Krusell and Smith

Diego de Sousa Rodrigues

Sciences Po

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1 Aiyagari model with aggregate productivity shocks

In this section we are going to solve the **Aiyagari model with aggregate productivity shocks**.

- The **Aggregate productivity shock** z only takes two values, $z \in Z \equiv \{z_b, z_g\} = \{0.99, 1.01\}$.
- The **Idiosyncratic productivity shock** e only take two values, $e \in E \equiv \{e_b, e_g\} = \{0, 1\}$.
- We are going to let

  $$\pi(z', e'|z, e) = P_t(z_{t+1} = z', e_{t+1} = e'|z_t = z, e_t = e)$$

  be the Markov chain that describes the joint evolution of the exogenous shocks.
- In the **Production side**, the firm's optimization problem will give us:

  $$w(K_t, H_t, Z_t) = (1 - \alpha) Z_t K_t^{\alpha} H_t^{-\alpha}$$

  $$R(K_t, H_t, Z_t) = 1 + \alpha Z_t K_t^{\alpha-1} H_t^{1-\alpha} - \delta$$

,

where the firm uses capital K_t and labor H_t as inputs This firm also faces a productivity shock Z_t.Observe the wage and return in capital depends on the aggregate state variables, K_t and H_t.

- There is also also a continuum of ex-ante identical **Households** with unit mass. In each period the household faces an idiosyncratic shock e that is going to determine whether they are employed $e = 1$ or unemployed $e = 0$. An employed earns w per unit of labor. The markets will be incomplete and the agents will save through capital accumulation denoted by a and R is going to be the rate of return net of depreciation. Since there is aggregate uncertainty but no aggregation, we will use the Krusell and Smith algorithm. Notice labor is endogenous, and the labor supply is not fixed. This fact implies that capital is not a sufficient forecast for all future prices next period. In other words, the agent has to forecast R and w in order to optimize. In a setting with fixed labor supply, a forecast of K gives us a forecast of K/H, which in turn allows the agent to forecast marginal productivities and both prices. Now, H is no longer fixed. This implies that two forecasts are required. I am going to assume that agents use the following forecasting rule:

$$logK' = b^0_{z_b} + b^1_{z_b} logK$$

$$logH' = d^0_{z_b} + d^1_{z_b} logK$$

$$logK' = b^0_{z_g} + b^1_{z_g} logK$$

$$logH' = d^0_{z_g} + d^1_{z_g} logK$$

- The agent recursive problem is given by

$$V(a,e,K,H,Z) = max_{c,a',n} \frac{\left(c^{\eta}(1-n)^{1-\eta}\right)^{1-\mu}}{1-\mu} + \beta E\left[V(a',e',K',H',Z')|e,K,H,Z\right]$$

subject to

$$c \leq R(K,H,Z)a - a' + w(K,H,Z)en$$

$$a' \geq 0$$

$$logK' = b^0_{z_b} + b^1_{z_b} logK$$

$$logH' = d^0_{z_b} + d^1_{z_b} logK$$

$$logK' = b^0_{z_g} + b^1_{z_g} logK$$

$$logH' = d^0_{z_g} + d^1_{z_g} logK$$

The parameters of the model were set such that $\beta = 0.99$, $\delta = 0.0025$, and $\mu = 1$. The stochastic process for (Z, e) was set so that unemployment rate in expansions is 0.04 and 0.1 in recessions. The average duration of both an expansion and a recession is 8 quarter and the average duration of unemployment during expansions is 1.5 quarters and 2.5 quarters in recessions.

- To solve the household's problem we used the Carroll's endogenous grid method, since otherwise it would take a lot of time to solve due to the presence of high number of state variables.

