# Department of Economics - Sciences Po
# Macroeconomics III

**Problem Set 3 - Dynamic Programming**

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Fall 2023

## Question 1

Consider an economy in which the representative consumer lives forever. There is a good in each period that can be consumed or saved as capital as well as labor. The consumer's utility function is

$$\sum_{t=0}^{\infty} \beta^t \left(\log c_t + \gamma \log x_t\right)$$

Here $\beta \in (0,1)$ and $\gamma \in (0,1)$. The consumer is endowed with 1 unit of labor in each period, some of which can be consumed as leisure, $x_t$, and some of which is supplied as labor, $l_t$. The consumer is also endowed with $\bar{k}_0$ units of capital in the first period. The feasible allocations satisfy:

$$c_t + k_{t+1} \leq \theta k_t^{\alpha} l_t^{1-\alpha},$$

where $\theta > 0$ and $0 < \alpha < 1$. Notice we also have the following constraints:

$$x_t + l_t \leq 1,$$

$$c_t, x_t, l_t, k_t \geq 0,$$

$$k_0 \leq \bar{k}_0.$$

a) Write down the **Bellman equation** for this problem.

b) **Guessing that the value function** $V(k)$ has the form $a_0 + a_1 \log k$ and that the policy function for labor $l(k)$ is constant, find **analytical solutions** for the value function $V(k)$ and the policy functions $c(k), x(k), l(k), k'(k)$.

c) Define a competitive equilibrium for this economy (either an Arrow-Debreu or a Sequential Market Equilibrium). Explain how would you be able to use the results in (b) to find the competitive equilibrium for this economy.

## Question 2

Consider a single agent problem where each period, $w$ total output is produced and can be divided into consumption of a perishable good, $c_t$, and investment in a durable good, $d_{xt}$. The durable depreciates like a capital good, but is not directly productive. The stock of durables at any date, $d_t$, produces a flow of services that enters the utility function. Thus, the problem faced by the household with initial stock $d_0$ is:

$$\begin{aligned}\max_{\{(c_t,d_t,d_{xt})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[u_1(c_t) + u_2(d_t)\right] \\ \text{s.t. } c_t + d_{xt} \le w \quad \forall t \\ d_{t+1} \le (1-\delta)d_t + d_{xt} \quad \forall t \\ c_t, d_t, d_{xt} \ge 0 \quad \forall t \\ d_0 \text{ given}\end{aligned}$$

where both $u_1$ and $u_2$ are strictly increasing and continuous. Ignore the non-negativity constraints on $d_{xt}$ while solving this problem.

a) State a condition on either $u_1$ or $u_2$ (or both) such that you can write an equivalent problem in the following form:

$$\begin{aligned}\max_{\{d_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(d_t, d_{t+1}) \\ \text{s.t. } d_{t+1} \in \Gamma(d_t) \\ d_0 \text{ given}\end{aligned},$$

where $\Gamma(d_t) \in \mathbb{R}^+$. What is $F$? What is the correspondence $\Gamma$ (i.e., this is the set of possible values where the variable $d_{t+1}$ can be chosen)?

b) Write the **Bellman equation** for this problem.

c) State additional conditions on $u_1$ and $u_2$ such that the value function $v(d)$ you found previously is both strictly increasing and strictly concave. Prove or argument with words these two properties.

*For the remaining questions, assume that both $u_1$ and $u_2$ satisfy the Inada conditions and are continuously differentiable.*

d) State the **envelope condition** and the **F.O.C.** for the functional equation problem in (b).

e) Find the **Euler equation** of this problem.

f) Show that **there is a unique steady state** value of the stock, $d^*$, such that if $d_0 = d^*$, then $d_t = d^* \forall t$. Show that $d^* > 0$.

g) Show that the policy functions for the solution, $c^*(d)$ and $d' = g^*(d)$, are increasing (i.e., you can use the fact that the value function is concave to prove this result).

h) Show or argue that the system is **globally stable**, i.e., for any value $d > d^*$ the value of $d$ decreases until it reaches $d^*$ and for any value $d < d^*$, the value of $d$ increases until it reaches $d^*$. You have to use the policy functions and assume that they are differentiable.

