Neutron Stars in General Relativity

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In this work we study how the theory of general relativity (GR) is able to describe stellar structure of neutron stars (NS). We first justify the need for GR by developing the equations of stellar structure for both Newtonian gravity and GR, and then comparing them on a toy model of a NS of constant density. After this, we discuss two model for the equations of state (EoS) of NS, a free Fermi gas and one obtained via chiral effective field theory. We then solve the equations of stellar structure for both models obtaining the Mass-Radius (MR) relations for them and we compare them to current observations of NS.

1 Introduction

Neutron stars stand as one of the clearest examples of astrophysical systems where the interplay between gravity and the microscopic properties of matter becomes essential for understanding their structure. At the densities reached in their interiors, where nucleons are forced into extremely compact configurations, the Newtonian approximation ceases to be reliable, and the relativistic description of gravity must be employed. In this work we present a detailed comparison between the Newtonian and relativistic equations of stellar structure, illustrating their differences through a simple constant-density model. We then introduce two representative equations of state for neutron-star matter—one based on a degenerate Fermi gas and another obtained from chiral effective field theory—and use them to study the resulting hydrostatic solutions. This allows us to analyze how the choice of equation of state affects the mass–radius relation and how these theoretical predictions compare with present observations of neutron stars.

2 Equations of stellar structure

In this section we will develop the equations of stellar structure for both Newtonian gravity and general relativity. Once derived we will compare their behavior by solving them for a toy model of a neutron star of constant density. By doing this we will see how, at such high densities, GR is necessary for the description of compact objects.

2.1 Newtonian equation

Let our star be spherically symmetric, and have mass M and radius R. The mass enclosed by a shell of radius r (for $r \leq R$) is [1]:

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr \Rightarrow \frac{dm(r)}{dr} = 4\pi r^2 \rho(r)\,, \tag{1}$$

where we have taken the limit of the energy density ρ being equal to the mass density ρ_0 [2].

To calculate the equation of hydrostatic equilibrium we equate the forces generated by stars on gravity to the pressure generated by the fluid. The total pressure force generated on a shell of radius r and thickness dr is

$$F_P = -\left[p(r+dr) - p(r)\right] 4\pi r^2 = -4\pi r^2 \frac{dp}{dr} dr\,. \tag{2}$$

In order for the star to be in hydrostatic equilibrium, we need this force to be equal to that of gravity. By doing this we derive the equation of hydrostatic equilibrium[1] to be

$$4\pi r^2 \frac{dp}{dr} dr = -\frac{G m(r) dm}{r^2} \Rightarrow \frac{dp}{dr} = -\frac{G m(r) \rho(r)}{r^2}\,, \tag{3}$$

where in the second step we used (1) to rewrite dm.

2.2 TOV equation

In GR to find the equations of stellar structure we need to solve Einstein's equation for the interior of a solid object where $T_{\mu\nu} \neq 0$. We start again by considering a spherically symmetric static star. The general metric describing such a space is [3, 4]

$$ds^2 = -e^{2\Phi(r)} dt^2 + e^{2\Lambda(r)} dr^2 + r^2 d\Omega^2\,. \tag{4}$$

From this metric we compute the components of the Einstein tensor to be [3]

$$\begin{aligned} G_{tt} &= \frac{1}{r^2} e^{2(\Phi-\Lambda)} \left(2r\partial_r\Lambda - 1 + e^{2\Lambda}\right)\,, \\ G_{rr} &= \frac{1}{r^2}\left(2r\partial_r\Phi + 1 - e^{2\Lambda}\right)\,, \\ G_{\theta\theta} &= r^2 e^{-2\Lambda}\left[\partial_r^2\Phi + (\partial_r\Phi)^2 - \partial_r\Phi\partial_r\Lambda \right. \\ &\qquad \left. + \frac{1}{r}\left(\partial_r\Phi - \partial_r\Lambda\right)\right]\,, \\ G_{\phi\phi} &= \sin^2\theta\, G_{\theta\theta}\,. \end{aligned} \tag{5}$$

[1]Notice that (3) is $\rho\frac{d\mathbf{v}}{dr} = -\nabla p - \rho\nabla\Phi$, when $\mathbf{v} = 0$.

