Department of Economics - Sciences Po
Macroeconomics III

Problem Set 2

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Part 1 - Stochastic Environment and Finite Markov Chain

Question 1

Consider the following neoclassical growth model with uncertainty; in each period, there is a realization of an exogenous random shock, s, that takes values in a finite set, S. Let us denote the history of shocks up to period t by $s^t = \{s_0, s_1, ..., s_t\}$ for $t = 1, 2, ...$. There is an infinitely-lived representative household in the economy, whose preferences over consumption allocations are represented by the following utility function:

$$E_0 \sum_{t=0}^{\infty} u\left(c_t\left(s^t\right)\right),$$

where the expectation is taken with respect to the stochastic process governing exogenous shocks, and the initial realization of s_0. Household is endowed with k_0 units of physical capital at the beginning of time, and one unit of time in each period. There is a firm that has access to a constant returns to scale technology, denoted by $A(s)F(K, H)$, that transforms capital services and labor into a composite good. This good can then be sold to the household. $A(s)$ is a productivity shock that is a functional of the realization of s.
Suppose household sells all its endowment of capital to the firm at the beginning of time, and firm makes all the investment decisions afterwards. The resource constraint in the economy is:

$$C_t + K_{t+1} = (1-\delta)K_t + A(s)F(K, H)$$

a) Define the Arrow-Debreu equilibrium for this economy.

b) Suppose, instead of selling the initial endowment of capital to the firms, households keep the capital and make the investment decisions. Prove this leads to the same equilibrium allocation as in item (a).

c) Now, assume household holds the capital in the first period, and make investment decision for one period. After the shock in period $t = 1$ is realized, s_1, markets open once more, and household sells the capital to the firm. What can you say about the allocation that results in the equilibrium from history $s_1 = (s_0, s_1)$, onward?

Question 2

Consider a two-period endowment economy where there are M types of agents. There is one good in the first period, and N goods in the second period. Agent $i \in \{1, ..., M\}$ receives endowments given as:

$$e_i^0, e_i^1(s),$$

where $s \in \{s_1, ..., s_N\}$ is a stochastic shock.

a) Define an Arrow-Debreu Competitive Equilibrium.

b) Suppose the only markets are Arrow securities $a(s)$, $\forall s$ that promise to provide a units of goods in terms of period 0's good at state s. Specifically,

$$a(s) = (0, \ldots, a, 0, \ldots 0),$$

$\forall i \in \{1, 2\}$, the sequential budget constraints are:

$$c_{i0} + \sum_s q(s)a_i(s) \leq e_0$$
$$p(s)c_i(s) \leq a_i(s) + e_i(s)$$

Define a sequential markets competitive equilibrium with one-period Arrow securities.

c) Show that an Arrow-Debreu equilibrium outcome is also an equilibrium outcome with Arrow securities.

Question 3

There is only one period, two types of agents (1 and 2) and two goods (A and B). The representative agent of each type has a utility function given by:

$$u\left(c^A, c^B\right) = \frac{\left(c^A c^B\right)^{\frac{1-\rho}{2}}}{1-\rho},$$

where $\rho > 0$ and where c^A represents the consumption of good A and similarly for c^B . There is a state of world indexed by $s \in S$. In state $s \in S$, which occurs with probability $\pi(s) > 0$, agent type 1 is endowed with $e_A(s)$ units of good A while agent type 2 is endowed with $e_B(s)$ units of good B (Note agent type 1 has no endowment of good B, and agent type 2 has no endowment of good A).

a) Solve the Social Planner's problem and find the Pareto optimal allocations.

b) Suppose that there are no financial markets but agents of different types can trade goods after the state of the world is realized. Show that the competitive equilibrium allocation that results without financial assets is Pareto efficient ex-ante (that is, before the state of the world is known).

c) Given your answer to part (b), what happens to the relative price of good A in states where $e_A(s)/e_B(s)$ is low? Explain in words how risk sharing is achieved.

d) Why is (b) an important result to consider when thinking about risk sharing across agents? Could you think about a situation where this type of problem can be applied?

Question 4 (Programming)

In numerical work it is sometimes convenient to replace a continuous model with a discrete one. In particular, Markov chains are routinely generated as discrete approximations to AR(1) processes of the form:

$$y_{t+1} = \mu(1-\rho) + \rho y_t + \varepsilon_{t+1},$$

where ε_{t+1} is assumed to be an i.i.d. and $N \sim (0, \sigma^2)$. Tauchen's method is the most common to approximate this continuous process with a finite Markov chain. [1] In this exercise you task is to approximate the continuous AR(1) process with a discrete-first order Markov Chain. The goal here is to discretize y so that the resulting process resembles a continuous AR(1) process.

In what follows we will describe the steps to discretize this continuous process.

[1] Tauchen, G. (1986). Finite state markov-chain approximations to univariate and vector autoregressions. Economics letters, 20(2), 177-181.

Objective: Discretize the range (or state space) of y_t into points $y_i, i = 1, ..., N$ and give each point y_j an approximate probability of occuring π_{ij}, given that the previous period state was y_i. The matrix $\Pi = [\pi_{ij}]$ constitutes the transition matrix.

