Chapter 1: Limits

We start with the limit def. of a function of two variables (easily extended to three/more). Matches (in meaning, not in wording) the definition given by Stewart. Thomas is similar but the set is defined to be the domain of f, though the point can be a boundary point of the domain (which implies, but doesn't require, that P might not be part of the set). Larson/Edwards states that P must be in an open ball on which f is defined, though possibly not at P. Taalman/Kohn doesn't specify the set on which f is defined, etc., at all.

Definition 1 Limit of a Function of Two Variables

Let S be a set containing $P = (x_0, y_0)$ where every open disk centered at P contains points in S other than P, let f be a function of two variables defined on S, except possibly at P, and let L be a real number. The **limit of** $f(x, y)$ **as** (x, y) **approaches** (x_0, y_0) **is** L, denoted

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all (x, y) in S, where $(x, y) \neq (x_0, y_0)$, if (x, y) is in the open disk centered at (x_0, y_0) with radius δ, then $|f(x, y) - L| < \varepsilon$.

A key phrase is "all (x, y) in S". Allows for limits at the boundary of S. Larson/Edwards uses "all (x, y) in open ball centered at (x_0, y_0) ..." but later gives limits such as

$$\lim_{(x,y)\to(0,0)} \frac{x + y}{x^2 + y},$$

where the function is not defined on all points in any open ball containing $(0, 0)$, therefore technically we can't take the limit. (See note in 12.2 section later.)

The natural constriction of this definition to functions of one variable is given below, which doesn't match any texts I've seen. Key phrase: "for all x in I".

Definition 2 The Limit of a Function of One Variable

Let I be an interval containing c, let f be a function defined on I, except possibly at c, and let L be a real number. The **limit of** $f(x)$ **as** x **approaches** c **is** L, denoted by

$$\lim_{x\to c} f(x) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all x in I, where $x \neq c$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

As before, this allows for finding "the" limit at the boundary/endpoint of I.

The following is the current APEX def of the left-hand limit. It is wrong; c is in an open interval (a, b), therefore it doesn't allow for finding limit from the left at $x = b$.

Definition 3 One Sided Limits

Left-Hand Limit
Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. The **limit of $f(x)$, as x approaches c from the left, is** L, or, **the left–hand limit of f at c is** L, denoted by

$$\lim_{x \to c^-} f(x) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x < c$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

A proposed correction, which is similar to Thomas, though the Thomas definition lacks rigor. Larson/Edwards has a heuristic definition, stated informally within a discussion.

Definition 4 One Sided Limits

Left-Hand Limit
Let f be a function defined on (a, c) for some $a < c$ and let L be a real number.
The **limit of $f(x)$, as x approaches c from the left, is** L, or, **the left–hand limit of f at c is** L, denoted by

$$\lim_{x \to c^-} f(x) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $a < x < c$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

This is a natural restriction of the limit def. to a one-sided limit. The key phrase "for all x in I" from the limit definition is replaced with a simple $a < x < c$. Note also $|x - c|$ can be construed as unnecessary; we could write "$c - x$", but it is kept to model previous definitions. Opens up an opportunity for an exercise. The right hand limit would be defined similarly.

The change to "the" limit definition doesn't change the following theorem, as it is given on an open interval. Some of the content following may need to change; perhaps highlight the fact that we are only talking of points within an open interval.

Theorem 1 Limits and One Sided Limits

Let f be a function defined on an open interval I containing c and let L be a real number. Then

$$\lim_{x \to c} f(x) = L$$

if, and only if,

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

In Section 1.3, need to change part 8 to be more precise, matching limit thm in Chapter 12. (Check also emails for errata on # 9)

Theorem 2 **Basic Limit Properties**

Let b, c, L and K be real numbers, let n be a positive integer, and let f and g be functions defined on an interval I containing c with the following limits:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K.$$

The following limits hold.

