Department of Economics - Sciences Po
Macroeconomics III

Problem Set 4 - Value Function Iteration in Stochastic Environment

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Question 1

Consider the representative consumer has preferences given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right),$$

where $\beta \in (0,1)$, c_t denotes consumption, $u(.)$ is strictly increasing, strictly concave and twice differentiable, and E_0 is the expectation operator conditional on information at $t = 0$. Note here that, in general, c_t will be random. The representative consumer has 1 unit of labor available in each period, which is supplied inelastically. The production technology is given by:

$$y_t = z_t F\left(k_t, n_t\right),$$

where $F(.,.)$ is strictly quasiconcave, homogeneous of degree one, and increasing in both arguments. Here, k_t is the capital input, n_t is the labor input, and z_t is a random technology disturbance. That is, $\{z_t\}_{t=0}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables (each period z_t is an independent draw from a fixed probability distribution $G(z)$). In each period, the current realization, z_t, is learned at the beginning of the period, before decisions are made. The law of motion for the capital stock is:

$$k_{t+1} = i_t + (1-\delta)k_t,$$

where i_t is investment and δ is the depreciation rate, with $0 < \delta < 1$. The resource constraint for this economy is:

$$c_t + i_t = y_t.$$

a) Explain in words how the **Competitive Equilibrium** is found in this economy, i.e., explain how it works the Competitive Equilibrium in an Arrow-Debreu structure and in a Sequential Market structure. You should be able to say what are the objects that are tradeable in this economy and the moments the negotiation happens.

b) Set up the **Social Planner's Problem**.

c) Write down the **Bellman equation** for this problem.

d) What are the objects you need to determine in the problem you wrote in part (c)?

e) For this part let $F\left(k_t, n_t\right) = k_t^{\alpha} n_t^{1-c}$, with $0 < \alpha < 1$, $u(c_t) = ln(c_t)$, $\delta = 1$, and $E[ln z_t] = \mu$. **Guessing that the value function** $v(k_t, z_t)$ has the form:

$$v(k_t, z_t) = A + B ln k_t + D ln z_t,$$

find the **analytical solutions** for the value function $v(k_t, z_t)$ and the policy functions $c_t(k_t, z_t)$ and $k_{t+1}(k_t, z_t)$.

f) Does the economy described above converge to a steady state?

g) Explain in words how would you use the policy functions you obtained above to find the competitive equilibrium for this economy.

h) **Programming.** Using the policy functions you obtained in part (e) (i.e., $c_t(k_t, z_t)$ and $k_{t+1}(k_t, z_t)$) and assuming any initial value k_0, determine a sequence $\{z_t\}_{t=0}^T$ using a random number generator and fixing T. Use this sequence to find time series for consumption and investment. Observe you can also obtain time series for y_t.

i) What can you say about the $var(lnk_{t+1})$, $var(lnc_t)$, and $var(lny_t)$? Does this model fits the trend we observe in the data for consumption, investment, employment and output? If not, how could you modify this model such that it matches the moments found in the data?

Question 2 (Programming)

Consider the following growth problem:

$$\max_{\{c_t, x_t, \ell_t\}} E \sum_{t=0}^{\infty} \beta^t \{\log(c_t) + \psi \log(\ell_t)\}$$

subject to:

$$c_t + x_t = k_t^\theta (z_t h_t)^{1-\theta}$$

$$k_{t+1} = (1-\delta)k_t + x_t$$

$$\log z_t = \rho \log z_{t-1} + \epsilon_t, \quad \epsilon \sim N\left(0, \sigma_\epsilon^2\right)$$

$$h_t + l_t = 1$$

$$c_t, x_t \geq 0 \quad \text{in all states}$$

a) Write the F.O.C. of the problem above with respect to k_{t+1} and h_t (i.e., hours worked).

b) Find the values of the steady state for capital and labor. Notice in this case it is possible to find analytical expressions for k_{ss} and h_{ss}, but this will not always be true. If you want you can use a numerical method to find those values (e.g., Newton method).

c) Use the value for the steady state for capital to build the grid points for the capital (i.e., $[0.25 * k_{ss}, 1.25 * k_{ss}]$, where k_{ss} denotes the capital in the steady state). Notice for the labor you can use a grid point between 0 and 1.

d) Set the parameters for you model, as well as the grid points and do a code where you solve the problem above by **Value Function Iteration.**

   Tip: Notice in this case the planner is choosing the following variables: c_t, k_{t+1} and h_t. It is easy to observe that c_t will be a function of (k_t, k_{t+1}, z_t, h_t). We can then separate the problem in two pieces: a function $log(c_t)$ which will be in the space (k_t, k_{t+1}, z_t, h_t) and a function $\phi log(1 - ht)$ which will be in the space h_t. Different from the Value Function Iteration we had seen (i.e., $\psi = 0$), in the first step you can find an auxiliary function in the space (k_t, k_{t+1}, z_t), which maximizes the following $log(c_t(k_t, k_{t+1}, z_t, :)) + \phi log(:)$ (i.e., find where the optimal labor decisions are in the space (k_t, k_{t+1}, z_t)) and after that use this function in the value function iteration and proceed as usual.