In order to implement this process we need first to specify the grid, i.e., the possible values that y can assume. First of all, define two bounds y_1 and y_N by:

$$y_1 = \mu - m\sqrt{\frac{\sigma^2}{1-\rho^2}}, \quad y_N = \mu + m\sqrt{\frac{\sigma^2}{1-\rho^2}},$$

where m is a scaling parameter (in practice is set at 3 or 4). Then, the other values $y_2, ..., y_{N-1}$ are defined by an equispaced grid:

$$\Delta = \frac{y_N - y_1}{N-1},$$

i.e., each $y_i \quad \forall i \in \{2, ..., N-1\}$ is defined as:

$$y_i = y_1 + (i-1)\Delta.$$

Now create the borders of each interval $[y_1, y_{i+1}]$:

$$m_i = \frac{y_{i+1} + y_i}{2} = y_1 + (2i-1)\frac{\Delta}{2} = y_i + \frac{\Delta}{2}$$

Observe:

$$y_i \in \begin{cases} (-\infty, m_1] & \text{if } i = 1 \\ (m_{i-1}, m_i] & \text{if } 1 < i < N \\ (m_{N-1}, \infty) & \text{if } i = N \end{cases}$$

Therefore it is easy to observe the following is true:

- If $j = 2, ..., N-1$:

$$\begin{aligned} \pi_{ij} &= \Pr\left(y_{t+1} = y_j \mid y_t = y_i\right) = \Pr\left(\mu(1-\rho) + \rho y_i + \varepsilon_{t+1} = y_j\right) \\ &\approx \Pr\left(m_{j-1} \leq \mu(1-\rho) + \rho y_i + \varepsilon_{t+1} \leq m_j\right) \\ &= \Phi\left(\frac{m_j - \rho y_i - \mu(1-\rho)}{\sigma}\right) - \Phi\left(\frac{m_{j-1} - \rho y_i - \mu(1-\rho)}{\sigma}\right), \end{aligned}$$

  where Φ is the c.d.f. of the normal distribution.

- If $j = 1$

$$\begin{aligned} \pi_{i1} &= \Pr\left(y_{t+1} = y_1 \mid y_t = y_i\right) \\ &= \Pr\left(\mu(1-\rho) + \rho y_t + \varepsilon_{t+1} = y_1 \mid y_t = y_i\right) \\ &= \Pr\left(\mu(1-\rho) + \rho y_i + \varepsilon_{t+1} = y_1\right) \\ &\approx \Pr\left(\mu(1-\rho) + \rho y_i + \varepsilon_{t+1} \leq m_1\right) \\ &= \Phi\left(\frac{m_1 - \rho y_i - \mu(1-\rho)}{\sigma}\right) \end{aligned}$$

- If $j = N$

$$\begin{aligned} \pi_{iN} &= \Pr\left(y_{t+1} = y_N \mid y_t = y_i\right) \\ &= 1 - \Phi\left(\frac{m_{N-1} - \rho y_i - \mu(1-\rho)}{\sigma}\right) \end{aligned}$$

The result of this discretization process is a grid $\{y_i\}_{i=1}^N$, and the transition probability Π.

Your task is to write a function that has as inputs μ, ρ, σ, m and N and give as output the state space $\{y_i\}_{i=1}^N$, and the transition probability Π. For example, consider the following case: $N = m = 3, \mu = 1, \rho = 0.9, \sigma = 0.5$

The output of your code should be:

$$y_i \in \{-2.44, 1, 4.44\}$$
$$\Pi = \begin{pmatrix} 0.997 & 0.003 & 0 \\ 0.0003 & 0.9994 & 0.0003 \\ 0 & 0.003 & 0.997 \end{pmatrix}.$$

Part 2 - Basic Asset Pricing

Question 5

Consider an endowment economy with a representative agent who has expected utility preferences and power utility (CRRA):

$$U = \mathbb{E}\sum_{t\geq 0}\beta^t \frac{C_t^{1-\gamma}}{1-\gamma}.$$

Assume that the endowment follows the process

$$\Delta \log C_{t+1} = \mu_c - \frac{\sigma_c^2}{2} + \sigma_c \varepsilon_{t+1},$$

where ε_{t+1} is *iid* $N(0,1)$. Consider N assets, $i = 1...N$ which pay respectively dividends D_{it} where each D_{it} follows a different process:

$$\Delta \log D_{it+1} = \mu_i - \frac{\lambda_i^2}{2} - \frac{\chi_i^2}{2} + \lambda_i \varepsilon_{t+1} + \chi_i\, \eta_{i.t+1},$$

with η_{it+1} *iid* $N(0,1)$. Assume $\eta_{i,t+1}$ is uncorrelated with ε_{t+1} at all leads and lags, i.e.

$$\mathbb{E}(\varepsilon_{t+1-k}\eta_{i,t+1}) = 0 \text{ for all } k \geq 0, \text{ and all } k \leq 0.$$

a) Compute the mean of $\frac{C_{t+1}}{C_t}$ and $\frac{D_{it+1}}{D_{it}}$.
Hint : recall that for any real or complex number n, the n-th moment of a log-normally distributed variable z is given by $\mathbb{E}(exp(z^n)) = exp\left[n\mathbb{E}(z) + \frac{1}{2}n^2 Var(z)\right]$

b) Compute the (log) risk free rate.

Let $\beta = e^{-\delta}$ (why ?)

c) Compute the price-dividend ratio on asset i. Discuss.

Hint : in this iid environment, price-dividend ratios are constant.

d) (i) Compute an approximation of the expected excess return (over the riskfree rate) on asset i.
(ii) Explain intuitively how it depends on μ_i, λ_i and χ_i.
(iii) Discuss the statement "idiosyncratic risk is not priced".
(iv) Is it true that more volatile assets ("more risky assets") have higher average returns?

e) What is the effect of a shock ε_{t+1} on consumption, dividends, returns, and the price-dividend ratio?