1. Constants: $\lim_{x\to c} b = b$
2. Identity $\lim_{x\to c} x = c$
3. Sums/Differences: $\lim_{x\to c} \big(f(x) \pm g(x)\big) = L \pm K$
4. Scalar Multiples: $\lim_{x\to c} \big(b \cdot f(x)\big) = bL$
5. Products: $\lim_{x\to c} \big(f(x) \cdot g(x)\big) = LK$
6. Quotients: $\lim_{x\to c} \big(f(x)/g(x)\big) = L/K$, (where $K \neq 0$)
7. Powers: $\lim_{x\to c} \big(f(x)^n\big) = L^n$
8. Roots: $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$
   (If n is even then require $f(x) \geq 0$ on I.)
9. Compositions: Adjust our previously given limit situation to:

   $$\lim_{x\to c} f(x) = L,\ \lim_{x\to L} g(x) = K \text{ and } g(L) = K.$$

   Then $\lim_{x\to c} g(f(x)) = K$.

The definition of continuity changes to remove the restriction of an open interval. This allows for continuity on a closed interval. We effectively get to this point later, anyway, by incorporating left/right hand limits on closed intervals. This will immediately change the outcome of the example in the text following the definition. This will actually remove some ambiguity; in that example, we state that a function is continuous on $(0, 1)$ and $(1, 3)$, whereas later we would have said $[0, 1)$ and $(1, 3]$ once we defined continuity at an endpoint.

Definition 5 **Continuous Function**

Let f be a function defined on an interval I containing c.

1. f is **continuous at** c if $\lim_{x\to c} f(x) = f(c)$.
2. f is **continuous on** I if f is continuous at c for all values of c in I. If f is continuous on $(-\infty, \infty)$, we say f is **continuous everywhere**.

We now remove the following definition from the book, as it is unnecessary.

Definition 6 Continuity on Closed Intervals

Let f be defined on the closed interval $[a, b]$ for some real numbers a, b. f is **continuous on** $[a, b]$ if:

1. f is continuous on (a, b),
2. $\lim_{x\to a^+} f(x) = f(a)$ and
3. $\lim_{x\to b^-} f(x) = f(b)$.

We have always wanted continuity on a closed interval; we now have it by "initial" definition, not by "additional" definition. We use continuity on a closed interval for the Intermediate Value Theorem, which does not change.

Limits at c that involve infinity needs to change. The current APEX definition is ok in spirit, but imprecise. It doesn't give a domain for f. (Thomas actually doesn't formally define this limit; it is discussed in a paragraph. Larson/Edwards defines this limit formally on open intervals.) Suggested change, which formally adds limit approaching $-\infty$:

Definition 7 Limit of Infinity, ∞, Vertical Asymptote

Let I be an interval containing c, and let f be a function defined on I, except possibly at c.

- We say **the limit of** $f(x)$**, as** x **approaches** c**, is infinity**, denoted by $\lim_{x\to c} f(x) = \infty$, if for every $N > 0$ there exists $\delta > 0$ such that for all x in I, where $x \neq c$, if $|x - c| < \delta$, then $f(x) > N$.
- We say **the limit of** $f(x)$**, as** x **approaches** c**, is negative infinity**, denoted by $\lim_{x\to c} f(x) = -\infty$, if for every $N < 0$ there exists $\delta > 0$ such that for all x in I, where $x \neq c$, if $|x - c| < \delta$, then $f(x) < N$.
- If $\lim_{x\to c} f(x) = \pm\infty$, we say the line $x = c$ is a **vertical asymptote** of f.

Note that this allows for c to be an endpoint of I; it does not need one-sided limits. Allows us to immediately say that $x = 0$ is a vertical asymptote of $\ln x$.

The current APEX does not formally define vertical asymptotes, though it does in a paragraph. It formally defines horizontal asymptotes so this change aligns well.

The definition of limits at infinity are ok in spirit but need to be made more precise.

Definition 8 Limits at Infinity and Horizontal Asymptote

Let L be a real number.

1. Let f be a function defined on (a, ∞) for some number a. We say $\lim_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M > a$ such that if $x > M$, then $|f(x) - L| < \varepsilon$.
2. Let f be a function defined on $(-\infty, b)$ for some number b. We say $\lim_{x\to-\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M < b$ such that if $x < M$, then $|f(x) - L| < \varepsilon$.
3. If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, we say the line $y = L$ is a **horizontal asymptote** of f.

Will add a line to the text and an exercise along the lines of "We can combine definitions 7 & 8 to give meaning to the expression '$\lim_{x\to\infty} f(x) = \infty$.' We leave this to the reader as a problem in the Exercises."

