Macroeconomics III

Stochastic Environment and Finite Markov Chains

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Environment

- The agents make decision at date $t = 0$;
- There are infinite periods;
- At each date $t \geq 0$ there is a realization of a stochastic event $s_t \in \mathcal{S}$;
- The history of these events until period t is denoted by $s^t = [s_0, s_1, s_2, \dots, s_t]$;
- A particular sequence of events occur with probability $\pi_t(s^t)$;
- The chance that a sequence of events s^τ occurs given the sequence s^t occurred is given by $\pi_t(s^\tau | s^t)$;
- There is a set of I agents that receive a perfectly anticipated endowment $y_t^i(s^t)$ in each period. This endowment is perishable, but the agents can trade among themselves;
- The individual savings can be positive or negative, but the aggregate one will always be zero;

Environment

- Agents have utility $u(c)$ that is increasing and concave, i.e., $u'(c) > 0$ and $u^{''}(c) < 0$;
- Agents will draw a consumption plan for all his life $\left\{c_t^i\left(s^t\right)\right\}_{t=0}^{\infty} \equiv C^i$, which will be subject to a discount factor $\beta \in (0,1)$ such that:

$$U\left(C^i\right) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t\left(s^t\right) u\left[c_t^i\left(s^t\right)\right]$$

- First of all we will solve the **social planner's problem** and then verify the **conditions such that the solution of the competitive equilibrium is efficient**.

The Social Planner's problem

$$\max_{\{C^i\}_{i=1}^{I}} \sum_{i=1}^{I} \lambda^i U\left(C^i\right) \quad \text{s.a.} \quad \sum_{i=1}^{I} c_t^i\left(s^t\right) \leq \sum_{i=1}^{I} y_t^i\left(s^t\right), \quad \forall t, \forall s^t$$

- The problem of the Social Planner is then:

$$\mathcal{L} = \sum_{i=1}^{I} \lambda^i \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t\left(s^t\right) u\left[c_t^i\left(s^t\right)\right] + \sum_{t=0}^{\infty} \sum_{s^t} \theta_t\left(s^t\right)\left[\sum_{i=1}^{I}\left(y_t^i\left(s^t\right) - c_t^i\left(s^t\right)\right)\right]$$

Let's derive the F.O.C.

- $\left[c_t^i\left(s^t\right)\right]:$

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- $\left[c_t^i\left(s^t\right)\right]$:

- By defining the aggregate endowment as $Y_t\left(s^t\right) \equiv \sum_{i=1}^{I} y_t^i\left(s^t\right)$ and assuming the **aggregate endowment between any two periods t and τ is the same**, we can show:
$Y_t\left(s^t\right) = Y_\tau\left(s^\tau\right) \Longrightarrow c_t^i\left(s^t\right) = c_\tau^i\left(s^\tau\right), \forall i \in I$

Agent's problem in an Arrow-Debreu structure

- Assume there are only two possible states of nature: 0 and 1, i.e., $\mathcal{S} = \{0, 1\}$
- Let the economy always begin in state 0 (i.e., the initial state is deterministic $\pi_0(0) = 1$)

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- Let the economy always begin in state 0 (i.e., the initial state is deterministic $\pi_0(0) = 1$)
- The structure for an economy with 3 periods can be represented by the following tree:

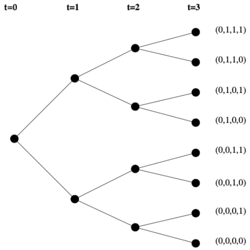

Figure 1: The Arrow-Debreu commodity space for a two-state.

Agent's problem in an Arrow-Debreu structure

- The **market structure is complete** in the sense that the agent can acquire **rights of consumption for any period and for any possible history**.

