Neutron Stars in General Relativity

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1 Introduction

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2 Equations of stellar structure

In this section we will develop the equations of stellar structure for both Newtonian gravity and general relativity. Once derived we will compare their behavior by solving them for a toy model of a neutron star of constant density. By doing this we will see how, at such high densities, GR is necessary for the description of compact objects.

2.1 Newtonian equation

Let our star be spherically symmetric, and have mass M and radius R. The mass enclosed by a shell of radius r (for $r \leq R$) is [1]:

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr \Rightarrow \frac{dm(r)}{dr} = 4\pi r^2 \rho(r)\,, \quad (1)$$

where we have taken the limit of the energy density ρ being equal to the mass density ρ_0 [2].

To calculate the equation of hydrostatic equilibrium we equate the forces generated by stars on gravity to the pressure generated by the fluid. The total pressure force generated on a shell of radius r and thickness dr is

$$F_P = -\left[p(r+dr) - p(r)\right] 4\pi r^2 = -4\pi r^2 \frac{dp}{dr} dr\,. \quad (2)$$

In order for the star to be in hydrostatic equilibrium, we need this force to be equal to that of gravity. By doing this we derive the equation of hydrostatic equilibrium[1] to be

$$4\pi r^2 \frac{dp}{dr} dr = -\frac{Gm(r)dm}{r^2} \Rightarrow \frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2}\,, \quad (3)$$

where in the second step we used (1) to rewrite dm.

2.2 TOV equation

In GR to find the equations of stellar structure we need to solve Einstein's equation for the interior of a solid object where $T_{\mu\nu} \neq 0$. We start again by considering a spherically symmetric static star. The general metric describing such a space is [3, 4]

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2 d\Omega^2\,. \quad (4)$$

From this metric we compute the components of the Einstein tensor to be [3]

$$\begin{aligned}
G_{tt} &= \frac{1}{r^2} e^{2(\Phi-\Lambda)} \left(2r\partial_r\Lambda - 1 + e^{2\Lambda}\right)\,, \\
G_{rr} &= \frac{1}{r^2} \left(2r\partial_r\Phi + 1 - e^{2\Lambda}\right)\,, \\
G_{\theta\theta} &= r^2 e^{-2\Lambda} \Big[\partial_r^2\Phi + (\partial_r\Phi)^2 - \partial_r\Phi\partial_r\Lambda \\
&\qquad + \frac{1}{r}\left(\partial_r\Phi - \partial_r\Lambda\right)\Big]\,, \\
G_{\phi\phi} &= \sin^2\theta\, G_{\theta\theta}\,.
\end{aligned} \quad (5)$$

[1]Notice that (3) is $\rho\frac{d\mathbf{v}}{dr} = -\nabla p - \rho\nabla\Phi$, when $\mathbf{v} = 0$.

Modeling the star as a perfect fluid, in its commoving frame, the energy-momentum tensor is

$$T_{\mu\nu} = (\rho - p)U_\mu U_\nu + p g_{\mu\nu} \,. \tag{6}$$

Normalizing the four-velocity to be timelike, we find that in this frame it becomes $U_\mu = (e^\Phi, 0, 0, 0)$. Inserting this into (6) we obtain that

$$\begin{aligned} T_{tt} &= \rho e^{2\Phi} \,, \\ T_{rr} &= p e^{2\Lambda} \,, \\ T_{\theta\theta} &= p r^2 \,, \\ T_{\phi\phi} &= \sin^2\theta T_{\theta\theta} \,. \end{aligned} \tag{7}$$

With (5) and (7) we find three independent components of Einstein's equation. the tt component

$$\frac{1}{r^2} e^{-2\Lambda} \left(2r\partial_r\Lambda - 1 + e^{2\Lambda}\right) = 8\pi G\rho \,, \tag{8}$$

the rr component

$$\frac{1}{r^2} e^{-2\Lambda} \left(2r\partial_r\Phi + 1 - e^{2\Lambda}\right) = 8\pi G p \,, \tag{9}$$

and the $\theta\theta$ component

$$e^{-2\Lambda}\left[\partial_r^2\Phi + (\partial_r\Phi)^2 - \partial_r\Phi\partial_r\Lambda + \frac{1}{r}\left(\partial_r\Phi - \partial_r\Lambda\right)\right] = 8\pi G p \,. \tag{10}$$

