Neutron Stars in General Relativity

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1 Equations of stellar structure

In this section we will develop the equations of stellar structure for both Newtonian gravity and General Relativity. This list of equations include: the hydrostatic equilibrium equation, the mass equation, and the equation of state. In Addition to this we will also solve them for a simple model of a neutron star of constant density, by doing this we will see the necessity of the GR equations and also develop and understanding for the behavior of the solutions.

1.1 Newtonian Gravity

Let our star be spherically symmetric, then the mass inside a shell of radius r is given by

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho_0(r) \Leftrightarrow m(r) = \int_0^r 4\pi r^2 \rho_0(r) dr \approx \int_0^r 4\pi r^2 \rho(r) dr\,, \tag{1}$$

where in the last step we implied that the mass density ρ_0 is equivalent to the energy density ρ because we are in Newtonian gravity [1]. This substitution is useful as it provides a way of comparing both frameworks directly.

Now, consider the same shell of thickness dr. The forces acting on that shell are gravity and the fluids on pressure force. The outwards force generated by the fluid on the inside is given by $F_{\rightarrowtail} = p(r)dA$, while the force generated by the fluid outside is $F_{\leftarrowtail} = p(r+dr)dA$ [1,2]. By adding both of this pressures we get that the total outwards pressure force generated by the fluid on all the shell ($dA \rightarrow 4\pi r^2$) is

$$F_P = -\left[p(r+dr) - p(r)\right] 4\pi r^2 = -4\pi r^2 \frac{dp}{dr} dr\,. \tag{2}$$

In order for the star to be in hydrostatic equilibrium, we need this force to be equal to that of gravity so we write

$$4\pi r^2 \frac{dp}{dr} dr + \frac{Gm(r)dm}{r^2} = 0 \Rightarrow 4\pi r^2 \left[\frac{dp}{dr} + \frac{Gm(r)\rho(r)}{r^2}\right] dr = 0 \Rightarrow \frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2}\,, \tag{3}$$

where in the second step we used (1) to rewrite dm.

Equation 3 is called the equation of hydrostatic equilibrium. The set of equations given by (1), (3) and an Equation of State (EoS) that relates pressure and density ($P(\rho)$) [3], are called the equations of stellar structure [4].

Another way of deriving equation (3) comes form fluid mechanics [5]. Start from the Euler equation

$$\rho \frac{d\mathbf{v}}{dr} = -\nabla p - \rho \nabla \Phi\,. \tag{4}$$

Substitute $\mathbf{v} = 0$ because the star is static, and set $\nabla \to d/dr$ due to the spherical symmetry

$$0 = -\frac{dp}{dr} - \rho(r)\frac{d\Phi}{dr} \Rightarrow \frac{dp}{dr} = -\rho(r)\frac{d\Phi}{dr}\,. \tag{5}$$

Finally, substitute $d\Phi/dr$ by the expression of Newtonian gravity and once again we obtain the equation of hydrostatic equilibrium

$$\frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2}\,. \tag{6}$$

1.2 Tolman-Oppenheimer-Volkoff equation

We start again by considering a spherically symmetric static star. The general metric describing such a space is [3,4]

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\,. \tag{7}$$

We will divide spacetime in two spatial regions: outside the star where will have vacuum, and inside the star where $T_{\mu\nu} \neq 0$.

For the vacuum (outside) solution we know, thanks to Birkhoff's theorem, that the Schwarzschild metric (8) is the unique vacuum solution for a spherically symmetric spacetime [3].

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\,. \tag{8}$$

For the inner solution we need to solve Einstein's field equations (9) for a non zero energy-momentum tensor $T_{\mu\nu}$.

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,. \tag{9}$$

The components of Einstein's tensor for the metric (7) are

$$\begin{aligned}
G_{tt} &= \frac{1}{r^2}e^{2(\Phi-\Lambda)}\left(2r\partial_r\Lambda - 1 + e^{2\Lambda}\right)\,,\\
G_{rr} &= \frac{1}{r^2}\left(2r\partial_r\Phi + 1 - e^{2\Lambda}\right)\,,\\
G_{\theta\theta} &= r^2 e^{-2\Lambda}\left[\partial_r^2\Phi + (\partial_r\Phi)^2 - \partial_r\Phi\partial_r\Lambda + \frac{1}{r}\left(\partial_r\Phi - \partial_r\Lambda\right)\right]\,,\\
G_{\phi\phi} &= \sin^2\theta G_{\theta\theta}\,.
\end{aligned} \tag{10}$$

We model the star as a perfect fluid. In the fluids co-moving frame, the energy-momentum tensor has the following form

$$T_{\mu\nu} = (\rho - p)U_\mu U_\nu + p g_{\mu\nu}\,, \tag{11}$$

where U_μ, because we are in the co-moving frame, only has a time component guiven by

$$-1 = g^{\mu\nu}U_\mu U_\nu = -e^{-2\Phi}U_0^2 \Rightarrow U_0 = e^\Phi\,. \tag{12}$$

