Macroeconomics 1

Lecture - The neoclassical growth model

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Introduction

- In the **Solow model** we have a **constant saving rate**.
  - Now, we will study the most famous model of growth, the **Ramsey model**.
  - The **rate of growth** is determined by the **evolution of technology assumed exogenous**.
  - Agents will **choose their consumption and investment decisions** through intertemporal maximization.

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  - The **rate of growth** is determined by the **evolution of technology assumed exogenous**.
  - Agents will **choose their consumption and investment decisions** through intertemporal maximization.

- Households are represented as a single dynasty of infinitely lived households.
- All of them have the same utility function and the same budget constraint.

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- **Positive economics**: individual demands are such that the **demand side of the economy can be represented** as if a single household is doing all the aggregate decisions subject to a budget constraint.
- **Normative economics**: the demand is studied such that it can be represented by a single consumer and **the welfare of individuals** can be analyzed with the representative utility function.

What about the infinite horizon assumption?

- Most models assume individuals have an **infinite planning horizon**.
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- How do we give a micro foundation for this infinite horizon perspective:
  1. **Perpetual youth model**.
  2. **Finite lives but there is perfect altruism across generations**.

Parenthesis on growth rate in continuous time

- Imagine $g(t)$ is the growth rate of $X(t)$:

$$\lim_{\Delta t \to 0} X(t+\Delta t) = X(t)(1+g(t)\Delta t),$$

$$\lim_{\Delta t \to 0} \left(\frac{X_{t+\Delta t} - X_t}{\Delta t} \cdot \frac{1}{X_t} \right) = g(t),$$

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- Using $\frac{d \ln X(t)}{dt} = \frac{\dot{X}(t)}{X(t)}$:

$$ln(X(t)) - ln(X(0) = \int_0^t \frac{\dot{X}(s)}{X(s)} ds = \int_0^t g(s) ds,$$
$$X(t) = X(0) e^{\int_0^t g(s) ds}.$$

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- In particular case of constant growth rate: $g(s) = g$: $X(t) = X(0)e^{gt}$.

THE RAMSEY MODEL

Production

Production function

- Producers will have a **constant returns technology** given by:

$$Y_t = F\left(K_t, Z_t L_t\right).$$

- By now assume $Z_t = 1$, so the production function will become:

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- The **evolution of capital** will be:
$$\dot{K}_t = I_t - \delta K_t = Y_t - C_t - \delta K_t.$$
- Assume the **workforce grows at the rate n**:
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- Assume the **workforce grows at the rate** n:
$$L_t = L_0 e^{nt}.$$
- Define the following **per capita** variables:
$$y_t = \frac{Y_t}{L_t}, \quad k_t = \frac{K_t}{L_t}, \quad c_t = \frac{C_t}{L_t}.$$

Production function

- The associated production function will be:

$$y_t = f\left(k_t\right) = F\left(k_t, 1\right).$$

- We can write $\dot{K}_t = I_t - \delta K_t = Y_t - C_t - \delta K_t$ in **per capita** terms
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$$\dot{k}_t = y_t - c_t - (\delta + n)k_t.$$

Households

- Households will maximize **discounted utility**:[1]

$$V = \int_0^{\infty} e^{-\rho t} U(c_t)\, dt,$$

and their **budget constraint** is:

$$C_t + \dot{K}_t = \omega_t L_t + r_t K_t.$$

[1]Some people assume that the per capita utilities are weighted by the size of the population, that is they maximize $e^{-\rho t} L_t U(c_t)$ instead of $e^{-\rho t} U(c_t)$.

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- In **per capita the budget constraint becomes**:

$$c_t + \frac{dk_t}{dt} + nk_t = \omega_t + r_t k_t.$$

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Market equilibrium

First-order conditions

- **Equilibrium of the firms implies**:

$$r_t = f'(k_t) - \delta,$$
$$\omega_t = f(k_t) - k_t f'(k_t).$$

- The **problem of the household** is:

$$\text{Maximize } \int_0^\infty e^{-\rho t} U(c_t)\, dt, \qquad \text{s.t.}$$
$$c_t + \frac{dk_t}{dt} + nk_t = \omega_t + r_t k_t.$$

Parenthesis on Dynamic Optimization

- Consider the following **dynamic optimization problem**:

$$\text{Maximize } \int_0^T F(x_t, u_t, t)\, dt, \quad \text{s.t.}$$
$$\dot{x}_t = g(x_t, u_t, t),$$

where x_t **is called the state variable**, u_t **is the control variable**, $\dot{x}_t = dx_t/dt$ and the initial value x_0 is given.

