Department of Economics - Sciences Po
Macroeconomics III

Problem Set 1 - Complete Markets and Simple Incomplete Markets Models

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Question 1

Consider the following savings problem. An infinitely-lived household has preferences over consumption at each date given by

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

The household has wealth given by W_0 in period 0. The budget constraint in each period is

$$c_t + W_{t+1} \leq RW_t$$

Also assume $c_t, W_{t+1} \geq 0 \quad \forall t \geq 0$.

a) Suppose $\beta R = 1$. Set up and characterize the solution to the household's problem.

b) Suppose $\beta R \neq 1$. Characterize the solution to the household's problem (i.e., characterize the solution for the case in which we have $\beta R < 1$ and for the case in which we have $\beta R > 1$).

Question 2

A pure endowment economy consists of two types of infinitely lived consumers, each of whom has the same utility function,

$$u\left(c_0^i, c_1^i, \ldots\right) = \sum_{t=0}^{\infty} \beta^t \log c_t^i,$$

where $0 < \beta < 1$ is a common discount factor. Suppose that consumer 1 has the endowments $\left(e_0^1, e_1^1, e_2^1, e_3^1, \ldots\right) = (5, 3, 5, 3, \ldots)$ and that consumer 2 has the endowments $\left(e_0^2, e_1^2, e_2^2, e_3^2, \ldots\right) = (3, 5, 3, 5, \ldots)$.

a) Describe an Arrow-Debreu structure for this economy, explaining when markets are open, who trades with whom, and so on. Define an Arrow-Debreu Equilibrium. Be careful to include definitions of all the objects of which the competitive equilibrium is defined.

b) Compute the Arrow-Debreu Equilibrium of this economy.

c) Describe a Sequential Market structure for this economy, explaining when markets are open, who trades with whom, and so on. Define a Sequential Market Equilibrium. Be careful to include definitions of all the objects of which the competitive equilibrium is defined.

d) Compute the Sequential Market Equilibrium for this economy (including the one-period gross interest rates).

e) Redo the calculation in b) by considering the following stream of endowments (in this case there will be a price for when the aggregate endowment is high and a different one for when the aggregate endowment is low):

$$\left(e_0^1, e_1^1, e_2^1, e_3^1, \ldots\right) = (5, 3, 5, 3, \ldots)$$
$$\left(e_0^2, e_1^2, e_2^2, e_3^2, \ldots\right) = (4, 4, 4, 4, \ldots)$$

Question 3

A pure endowment economy consists of two types of consumers. Consumers of type 1 order consumption streams of the good according to the utility function,

$$\sum_{t=0}^{\infty} \beta^t c_t^1,$$

and consumer of type 2 order consumption streams according to,

$$\sum_{t=0}^{\infty} \beta^t \ln\left(c_t^2\right),$$

where $c_t^i \geq 0$ is the consumption of a type i consumer and $0 < \beta < 1$ is a common discount factor. The consumption good is tradable but non-storable. There are equal numbers of the two types of consumers. The consumer of type 1 is endowed with the consumption sequence:

$$e_t^1 = \mu > 0 \quad \forall t \geq 0$$

The consumer of type 2 is endowed with the consumption sequence:

If t is even (pair), $e_t^2 = 0$
If t is odd (impair), $e_t^2 = \alpha$,

where $\alpha = \mu\left(1 + \beta^{-1}\right)$.

a) Define an Arrow-Debreu Equilibrium. Be careful to include definitions of all the objects of which the competitive equilibrium is defined.

b) Compute the Arrow-Debreu Equilibrium of this economy.

c) Compute the time 0 wealths of the two types of consumers using the equilibrium prices found in the previous item.

d) Define a Sequential Market Equilibrium. Be careful to include definitions of all the objects of which the competitive equilibrium is defined.

e) Compute the Sequential Market Equilibrium for this economy (including the one-period gross interest rates).

