Macroeconomics I

Recitation 4

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1 The Solow-Swan Model with Technological Progress

1.1 The Model

Assume that the production function takes the form:

$$Y_t = F(K_t, A_t L_t) \tag{1}$$

where $F(.)$ satisfies the neoclassical axioms. Technological progress is modeled as a labor-augmenting process A_t. Assume as well that A_t grows at a constant rate, $x = \dot{A}_t / A_t$.

Consider: $Y_t/L_t = F(K_t/L_t, A_t)$. Since A_t is growing, Y_t/L_t is probably not going to be stable over time. Therefore, to find a steady state, we normalize variables by $A_t L_t$. $A_t L_t$ is called the efficient amount of labor. It conveys the idea that, if productivity has been multiplied by 2 compared to 50 years ago, then a worker today is as productive as 2 workers from 50 years ago.

In practice this means that we study:

$$\frac{Y_t}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, 1\right) \tag{2}$$

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And we introduce the notations[1]:

$$\hat{y}_t \equiv \frac{Y_t}{A_t L_t}$$
$$\hat{k}_t \equiv \frac{K_t}{A_t L_t}$$
$$f(\hat{k}_t) \equiv F\left(\frac{K_t}{A_t L_t}, 1\right) = \hat{y}_t$$

Note that this slightly complicates the time derivative of $\hat{k}_t$:

$$\dot{\hat{k}}_t = \frac{\dot{K}_t A_t L_t - \left(\dot{A}_t K_t L_t + \dot{L}_t A_t K_t\right)}{(A_t L_t)^2} = \frac{\dot{K}_t}{A_t L_t} - \left(\frac{\dot{A}_t}{A_t}\frac{K_t}{A_t L_t} + \frac{\dot{L}_t}{L_t}\frac{K_t}{A_t L_t}\right) = \frac{\dot{K}_t}{A_t L_t} - (x+n)\hat{k}_t \tag{3}$$

We can now invoke the usual capital accumulation constraint:

$$\dot{K}_t = sF(K_t, A_t L_t) - \delta K_t \tag{4}$$

Dividing by $A_t L_t$:

$$\frac{\dot{K}_t}{A_t L_t} = sF\left(\frac{K_t}{A_t L_t}, 1\right) - \delta \frac{K_t}{A_t L_t} = sf(\hat{k}_t) - \delta \hat{k}_t$$

Plugging equation (3):

$$\dot{\hat{k}}_t = sf(\hat{k}_t) - (x+n+\delta)\,\hat{k}_t$$

Equivalently:

$$\frac{\dot{\hat{k}}_t}{\hat{k}_t} = s\frac{f(\hat{k}_t)}{\hat{k}_t} - (x+n+\delta) \tag{5}$$

Equation (5) is the dynamic equation for the capital stock per efficient unit of labor. Figure 1 plots the dynamics and shows that there is a unique stable steady state.

Even if $\hat{k}_t$ and $\hat{y}_t$ are constant in steady state, variables per capita are not. In fact, output per capita grows at a constant rate:

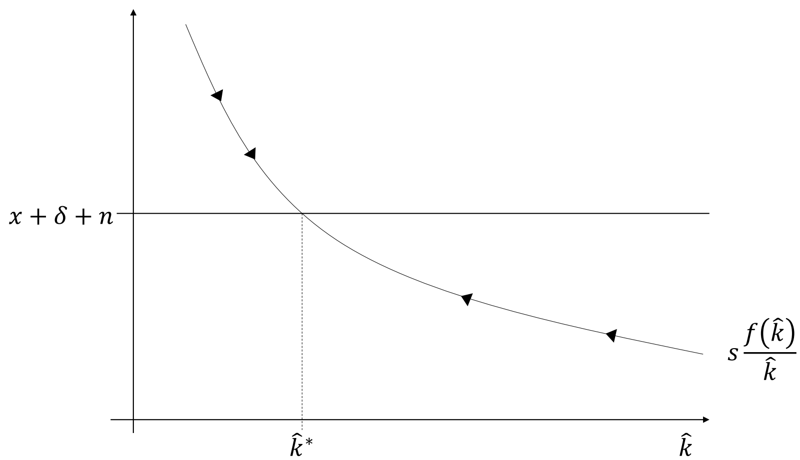
$$Y_t/L_t = A_t \hat{y^*} \Rightarrow \frac{\partial\,(Y_t/L_t)}{\partial t}\frac{L_t}{Y_t} = \frac{\dot{A}_t}{A_t} = x$$

The same is true of capital per capita.

