Macroeconomics I

Recitation 1

Diego Rodrigues*

Fall 2023

1 First-Order Ordinary Differential Equations

A differential equation is an equation that features derivatives. It is called ordinary differential equation (ODE) if there is only one independent variable. The equations that we will deal with in the class usually have the following form:

$$\dot{y}(t) = f[y(t), t] \tag{1}$$

where $y(t)$ can be scalar or column vector, and where the dot on top of $y(t)$ denotes the derivative with respect to time t. This equation is a first-order ODE since we only have first-order derivatives. An ODE is said to be autonomous when f does not depend on t.

Example 1 *The following expression is an autonomous ODE:*

$$\dot{y}(t) = f[y(t)] = a \times y(t) - x$$

For reasons that will become clear later in the semester, macroeconomics is often preoccupied with the steady state.

Definition 1 (Steady state) *A steady state of an ordinary differential equation is a point where $\dot{y}(t) = 0$.*

*Sciences Po, Department of Economics.

1.1 Graphical Solutions

1.1.1 The Linear Case

We can solve Example 1 with a simple chart. On figure 1, the dashed line represents $\dot{y}(t)$ as a function of $y(t)$. $y^* = x/a$ is the steady state. The slope of the dotted line depends on the sign of a. On the first subplot of Figure 1, $a > 0$ so the relationship is increasing. As a result, when $y(t)$ is to the right of y^*, we have:

$$\dot{y}(t) = a \times y(t) - x > a \times \frac{x}{a} - x = 0$$

$\dot{y}(t)$ is positive, which means that $y(t)$ grows and goes further from y^*. Similarly, when $y(t)$ is to the left of y^*, it becomes smaller, thus moving away from y^*. When a solution exhibits this behavior, the equation is called unstable: unless we start exactly at y^*, $y(t)$ never converges. This behavior is summarized by the arrows that are drawn on the chart. The behavior of $y(t)$ is different when $a < 0$: if $y(t)$ starts above y^*, $y(t)$ decreases and vice versa. Wherever we start, we end up at the steady state: the equation is stable.

1.1.2 A Non-Linear Case

We can now switch to the less trivial example of a non-linear function:

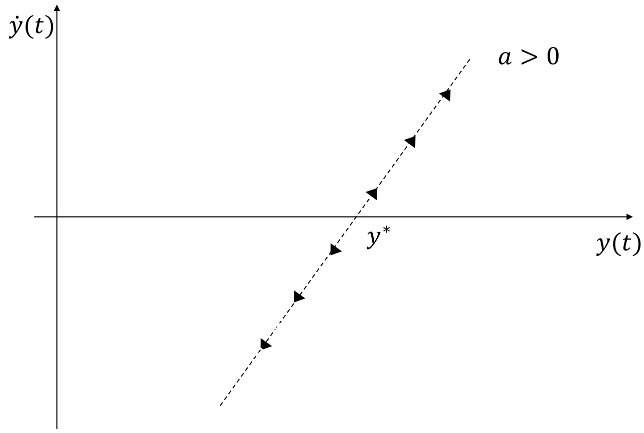
Example 2 *The following expression is a non-linear ODE:*

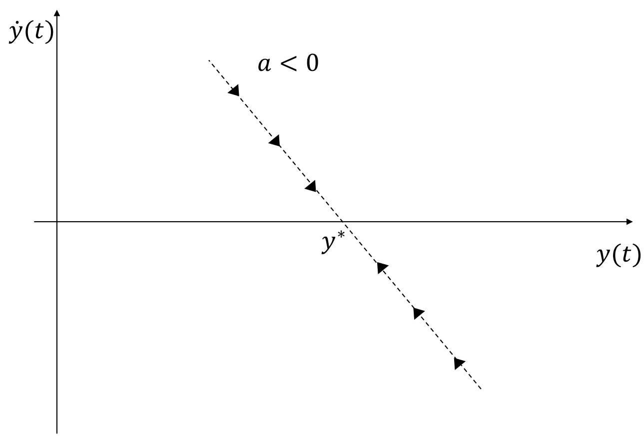
$$\begin{aligned}\dot{k}(t) = f[k(t)] = s \times k(t)^\alpha - \delta \times k(t) \\ 0 < \alpha < 1 \\ \delta > 0 \\ s > 0\end{aligned}$$