Chapter 2: Derivatives

This chapter begins with an example of approximating the velocity of a falling object when it lands. Strictly speaking, by our old definitions of limits (and derivatives), our answer should be "We don't know as we can't find the derivative at the endpoint of an interval."

A new definition of derivative: I is no longer an open interval (that was required by the limit definition). Opens the door for the derivative at the endpoint.

Definition 9 Derivative at a Point

Let f be a continuous function on an interval I containing c. The **derivative of f at c**, denoted $f'(c)$, is

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h},$$

provided the limit exists. If the limit exists, we say that f **is differentiable at** c; if the limit does not exist, then f **is not differentiable at** c. If f is differentiable at every point in I, then f **is differentiable on** I.

This changes the definitions of tangent/normal lines by not restricting to open intervals. We may as well go further: instead of "Let f be continuous on an interval I and differentiable at c, for some c in I." we can say "Let f be a differentiable function on an interval I containing c.", as done below.

Definition 10 Tangent Line

Let f be a differentiable function on an interval I containing c. The line with equation $\ell(x) = f'(c)(x-c) + f(c)$ is the **tangent line** to the graph of f at c; that is, it is the line through $(c, f(c))$ whose slope is the derivative of f at c.

Definition 11 Normal Line

Let f be a differentiable function on an interval I containing c. The **normal line** to the graph of f at c is the line with equation

$$n(x) = \frac{-1}{f'(c)}(x-c) + f(c),$$

where $f'(c) \neq 0$. When $f'(c) = 0$, the normal line is the vertical line through $\big(c, f(c)\big)$; that is, $x = c$.

Next is the definition of the derivative function; we again remove the open interval restriction.

Definition 12 Derivative Function

Let f be a differentiable function on an interval I. The function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is **the derivative of** f.

Notation:
Let $y = f(x)$. The following notations all represent the derivative of f:

$$f'(x) \ = \ y' \ = \ \frac{dy}{dx} \ = \ \frac{df}{dx} \ = \ \frac{d}{dx}(f) \ = \ \frac{d}{dx}(y).$$

Nothing in Section 2.2 changes. In Section 2.3, we use “an interval” as before.

Theorem 3 Properties of the Derivative

Let f and g be differentiable functions on an interval I and let c be a real number. Then:

1. **Sum/Difference Rule:**
$$\frac{d}{dx}\Big(f(x) \pm g(x)\Big) = \frac{d}{dx}\Big(f(x)\Big) \pm \frac{d}{dx}\Big(g(x)\Big) = f'(x) \pm g'(x)$$
2. **Constant Multiple Rule:**
$$\frac{d}{dx}\Big(c \cdot f(x)\Big) = c \cdot \frac{d}{dx}\Big(f(x)\Big) = c \cdot f'(x).$$

Definition 13 Higher Order Derivatives

Let $y = f(x)$ be a differentiable function on an interval I.

1. The *second derivative* of f is:
$$f''(x) = \frac{d}{dx}\Big(f'(x)\Big) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''.$$
2. The *third derivative* of f is:
$$f'''(x) = \frac{d}{dx}\Big(f''(x)\Big) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = y'''.$$
3. The n^{th} *derivative* of f is:
$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right) = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = y^{(n)}.$$

There is a marginal note for this last definition that gives the caveat “where the corresponding limits exist.” It is a necessary note, but not needed in the definition, per se.

Section 2.4: Remove “open”.

Theorem 4 Product Rule

Let f and g be differentiable functions on an interval I. Then fg is a differentiable function on I, and

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = f(x)g'(x) + f'(x)g(x).$$

The APEX Quotient rule wasn't good. f and g were "functions defined on an open interval I, not even mentioning that they were differentiable.

Theorem 5 Quotient Rule

Let f and g be differentiable functions on an interval I, where $g(x) \neq 0$ on I. Then f/g is differentiable on I, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

Section 2.5: The definition of the Chain Rule needed some cleaning up anyway to talk about the domains of the functions. It gets a bit wordy.

