Macroeconomics III

Economies with Idiosyncratic Risk and Incomplete Markets: Stationary Equilibrium

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Main ideas

We are interested in building a class of models whose equilibrium feature a **nontrivial endogenous distribution of income and wealth** across agents in order to analyze questions such as:

1. What is the fraction of aggregate savings due to the **precautionary motive**?
2. How much of the **observed wealth inequality** can one explain through **uninsurable earnings** variation across agents?
3. What are the **redistributional implications of various fiscal policies**? How are **inequality and welfare** affected by such policies?
4. Can we generate a reasonable **equity premium** (i.e., excess return of stocks over a risk-free bonds), once we introduce a risky asset?
5. How large are the **welfare losses** from individual-level labor market risk (e.g., unemployment)?

Construction of the model

The model is built under the following building blocks:

1. The **"income-fluctuation problem"**.
2. The **aggregate neoclassical production function**.
3. The **equilibrium of the asset market**.

1. Income fluctuation problem

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- Continuum of agents subject to different shocks will give rise to a **wealth distribution**.
- Integrating wealth holdings across all agents will give rise to an **aggregate supply of capital** .

2. Aggregate production function

- Profit maximization of the competitive representative firm operating a C.R.S. technology will give rise to an **aggregate demand for capital**.

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- When we let demand and supply interact in an asset market, an **equilibrium interest rate will arise endogenously**.
- Notice that if a full set of Arrow-Debreu contingent claims were available, the economy would collapse to a **representative agent** model with a stationary amount of savings such that $(1+r)\beta = 1$.
- With uninsurable risk, the supply of savings is larger (r is lower) because of precautionary saving, and consequently $(1+r)\beta < 1$. We like this because we know that this is a **necessary condition for the income-fluctuation problem to have a bounded consumption sequence as solution.**

The Economy

Demographics

The economy is populated with a **continuum of measure one** of infinitely lived, ex-ante identical agents.

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Preferences

The individual has time-separable preferences over streams of consumption,

$$U\left(c_0, c_1, c_2, \ldots\right) = E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

where the period utility function $u(c_t)$ satisfies $u' > 0, u'' < 0$ and the discount factor $\beta \in (0,1)$. The **expectation is over future sequences of shocks**, conditional to the realization at time 0. The individual supplies labor inelastically.

The Economy

Endowment

Each individual has a **stochastic endowment of efficiency units of labor**,

$$\varepsilon_t \in E \equiv \left\{\varepsilon^1, \varepsilon^2, \ldots, \varepsilon^{N-1}, \varepsilon^N\right\}.$$

The shocks follow a Markov process with transition probabilities:

$$\pi\left(\varepsilon', \varepsilon\right) = \Pr\left(\varepsilon_{t+1} = \varepsilon' \mid \varepsilon_t = \varepsilon\right).$$

Those Shocks are $i.i.d.$ across individuals. We assume a law of large numbers to hold, so that

$$\pi\left(\varepsilon', \varepsilon\right),$$

represents a fraction of the population subject to this particular transition. The Markov transition is well-behaved, so there is a unique invariant distribution $\Pi^*(\varepsilon)$.

The Economy

Endowment

As a result of the above discussion the **aggregate endowment of efficiency units** will be:

$$H_t = \sum_{i=1}^{N} \varepsilon_i \Pi^* \left(\varepsilon_i \right), \text{ for all } t$$

This value is constant over time, i.e., **there is no aggregate uncertainty**. Note in particular, that H_t is **exogenously determined**.

The Economy

Budget Constraint

For each individual i at time t, the budget constraint reads:

$$c_t + a_{t+1} = (1 + r_t)\, a_t + w_t \varepsilon_t$$

Wealth is held in the form of a one-period risk-free bond whose price is one and whose return, next period, will be $(1 + r_{t+1})$, independently of the individual state (i.e., r_{t+1} does not depend on the realization of ε_{t+1}). In this sense, the asset a is **non state-contingent**.

The Economy

Liquidity constraint

At every t, agents face the borrowing limit:

$$a_{t+1} \geq -b,$$

where b is exogenously specified. Alternatively, we could assume agents face the **"natural" borrowing constraint, which is the present value of the lowest possible realization of her future earnings**.

The Economy

Technology

The representative competitive firm produces with **C.R.S. production function** $Y_t = F(K_t, H_t)$ with decreasing marginal returns in both inputs and standard Inada conditions. Physical capital depreciates geometrically at rate $\delta \in (0, 1)$.