You may have recognized the equation that describes capital accumulation in the Solow growth model. Deriving f twice:

$$\begin{aligned}f'[k(t)] &= \alpha \times s \times k(t)^{\alpha-1} - \delta \\ f''[k(t)] &= \alpha(\alpha - 1) \times s \times k(t)^{\alpha-1}\end{aligned}$$

Figure 1: Example 1





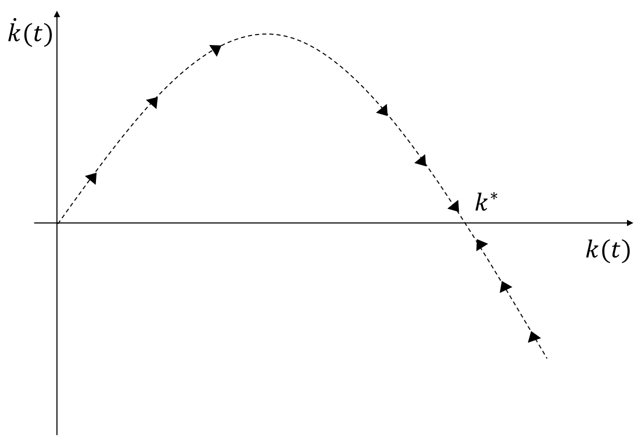
Also note that:

$$\lim_{k(t)\to 0} f'[k(t)] = +\infty$$

$$\lim_{k(t)\to +\infty} f'[k(t)] = -\delta$$

Since $f''[k(t)] < 0$, f' is strictly decreasing. Together with its limits in 0 and $+\infty$, this means that f' is first positive and then negative. Therefore f is first increasing and then decreasing. Remarking that $f[0] = 0$ and $\lim_{k(t)\to+\infty} f'[k(t)] = -\infty$, we can draw Figure 2. Using the same convention, k^* is the steady state. Since f is decreasing around k^*, $k(t)$ is increasing to the left of k^*, and decreasing to the right of k^*. The equation is stable at k^*: whatever $k_0 > 0$, we end up at k^*.

Figure 2: Example 2



A common theme in these two examples is the sign of $\partial \dot{y}/\partial y$ at steady state. It determines whether the steady state is stable or not.

Summary 1 (Stability) *If $\partial \dot{y}/\partial y|_{y^*} > 0$, y is locally unstable. If $\partial \dot{y}/\partial y|_{y^*} < 0$, y is locally stable.*

1.2 Analytical Solutions

1.2.1 General Method

In this section, we will study how to analytically solve linear first-order ODE with constant coefficients:

$$\dot{y}(t) + a \times y(t) + x(t) = 0 \tag{2}$$

First, put the y terms on the left hand side, the other on the right hand side:

$$\dot{y}(t) + a \times y(t) = -x(t)$$

Multiply by e^{at}:

$$e^{at}\left[\dot{y}(t) + a \times y(t)\right] = -e^{at}x(t)$$

Integrate over t between 0 and T (T thus becomes the time variable, t is the integration one):

$$\int_0^T e^{at}\left[\dot{y}(t) + a \times y(t)\right] dt = -\int_0^T e^{at} x(t)\, dt$$

Remark that the term in the integral is the time derivative of $e^{at}y(t)$, plus some time-independent term b_0:

$$\frac{d\left[e^{at}y(t) + b_0\right]}{dt} = a \times e^{at}y(t) + e^{at}\dot{y}(t)$$

The left-hand side integral is therefore solved:

$$e^{aT}y(T) + b_0 = -\int_0^T e^{at}x(t)\, dt$$

Rearranging this expression yields the general solution of the equation:

$$y(T) = -e^{-aT}\int_0^T e^{at}x(t)\, dt - e^{-aT}b_0 \tag{3}$$

$x(t)$ is a known function of time, so the integral is known as well (though it does not necessarily have an analytical expression). The only quantity that is still unknown here is b_0. We call Equation (3) the general solution of the differential equation. In applications, b_0 is going to be pinned down by some initial or terminal condition. A specific b_0 will give a particular solution of the differential equation.