Using the results of equations (11) and (12) we get that the components of the energy-momentum tensor are

$$\begin{aligned}
T_{tt} &= \rho e^{2\Phi}\,,\\
T_{rr} &= p e^{2\Lambda}\,,\\
T_{\theta\theta} &= p r^2\,,\\
T_{\phi\phi} &= p r^2\sin^2\theta\,.
\end{aligned} \tag{13}$$

Inserting the results of (10) and (13) onto (9) we get 4 different coupled differential equations that we need to solve in order to find the metric. It's easy to see that the $\phi\phi$ equation will be the same as $\theta\theta$ only multiplied by $\sin^2\theta$. With this said, three independent components of Einstein's equations are: the tt component

$$\frac{1}{r^2}e^{-2\Lambda}\left(2r\partial_r\Lambda - 1 + e^{2\Lambda}\right) = 8\pi G\rho\,, \tag{14}$$

the rr component

$$\frac{1}{r^2}e^{-2\Lambda}\left(2r\partial_r\Phi + 1 - e^{2\Lambda}\right) = 8\pi Gp\,, \tag{15}$$

and the $\theta\theta$ component

$$e^{-2\Lambda}\left[\partial_r^2\Phi + (\partial_r\Phi)^2 - \partial_r\Phi\partial_r\Lambda + \frac{1}{r}\left(\partial_r\Phi - \partial_r\Lambda\right)\right] = 8\pi Gp\,. \tag{16}$$

Motivated by Schwarzschild's solution we make the following change of variable

$$m(r) = \frac{1}{2G}\left(r - re^{-2\Lambda}\right) \Leftrightarrow e^{2\Lambda} = \left[1 - \frac{2Gm(r)}{r}\right]^{-1}\,. \tag{17}$$

Substituting this new definition onto equation (14) allows us to simplify it to

$$\frac{dm(r)}{dr} = 4\pi\rho(r)r^2\,. \tag{18}$$

We see that this equation is identical to the mass equation (1) of the Newtonian limit. Equation (18) is the relativistic version of the mass equation. The main difference is that now we integrate the energy density instead of the matter density, this implies that the binding energy of the system is taken into account making the star lighter than it could be if we just summed the masses independently[1].

Using (17) in (15) we find that it can be rewritten as

$$\frac{d\Phi(r)}{dr} = \frac{4\pi Gr^3p(r) + Gm(r)}{r\left[r - 2Gm(r)\right]}\,. \tag{19}$$

This equations gives us the differential equation to solve the potential (tt) part of the metric. For our purposes though we want to find the equation of hydrostatic equilibrium and for that we need a relation between Φ and p.

From the conservation of Energy-Momentum ($\nabla_\mu T^{\mu\nu} = 0$), where the only non null component will be that with $\nu = r$ [3], we obtain the following relation

$$(\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr}\,. \tag{20}$$

Inserting this result back onto (19) we finally obtain the equation of hydrostatic equilibrium for general relativity, also called the Tolman-Oppenheimer-Volkoff equation.

$$\frac{dp}{dr} = -(\rho + p)\frac{4\pi Gr^3p(r) + Gm(r)}{r\left[r - 2Gm(r)\right]}\,. \tag{21}$$

1.3 Comparison of formalisms

In this subsection we will solve both systems of equations for a star with constant density. This baby stellar model will prove useful to see the general phenomenology hidden on both formalisms and also analyze how both systems differ when we move onto higher densities like those seen on neutron stars.

[1] Excellent discussions of this effect can be found in [3, 4].

Let us consider a star made up of an incompressible fluid with constant density $\rho_\star$

$$\rho(r) = \begin{cases} \rho_\star & , \quad r < R \\ 0 & , \quad r > R \end{cases} . \tag{22}$$

Integrating equations (1) and (18) we get that the mass profile, for both formalisms, will be

$$m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho_\star & , \quad r < R \\ \frac{4}{3}\pi R^3 \rho_\star = M & , \quad r > R \end{cases} . \tag{23}$$

Now, with the mass equations solved we solve the equation for hydrostatic equilibrium for Newtonian gravity (6)

$$p(r) = \frac{2\pi}{3}\, G\, \rho_*^2 \left(R^2 - r^2\right) , \tag{24}$$

and we solve the TOV equation for GR (21) [3, 4]

$$p(r) = \rho_\star \left(\frac{\sqrt{R^3 - 2GMR^2} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3\sqrt{R^3 - 2GMR^2}} \right) . \tag{25}$$

In solving both of this equations we used the boundary conditions of $p_c = p(0)$ and $p(R) = 0$. The physical meaning of this conditions is straight forward as we have that p_c is the maximum value of the pressure and as we advance in r its value drops to 0, when $r = R$.

References

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