Macroeconomics III

Stochastic Environment and Finite Markov Chains

Diego Rodrigues

SciencesPo
diego.desousarodrigues@sciencespo.fr

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1. Stochastic Environment

Environment

- The agents make decision at date $t = 0$;
- There are infinite periods;
- At each date $t \geq 0$ there is a realization of a stochastic event $s_t \in \mathcal{S}$;
- The history of these events until period t is denoted by $s^t = [s_0, s_1, s_2, \ldots, s_t]$;
- A particular sequence of events occur with probability $\pi_t(s^t)$;
- The chance that a sequence of events s^τ occurs given the sequence s^t occurred is given by $\pi_t(s^\tau | s^t)$;
- There is a set of I agents that receive a perfectly anticipated endowment $y_t^i(s^t)$ in each period. This endowment is perishable, but the agents can trade among themselves;
- The individual savings can be positive or negative, but the aggregate one will always be zero;

Environment

- Agents have utility $u(c)$ that is increasing and concave, i.e., $u'(c) > 0$ and $u^{''}(c) < 0$;
- Agents will draw a consumption plan for all his life $\left\{c_t^i\left(s^t\right)\right\}_{t=0}^{\infty} \equiv C^i$, which will be subject to a discount factor $\beta \in (0,1)$ such that:

$$U\left(C^i\right) = \sum_{t=0}^{\infty}\sum_{s^t}\beta^t \pi_t\left(s^t\right) u\left[c_t^i\left(s^t\right)\right]$$

- First of all we will solve the **social planner's problem** and then verify the **conditions such that the solution of the competitive equilibrium is efficient**.

The Social Planner's problem

$$\max_{\{C^i\}_{i=1}^{I}} \sum_{i=1}^{I} \lambda^i U\left(C^i\right) \quad \text{s.a.} \quad \sum_{i=1}^{I} c_t^i\left(s^t\right) \leq \sum_{i=1}^{I} y_t^i\left(s^t\right), \quad \forall t, \forall s^t$$

- The problem of the Social Planner is then:

$$\mathcal{L} = \sum_{i=1}^{I} \lambda^i \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t\left(s^t\right) u\left[c_t^i\left(s^t\right)\right] + \sum_{t=0}^{\infty} \sum_{s^t} \theta_t\left(s^t\right)\left[\sum_{i=1}^{I}\left(y_t^i\left(s^t\right) - c_t^i\left(s^t\right)\right)\right]$$

Let's derive the F.O.C.

- $\left[c_t^i\left(s^t\right)\right] : \lambda^i \beta^t \pi_t\left(s^t\right) u'\left[c_t^i\left(s^t\right)\right] = \theta_t\left(s^t\right), \quad \forall i \in I, \forall t, \forall s^t$

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- By defining the aggregate endowment as $Y_t\left(s^t\right) \equiv \sum_{i=1}^I y_t^i\left(s^t\right)$ and assuming the **aggregate endowment between any two periods t and τ is the same**, we can show:
  $Y_t\left(s^t\right)=Y_\tau\left(s^\tau\right) \Longrightarrow c_t^i\left(s^t\right)=c_\tau^i\left(s^\tau\right), \forall i \in I$

Let's derive the F.O.C.

Divide the F.O.C. of the agent i by the F.O.C. of a generic agent 1:

$$\frac{\lambda^i \beta^t \pi_t\left(s^t\right) u'\left[c_t^i\left(s^t\right)\right] = \theta_t\left(s^t\right)}{\lambda^1 \beta^t \pi_t(s^t) u'\left[c_t^1(s^t)\right] = \theta_t(s^t)}$$
$$\implies u'\left[c_t^i\left(s^t\right)\right] = \frac{\lambda^1}{\lambda^i} u'\left[c_t^1\left(s^t\right)\right]$$

By assuming the derivative of the utility function admits inverse we can have:

$$c_t^i\left(s^t\right) = u'^{-1}\left(\frac{\lambda^1}{\lambda^i} u'\left[c_t^1\left(s^t\right)\right]\right)$$
$$\implies c_t^i\left(s^t\right) = h^i\left[c_t^1\left(s^t\right)\right]$$

The feasibility conditions is going to imply that:

$$\sum_{i=1}^{I} c_t^i\left(s^t\right) = \sum_{i=1}^{I} y_t^i\left(s^t\right) \implies \sum_{i=1}^{I} h^i\left[c_t^1\left(s^t\right)\right] = \sum_{i=1}^{I} y_t^i\left(s^t\right)$$

Let's derive the F.O.C.

