Department of Economics - Sciences Po
Macroeconomics III

Complete markets economy

Diego Rodrigues

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Abstract

This material contains two examples. In Example 1 we go through the resolution of a Sequential markets equilibrium in an environment where the total output is constant across all periods, i.e., the economy is not experiencing the downwards and upwards in terms of total production. We will show that the consumers will smooth consumption across periods through the mechanism of borrowing and lending. In order to solve the Sequential markets we will recover the prices of the Arrow-Debreu structure and then we will use those prices to find the allocations and the assets.

In Example 2 we go through a case where the endowment varies among periods, i.e., this can be interpreted as an economy experiencing *booms* and *recessions*. We will show that the consumer will still smooth consumption, but in this case the consumption bundle in *recessions* is not going to be the same as in *booms*.

Example 1

Consider the following economy

$$u\left(c_0^i, c_1^i, \cdots\right) = \sum_{t=0}^{\infty} \beta^t \log c_t^i,$$

$$\text{where } \left(e_0^1, e_1^1, e_2^1, e_3^1, \cdots\right) = (6, 4, 6, 4, \cdots) \quad \text{and}$$
$$\left(e_0^2, e_1^2, e_2^2, e_3^2, \cdots\right) = (4, 6, 4, 6, \cdots)$$

a) Define a **Sequential Market Equilibrium** and find it.

b) Find the **Arrow-Debreu Equilibrium** in this economy.

The problem is given by:

$$\max_{\{(c_t^i, a_{t+1}^i)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u_i(c_t^i) \tag{SM}$$

subject to

$$\begin{aligned} &c_t^i + a_{t+1}^i \leq e_t^i + (1 + r_t)a_t^i \quad \forall t \geq 0 \\ &a_0^i = 0 \end{aligned}$$

The Sequential Market Equilibrium is a sequence of allocations $\{(c_t^i, a_{t+1}^i)\}_{t=0}^{\infty}$ for all agents $i \in I = \{1, 2\}$ and a price system $\{(r_t)\}_{t=0}^{\infty}$ such that:

1. Given the price system $\{(r_t)\}_{t=0}^{\infty}$, each agent i solves the (SM) problem.
2. The good and asset markets clear:

$$\sum_{i \in I} c_t^i = \sum_{i \in I} e_t^i \Rightarrow c_t^1 + c_t^2 = e_t^1 + e_t^2 = 10 \quad \forall t \geq 0 \tag{1}$$

$$\sum_{i \in I} a_t^i = 0 \Rightarrow a_t^1 + a_t^2 = 0 \quad \forall t \geq 0 \tag{2}$$

3. There is a Non-Ponzi scheme and a transversality condition:

$$a_{t+1}^i \geq -\overline{A},$$

with $\overline{A} \in \mathcal{R}_+$ and

$$\lim_{t \to \infty} p_t a_{t+1}^i \leq 0,$$

such that the consumer does not over accumulate assets.

By writing the Lagrangian of the consumer problem we will have the following:

$$\mathcal{L} : \sum_{t=0}^{\infty} \beta^t u_i(c_t^i) + \sum_{t=0}^{\infty} \lambda_t (e_t^i + (1 + r_t) a_t^i - c_t^i - a_{t+1}^i).$$

Notice we have one Lagrangian multiplier for each period of time, since the consumer is choosing in every $t \geq 0$ the consumption good, c_t^i, and the asset, a_{t+1}^i.

By taking F.O.C. we have the following:

$$[c_t^i] : \frac{B^t}{c_t^i} = \lambda_t \quad \forall t \geq 0, \forall i \in I \tag{3}$$

$$[a_{t+1}^i] : -\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0 \quad \forall t \geq 0 \tag{4}$$

$$[\lambda_t] : e_t^i + (1 + r_t) a_t^i = c_t^i + a_{t+1}^i \tag{5}$$

Replace (3) into (4) and observe we can reach the following Euler Equation:

$$\frac{c_{t+1}^i}{c_t^i} = \beta(1 + r_{t+1}) \tag{EE}$$

Now take (EE) and observe we have the following:

$$c_{t+1}^i = c_t^i \beta (1 + r_{t+1})$$

Sum up across all agents and we will have:

$$\sum_{i \in I} c_{t+1}^i = \sum_{i \in I} c_t^i \beta (1 + r_{t+1}),$$

By equation (1) we know $\sum_{i \in I} c_t^i = \sum_{i \in I} c_{t+1}^i = 10$, which implies that:

$$\beta = \frac{1}{1 + r_{t+1}}$$

Observe for any $t \geq 0$, $r_{t+1} = r$. Then by (EE) we have the following is true:

$$c_{t+1}^i = c_t^i \equiv c^i \quad \forall t \geq 0$$

.

$$\beta = \frac{1}{1 + r} \tag{6}$$

The above highlights the consumption is going to be the same in every period (observe (EE)). This model empathizes all our discussion about smooth consumption.