  Observe by solving the F.O.C. of the recursive problem above we obtain:

$$\frac{u_n}{u_c} = w$$

Also notice we have the following:

$$n^*(c, e, K, H, Z) = \begin{cases} 1 - \frac{1-\eta}{\eta}\frac{c}{w(K,H,Z)}, & \text{if } e = 1 \\ 0, & \text{otherwise} \end{cases}$$

By using the inter temporal Euler equation we obtain:

$$u_c(c, n) = \beta E[R(K', H', Z')u_c(c', n')|e, K, H, Z]$$

The goal here is define a grid for assets in $t+1$ and to guess a functional form for c such that the right hand side of the equation above is a constant (C). By using this idea we have that:

$$c = \begin{cases} \left(\frac{C}{\eta}\left(\frac{(1-\eta)}{\eta w(K,H,Z)}\right)^{-(1-\mu)(1-\eta)}\right)^{\frac{-1}{\mu}}, & \text{if } e = 1 \\ \left(\frac{C}{\eta}\right)^{\frac{-1}{\mu}}, & \text{otherwise} \end{cases}$$

By using the equations above and the idea we can implement the endogenous grid method. The code that runs this is called **KS.jl**.

2 The Krusell and Smith algorithm

Now we will use the Krussell and Smith algorithm. The code that runs this is called **KS.jl**:

1. Specify a functional form for the laws of motion:

$$logK' = b_z^0 + b_z^1 logK$$

$$logH' = d_z^0 + d_z^1 logK,$$

where $z \in \{z_b, z_g\}$.

2. Guess the coefficients $b^0_{z_b}$, $b^1_{z_b}$, $b^0_{z_g}$, $b^1_{z_g}$, $d^0_{z_b}$, $d^1_{z_b}$, $d^0_{z_g}$, and $d^1_{z_g}$.

3. Solve the household problem and obtain the decision rules $(c, a', n)(a, e, K, H, Z)$ using the method described above. Notice that forecasts of (R', w') are required. Observe that our laws of motions are enough to forecast R' and w'.

4. Simulate the economy for N individuals and T time periods. Draw first a sequence of aggregate shocks $\{Z_t\}_{t=1}^T$. Then, draw individual productivity shocks $\{e_{it}\}_{i=1,t=1}^{N,T}$ (potentially conditional on the path of the aggregate shocks, if there were correlation). Given initial conditions for each agent $\{a_{i0}\}_{i=1}^N$ compute the path of policies $\{(a_{it}, n_{it})\}_{i=1,t=1}^{N,T}$. For each period, compute the average capital stock and labor supply

$$\hat{K}_t = \frac{1}{N}\sum_{i=1}^{N} a_{it}$$

$$\hat{H}_t = \frac{1}{N}\sum_{i=1}^{N} e_{it}n_{it}$$

5. Labor market need to clear every period. Observe that capital is predetermined and therefore this is not going to be an issue. To do so, we must in each period guess an initial aggregate state for labor and then loop until the convergence. In order words, if $\hat{H}_t$ does not coincide with the forecast, update w_t until the labor market clears. After doing this inner loop store the sequence $\{\hat{K}_t, \hat{H}_t\}_{t=0}^{\infty}$.

6. Discard the first S periods in order to avoid dependence on initial conditions. Using the remaining sequence regress

$$log\hat{K}_{t+1} = B^0_z + B^1_z log\hat{K}_t$$

$$log\hat{H}_{t+1} = D^0_z + D^1_z log\hat{H}_t,$$

   Observe the law of motion is time-invariant, so we can separate the dates t in the sample where the state is z_b from those where the state is z_g and run two distinct regressions. Now estimate $\{B^0_z, B^1_z, D^0_z, D^1_z\}_{z\in Z}$

7. Now for a given tolerance level ϵ, check if

$$\max_{z\in Z}\{\max_{i=0,1}|b_z^i - B_z^i|, \max_{i=0,1}|d_z^i - D_z^i|\} < \epsilon$$

   In case the convergence is achieved stop. Otherwise, replace $b = B$, $d = D$ and come back to step 3.

8. Assess whether the solution is accurate enough. Compute a measure to the fit of the regression, for example by using R^2. If fit at the solution is not satisfactory add moments of the distribution or try a different functional form for the law of motion and repeat all the steps above. We need to verify this because this is only an approximation to the fully rational-expectation equilibrium.