Modeling the star as a perfect fluid, in its commoving frame, the energy-momentum tensor is

$$T_{\mu\nu} = (\rho - p)U_\mu U_\nu + p g_{\mu\nu} \,. \tag{6}$$

Normalizing the four-velocity to be timelike, we find that in this frame it becomes $U_\mu = (e^\Phi, 0, 0, 0)$. Inserting this into (6) we obtain that

$$\begin{aligned} T_{tt} &= \rho e^{2\Phi} \,, \\ T_{rr} &= p e^{2\Lambda} \,, \\ T_{\theta\theta} &= p r^2 \,, \\ T_{\phi\phi} &= \sin^2\theta T_{\theta\theta} \,. \end{aligned} \tag{7}$$

With (5) and (7) we find three independent components of Einstein's equation. the tt component

$$\frac{1}{r^2} e^{-2\Lambda} \left(2r\partial_r\Lambda - 1 + e^{2\Lambda}\right) = 8\pi G\rho \,, \tag{8}$$

the rr component

$$\frac{1}{r^2} e^{-2\Lambda} \left(2r\partial_r\Phi + 1 - e^{2\Lambda}\right) = 8\pi G p \,, \tag{9}$$

and the $\theta\theta$ component

$$e^{-2\Lambda} \Big[\partial_r^2\Phi + (\partial_r\Phi)^2 - \partial_r\Phi\partial_r\Lambda + \frac{1}{r}\left(\partial_r\Phi - \partial_r\Lambda\right)\Big] = 8\pi G p \,. \tag{10}$$

Motivated by Schwarzschild's exterior solution we make the following change of variable

$$m(r) = \frac{1}{2G}\left(r - r e^{-2\Lambda}\right) \Leftrightarrow e^{2\Lambda} = \left[1 - \frac{2Gm(r)}{r}\right]^{-1} \,. \tag{11}$$

Substituting this new definition onto equation (8) allows us to simplify it to

$$\frac{dm(r)}{dr} = 4\pi\rho(r)r^2 \,. \tag{12}$$

We see that this equation is identical to (12), with the caveat that $\rho \neq \rho_0$ outside the NR limit[2]. Using (11) in (9) we find that it can be rewritten as

$$\frac{d\Phi(r)}{dr} = \frac{4\pi G r^3 p(r) + Gm(r)}{r\left[r - 2Gm(r)\right]} \,. \tag{13}$$

From the conservation of Energy-Momentum ($\nabla_\mu T^{\mu\nu} = 0$), where the only non null component will be that with $\nu = r$ [3], we obtain the following relation

$$(\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr} \,. \tag{14}$$

Inserting this result back onto (13) we finally obtain the equation of hydrostatic equilibrium for general relativity, also called the Tolman-Oppenheimer-Volkoff equation.

$$\frac{dp}{dr} = -(\rho + p)\frac{4\pi G r^3 p(r) + Gm(r)}{r\left[r - 2Gm(r)\right]} \,. \tag{15}$$

[2] See [3, 4] for the effects of this difference.

2.3 Constant density star

Now, in order to compare both formalisms let us assume that our star has mass M, radius R and a constant density $\rho_\star$ for $r \leq R$. Under this assumptions both (1) and (12) result in

$$m(r) = \begin{cases} \frac{4}{3}\pi\rho_\star r^3 & \text{for} \quad r \leq R \\ \frac{4}{3}\pi\rho_\star R^3 & \text{for} \quad r > R \end{cases} \,. \tag{16}$$

With this, we now solve the differential equations (3) and (15). The integration is left as an exercise to the reader, its important to note that the boundary condition is $p(R) = 0$. This condition is needed in order to match the interior metric to Schwarzschild at $r = R$, but physically it also makes sense as the limit of the star will be where the is no pressure due to the gas. The solution for the Newtonian equation is

$$p(r) = \frac{2\pi}{3} G\rho_\star^2 \left(R^2 - r^2\right) \,, \tag{17}$$

while the solution for the TOV equation (15) is

$$p(r) = \rho_\star \left(\frac{\sqrt{R^3 - 2GMR^2} - \sqrt{R^3 - 2GMr^2}}{\sqrt{R^3 - 2GMr^2} - 3\sqrt{R^3 - 2GMR^2}}\right) \,. \tag{18}$$

