Macroeconomics 1

Lecture - Overlapping generations model

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Introduction

Moving away from the Representative agent model

- **Representative agent implies a disregard for life-cycle motives for saving**, i.e., saving for old age.
- All agents are the same and age does not affect saving decisions.
- How would allowing for some life-cycle motives for savings change the conclusions we had before?

Overlapping generations model

- We introduce a discrete-time model developed by Allais (1947), Samuelson (1958), and Diamond (1965).
- **Households** are not pictured as a single dynasty but as a **sequence of overlapping families**, each one with its own utility and budget constraint.

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- We introduce a discrete-time model developed by Allais (1947), Samuelson (1958), and Diamond (1965).
- **Households** are not pictured as a single dynasty but as a **sequence of overlapping families**, each one with its own utility and budget constraint.

- Each family lives for **two periods and overlaps with the family from the previous period (when young) and the next one (when old)**.
- This setting has an equilibrium that could **Pareto dominated the market one** and the failure of the Ricardian equivalence.

Why is this interesting?

- New economic interactions: **decisions made by older generations will affect the prices faced by younger generations**;
- Provide a tractable alternative to infinite-horizon representative agent models;

Why is this interesting?

- New economic interactions: **decisions made by older generations will affect the prices faced by younger generations**;
- Provide a tractable alternative to infinite-horizon representative agent models;

- Some key implications are different from a neoclassical growth model, **where there is an equilibrium that is better than the market outcome**;
- It gives insights into the role of National debt, i.e., **Fiscal Policy and Social Security in the economy**.

The Model

Households

- Each household lives for **two periods**.
- There are N_t **young households** born in period t and each one supplies labor inelastically so that the total **labor supply** L_t **is equal to** N_t. Assume the **labor force grows at the rate** n:

$$\frac{L_{t+1}}{L_t} = \frac{N_{t+1}}{N_t} = 1 + n.$$

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- The **budget constraint of a household born in** t is:

$$c_{1t} + s_t = \omega_t,$$
$$c_{2t+1} = R_{t+1} s_t.$$

- They will maximize their **utility**:

$$U(c_{1t}) + \beta U(c_{2t+1}).$$

Firms

- Assume a **production function** homogeneous of degree one:

$$Y_t = F\left(K_t, L_t\right),$$

  where we could also have the format: $Y_t = F\left(K_t, A_t L_t\right)$.
- **Capital accumulation** is:

$$K_{t+1} = (1-\delta)K_t + I_t.$$

[1] Notice there are N_t workers but $N_t + N_{t+1}$ households alive.

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- Define the **per worker variables** as:[1]

$$y_t = f\left(k_t\right), \quad y_t = \frac{Y_t}{L_t}, \quad k_t = \frac{K_t}{L_t}.$$

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Firms

- The **derivative of the original and per worker functions** F **and** f are related by:

$$\frac{\partial F(K_t, L_t)}{\partial K_t} = f'(k_t),$$
$$\frac{\partial F(K_t, L_t)}{\partial L_t} = f(k_t) - k_t f'(k_t).$$

Market equilibrium

Households

- The **savings** s_t are the solution to the following program:

$$\begin{aligned} &\text{Max }_{s_t} U\left(c_{1t}\right) + \beta U\left(c_{2t+1}\right), \\ &c_{1t} + s_t = \omega_t, \\ &c_{2t+1} = R_{t+1} s_t. \end{aligned}$$

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- Therefore, the **savings function** will be:

$$s_t\equiv S\left(\omega_t,R_{t+1}\right).$$

Optimal savings – effect of wages

Savings $S(\omega, R)$ solve:

$$U'(\underbrace{\omega - S(\omega, R)}_{=c_1}) = \beta R U'(\underbrace{RS(\omega, R)}_{=c_2}).$$

How do **savings depend on wages (income)**?

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How do **savings depend on wages (income)**?

1. **Differentiate in** ω the above implicit equation:

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$$\frac{dS}{d\omega} = \frac{U''(c_1)}{\beta R^2 U''(c_2) + U''(c_1)} > 0.$$

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3. An **increase in the wage level implies an increase in consumption at both periods**, thus **savings** (to transfer consumption to the second period).

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3. Effect of interest rate on savings is **ambiguous**.

