Average person:

needs d_i units of nutrient i, $i \in \{1, ..., 13\}$

One unit of food $j \in \{1, ..., 77\}$ contains N_{ij} in nutrient i.

We are looking for a diet that is a combination of foods such that we meet all the nutrient requirements.

Look for $q_j \geq 0$ such that $\sum_j q_j N_{ij} \geq d_i$ for all i that is in vector notation

$$Nq \geq d$$

Let c_j be the cost of one unit of food j, we are looking for

$$\begin{aligned} V_P = \min_{(q_j) \geq 0} & \quad \sum_j q_j c_j \\ s.t. & \quad \sum_j q_j N_{ij} \geq d_i \ \ [y_i \geq 0] \end{aligned}$$

or in vector notations

$$\begin{aligned} V_P = \min_{(q_j) \geq 0} & \quad q^\top c \\ s.t. & \quad Nq \geq d. \end{aligned}$$

This is the optimal diet problem.

Duality: we rewrite the problem as

$$V_P = \min_{(q_j) \geq 0} q^\top c + \sum_i L\left(d_i - \sum_j q_j N_{ij}\right)$$

where $L(z) = 0$ if $z \leq 0$ and $+\infty$ otherwise. Claim: we can write L as

$$L(z) = \max_{y \geq 0} zy.$$

The program becomes

$$\begin{aligned} & \min_{(q_j) \geq 0} q^\top c + \sum_i \max_{y_i \geq 0} y_i \left(d_i - \sum_j q_j N_{ij}\right) \\ = & \min_{(q_j) \geq 0} \max_{(y_i) \geq 0} \left\{q^\top c + y^\top (d - Nq)\right\} = V_P \end{aligned}$$

The weak duality inequality always holds, that is

$$\min_q \max_y f(q, y) \geq \max_y \min_q f(q, y)$$

Exercise: prove it.

We have

$$V_P \geq V_D = \max_{(y_i) \geq 0} \min_{(q_j) \geq 0} \left\{q^\top c + y^\top (d - Nq)\right\}$$

Let's study V_D. We have

$$\begin{aligned} V_D &= \max_{(y_i)\geq 0} y^\top d + \min_{(q_j)\geq 0} \left\{q^\top c - y^\top N q\right\} \\ &= \max_{(y_i)\geq 0} y^\top d + \min_{(q_j)\geq 0} \left\{q^\top c - q^\top N^\top y\right\} \\ &= \max_{(y_i)\geq 0} y^\top d + \min_{(q_j)\geq 0} \left\{q^\top \left(c - N^\top y\right)\right\} \\ &= \max_{(y_i)\geq 0} y^\top d + \sum_j \min_{q_j\geq 0} \left\{q_j \left(c_j - \left(N^\top y\right)_j\right)\right\} \end{aligned}$$

We need to evaluate

$$\begin{aligned} \min_{q\geq 0} qw &= -\infty \text{ if } w < 0 \\ &= 0 \text{ if } w \geq 0 \end{aligned}$$

We have thus

$$\begin{aligned} V_D = \quad & \max_{(y_i)\geq 0} y^\top d \\ s.t. \quad & c_j - \left(N^\top y\right)_j \geq 0 \end{aligned}$$

that is

$$\begin{aligned} V_D &= \max_{(y_i)\geq 0} y^\top d \\ s.t. \quad & \left(N^\top y\right)_j \leq c_j \end{aligned}$$

which we can write as

$$\begin{aligned} V_D &= \max_{(y_i)\geq 0} \sum_i y_i d_i \\ s.t. \quad & \sum_i N_{ij} y_i \leq c_j \ \ [q_j \geq 0] \end{aligned}$$

Interpretation of y_i.

We say that q is a primal feasible solution if it satisfies the constraints of the primal problem, ie $q \geq 0$ and $Nq \geq d$.

When q is a solution to the primal problem, we say that q is a primal optimal solution.

Same notions for the dual.

Theorem (duality). (i) If the primal or the dual is feasible (that is, if there are primal or dual feasible solutions) then strong duality holds, i.e. $V_P = V_D$.

(ii) If BOTH the primal and the dual are feasible, then $V_P = V_D$ are both finite.

Remark: if $V_P \neq V_D$ then $V_P = +\infty$ and $V_D = -\infty$. This happens if and only if both the primal and the dual are not feasible.

Exercise: give a simple example of such a situation.

Theorem (complementary slackness). Let q be a primal feasible solution and y a dual feasible solution. The following two statements are equivalent:

(i) q is optimal for the primal and y is optimal for the dual;

and

(ii) complementary slackness relations hold, that is: $q_j > 0 \implies c_j = \sum_i N_{ij} y_i$ and $y_i > 0 \implies d_i = \sum_j N_{ij} q_j$.

Back to the diet problem.

We have $N_{ij} \geq 0$, so the dual is feasible as $y = 0$ does the trick.

Thus we have $V_P = V_D$, and these values are either a finite number, or $+\infty$.

The primal is feasible if $\max_j N_{ij} > 0$ for every i.

y_i is the shadow cost of nutrient i..