Macroeconomics 1

Lecture - Dynamic Programming

Diego de Sousa Rodrigues
diego.desousarodrigues@sciencespo.fr

Sciences Po

2022 Fall

Infinite Horizon Optimization

1. Time is discrete and the horizon is infinite: $t \in \{0, 1, \ldots\}$.
2. Single infinitely lived household.
3. Single good representing capital and consumption.
4. Production takes one period.
5. The amount y_t of good produced in period t depends on the stock of capital k_t in this period.

The Model

- We assume that $y_t = F(k_t)$ where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous on $\mathbb{R}^+$, continuously differentiable on $(0, \infty)$, strictly concave, with

$$F(0) = 0, \quad \forall k > 0, \quad F'(k) > 0, \quad \text{and} \quad \lim_{k \to \infty} F'(k) = 0$$

The Model

- We assume that $y_t = F(k_t)$ where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous on $\mathbb{R}^+$, continuously differentiable on $(0, \infty)$, strictly concave, with

$$F(0) = 0, \quad \forall k > 0, \quad F'(k) > 0, \quad \text{and} \quad \lim_{k \to \infty} F'(k) = 0$$

- In each period t, the output y_t combined with the depreciated capital stock $(1 - \delta)k_t$ can be divided between consumption c_t and new capital stock k_{t+1}.

The Model

- We assume that $y_t = F(k_t)$ where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous on $\mathbb{R}^+$, continuously differentiable on $(0, \infty)$, strictly concave, with

$$F(0) = 0, \quad \forall k > 0, \quad F'(k) > 0, \quad \text{and} \quad \lim_{k \to \infty} F'(k) = 0$$

- In each period t, the output y_t combined with the depreciated capital stock $(1 - \delta)k_t$ can be divided between consumption c_t and new capital stock k_{t+1}.
- Implicitly, there is a **one-to-one technology transforming capital in good**.

The Model

- The householdâs initial stock of capital is $k_0 > 0$.

The Model

- The householdâs initial stock of capital is $k_0 > 0$.
- His utility from an infinite consumption stream $c = (c_t)_{t \geq 0}$ is

$$U(c) := \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $\beta \in (0,1)$ and $u : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded, continuous on $\mathbb{R}^+$, continuously differentiable on $(0,\infty)$, strictly increasing, and strictly concave.

The Set of Alternatives

- We let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by:

$$f(k) := F(k) + (1 - \delta)k$$

The Set of Alternatives

- We let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by:

$$f(k) := F(k) + (1 - \delta)k$$

- The household maximizes his discounted utility $U(c)$ among all consumption streams c satisfying the resource feasibility restrictions

$$c_0 + k_1 \leqslant f(k_0)$$

and for every $t \geqslant 1$,

$$c_t + k_{t+1} \leqslant f(k_t)$$

where $c_t \geqslant 0$ and $k_{t+1} \geqslant 0$.

The Set of Alternatives

- We let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by:

$$f(k) := F(k) + (1-\delta)k$$

- The household maximizes his discounted utility $U(c)$ among all consumption streams c satisfying the resource feasibility restrictions

$$c_0 + k_1 \leqslant f(k_0)$$

  and for every $t \geqslant 1$,

$$c_t + k_{t+1} \leqslant f(k_t)$$

  where $c_t \geqslant 0$ and $k_{t+1} \geqslant 0$.
- The initial capital stock $k_0 > 0$ is given.
- A key feature in this problem is that it involves **choosing an infinite sequence of consumption or capital stock, one for each** t.

The Set of Alternatives

1. Since the utility function is strictly increasing, we can restrict attention to consumption streams $c = (c_t)\, t \geqslant 0$ satisfying

$$c_t = f\left(k_t\right) - k_{t+1}$$

The Set of Alternatives

1. Since the utility function is strictly increasing, we can restrict attention to consumption streams $c = (c_t)\, t \geqslant 0$ satisfying

$$c_t = f(k_t) - k_{t+1}$$

2. Therefore, the householdâs problem is

$$\max \sum_{t=0}^{\infty} \beta^t u\left(f(k_t) - k_{t+1}\right), \tag{SP}$$

among all capital stock streams $(k_{t+1})_t \geqslant 0$ satisfying
$\forall t \geqslant 0, \quad 0 \leqslant k_{t+1} \leqslant f(k_t)$, where k_0 is given.