Income and substitution effect

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- **Income effect**: because the young are **savers**, an increase in the interest rate makes them richer – the future return on their savings will be higher – this increases early consumption and hence **reduces the incentives to save**.

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- **Income effect**: because the young are **savers**, an increase in the interest rate makes them richer – the future return on their savings will be higher – this increases early consumption and hence **reduces the incentives to save**.

- **Substitution effect**: an increase in the interest rate makes **future consumption relatively cheaper** (increases the returns on savings), this **increases the incentives to save**.

Optimal savings

1. Take as an example the **isoelastic utility function**:

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$$S\left(\omega_t, R_{t+1}\right) = \frac{\omega_t}{1 + \left(\beta R_{t+1}^{1-\theta}\right)^{-1/\theta}}.$$

3. If $\theta = 1$, **savings are a constant fraction of real wage income**:

$$S\left(\omega_t, R_{t+1}\right) = \frac{\beta \omega_t}{1+\beta}.$$

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- Consequently R_t, the **marginal rate of return on K_t** is:

$$R_t = \frac{\partial}{\partial K_t}\left[(1-\delta)K_t + F\left(K_t, L_t\right)\right] = 1 - \delta + f'\left(k_t\right).$$

Capital dynamics

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- Using $\omega_t = f(k_t) - k_t f'(k_t)$ and $R_t = 1 - \delta + f'(k_t)$ we have:

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- We cannot say that much, since $S(.)$ can take many forms:
  1. There can be **multiple steady states**.
  2. There can be **no steady state**.
  3. There can be a **unique steady state**.

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  4. **No over-saving**.

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  1. **We cannot say much** in general.

Comparison with Neoclassical Growth Model

- Explanation is in the **income and substitution effect**.
  - **Stability in Neoclassical Growth Model** because on the equilibrium path:

$$k_t > k^* \Leftrightarrow r_t < r^* \Rightarrow \text{ dissave} \Rightarrow \dot{c}_t < 0 \;\&\; \dot{k}_t < 0.$$

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  - With O.L.G. if income effect is strong, then **low-interest rates can increase savings** (saving does not have as high a return as before, so need to save more to compensate). This would **increase capital accumulation**.

Income and substitution effect with CRRA

Assume the utility is CRRA:

$$U(c_t) = \frac{c_t^{1-\theta} - 1}{1-\theta}.$$

1. Find the **optimal saving rates function, s_t**.
2. Analyse the **impact of ω and R_{t+1} in the saving rates**. How does this impact depend on the parameter θ (which effects dominate for each possible scenario)? Explain your results.

Using an example to evaluate capital dynamics

1. Let the **utility function be**:

$$U(c_{1t}) + \beta U(c_{2t+1}) = \log_{1t} + \beta \log c_{2t+1}.$$

2. Assume:

$$f(k) = Ak^{\alpha}.$$

3. Remember the **capital dynamics** is:

$$k_{t+1} = \frac{S(\omega_t, R_{t+1})}{1+n}$$

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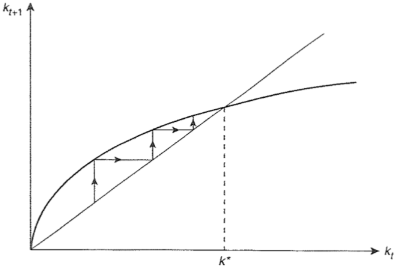

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Optimality

How does this result compare to the planner problem?

- Question: what would the **social planner's choice be if she were to maximize the weighted average of all generations' utilities**?
- What **aspects** the Social Planner should take into consideration?
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  - Allocation between **generations**.

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  - Allocation between two periods of one household (**life-cycle allocation**).
  - Allocation between **generations**.
- It is possible to show that the **planner allocates life-cycle consumption exactly as the agent would do** (i.e. following the Euler equation).
- However, the **allocation between generations is not Pareto-optimal**: there is **over-accumulation of capital** because older generations do not take into account the negative impact of their capital accumulation on the interest rate faced by following generations.