Proposition 1 *In a Solow-Swan model with growth, at steady state, variables per capita grow at the rate of technological progress.*

[1] In macroeconomics, hats are often used to refer to log-deviations from steady state. It is not the case here.

Figure 1: Steady state with technological progress



1.2 The Kaldor (1961) Facts

Fact 1: per capita output grows over time, and its growth rate does not tend to diminish. Proposition 1 implies that this is true in the model that we have just studied.

Fact 2: physical capital per worker grows over time. Also true.

Fact 3: the rate of return to capital is nearly constant. If capital is paid its marginal product:

$$r_t = F_K(K_t, A_tL_t) - \delta = F_K\left(\frac{K_t}{A_tL_t}, 1\right) - \delta = f'(\hat{k}^*) - \delta$$

where the second equality used the homogeneity of degree 0 of F_K.

Fact 4: the ratio of physical capital to output is nearly constant. In steady state, by equation (5):

$$\frac{x+n+\delta}{s} = \frac{f(\hat{k}^*)}{\hat{k}^*} = \frac{Y_t}{K_t}$$

Fact 5: the shares of output accruing to labor and physical capital are nearly constant. Assume a Cobb-Douglas production function for that one. If labor is paid its marginal product, the share of output accruing to labor is:

$$\frac{w_tL_t}{Y_t} = \frac{(1-\alpha)\,K_t^{\alpha}A_t^{1-\alpha}L_t^{-\alpha}\times L_t}{Y_t} = \frac{(1-\alpha)\,Y_t}{Y_t} = 1-\alpha$$

What is not going to labor is going to capital, so the capital share is equal to α, hence constant. As long as the production function is Cobb-Douglas, this is true in and out of steady state[2]. With a more general

[2] See section A.3 for more details on Cobb-Douglas.

production function like (1), this will only be true in steady state.

Fact 6: The growth rate of output per worker differs substantially across countries. x could differ by countries, but that is more an assumption than a result... Otherwise, we could say that some countries are not in steady state. See section 3 of the last recitation.

Recently, some of these facts have been disputed. For instance, the persistent decline in the interest rate that took place in the last decade in the US, Europe and Japan contradicts fact 3. Piketty (2014) contends that fact 5 is no longer true: capital share in national income is increasing.

2 The Solow-Swan Model with Human Capital

Assume that the production function takes the form:

$$Y_t = K_t^{\alpha} H_t^{\eta} \left(A_t L_t\right)^{1-\alpha-\eta} \tag{6}$$

H_t is human capital: education, training, experience...

2.1 Equalized Marginal Products

The first possibility is to assume that the law of motion of capital is:

$$\dot{K}_t + \dot{H}_t = sY_t - \delta\left(K_t + H_t\right) \tag{7}$$

With the same kind of computations that led to equation (3), we can normalize by $A_t L_t$:

$$\dot{\hat{k}}_t + \dot{\hat{h}}_t = s\hat{k}_t^{\alpha}\hat{h}_t^{\eta} - (x + \delta + n)\left(\hat{k}_t + \hat{h}_t\right) \tag{8}$$

To pin down the level of human capital, we can assume that marginal products of both kinds of capital are equalized:

$$\eta\frac{\hat{k}_t^{\alpha}\hat{h}_t^{\eta}}{\hat{h}_t} - \delta = \alpha\frac{\hat{k}_t^{\alpha}\hat{h}_t^{\eta}}{\hat{k}_t} - \delta \tag{9}$$

Equivalently:

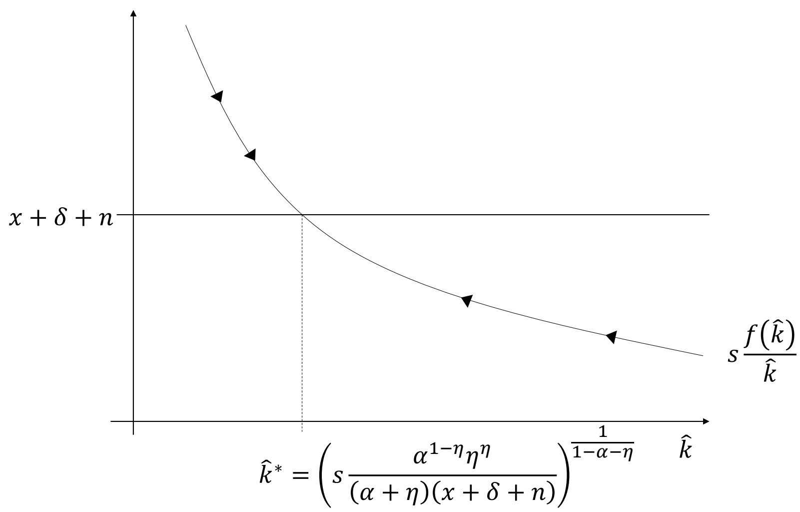
$$\hat{h}_t = \frac{\eta}{\alpha}\hat{k}_t \tag{10}$$

Plugging the latter into equation (8):

$$\frac{\dot{\hat{k}}_t}{\hat{k}_t} = s\left(\frac{\alpha^{1-\eta}\eta^{\eta}}{\alpha+\eta}\right)\hat{k}_t^{-(1-\alpha-\eta)} - (x+\delta+n) \tag{11}$$

One can then solve for the steady state and dynamics. See figure 2.

Figure 2: Steady state with human capital



2.2 Exogenous Savings in Human Capital

Otherwise, we can assume that physical and human capital accumulation have distinct exogenous savings rate s_k and s_h:

$$\dot{K}_t = s_k Y_t - \delta K_t \tag{12}$$

$$\dot{H}_t = s_h Y_t - \delta H_t \tag{13}$$

Normalizing by $A_t L_t$:

$$\frac{\dot{\hat{k}}_t}{\hat{k}_t} = s_k\frac{\hat{k}_t^{\alpha}\hat{h}_t^{\eta}}{\hat{k}_t} - (x+\delta+n) \tag{14}$$

$$\frac{\dot{\hat{h}}_t}{\hat{h}_t} = s_h\frac{\hat{k}_t^{\alpha}\hat{h}_t^{\eta}}{\hat{h}_t} - (x+\delta+n) \tag{15}$$

where $\hat{h}_t = H_t/\left(A_t L_t\right)$ and other notations are similar to the previous section.

As usual the steady state satisfies $\dot{\hat{k}}_t = \dot{\hat{h}}_t = 0$:

$$s_k \frac{\hat{k}_t^{\alpha} \hat{h}_t^{\eta}}{\hat{k}_t} = (x + \delta + n) \tag{16}$$

$$s_h \frac{\hat{k}_t^{\alpha} \hat{h}_t^{\eta}}{\hat{h}_t} = (x + \delta + n) \tag{17}$$

Take the ratio of these two equations:

$$\hat{h}_t = \frac{s_h}{s_k} \hat{k}_t \tag{18}$$

Substituting $\hat{h}_t$ in equation (16) or (17), we get:

$$\hat{k}^* = \left(\frac{s_k^{1-\eta} s_h^{\eta}}{x + \delta + n} \right)^{\frac{1}{1-\alpha-\eta}} \tag{19}$$

$$\hat{h}^* = \left(\frac{s_k^{\alpha} s_h^{1-\alpha}}{x + \delta + n} \right)^{\frac{1}{1-\alpha-\eta}} \tag{20}$$

$$\hat{y}^* = \left(\frac{s_k^{\alpha} s_h^{\eta}}{(x + \delta + n)^{\alpha+\eta}} \right)^{\frac{1}{1-\alpha-\eta}} \tag{21}$$

