ONE SECTOR GROWTH MODEL

- REPRESENTATIVE CONSUMER WHO LIVES FOREVER
- $C(t)$ IS CONSUMPTION AT TIME t.
- $$\boxed{U = \int_0^\infty e^{-\rho \cdot t} \cdot u(c(t))\,dt}$$

$\rho > 0$, $u(\cdot)$ IS STRICTLY INCREASING, CONCAVE AND DEFINED ON $[0, \infty)$

- ρ IS DISCOUNT FACTOR
- $$\left.\begin{aligned} &c(t) + x(t) \leq f(k(t)) \\ &\dot{k}(t) = -\delta \cdot k(t) + x(t) \end{aligned}\right\}$$

$c(t), k(t) \geq 0$
$\forall t \in [0, \infty)$

- $k(0)$ IS GIVEN

$$\boxed{c(t) + \dot{k}(t) \leq f(k(t)) - \delta \cdot k(t)}$$

$$\mathcal{L} = \int_0^\infty e^{-\delta \cdot t} \cdot \left[u(c(t)) + \lambda(t) \cdot (f(k(t)) - \delta \cdot k(t) - \dot{k}(t) - c(t)) \right] dt$$

$$\int_a^b f'(t)\, g(t)\, dt = f(t)\, g(t) \Big|_a^b - \int_a^b f(t)\, g'(t)\, dt$$

$$\int_0^\infty e^{-\delta t} \cdot \lambda(t) \cdot \dot{k}(t)\, dt$$

$f'(t) = \dot{k}(t)$ $\quad f(t) = k(t)$

$g(t) = e^{-\delta t} \cdot \lambda(t)$

$g'(t) = -\delta \cdot e^{-\delta t} \cdot \lambda(t) + e^{-\delta t} \cdot \dot{\lambda}(t)$

$g'(t) = e^{-\delta t} \cdot (-\delta \cdot \lambda(t) + \dot{\lambda}(t))$

$$\int_0^\infty e^{-st}\cdot x(t)\cdot \dot{k}(t)\,dt =$$

$$e^{-st}\cdot x(t)\cdot k(t)\Big|_0^\infty -$$

$$\int_0^\infty e^{-st}\cdot\left(-s\cdot x(t)+\dot{x}(t)\right)\cdot k(t)\,dt$$

$$\lim_{t\to\infty} e^{-s\cdot T}\cdot x(T)\cdot k(T) -$$

$$x(0)\cdot k(0)$$

$$\boxed{\begin{aligned}&\int_0^\infty e^{-st}\cdot x(t)\cdot \dot{k}(t)\,dt = \\ &\lim_{t\to\infty} e^{-sT}\, x(T)\cdot k(T) - \qquad (2)\\ &x(0)\cdot k(0) - \int_0^\infty e^{-st}\cdot\big(\\ &-s\cdot x(t) + \dot{x}(t)\big)\cdot k(t)\,dt\end{aligned}}$$

$$\mathcal{L} = \int_0^\infty e^{-\delta t} \cdot [u(c(t)) + \lambda(t) \cdot (\phi(k(t)) - \delta \cdot k(t) - \rho \cdot k(t) - c(t)) + \dot{\lambda}(t) \cdot k(t)]\, dt + \lambda(0) \cdot k(0) - \lim_{T \to \infty} e^{-\delta \cdot T} \cdot \lambda(T) \cdot k(T) \quad (3)$$

We are choosing $c(t)$ and $k(t)$

TRANSVERSALITY CONDITION

$$\boxed{\lim_{T \to \infty} e^{-\delta \cdot T} \cdot \lambda(T) \cdot k(T) = 0 \quad (a)}$$

$$(c(t)): \; e^{-\delta t} \cdot u'(c(t)) - e^{-\delta t} \cdot \lambda(t) = 0 \Rightarrow$$

$$\boxed{u'(c(t)) = \lambda(t) \quad (b)}$$

$(k(t)):$

$$\lambda(t) \cdot [Q'(k(t)) - (\rho + \delta)] + \dot{\lambda}(t) = 0 \Rightarrow$$

$$\boxed{-\frac{\dot{\lambda}(t)}{\lambda(t)} = Q'(k(t)) - (\rho + \delta) \quad (c)}$$

$c(t), k(t) \wedge \lambda(t)$ THAT SATISFIES $(a), (b) \wedge (c)$

$$\lambda(t) = u'(c(t)) \qquad (1)$$

$$\dot{\lambda}(t) = u''(c(t)) \cdot \dot{c}(t) \qquad (2)$$

$$\boxed{-\frac{u''(c(t)) \cdot \dot{c}(t)}{u'(c(t))} = Q'(k(t)) - (\rho + \delta)}$$

