{t+\Delta t} - X_t}{\Delta t} \cdot \frac{1}{X_t}\right) &= \frac{dX(t)}{dt} \cdot \frac{1}{X(t)} \\ &= \frac{\dot{X}(t)}{X(t)}.\end{aligned}$$

- Note that $\frac{d \ln X(t)}{dt} = \frac{d \ln X(t)}{dX(t)} \cdot \frac{dX(t)}{dt} = \frac{1}{X(t)} \cdot \frac{dX(t)}{dt} = \frac{\dot{X}(t)}{X(t)}$.

Back to the model

- Nothing has changed on the production side:
  $w_t = MPL$, $r_t = MPK$, $\pi_t = 0$.
- People still save s fraction of their income, so $S_t = sY_t$.
- The capital accumulation equation is now:

$$\dot{K}_t = sF(K_t, A_t L_t) - \delta K_t.$$

Back to the model

- Nothing has changed on the production side:
  $w_t = MPL$, $r_t = MPK$, $\pi_t = 0$.
- People still save s fraction of their income, so $S_t = sY_t$.
- The capital accumulation equation is now:

$$\dot{K}_t = sF(K_t, A_t L_t) - \delta K_t.$$

- Let $\widehat{k}(t)$ be capital **per unit of effective labor** (or effective capital-labor ratio):
  ...

Back to the model

- Nothing has changed on the production side:
  $w_t = MPL$, $r_t = MPK$, $\pi_t = 0$.
- People still save s fraction of their income, so $S_t = sY_t$.
- The capital accumulation equation is now:

$$\dot{K}_t = sF(K_t, A_t L_t) - \delta K_t.$$

- Let $\widehat{k}(t)$ be capital **per unit of effective labor** (or effective capital-labor ratio):
  ...

$$\widehat{k}_t = \frac{K_t}{A_t L_t},$$

$$\dot{\widehat{k}}_t = \frac{\dot{K}_t}{A_t L_t} - (\dot{A}_t L_t + A_t \dot{L}_t)\frac{K_t}{(A_t L_t)^2} = \frac{\dot{K}_t}{A_t L_t} - \gamma_A \widehat{k}_t - n\widehat{k}_t.$$

- Dividing by $\widehat{k}_t$: $\frac{\dot{\widehat{k}}_t}{\widehat{k}_t} = \frac{\dot{K}_t}{K_t} - \gamma_A - n$.

Back to the model

- **Output per unit of effective labor** can be expressed as

$$\widehat{y}_t \equiv \frac{Y_t}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, 1\right) \equiv f(\widehat{k}_t).$$

Back to the model

- **Output per unit of effective labor** can be expressed as

$$\widehat{y}_t \equiv \frac{Y_t}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, 1\right) \equiv f(\widehat{k}_t).$$

- Using that $\dot{K}_t = sF(K_t, A_t L_t) - \delta K_t$ to substitute:

$$\begin{aligned}
\frac{\dot{\widehat{k}}_t}{\widehat{k}_t} &= \frac{sF(K_t, A_t L_t) - \delta K_t}{K_t} - \gamma_A - n \\
&= \frac{sF(K_t, A_t L_t)}{K_t} - (\delta + \gamma_A + n) \\
&= \frac{sf(\widehat{k}_t)}{\widehat{k}_t} - (\delta + \gamma_A + n). \\
&\Leftrightarrow \\
\dot{\widehat{k}}_t &= sf(\widehat{k}_t) - (\delta + \gamma_A + n)\widehat{k}_t.
\end{aligned}$$

- Same structure as the baseline model, with **effective depreciation rate**.