Solving SP

- We want to solve (SP) by identifying the optimal capital stock stream k^* and the corresponding maximum utility

$$V^*(k_0) = \sum_{t=0}^{\infty} \beta^t u\left(f(k_t^*) - k_{t+1}^*\right)$$

Solving SP

- We want to solve (SP) by identifying the optimal capital stock stream k^* and the corresponding maximum utility

$$V^*(k_0) = \sum_{t=0}^{\infty} \beta^t u\left(f\left(k_t^*\right) - k_{t+1}^*\right)$$

- The problem we need to solve is the following one

$$\underbrace{V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})}_{}$$

Solving SP

- We want to solve (SP) by identifying the optimal capital stock stream k^* and the corresponding maximum utility

$$V^*(k_0) = \sum_{t=0}^{\infty} \beta^t u\left(f\left(k_t^*\right) - k_{t+1}^*\right)$$

- The problem we need to solve is the following one

$$\underbrace{V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(f\left(k_t\right) - k_{t+1}\right)}_{V(k_0)=\max_{\{k_{t+1}\}_{t=0}^{\infty}}[u(f(k_0)-k_1)+\beta u(f(k_1)-k_2)+\beta^2 u(f(k_2)-k_3)...]}$$

Solving SP

- Observe

$$V\left(k_0\right)=\max_{\{k_{t+1}\}_{t=0}^{\infty}}\left[u\left(f\left(k_0\right)-k_1\right)+\beta u\left(f\left(k_1\right)-k_2\right)+\beta^2 u\left(f\left(k_2\right)-k_3\right)\ldots\right]$$

- Therefore,

$$V\left(k_0\right)=\max_{\{0\leq k_{t+1}\leq f(k_t)\}_{t=0}^{\infty}}\left\{u\left(f\left(k_0\right)-k_1\right)+\sum_{t=1}^{\infty}\beta^t u\left(f\left(k_t\right)-k_{t+1}\right)\right\}$$

Ideas Behind the Dynamic Programming

- Since k_t for $t \geq 2$ does not appear in $u(f(k_0 - k_1))$, we can rewrite the problem as

$$V(k_0) = \max_{k_1} \left\{ u(f(k_0) - k_1) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \left[\sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1}) \right] \right\}$$

Ideas Behind the Dynamic Programming

- Since k_t for $t \geq 2$ does not appear in $u(f(k_0 - k_1))$, we can rewrite the problem as

$$V\left(k_0\right) = \max_{k_1} \left\{ u\left(f\left(k_0\right) - k_1\right) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \left[\sum_{t=1}^{\infty} \beta^{t-1} u\left(f\left(k_t\right) - k_{t+1}\right) \right] \right\}$$

- Observe:

$$\max_{\{k_{t+1}\}_{t=1}^{\infty}} \left[\sum_{t=1}^{\infty} \beta^{t-1} u\left(f\left(k_t\right) - k_{t+1}\right) \right] =$$

$$\max_{\{k_{t+1}\}_{t=1}^{\infty}} \left[u\left(f\left(k_1\right) - k_2\right) + \beta u\left(f\left(k_2\right) - k_3\right) + \ldots \right] = V\left(k_1\right)$$

Ideas Behind the Dynamic Programming

- Since k_t for $t \geq 2$ does not appear in $u(f(k_0 - k_1))$, we can rewrite the problem as

$$V(k_0) = \max_{k_1} \left\{ u(f(k_0) - k_1) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \left[\sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1}) \right] \right\}$$

- Observe:

$$\max_{\{k_{t+1}\}_{t=1}^{\infty}} \left[\sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1}) \right] =$$

$$\max_{\{k_{t+1}\}_{t=1}^{\infty}} \left[u(f(k_1) - k_2) + \beta u(f(k_2) - k_3) + \ldots \right] = V(k_1)$$

- Therefore,

$$\boldsymbol{V(k_0) = \max_{0 \leq k_1 \leq f(k_0)} \{ u(f(k_0) - k_1) + \beta V(k_1) \}}$$

Ideas Behind the Dynamic Programming

- But the dates $t = 0$ and $t = 1$ are arbitrary,so it is common to simply denote the capital stock today by k and the capital stock in the next period by $k^{'}$. Then

$$\boldsymbol{V(k) = \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta V\left(k'\right) \right\}}, \tag{BE}$$

  where (') denotes for future variables.
- The main idea of dynamic programming was to transform an **inter-temporal program** into a **two-period program : today and the future**.
- If function $V(k)$ were known, we could use (BE) to define policy function $k^{'}$ that attains the maximum in (BE). Given the capital stock today, agents decide about capital stock tomorrow.