Agent's problem in an Arrow-Debreu structure

The problem of the agents is

$$\max_{\left\{c_t^i(s^t)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t\left(s^t\right) u\left[c_t^i\left(s^t\right)\right]$$

subject to

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0\left(s^t\right) y_t^i\left(s^t\right) \geq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0\left(s^t\right) c_t^i\left(s^t\right)$$

Agent's problem in an Arrow-Debreu structure

- Since the markets open only once, in $t = 0$, we will have just one Lagrange multiplier in the problem below:

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t \left(s^t\right) u \left[c_t^i \left(s^t\right)\right] + \gamma^i \sum_{t=0}^{\infty} \sum_{s^t} q_t^0 \left(s^t\right) \left[y_t^i \left(s^t\right) - c_t^i \left(s^t\right)\right]$$

- The F.O.C.
  $\left[c_t^i \left(s^t\right)\right]$:

Agent's problem in an Arrow-Debreu structure

- Using the same idea we used for the Social Planner's problem we can divide the F.O.C. of the agent i by the F.O.C. of an agent 1 generic and obtain:

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- In other words, by setting $\lambda^i = \frac{\gamma^i}{\gamma^1}\lambda^1$ and using those weights to solve the SP, we will reach the same allocation as the competitive equilibrium. In other words, **there are weights such that the allocation of the social planner is a competitive equilibrium.**

Agent's problem in an Arrow-Debreu structure

- So far, we obtained 4 important equations, which are:

$$\lambda^i \beta^t \pi_t \left(s^t\right) u' \left[c_t^i \left(s^t\right)\right] = \theta_t \left(s^t\right), \quad \forall i \in I, \forall t, \forall s^t \qquad \text{(F.O.C. SP)}$$

$$\beta^t \pi_t \left(s^t\right) u' \left[c_t^i \left(s^t\right)\right] = \gamma^i q_t^0 \left(s^t\right), \quad \forall i \in I, \forall t, \forall s^t \qquad \text{(F.O.C. AD)}$$

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0 \left(s^t\right) \left[y_t^i \left(s^t\right) - c_t^i \left(s^t\right)\right] = 0 \qquad \text{(BC)}$$

$$Y_t \left(s^t\right) = \sum_{i=1}^{I} y_t^i \left(s^t\right) = \sum_{i=1}^{I} c_t^i \left(s^t\right) \qquad \text{(F)}$$

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- By assuming $q_0^0(s^0) = 1$ we can then reach the Euler Equation:

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$$u'\left[c_0^i\left(s^0\right)\right] = \frac{\beta^t \pi_t\left(s^t\right) u'\left[c_t^i\left(s^t\right)\right]}{q_t^0\left(s^t\right)}$$

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- From the Euler Equation we can have idea about the prices in this economy:

$$q_t^0\left(s^t\right) = \beta^t \pi_t\left(s^t\right) \frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_0^i\left(s^0\right)\right]}$$

Agent's problem in an Arrow-Debreu structure

- Suppose **there is no aggregate uncertainty**, $Y_t\left(s^t\right)=\bar{Y}, \quad \forall t, \forall s^t$
- By the analysis we already made we can conclude $u'\left[c_t^i\left(s^t\right)\right]=u'\left[c_0^i\left(s^0\right)\right]$, which will lead us to a completely exogenous price:

$$q_t^0\left(s^t\right)=\beta^t \pi_t\left(s^t\right)$$

Agent's problem in an Arrow-Debreu structure

- Suppose $u(c) = \log(c) \Longrightarrow u'(c) = \frac{1}{c}$
- **Show that the consumption of the agent is always a function of the total endowment.**

Agent's problem in an Arrow-Debreu structure

- By still considering the case where there is no aggregate uncertainty $q_t^0\left(s^t\right)=\beta^t\pi_t\left(s^t\right)\frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_0^i\left(s^0\right)\right]}$

$$\sum_{t=0}^{\infty}\sum_{s^t}q_t^0\left(s^t\right)y_t^i\left(s^t\right)=\sum_{t=0}^{\infty}\sum_{s^t}q_t^0\left(s^t\right)c_t^i\left(s^t\right)\Longrightarrow\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)y_t^i\left(s^t\right)=\sum_{t=0}^{\infty}\sum_{s^t\in S}\beta^t\pi_t\left(s^t\right)c_t^i\left(s^t\right)$$