If the total endowment is the same we have $Y_t(s^t) = Y_\tau(s^\tau)$, where $Y_t(s^t) = \sum_{i=1}^{I} y_t^i(s^t)$ we can show by using the previous equation that:

$$\sum_{i=1}^{I} h^i \left[c_t^1 \left(s^t\right)\right] = \sum_{i=1}^{I} h^i \left[c_\tau^1 \left(s^\tau\right)\right]$$

The only way for the above to be true is such that $c_t^1(s^t) = c_\tau^1(s^\tau)$. Since agent 1 was chose arbitrarily, the above is valid for all agents $i \in I$

Agent's problem in an Arrow-Debreu structure

- Assume there are only two possible states of nature: 0 and 1, i.e., $\mathcal{S} = \{0, 1\}$
- Let the economy always begin in state 0 (i.e., the initial state is deterministic $\pi_0(0) = 1$)

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- Let the economy always begin in state 0 (i.e., the initial state is deterministic $\pi_0(0) = 1$)
- The structure for an economy with 3 periods can be represented by the following tree:

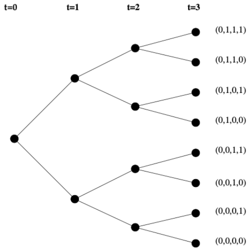

Figure 1: The Arrow-Debreu commodity space for a two-state.

Agent's problem in an Arrow-Debreu structure

- The **market structure is complete** in the sense that the agent can acquire **rights of consumption for any period and for any possible history**.

Agent's problem in an Arrow-Debreu structure

The problem of the agents is

$$\max_{\left\{c_t^i(s^t)\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t\left(s^t\right) u\left[c_t^i\left(s^t\right)\right]$$

subject to

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0\left(s^t\right) y_t^i\left(s^t\right) \geq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0\left(s^t\right) c_t^i\left(s^t\right)$$

Agent's problem in an Arrow-Debreu structure

- Since the markets open only once, in $t=0$, we will have just one Lagrange multiplier in the problem below:

$$\mathcal{L}=\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)u\left[c_t^i\left(s^t\right)\right]+\gamma^i\sum_{t=0}^{\infty}\sum_{s^t}q_t^0\left(s^t\right)\left[y_t^i\left(s^t\right)-c_t^i\left(s^t\right)\right]$$

- The F.O.C.
$\left[c_t^i\left(s^t\right)\right]:\beta^t\pi_t\left(s^t\right)u'\left[c_t^i\left(s^t\right)\right]=\gamma^i q_t^0\left(s^t\right),\quad\forall i\in I,\forall t,\forall s^t$

Agent's problem in an Arrow-Debreu structure

- Using the same idea we used for the Social Planner's problem we can divide the F.O.C. of the agent i by the F.O.C. of an agent 1 generic and obtain:

$$\frac{\beta^t \pi_t \left(s^t\right) u' \left[c_t^i \left(s^t\right)\right] = \gamma^i q_t^0 \left(s^t\right)}{\beta^t \pi_t \left(s^t\right) u' \left[c_t^1 \left(s^t\right)\right] = \gamma^1 q_t^0 \left(s^t\right)} \Longrightarrow u' \left[c_t^i \left(s^t\right)\right] = \frac{\gamma^i}{\gamma^1} u' \left[c_t^1 \left(s^t\right)\right]$$