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- Define the Hamiltonian as:

$$\mathcal{H}_t(x_t, u_t, \lambda_t, t) = F(x_t, u_t, t) + \lambda_t g(x_t, u_t, t),$$

where λ_t is a multiplier similar to the **Lagrange multiplier**.

Parenthesis on Dynamic optimization

- The **necessary conditions** for a maximum are:

$$\frac{\partial \mathcal{H}_t}{\partial u_t} = 0,$$
$$\dot{x}_t = \frac{\partial \mathcal{H}_t}{\partial \lambda_t},$$
$$\dot{\lambda}_t = -\frac{\partial \mathcal{H}_t}{\partial x_t}.$$

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- If the Hamiltonian is concave in (x_t, u_t), then the conditions above are **sufficient for a maximum**.

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- If not, there is a **weaker condition**. Let $u^* (x_t, \lambda_t, t)$ be the value of the control variable that maximizes $\mathcal{H}_t (x_t, u_t, \lambda_t, t)$ for given (x_t, λ_t, t) and define:

$$\mathcal{H}_t^* (x_t, \lambda_t, t) = \mathcal{H}_t [x_t, u^* (x_t, \lambda_t, t), \lambda_t, t] .$$

- Then, a **sufficient condition** for the above to yield a maximum is that the maximized Hamiltonian be concave in x_t.

Parenthesis on Dynamic Optimization: The current Value Hamiltonian

- In many applications the function F is given by:

$$F\left(x_t, u_t, t\right) = e^{-\rho t} f\left(x_t, u_t, t\right).$$

- So, the maximization problem can be written as:

$$\text{Maximize } \int_0^T e^{-\rho t} f\left(x_t, u_t, t\right) dt \quad \text{s.t.}$$
$$\dot{x}_t = g\left(x_t, u_t, t\right).$$

- Now define the Hamiltonian multiplier as $u_t = \lambda_t e^{\rho t}$ and the **current value Hamiltonian** will be:

$$\mathcal{H}_t^c = f\left(x_t, u_t, t\right) + \mu_t g\left(x_t, u_t, t\right).$$

Parenthesis on Dynamic Optimization: The current Value Hamiltonian

- Then, the **necessary conditions** for optimality are:

$$\frac{\partial \mathcal{H}_t^c}{\partial u_t} = 0, \tag{1}$$

$$\dot{x}_t = \frac{\partial \mathcal{H}_t^c}{\partial \mu_t}, \tag{2}$$

$$\dot{\mu}_t = \rho\mu_t - \frac{\partial \mathcal{H}_t^c}{\partial x_t}. \tag{3}$$

- There is also a set of **Transversality conditions**, but this would be easier to see in an example.

Example: one sector growth model

- Consider a **one-sector continuous-time growth model**.
- The representative consumer lives forever and consumption at time t is given by c_t and the preferences are given by:

$$\int_0^\infty e^{-\rho t} U(c_t)\, dt,$$

with $\rho > 0$ and $u(.)$ is strictly increasing, concave and sufficiently smooth period utility function is defined on $[0, \infty)$. The **resource constraint** is given by:

$$c_t + i_t \leq f(k_t),$$
$$\dot{k}_t \leq i_t - \delta k_t,$$
$$c_t, k_t \geq 0 \quad \forall t \in [0, \infty).$$

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- We shall assume that the optimal path for capital is such that $\dot{k}_t$ exists. The initial capital stock k_0 is given and assumed to be positive.
- Now let's solve the problem using the **Lagrangian approach**...

Coming back to our problem with population growth

- Recall the problem was given by:

$$\text{Maximize } \int_0^\infty e^{-\rho t} U(c_t)\, dt, \qquad \text{s.t.}$$
$$c_t + \frac{dk_t}{dt} + nk_t = \omega_t + r_t k_t.$$

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- The **current value Hamiltonian** is:

$$\mathcal{H}_t \doteq U(c_t) + \lambda_t (\omega_t + r_t k_t - c_t - nk_t).$$

- Using the **optimality conditions** in (1), (2), and (3), the F.O.C. are:

$$\lambda_t = U'(c_t),$$
$$\dot{\lambda}_t = \rho\lambda_t - \frac{\partial \mathcal{H}_t}{\partial k_t} = \lambda_t (\rho + n - r_t).$$

Coming back to our problem with population growth

- By **combining both equations** we have:

$$\frac{U''(c_t)}{U'(c_t)}\frac{dc_t}{dt} = \rho + n - r_t.$$

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$$\lim_{t\to\infty} e^{-\rho t} k_t U'(c_t) = 0.$$

- What is the problem if the condition above is positive?

