Virtual whiteboard for Jan'22 math+econ+code

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1 The diet problem

Consider N_{ij}=amount of nutrient i that is brought by one dollar's worth of food j.

Look for q_j = \$ invested in food j
such that we minimize total cost
$\min \sum_j q_j$

subject to the constraint that the minimal intake in each food i is met
$\sum_j N_{ij} q_j \geq d_i$ for each nutrient i
where d_i is the minimum quantity of nutrient i.

To summarize, we need to solve
$\min_{q \geq 0} \sum_j q_j$
s.t. $\sum_j N_{ij} q_j \geq d_i$

In matrix form this is
$\min_{q \geq 0} c^\top q$
s.t. $Nq \geq d$
where $c_j = 1$.

1.1 Duality worked out by hand

$\min_{q \geq 0} \sum_j c_j q_j$
s.t. $\sum_j N_{ij} q_j \geq d_i$
transform into an unconstrained minimization problem
$\min_{q \geq 0} \sum_j c_j q_j + \sum_i F\left(d_i - \sum_j N_{ij} q_j\right)$
where $F(z) = 0$ if $z \leq 0$, and $F(z) = +\infty$ if $z > 0$.
How to represent F? How about
$F(z) = \max_{\pi \geq 0} \{\pi z\}$.

$\min_{q \geq 0} \sum_j c_j q_j + \sum_i \max_{\pi_i \geq 0} \pi_i \left(d_i - \sum_j N_{ij} q_j\right)$

$\min_{q\geq 0} \sum_j c_j q_j + \max_{\pi_i \geq 0} \sum_i \pi_i \left(d_i - \sum_j N_{ij} q_j \right)$

$\min_{q_j\geq 0} \max_{\pi_i \geq 0} \sum_j c_j q_j + \sum_i \pi_i \left(d_i - \sum_j N_{ij} q_j \right)$
Assuming $\min\max = \max\min$, we can reformulate this as
$\max_{\pi_i \geq 0} \min_{q_j\geq 0} \sum_j c_j q_j + \sum_i \pi_i \left(d_i - \sum_j N_{ij} q_j \right)$
$\max_{\pi_i \geq 0} \min_{q_j\geq 0} \sum_j c_j q_j + \sum_i \pi_i d_i - \sum_{i,j} \pi_i N_{ij} q_j$
going through the same steps in the opposite direction:
$\max_{\pi_i \geq 0} \sum_i \pi_i d_i + \min_{q_j\geq 0} \sum_j c_j q_j - \sum_{i,j} \pi_i N_{ij} q_j$
$\max_{\pi_i \geq 0} \sum_i \pi_i d_i + \min_{q_j\geq 0} \sum_j q_j \left(c_j - \sum_i \pi_i N_{ij} \right)$
$\max_{\pi_i \geq 0} \sum_i \pi_i d_i + \sum_j \min_{q_j\geq 0} q_j \left(c_j - \sum_i \pi_i N_{ij} \right)$
But $\left(\min_{q_j \geq 0} q_j w_j \right) = -\infty$ if $w_j < 0$ and $= 0$ if $w_j \geq 0$

Therefore
$\max_{\pi_i \geq 0} \sum_i \pi_i d_i$
$s.t.\ c_j \geq \sum_i \pi_i N_{ij}$
that is
$\max_{\pi \geq 0} \pi^\top d$
$s.t.\ N^\top \pi \leq c$

Theorem. Consider the "primal problem"
$\min_{q\geq 0} \sum_j c_j q_j$
s.t. $\sum_j N_{ij} q_j \geq d_i$ $[\pi_i \geq 0]$
and the "dual problem"
$\max_{\pi_i \geq 0} \sum_i \pi_i d_i$
$s.t.\ \sum_i \pi_i N_{ij} \leq c_j$ $[q_j \geq 0]$

Then if either of them is feasible [i.e. there is a variable that meets the constraints] then

(1) the value of the primal problem is equal to the value of the dual problem
(2) complementary slackness:
Assume q is a solution to the primal problem and π is a solution to the dual problem
$\pi_i > 0$ implies $d_i = \sum_j N_{ij} q_j$
$q_j > 0$ implies $c_j = \sum_i \pi_i N_{ij}$

Theorem. If q is feasible for the primal and π is feasible for the dual and if complementary slackness holds, then q is optimal for the primal and π is optimal for the dual.

2 The optimal assigment problem

Joint surplus matrix
$\Phi_{xy} = x^\top A y = \sum_{k,l} A_{kl} x_k y_l$

Assume n_x men of type x and m_y women of type y.
Becker-Shapley-Shubik's model of matching.
Assume if man x matches with woman y, then:
x gets surplus α_{xy}
y gets surplus γ_{xy}

Assume utility is transferable, ie if w_{xy} is the transfer from the woman to the man (either positive or negative),
x gets surplus $\alpha_{xy} + w_{xy}$
y gets surplus $\gamma_{xy} - w_{xy}$
This is the Transferable Utility assumption.

w_{xy} is determined at equilibrium.
Regardless of what w_{xy} is, the joint surplus
$\Phi_{xy} = (\alpha_{xy} + w_{xy}) + \left(\gamma_{xy} - w_{xy}\right) = \alpha_{xy} + \gamma_{xy}$ is the same.