Agent's problem in an Arrow-Debreu structure

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$$\frac{\beta^t \pi_t\left(s^t\right) u'\left[c_t^i\left(s^t\right)\right] = \gamma^i q_t^0\left(s^t\right)}{\beta^t \pi_t\left(s^t\right) u'\left[c_t^1\left(s^t\right)\right] = \gamma^1 q_t^0\left(s^t\right)} \Longrightarrow u'\left[c_t^i\left(s^t\right)\right] = \frac{\gamma^i}{\gamma^1} u'\left[c_t^1\left(s^t\right)\right]$$

- In other words, by setting $\lambda^i = \frac{\gamma^i}{\gamma^1}\lambda^1$ and using those weights to solve the SP, we will reach the same allocation as the competitive equilibrium. In other words, **there are weights such that the allocation of the social planner is a competitive equilibrium.**

Agent's problem in an Arrow-Debreu structure

- So far, we obtained 4 important equations, which are:

$$\lambda^i \beta^t \pi_t \left(s^t\right) u' \left[c_t^i \left(s^t\right)\right] = \theta_t \left(s^t\right), \quad \forall i \in I, \forall t, \forall s^t \qquad \text{(F.O.C. SP)}$$

$$\beta^t \pi_t \left(s^t\right) u' \left[c_t^i \left(s^t\right)\right] = \gamma^i q_t^0 \left(s^t\right), \quad \forall i \in I, \forall t, \forall s^t \qquad \text{(F.O.C. AD)}$$

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0 \left(s^t\right) \left[y_t^i \left(s^t\right) - c_t^i \left(s^t\right)\right] = 0 \qquad \text{(BC)}$$

$$Y_t \left(s^t\right) = \sum_{i=1}^{I} y_t^i \left(s^t\right) = \sum_{i=1}^{I} c_t^i \left(s^t\right) \qquad \text{(F)}$$

Agent's problem in an Arrow-Debreu structure

- By assuming $q_0^0(s^0) = 1$ we can then reach the Euler Equation:

Agent's problem in an Arrow-Debreu structure

- By assuming $q_0^0(s^0) = 1$ we can then reach the Euler Equation:

$$u'\left[c_0^i\left(s^0\right)\right] = \frac{\beta^t \pi_t\left(s^t\right) u'\left[c_t^i\left(s^t\right)\right]}{q_t^0\left(s^t\right)}$$

Agent's problem in an Arrow-Debreu structure

- By assuming $q_0^0(s^0) = 1$ we can then reach the Euler Equation:

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- From the Euler Equation we can have idea about the prices in this economy:

$$q_t^0\left(s^t\right) = \beta^t \pi_t\left(s^t\right) \frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_0^i\left(s^0\right)\right]}$$

Agent's problem in an Arrow-Debreu structure

- Suppose **there is no aggregate uncertainty**, $Y_t\left(s^t\right)=\bar{Y}, \quad \forall t, \forall s^t$
- By the analysis we already made we can conclude $u'\left[c_t^i\left(s^t\right)\right]=u'\left[c_0^i\left(s^0\right)\right]$, which will lead us to a completely exogenous price:

$$q_t^0\left(s^t\right)=\beta^t\pi_t\left(s^t\right)$$

Agent's problem in an Arrow-Debreu structure

- Suppose $u(c) = \log(c) \Longrightarrow u'(c) = \frac{1}{c}$
- **Show that the consumption of the agent is always a function of the total endowment.**

Agent's problem in an Arrow-Debreu structure

- By still considering the case where there is no aggregate uncertainty $q_t^0\left(s^t\right)=\beta^t\pi_t\left(s^t\right)\frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_0^i\left(s^0\right)\right]}$

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) y_t^i\left(s^t\right)=\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) c_t^i\left(s^t\right) \Longrightarrow \sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right) y_t^i\left(s^t\right)=\sum_{t=0}^{\infty}\sum_{s^t\in S}\beta^t\pi_t\left(s^t\right) c_t^i\left(s^t\right)$$