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  By **selling all or some of the capital accumulated, the household could increase its discounted utility by a positive number**, so the initial path could not have been optimum.

The dynamics equation

- The equation of **evolution of capital per capita** is given by:

$$\dot{k}_t = f\left(k_t\right) - (\delta + n)k_t - c_t.$$

- The **dynamic optimality** equation requires:

$$\frac{U''\left(c_t\right)}{U'\left(c_t\right)}\frac{dc_t}{dt} = \rho + n - r_t.$$

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- As before define:

$$\theta(c_t) = -\frac{c_t U''(c_t)}{U'(c_t)},$$

which is the inverse of the intertemporal elasticity of substitution (i.e., regulates the willingness to substitute consumption over time).

- In the end we have:

$$\frac{1}{c_t}\frac{dc_t}{dt} = \frac{f'(k_t) - \rho - \delta - n}{\theta(c_t)},$$

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- The **dynamic equations** are:

$$\dot{k}_t = f(k_t) - (\delta + n)k_t - c_t.$$

$$\frac{\dot{c}_t}{c_t} = \frac{f'(k_t) - \rho - \delta - n}{\theta(c_t)}.$$

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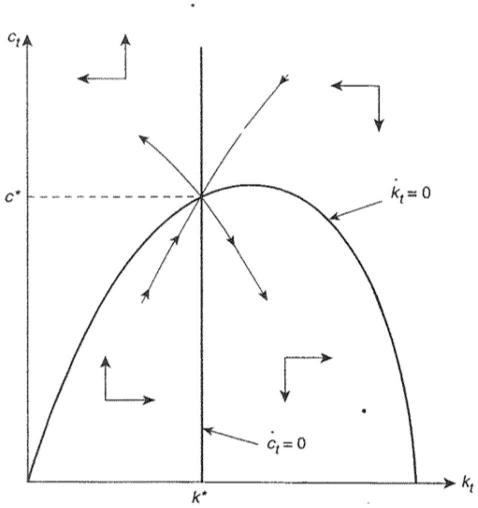
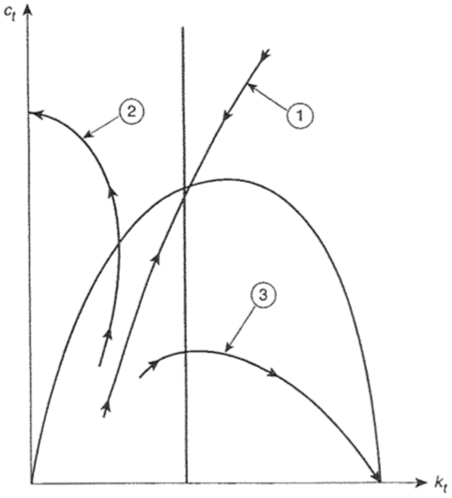
- With those equations we can determine the **dynamics**.

- The **vertical line** is the locus $\dot{c}_t = 0$, where we have $f'\left(k_t\right) = \rho + \delta + n$.
- The **bell-shaped curved** is the locus $\dot{k}_t = 0$, where we have $c_t = f\left(k_t\right) - (\delta + n)k_t$.
- Denote c^* and k^* the steady-state values.

Representation in the (k, c) space



Representation in the (k, c) space



Representation in the (k, c) space

1. Imagine that the economy follows a saddle path as in (2). In **finite time the economy will reach a point in the $k_t = 0$ with positive consumption** and since $\dot{k}_t = -c_t$, the capital would have to become negative.

Representation in the (k, c) space

1. Imagine that the economy follows a saddle path as in (2). In **finite time the economy will reach a point in the $k_t = 0$ with positive consumption** and since $\dot{k}_t = -c_t$, the capital would have to become negative.

2. Now let the economy follows (3) this case the economy will **save too much capital and the transversality condition will not hold**:
   - $$d \log \left[e^{-\rho t} k_t U'(c_t) \right] = -\rho + \frac{\dot{k}_t}{k_t} + \frac{U''(c_t)}{U'(c_t)} \dot{c}_t .$$
   - $$d \log \left[e^{-\rho t} k_t U'(c_t) \right] = \frac{f(k_t) - c_t}{k_t} - r_t - \delta$$
   - Since $f(k_t) > r_t k_t$, the transversality condition will not go to zero.

Transitional dynamics

- **Dynamics in Solow**:

  Start at k_0 and then go forwards, **system monotonically converges to the balanced growth path (BGP)**.

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- **Dynamics in Solow**:

  Start at k_0 and then go forwards, **system monotonically converges to the balanced growth path (BGP)**.

- **Dynamics in Neoclassical Growth Model**:

  The value c_0 is **not determined from an initial condition**.