1.2.2 An Example

Example 3 *Consider the following ODE:*

$$\dot{y}(t) - y(t) - 1 = 0$$

This is a particular case of Equation (2) with $a = -1$ and $x(t) = -1$. We could plug these values in the general solution (Equation (3)). I would rather have you be able to derive this solution. Put terms that do not involve $y(t)$ on the right-hand side and multiply by e^{-t}:

$$e^{-t}\left[\dot{y}(t) - y(t)\right] = e^{-t}$$

Integrate from 0 to T:

$$\int_0^T e^{-t}\left[\dot{y}(t) - y(t)\right]\, dt = \int_0^T e^{-t}\, dt$$

Solve the left-hand side:

$$e^{-T} y(T) + b_0 = \int_0^T e^{-t}\, dt$$

Rearrange and compute the remaining integral:

$$\begin{aligned} y(T) &= e^T \int_0^T e^{-t}\, dt - e^T b_0 \\ &= e^T(-e^{-T} + 1) - e^T b_0 \\ &= -1 + e^T(1 - b_0) \end{aligned}$$

Once again, there is no way to determine b_0 with the information that we presently have. Now suppose that we knew that $y(0) = 0$, we could solve for b_0:

$$0 = -1 + e^0(1 - b_0) \Rightarrow b_0 = 0$$

This is called an initial condition. We could also have a terminal condition. For instance, if we know that $\lim_{T\to+\infty} y(T)$ is finite, it implies that:

$$1 - b_0 = 0 \Rightarrow b_0 = 1$$

1.2.3 Time-Dependant Coefficients

We can complicate this problem slightly by having the coefficient a depend on t:

$$\dot{y}(t) + a(t) \times y(t) + x(t) = 0$$

The procedure is identical except that we now multiply by $\exp\left(\int_0^t a(\tau)\, d\tau\right)$. The primitive of the left-hand side is $y(t) \times \exp\left(\int_0^t a(\tau)\, d\tau\right)$. The final solution has the form:

$$y(T) = -\exp\left(-\int_0^t a(\tau)\, d\tau\right) \int_0^T \left[\exp\left(\int_0^t a(\tau)\, d\tau\right) x(t)\right] dt - b_0 \exp\left(-\int_0^t a(\tau)\, d\tau\right)$$

Notice that if $a(\tau)$ is a constant, this expression collapses to Equation (3).

2 Systems of Linear Ordinary Differential Equations

We will now study systems of linear ODE of the form:

$$\begin{cases} \dot{y}_1(t) = a_{11}y_1(t) + ... + a_{1n}y_n(t) + x_1(t) \\ ... \\ \dot{y}_n(t) = a_{n1}y_1(t) + ... + a_{nn}y_n(t) + x_n(t) \end{cases}$$

The latter can be rewritten in matrix form:

$$\dot{y}(t) = A.y(t) + x(t) \tag{4}$$

2.1 Phase Diagram

The phase diagram is a very common device to solve a system of two ODE with two variables. It will show up repeatedly during the class, so it is important that you master it. I will start with the study of a simple example, to then expose a more general method.

Example 4 *Consider the following system of linear ODE:*

$$\dot{y}_1(t) = .06y_1(t) - y_2(t) + 1.4 \tag{5}$$

$$\dot{y}_2(t) = -.004y_1(t) + 0.04 \tag{6}$$

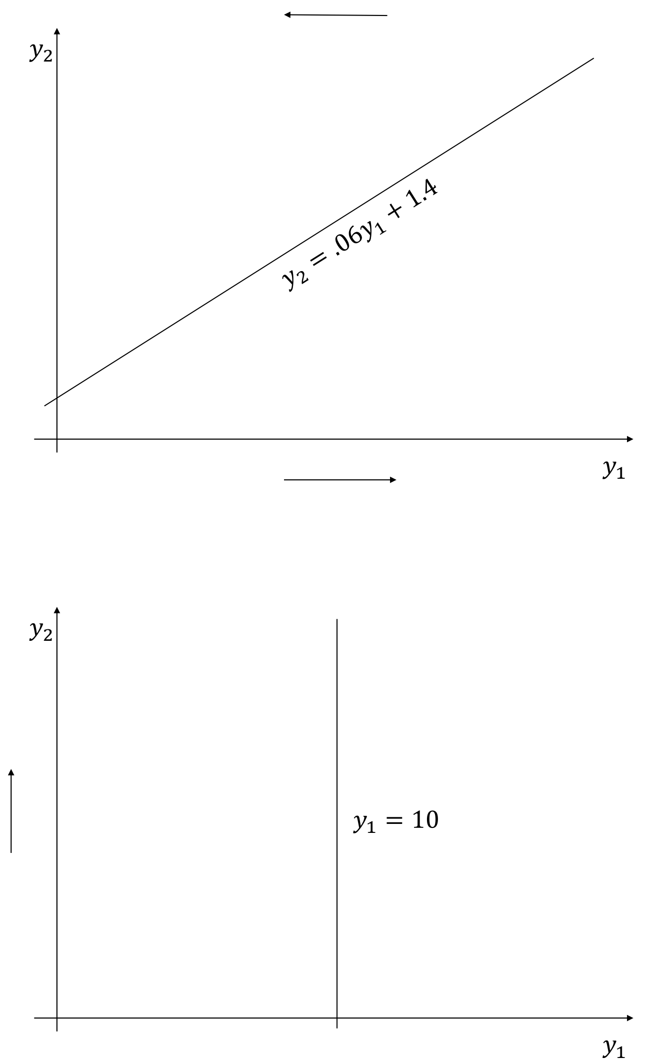
By definition, in steady state, we have: $\dot{y}_1(t) = 0$ and $\dot{y}_2(t) = 0$. Each of these conditions defines a locus where one of the variables is constant. Using Equations (5) and (6), the $\dot{y}_1(t) = 0$ and $\dot{y}_2(t) = 0$ loci can be written:

$$y_2(t) = .06y_1(t) + 1.4 \tag{7}$$

$$y_1(t) = 10 \tag{8}$$

Equation (7) can be drawn in the (y_1, y_2) space (first plot of Figure 3). Note that if we are above this line, that is if $y_2(t) > .06y_1(t) + 1.4$, Equation (5) tells us that $\dot{y}_1(t) < 0$. If we are below, $\dot{y}_1(t) > 0$. I represent this with two arrows: for any point that is above the line, $y_1(t)$ gets smaller, for any point that is below, it gets bigger. We can do the same with Equation (8) (second plot of Figure 3). This time the line is vertical. To its left, $y_2(t)$ increases; to its right, it decreases.

Figure 3: Example 4, loci



In the first plot of Figure 4, I put together these pictures. The (y_1, y_2) space is now divided into four parts:

- in the NW (North-West) quadrant, we are above the $\dot{y}_1(t) = 0$ line and to the left of the $\dot{y}_2(t) = 0$ one. $y_1(t)$ gets smaller and $y_2(t)$ gets larger: we move away from the steady state;
- in the NE quadrant, we are above the first line and to the right of the second one. $y_1(t)$ and $y_2(t)$ get smaller: we *might* move toward the steady state;
- in the SE quadrant, we are below the first line and to the right of the second one. $y_1(t)$ gets larger and

$y_2(t)$ gets smaller: we move away from the steady state;

- in the SW quadrant, we are below the first line and to the left of the second one. $y_1(t)$ and $y_2(t)$ get larger: we *might* move toward the steady state.

At this point, you should admit the following proposition[1]: the points which take us to the steady state are a line in the (y_1, y_2) space. This line is called the stable arm of the saddle path. Starting anywhere below or above the stable arm would eventually take us away from the steady state. Why is that? Because if we do not start on the stable arm, the system drifts toward the NW or SE quadrants, at which point it irremediably diverges away from the steady state. This behavior is summarized by the second plot of Figure 4. The system is said to be saddle-path stable.

The phase diagram technique also works with non-linear functions:

Example 5 *Consider the following system of non-linear ODE:*

$$\dot{k}(t) = k(t)^{.3} - c(t)$$
$$\dot{c}(t) = c(t)\left[.3 \times k(t)^{-.7} - .06\right]$$

The loci are:

$$\dot{k}(t) = 0 \Rightarrow c(t) = k(t)^{.3}$$
$$\dot{c}(t) = 0 \Rightarrow k(t) = \left(\frac{.06}{.3}\right)^{-1/.7} = 5^{1/.7}$$