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  $q_t^0\left(s^t\right) = \beta^t \pi_t\left(s^t\right) \frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_0^i\left(s^0\right)\right]}$

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) y_t^i\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) c_t^i\left(s^t\right) \Longrightarrow \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi_t\left(s^t\right) y_t^i\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t \in S} \beta^t \pi_t\left(s^t\right) c_t^i\left(s^t\right)$$

Define $\bar{C}^i \equiv \sum_{t=0}^{\infty}\sum_{s^t} c_t^i\left(s^t\right)$

Agent's problem in an Arrow-Debreu structure

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$$\sum_{t=0}^{\infty}\sum_{s^t}q_t^0\left(s^t\right)y_t^i\left(s^t\right)=\sum_{t=0}^{\infty}\sum_{s^t}q_t^0\left(s^t\right)c_t^i\left(s^t\right)\Longrightarrow\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)y_t^i\left(s^t\right)=\sum_{t=0}^{\infty}\sum_{s^t\in S}\beta^t\pi_t\left(s^t\right)c_t^i\left(s^t\right)$$

Define $\bar{C}^i\equiv\sum_{t=0}^{\infty}\sum_{s^t}c_t^i\left(s^t\right)$

$$=\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)c_t^i\left(s^t\right)=\bar{C}^i\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi_t\left(s^t\right)=\bar{C}^i\cdot\frac{1}{1-\beta}\cdot 1$$

Agent's problem in an Arrow-Debreu structure

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$$q_t^0\left(s^t\right)=\beta^t\pi_t\left(s^t\right)\frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_0^i\left(s^0\right)\right]}$$

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) y_t^i\left(s^t\right)=\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) c_t^i\left(s^t\right)\Longrightarrow\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right) y_t^i\left(s^t\right)=\sum_{t=0}^{\infty}\sum_{s^t\in S}\beta^t\pi_t\left(s^t\right) c_t^i\left(s^t\right)$$

Define $\bar{C}^i\equiv\sum_{t=0}^{\infty}\sum_{s^t} c_t^i\left(s^t\right)$

$$=\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right) c_t^i\left(s^t\right)=\bar{C}^i\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi_t\left(s^t\right)=\bar{C}^i\cdot\frac{1}{1-\beta}\cdot 1$$

$$\frac{\bar{C}^i}{1-\beta}=\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right) y_t^i\left(s^t\right)\Longrightarrow\bar{C}^i=(1-\beta)\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi_t\left(s^t\right) y_t^i\left(s^t\right)$$

- Therefore, **the present value of consumption is equal to the expected value of the endowment of the agent.**

Agent's problem in a Sequential structure

- First of all define $Q(s_{t+1}|s^t)$ as being the price in t, given the history s^t of an unit of consumption good in period $t+1$ contingent to the realization s_{t+1}.

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- First of all define $Q(s_{t+1}|s^t)$ as being the price in t, given the history s^t of an unit of consumption good in period $t+1$ contingent to the realization s_{t+1}.
- The problem of the agents will be given by:

$$\max_{\left\{c_t^i(s^t),a_{t+1}^i(s^{t+1})\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t\left(s^t\right) u\left[c_t^i\left(s^t\right)\right]$$

subject to:

$$c_t^i\left(s^t\right)+\sum_{s_{t+1}} Q_t\left(s_{t+1} \mid s^t\right) a_{t+1}^i\left(s^{t+1}\right) \leq y_t^i\left(s^t\right)+a_t^i\left(s^t\right) \quad \forall t, \forall s^t$$

Agent's problem in a Sequential structure

- Let us define now a **natural limit for the debt** of the agent:

$$-a_t^i\left(s^t\right) \leq \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{t+1}} q_\tau^0\left(s^\tau\right) y_\tau^i\left(s^\tau\right)$$