  The transversality condition determines it, which is a boundary condition at infinity.

Transitional dynamics 2

Stability concept: **saddle-path stability**.

- As we have seen there is a unique stable arm of the system that satisfies:

$$\lim_{t\to\infty}(k_t, c_t) \to (k^*, c^*).$$

  - Given k_0 there is a unique c_0 such that the equilibrium conditions are satisfied.
  - **Given** (k_0, c_0) the system is governed by the **two differential equations, and converges** to the BGP.

Phase diagram

- 2-dimensional space (k, c).
- At each point (k, c), we can use the differential equations to determine the sign of $(\dot{k}, \dot{c})$.

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- Analyze behavior of k when c is below/above the locus $\dot{k} = 0$:

$$\dot{k} = 0 \Leftrightarrow c^*(k) = f(k) - (n + \delta)k.$$

  - if $c < c^*(k) \Rightarrow \dot{k} > 0$.
  - if $c > c^*(k) \Rightarrow \dot{k} < 0$.

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- Analyze the behavior of c when k is to the right/left of the locus $\dot{c} = 0$:

$$\dot{c} = 0 \Leftrightarrow f'(k^*(c)) = \delta + \rho + n.$$

  - if $k < k^*(c) \Rightarrow \dot{c} > 0$.
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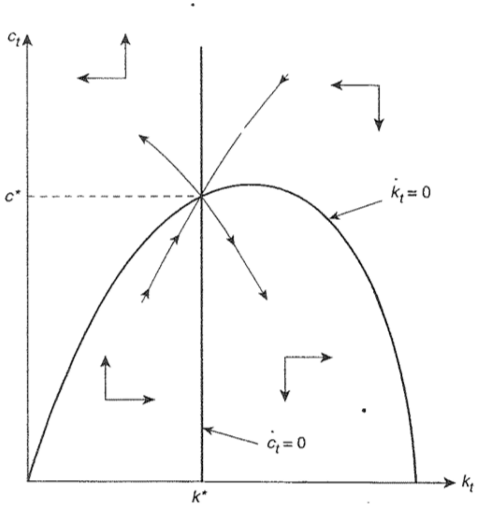
On the stable arm



Efficiency

Is the competitive equilibrium Pareto optimum?

- The **first welfare theorem says that a market equilibrium, when it exists, is Pareto optimum**.
- We can investigate **efficiency** directly for the Ramsey model, by letting a **Planner doing the choices** instead of individuals and firms.

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- The **first welfare theorem says that a market equilibrium, when it exists, is Pareto optimum**.
- We can investigate **efficiency** directly for the Ramsey model, by letting a **Planner doing the choices** instead of individuals and firms.

- Notice the **optimum** will be a solution to the following problem:

$$\text{Maximize } \int_0^\infty e^{-\rho t} U(c_t)\, dt, \quad \text{s.t.}$$
$$\dot{k}_t = f(k_t) - (\delta + n)k_t - c_t$$

Is the competitive equilibrium Pareto optimum?

- The **current-value Hamiltonian** for this problem is:

$$\mathcal{H}_t = U(c_t) + \lambda_t \left[f(k_t) - (\delta + n) k_t - c_t \right].$$

- The **F.O.C.s** are:

$$U'(c_t) = \lambda_t,$$
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- Combining the equations we have:

$$\frac{U''(c_t)}{U'(c_t)} \frac{dc_t}{dt} = \delta + n + \rho - f'(k_t).$$

- The transversality conditions are the same, and so **the market equilibrium is optimum**.

Ricardian equivalence

Introducing the government

- Assumes the **government spends G_t, taxes T_t, and has a debt D_t**. The government debt evolves according to:

$$\dot{D}_t = r_t D_t + G_t - T_t.$$

- Notice in this setting the **consumer can now save and accumulate both capital K_t and debt D_t**. The budget constraint will be:

$$C_t + \dot{K}_t + \dot{D}_t = \omega_t L_t + r_t K_t + r_t D_t - T_t.$$

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- Use the real discount factor:

$$\Delta_t = \exp\left(-\int_0^t r_s ds\right)$$

and **integrate the government budget constraint and the budget constraint of the household**.