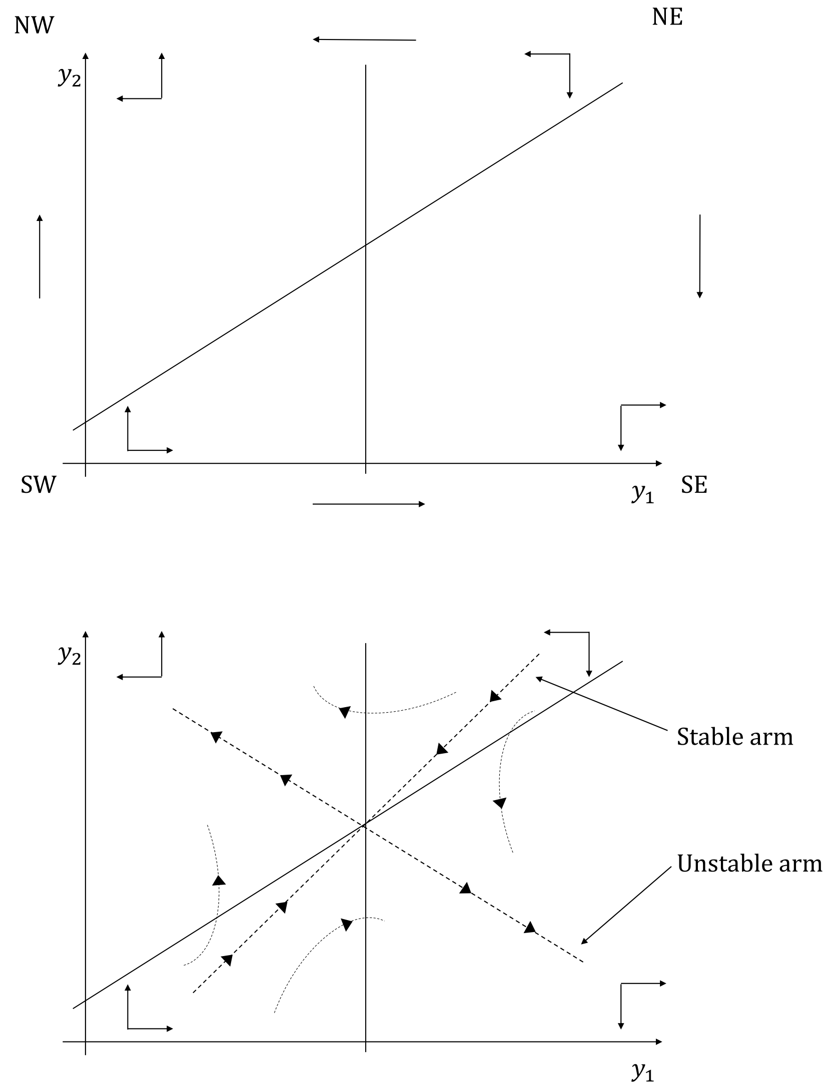
The phase diagram is drawn in Figure 5. Note that neither the $\dot{k} = 0$ locus, nor the saddle path are linear this time.

Summary 2 (Phase diagram) *To draw the phase diagram of a system of two ODE:*

- *draw the* $\dot{y}_1 = 0$ *and* $\dot{y}_2 = 0$ *loci in the* (y_1, y_2) *space;*
- *determine the direction of* $y_1(t)$ *and* $y_2(t)$ *in each of the four quadrants;*
- *draw the stable and unstable arms of the saddle path (if they exist).*

[1] We will see a proof in Example 6.

Figure 4: Example 4, the saddle path



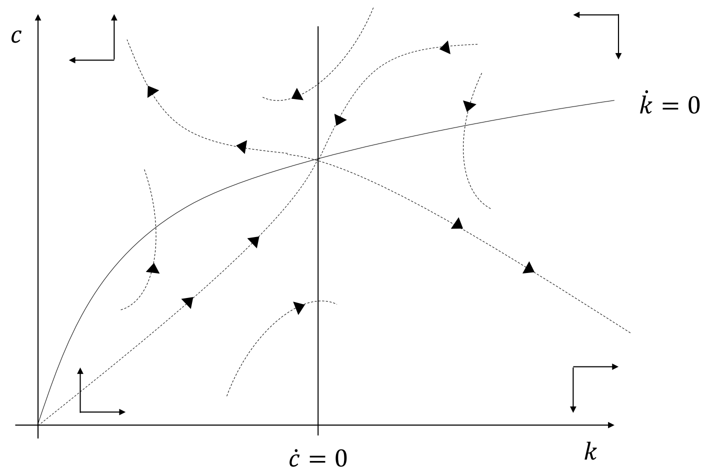
2.2 Analytical Solutions

Consider the system described by Equation 4:

$$\dot{y}(t) = A.y(t) + x(t)$$

If A is diagonalizable, we can write it as VDV^{-1}, where D is a diagonal matrix and V an invertible matrix. This implies that: $D = V^{-1}AV$. The coefficients of D are the eigenvalues of A, and the columns of V are the associated eigenvectors. We are going to work with a transformation of $y(t)$, $z(t) = V^{-1}y(t)$. Notice