Now what we are going to do is to go from the Sequential Market Equilibrium to the Arrow-Debreu structure and find the prices in the Arrow-Debreu in a direct approach, different from the proof by contradiction we had seen in class. For this notice by using the fact that $c^i_{t+1} = c^i_t \equiv c^i \quad \forall t \geq 0$ and $r_{t+1} = r \quad \forall t \geq 0$ that each consumer faces a sequence of budget constraints, one for each period of time. The sequence can be described as follows (Notice in period 0 as defined in (SM), $a^i_0 = 0$):

$$c^i + a^i_1 = e^i_0 \tag{7}$$
$$c^i + a^i_2 = e^i_1 + (1+r)a^i_1 \tag{8}$$
$$c^i + a^i_3 = e^i_2 + (1+r)a^i_2 \tag{9}$$

...

Take equation (8), isolate a^i_1 and observe that:

$$a^i_1 = \frac{c^i}{1+r} + \frac{a^i_2}{1+r} - \frac{e^i_1}{1+r} \tag{10}$$

Replace (10) in (7) and notice we obtain the following:

$$c^i + \frac{c^i}{1+r} + \frac{a^i_2}{1+r} = e^1_0 + \frac{e^i_1}{1+r} \tag{11}$$

Now take equation (9), isolate a^i_2 and observe that:

$$a^i_2 = \frac{c^i}{1+r} + \frac{a^i_3}{1+r} - \frac{e^i_2}{1+r} \tag{12}$$

Replace (12), in (11) and notice we obtain the following:

$$c^i + \frac{c^i}{1+r} + \frac{c^i}{(1+r)^2} + \frac{a^i_3}{(1+r)^2} = e^1_0 + \frac{e^i_1}{1+r} + \frac{e^i_2}{(1+r)^2} \tag{13}$$

From the equation above you can perceive we already have a pattern, in the sense that if we continue doing this substitutions until an arbitrary period T we will obtain the following:

$$\sum_{t=0}^{T} \frac{c^i}{\prod_{j=0}^{t}(1+r)} + \frac{a^i_{T+1}}{\prod_{j=0}^{T}(1+r)} = \sum_{t=0}^{T} \frac{e^i_t}{\prod_{j=0}^{t}(1+r)} \tag{14}$$

Notice from (14) that for the case we set $T = 2$ we obtain exactly the expression (13).
The limit on debt is going to guarantee us that:

$$\lim_{T\to\infty} \frac{a^i_{T+1}}{\prod_{j=0}^{T}(1+r)} = 0$$

Observe that in the limit, (14) will become the following:

$$\sum_{t=0}^{\infty} \frac{c^i}{\prod_{j=0}^{t}(1+r)} = \sum_{t=0}^{\infty} \frac{e^i_t}{\prod_{j=0}^{t}(1+r)} \tag{15}$$

We want to find the prices $\{p_t\}_{t=0}^{\infty}$ such that we have the following structure in the Arrow-Debreu equilibrium:

$$\sum_{t=0}^{\infty} p_t c^i = \sum_{t=0}^{\infty} p_t e^i_t \tag{16}$$

Now it is trivial to observe that for (15) to be equal to (16) we need to set the prices such that:

$$p_t = \frac{1}{\prod_{j=0}^{t}(1+r)} \tag{17}$$

Observe from (6) we are able to show that $\dfrac{1}{(1+r)} = \beta$. Therefore the equation (17) becomes exactly:

$$p_t = \beta^t \tag{18}$$

Now using the price (18) and the budget constraint (16) we can find the consumption bundles for each agent as we proceeded in class. First we will find the consumption of agent 1 (do not forget to use his endowment):

$$c^1 \underbrace{\sum_{t=0}^{\infty} p_t}_{\frac{1}{1-\beta}} = \sum_{t=0}^{\infty} p_t e_t^i = \underbrace{6 + 4\beta + 6\beta^2 + 4\beta^3 + ...}_{\frac{6+4\beta}{1-\beta^2}}$$

From the above we will get that:

$$c^1 = \frac{6+4\beta}{1+\beta}$$

.

We can proceed in the same fashion for agent 2 and obtain:

$$c^2 \underbrace{\sum_{t=0}^{\infty} p_t}_{\frac{1}{1-\beta}} = \sum_{t=0}^{\infty} p_t e_t^i = \underbrace{4 + 6\beta + 4\beta^2 + 6\beta^3 + ...}_{\frac{4+6\beta}{1-\beta^2}}$$

From the above we will get that:

$$c^2 = \frac{4+6\beta}{1+\beta}$$

Verify that $c^1 + c^2$ is indeed equal to 10.

