Department of Economics - Sciences Po
Macroeconomics I

Problem Set 2

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Fall 2022

Question 1 – The transversality condition

Consider the following problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln c_t$$

subject to:

$$\begin{gathered} \textstyle\sum_{t=0}^{\infty} c_t \leq k_0 \\ c_t \geq 0 \ \text{ for } \ \forall t \geq 0 \\ k_0 \ \text{ is given} \end{gathered}$$

Use the Lagrangian method to derive the optimal consumption plan for the *cake eating* problem. The name of the problem comes from the first constraint: there is a certain amount of capital, that is not productive and can be consumed over time.

Hint: Follow these steps:

1. Set up the Lagrangian.
2. Combine the FOCs to get the Euler equation, which relates consumption in two consecutive periods (period t and $t+1$).
3. Use the complementary slackness constraint (i.e. that the resource constraint is satisfied with equality).
4. How much of the cake is left at time T?
5. Is the transversality condition satisfied?
6. What if the resource constraint is not satisfied?

Question 2 – End-of-the-world model

Question 2.5 from the Barro book.

Suppose that everyone knows that the world will end deterministically at time $T > 0$. We worked out this problem in the text when we discussed the importance of the transversality condition. Go through the analysis here in the following steps:

a. How does this modification affect the transition equations for $\widehat{k}$ and $\widehat{c}$?

b. How does the modification affect the transversality condition?

c. Use the phase diagram to describe the new transition path for the economy.

d. As T gets larger, how does the new transition path relate to the one shown in the phase diagram? What happens as T approaches infinity?

Question 3 – Exponential utility

Question 2.3 from the Barro book.

Assume that infinite-horizon households maximize its present discounted utility, where $u(c)$ is now given by the exponential form,

$$u(c) = -\frac{1}{\theta}e^{-\theta c}$$

where $\theta > 0$. The behavior of firms is the same as in the Ramsey model, with zero technological progress.

a. Relate θ to the concavity of the utility function and to the desire to smooth consumption over time. Compute the inter-temporal elasticity of substitution. How does it relate to the level of per capita consumption, c?

b. Find the first-order conditions for a representative household with preferences given by this form of $u(c)$.

c. Combine the first-order conditions for the representative household with those of firms to describe the behavior of c and k over time. [Assume that $k(0)$ is below its steady-state value.]

d. How does the transition depend on the parameter θ? Compare this result with the one in the model discussed in the text.

Question 4 – Two-period model

Consider a two-period discrete time model of consumption-saving choice. In particular assume that the household lives for 2 periods ($t = 0, 1$), has labor income w_t in period $t = 0, 1$. Assume that there is a risk-free asset, which pays r_t interest rate at time t. Denote asset holdings at time t by a_t (this is the amount that pays returns in period t), and assume that initial asset holdings are zero, $a_0 = 0$. Consumer's maximize the following discounted lifetime utility:

$$U(c_0, c_1) = u(c_0) + \beta u(c_1)$$

subject to the following per period budget constraints:

$$\begin{aligned} c_0 + a_1 &= w_0 \\ c_1 + a_2 &= (1 + r_1)a_1 + w_1 \end{aligned}$$

by choosing c_0, c_1, a_1, a_2.

a. Is there a limit on how much a household can borrow (i.e. is there a lower bound on a_1 and a_2)?

b. What is the optimal a_2 for the household?

c. Using what you found for a_2 eliminate a_1 by combining the two per period budget constraints to get the inter-temporal budget constraint.

d. Set up the Lagrangian for the household using the inter-temporal budget constraint. (Here the only choice variables are c_0 and c_1, as we found the optimal a_2 and have eliminated a_1.) Derive the Euler equation (which relates the optimal c_0 to the optimal c_1).

e. What does the Euler equation imply depending on the value of $(1 + r_1)\beta$?

f. Now use the Euler equation and the inter-temporal budget constraint to express c_0 and c_1 in terms of w_0, w_1 and r_1 in case $u(c) = \ln c$ and in case $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$.