Introducing the government

- The **intertemporal budget constraint of the government** will be:

$$\int_0^\infty \Delta_t T_t dt = D_0 + \int_0^\infty \Delta_t G_t dt.$$

- The **intertemporal budget constraint of the household** will be:

$$\int_0^\infty \Delta_t C_t dt = D_0 + K_0 + \int_0^\infty \Delta_t \omega_t L_t dt - \int_0^\infty \Delta_t T_t dt.$$

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- The **intertemporal budget constraint of the household** will be:

$$\int_0^\infty \Delta_t C_t dt = D_0 + K_0 + \int_0^\infty \Delta_t \omega_t L_t dt - \int_0^\infty \Delta_t T_t dt.$$

- Notice in the end:

$$\int_0^\infty \Delta_t C_t dt = K_0 + \int_0^\infty \Delta_t \omega_t L_t dt - \int_0^\infty \Delta_t G_t dt.$$

Ideas about introducing the government

1. Both the government debt and taxes disappeared, i.e., **all that matters is the sequence of government spending G_t, how it is financed is irrelevant** (i.e., as long as the government plans to balance its budget in the long run).

2. The idea is that the **value of government debt is exactly compensated by the discounted value of taxes that the government will have to levy** to repay the debt.

Government spending and dynamics

- Let $\dot{K}_t = I_t - \delta K_t = Y_t - C_t - G_t - \delta K_t$.
- In per-capita terms we will have:

$$\dot{k}_t = f(k_t) - (\delta + n)k_t - c_t - g_t.$$

- If the government starts from an initial $g = 0$ and suddenly the **government spending goes to $g > 0$** the locus $\dot{k}_t = 0$ goes down by g, whereas the locus $\dot{c}_t$ does not need to move.

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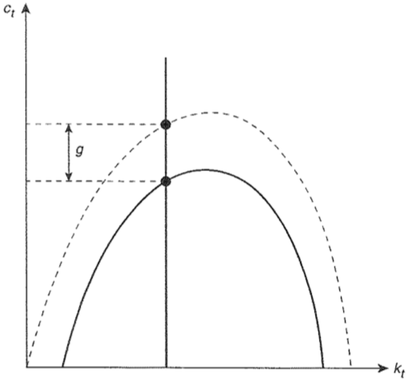
- In per-capita terms we will have:

$$\dot{k}_t = f(k_t) - (\delta + n)k_t - c_t - g_t.$$

- If the government starts from an initial $g = 0$ and suddenly the **government spending goes to $g > 0$** the locus $\dot{k}_t = 0$ goes down by g, whereas the locus $\dot{c}_t$ does not need to move.



The Ramsey Model in Discrete time

How to solve the model in discrete time?

- Suppose the **population increases at the rate n**:

$$L_t = L_0(1+n)^t.$$

- The **capital depreciation** will be the same as before:

$$K_{t+1} = (1-\delta)K_t + I_t.$$

- The **household will maximize** the discounted utility, $\sum_0^\infty \beta^t U(C_t)$ subject to the constraint: $C_t + K_{t+1} = \omega_t L_t + R_t K_t - T_t$.

Ramsey in Discrete Time

- The Lagrangian will be:

$$\sum_{0}^{\infty} \beta^t \left[U\left(C_t\right) + \lambda_t\left(\omega_t L_t + R_t K_t - T_t - C_t - K_{t+1}\right)\right].$$

- So, the **F.O.C.s for consumption and capital** are:

$$\lambda_t = U'\left(C_t\right),$$
$$\lambda_t = \beta R_{t+1} \lambda_{t+1}.$$

- Combining those equations we will have:

$$U'\left(C_t\right) = \beta R_{t+1} U'\left(C_{t+1}\right)$$

Idea behind the Euler equation

- Imagine the consumer decides to consume less at time t by an amount ε. The immediate loss in utility is given by $U'(c_t)\varepsilon$. In the next period it will obtain an extra income $R_{t+1}\varepsilon$, and thus a utility gain of $\beta U'(c_{t+1})R_{t+1}\varepsilon$:

$$\beta R_{t+1} U'\left(C_{t+1}\right)\varepsilon = U'\left(C_t\right)\varepsilon.$$

- The **gain and loss** in the utility exactly compensate each other.

Government spending with distortionary taxation

Government spending is financed by a tax

- Assume government spending is financed by a **proportional distortionary tax on income**, given by:
$$G_t = \tau\left(\omega_t L_t + r_t K_t\right),$$
where τ represents the level of taxation.