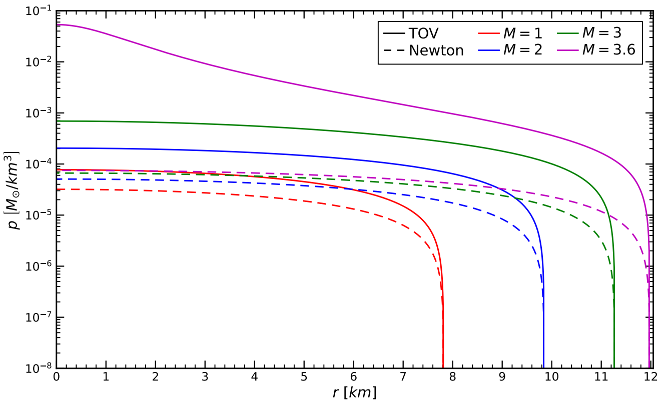


Figure 1: Pressure profiles of various constant density stars obtained in both Newtonian gravity and General Relativity. The different line style indicates wether is was obtained from the Newtonian equation (dashed) or the TOV equation (solid). The mass of the star is given in units of solar masses.

While the equation governing the mass is the same for both models, therefore giving the same Mass-Radius values, the pressure profiles differ greatly. From figure 1 we can see that the general relativity solution gives higher pressure values for the same points r. This difference in pressure is greater ass we increase decrease in r and we increase in M. This is due to the fact that equation 18 reaches a singularity for any $M \geq 4R/9G$. This is called Buchdahl's theorem [3] and its a general feature in GR.

When we solve the TOV equations for different stars we see that there is a maximum allowed mass before they collapse. This feature is not built into Newtonian gravity, we can set a limit of which the matter inside the star collapses but this would be due to matter degeneracy pressure where as this limit is entirely due to gravity.

This reason is why we need GR in order to describe Neutron Star, and other compact objects, the pressures are so high that the Newtonian approximation would not be valid. For the same masses and radii the pressure would be so low that we could not be able to explain how a neutron star can come to be in the first place [1–5].

3 Equations of State

In this section we will look at two possible Equations of State (EoSs) to describe Neutron Stars. The first model is the first ever considered to describe neutron stars [5], it consists on a completely degenerate ideal fermi gas (where the fermions are neutrons). The second model is a more realistic one obtained from chiral effective field theory [6], due to the complex nature of this, we will take it as god-given and only analyze the phenomenology that it brings.

3.1 Free Fermi Gas

From statistical mechanics we know that the number density of a gas can be computed as

$$n = \int \frac{gf}{h^3} d^3k \, , \tag{19}$$

where f is the distribution function, g indicates the degeneracy of spin $g = 2S + 1$, k is the particles momenta and h is Planck's constant [1, 2, 7, 8].

The energy density will be [2, 8]

$$\rho = \int E \frac{gf}{h^3} d^3k \, , \tag{20}$$

where E is the particle's energy ($E^2 = k^2 + m^2$).

Finally the pressure is given by [1, 2, 7]

$$p = \frac{g}{3h^3} \int kvf d^3k \, , \tag{21}$$

where v is the particles velocity ($v = k/E$).

If we have a completely degenerate Fermi Gas[3] the distribution function becomes

$$f(E) = \begin{cases} 1, & E \leq E_F \\ 0, & E \geq E_F \end{cases} \, , \tag{22}$$

where E_F is the Fermi energy [8]. Substituting this into the previous expressions we get the following relations for: the number density

$$n = \int_0^{k_F} \frac{g}{h^3} 4\pi k^2 dk \Rightarrow n = \frac{8\pi k_F^3}{3h^3} = \frac{8\pi m_\chi^3 \xi_F^3}{3h^3} \, , \tag{23}$$

the energy density

$$\rho = \frac{8\pi}{h^3} \int_0^{k_F} \sqrt{m_\chi^2 + k^2} k^2 dk = \frac{\pi m_\chi^4}{h^3} \psi\left(\xi_F\right) \, , \tag{24}$$

and the pressure

$$p = \frac{8\pi}{3h^3} \int_0^{k_F} \frac{k^4 dk}{\sqrt{k^2 + m_\chi^2}} = \frac{\pi m_\chi^4}{h^3} \phi\left(\xi_F\right) \, , \tag{25}$$

[3] We can consider a NS to be degenerate as its temperature is smaller than the Fermi Temperature [4].

where

$$\psi = \sqrt{1 + \xi_F^2} \left(2\xi_F^3 + \xi_F\right) - \ln\left(\xi_F + \sqrt{1 + \xi_F^2}\right) , \tag{26}$$

$$\phi = \sqrt{1 + \xi_F^2} \left(\frac{2}{3}\xi_F^3 - \xi_F\right) + \ln\left(\xi_F + \sqrt{1 + \xi_F^2}\right) . \tag{27}$$