Last observe that in this setting we can also recover how the assets are going to be in each period. The only restriction we need to see is to the budgets constraints (7), (8), and (9). So, for instance, pick equation (7) for agent 1 and notice:

$$a_1^1 = e_0^1 - c^1 = 6 - \frac{6+4\beta}{1+\beta} = \frac{2\beta}{1+\beta}$$

For agent 2 we have:

$$a_1^2 = e_0^2 - c^1 = 4 - \frac{6+4\beta}{1+\beta} = \frac{-2\beta}{1+\beta}$$

This shows that in period 1 agent 1, the *rich* agent (this agent has an endowment of 6), is saving, since his asset position is positive and agent 2, the *poor* agent (this agent has an endowment of 4), is borrowing, since his asset position is negative. In other words, agent 1 is lending to agent 2. We can continue this process and discover actually how the assets are going to evolve. We will go just one more period further. Take equation (8) for agent 1 and notice:

$$a_2^1 = e_1^1 + (1+r)a_1^1 - c^1$$

Replace the elements in the above equation and notice that

$$a_2^1 = 4 + \frac{1}{\beta}\Big(\frac{2\beta}{1+\beta}\Big) - \frac{6+4\beta}{1+\beta} = 0$$

Therefore, finally we found that the Sequential Markets Equilibrium is given by the elements (6), (18) and the allocations c^1, c^2, a_1^1, a_1^2, $a_2^1, a_2^2, ...$

Example 2

Consider the following economy

$$u\left(c_0^i, c_1^i, \cdots\right) = \sum_{t=0}^{\infty} \beta^t \log c_t^i,$$

$$\text{where } \left(e_0^1, e_1^1, e_2^1, e_3^1, \cdots\right) = (3,1,0,3,1,0,3,1,0,\cdots) \quad \text{and}$$

$$\left(e_0^2, e_1^2, e_2^2, e_3^2, \cdots\right) = (1,1,2,1,1,2,1,1,2,\cdots)$$

a) Find the **Arrow-Debreu Equilibrium** for this economy

First thing to notice in this economy different from the previous one is that the total endowment is not constant anymore. Observe

$Y_t = e_t^1 + e_t^2 \quad \forall t \geq 0$ is such that

$$Y = (4,2,2,4,2,2,4,2,2,\cdots)$$

The above can be seen as an economy which has periods of *booms* (High endowment) and *recessions* (Low endowment). By denoting the periods with High endowment by H, when $Y_t = 4$ and the periods of Low endowment by L, when $Y_t = 2$ we have the following structure:

$$Y = (Y_H, Y_L, Y_L, Y_H, Y_L, Y_L, Y_H, Y_L, Y_L, \cdots)$$

What we expect in this case is that we will have consumption smooth, but the bundles will be such that, agents are going to consume the same bundle every time the economy is H and another consumption bundle every time the economy is L. For the previous point to happen what we need is to have a price system when the economy is in H and another one when the economy is in L. From a theoretical point of view, we can expect that the price when the economy is in L will be higher when the economy is in H, since the consumption good will be more valuable in *recessions* than in *booms*.

First notice the Lagrangian of the problem will be given by:

$$\mathcal{L} : \sum_{t=0}^{\infty} \beta^t \log c_t^i + \lambda^i [\sum_{t=0}^{\infty} p_t(e_t^i - c_t^i)]$$

In this case we just have one Lagrangian multiplier since we are solving the Arrow-Debreu problem (i.e., agents are choosing all consumption bundles in period t=0 and they have information about the prices)

By solving the F.O.C. we have:

$$[c_t^i] : \frac{\beta^t}{c_t^i} = \lambda^i p_t \quad \forall t \geq 0 \tag{1}$$

The equation above is valid for all t, including $t = 0$, which has a result:

$$\frac{\beta^0}{c_0^i} = \lambda^i p_0 \tag{2}$$

.

In class we had discussed that what matters is the relative price in the economy, so we can set p_0 to be the *numeraire* and be such that $p_0 = 1$. Divide (1) by (2) and obtain the following:

$$p_t = \frac{\beta^t c_0^i}{c_t^i} \quad \forall i \in I \tag{3}$$

By equation (3) and since we are in competitive equilibrium we know that :

$$p_t = \frac{\beta^t c_0^1}{c_t^1} = \frac{\beta^t c_0^2}{c_t^2} \tag{4}$$

Notice that by the way we defined the vector

$$Y = (Y_H, Y_L, Y_L, Y_H, Y_L, Y_L, Y_H, Y_L, Y_L, \cdots)$$

our periods are such that we have periods H and periods L. Observe the period $t=0$ is the period where we have High Endowment (H), so that $c_0^i = c_H^i$. The next period t is the period of Low Endowment (L), so that $c_t^i = c_L^i$.