STEADY STATE

$\dot{\lambda}(t) = 0, \quad \dot{k}(t) = 0$

$\dot{c}(t) = 0$

$g(k^*) = (\rho + \delta) \implies$

$$\boxed{k^* = g^{-1}(\rho + \delta)}$$

$$\boxed{c^* = g(k^*) - \delta \cdot k^*}$$

Macroeconomics I

Recitation 2

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Fall 2023

1 The Hamiltonian: A Pratical Guide and Some Examples

The Hamiltonian is the handiest way to solve a problem of dynamic optimization in continuous time. The typical problem is of the form:

$$\max_{c(t),k(t)} V(0) = \int_0^{\infty} e^{-rt}.V\left[c(t), k(t)\right]\, dt \tag{1}$$

subject to:

$$\dot{k}(t) = g\left[c(t), k(t), t\right] \tag{2}$$

$$k(0) = k_0 \tag{3}$$

$$\lim_{t\to\infty} e^{-rt}k(t) \geq 0 \tag{4}$$

The formula of the Hamiltonian is:

$$\mathcal{H} = e^{-rt}.V\left[c(t), k(t)\right] + \lambda(t).g\left[c(t), k(t), t\right] \tag{5}$$

For practical purposes, *learn Equation (5) by heart.* Once you know it, you are equipped to apply the four-step method. In problem sets or exams, you need not (and should not) redo the proof that we covered during the lecture. Just apply the method!

Step 1. Determine the state and control variables. State variable(s) have dots on top of them in the constraint(s).

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Step 2. Write the Hamiltonian. Here, $c(t)$ is the control, and $k(t)$ the state.

Step 3. Set the derivative(s) of the Hamiltonian with respect to control variable(s) to zero, and its derivative(s) with respect to state variable(s) to $-\dot{\lambda}(t)$:

$$\frac{\partial \mathcal{H}}{\partial c(t)} = 0 \tag{6}$$

$$\frac{\partial \mathcal{H}}{\partial k(t)} = -\dot{\lambda}(t) \tag{7}$$

Step 4. Add the transversality condition:

$$\lim_{t \to +\infty} e^{-rt}.k(t) = 0 \tag{8}$$

Example 1 (Investment)

$$\max_{I(t),L(t),K(t)} V(0) = \int_0^T e^{-rt}.\left[F(K(t), L(t), A) - w(t).L(t) - I(t)\right] dt$$

subject to:

$$\dot{K}(t) = I(t) - \delta K(t)$$

$$K(0) = K_0 > 0$$

$$\lim_{t \to +\infty} e^{-rt}.K(t) \geq 0$$

$I(t)$ and $L(t)$ are the control variables. $K(t)$ is the state. The Hamiltonian writes:

$$\mathcal{H} = e^{-rt}\left[F(K(t), L(t), A) - w(t).L(t) - I(t)\right] + \lambda(t)\left[I(t) - \delta.K(t)\right]$$

The first-order conditions are:

$$\frac{\partial \mathcal{H}}{\partial L(t)} = 0 \iff \frac{\partial F}{\partial L(t)} = w(t)$$

$$\frac{\partial \mathcal{H}}{\partial I(t)} = 0 \iff \lambda(t) = e^{-rT}$$

$$\frac{\partial \mathcal{H}}{\partial K(t)} = -\dot{\lambda}(t) \iff e^{-rT}\frac{\partial F}{\partial K(t)} - \lambda(t).\delta = -\dot{\lambda}(t)$$

The transversality condition is:

$$\lim_{t\to+\infty} \lambda(t).K(t) = 0$$

Example 2 (Consumption)

$$\max_{c(t),a(t)} V(0) = \int_0^{\infty} e^{-rt}.u\left[c(t)\right]\, dt$$

subject to:

$$\dot{a}(t) = w(t) + r.a(t) - c(t)$$

$$a(0) = a_0$$

$$\lim_{t\to+\infty} e^{-rt}.a(t) \geq 0$$

with $w(t)$ exogenous.

$c(t)$ is the control, $a(t)$ the state. The Hamiltonian writes:

$$\mathcal{H} = e^{-rt}.u\left[c(t)\right] + \lambda(t).\left[w(t) + r.a(t) - c(t)\right]$$

The first-order conditions are:

$$\frac{\partial \mathcal{H}}{\partial c(t)} = 0 \iff e^{-rt}.u'(c(t)) = \lambda(t)$$
$$\frac{\partial \mathcal{H}}{\partial a(t)} = -\dot{\lambda}(t) \iff \lambda(t).r = -\dot{\lambda}(t)$$

The transversality condition is:

$$\lim_{t\to+\infty} \lambda(t).a(t) = 0$$

2 The Transversality Condition

The transversality condition is usually puzzling to the first-year graduate student. In particular, it is often confused with the no-Ponzi condition. In fact, the transversality condition is a mix of two things:

- an external constraint, imposed by financial markets, that forbids the agent to die in debt. This constraint is called the no-Ponzi condition;
- an optimality condition that pushes the agent to leave as much debt as she can.

To understand this, remember the finite horizon problem studied in class:

$$\max_{c(t),k(t)} V(0) = \int_0^T e^{-rt}.V\left[c(t), k(t)\right]\, dt \tag{9}$$

subject to:

$$\dot{k}(t) = g\left[k(t), c(t), t\right] \tag{10}$$

$$k(0) = k_0 > 0 \tag{11}$$

$$e^{-rT}.k(T) \geq 0 \tag{12}$$

Equation (12) is the no-Ponzi condition.