Theorem 6 The Chain Rule

Let g be a differentiable function on an interval I, let the range of g be a subset of the interval J, and let f be a differentiable function on J. Then $y = f(g(x))$ is a differentiable function on I, and

$$y' = f'(g(x)) \cdot g'(x).$$

Theorem 7 Generalized Power Rule

Let $g(x)$ be a differentiable function on an interval I and let $n \neq 0$ be an integer. Then $g(x)^n$ is differentiable on I, and

$$\frac{d}{dx}\Big(g(x)^n\Big) = n \cdot \big(g(x)\big)^{n-1} \cdot g'(x).$$

Slight rewording here, using "differentiable everywhere":

Theorem 8 Derivatives of Exponential Functions

Let $f(x) = a^x$, for $a > 0$ and $a \neq 1$. Then f is differentiable everywhere and

$$f'(x) = \ln a \cdot a^x.$$

2.6: Implicit Differentiation gives a final version of the Power Rule. Changed to "diff on its domain except possibly at $x = 0$."

Theorem 9 Power Rule for Differentiation

Let $f(x) = x^n$, where $n \neq 0$ is a real number. Then f is differentiable on its domain, except possibly at $x = 0$, and $f'(x) = n \cdot x^{n-1}$.

2.7: Inverse Functions. The original theorem restricts to open intervals which we remove here. In practice, open intervals are the result of one-to-one functions on as large an interval as possible, as, for instance, $f(x) = \sin x$ is one to one on $[-\pi/2, \pi/2]$ but has $f'(x) = 0$ at the endpoints.

Theorem 10 Derivatives of Inverse Functions

Let f be differentiable and one to one on an interval I, where $f'(x) \neq 0$ for all x in I, let J be the range of f on I, let g be the inverse function of f, and let $f(a) = b$ for some a in I. Then g is a differentiable function on J, and in particular,

1. $\left(f^{-1}\right)'(b) = g'(b) = \dfrac{1}{f'(a)}$ and 2. $\left(f^{-1}\right)'(x) = g'(x) = \dfrac{1}{f'(g(x))}$

The following theorem will be unchanged. It does specify open sets, but this is also specific to these functions. Some have closed domains, but they are not differentiable at their endpoints.

Theorem 11 Derivatives of Inverse Trigonometric Functions

The inverse trigonometric functions are differentiable on all open sets contained in their domains (as listed in Figure **??**) and their derivatives are as follows:

1. $\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$
2. $\dfrac{d}{dx}\left(\sec^{-1}(x)\right) = \dfrac{1}{|x|\sqrt{x^2-1}}$
3. $\dfrac{d}{dx}\left(\tan^{-1}(x)\right) = \dfrac{1}{1+x^2}$
4. $\dfrac{d}{dx}\left(\cos^{-1}(x)\right) = -\dfrac{1}{\sqrt{1-x^2}}$
5. $\dfrac{d}{dx}\left(\csc^{-1}(x)\right) = -\dfrac{1}{|x|\sqrt{x^2-1}}$
6. $\dfrac{d}{dx}\left(\cot^{-1}(x)\right) = -\dfrac{1}{1+x^2}$

Chapter 3: Graphical Behavior of Functions

3.1: Extreme Values

The definition of the extreme values (absolute max/min) does not change. The Extreme Value Theorem does not change.

The definition of Relative Max/Min changes. It was always defined on an interval I (not necessarily open). Allows for a rel max/min to occur at an endpoint. Examples and exercises will need to be changed to follow.

Definition 14 **Relative Minimum and Relative Maximum**

Let f be defined on an interval I containing c.

1. If there is a $\delta > 0$ such that $f(c) \leq f(x)$ for all x in I where $|x - c| < \delta$, then $f(c)$ is a **relative minimum** of f. We also say that f has a relative minimum at $(c, f(c))$.
2. If there is a $\delta > 0$ such that $f(c) \geq f(x)$ for all x in I where $|x - c| < \delta$, then $f(c)$ is a **relative maximum** of f. We also say that f has a relative maximum at $(c, f(c))$.

The relative maximum and minimum values comprise the **relative extrema** of f.

The definition of a critical number/point does not change.
The following theorem is rewritten to specify we only consider rel extrema on open intervals.

Theorem 12 **Relative Extrema and Critical Points**

Let a function f be defined on an open interval I containing c, and let f have a relative extrema at the point $(c, f(c))$. Then c is a critical number of f.