Agent's problem in an Arrow-Debreu structure

- By still considering the case where there is no aggregate uncertainty $q_t^0\left(s^t\right)=\beta^t\pi_t\left(s^t\right)\frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_0^i\left(s^0\right)\right]}$

$$\sum_{t=0}^{\infty}\sum_{s^t}q_t^0\left(s^t\right)y_t^i\left(s^t\right)=\sum_{t=0}^{\infty}\sum_{s^t}q_t^0\left(s^t\right)c_t^i\left(s^t\right)\Longrightarrow\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)y_t^i\left(s^t\right)=\sum_{t=0}^{\infty}\sum_{s^t\in S}\beta^t\pi_t\left(s^t\right)c_t^i\left(s^t\right)$$

Define $\bar{C}^i\equiv\sum_{t=0}^{\infty}\sum_{s^t}c_t^i\left(s^t\right)$

Agent's problem in an Arrow-Debreu structure

- By still considering the case where there is no aggregate uncertainty
  $q_t^0\left(s^t\right) = \beta^t \pi_t\left(s^t\right) \frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_0^i\left(s^0\right)\right]}$

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) y_t^i\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t} q_t^0\left(s^t\right) c_t^i\left(s^t\right) \Longrightarrow \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi_t\left(s^t\right) y_t^i\left(s^t\right) = \sum_{t=0}^{\infty}\sum_{s^t\in S} \beta^t \pi_t\left(s^t\right) c_t^i\left(s^t\right)$$

Define $\bar{C}^i \equiv \sum_{t=0}^{\infty}\sum_{s^t} c_t^i\left(s^t\right)$

$$= \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi_t\left(s^t\right) c_t^i\left(s^t\right) = \bar{C}^i \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t\left(s^t\right) = \bar{C}^i \cdot \frac{1}{1-\beta} \cdot 1$$

Agent's problem in an Arrow-Debreu structure

- By still considering the case where there is no aggregate uncertainty
$$q_t^0\left(s^t\right)=\beta^t\pi_t\left(s^t\right)\frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_0^i\left(s^0\right)\right]}$$

$$\sum_{t=0}^{\infty}\sum_{s^t}q_t^0\left(s^t\right)y_t^i\left(s^t\right)=\sum_{t=0}^{\infty}\sum_{s^t}q_t^0\left(s^t\right)c_t^i\left(s^t\right)\Longrightarrow\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)y_t^i\left(s^t\right)=\sum_{t=0}^{\infty}\sum_{s^t\in S}\beta^t\pi_t\left(s^t\right)c_t^i\left(s^t\right)$$

Define $\bar{C}^i\equiv\sum_{t=0}^{\infty}\sum_{s^t}c_t^i\left(s^t\right)$

$$=\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)c_t^i\left(s^t\right)=\bar{C}^i\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi_t\left(s^t\right)=\bar{C}^i\cdot\frac{1}{1-\beta}\cdot 1$$

$$\frac{\bar{C}^i}{1-\beta}=\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi_t\left(s^t\right)y_t^i\left(s^t\right)\Longrightarrow\bar{C}^i=(1-\beta)\sum_{t=0}^{\infty}\beta^t\sum_{s^t}\pi_t\left(s^t\right)y_t^i\left(s^t\right)$$

- Therefore, **the present value of consumption is equal to the expected value of the endowment of the agent.**

Agent's problem in a Sequential structure

- First of all define $Q(s_{t+1}|s^t)$ as being the price in t, given the history s^t of an unit of consumption good in period $t+1$ contingent to the realization s_{t+1}.

Agent's problem in a Sequential structure

- First of all define $Q(s_{t+1}|s^t)$ as being the price in t, given the history s^t of an unit of consumption good in period $t+1$ contingent to the realization s_{t+1}.
- The problem of the agents will be given by:

$$\max_{\left\{c_t^i(s^t), a_{t+1}^i(s^{t+1})\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t\left(s^t\right) u\left[c_t^i\left(s^t\right)\right]$$

subject to:

$$c_t^i\left(s^t\right) + \sum_{s_{t+1}} Q_t\left(s_{t+1} \mid s^t\right) a_{t+1}^i\left(s^{t+1}\right) \leq y_t^i\left(s^t\right) + a_t^i\left(s^t\right) \quad \forall t, \forall s^t$$