Remember that $\hat{y}^*$ is the steady state value of $Y_t/(A_t L_t)$. Therefore, Equation (21) can be rewritten:

$$\log \left(\frac{Y_t}{L_t} \right) = \log A_t + \frac{\alpha}{1-\alpha-\eta} \log s_h + \frac{\eta}{1-\alpha-\eta} \log s_k - \frac{\alpha+\eta}{1-\alpha-\eta} \log(x + \delta + n) \tag{22}$$

Since Y_t/L_t, s_k, s_h and $(x + \delta + n)$ can be approximated empirically, α and η can be estimated with an OLS regression in a cross-section of countries ($\log A_t$ is treated as a constant). Moreover, one can test whether the coefficients on $\log s_h$, $\log s_k$ and $\log(x + \delta + n)$ sum to zero. Table 1 shows that the model explains almost 80% of cross-country differences! The restriction can never be rejected.

3 The Euler Equation

3.1 Derivation

Households maximize the utility function:

$$U = \int_0^{\infty} e^{-(\rho-n)t} \frac{c_t^{1-\theta} - 1}{1-\theta} \, dt$$

Table 1: Estimation of the augmented Solow model

ESTIMATION OF THE AUGMENTED SOLOW MODEL

Dependent variable: log GDP per working-age person in 1985			
Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	6.89	7.81	8.63
	(1.17)	(1.19)	(2.19)
ln(I/GDP)	0.69	0.70	0.28
	(0.13)	(0.15)	(0.39)
$\ln(n + g + \delta)$	−1.73	−1.50	−1.07
	(0.41)	(0.40)	(0.75)
ln(SCHOOL)	0.66	0.73	0.76
	(0.07)	(0.10)	(0.29)
$\overline{R}^2$	0.78	0.77	0.24
s.e.e.	0.51	0.45	0.33
Restricted regression:			
CONSTANT	7.86	7.97	8.71
	(0.14)	(0.15)	(0.47)
$\ln(I/GDP) - \ln(n + g + \delta)$	0.73	0.71	0.29
	(0.12)	(0.14)	(0.33)
$\ln(SCHOOL) - \ln(n + g + \delta)$	0.67	0.74	0.76
	(0.07)	(0.09)	(0.28)
$\overline{R}^2$	0.78	0.77	0.28
s.e.e.	0.51	0.45	0.32
Test of restriction:			
p-value	0.41	0.89	0.97
Implied α	0.31	0.29	0.14
	(0.04)	(0.05)	(0.15)
Implied β	0.28	0.30	0.37
	(0.03)	(0.04)	(0.12)

Note. Standard errors are in parentheses. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05. SCHOOL is the average percentage of the working-age population in secondary school for the period 1960–1985.

Source: Gregory et al. (1992). Note: x is denoted g by Gregory et al.. s_k is approximated by investment as a fraction of GDP and s_h by the percentage of the working-age population that is in secondary school.

subject to:

$$\dot{a}_t = (r_t - n)a_t + w_t - c_t$$

$$a_0 \text{ is given}$$

$$\lim_{t\to\infty} \exp\left(-\int_0^t (r_t - n)\, dt\right) a_t \geq 0$$

The Hamiltonian writes:

$$\mathcal{H} = e^{-(\rho-n)t}\frac{c_t^{1-\theta} - 1}{1-\theta} + \lambda_t\left[(r_t - n)a_t + w_t - c_t\right]$$

c_t is the control variable and a_t is the state. The FOC are:

$$\frac{\partial \mathcal{H}}{\partial c_t} = 0$$

$$\frac{\partial \mathcal{H}}{\partial a_t} = -\dot{\lambda}_t$$

$$\lim_{t\to\infty} \lambda_t a_t = 0$$

They can be rewritten:

$$e^{-(\rho-n)t}c_t^{-\theta} = \lambda_t \tag{23}$$

$$\lambda_t(r_t - n) = -\dot{\lambda}_t \tag{24}$$

$$\lim_{t\to\infty} \lambda_t a_t = 0 \tag{25}$$

We want to eliminate λ_t. To do so, we will combine equations (23) and (24). In practice, we take the logarithm of equation (23):

$$-(\rho - n)t - \theta \log c_t = \log \lambda_t$$

And derive it with respect to time:

$$-(\rho - n) - \theta\frac{\dot{c}_t}{c_t} = \frac{\dot{\lambda}_t}{\lambda_t} \tag{26}$$