Ideas Behind the Dynamic Programming

- The main idea was to transform an inter-temporal program into a two period program: **today and the future.**
- Notice that with the Lagrangian approach we had introduced a set of Lagrange Multipliers .
- With the Bellman Equation we introduced a new function, the function V, which is **inter-temporal welfare**.

Recursive structure

- Notice

$$V(k) = \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta V\left(k'\right) \right\}, \tag{BE}$$

Recursive structure

- Notice

$$\boldsymbol{V(k) = \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta V\left(k'\right) \right\},} \tag{BE}$$

- This determines tomorrow's capital stock as a function of today's capital stock. Specifically,

$$k' = g(k) \in \arg \max_{0 \leq x \leq f(k)} \{u[f(k) - x] + \beta V(x)\}$$

Recursive structure

- Notice

$$\boldsymbol{V(k) = \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta V\left(k'\right)\right\},} \tag{BE}$$

- This determines tomorrow's capital stock as a function of today's capital stock. Specifically,

$$k' = g(k) \in \arg \max_{0 \leq x \leq f(k)} \{u[f(k) - x] + \beta V(x)\}$$

- This function g is know as a **policy function**.

Bellman Equation in Practice

- Suppose further that V is **concave** and **differentiable**. Then the **objective function is the sum of two concave functions and hence is also concave**. So the **optimal policy is characterized by the first order condition**:

$$u'(c) = \beta V\left(k'\right)$$

Bellman Equation in Practice

- Suppose further that V is **concave** and **differentiable**. Then the **objective function is the sum of two concave functions and hence is also concave**. So the **optimal policy is characterized by the first order condition**:

$$u'(c) = \beta V\left(k'\right)$$

- If V were known, we could solve for the policy function by finding the appropriate $k' = g(k)$ for each k such that;

$$u'[f(k) - g(k)] = \beta V'(g(k))$$

Bellman Equation in Practice

- With the policy function g, we can then construct the entire sequence of capital stocks $k_{t+1} = g(k_t)$ starting with the given initial condition k_0. Then we can back out consumption with $c_t = f(k_t) - g(k_t)$.

Bellman Equation in Practice

- With the policy function g, we can then construct the entire sequence of capital stocks $k_{t+1} = g(k_t)$ starting with the given initial condition k_0. Then we can back out consumption with $c_t = f(k_t) - g(k_t)$.
- Note that **today's capital stock is all that we need to know in order to choose optimal consumption**. In this regard, today's capital stock is the only **state variable** in this problem.

Bellman Equation in Practice

- As already discussed, an optimal policy g(k) is characterized by the FOC:

$$u'[f(k) - g(k)] = \beta V'(g(k))$$

Bellman Equation in Practice

- As already discussed, an optimal policy g(k) is characterized by the FOC:

$$u'[f(k) - g(k)] = \beta V'(g(k))$$

- We can make progress by using the **envelope theorem**.

$$V'(k) = u'[f(k) - g(k)]f'(k)$$

The above is called the envelope condition.

Bellman Equation in Practice

- As already discussed, an optimal policy g(k) is characterized by the FOC:

$$u'[f(k) - g(k)] = \beta V'(g(k))$$

- We can make progress by using the **envelope theorem**.

$$V'(k) = u'[f(k) - g(k)]f'(k)$$

  The above is called the envelope condition.
- This can be written as:

$$V'(g(k)) = u'[f(g(k)) - g(g(k))]f'(g(k))$$

Bellman Equation in Practice

- Plugging this into the FOC gives:

$$u'[f(k) - g(k)] = \beta u'[f(g(k)) - g(g(k))]f'(g(k))$$

Bellman Equation in Practice

- Plugging this into the FOC gives:

$$u'[f(k) - g(k)] = \beta u'[f(g(k)) - g(g(k))]f'(g(k))$$

- The above is just the standard consumption Euler Equation, which is

$$u'\left(c_t\right) = \beta u'\left(c_{t+1}\right) f'\left(k_{t+1}\right)$$

Bellman Equation in Practice

- Plugging this into the FOC gives:

$$u'[f(k) - g(k)] = \beta u'[f(g(k)) - g(g(k))]f'(g(k))$$

- The above is just the standard consumption Euler Equation, which is

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1})$$

- We can solve for the steady state of the model without knowing either the value function or the policy function. In the steady state $k' = k$ so that $k^* = g(k^*)$. Therefore, we have:

$$1 = \beta f'(k^*),$$

which we can do without knowing either V or g.