Roadmap:
1. Optimality
2. Equilibrium

2.1 Optimality

A matching is a $\mu_{xy} \geq 0$ which is the number of men of type x matched with women of type y.
Constraint on μ_{xy}:
$\sum_y \mu_{xy} = n_x$
$\sum_x \mu_{xy} = m_y$

Optimal matching consists of
$\max_{\mu \geq 0} \sum_{xy} \mu_{xy} \Phi_{xy}$
s.t.
$\sum_y \mu_{xy} = n_x \;\; [u_x]$
$\sum_x \mu_{xy} = m_y \;\; [v_y]$

Duality:
$\max_{\mu \geq 0} \sum_{xy} \mu_{xy} \Phi_{xy} + \sum_x \min_{u_x} u_x \left(n_x - \sum_y \mu_{xy}\right) + \sum_y \min_{v_y} v_y \left(m_y - \sum_x \mu_{xy}\right)$
$\min_{u_x, v_y} \sum_x n_x u_x + \sum_y m_y v_y + \max_{\mu \geq 0} \sum_{xy} \mu_{xy} \left(\Phi_{xy} - u_x - v_y\right)$
that is, the dual problem is
$\min_{u_x, v_y} \sum_x n_x u_x + \sum_y m_y v_y$
s.t. $u_x + v_y \geq \Phi_{xy} \;\; \left[\mu_{xy} \geq 0\right]$

By complementary slackness, we have $\mu_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$.

2.2 Interpretation as a stable outcome.

We now consider the decentralized version of the problem.

Definition: (μ, u, v) is a stable outcome if

(1) μ is a matching:

$\sum_y \mu_{xy} = n_x \;\; [u_x]$

$\sum_x \mu_{xy} = m_y \;\; [v_y]$

(2) Stability: we have for all x and y that

$u_x + v_y \geq \Phi_{xy}$

[otherwise we would have $u_x + v_y < \Phi_{xy}$ and xy would be a blocking pair, ie there is a way for x to have more than u_x and y to have more than v_y if x and y match together]

(3) Feasibility: if $\mu_{xy} > 0$, then $u_x + v_y = \Phi_{xy}$.

Note that (1) means that μ is feasible for the primal problem

(2) means that (u, v) is feasible for the dual problem

(3) means complementary slackness.

Hence μ is a solution to the primal problem

$\max_{\mu \geq 0} \sum_{xy} \mu_{xy} \Phi_{xy}$

s.t.

$\sum_y \mu_{xy} = n_x \;\; [u_x]$

$\sum_x \mu_{xy} = m_y \;\; [v_y]$

and (u, v) is a solution to the dual problem

$\min_{u_x, v_y} \sum_x n_x u_x + \sum_y m_y v_y$

s.t. $u_x + v_y \geq \Phi_{xy} \;\; \left[\mu_{xy} \geq 0\right]$

Recovering the transfers. We had that if w_{xy} is the transfer from y to x, then

x gets surplus $\alpha_{xy} + w_{xy}$

y gets surplus $\gamma_{xy} - w_{xy}$

Hence the payoff of x at equilibrium is

$u_x = \max_y \left\{\alpha_{xy} + w_{xy}\right\}$

and

$v_y = \max_x \left\{\gamma_{xy} - w_{xy}\right\}$

Hence

$u_x \geq \alpha_{xy} + w_{xy}$ for all x and y

$v_y \geq \gamma_{xy} - w_{xy}$ for all x and y

This yields

$\gamma_{xy} - v_y \leq w_{xy} \leq u_x - \alpha_{xy}$

We have $u_x - \alpha_{xy} \geq \gamma_{xy} - v_y$ – indeed $u_x + v_y \geq \alpha_{xy} + \gamma_{xy}$

When $\mu_{xy} > 0$, we have $w_{xy} = \gamma_{xy} - v_y = u_x - \alpha_{xy}$

When partners can remain unmatched.

Assume that individuals get utility zero if they remain unmatched.