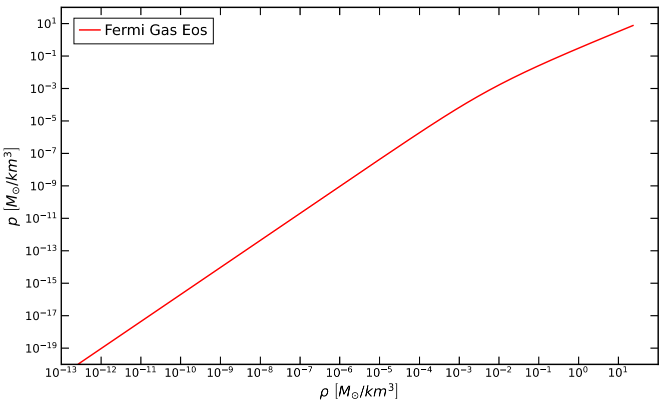


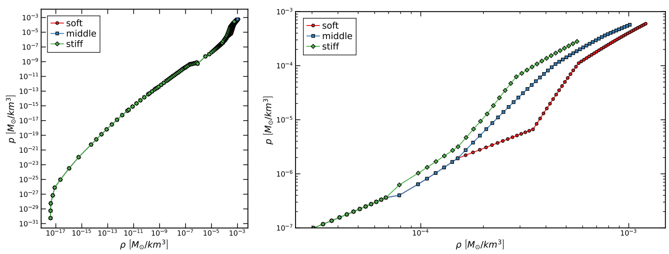
Figure 2: Equation of State for a Completely Degenerate, Ideal Fermi Gas of particle mass $m_n = 0.93956542$GeV [9].

3.2 Nuclear Matter

The previous EoS, although it was the fist ever proposed [5], has proven to be a great simplification of the problem. Sin the discovery of Neutron Stars [10], various observations of these objects have classified the Mass-Radius relations of various pulsars [11].

Through these results and our improved understanding of nucleon nucleon interactions [4, 6], he have come to realize that the determination of an EoS is a subject that requires great care and at the time of writing there is still great uncertainty on this area of nuclear astrophysics.

Due to the complexity of this, for this work we will just limit ourselves to presenting the EoS of [6] and using it for our stellar model calculations.



(a) (b)

Figure 3: Equation of State taken from [6]. Different colors indicate different stiffness. The "stiff" and "soft" EoS represent the upper and lower limits while "middle" is used as a good estimate for a NS. (a) Complete EoS. (b) EoS at high densities.

The EoS [6] of figure 3 is calculated using chiral effective field theory. When calculating it, two conditions where imposed: the speed of sound can't be super-luminal ($\sqrt{dp/d\rho} < 1$)and the EoS mus allow for $M \geq 1.97$ as per the observation of [12].

4 Solutions for the TOV equation

In section 2.3 we solved the TOV equation analytically for a star of constant density. On section 3 we presented two, more realistic, EoS for a neutron star: one obtained from a free Fermi gas of neutrons and the other from chiral effective field theory. Solving the TOV equations for these EoS can´t be done analytically so in this section we will show how it can be done by numerical integration.

4.1 Numerical integration of the TOV equation

To numerically solve the TOV equation one proceeds as follows. First specify the EoS, this can be done analytically or (like we do) through data points that can be interpolated. The you start with your initial conditions $p(r=0) = p_c$ and $m(r=0) = 0$, by virtue of (12) and (15) the whole star is determined by it's central pressure[4] p_c. Now, you integrate equations (12) and (15) from $r = 0$ to when $p = 0$. This value of r is the star's surface, so we say that the star's radius is R and the final value of mass it's the total mass of the star $m(R) = M$ [4].

For this work we do this integration with a Runge Kutta 4 method in Python. The code for this can be found in the following repository [13], while this repository itself is a reduced version of the DANTE software used for solving the TOV equations for neutron stars with a dark matter component [14].

4.2 Fermionic Star

Solving the equations (12) and (15) for the EoS of figure 2 we obtain the following hydrostatic solutions for different values of p_c.