- The **debt of the agent has to be lower than the present value of the income of the agent for the remaining of his/her life.**

Agent's problem in a Sequential structure

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t \left(s^t\right) u \left[c_t^i \left(s^t\right)\right] + \sum_{t=0}^{\infty} \sum_{s^t} \eta_t^i \left(s^t\right) [y_t^i \left(s^t\right) + a_t^i \left(s^t\right) - c_t^i \left(s^t\right) - \sum_{s_{t+1}} Q_t \left(s_{t+1} \mid s^t\right) a_{t+1}^i \left(s^{t+1}\right)]$$

In this case we will have the following F.O.C.

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  $\left[c_t^i \left(s^t\right)\right]$:

  $\left[a_{t+1}^i \left(s^{t+1}\right)\right]$:

Agent's problem in a Sequential structure

- So far, we obtained 4 important equations, which are:

$$Q_t\left(s_{t+1} \mid s^t\right) = \beta \pi_t\left(s^{t+1} \mid s^t\right) \frac{u'\left[c_{t+1}^i\left(s^{t+1}\right)\right]}{u'\left[c_t^i\left(s^t\right)\right]} \tag{EE}$$

$$c_t^i\left(s^t\right) + \sum_{s_{t+1}} Q_t\left(s_{t+1} \mid s^t\right) a_{t+1}^i\left(s^{t+1}\right) = y_t^i\left(s^t\right) + a_t^i\left(s^t\right) \quad \forall t, \forall s^t \tag{BC}$$

$$Y_t\left(s^t\right) = \sum_{i=1}^{I} y_t^i\left(s^t\right) = \sum_{i=1}^{I} c_t^i\left(s^t\right) \tag{F}$$

$$\sum_{i=1}^{I} a_t^i\left(s^t\right) = 0 \tag{A}$$

Equivalence of equilibrium in both structures

Remember we have the following important equations in the two structures:

$$q_t^0\left(s^t\right)=\beta^t \pi_t\left(s^t\right) \frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_0^i\left(s^0\right)\right]}$$

$$Q_t\left(s_{t+1} \mid s^t\right)=\beta \pi_t\left(s^{t+1} \mid s^t\right) \frac{u'\left[c_{t+1}^i\left(s^{t+1}\right)\right]}{u'\left[c_t^i\left(s^t\right)\right]}$$

How do we guarantee that the allocations are the same ?

Equivalence of equilibrium in both structures

Remember we have the following important equations in the two structures:

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How do we guarantee that the allocations are the same ?
Just set

Equivalence of equilibrium

$$\frac{q_{t+1}^0\left(s^{t+1}\right)}{q_t^0\left(s^t\right)} \equiv Q_t\left(s_{t+1} \mid s^t\right)$$

Equivalence of equilibrium in both structures

Now we need to check whether the allocation in AD is comparable to that in a Sequential Market Equilibrium:
Set the following value and we will be done:

$$a_t^i\left(s^t\right)=\sum_{\tau=t}^{\infty}\sum_{s^\tau|s^t}q_\tau^t\left(s^\tau\right)\left[c_\tau^i\left(s^\tau\right)-y_\tau^i\left(s^\tau\right)\right]$$

1. Stochastic Environment
2. Finite Markov Chains

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- simple, flexible and supported by many elegant theoretical results;
- valuable for building intuition about random dynamic models;
- central to quantitative modeling in their own right.

Definition

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A Markov chain is a **stochastic process** - a sequence of random variables - on S, a discrete set, with **Markov property**.

A Markov chain (x, P, π) is characterized by a triple of three objects: a **state space** identified with an n-vector x, an $n \times n$ **transition matrix** P and an **initial distribution**, a n-vector π_0.