- The **budget constraint of the household** now:
$$C_t + \dot{K}_t = (1-\tau)\left(\omega_t L_t + r_t K_t\right).$$

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- In **per capita** terms:

$$c_t + \frac{dk_t}{dt} + nk_t = (1-\tau)\left(\omega_t + r_t k_t\right).$$

- The one for the **government** will be:

$$g_t = \tau\left(\omega_t + r_t k_t\right).$$

First-order conditions

- The **equilibrium conditions** will be:

$$r_t = f'(k_t) - \delta,$$
$$\omega_t = f(k_t) - k_t f'(k_t).$$

- The **household's problem** is:

$$\text{Maximize } \int_0^\infty e^{-\rho t} U(c_t)\, dt \quad \text{s.t.}$$
$$c_t + \frac{dk_t}{dt} + nk_t = (1-\tau)(\omega_t + r_t k_t)$$

The current-value Hamiltonian

- The **current-value Hamiltonian** is:

$$\mathcal{H}_t = U\left(c_t\right) + \lambda_t\left[\left(1-\tau\right)\left(\omega_t + r_t k_t\right) - c_t - nk_t\right].$$

- The **F.O.C.** for this problem will be:

$$\begin{aligned} \lambda_t &= U'\left(c_t\right), \\ \dot{\lambda}_t &= \rho\lambda_t - \frac{\partial \mathcal{H}_t}{\partial k_t} = \lambda_t\left[\rho + n - (1-\tau)r_t\right]. \end{aligned}$$

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$$\dot{\lambda}_t = \rho \lambda_t - \frac{\partial \mathcal{H}_t}{\partial k_t} = \lambda_t \left[\rho + n - (1-\tau) r_t \right].$$

- Combining **both equations** we obtain:

$$\frac{U''(c_t)}{U'(c_t)} \frac{dc_t}{dt} = \rho + n - (1-\tau) r_t.$$

The Dynamic Equations

- The evolution of capital is:

$$\dot{k}_t = f(k_t) - (\delta + n)k_t - c_t - g_t.$$

- Using the definition of $\theta(c_t) = -\frac{c_t U''(c_t)}{U'(c_t)}$ we have:

$$\frac{\dot{c}_t}{c_t} = \frac{(1-\tau)\left[f'(k_t) - \delta\right] - \rho - n}{\theta(c_t)}$$

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- Notice:

$$g_t = \tau(\omega_t + r_t k_t) = \tau[f(k_t) - \delta k_t].$$

- In the end we have:

$$\dot{k}_t = (1-\tau)[f(k_t) - \delta k_t] - nk_t - c_t.$$

The effects of a Tax Increase

- We have two dynamic equations in $\dot{c}_t$ and $\dot{k}_t$.
- First observe:

$$(1-\tau)\left[f'(k_t)-\delta\right]=\rho+n,$$

- And also:

$$c_t=(1-\tau)\left[f(k_t)-\delta k_t\right]-nk_t$$

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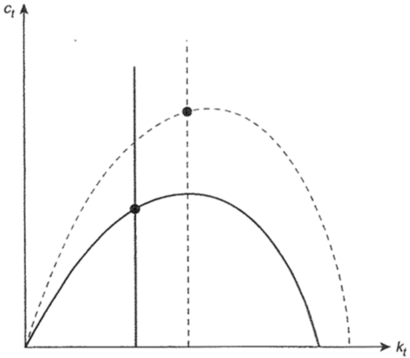

Government Spending

- Let the households have the intertemporal utility function:

$$U = \sum_{t=0}^{\infty} \beta^t \log C_t.$$

- Assume firms have the production function of the form:

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha}.$$

- Capital fully depreciates such that:

$$K_{t+1} = I_t.$$

- The government wishes public spending in the following way:

$$G_t = \zeta Y_t,$$

  which are financed by a lump-sum tax $T_t = G_t$.
  1. Write the F.O.C.s of the households.
  2. Compute consumption and investment in the steady state. Comment your results.

Extension with technological growth and a generic asset

Assumptions

- Unique **final good**.
- Firm side exactly the same, **representative neoclassical firm** $F(K_t, Z_t L_t)$ with same assumptions as in Solow.
- Markets:
  1. **Spot market for final good (price = 1)**.
  2. **Spot market for labor (price = w_t)**.
  3. **Asset market (price = r_t)**.
- All markets are **perfectly competitive**.

Assumptions

- Households supply all **labor inelastically, consume and accumulate assets**.
- A_t denotes the amount of assets accumulated at t:
- What could these assets represent?
  - **Bonds** D_t.
  - **Capital stock** K_t.

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- Return r_t on assets.
- Households are **price takers**, the wages w_t and the interest rate r_t are given.

Household's constraints, flow budget constraint

- A each period **change in assets is a change in income minus change in consumption** (which translates into assets accumulation/decumulation), flow of asset change.

$$\frac{dA_t}{dt} = \underbrace{r_t A_t}_{\text{returns}} + \underbrace{w_t L_t}_{\text{income}} - \underbrace{c_t L_t}_{\text{consumption}} .$$

- Not so much an assumption, but rather an accounting identity.