3 Results of the Regression

Capital

The coefficients we obtain for capital:

Bad times/ Recessions

$$lnK_{t+1} = 0.11495 + 0.95422 lnK_t$$

$R^2 = 0.999$

Good times/ Expansions

$$lnK_{t+1} = 0.101475 + 0.957641 lnK_t$$

$R^2 = 0.999$

Krusell and Smith results:

Bad times/ Recessions

$$lnK_{t+1} = 0.114 + 0.953lnK_t$$

Good times/ Expansions

$$lnK_{t+1} = 0.123 + 0.951lnK_t$$

Labor

The coefficients we obtain for labor:

Bad times/ Recessions

$$lnH_t = -0.68278 - 0.217856lnK_t$$

$R^2 = 0.995$

Good times/ Expansions

$$lnH_t = -0.605365 - 0.226267lnK_t$$

$R^2 = 0.997$

Krusell and Smith results:

Bad times/ Recessions

$$lnH_t = -0.592 - 0.255lnK_t$$

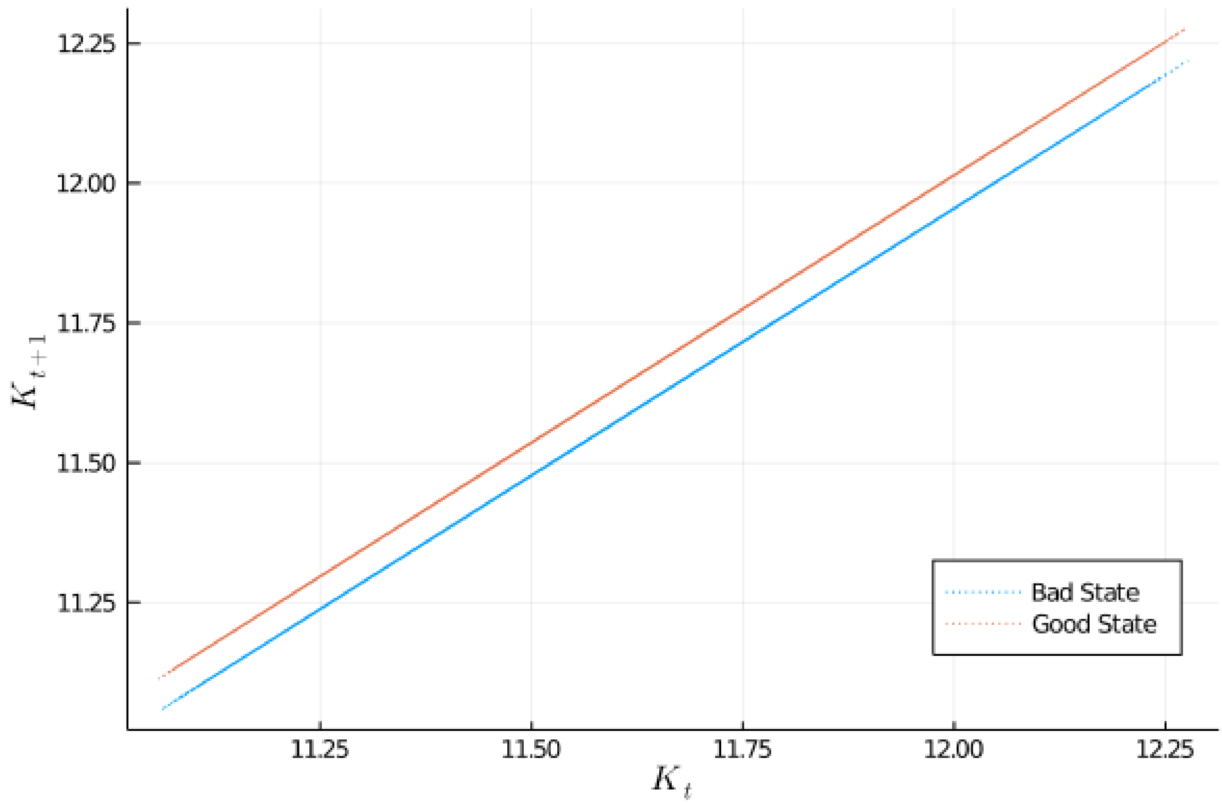
Good times/ Expansions

$$lnH_t = -0.544 - 0.252lnK_t$$

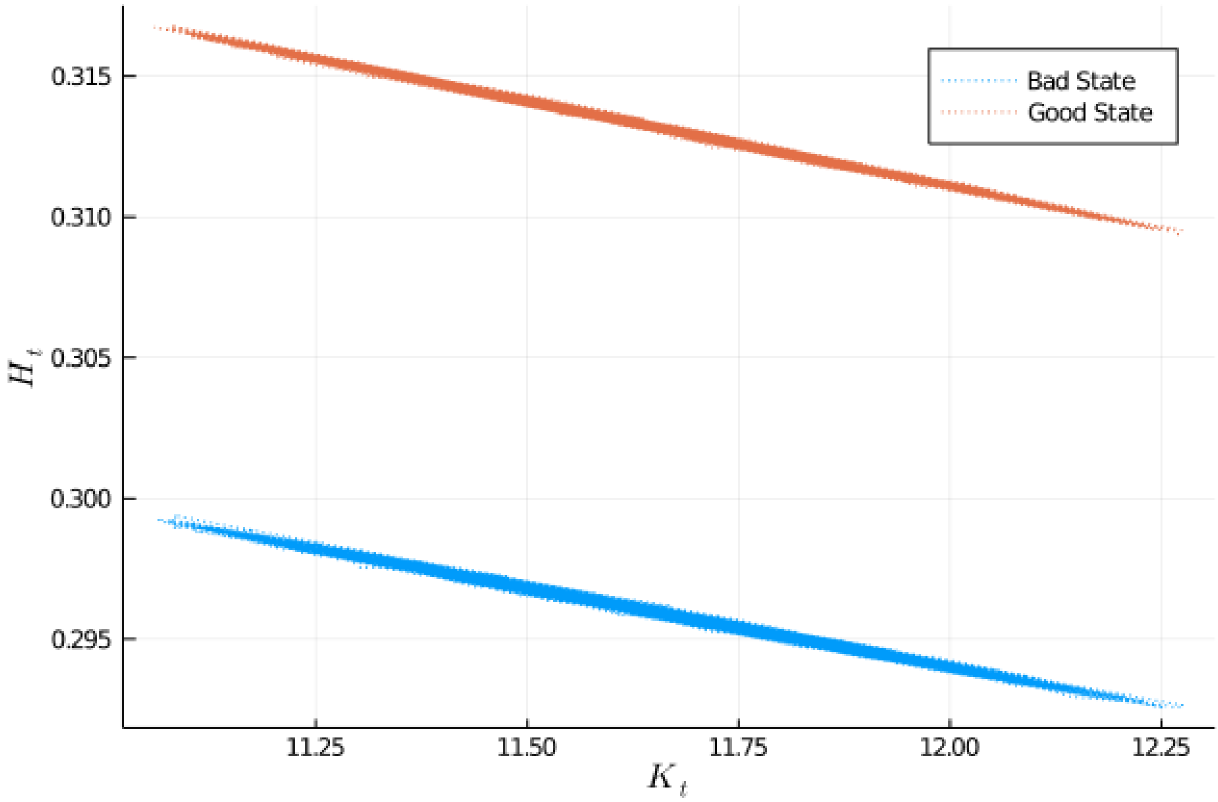
The paper we used to get the results here is the one **Income and Wealth Heterogeneity in the Macroeconomy**.

Observe that the fit of the regression in both cases are good, so we do not need to change the functional form or to add moments.