Balanced growth and steady state

- **Steady state** at which the **capital per effective unit of labor is constant** (not at which capital per capita is constant) is given by

$$\frac{f(\widehat{k}^*)}{\widehat{k}^*} = \frac{\delta + \gamma_A + n}{s}.$$

- **Existence, uniqueness, stability** same as before
- Note that

$$\begin{aligned} w_t &= A_t F_L(K_t, A_t L_t), \\ &= f(\widehat{k}_t)A_t - f'(\widehat{k}_t)\widehat{k}_t A_t. \end{aligned}$$

Balanced growth and steady state

- Thus in this **steady state**:

$$y_t = \frac{Y_t}{L_t} = A_t f(\widehat{k}^*),$$
$$w_t = A_t[f(\widehat{k}^*) - f'(\widehat{k}^*)\widehat{k}^*].$$

  wages and income per capita grow at rate γ_A.
- $Y_t = L_t y_t$ *grows at rate* $\gamma_A + n$
- This steady state is also called a **balanced growth path**.

Balanced Growth

- Consider the model with **labor augmenting** type in the form:

$$Y_t = F\left(K_t, A_t L_t\right) \quad \frac{\dot{A}_t}{A_t} = \gamma_A.$$

- It is not necessary to redo all the analyses. The only thing we need to do is to set up the following variable $k_t \equiv \dfrac{K_t}{A_t L_t}$ and replace n by $n + \gamma_A$.
- Redo the analyses for this case and **show that in the steady state the per capita variables will grow at the rate γ_A**.

Convergence and Divergence in the Growth Model

Convergence

- Does the Solow model predicts **convergence among economies**?

Convergence

- Does the Solow model predicts **convergence among economies**?
- Consider a set of countries indexed by i, then the formula we derived will be:

$$\frac{\dot{k}_{it}}{k_{it}} = \frac{s_i f\left(k_{it}\right)}{k_{it}} - \left(\delta_i + n_i\right),$$

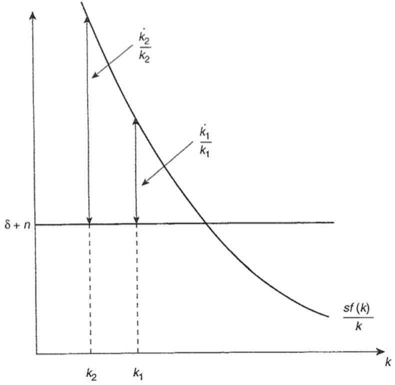
  and assume all countries have the production function: $f\left(k_{it}\right) = A_i k_i^{\alpha}$.
- Therefore, the **rate of growth** will be given by:

$$\frac{\dot{k}_{it}}{k_{it}} = s_i A_i k_{it}^{\alpha-1} - \left(\delta_i + n_i\right),$$

  and the dynamics of each country will thus be characterized by a vector $z_i = \left(s_i, A_i, \dot{\delta}_i, n_i\right)$.

Absolute convergence

- If all the countries have the same **fundamental parameters** $z_i = z$, then there is **absolute convergence**, which means that countries with lower levels of capital per capita will also have higher rates:



Conditional convergence

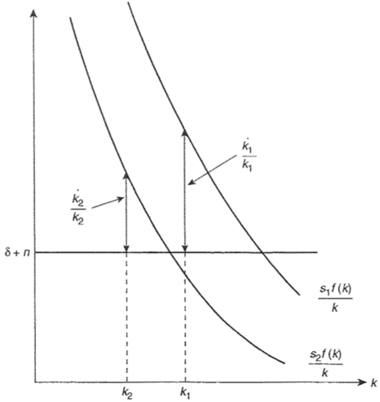
- Consider countries with the same δ, A and n but with **different saving rates**:

  ...

Conditional convergence

- Consider countries with the same δ, A and n but with **different saving rates**:

  ...