Figure 5: Example 5



that:

$$\dot{z}(t) = V^{-1}\dot{y}(t) = V^{-1}(Ay(t) + x(t)) = V^{-1}AVV^{-1}y(t) + V^{-1}x(t) = Dz(t) + V^{-1}x(t)$$

The latter equation is in matrix form. Each of its rows is an ODE of the form:

$$\dot{z}_i(t) = \alpha_i z_i(t) + V_i^{-1}x(t) \tag{9}$$

where the α_i are the diagonal coefficients of D (equivalently, the eigenvalues of A) and the V_i^{-1} the rows of matrix V^{-1}. Each equation involves a single unknown variable ($z_i(t)$): the equations can be solved separately. Moreover, each equation is a version of Equation (2) with $a = -\alpha_i$ and $-V_i^{-1}x(t)$ instead of $x(t)$. We can immediately apply Equation (3):

$$z_i(T) = e^{\alpha_i T}\int_0^T e^{-\alpha_i t}V_i^{-1}x(t)\,dt + e^{\alpha_i T}b_i$$

where b_i is some constant that does not depend on time. $z(T)$ is solved up to the b_i, which will be solved by looking at the initial or terminal conditions. Finally, we can recover $y(T)$ by applying the inverse transform: $y(T) = V.z(t)$.

Let's illustrate this by focusing on the 2×2 case:

Example 6 *Consider the* 2×2 *case:*

$$\dot{y}_1(t) = a_{11}y_1(t) + a_{12}y_2(t)$$
$$\dot{y}_2(t) = a_{21}y_1(t) + a_{22}y_2(t)$$

Note that as long as A is invertible, the single steady state is $(0, 0)$. Now, after remarking that $x(t) = 0$ in this case, apply Equation (9):

$$\dot{z}_1(t) = \alpha_1 z_1(t)$$
$$\dot{z}_2(t) = \alpha_2 z_2(t)$$

From Equation (3), the solution has the form:

$$z_1(t) = b_1 e^{\alpha_1 t}$$
$$z_2(t) = b_2 e^{\alpha_2 t}$$

where b_1 and b_2 are two undetermined constants. And therefore:

$$y_1(t) = V_1.z(t) = v_{11}b_1e^{\alpha_1 t} + v_{12}b_2e^{\alpha_2 t}$$
$$y_2(t) = V_2.z(t) = v_{21}b_1e^{\alpha_1 t} + v_{22}b_2e^{\alpha_2 t}$$

There are several possibilities:

- both eigenvalues are real and positive: the system is unstable;
- both eigenvalues are real and negative: the system is stable;
- both eigenvalues are real, one is positive and the other one negative: the system is saddle path stable;
- eigenvalues are complex: the system can exhibit cyclical behavior.

To prove the first two, just send t to infinity. If you want to learn more about the fourth one, I invite you to have a look at the textbook Barro and Sala-i Martin (Barro and Sala-i Martin, pp. 587-588). In classroom situations, I have never encountered anything else than the third one, on which I will focus.

Suppose that $\alpha_1 > 0$ and $\alpha_2 < 0$. A solution which does not go to infinity over time requires $b_1 = 0$, otherwise $y(t)$ cannot converge. This is the stable arm of the saddle path that we drew on Figure 4. Symmetrically, the unstable arm is characterized by $b_2 = 0$. Note that on both of these arms:

$$y_2(t) = v_{2i} b_i e^{\alpha_i t} = \frac{v_{2i}}{v_{1i}} v_{1i} b_i e^{\alpha_i t} = \frac{v_{2i}}{v_{1i}} y_1(t)$$

This justifies our drawing the saddle paths as lines in the linear case (Figure 4).