Question 4

Time is denoted $t = 0, 1, 2, \dots \infty$. The economy is composed of 2 types of households and by entrepreneurs.
Households. There are two types of infinitely lived households (Type A and B). The number of each type of households is normalized to 1. Type A households have endowment 1 in all even (pair) period and 0 in all odd (impair) period. Type B households have endowment 0 in all even period and 1 in all odd period. Households consume and they have the opportunity to save each period by lending to entrepreneurs at a net interest rate r_t. Let $a_t > 0$ be the agent's saving at the beginning of period t. All agents seek to maximize :

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t^i\right),$$

where $u' > 0$, $u'' < 0$.
Remark : All type A households consume c_t^A and save a_t^A in period t, and all type B households consume c_t^B and save a_t^B in period t.
Entrepreneurs. Entrepreneurs are price-taker and they have access to a production function

$$y_t = k_t^\alpha$$

where k_t is the period t capital stock. Capital fully depreciate in production.

a) Show that

$$\alpha k_t^{\alpha-1} = 1 + r_t$$

and that the consumption of entrepreneurs is $c_t = (1-\alpha)\, k_t^{\alpha}$.

b) The goods market equilibrium is :

$$c_t^A + c_t^B + k + c = 1 + f(k)$$

Explain.

From this question we will first consider a standard case, similar to the ones we worked previously, and we will then assume that financial markets are imperfect.

No credit constraints

c) Define q_t as the wealth at beginning of period t. The program of an agent A in even period or of agent B in odd period is the same and is denoted R program (for rich).

   (i) Write the program of the rich agent.

   (ii) Assume that the constraint $a_{t+1}^R \geq 0$ does not bind. Show that $u'\left(c_t^R\right) = \lambda_t^R$ and $u'\left(c_t^R\right) = \beta\lambda_{t+1}^P(1 + r_{t+1})$, with λ_t^i the lagrange multiplier associated to the budget constraint of the agent $i \in (R, P)$ at time t.

d) Again, define q_t as the wealth at beginning of period t. The program of an agent A in odd period or of agent B in even period is the same and is denoted P program (for poor).

   (i) Write the program of the poor agent.

   (ii) Assume that the constraint $a_{t+1}^P \geq 0$ does not bind. Show that $u'\left(c_t^P\right) = \lambda_t^P$ and $u'\left(c_t^P\right) = \beta\lambda_{t+1}^R(1 + r_{t+1})$, with λ_t^i the lagrange multiplier associated to the budget constraint of the agent $i \in (R, P)$ at time t.

e) Assume that the economy is at the steady state (all variables are constant). Use the results in c) and d) to show that $1 + r = 1/\beta$. Show that $c^P = c^R$.

f) Explain the previous results comparing them to the RBC model.

g) Assume that the State taxes the agents A in even period and agents B in odd period, by an amount T. Would it change the interest rate $1 + r$? (Explain without equations).

Credit constraints

h) Assume now that agent A in odd period or agent B in even period are credit constrained. They consume all their income $c_t^P = q_t$ and save 0. ($a_{t+1}^P = 0$, i.e. the credit constraint is binding). Write the program of the poor agent. Assume that the economy is in steady state. Show that $u'\left(c^R\right)/u'\left(c^P\right) = \beta(1+r)$ and that $u'\left(c^P\right)/u'\left(c^R\right) > \beta(1+r)$.

i) Show that those 3 equations hold :

$$\begin{aligned} c^R + a^R &= 1 \\ c^P &= a^R(1+r) \\ a^R &= k \end{aligned}$$

Explain.

j) Assume that $u(c) = \log(c)$, and $u\prime(c) = 1/c$. Using the previous 4 equations, show that

$$k = \frac{\beta}{1+\beta}$$

k) Using the the result in a), determine the equilibrium interest rate $1+r$. Why it is different from $1/\beta$?

l) Assume that the state taxes the agents A in even period and agents B in odd period, by an amount T. Would it change the interest rate $1+r$? (Explain without equations).

m) Explain the difference between the results in g) and the results in k).