-

A Model with two accumulated factors

A Model with two accumulated factors

- Assume right now that there are two accumulated factors, **physical and human capital**.
- The model will be given in the following way, where L_t will be raw labor, which evolution is exogenous as before:

$$Y_t = K_t^{\alpha} H_t^{\beta} \left(A_t L_t\right)^{1-\alpha-\beta},$$

with $\frac{\dot{A}_t}{A_t} = \gamma_A$.
- The **accumulation equations** will be given by:

$$\dot{K}_t = s_K Y_t - \delta K_t, \tag{9}$$
$$\dot{H}_t = s_H Y_t - \delta H_t. \tag{10}$$

- Define the **quantities per unit of effective raw labor**:

$$y_t = \frac{Y_t}{A_t L_t}, \quad k_t = \frac{K_t}{A_t L_t}, \quad h_t = \frac{H_t}{A_t L_t}.$$

A Model with two accumulated factors

1. Rewrite equations (9) and (10) using the dynamics and in terms of effective labor, i.e., $\dot{k}_t$ and $\dot{h}_t$.
2. Compute the steady-state values.
3. Insert the two values found in the income per capita in the steady-state and show you can find the following expression:

$$\log\left(\frac{Y_t}{L_t}\right) = \log A_0 + \gamma_A t + \frac{\alpha}{1-\alpha-\beta}\left[\log s_K - \log\left(n+\gamma_A+\delta\right)\right] + \frac{\beta}{1-\alpha-\beta}\left[\log s_H - \log\left(n+\gamma_A+\delta\right)\right].$$

4. If you had to estimate this equation for a panel of countries, how would you proceed? What is the approximate value you would think to find in the data for α and β?

Conclusions

Conclusions about the Solow model

- **Canonical model**:
  - **No long run growth**: capital accumulation alone does not make an economy grow in the long run.
  - **Conditional convergence**: if countries share $s, \delta, A, F(\cdot)$, they will converge to the same income per capita (and to the same steady state).

Conclusions about the Solow model

- **Canonical model**:
  - **No long run growth**: capital accumulation alone does not make an economy grow in the long run.
  - **Conditional convergence**: if countries share $s, \delta, A, F(\cdot)$, they will converge to the same income per capita (and to the same steady state).
  - Countries further from the steady state, i.e. **poorer countries will grow faster**.
  - **Crucial mechanism**: decreasing marginal product of capital.

Conclusions about the Solow model

- **Canonical model**:
  - **No long run growth**: capital accumulation alone does not make an economy grow in the long run.
  - **Conditional convergence**: if countries share $s, \delta, A, F(\cdot)$, they will converge to the same income per capita (and to the same steady state).
  - Countries further from the steady state, i.e. **poorer countries will grow faster**.
  - **Crucial mechanism**: decreasing marginal product of capital.
- **Extended model, with productivity and population growth**:
  - **Long run growth possible** if productivity grows, $\gamma_A > 0$.
  - With **constant growth**, model is consistent with balanced growth.

Conclusions about the Solow model

- **Canonical model**:
  - **No long run growth**: capital accumulation alone does not make an economy grow in the long run.
  - **Conditional convergence**: if countries share $s, \delta, A, F(\cdot)$, they will converge to the same income per capita (and to the same steady state).
  - Countries further from the steady state, i.e. **poorer countries will grow faster**.
  - **Crucial mechanism**: decreasing marginal product of capital.
- **Extended model, with productivity and population growth**:
  - **Long run growth possible** if productivity grows, $\gamma_A > 0$.
  - With **constant growth**, model is consistent with balanced growth.

$\rightarrow$ s and γ_A, which are the most interesting, are **exogenous**.

$\rightarrow$ Understand what drives **saving rate and technological progress**.

$\rightarrow$ This is a **simple theory** that can be confronted with the data.

Growth and convergence in the Solow model

- Solow model contains two sources of growth:
  1. **Transitional growth** due to **capital accumulation – endogenous**.
  2. **Balanced growth** due to **technology – exogenous**.

- It is important to distinguish these two:
  - When interpreting cross-country data.
  - And to answer questions such as: will China catch up to the US (in output per capita)?

- Solow model gives a good framework to answer these questions.