Then a feasible matching imposes

$\sum_y \mu_{xy} \leq n_x$
$\sum_x \mu_{xy} \leq m_y$

An optimal matching solves
$\max_{\mu \geq 0} \sum_{xy} \mu_{xy} \Phi_{xy}$
s.t.
$\sum_y \mu_{xy} \leq n_x \;\; [u_x \geq 0]$
$\sum_x \mu_{xy} \leq m_y \;\; [v_y \geq 0]$

Duality:
$\max_{\mu \geq 0} \sum_{xy} \mu_{xy} \Phi_{xy} + \sum_x \min_{u_x \geq 0} u_x \left(n_x - \sum_y \mu_{xy} \right) + \sum_y \min_{v_y \geq 0} v_y \left(m_y - \sum_x \mu_{xy} \right)$
$\min_{u_x \geq, v_y \geq} \sum_x n_x u_x + \sum_y m_y v_y + \max_{\mu \geq 0} \sum_{xy} \mu_{xy} \left(\Phi_{xy} - u_x - v_y \right)$
that is, the dual problem is
$\min_{u_x \geq, v_y \geq 0} \sum_x n_x u_x + \sum_y m_y v_y$
s.t. $u_x + v_y \geq \Phi_{xy} \;\; \left[\mu_{xy} \geq 0 \right]$

By complementary slackness, we have $\mu_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$.
The dual is
$\min_{u_x \geq 0, v_y \geq 0} \sum_x n_x u_x + \sum_y m_y v_y$
s.t. $u_x + v_y \geq \Phi_{xy} \;\; \left[\mu_{xy} \geq 0 \right]$

Alternatively, the primal can be expresses as
$\max_{\mu \geq 0} \sum_{xy} \mu_{xy} \Phi_{xy}$
s.t.
$\sum_y \mu_{xy} + \mu_{x0} = n_x \;\; [u_x \geq 0]$
$\sum_x \mu_{xy} + \mu_{0y} = m_y \;\; [v_y \geq 0]$
where μ_{x0} and μ_{0y} act as slackness variable.

Stability interpretation: a stable outcome (μ, u, v) in the problem with singles is such that

(1) μ is a feasible partial matching:
$\sum_y \mu_{xy} + \mu_{x0} = n_x$
$\sum_x \mu_{xy} + \mu_{0y} = m_y$
(2) Stability holds
$u_x + v_y \geq \Phi_{xy}$
$u_x \geq 0, v_y \geq 0$
(3) Complementary slackness
$\mu_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$
$\mu_{x0} > 0$ i.e $\left(\sum_y \mu_{xy} < n_x \right) \implies u_x = 0$
$\mu_{0y} > 0$ i.e. $\left(\sum_x \mu_{xy} < m_y \right) \implies v_y = 0$

2.3 Indivisibilities (finite population)

When $n_x = 1$ for each x and $m_y = 1$ for each y, we should in principle impose an integrality constraint, that is

$$\mu_{xy} \in \{0, 1\}.$$

Then the problem is no longer a linear programming problem, but an integer programming problem.

However, in the bipartite case, one can abstract away from the integrality constraint.

2.4 Computation

Consider the problem
$\max_{\mu \geq 0} \sum_{xy} \mu_{xy} \Phi_{xy}$
s.t.
$\sum_y \mu_{xy} \leq n_x \;\; [u_x \geq 0]$
$\sum_x \mu_{xy} \leq m_y \;\; [v_y \geq 0]$
this is of the form
$\max_{\mu \geq 0} \mu^\top \Phi$
$M\mu \leq \binom{n}{m}$

How to convert a matrix into a vector? Take a matrix M, we call $vec\,(M)$ its vectorized version,

This can be done by:

* stack the columns: Matlab, Julia, R, Fortran column-major ordering, or Fortran ordering.

* stack the rows: done by NumPy by default, as well as C as well as some other languages: row-major ordering, or C ordering – primary convetion in this course.

Take the first set of constraints $\sum_y \mu_{xy} \leq n_x$. If μ is understood as a matrix, this is

$$\mu 1_Y \leq n$$

by pre-multuplying by the identity this yields

$$I_X \mu 1_Y \leq n$$

And we need to look at $vec\,(I_X \mu 1_Y) = matrix.vec\,(\mu)$

Fundamental identity is (assuming row-major ordering)

$$vec\left(AXB^\top\right) = (A \otimes B)\, vec\,(X).$$

Here, our constraints become

$$vec\,(I_X \mu 1_Y) = \left(I_X \otimes 1_Y^\top\right) vec\,(\mu) \leq n$$

Similarly, we can vectorize the other constraints $\sum_x \mu_{xy} \leq m_y$ into $1_X^\top \mu I_Y \leq m$, that is

$$vec\left(1_X^\top \mu I_Y\right) = \left(1_X^\top \otimes I_Y\right) vec\,(\mu) \leq m$$

Hence our optimal assignment problem becomes in a vectorized fashion

$$
\begin{aligned}
\max_{vec(\mu)\geq 0} \quad & vec(\mu)^\top vec(\Phi) \\
s.t. \quad & \left(I_X \otimes 1_Y^\top\right) vec(\mu) \leq n \\
& \left(1_X^\top \otimes I_Y\right) vec(\mu) \leq m
\end{aligned}
$$

that is

$$
\begin{aligned}
\max_{v\geq 0} \quad & v^\top vec(\Phi) \\
s.t. \quad & Mv \leq \begin{pmatrix} n \\ m \end{pmatrix}
\end{aligned}
$$

where

$$
M = \begin{pmatrix} I_X \otimes 1_Y^\top \\ 1_X^\top \otimes I_Y \end{pmatrix}
$$

is called the margining matrix.