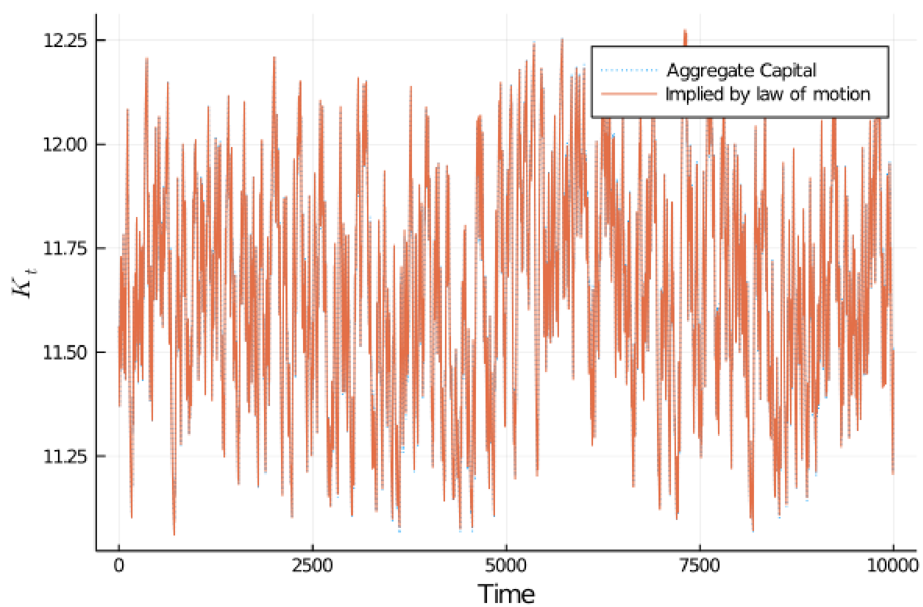
Below we plot the figure for the **Aggregate capital in period** $t + 1$ **versus the Aggregate capital in period** t for both states of the economy:

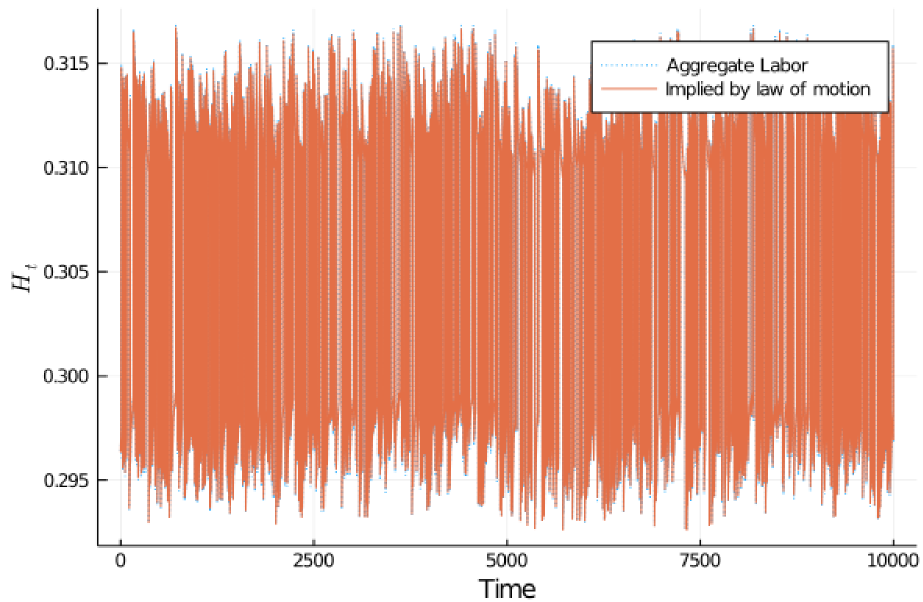


Now we plot the **Aggregate Labor versus the Capital** in both states of the economy.