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Final good market (consumption and investment goods), labor market, and capital market are all **competitive**.

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Aggregate resource

The **aggregate feasibility** condition in this economy reads:

$$F(K_t, H_t) = C_t + I_t = C_t + K_{t+1} - (1 - \delta)K_t,$$

where the capital letters denote aggregate variables

Stationary Equilibrium

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- Moreover, in the stationary equilibrium of this economy we require the distribution of agents across states to be **invariant**.
- However, **individuals move up and down in the earnings and wealth distribution, so "social mobility" can be meaningfully defined**. Recall that with complete markets, there is no social mobility: initial rankings persist forever.
- The probability measure will permanently reproduce itself. It is in this sense that the economy is in a rest-point, i.e., a steady state.

Stationary Equilibrium

Some Mathematical Preliminaries

The individual is characterized by the pair (a, ε) —the **individual states**. Let λ be the **distribution of agents over states**. We would like this object to be a **probability measure**, so we need to define an appropriate mathematical structure.

Stationary Equilibrium

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- The **state space** set $S = A$ X E
- Now let $\sigma - algebra \sum_s$ be defined as $B(A) \otimes P(E)$ where $B(A)$ is the Borel sigma-algebra on A and and $P(E)$ is the power set of E.
- The space $(S, \sum_s)$ is a measurable space. Let $\mathcal{S} = (\mathcal{A} X \mathcal{E})$ be the typical subset of $\sum_s$. For any element of the sigma algebra $\mathcal{S} \in \Sigma_s, \lambda(\mathcal{S})$ is the **measure of agents** in the set $\mathcal{S}$.

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- Define $Q((a,\varepsilon),\mathcal{A}\times\mathcal{E})$, as the (conditional) probability that an individual with current state (a,ε) transits to the set $\mathcal{A}X\mathcal{E}$ next period, formally: $Q : S\times\Sigma_s \rightarrow [0,1]$, and

$$Q((a,\varepsilon),\mathcal{A}\times\mathcal{E}) = I_{\{a'(a,\varepsilon)\in\mathcal{A}\}}\sum_{\varepsilon'\in\mathcal{E}}\pi\left(\varepsilon',\varepsilon\right) \tag{1}$$

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where $I_{\{.\}}$ is the indicator function, and $a'(a,e)$ is the optimal saving policy. Then Q **is the transition function** and the associated T^* operator yields

$$\lambda_{n+1}(\mathcal{A}X\mathcal{E}) = T^*(\lambda_n) = \int_{AXE} Q((a,e),\mathcal{A}X\mathcal{E})d\lambda_n, \tag{2}$$

where $d\lambda_n$ stands for $\lambda_n(da,de)$.

Recursive formulation

The **problem of the individual in recursive form** is going to be given by

$$v(a, e; \lambda) = \max_{c,a'} \{u(c) + \beta \sum_{e' \in E} \pi(e', e) v(a', e'; \lambda)\} \tag{3}$$

subject to

$$\begin{aligned} & c + a' \leq R(\lambda)a + w(\lambda)e \\ & a' \geq -b \end{aligned}$$

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where, for clarity, we have made explicit the dependence of prices from the distribution of agents (although, strictly speaking this dependence is redundant in a stationary environment and it can be omitted since the probability measure λ is **constant**).

Definition of a Stationary RCE

Definition of Stationary Recursive Competitive Equilibrium: A stationary recursive competitive equilibrium is a value function $v : S \to R$; policy functions for the household $a' : S \to R$ and $c : S \to R$; firm's choice H and K, prices r and w; and a stationary measure λ^* such that:

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- the **goods markets clears**: $\int_{AXE} c(a, e) d\lambda^* + \delta K = F(K, H)$
- for all $(\mathcal{A} X \mathcal{E}) \in \sum_s$, the **invariant probability measure** λ^* satisfies

  $$\lambda^*(\mathcal{A} X \mathcal{E}) = \int_{AXE} Q((a, e), \mathcal{A} X \mathcal{E}) d\lambda^*,$$

  where Q is the transition function defined in (1)