A stochastic process $\{X_t\}$ has the **Markov property** if, knowing its **current** state is enough to know probabilities for its future states :

$$\mathbb{P}\{X_{t+1} = y | X_t\} = \mathbb{P}\{X_{t+1} = y | X_t, X_{t-1}, ...\}$$

The dynamics of a Markov chain are fully determined by the set of values

$$P(x, y) := \mathbb{P}\{X_{t+1} = y | X_t = x\} \quad (x, y \in S)$$

We can view P as a stochastic matrix with $P_{ij} = \mathbb{P}(X_{t+1}= x_j | X_t = x_i)$, $1 \leqslant i, j \leqslant n$. **A stochastic matrix defines the probability of moving from each value of the state to any other in one period.**

Definition

Markov property

A stochastic matrix (or **Markov matrix**) is an nXn square matrix P such that:

- each element of P is non-negative;
- each row of P sums to one.

Each row of P can be regarded as the probability mass function over n possible outcomes.

A **probability mass function** is a function that gives the probability that a discrete random variable is exactly equal to some value

With $P_{ij} = \mathbb{P}(x_i, x_j)$, fix a row i, then the elements in each of the j columns give the **conditional probabilities of transiting from state** x_i **to state** x_j.

With $P_{ij} = \mathbb{P}(x_i, x_j)$, fix a row i, then the elements in each of the j columns give the **conditional probabilities of transiting from state** x_i **to state** x_j. Example :

At any given time t, a worker is either **unemployed (state 1)** or **employed (state 2)**. Over a one month period, a **u worker** finds a job with probability $\alpha \in (0,1)$ and a **e worker** loses her job and with probability $\beta \in (0,1)$

$$P = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Here : $S =$; $P(1,2) =$; $P(2,1) =$

Example :

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

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Example :

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

Here : $S = \{1, 2\}$; $P(1, 2) = \alpha$; $P(2, 1) = \beta$
Once we have the values of α and β we will be able to answer the following questions:

- What is the **average duration of unemployment**?
- Over the long-run, **what fraction of time does a worker find herself unemployed**?
- **Conditional on employment, what is the probability of becoming unemployed at least once over the next 12 months**?

The initial distribution $\pi_0 = [\pi_{0,i}]$ has elements with the interpretation

$$\pi_{0,i} = \mathbb{P}\{X_0 = x_i\}$$

The initial distribution is often set to have $\pi_i = 1$ for some i and zero everywhere else so that the chain is started in state i with probability 1

Marginal Distributions

Suppose that $\{X_t\}$ is a **Markov chain** with stochastic matrix P and a distribution π_t. You would like to know the distribution of $\{X_{t+1}\}$ i.e., **to know π_{t+1} given π_t and P (and more generally, the distribution of X_{t+m})**. Using the law of total probability, we can decompose the probability that $X_{t+1} = y$ (for any given $y \in S$) :

$$\mathbb{P}\{X_{t+1} = y\} = \sum_{x \in S} \mathbb{P}\{X_{t+1} = y | X_t = x\} . \mathbb{P}\{X_t = x\}$$

To get the probability of being at y tomorrow, **we account for all ways this can happen and sum their probabilities.**

Marginal Distributions

Rewriting the previous statement in terms of **marginal and conditional probabilities gives**:

$$\pi_{t+1}(y) = \sum_{x \in S} P(x, y)\pi_t(x)$$

Or (in matrix form, with π a row vector) :

$$\pi_{t+1} = \pi_t P$$

Marginal Distributions

To move the distribution forward (1 unit of time), we multiply by P

$$\begin{aligned}\pi_{t+2} &= \pi_{t+1}P \\ &= \pi_t P \times P = \pi_t P^2\end{aligned}$$

Then,

$$\pi_{t+m} = \pi_t P^m$$

Marginal Distributions
Intuition

The marginal distributions can be viewed either **as probability or as cross-sectional frequencies in large samples**, **it records the fractions of workers e and u at a given moment**.

Example : consider a large population of workers; let π be the current cross-sectional distribution over$\{u, e\}$. Then, $\pi(1)$ **is the unemployment rate**.