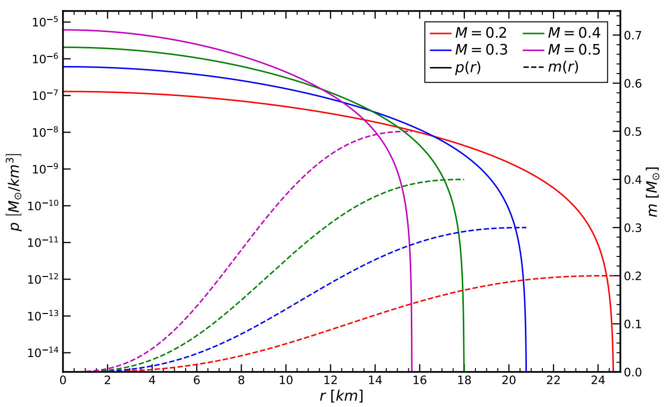


Figure 4: Pressure and mass profiles of different solutions for a neutron star described by a free Fermi Gas. The pressure $p(r)$ is described by solid lines while the mass is described by dashed ones $m(r)$. The different colors indicated different total mass M values.

From figure 4 we see how as we advance in r, m increases uniformly while p decreases till it reaches 0 at $r = R$. Another behavior we notice is how by changingp_c, we obtain different values of M and R, doing this for different values of p_c we can construct the Mass-Radius (MR) curves for this stellar model.

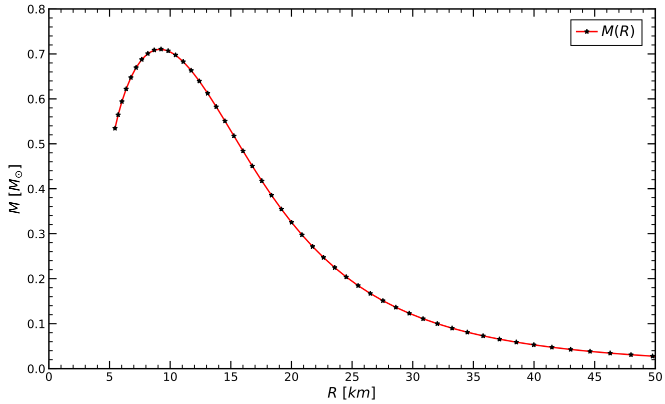


Figure 5: Mass-Radius relation for a stellar model of a neutron star made up of a free Fermi gas of neutrons.

From figure 5 we see how the MR affects the stellar structure. As the star increases its mass, the central pressure and the radius decrease till a point of maximum compactness at $M \sim 0.7\,\mathrm{M}_{\odot}$. From this point, by increasing the central pressure we find more compact solutions for the same value of mass. Although this solutions are valid, they are unstable and eventually they will collapse [5,15].

The curve of figure 5 is similar to that obtained by Oppenheimer and Volkoff [5]. Although this model helps us understand the order of magnitude for the size of neutron stars, it predict that the maximum mass is of the order of $M \sim 0.7\,\mathrm{M}_{\odot}$. Through observations [11,12] we have found more massive neutrons stars so we know that this model is not adequate to describe these results.

4.3 Neutron Star

The Equation of State of figure 3 differentiates between three different stiffness. It has two limiting cases "soft" and "stiff" that represent the limits of our uncertainty in the EoS of neutron stars [6]. The "middle" EoS is a representative option [6] and it is the one we will use for our calculations. But first, let us analyze how the stiffens of the EoS affects the hydrostatic solution for a neutron star.

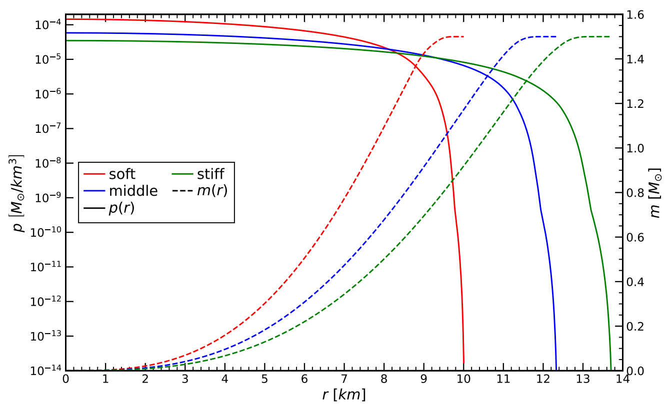


Figure 6: Hydrostatic solutions for a neutron star of mass $M = 1.5\,\mathrm{M}\odot$. Each color depicts the pressure and mass profiles of the star for the different stiffness's allowed by the EoS of [6].

[4]this is important as we will see that M is not enough to classify the star, because for a star of mass M one can find two solutions with different R, one stable and another unstable.