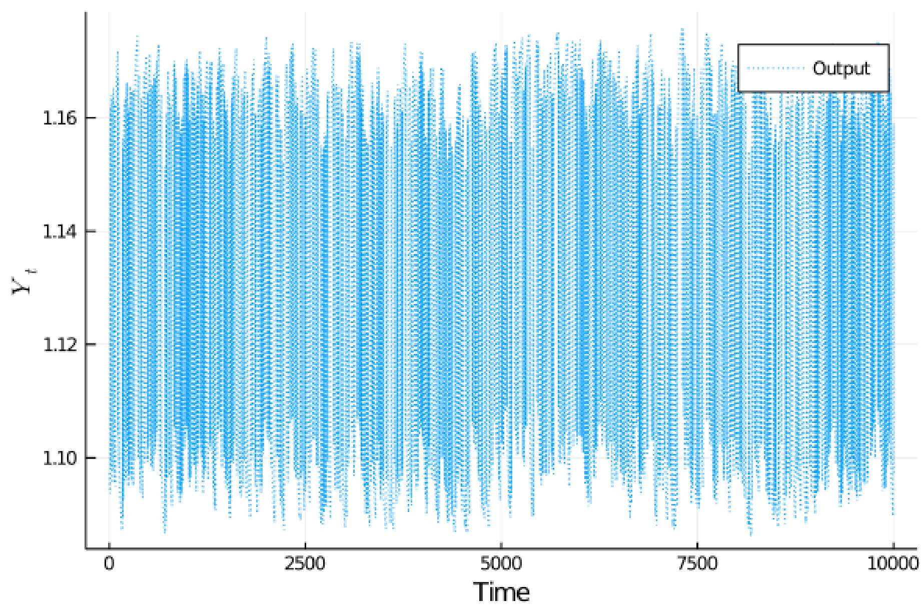


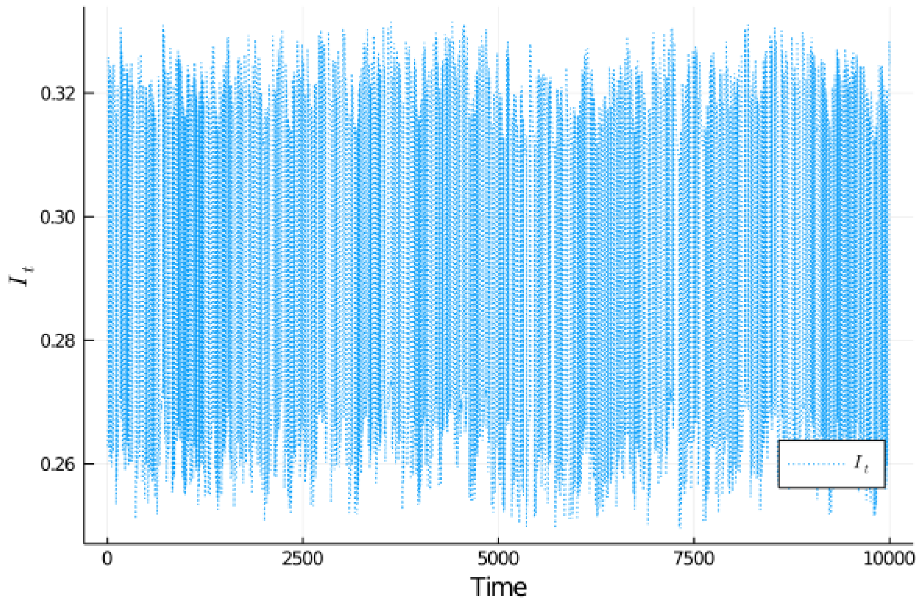
Finally, below we plot the evolution of the actual Aggregate Capital and Labor versus the one implied by the Laws of motion. Notice that the fit is quite good in both cases