Motivated by Schwarzschild's exterior solution we make the following change of variable

$$m(r) = \frac{1}{2G}\left(r - r e^{-2\Lambda}\right) \Leftrightarrow e^{2\Lambda} = \left[1 - \frac{2Gm(r)}{r}\right]^{-1} \,. \tag{11}$$

Substituting this new definition onto equation (8) allows us to simplify it to

$$\frac{dm(r)}{dr} = 4\pi\rho(r) r^2 \,. \tag{12}$$

We see that this equation is identical to (12), with the caveat that $\rho \neq \rho_0$ outside the NR limit[2]. Using (11) in (9) we find that it can be rewritten as

$$\frac{d\Phi(r)}{dr} = \frac{4\pi G r^3 p(r) + Gm(r)}{r\left[r - 2Gm(r)\right]} \,. \tag{13}$$

From the conservation of Energy-Momentum ($\nabla_\mu T^{\mu\nu} = 0$), where the only non null component will be that with $\nu = r$ [3], we obtain the following relation

$$(\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr} \,. \tag{14}$$

Inserting this result back onto (13) we finally obtain the equation of hydrostatic equilibrium for general relativity, also called the Tolman-Oppenheimer-Volkoff equation.

$$\frac{dp}{dr} = -(\rho + p)\frac{4\pi G r^3 p(r) + Gm(r)}{r\left[r - 2Gm(r)\right]} \,. \tag{15}$$

[2]See [3, 4] for the effects of this difference.

2.3 Constant density star

Now, in order to compare both formalisms let us assume that our star has mass M, radius R and a constant density $\rho_\star$ for $r \leq R$. Under this assumptions both (1) and (12) result in

$$m(r) = \begin{cases} \frac{4}{3}\pi\rho_\star r^3 & \text{for} \quad r \leq R \\ \frac{4}{3}\pi\rho_\star R^3 & \text{for} \quad r > R \end{cases} \,. \tag{16}$$

With this, we now solve the differential equations (3) and (15). The integration is left as an exercise to the reader, its important to note that the boundary condition is $p(R) = 0$. This condition is needed in order to match the interior metric to Schwarzschild at $r = R$, but physically it also makes sense as the limit of the star will be where the is no pressure due to the gas. The solution for the Newtonian equation is

$$p(r) = \frac{2\pi}{3} G\rho_\star^2 \left(R^2 - r^2\right) \,, \tag{17}$$

while the solution for the TOV equation (15) is

$$p(r) = \rho_\star \left(\frac{\sqrt{R^3 - 2GMR^2} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3\sqrt{R^3 - 2GMR^2}}\right) \,. \tag{18}$$

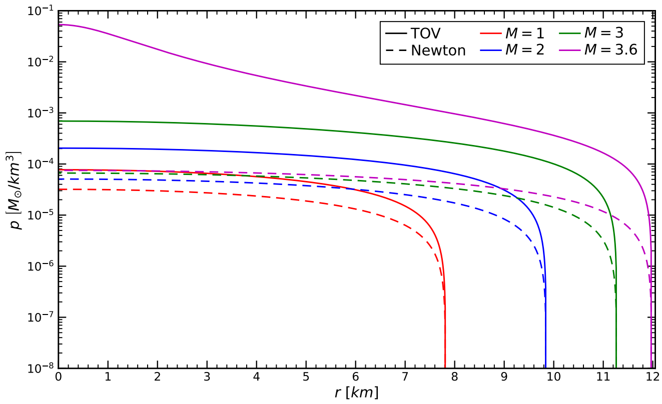

Figure 1: Pressure profiles of various constant density stars obtained in both Newtonian gravity and General Relativity. The different line style indicates wether is was obtained from the Newtonian equation (dashed) or the TOV equation (solid). The mass of the star is given in units of solar masses.