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- Plugging this into **flow budget constraint**:

$$\dot{a}_t = w_t + (r_t - n)a_t - c_t.$$

- How much will households borrow?

Household's constraints, no-Ponzi condition

- The flow budget constraint does not generate a lifetime budget constraint.
- Finite horizon: no-Ponzi condition is $A_T \geq 0$ (household cannot die in debt).
- Infinite horizon equivalent of $A_T \geq 0$ is:

$$0 \leq \lim_{t\to\infty} \left[A_t e^{-\int_0^t r_s ds} \right] = \lim_{t\to\infty} \left[a_t e^{-\int_0^t (r_s - n) ds} \right],$$

  i.e., the present value of **future assets at an infinite horizon cannot be negative**.

Household's constraints, no-Ponzi condition

- Implies an intertemporal budget constraint.
- Integrating the flow budget constraints over $[0, T]$:

$$A_T = \left(A_0 + \int_0^T w_t L_t - \int_0^T c_t L_t \right) e^{\int_0^T r_s ds}.$$

- The constraint on $A_T / e^{\int_0^T r_s ds} \geq 0$ when $T \to \infty$ gives the **intertemporal budget constraint**.

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- No Ponzi condition gives you this and provides an **intertemporal borrowing constraint that relates total lifetime consumption with initial assets and total expected assets**.

Equilibrium in the neoclassical growth model

Equilibrium in this Neoclassical Growth Model is a **sequence** of:

- **prices** $\{w_t, R_t\}_{t=0}^{\infty}$,
- **quantities** $\{C_t, A_t, K_t\}_{t=0}^{\infty}$,
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given the **path**,

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such that:

- **Firms maximize profits**,
- **Consumers maximize lifetime utility**,
- **Markets clear**.

Household's necessary and sufficient conditions

The present value Hamiltonian for this problem is:

$$\mathcal{H}(a, c, \lambda) = u(c_t)e^{(n-\rho)t} + \lambda_t \underbrace{((r_t - n)a_t + w_t - c_t)}_{=\dot{a}_t \text{ from the flow budget constraint}}.$$

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Optimality conditions:

- the first order conditions

$$\frac{\partial \mathcal{H}}{\partial c} = 0 \qquad \Leftrightarrow u'(c_t)e^{(n-\rho)t} - \lambda_t = 0.$$
$$\frac{\partial \mathcal{H}}{\partial a} = -\dot{\lambda}_t \qquad \Leftrightarrow \lambda_t(r_t - n) = -\dot{\lambda}_t.$$

- The transversality condition:

$$\lim_{t\to\infty} a_t \lambda_t \leq 0.$$

Manipulating the Hamiltonian

The **first order conditions** again:

$$u'(c_t)e^{(n-\rho)t} = \lambda_t.$$
$$\lambda_t(r_t - n) = -\dot{\lambda}_t.$$

Manipulating the Hamiltonian

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$$
\begin{aligned}
u'(c_t)e^{(n-\rho)t} &= \lambda_t. \\
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\end{aligned}
$$

Combine the two expressions to get:

$$\frac{\dot{c}_t}{c_t} = \frac{u'(c_t)}{u''(c_t)c_t}(\rho - r_t).$$

Continuous time Euler equation

$$\frac{\dot{c}_t}{c_t} = -\frac{u'(c_t)}{u''(c_t)c_t}(r_t - \rho).$$

- Consumption grows over time (i.e. $\frac{\dot{c}_t}{c_t} > 0$) if:
    1. The discount rate ρ (impatience) is less than the rate of return on assets r_t. Why
    2. If θ is small this means that consumers are not that risk averse, so the optimal rate at which consumption grows must be high.

Specify the utility function: constant IES

- Isoelasic utility:

$$u(c) = \frac{c^{1-\theta} - 1}{1-\theta}, \quad \text{where} \quad \text{if } \theta \geq 0, \theta \neq 1$$

$$u(c) = ln(c) \text{ if } \theta = 1$$

- For this utility function the intertemporal elasticity of substitution is:

$$-\frac{u'(c)}{u''(c)c} = \frac{1}{\theta}.$$

$\Rightarrow$ **Constant IES, or constant relative risk aversion (CRRA)**.