Let's now pick a solution by adding some boundary conditions:

$$\begin{aligned} y_1(0) &= c \\ \lim_{t \to \infty} e^{-\lambda t} y_1(t) &= 0 \\ \alpha_2 < \lambda &< \alpha_1 \end{aligned}$$

Now:

$$\begin{aligned} e^{-\lambda t} y_1(t) = v_{11} b_1 e^{(\alpha_1 - \lambda)t} + v_{12} b_2 e^{(\alpha_2 - \lambda)t} \\ \alpha_1 - \lambda > 0 \\ \alpha_2 - \lambda < 0 \end{aligned}$$

$e^{-\lambda t} y_1(t)$ goes to infinity unless $b_1 = 0$. Therefore the terminal condition pins down the solution on the stable arm of the saddle path: $b_1 = 0$. The initial condition is just going to help us find the point we start from on this line:

$$y_1(0) = c \Rightarrow b_2 = \frac{c}{v_{12}}$$

3 Linearization of Non-Linear Systems

3.1 Method

Consider the following system of non-linear ODE:

$$\begin{cases} \dot{y}_1(t) = f^1 \left[y_1(t), ..., y_n(t) \right] \\ ... \\ \dot{y}_n(t) = f^n \left[y_1(t), ..., y_n(t) \right] \end{cases}$$

The f^i, $1 \le i \le n$, can be non-linear functions of $y(t)$.

To solve this system, we conduct a first-order Taylor expansion around the relevant steady state y^*. That is, for every i, we write:

$$\dot{y}_i(t) = f^i(\bullet) + (f^i)_{y_1}(\bullet) \times (y_1(t) - y_1^*) + ... + (f^i)_{y_n}(\bullet) \times (y_n(t) - y_n^*) + R_i$$

where $f^i(\bullet)$ is the value of f^i at steady state, $(f^i)_{y_k}(\bullet)$ the value at steady state of the partial derivative of f^i with respect to y_k, and R_i the residual of the Taylor expansion. By definition of the steady state:

$$f^i(\bullet) = \dot{y}_i(t)|_{y(t)=y^*} = 0$$

So the equations simplify to:

$$\begin{aligned}\dot{y}_i(t) &= (f^i)_{y_1}(\bullet) \times (y_1(t) - y_1^*) + ... + (f^i)_{y_n}(\bullet) \times (y_n(t) - y_n^*) + R_i \\ &\approx (f^i)_{y_1}(\bullet) \times (y_1(t) - y_1^*) + ... + (f^i)_{y_n}(\bullet) \times (y_n(t) - y_n^*)\end{aligned}$$

We can now write the system as:

$$\dot{y}(t) \approx A.(y(t) - y^*)$$

where:

$$A = \begin{bmatrix} (f^1)_{y_1}(\bullet) & ... & (f^1)_{y_n}(\bullet) \\ ... & & \\ (f^n)_{y_1}(\bullet) & ... & (f^n)_{y_n}(\bullet) \end{bmatrix}$$

This linear system can be solved with the method described in Section 2.2.

3.2 Example

Back to Example 5:

$$\begin{aligned}\dot{k}(t) &= k(t)^{.3} - c(t) \\ \dot{c}(t) &= c(t)\left[.3 \times k(t)^{-.7} - .06\right]\end{aligned}$$

Solving for the steady state:

$$k^* = \left(\frac{.06}{.3}\right)^{-1/.7} = 5^{1/.7} \approx 10$$
$$c^* = (k^*)^{.3} = 5^{.3/.7} \approx 2$$

We can Taylor expand these equations:

$$\dot{k}(t) = .3(k^*)^{-.7} \times (k(t) - k^*) - (c(t) - c^*)$$
$$\dot{c}(t) = c^* \times .3(-.7)(k^*)^{-1.7} \times (k(t) - k^*) - \left[.3 \times (k^*)^{-.7} - .06\right] \times (c(t) - c^*)$$

Note that by definition of k^*, $.3 \times (k^*)^{-.7} - .06 = 0$; so that the second equation becomes:

$$\dot{c}(t) = c^* \times .3(-.7)(k^*)^{-1.7} \times (k(t) - k^*)$$

Plugging in the values k^* and c^*:[2]

$$\dot{k}(t) \approx .06k(t) - c(t) + 1.4$$
$$\dot{c}(t) \approx -.008k(t) + .08$$

Solving this system with the method of Section 2.2 is left as an exercise. Barro and Sala-i Martin (Barro and Sala-i Martin, pp. 590-591) have an almost identical example.

References

Barro, R. J. and X. Sala-i Martin. Economic growth (second ed.). *Cambridge: MIT Press*.

[2]Values are rounded.