## Question 3

Assume that agents have the period utility function:

$$u(c_t - hc_{t-1}, l_t), \text{ with } h < 1$$

where $l_t$ is labor and $c_t$ denotes consumption. They are assumed to have a discount factor $\beta$ and they understand now that consumption affects their habit. Budget constraint is ($w$ is constant to simplify the problem) :

$$c_t + a_{t+1} = (1 + r_t) a_t + wl_t$$

a) Write the **Bellman equations** and derive the pricing kernel.

b) What is the **steady state interest rate**?

c) Assume that

$$u(c,l) = \frac{c^{1-\sigma} - 1}{1-\sigma} - \frac{l^{1+\varepsilon}}{1+\varepsilon}$$

and that $a_t = 0$. How does the steady-state labor supply change when agents care more about habits?

## Question 4

Houses are durable goods from which households derive some utility. To model the demand for houses, a simple shortcut consists in introducing houses in the utility function. The goal of this exercise is to be able to use data on house prices and interest rates to derive properties of the demand for houses. Households thus derive utility from consumption and from having houses. The instantaneous utility function is $u(c_t, H_t)$ where $H_t$ is the amount of housing. Households also have access to financial savings denoted $b_t$ at period $t$, remunerated at a real rate $r_t$ between period $t$ and $t+1$. The program of the households is

$$\max_{\{c_t, H_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t, H_t)$$
$$c_t + b_t + P_t H_t = W_t + (1 + r_{t-1}) b_{t-1} + P_t (1 - \delta) H_{t-1}$$

where $\delta$ is the depreciation rate for houses.

a) Write the **transversality conditions** for this problem for the financial wealth and the stock of housing. Explain the intuition for these conditions.

b) Write the **Bellman equation** for this problem, with the value function denoted $V(b_{t-1}, H_{t-1})$.

c) Compute the **F.O.C.s**

d) Derive the **envelope conditions** and find the **two Euler equations.**

e) We assume that the utility function is

$$u(c_t, H_t) = \left(c_t^\rho + H_t^\rho\right)^{\frac{1}{\rho}} \text{ with } \rho < 1$$

Explain what is the economic meaning of the $\rho$ coefficient. Using the two Euler equations, express $\frac{c_t}{H_t}$ as a function of $P_t$, $P_{t+1}$ and $r_t$. How can we get $\rho$ from the data?

## Question 5 (Programming)

**For this question you need to use a computer.** Feel free to use the language you are more comfortable with. Consider the optimal growth problem:

$$\begin{aligned} &\max \textstyle\sum_{t=0}^{\infty} (0.6)^t \log c_t \\ &\text{s.t. } c_t + k_{t+1} \leq 10 k_t^{0.4} \\ &c_t, k_t \geq 0 \\ &k_0 = \bar{k}_0 \end{aligned}$$

a) Write down the **Euler conditions** and the **transversality condition** for this problem. Calculate the steady state values of $c$ and $k$.

b) Write down the functional equation that defines the value function for this problem. Guess that the value function has the form $a_0 + a_1 \log k$. Calculate the value function and the policy function. Verify that the policy function generates a path for capital that satisfies the Euler conditions and transversality condition in part (a).

c) Let capital take values for the discrete grid $(2, 4, 6, 8, 10)$. Make the original guess $V_0(k) = 0 \forall k$, and perform the first ten steps of the value function iteration below:

$$V_{i+1}(k) = \max \log\left(10k^{0.4} - k'\right) + 0.6 V_i(k')$$

Plot your results (value function in terms of the grid on capital and the policy functions for consumption and capital) for each of the steps and comment the pattern you can observe.

d) Using the same grid discrete grid $(2, 4, 6, 8, 10)$ plot the analytical result you obtained in part (b) and compare with the results obtained previously.

e) Now, perform the value function iterations until

$$\max_k |V_{i+1}(k) - V_i(k)| < 10^{-5}$$

Report the value function and the policy function that you obtain. Compare these results with what you obtained in (c) and with the analytical solution in (b).

f) Repeat part (e) for the grid of capital stocks (0.05, 0.10, .., 9.95, 10). What can you observe in this case?