Existence and Uniqueness of the Stationary Equilibrium

Demand for capital: Observe we obtain for the optimal choice of the firm that

$$K(r) = F_K^{-1}(r + \delta)$$

Notice if $r = -\delta$, then $K \to \infty$, while $r \to +\infty$, $K \to 0$. In case $F(K, H) = K^{\alpha} H^{1-\alpha}$, then

$$K(r) = \left(\frac{\alpha H}{\delta + r}\right)^{\frac{1}{1-\alpha}}$$

Supply for capital: The aggregate supply function is going to be given by

$$A(r) = \int_{AXE} a'(a, e; r) d\lambda_r^*$$

Existence and Uniqueness of the Stationary Equilibrium

Standard results in dynamic programming is going to ensure that if u is continuous, $u' > 0$ and $u'' < 0$, the solution to the household problem is unique and the policy function $a'(a, e; r)$ is continuous in r (by the **Theorem of the Maximum**). Observe that if $(1 + r)\beta = 1$, then the aggregate supply of assets $A(\frac{1}{\beta} - 1) \to \infty$. In case $r = -1$, the individual would like to borrow until the limit, as every unit of capital saved will vanish $A(-1) \to -b$.

An Algorithm for the Computation of the Equilibrium

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3. Given prices $(r^0, w(r^0))$, now solve the dynamic programming problem of agent in (3) to obtain $a'(a, e; r^0)$ and $c(a, e; r^0)$.
4. Given the policy function $a'(a, e; r^0)$ and the Markov transition over productivity shock $\pi(e', e)$ we can construct the transition functions $Q(r^0)$ and, by successive iterations over (2), we obtain the fixed point distribution $\lambda(r^0)$, conditional on the candidate interest rate r^0.

An Algorithm for the Computation of the Equilibrium

5. Compute the aggregate demand of capital $K(r^0)$ from the optimal choice of the firm who takes as given r^0, i.e.,

$$K(r^0) = F_K^{-1}(r^0 + \delta)$$

6. Compute the integral

$$A(r^0) = \int_{AXE} a'(a, e; r^0) d\lambda(a, e; r^0),$$

which gives the aggregate supply of assets.

An Algorithm for the Computation of the Equilibrium

7. Compute $K(r^0)$ with $A(r^0)$ to verify the assets market clearing condition. If $A(r^0) > (<) K(r^0)$, then the next guess of the interest rate should be lower (higher), i.e., $r^1 < (>) r^0$. To obtain the new candidate r^1 a good choice is, for example,

$$r^1 = \frac{1}{2}\{r^0 + [F_K(A(r^0), H) - \delta]\},$$

which is the bi section method. Note that r^0 and $F_K(A(r^0), H) - \delta$ are, by construction, on opposite sides of the steady-state interest rate r^*.

8. Update the guess to r^1 and come back to step (1). Keep iterating until one reaches convergence of the interest rate, i.e., until

$$|r^{n+1} - r^n| < \epsilon, \tag{4}$$

for ϵ small enough

Calibration of the Model

Technology

With Cobb-Douglas production function, pick the capital share α to be equal to $1/3$ Set the depreciation rate δ to 6%.

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Preferences

Typically, we work with CRRA utility. Let γ be the **coefficient of relative risk aversion. Typical values, in this type of applications, range between 1 and 5**. Choose β such that replicates the **aggregate wealth-income**, e.g., for the U.S. economy which is around 3. Imagine that you're in **complete markets**, then you know that:

$$\alpha K^{\alpha-1}H^{1-\alpha} - \delta = \left(\frac{1}{\beta} - 1\right) \Rightarrow \alpha\left(\frac{Y}{K}\right) - \delta = \frac{1}{\beta} - 1 \Rightarrow$$

$$\beta = \frac{1}{1 + \alpha\left(\frac{Y}{K}\right) - \delta} = \frac{1}{1 + 0.33(0.33) - 0.06} = 0.951$$

Calibration of the Model

Labor Income

Choose the labor endowment shocks to replicate the typical dynamics of individual earnings in the U.S. economy. The right source of data for this purpose are panel-data with information on labor income, such as the **Panel Study of Income Dynamics**. A decent approximation to U.S. individual earnings dynamics is an AR(1) process like

$$\ln y_t = \rho \ln y_{t-1} + \varepsilon_t, \quad \text{with } \varepsilon_t \sim N(0, \sigma_\varepsilon)$$

where the auto-correlation coefficient is $\rho = 0.95$ and the standard deviation of the shocks is 0.20 at a annual frequency. More sophisticated estimates include a transitory component to capture measurement error, as well as less persistent shocks, and a fixed individual component to capture the effect of education, ability, etc.

Calibration of the Model

Borrowing Constraint

If the natural borrowing constraint is not a good choice for the problem at hand, one could calibrate the borrowing constraint in order to match, say, **the fraction of agents with negative net worth** which is around 13% in the U.S. economy.