Now, rearrange equation (24):

$$\frac{\dot{\lambda}_t}{\lambda_t} = -(r_t - n)$$

And plug equation (26):

$$-(\rho - n) - \theta\frac{\dot{c}_t}{c_t} = -(r_t - n)$$

Equivalently:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r_t - \rho) \tag{27}$$

Finally, using equation (23) to substitute out λ_t, the transversality condition can be rewritten into:

$$\lim_{t\to\infty} e^{-(\rho-n)t} c_t^{-\theta} a_t = 0 \tag{28}$$

Summary 1 *To derive the Euler equation, (i) set up the Hamiltonian, (ii) find the FOC, (iii) take the log-derivative of the FOC for consumption, and (iv) use it to substitute out the shadow price in the FOC for capital.*

3.2 Economic Meaning

The Euler equation (equation (27)) is at the heart of almost every model in macroeconomics[3]. It is very important that you be able to derive it and that you understand it.

The Euler equation is first and foremost a marginal rate of substitution between present and future consumption. To see this[4], rewrite $\dot{c}_t/c_t$:

$$\frac{\dot{c}_t}{c_t} = \frac{dc_t/dt}{c_t} \approx \frac{(c_{t+dt} - c_t)/dt}{c_t} = \left(\frac{c_{t+dt}}{c_t} - 1\right)\frac{1}{dt}$$

Substituting into equation (27):

$$\left(\frac{c_{t+dt}}{c_t} - 1\right)\frac{1}{dt} = \frac{1}{\theta}(r_t - \rho)$$

Rearranging:

$$\theta\left(\frac{c_{t+dt}}{c_t} - 1\right) = (r_t dt - \rho dt)$$

Using the approximation $\log(1+x) \approx x$ when x is close to zero:

$$\theta \log\left(1 + \frac{c_{t+dt}}{c_t} - 1\right) \approx (\log(1 + r_t dt) - \rho dt)$$

Taking the exponential:

$$\left(\frac{c_{t+dt}}{c_t}\right)^{\theta} \approx (1 + r_t dt)e^{-\rho dt}$$

[3] In future classes, you will mostly encounter its discrete time version. Needless to say, both mean the same thing.

[4] The mathematics that follow are loose and meant to build intuitions. It is unlikely that you be asked to do that kind of computations during an exam.

Rearranging:

$$\frac{c_t^{-\theta}}{e^{-\rho dt}c_{t+dt}^{-\theta}} \approx 1 + r_t dt \tag{29}$$

On the left hand side of equation (29), we have the ratio of marginal utilities of present and future consumptions. The marginal utility of future consumption is adjusted by the discount rate to take into account the cost of waiting. On the right hand side, we have the relative price of consuming now versus consuming in dt: the household forgoes the proceeds of savings $r_t dt$. This is micro 101! If a consumer allocates her budget between goods i and j, her optimal consumption bundle satisfies:

$$\frac{\partial U/\partial c_i}{\partial U/\partial c_j} = \frac{p_i}{p_j}$$

The Euler equation can also be seen as a pricing equation for assets. Rewriting equation (27):

$$r_t = \rho + \theta\gamma_{ct} \tag{30}$$

where $\gamma_{ct} = \dot{c}_t/c_t$. First, consider the case where $\gamma_{ct} = 0$. We have $r_t = \rho$: if consumption is perfectly smooth ($\gamma_{ct} = 0$), the interest rate must compensate the household for the cost of postponing consumption. Second, consider the case where $\gamma_{ct} > 0$. We have $r_t > \rho$. Indeed, since the utility function is strictly concave, having higher consumption in the future than in the present ($\gamma_{ct} > 0$) is costly. Households would like to transfer consumption from the future to the present. Hence, the interest rate must be high enough for them not do so. Third, consider the case where $\gamma_{ct} < 0$. We have $r_t < \rho$. Indeed, since the utility function is strictly concave, having lower consumption in the future than in the present ($\gamma_{ct} < 0$) is costly. Households would like to transfer consumption from the present to the future. Hence, the interest rate must be low enough for them not do so.

