Department of Economics - Sciences Po
Macroeconomics III

Problem Set 5 - Incomplete Markets - Economies without aggregate uncertainty

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Question 1

Assume that an agent i has a utility function that depends positively on consumption, c^i, and negatively on labor supplied to work l^i.

$$u\left(c^i\right) - l^i$$

Assume that the hourly wage is w, such that the agent's total labor income is $w^i l^i$. Moreover assume that this agent has an endowment A^i, such that his budget constraint is

$$c^i = wl^i + A^i$$

Solve:

$$\begin{aligned} &\max_{c^i, l^i} u\left(c^i\right) - l^i \\ \text{s.t.} \quad c^i &= wl^i + A^i \end{aligned}$$

Show that consumption does not depend on wealth, but that hours worked depend on wealth. Explain this result.

Question 2

Using the same model as above. Now assume the agent lives for two periods, he consumes in the two periods, but can only work when he is young. Young agents can save an amount a_1, remunerated at an interest rate $R_1 < 1/\beta$. With obvious notations, the agent now maximizes

$$\begin{aligned} &\max_{c_1 c_2 a_1, l_1} u\left(c_1\right) - l_1 + \beta u\left(c_2\right) \\ a_1 + c_1 &= wl_1 \\ c_2 &= R_1 a_1 \end{aligned}$$

a) Show that c_2 is determined by

$$c_2 = u'^{-1}\left(\frac{1}{\beta R_1 w}\right)$$

How does c_2 evolve when w increases? Explain the intuition.

b) Find the expression for a_1. How does the saving rate a_1 evolve when w increases? Explain.

Question 3

Assume that the agent lives for two periods as before, but can work with a probability α (and is unemployed with a probability $1-\alpha$). When unemployed he earns a income δ.

His budget constraint in period 2 is

$$\begin{aligned} c_2^{employ} &= wl_2 + R_1 a_1 \\ c_2^{unemploy} &= \delta + R_1 a_1 \end{aligned}$$

The program is now

$$\begin{aligned} & \max_{c_1 c_2, a_1, l_1, l_2} u(c_1) - l_1 + \beta \left(\alpha \left[u\left(c_2^{employ}\right) - l_2 \right] + (1-\alpha) u\left(c_2^{unemploy}\right) \right) \\ a_1 + c_1 &= wl_1 \\ c_2^{employ} &= wl_2 + R_1 a_1 \\ c_2^{unemploy} &= \delta + R_1 a_1 \end{aligned}$$

a) Solve this problem. Show that $u'(c_1) = u'\left(c_2^{employ}\right) = 1/w$.

b) Find the Euler equation. Find the value of a_1. show that

$$\frac{1}{w}\frac{1}{\beta R} = \frac{\alpha}{w} + (1-\alpha) u'(\delta + a_1 R_1)$$

c) Notice α measures the unemployment risk. How does the savings a_1 evolve with α? with R? Explain.

Question 4

We consider an endowment economy. The economy is only composed of a unit mass of agents (no firm).

1. *Risk*: Each agent has labor income $\varepsilon \in \{\varepsilon_h, \varepsilon_l\}$. $\varepsilon_h > \varepsilon_l$.

$$\Pr[\varepsilon = \varepsilon_{s'} | \varepsilon = \varepsilon_s] = \pi_{ss''}$$

2. *Preferences*: There is a continuum of mass 1 of agents. Each agent maximizes

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \right]$$

3. *Assets and budget constraints*: Denote as a the current beginning of period wealth. The wealth at the beginning of next period is denoted a' (think of it as a_{t+1}). The price today of one unit of wealth in the future is q. The budget constraint is

$$c + qa' = a + \varepsilon$$

To simplify, it is assumed that agents can not borrow (in the class we have seen: credit limit $-\bar{b}$)

$$a' \geq 0$$

Denote $V_h(a)$ and $V_l(a)$ the value function for high income (h) and low income (l) agents respectively.

Now answer the following questions:

a) Write the Bellman equation for agents h and l.

b) What is the asset supply? What is the financial market equilibrium? Derive the equilibrium values of $a\prime$ and c_s $(s \in (h,l))$.

c) Denote λ_h and λ_l the Lagrange coefficient on the credit constraint in state h and l. Write the first order and envelop conditions in each case.

d) Write the two Euler equations.

e) Show that

$$\frac{\lambda_h}{\beta \varepsilon_h^{-\sigma}} < \frac{\lambda_l}{\beta \varepsilon_l^{-\sigma}}$$

What do you conclude?

f) What is the interest rate? How does it evolve when π_{hh} decreases? Why?

Question 5

Now assume agents live an infinite amount of periods. Small-heterogeneity models are classes of equilibria where agents do save but where the equilibrium distribution of wealth endogenously features a finite state space. Three classes of equilibria can be found in the literature. Each type of equilibrium has its own merit according to the question under scrutiny. This first class of equilibria is based on two assumptions :

1) Agents choose their labor supply when employed and that the disutility of labor supply is linear. With c consumption and l the labor supply, the period utility function is $U(c,l) = u(c) - l$. The trick here is that the first-order condition for labor supply pins down the marginal utility of consumption of employed agents. unemployed agents have home production $\delta < 1$.