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Example : consider a large population of workers; let π be the current cross-sectional distribution over$\{u, e\}$. Then, $\pi(1)$ **is the unemployment rate**.

The **same distribution also describes the fractions of a particular worker's career spent being employed and unemployed.**

Moments of the Markov chain

$$\mathbb{E}\{X_1\} =$$
$$\mathbb{E}\{X_2\} =$$
$$\dots$$
$$\mathbb{E}\{X_t\} =$$

And,

$$\mathbb{E}\{X_{t+m}|X_t = x\} =$$

Moments of the Markov

$$
\begin{aligned}
&\mathbb{E}\{X_1\} = \pi_0 x \\
&\mathbb{E}\{X_2\} = \pi_0 P x \\
&... \\
&\mathbb{E}\{X_t\} = \pi_0 P^t x
\end{aligned}
$$

And,

$$\mathbb{E}\{X_{t+m} | X_t = x\} = P^m x$$

Moments of the Markov

The probability that the economy is in state x is estimated to be $\pi(x)$.

Considering the example we are working, what is the probability of being **employed** after 1 month given a initial distribution π_0?

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and after **6 months**?

$$\pi_0 P^6 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Few Properties

A matrix is said to be **irreducible** if we can reach any state from any other state (eventually)

If certain regions of the state space cannot be accessed from other regions a matrix is said to have **infinite persistence**.

A Markov chain is called **periodic** if it cycles in a **predictible** way, and **aperiodic** otherwise.

Few Properties

Consider the following case:

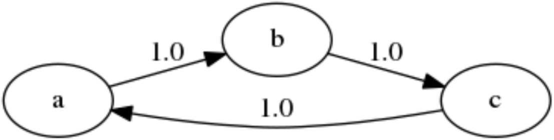

Figure 2: A chain cycle with period 3.

In this case what is the **matrix P**?

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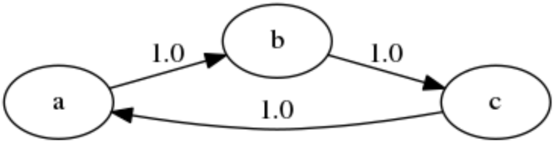

Figure 2: A chain cycle with period 3.

In this case what is the **matrix P**?

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Stationary probability distribution

A **stationary probability distribution**[1] on S is a vector $\bar{\pi}$ such that $\bar{\pi} = \bar{\pi}P$, ie the distribution is invariant under the updating process.

$$\bar{\pi} = \bar{\pi}P \iff \bar{\pi}' = P'\bar{\pi}' \iff (P' - I_n)\bar{\pi} = \underset{n \times n}{0}$$

(P, π) is a stationary Markov chain if the initial distribution π_0 is such that this equation holds. The stationary probability distribution is an eigenvector associated with a unit eigenvalue of P'.

[1] A stationary distribution is a fixed point of P when P is thought of as the map $\pi \to \pi P$ from (row) vectors to (row) vectors

Stationary probability distribution

P must have [2] at least one unit eigenvalue, but there may be more than one such eigenvalue. **Then, every stochastic matrix P (on a finite state space S) has at least one stationary distribution.**

Stationary distributions have a natural interpretation as **stochastic steady states**.

[2] Because $1 \leqslant p_{ij} \leqslant 0$ and $\sum_{i=1}^{n} p_{ij} = 1$

Stationary probability distribution

Example of a stationary distribution :

$$P = \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix} \quad \pi = \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix} \quad \pi P = \begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}$$

And,

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{5} & \frac{1}{2} & \frac{3}{10} \\ 0 & 0 & 1 \end{pmatrix} \quad \pi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \pi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Stationary probability distribution

If P is both aperiodic and irreducible : (i) P has exactly one stationary distribution $\bar{\pi}$ (ii) the chain will asymptotically converge to this unique stationary distribution for all initial conditions $\|\pi_0 P^t - \bar{\pi}\|$ as $t \to \infty$.