3.3 The Inter-Temporal Elasticity of Substitution (IES)

By definition the inter-temporal elasticity of substitution is the derivative of the growth rate of consumption with respect to the relative price of postponing consumption: IES $= \partial\gamma_c/\partial r_t$. From equation (27):

$$\text{IES} = \frac{\partial \gamma_c}{\partial r_t} = \frac{1}{\theta}$$

θ is the inverse of the IES!

Going back to our pricing equation (30), the IES is a determinant of the interest rate. For a higher IES (that is, a lower θ), the interest rate is less sensitive to consumption growth. Indeed, with a higher IES, households are more willing to substitute consumption across time. Hence, the interest rate does not need to move as much to entice households to save more or less.

References

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APPENDIX

A Some Mathematical Tips

Here are a few shortcuts that you might find helpful. You can use them without proof.

A.1 Growth Rates

A.1.1 Constant Growth Rate

When you are told that a variable, $A_t > 0$, grows at a constant rate, x, by definition it means:

$$\frac{\dot{A}_t}{A_t} = x \tag{31}$$

Note that equation (31) is a differential equation. It can be solved with the method that we saw in the first recitation. Applying the formula that we derived then, the general solution is:

$$A_t = be^{xt}$$

where b is some constant. Noting that $A_0 = be^0 = b$, we can rewrite this:

$$A_t = A_0 e^{xt}$$

Proposition 2 *If $A_t > 0$ grows at a constant rate x, then: $A_t = A_0 e^{xt}$.*

A.1.2 Shortcuts

To get the growth rate of a function of several variables, it is often convenient to take the logarithm and derive with respect to time. Here are a few particular cases.

The sum:

$$\begin{aligned} x = y + z \Rightarrow \log x = \log(y+z) \\ \Rightarrow \frac{\dot{x}}{x} = \frac{\dot{y}+\dot{z}}{y+z} = \frac{y}{y+z}\frac{\dot{y}}{y} + \frac{z}{y+z}\frac{\dot{z}}{z} = \frac{y}{x}\frac{\dot{y}}{y} + \frac{z}{x}\frac{\dot{z}}{z} \end{aligned}$$

The product:

$$\begin{aligned} x = yz \Rightarrow \log x = \log y + \log z \\ \Rightarrow \frac{\dot{x}}{x} = \frac{\dot{y}}{y} + \frac{\dot{z}}{z} \end{aligned}$$

The ratio:

$$\begin{aligned} x = \frac{y}{z} \Rightarrow \log x = \log y - \log z \\ \Rightarrow \frac{\dot{x}}{x} = \frac{\dot{y}}{y} - \frac{\dot{z}}{z} \end{aligned}$$

The power function:

$$\begin{aligned} x = y^\alpha z^\beta \Rightarrow \log x = \alpha \log y + \beta \log z \\ \Rightarrow \frac{\dot{x}}{x} = \alpha\frac{\dot{y}}{y} + \beta\frac{\dot{z}}{z} \end{aligned}$$

A.2 Homogeneity

The neo-classical production function is homogeneous of degree 1 in (K_t, L_t):

$$F(\lambda K_t, \lambda L_t, A) = \lambda F(K_t, L_t, A) \tag{32}$$

Take the derivative with respect to K_t:

$$\lambda F_K(\lambda K_t, \lambda L_t, A) = \lambda F_K(K_t, L_t, A)$$

where F_K denotes the derivative of F with respect to its first coordinate. The latter implies:

$$F_K(\lambda K_t, \lambda L_t, A) = F_K(K_t, L_t, A) \tag{33}$$

F_K is homogeneous of degree 0!