2) The second assumption is that the credit constraint is tighter than the natural borrowing limit (the loosest credit constraint, which ensures that consumption is always positive). Then, $\bar{a} > \frac{-\delta}{r}$ with r the steady-state interest rate. This ensure that unemployed agents will hit the credit constraint after a finite number of periods of unemployment, it is a trick to reduce the state space.

The second class of borrowing constraints was the one suggested by Aiyagari (1994) we called it the "natural" borrowing constraint. Thus, this borrowing constraint is directly implied by the condition $c > 0$ holding in every state of the world. If we let ε_{min} be the lowest realization of individual productivity, then the natural borrowing limit becomes $a > \frac{\varepsilon_{min} w}{r}$. The natural borrowing limit will never bind when assuming Inada condition of the instantaneous utility function. Let's assume that the agent borrows up to the natural borrowing limit. In such case, the agent cannot consume at all for the rest of the period, since she has to use all her income to pay back her debt. The agent will end up having negative infinity lifetime utility due to the zero consumption for the rest of her life. Therefore, the agent never has an incentive to borrow up to the natural borrowing limit.

Notation : There is a continuum of agents of mass one. We study here simple partial equilibrium (the wage rate w and the real interest rate r are exogenous). e_t^i is the employment status, $e_t^i = 1$ when agents are employed and $e_t^i = 0$ when agents are unemployed. $e_{i,t}$ is the entire history of shocks $e_{i,t} = \{e_k^i\}_{k=0}^t$. a_t^i is the savings of agents i.

Agents can be either employed or unemployed with the transition matrix :

$$T = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\rho & \rho \end{pmatrix}$$

a) Write the program of an agent $i = \{e,u\}$ and the Bellman equations of each class of agents.

b) Find the Euler condition for both agents.

c) We know that the saving of unemployed agents decrease to reach the borrowing limit in a finite number of periods (not proved here, if interested in the proof see Huggett 1993). Assume here that the credit constraint binds after one period of unemployment. Moreover, to simplify the derivation, assume that $\bar{a} = 0$ and $w = 1$. How many different consumption levels are there in this economy? What are the consumption levels of each agents?

d) What is the condition for the employed agents to be borrowing constrained ?

e) What is the condition for unemployed agents to be credit constrained ?

f) Show that the employed agents consume the same amount. How is that possible ? Express the saving level of employed workers as a function of α.

g) You now almost have everything to characterise the description of the agent's decision. But you need to follow the number of employed and of each type of unemployed agent. What is the number of agents in each class ?

h) What is the total amount of savings in this economy (the sum of the savings of all unemployed plus all employed agents).

i) How does α affect the total savings? Show that there are two effects, the effect on the individual saving rate and the effect on the number of workers.

Question 6 (Programming)

Consider the Huggett (1993) economy we have seen in class. In this model, the environment is characterized by an infinitely-lived measure of households with preferences given by:

$$\mathrm{E}_0 \sum_{t=0} \beta^t u\left(c_t\right),$$

where $u(c_t) = \dfrac{c_t^{1-\sigma}}{1-\sigma}$.

We will assume that agents face an idiosyncratic shock each period given by $e_t \in \mathcal{E}$. There is a probability to change the idiosyncratic shock in each period given by $\pi(e'|e)$. For simplicity we will assume that in each period t, $e_t \in \{e_h, e_l\}$. As in class, also assume that there is no aggregate uncertainty.

Households are faced with the constraints:

$$a' + c = e + (1+r)a,$$
$$a' \geq -b.$$

Notice that in this case the productivity idiosyncratic shock is going to determine the income of the agents in this model.

Now use the following parameter values: period length is two months, $\beta = 0.96$ (annual basis), coefficient of relative risk-aversion $\sigma = 1.5$, $e_h = 1, e_l = 0, 1$, $\pi(e_h|e_h) = 0.925$, and $\pi(e_h|e_l) = 0.5$. Consider a stationary equilibrium. Answer the following questions:

a) In this equilibrium what is average income, and what is the average duration of an unemployment spell? (You will need to calculate the unconditional stationary distribution of the associated Markov Process.

   Set the borrowing constraint equal to one year's average income. Construct an equally-spaced grid on asset holdings, with a maximum value equal to $3X$ average income. Let the number of grid points $N = 20$. Suppose the interest rate r = 3.4% (annual basis).

b) Solve for optimal decision rules across the grid.

c) Imagine one agent, who starts out with zero assets and the high endowment. Simulate the evolution of the agent's wealth, income and consumption for 10,000 periods, each period drawing an endowment according to the Markov process described above. Plot a histogram for asset holdings over this simulation and the income distribution of agents on this economy.

d) Suppose the net supply of assets is zero. What is the market clearing interest rate?

e) Suppose we were to increase the value for the risk-aversion coefficient, σ, from 1.5 to 3. What would happen to the equilibrium interest rate? Can you provide some intuition for this result?