Pareto Optima

- In period t to **balance the goods markets** we should have:

$$N_t c_{1t} + N_{t-1} c_{2t} + K_{t+1} = F\left(K_t, L_t\right) + (1-\delta)K_t.$$

- In **per worker** terms we will have:

$$c_{1t} + \frac{c_{2t}}{1+n} + (1+n)k_{t+1} = f\left(k_t\right) + (1-\delta)k_t.$$

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- The **Social Planner Problem** can be written as:

$$\text{Maximize}_{\,c_{1t}, c_{2t+1}, k_{t+1}} \sum_{t=0}^{\infty} \zeta_t \left[U\left(c_{1t}\right) + \beta U\left(c_{2t+1}\right)\right], \qquad \text{s.t.}$$
$$c_{1t} + \tfrac{c_{2t}}{1+n} + (1+n)k_{t+1} = f\left(k_t\right) + (1-\delta)k_t. \quad \forall t$$

...

The Golden Rule

- Solving the previous problem we find:

$$U'\left(c_{1t}\right) = \beta R_{t+1} U'\left(c_{2t+1}\right).$$

- The idea is that an infinity of Pareto optima satisfies this condition.
- Let's use a **more discriminating criterion and compute the optimal level of capital in a steady state**.
- Denote it by $\hat{k}$, as the one that **maximizes the utility of the representative household**.

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1. The **problem** now will be:

$$\text{Maximize } U(c_1) + \beta U(c_2), \quad \text{s.t.}$$
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4. Since $R_t = 1 - \delta + f'(k_t)$, in the end, we have: $\hat{R} = 1 + n.$

Overaccumulation and Inefficiency

- Now we will explore the possibility of **inefficient equilibria**.
- We say that there is **overaccumulation of capital if the amount of capital per worker is superior to the value given by the golden rule**:

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- If this is the case, it is possible to **improve the situation of all generations**, so the initial one is not even Pareto optimal:

$$c_{1t} + \frac{c_{2t}}{1+n} + (1+n)k_{t+1} = f\left(k_t\right) + (1-\delta)k_t.$$

Overacculation and Inefficiency

- Imagine that starting at period $t = 1$, k_t **decreases to the level** k_1**, with** $\hat{k} \leq k_1 < k_0$.

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- Compared to the **reference trajectory corresponding to** k_0 we have:

$$\Delta \left(c_{1t} + \frac{c_{2t}}{1+n} \right) = \Delta \left[f\left(k_t\right) + (1-\delta)k_t - (1+n)k_{t+1} \right].$$

Overacculation and Inefficiency

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All generations benefit from it ,so the **initial equilibrium was not Pareto optimal**.

Pensions

Using the Overlapping generations models to study pensions

- We can study the debate between a **"fully funded"** pension system and a **"pay-as-you-go" pension system**. How do you explain those pension systems?
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$$

- The "fully-funded" and the "pay-as-you-go" system **differ basically in the way x_t and z_{t+1} are related**.
- Use the utility function:

$$
\log c_{1t} + \beta \log c_{2t+1}
$$

The Fully Funded System

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- The **household's budget constraint**:

$$c_{1t} + s_t = \omega_t - x_t,$$
$$c_{2t+1} = R_{t+1} s_t + R_{t+1} x_t.$$

- The **intertemporal budget constraint** is:

$$c_{1t} + \frac{c_{2t+1}}{R_{t+1}} = \omega_t.$$

- What is the **optimal level of savings**?

The Fully Funded System

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- Compared to the system without pensions, the introduction of a fully funded pension system is **neutral**.

The Pay-as-you-go System

- The **government collects premia from the young generation and distributes them immediately to the old generation**:

$$N_t x_t = N_{t-1} z_t,$$

and since $N_t = (1+n)N_{t-1}$ we have:

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and the **budget constraint** will be:

$$c_{1t} + s_t = \omega_t - x_t,$$
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The Pay-as-you-go System

- Assume that $x_{t+1} = x_t$. So, the **problem of the household** will be:

$$\text{Max}_{s_t} \log\left(\omega_t - x_t - s_t\right) + \beta \log\left[R_{t+1} s_t + (1+n)x_t\right].$$

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- By solving the problem the **optimal savings** will be:

$$s_t = \frac{\beta}{1+\beta}\left(\omega_t - x_t\right) - \frac{1+n}{(1+\beta)R_{t+1}} x_t.$$