After some algebra involving an integration by part, we showed that the Lagrangian could be written[1]:

$$\mathcal{L} = \int_0^T \Big(\mathcal{H}\left(c(t), k(t), t\right) + \dot{\lambda}(t).k(t)\Big) + \lambda(0).k_0 - \lambda(T).k(T) + \nu.e^{-rT}.k(T)$$

where: $\mathcal{H}\left(c(t), k(t), t\right) = e^{-rt}.V\left[c(t), k(t)\right] + \lambda(t).g\left[k(t), c(t), t\right]$. The first-order condition of this Lagrangian with respect to $k(T)$ is[2]:

$$\lambda(T) = \nu.e^{-rT} \tag{13}$$

To which one can add the complentary slackness condition that is associated with Equation (12):

$$\nu.e^{-rT}.k(T) = 0 \tag{14}$$

Equations (13) and (14) combine to give the transversality condition:

$$\lambda(T).k(T) = 0 \tag{15}$$

While Equation (14) is inherited from the no-Ponzi condition, Equation (13) comes from optimization: as long as k has a marginal value at the time of death ($\lambda(T) > 0$), the constraint is binding ($\nu > 0$). Indeed, the agent would like to leave some debt to increase consumption. Put another way, financial markets would be fine with $k(T) > 0$; while the agent would rather leave $k(T) < 0$ (as long as $\lambda(T) > 0$). Since both have to agree, they meet half way: $k(T) = 0$ (as long as $\lambda(T) > 0$).

[1] Equation (A.61) in Barro and Sala-i Martin (Barro and Sala-i Martin, p. 608).

[2] Since death happens at time T, $k(T)$ and $c(T)$ do not enter the objective function.

With an infinite horizon, the no-Ponzi and transversality conditions are:

$$\lim_{t \to +\infty} e^{-rt}.k(t) \geq 0 \tag{16}$$

$$\lim_{t \to +\infty} \lambda(t).k(t) = 0 \tag{17}$$

The intuition is the same as in finite time.

Of course, you can (and should) use Equations (15) and (17) without any proof.

3 The Rate of Depreciation

Consider the capital accumulation constraint:

$$\dot{k}(t) = I(t) - \delta k(t) \tag{18}$$

The general solution of this differential equation is:

$$k(t) = e^{-\delta t} \int_0^t e^{\delta \tau} I(\tau)\, d\tau + b e^{-\delta t} \tag{19}$$

where b is some constant.

There are two possible interpretations of δ. The first one stems from Equation (18):

$$\left. \frac{\dot{k}(t)}{k(t)} \right|_{I(t)=0} = -\delta \tag{20}$$

Equation (20) means that, without investment, the capital stock decreases at rate δ. Put another way, capital depreciates at rate δ. This does *not* mean that the economy (or the firm) loses δ% of its capital stock every millisecond. Indeed:

$$\frac{\dot{k}(t)}{k(t)} = \frac{dk(t)}{dt} \times \frac{1}{k(t)} \Rightarrow \left. \frac{dk(t)}{k(t)} \right|_{I(t)=0} = -\delta dt \tag{21}$$

So $k(t)$ decreases by ($\delta\mathbf{dt}$)% at every infinitesimal time period dt. Since dt is infinitesimal, the percentage decrease is also infinitesimal.

Suppose now that, without investment, the capital stock shrinks by x% every unit of time. Mathematically, this translates into:

$$\left. \frac{k(t+1)}{k(t)} \right|_{I(\tau)=0} = 1 - x\% \tag{22}$$

Note that we are still in continuous time: t does not need to be an integer. Using Equation (19), Equation (22) is equivalent to:

$$1 - x\% = \frac{e^{-\delta(t+1)}b}{e^{-\delta t}b} = e^{-\delta} \tag{23}$$

Taking logarithms:

$$\delta = -\log(1 - x\%) \approx x\% \tag{24}$$

So δ is approximately the share of the capital stock that is lost to depreciation in one unit of time.

Example 3 (Calibration) *Time is counted in days. Say that we know from the data that capital depreciates by 10% every year:*

$$1 - 10\% = \left.\frac{k(t+365)}{k(t)}\right|_{I(\tau)=0}$$

Using Equation (19):

$$1 - 10\% = \frac{e^{-\delta(t+365)}b}{e^{-\delta(t)}b} = e^{-365\times\delta}$$

Therefore:

$$\delta = \frac{-\log(1-10\%)}{365} \approx \frac{10\%}{365}$$

Table 1: Depreciation in the US in 2016

Type	Depreciation rate	Depreciation as a share of GDP
Fixed assets	5%	16%
Private nonresidential	9%	10%
Private residential	2%	3%
Government	4%	3%
Consumer durable goods	20%	6%
Total	6%	21%

Source: BEA. Note: the third column does not necessarily add up because of rounding

References

Barro, R. J. and X. Sala-i Martin. Economic growth (second ed.). *Cambridge: MIT Press*.