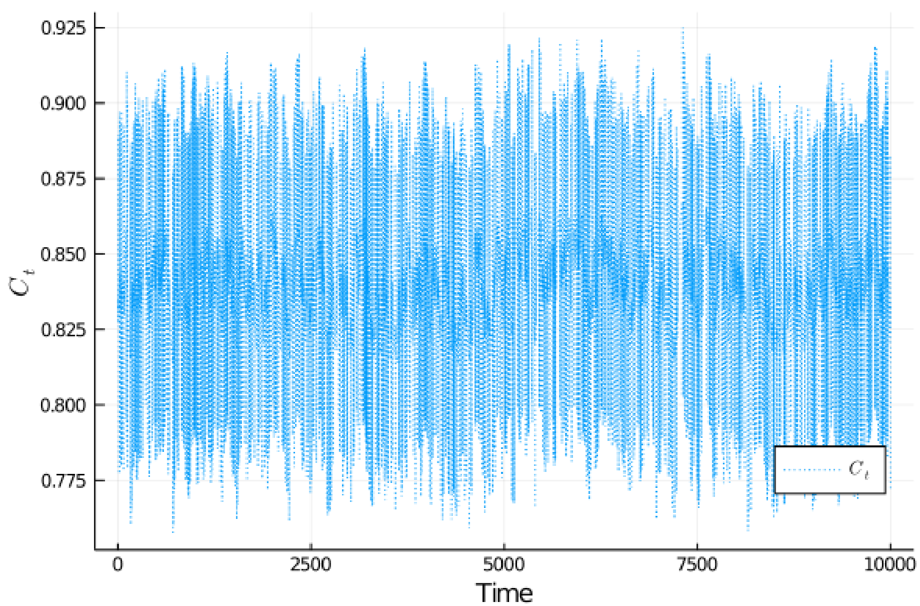




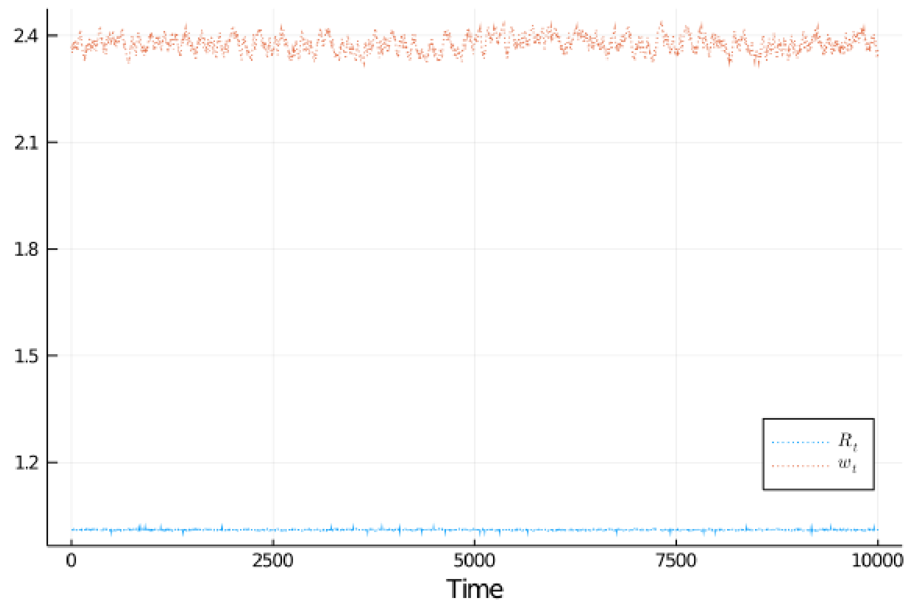
We also plot the **Aggregate output, Aggregate Investment, and Aggregate Consumption** respectively:







Below we plot the returns R and w:



The table below shows respectively the **Correlation matrix between the Output, Consumption, and Investment**, the **mean** and the **standard deviations** of these variables:

Correlation matrix	Y_t	C_t	I_t
Y_t	*1.0*	*0.570165*	0.744169
C_t	*0.570165*	*1.0*	-0.124477
I_t	0.744169	-0.124477	1.0

Mean	Y_t	C_t	I_t
	1.13097	0.839699	0.29127

Standard deviation	Y_t	C_t	I_t
	0.0341302	0.0229774	0.0282588