Firm optimality – same as in the Solow model

$$Y_t = F(K_t, Z_t L_t).$$

- Denote by $\widehat{k}_t \equiv \frac{K_t}{Z_t L_t}$ **capital per units of effective labor (or efficiency/effective units of labor)**.
- Now you can express output per unit of effective labor as:

$$\widehat{y}_t \equiv \frac{Y_t}{Z_t L_t} = F\left(\frac{K_t}{Z_t L_t}, 1\right) = F(\widehat{k}_t, 1) \equiv f(\widehat{k}_t).$$

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- **Rental rates and wage rates** are:

$$R_t = f'(\widehat{k}_t).$$

$$w_t = (f(\widehat{k}_t) - f'(\widehat{k}_t)\widehat{k}_t)A_t = (f(\widehat{k}_t) - f'(\widehat{k}_t)\widehat{k}_t)e^{gt}.$$

Asset market clearing

- The **households' assets are the firm's machines**.

$$a_t = k_t.$$

- **Return on assets** is:

$$r_t = f'(\widehat{k}_t) - \delta$$

- Households get R_t from the firms, but lose δ as depreciation.

Equilibrium conditions

From resource constraint (or from flow budget constraint):

$$\dot{K}_t = F(K_t, Z_t L_t) - C_t - \delta K_t.$$

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Capital accumulation in units of effective labor terms, with $\widehat{c}_t = \frac{C_t}{Z_t L_t}$:

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Euler equation in units of effective labor terms:

$$\frac{\dot{\widehat{c}}_t}{\widehat{c}} = \frac{1}{\theta(c_t)}(r_t - \rho) - g = \frac{1}{\theta}(f'(\widehat{k}_t) - \delta - \rho - \theta g).$$

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Consumption growth in units of effective labor terms:

$$\frac{d\widehat{c}_t}{dt} = \frac{1}{\theta}(f'(\widehat{k}_t) - \delta - \rho - \theta g)\widehat{c}_t.$$

Equilibrium: a system of nonlinear differential equations

Two differential equations in $\widehat{k}$ and $\widehat{c}$:

$$\frac{d\widehat{k}_t}{dt} = \dot{\widehat{k}}_t = f(\widehat{k}_t) - \widehat{c}_t - (g+n+\delta)\widehat{k}_t = 0.$$
$$\frac{d\widehat{c}_t}{dt} = \dot{\widehat{c}}_t = \frac{1}{\theta}(f'(\widehat{k}_t) - \delta - \rho - \theta g)\widehat{c}_t = 0.$$

Need boundary conditions on $\widehat{k}$:

- initial condition: $\widehat{k}_0$ given.
- transversality condition (using that $a_t = k_t$):

$$\lim_{t\to\infty} \widehat{k}_t e^{-\int_0^t (f'(\widehat{k}_s) - \delta - g - n)ds} = 0.$$

Steady state – balanced growth path

- The steady state is characterized by

$$\dot{\widehat{k}}^* = 0 \Rightarrow \qquad \widehat{c}^* = f(\widehat{k}^*) - (g+n+\delta)\widehat{k}^*.$$

$$\dot{\widehat{c}}^* = 0 \Rightarrow \qquad f'(\widehat{k}^*) = \delta + \rho + \theta g.$$

- In the steady state $\widehat{k}, \widehat{c}$ are **constant**, and so is $\widehat{y} = f(\widehat{k})$.
- k, c, y **and** w **grow at rate** g, and R is constant (because $f'(.)$ is homogeneous of degree zero).
- Parameters have level effects i.e. **the level of capital and income depend on the parameters**.

Conclusions

Was it worth it?

We did a lot of work to get similar results as with Solow but:

- Good to know the results **did not depend on the possibly problematic assumption of exogenous saving rates**.
- Opens the black box of **capital accumulation by specifying the preferences of consumers**: paves the way for further **analysis of capital accumulation, human capital, and endogenous technological progress**.
- By specifying individual preferences we can explicitly **compare equilibrium and optimal growth**.
- **No inefficient over-saving**.

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- **No inefficient over-saving**.

However:

- Did not generate real new insights about the sources of cross-country income differences and economic growth.

Extensions

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- Do the conclusions also strongly and critically depend on these assumptions?

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- Do the conclusions also strongly and critically depend on these assumptions?

- Results are **robust to a much more general framework** in which each household subject to idiosyncratic labor and potentially capital income risk, and with different preferences and biases (e.g., hyperbolic discounting, time-inconsistency, systematic mistakes, non-separable preferences, temptation etc.) as well as possibly borrowing and lending constraints.
- Results in Acemoglu and Jensen (2019).

Proximate vs fundamental causes

- Before, the growth of per capita consumption and output per worker is determined exogenously.
- The levels of income, consumption, etc depend on the parameters δ, θ, ρ, n and on the form of the production function $f(.)$.

- In this model, however, the **causes of the differences in income per capita are explained only in terms of preference and technology parameters**.
- Link between proximate and potential fundamental causes: e.g. **intertemporal elasticity of substitution and the discount rate** can be as related to **cultural or geographic factors**.