Agent's problem in a Sequential structure

- Let us define now a **natural limit for the debt** of the agent:

$$-a_t^i\left(s^t\right) \leq \sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^{t+1}} q_\tau^0\left(s^\tau\right) y_\tau^i\left(s^\tau\right)$$

- The **debt of the agent has to be lower than the present value of the income of the agent for the remaining of his/her life.**

Agent's problem in a Sequential structure

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t \left(s^t\right) u \left[c_t^i \left(s^t\right)\right] + \sum_{t=0}^{\infty} \sum_{s^t} \eta_t^i \left(s^t\right) \left[y_t^i \left(s^t\right) + a_t^i \left(s^t\right) - c_t^i \left(s^t\right) - \sum_{s_{t+1}} Q_t \left(s_{t+1} \mid s^t\right) a_{t+1}^i \left(s^{t+1}\right)\right]$$

In this case we will have the following F.O.C.

- The F.O.C.
  $\left[c_t^i \left(s^t\right)\right] : \beta^t \pi_t \left(s^t\right) u' \left[c_t^i \left(s^t\right)\right] = \eta_t^i \left(s^t\right)$

  $\left[a_{t+1}^i \left(s^{t+1}\right)\right] : \eta_t^i \left(s^t\right) Q_t \left(s_{t+1} \mid s^t\right) = \eta_{t+1}^i \left(s^{t+1}\right)$

- So far, we obtained 4 important equations, which are:

$$Q_t\left(s_{t+1} \mid s^t\right) = \beta \pi_t\left(s^{t+1} \mid s^t\right) \frac{u'\left[c_{t+1}^i\left(s^{t+1}\right)\right]}{u'\left[c_t^i\left(s^t\right)\right]} \tag{EE}$$

$$c_t^i\left(s^t\right) + \sum_{s_{t+1}} Q_t\left(s_{t+1} \mid s^t\right) a_{t+1}^i\left(s^{t+1}\right) = y_t^i\left(s^t\right) + a_t^i\left(s^t\right) \quad \forall t, \forall s^t \tag{BC}$$

$$Y_t\left(s^t\right) = \sum_{i=1}^{I} y_t^i\left(s^t\right) = \sum_{i=1}^{I} c_t^i\left(s^t\right) \tag{F}$$

$$\sum_{i=1}^{I} a_t^i\left(s^t\right) = 0 \tag{A}$$

Equivalence of equilibrium in both structures

Remember we have the following important equations in the two structures:

$$q_t^0\left(s^t\right)=\beta^t\pi_t\left(s^t\right)\frac{u'\left[c_t^i\left(s^t\right)\right]}{u'\left[c_0^i\left(s^0\right)\right]}$$

$$Q_t\left(s_{t+1}\mid s^t\right)=\beta\pi_t\left(s^{t+1}\mid s^t\right)\frac{u'\left[c_{t+1}^i\left(s^{t+1}\right)\right]}{u'\left[c_t^i\left(s^t\right)\right]}$$

How do we guarantee that the allocations are the same ?

Equivalence of equilibrium in both structures

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$$Q_t\left(s_{t+1}\mid s^t\right)=\beta\pi_t\left(s^{t+1}\mid s^t\right)\frac{u'\left[c_{t+1}^i\left(s^{t+1}\right)\right]}{u'\left[c_t^i\left(s^t\right)\right]}$$

How do we guarantee that the allocations are the same ?
Just set

Equivalence of equilibrium

$$\frac{q_{t+1}^0\left(s^{t+1}\right)}{q_t^0\left(s^t\right)}\equiv Q_t\left(s_{t+1}\mid s^t\right)$$

Equivalence of equilibrium in both structures

Now we need to check whether the allocation in AD is comparable to that in a Sequential Market Equilibrium:
Set the following value and we will be done:

$$a_t^i\left(s^t\right) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t\left(s^\tau\right)\left[c_\tau^i\left(s^\tau\right) - y_\tau^i\left(s^\tau\right)\right]$$