While the equation governing the mass is the same for both models, therefore giving the same Mass-Radius values, the pressure profiles differ greatly. From figure 1 we can see that the general relativity solution gives higher pressure values for the same points r. This difference in pressure is greater ass we increase decrease in r and we increase in M. This is due to the fact that equation 18 reaches a singularity for any $M \geq 4R/9G$. This is called Buchdahl's theorem [3] and its a general feature in GR.

When we solve the TOV equations for different stars we see that there is a maximum allowed mass before they collapse. This feature is not built into Newtonian gravity, we can set a limit of which the matter inside the star collapses but this would be due to matter degeneracy pressure where as this limit is entirely due to gravity.

This reason is why we need GR in order to describe Neutron Star, and other compact objects, the pressures are so high that the Newtonian approximation would not be valid. For the same masses and radii the pressure would be so low that we could not be able to explain how a neutron star can come to be in the first place [1–5].

3 Equations of State

In this section we will look at two possible Equations of State (EoSs) to describe Neutron Stars. The first model is the first ever considered to describe neutron stars [5], it consists on a completely degenerate ideal fermi gas (where the fermions are neutrons). The second model is a more realistic one obtained from chiral effective field theory [6], due to the complex nature of this, we will take it as god-given and only analyze the phenomenology that it brings.

3.1 Free Fermi Gas

From statistical mechanics we know that the number density of a gas can be computed as

$$n = \int \frac{gf}{h^3} d^3k \,, \tag{19}$$

where f is the distribution function, g indicates the degeneracy of spin $g = 2S + 1$, k is the particles momenta and h is Planck's constant [1, 2, 7, 8].

The energy density will be [2, 8]

$$\rho = \int E \frac{gf}{h^3} d^3k \,, \tag{20}$$

where E is the particle's energy ($E^2 = k^2 + m^2$).

Finally the pressure is given by [1, 2, 7]

$$p = \frac{g}{3h^3} \int kvf d^3k \,, \tag{21}$$

where v is the particles velocity ($v = k/E$).

If we have a completely degenerate Fermi Gas[3] the distribution function becomes

$$f(E) = \begin{cases} 1, & E \leq E_F \\ 0, & E \geq E_F \end{cases} \,, \tag{22}$$

where E_F is the Fermi energy [8]. Substituting this into the previous expressions we get the following relations for: the number density

$$n = \int_0^{k_F} \frac{g}{h^3} 4\pi k^2 dk \Rightarrow n = \frac{8\pi k_F^3}{3h^3} = \frac{8\pi m_\chi^3 \xi_F^3}{3h^3} \,, \tag{23}$$

the energy density

$$\rho = \frac{8\pi}{h^3} \int_0^{k_F} \sqrt{m_\chi^2 + k^2} k^2 dk = \frac{\pi m_\chi^4}{h^3} \psi \left(\xi_F\right) \,, \tag{24}$$

and the pressure

$$p = \frac{8\pi}{3h^3} \int_0^{k_F} \frac{k^4 dk}{\sqrt{k^2 + m_\chi^2}} = \frac{\pi m_\chi^4}{h^3} \phi \left(\xi_F\right) \,, \tag{25}$$

where

$$\psi = \sqrt{1 + \xi_F^2} \left(2\xi_F^3 + \xi_F\right) - \ln\left(\xi_F + \sqrt{1 + \xi_F^2}\right) , \tag{26}$$

$$\phi = \sqrt{1 + \xi_F^2} \left(\frac{2}{3}\xi_F^3 - \xi_F\right) + \ln\left(\xi_F + \sqrt{1 + \xi_F^2}\right) . \tag{27}$$

[3] We can consider a NS to be degenerate as its temperature is smaller than the Fermi Temperature [4].

3.2 Nuclear Matter

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4 Solutions for the TOV equation

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4.1 Numerical integration of the TOV equation

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4.2 Fermionic Star

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4.3 Neutron Star

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5 Conclusions

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