Bellman Equation in Practice

1. **Take the F.O.C.**

$$u'(c) = \beta V'\left(k'\right)$$

Bellman Equation in Practice

1. **Take the F.O.C.**

$$u'(c) = \beta V'\left(k'\right)$$

2. **Consider the Envelop condition**

$$V'(k) = \alpha k^{\alpha-1} u'(c)$$

Bellman Equation in Practice

1. **Take the F.O.C.**

$$u'(c) = \beta V'\left(k'\right)$$

2. **Consider the Envelop condition**

$$V'(k) = \alpha k^{\alpha-1} u'(c)$$

3. **Consider the Envelop condition the next period**

$$V'\left(k'\right) = \alpha k'^{\alpha-1} u'\left(c'\right)$$

Bellman Equation in Practice

1. **Take the F.O.C.**

$$u'(c) = \beta V'\left(k'\right)$$

2. **Consider the Envelop condition**

$$V'(k) = \alpha k^{\alpha-1} u'(c)$$

3. **Consider the Envelop condition the next period**

$$V'\left(k'\right) = \alpha k'^{\alpha-1} u'\left(c'\right)$$

4. **Substitute for $V^{'}$ in the envelop condition, using the F.O.C.**

$$u'(c) = \beta \alpha k'^{\alpha-1} u'\left(c'\right)$$

and find the Euler Equation. Sequential representation:

$$u'\left(c_t\right) = \beta \alpha k_{t+1}^{\alpha-1} u'\left(c_{t+1}\right).$$

It is always involves these four steps:

Example 1: Saving choice problem

- **Sequential problem:**

$$\begin{aligned} \max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ a_{t+1} + c_t & = a_t (1 + r_t) + w_t \end{aligned}$$

Example 1: Saving choice problem

- **Sequential problem:**

$$\begin{aligned} \max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \\ a_{t+1} + c_t \quad & = a_t\left(1 + r_t\right) + w_t \end{aligned}$$

- **Recursive problem:**

$$\begin{aligned} V(a) = & u(c) + \beta V\left(a'\right) \\ & \text{s.t. } a' + c = a(1+r) + w \end{aligned}$$

Example 1: Saving choice problem

1. **Take the F.O.C.**

2. **Consider the Envelop condition**

3. **Consider the Envelop condition the next period**

4. **Substitute for V' in the envelop condition, using the F.O.C.**

Example 2: Internal habit formation

- Consider the problem: $u\left(c_t - hc_{t-1}\right)$.

Example 2: Internal habit formation

- Consider the problem: $u\left(c_t - hc_{t-1}\right)$.
- Letâs write the Bellman equation recursively. Beginning-of-period wealth is a and past consumption is denoted c_{-1}. The Bellman equation is

$$V\left(a, c_{-1}\right) = \max_{a'} u\left(c - hc_{t-1}\right) + \beta V\left(a', c\right)$$
$$\text{s.t } a' + c = a(1+r) + w$$

Example 2: Internal Habit Formation

1. **Take the F.O.C.**

2. **Consider the Envelop condition**

3. **Consider the Envelop condition the next period**

4. **Substitute for $V^{'}$ in the envelop condition, using the F.O.C.**

Example 3: Stochastic Bellman Equations

$$V(a) = \max_{a'} u(c) + \beta \mathbb{E} V\left(a'\right)$$
$$a' + c = (1+r)a + w$$

Calculations are the same as before with an additional $\mathbb{E}$ operator. Find:

$$u'(c) = \beta \mathbb{E}(1+r) u'\left(c'\right)$$

Ways of Solving for the Value Function

In Macroeconomics III we will discuss two ways of solving dynamic programming problems. The two ways aim to solve the functional equation for V:

1. **Guess and verify:** this method involves a guess of V and verification of the Bellman equation. Inevitably, it assumes the uniqueness of the solution to the equation.

Ways of Solving for the Value Function

In Macroeconomics III we will discuss two ways of solving dynamic programming problems. The two ways aim to solve the functional equation for V:

1. **Guess and verify:** this method involves a guess of V and verification of the Bellman equation. Inevitably, it assumes the uniqueness of the solution to the equation.
2. **Value function iteration**