A sufficient condition for a matrix to be aperiodic and irreducibe is that every element of P is strictly positive.

Stationary probability distribution

Back to the **employed / unemployed** example : let $\bar{\pi} = (p, 1-p)$ be the stationary distribution. Then, you know that :

$$\begin{pmatrix} p & 1-p \end{pmatrix} \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} = \begin{pmatrix} p & 1-p \end{pmatrix}$$

$$\iff p(1-\alpha) + (1-p)\beta = p$$

$$\iff p = \frac{\beta}{\beta + \alpha}$$

In the cross-sectional interpretation, this is the fraction of people unemployed.

Example

Using US data, Hamilton [3] estimated the stochastic matrix (monthly frequency). The first state represents **"normal growth"**, the second **"mild recession"**, the third **"severe recession"**.

$$P = \begin{pmatrix} 0.971 & 0.029 & 0 \\ 0.145 & 0.778 & 0.077 \\ 0 & 0.508 & 0.492 \end{pmatrix}$$

[3] James D Hamilton. What's real about the business cycle? Federal Reserve Bank of St. Louis Review, (July-August):435–452, 2005.

Coming back to our stochastic environment

Remember we have the following price structure:

$$Q_t\left(s_{t+1} \mid s^t\right)=\beta \pi_t\left(s_{t+1} \mid s^t\right) \frac{u'\left[c_{t+1}^i\left(s^{t+1}\right)\right]}{u'\left[c_t^i\left(s^t\right)\right]}$$

Now assume that the probabilities follow a Markov Process

$$\pi\left(s_{t+1} \mid s_t\right) \Longrightarrow \pi\left(s' \mid s\right), \quad s, s' \in S$$

Stochastic Environment

As a consequence:

$$
\begin{gathered}
Q\left(s' \mid s\right)=\beta \pi\left(s' \mid s\right) \frac{u'\left[c^i\left(s'\right)\right]}{u'\left[c^i(s)\right]}, \quad \forall i, \forall s, \forall s' \\
c^i(s)+\sum_{s'} Q\left(s' \mid s\right) a^i\left(s'\right)=y^i(s)+a^i, \quad \forall i, \forall s, \forall s' \\
\sum_{i=1}^I c^i(s)=\sum_{i=1}^I y^i(s), \quad \forall s \\
\sum_{i=1}^I a^i(s)=0, \quad \forall s
\end{gathered}
$$

Stochastic Environment

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\end{gathered}$$

Now assume we have only two states $S=\{H, L\}$ and two agents $i=\{1,2\}$.
How the prices in this environment should behave?...

Stochastic Environment

The transition matrix will be given by:

$$\left[\begin{array}{cc} \pi(H \mid H) & \pi(L \mid H) \\ \pi(H \mid L) & \pi(L \mid L) \end{array}\right] = \left[\begin{array}{cc} \pi_{HH} & \pi_{LH} \\ \pi_{HL} & \pi_{LL} \end{array}\right]$$

Observe the endowments in each case will be such that:

$$y_{1L} + y_{2L} = Y_L \text{ e } y_{1H} + y_{2H} = Y_H$$

Stochastic Environment

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Observe the endowments in each case will be such that:

$$y_{1L} + y_{2L} = Y_L \text{ e } y_{1H} + y_{2H} = Y_H$$

Assume the following is true:

$$u(c) = \frac{-1}{c} \Longrightarrow u'(c) = \frac{1}{c^2} > 0 \Longrightarrow u''(c) = \frac{-2}{c^3} < 0$$

Stochastic Environment

Therefore, we have the final prices will be:

$$Q(H \mid H) = \beta \pi_{HH}$$

$$Q(L \mid L) = \beta \pi_{LL}$$

$$Q(H \mid L) = \beta \pi_{HL} \left(\frac{Y_L}{Y_H} \right)^2$$

$$Q(L \mid H) = \beta \pi_{LH} \left(\frac{Y_H}{Y_L} \right)^2$$