The Pay-as-you-go System

- Use the following relationships and **find the dynamic equation for capital accumulation**:

$$
\begin{aligned}
k_{t+1} &= \frac{s_t}{1+n}, \\
\omega_t &= f\left(k_t\right) - k_t f'\left(k_t\right), \\
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- This gives the following **dynamic equation for capital accumulation**:

$$k_{t+1} = \frac{\beta\left[f\left(k_t\right) - k_t f'\left(k_t\right) - x_t\right]}{(1+\beta)(1+n)} - \frac{x_t}{(1+\beta)\left[1 - \delta + f'\left(k_{t+1}\right)\right]}.$$

The Pay-as-you-go System

- Define

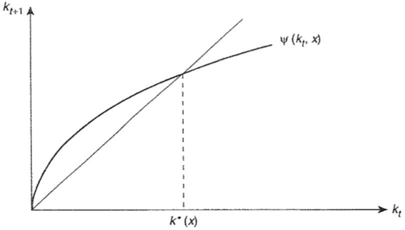
$$k_{t+1} = \frac{\beta \left[f\left(k_t\right) - k_t f'\left(k_t\right) - x_t \right]}{(1+\beta)(1+n)} - \frac{x_t}{(1+\beta)\left[1 - \delta + f'\left(k_{t+1}\right)\right]} \equiv \Theta(k_t, x_t).$$

- The derivatives are the following ones. **Prove it**:

$$\Theta_k > 0, \quad \Theta_x < 0.$$

The Pay-as-you-go System

- By assuming the steady state capital $k^*(x)$ is **dynamically efficient** we have:



Analysing Welfare

- The **fully funded system was neutral with respect to a system without pensions**.
- Observe we have:

$$k^*(x) = \Theta\left[k^*(x), x\right].$$

- We then have by taking derivative:

$$\frac{dk^*(x)}{dx} = \frac{\Theta_x}{1 - \Theta_k},$$

where at least at the steady state we have, $\Theta_k < 1$.

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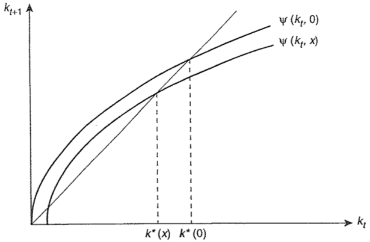

Taxes and Capital Accumulation

The way that **taxes are collected affects capital accumulation** in the O.L.G. economy. Let the households have utility function:

$$\log C_1 + \beta \log C_2.$$

Assume they receive an exogenous income Y when young and can invest an amount of capital K, which will give them an income AK in the second period. The government spends G, which is financed by lump-sum taxes.

1. Assume that the **government taxes young households in a lump-sum** by an amount $T_1 = G$. What is the level of capital accumulation?
2. Assume that the **government taxes old households in a lump-sum** by an amount $T_2 = G$. What is the level of capital accumulation?
3. Compare the answers.

Labor versus Capital Taxation

Let $Y_t = AK_t^{\alpha}L_t^{1-\alpha}$ and $K_{t+1} = I_t$. Suppose we have an O.L.G. structure with a constant population where the generation t has the utility function:

$$U_t = \log C_{1t} + \beta \log C_{2t+1}.$$

The government maximizes the discounted sum of utilities:

$$\sum_{t=s-1}^{\infty} \phi^t U_t = \sum_{t=s-1}^{\infty} \phi^t \left(\log C_{1t} + \beta \log C_{2t+1} \right).$$

The government spending is $G_t = \zeta Y_t$.
Assume the **government taxes labor and capital income at the constant rate τ_ℓ and τ_k**, so that the total taxes collected are:

$$T_t = \tau_\ell \omega_t L_t + \tau_k R_t K_t.$$

Assume also the government balances the budget period by period: $T_t = G_t$:

1. Compute the **dynamics of this economy**.
2. Find the **optimal tax rates** τ_ℓ and τ_k.

Conclusions

Summary

- O.L.G. models are more **realistic** than infinitely lived representative agent models.
- In O.L.G. models the **possibility of over-accumulation of capital leads to dynamic inefficiency**.
- Key to **overaccumulation**:
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  - Generations do not take into account the **negative impact of their savings on future interest rates**.

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