In figure 6 we see how for the same mass the softness of the EoS indicates how compressible the fluid is. This is reflected in how as we make the EoS softer, the central pressure p_c increases and the radius R decreases, indicating a compression of the fluid.

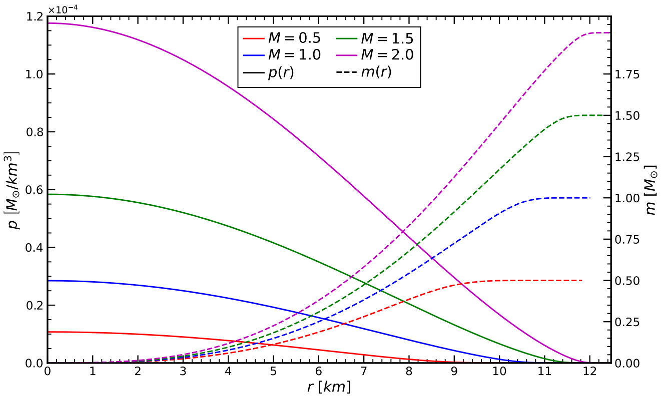


Figure 7: Hydrostatic solutions for neutron stars of different mass. The masses are given in units of $M_\odot$. And we used the "middle" EoS for the calculations [6].

From figure 7 we can see how changing the p_c of the star changes the mass and pressure profiles. In comparison to the results of the Fermi gas model (Fig. 4) we can see how the mass grows like r^3, something indicative of a constant density star (16), until it reaches a point of a few kilometers where the mass barely changes indicating a very low density, something not apparent in the free Fermi model.

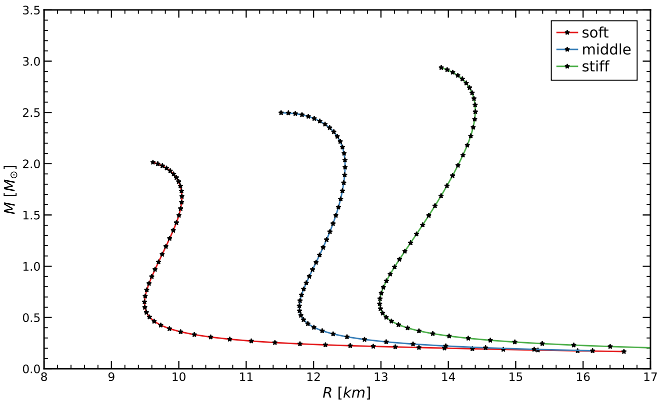


Figure 8: Mass-Radius relations for neutron stars described by the Equation of State of [6]. The different colors indicate the different stiffness's of the EoS.

Finally, we analyze this model's mass radius relations. We note that, by construction, the different EoS allow for $M > 2\,M_\odot$ as to allow for modern observations [11, 12]. Its also interesting to note how the uncertainty of the EoS reflects onto an uncertainty in the MR curve of figure 8. This uncertainty allows for deviations of $1\,M\odot$ on the value of maximum mass, and for measurements of radii for deviations of 4 km. This deviations are similar to the uncertainties of the observations [16] thus indicating that further development into this observation techniques might lead to better constraints on the EoS of neutron stars.

Lastly, another interesting observation of this EoS is that the way R evolves as a function of M is quite different from the one in figure 5. Until $M \sim 0.5\,M\odot$ the radius decreases rapidly. From then on, the radius increases, indicating that the degeneracy pressure is strong enough to maintain the pressure of added matter. Finally, there is a point at which adding more matter compacts the star again and from this point adding more matter actually leads to gravitational collapse.

5 Conclusions

Through this work we have seen how both Newtonian gravity and general relativity give different equations of hydrostatic equilibrium. This allowed us to see, via a baby toy model of a neutron star, why general relativity is needed to describe the stellar structure of neutron stars.

After this, we presented two EoS to describe the matter inside a neutron star. In the following section we saw how the first model, a Free fermi gas, was insufficient to explain the observations of [11, 12], so wee used a more advanced EoS that uses chiral effective field theory to describe the nucleon-nucleon interactions of the star [6].

This EoS allowed us to see how we expect the M-R relation of neutron stars to behave, but it also showed us how our current uncertainties in the theory describing the matter at such high densities, leads to lack of precision on our predictions. Thus showing us why this is an active area of research where the interplay between theory and experiment is essential to make progressively better models for neutron stars in general relativity.

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