By using what we just seen and (4) we can notice that:

$$\frac{c_H^1}{c_L^1} = \frac{c_H^2}{c_L^2} \tag{5}$$

The markets are gonna clear for each period (H) and (L) in a way such that:

$$c_H^1 + c_H^2 = 4 \tag{6}$$

$$c_L^1 + c_L^2 = 2 \tag{7}$$

From (6) observe $c_H^2 = 4 - c_H^1$ and from (7) observe $c_L^2 = 2 - c_L^1$. Now, replace those results into (5) and notice that:

$$\frac{c_H^1}{c_L^1} = \frac{4 - c_H^1}{2 - c_L^1} \tag{8}$$

Simple algebra in the above equation lead us to conclude:

$$\frac{c_H^1}{c_L^1} = 2 \tag{9}$$

Now take equation (4) again and notice that $c_0^i = c_H^1$, so in case we want to define the price when the endowment is H we will have

$$p_H = \frac{\beta^t c_H^i}{c_H^i} = \beta^t \tag{10}$$

Now in case we want to define the prices when the endowment is low we will have:

$$p_L = \frac{\beta^t c_H^i}{c_L^i}$$

The above is valid for all i, including when $i=1$, so we will have:

$$p_L = \frac{\beta^t c_H^1}{c_L^1} = 2\beta^t, \tag{11}$$

where the last equality comes from replacing equation (9) in the above. Therefore (10) and (11) define the prices.

In order to find the allocations observe we will have:

$$\sum_{t=0}^{\infty} p_t c_t^i = \sum_{t=0}^{\infty} p_t e_t^i$$

First we will find this for agent 1. Notice by using the structure H and L we defined we will have:

$$p_H c_H^1 + p_L c_L^1 + p_L c_L^1 + p_H c_H^1 + ... = p_H e_H^1 + p_L e_L^1 + p_L e_L^1 + p_H e_H^1 + ...$$

Now use (9) and observe $c_L^1 = \dfrac{c_H^1}{2}$. Replace this in the above and we will get:

$$p_H c_H^1 + p_L \frac{c_H^1}{2} + p_L \frac{c_H^1}{2} + p_H c_H^1 + ... = p_H e_H^1 + p_L e_L^1 + p_L e_L^1 + p_H e_H^1 + ...$$

Use now the prices (10) and (11) as well as the endowment of the agent 1 and notice:

$$\underbrace{p_H}_{\beta^0} c_H^1 + \underbrace{p_L}_{2\beta} \frac{c_H^1}{2} + \underbrace{p_L}_{2\beta^2} \frac{c_H^1}{2} + \underbrace{p_H}_{\beta^3} c_H^1 + ... = \underbrace{p_H}_{\beta^0} \underbrace{e_H^1}_{3} + \underbrace{p_L}_{2\beta} \underbrace{e_L^1}_{1} + \underbrace{p_L}_{2\beta^2} \underbrace{e_L^1}_{0} + \underbrace{p_H}_{\beta^3} \underbrace{e_H^1}_{3} + ...$$

By simple algebra we then get:

$$\frac{c_H^1}{1-\beta} = \frac{3+2\beta}{1-\beta^3}$$

Therefore

$$c_H^1 = \frac{(1-\beta)(3+2\beta)}{1-\beta^3} \tag{12}$$

Notice from (9) we can then find c_L^1 as being such that:

$$c_L^1 = \frac{c_H^1}{2} = \frac{(1-\beta)(3+2\beta)}{2(1-\beta^3)} \tag{13}$$

In order to find the allocations for the individual 2 we can use directly the feasibility condition or proceed in a similar fashion as we had done previously, just being careful that in the endowment part we need to consider the endowment structure for agent 2.

$$c_H^2 = 4 - c_H^1 = 4 - \frac{(1-\beta)(3+2\beta)}{1-\beta^3} \tag{14}$$

$$c_L^2 = 2 - c_L^1 = 2 - \frac{(1-\beta)(3+2\beta)}{2(1-\beta^3)} \tag{15}$$

A good exercise is do for agent 2 the same as we had done for agent 1 and then verify that indeed the markets are gonna clear. It is a good way also to verify you did the calculations correctly.

Therefore the equilibrium is defined by the prices (10) and (11) and the allocations c_H^i and c_L^i for each $i \in \{1, 2\}$ we had found previously.