Proposition 3 *If F is homogeneous of degree 1 in (K_t, L_t), F_K and F_L are homogeneous of degree 0 in (K_t, L_t).*

A.3 Cobb-Douglas Is Your Friend

Assume that F is a Cobb-Douglas function: $Y_t = AK_t^{\alpha}L_t^{1-\alpha}$. Assuming that labor is paid at its marginal product, the share of output going to labor is:

$$\frac{w_t L_t}{Y_t} = \frac{(1-\alpha)\, AK_t^{\alpha}L_t^{-\alpha} \times L_t}{Y_t} = \frac{(1-\alpha)\, Y_t}{Y_t} = 1-\alpha$$

As far as capital is concerned, its marginal product is: $r_t = \alpha A K_t^{\alpha-1}L_t^{1-\alpha} - \delta$. But the owners of capital must also replace its depreciated part. Therefore, the share of output accruing to capital is:

$$\frac{(r_t+\delta)K_t}{Y_t} = \frac{\alpha A K_t^{\alpha-1}L_t^{1-\alpha} \times K_t}{Y_t} = \frac{\alpha Y_t}{Y_t} = \alpha$$

Proposition 4 *With a Cobb-Douglas production function, if factors are paid their marginal product, the shares of output accruing to each factor are constant.*

What is the economics behind this result? Let's rewrite the labor share as:

$$w_t \times \frac{L_t}{Y_t} = (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha} \times \left(\frac{L_t}{K_t}\right)^{\alpha}$$

There are two effects of an increase in the labor to capital ratio: (i) the marginal product of labor falls, and (ii) the quantity of labor relative to output goes up. These two channels push the labor share in opposite directions. One can see them as a price vs. quantity effect — the price goes down but the quantity goes up. It turns out that, with a Cobb-Douglas production, these two channels perfectly compensate each other[5].

[5]Indeed, the Cobb-Douglas production function has a very special property: the elasticity of substitution between capital

A.4 Linearization vs. Log-Linearization

Let f be a function defined from $\mathbb{R}^2$ to $\mathbb{R}$. The Taylor formula for a linear expansion around (x^*, y^*) is:

$$f(x,y) \approx f(x^*,y^*) + f_x(x^*,y^*)(x-x^*) + f_y(x^*,y^*)(y-y^*) \tag{34}$$

where f_x is the derivative of f with respect to its first coordinate and f_y the derivative of f with respect to its second coordinate.

A closely linked approach is the log-linearization. We rewrite f into:

$$f(x,y) = f(e^{\log x}, e^{\log y}) \equiv f(e^X, e^Y)$$

where $X = \log x$ and $Y = \log y$. Then, apply the Taylor formula using (X, Y) as variables and not (x, y):

$$\begin{aligned} f(x,y) &= f(e^X, e^Y) \\ &\approx f(e^{X^*}, e^{Y^*}) + e^{X^*} f_x(e^{X^*}, e^{Y^*})(X - X^*) + e^{Y^*} f_y(e^{X^*}, e^{Y^*})(Y - Y^*) \\ &= f(x^*,y^*) + x^* f_x(x^*,y^*)(\log x - \log x^*) + y^* f_y(x^*,y^*)(\log y - \log y^*) \end{aligned} \tag{35}$$

What is the relation between equation (34) and (35)? They are equivalent once we use the approximation $\log(1+z) \approx z$ when z is close to 0:

$$\log x - \log x^* = \log\left(\frac{x}{x^*}\right) = \log\left(1 + \frac{x-x^*}{x^*}\right) \approx \frac{x-x^*}{x^*} \tag{36}$$

$$\log y - \log y^* = \log\left(\frac{y}{y^*}\right) = \log\left(1 + \frac{y-y^*}{y^*}\right) \approx \frac{y-y^*}{y^*} \tag{37}$$

Now, rewrite equation (34):

$$f(x,y) \approx f(x^*,y^*) + f_x(x^*,y^*)x^* \frac{x-x^*}{x^*} + f_y(x^*,y^*)y^* \frac{y-y^*}{y^*}$$

Plug the relations obtained in equations (36) and (37):

$$f(x,y) \approx f(x^*,y^*) + x^* f_x(x^*,y^*)(\log x - \log x^*) + y^* f_y(x^*,y^*)(\log y - \log y^*)$$

Which is nothing else than equation (35)!

In practice, you can use any of the two methods when you are asked to linearize a model.

and labor is 1. See Rognlie (2014) for more details.