The subsequent Key Idea on finding abs. extrema on a closed interval does not change.

3.2: The Mean Value Theorem
The Mean Value Thm and Rolle's Thm do not change. Both specify continuous on $[a, b]$ and diff. on (a, b) which is wanted; we don't need to require differentiability at the endpoints.

3.3 Increasing and Decreasing Functions
The definitions of increasing/decreasing are a bit of a mess. We remove "strictly incr/decr" and exchage the "$\leq$" for "$<$", etc.

Definition 15 **Increasing and Decreasing Functions**

Let f be a function defined on an interval I.

1. f is **increasing** on I if for every $a < b$ in I, $f(a) < f(b)$.
2. f is **decreasing** on I if for every $a < b$ in I, $f(a) > f(b)$.

The theorem following this definition is now correct; if $f'(x) > 0$, then f is increasing. The Key Idea that follows is also fine. The first derivative test is fine.

3.4/3.5: Concavity and the Second Derivative/Curve Sketching
All theorems and defs here are not affected by the changes.

Chapter 4: Applications of the Derivatives

No changes are needed.

Chapter 5: Integration

The Fundamental Theorem of Calc changes to be diff. on the closed interval.

Theorem 13 **The Fundamental Theorem of Calculus, Part 1**

Let f be continuous on $[a,b]$ and let $F(x) = \displaystyle\int_a^x f(t)\,dt$. Then F is a differentiable function on $[a,b]$, and

$$F'(x) = f(x).$$

The rest of chapter 5 is unchanged.

Chapter 6: Antidifferentiation

L'Hôpital's Rule (both parts) changes to remove the word "open" as before with limits. In "Part 2 of Part 2", we leave the word open because it really doesn't matter if the interval is (a,∞) or $[a,\infty)$. Added quotes to the indeterminate forms in that part, too.

Theorem 14 **L'Hôpital's Rule, Part 1**

Let $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = 0$, where f and g are differentiable functions on an interval I containing c, and let $g'(x) \neq 0$ on I except possibly at c. Then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

Theorem 15 **L'Hôpital's Rule, Part 2**

1. Let $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$, where f and g are differentiable on an interval I containing a. Then
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$
2. Let f and g be differentiable functions on the open interval (a,∞) for some value a, where $g'(x) \neq 0$ on (a,∞) and $\lim\limits_{x\to\infty} f(x)/g(x)$ returns either "0/0" or "∞/∞". Then
$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$
A similar statement can be made for limits where x approaches $-\infty$.

Chapter 7: Applications

The Arc Length Key Idea should be elevated to a Theorem. Also, it states that f should be diff. on an open interval containing $[a,b]$, but the first example is of the arc length of $f(x) = x^{3/2}$ on $[0,4]$. Changing wording to "differentiable on $[a,b]$. Same holds for surface area of solid of revolution.

Theorem 16 Arc Length

Let f be differentiable on $[a,b]$, where f' is also continuous on $[a,b]$. Then the arc length of f from $x=a$ to $x=b$ is

$$L=\int_a^b \sqrt{1+f'(x)^2}\,dx.$$

Theorem 17 Surface Area of a Solid of Revolution

Let f be differentiable on $[a,b]$, where f' is also continuous on $[a,b]$.

1. The surface area of the solid formed by revolving the graph of $y=f(x)$, where $f(x)\geq 0$, about the x-axis is
$$\text{Surface Area}=2\pi\int_a^b f(x)\sqrt{1+f'(x)^2}\,dx.$$
2. The surface area of the solid formed by revolving the graph of $y=f(x)$ about the y-axis, where $a,b\geq 0$, is
$$\text{Surface Area}=2\pi\int_a^b x\sqrt{1+f'(x)^2}\,dx.$$

Should Work and Fluid Force be given as Theorems or Key Ideas? They are currently Key Ideas. Volume found by integrating cross–sectional area is a Theorem; the subsequent Disk/Washer Methods are Key Ideas, as is the Shell Method. Should that change?

Chapter 8:Sequences and Series

Removed the first statement which says "bounded, monotonic sequences converge." All three are redundant. Will add the missing statement to the exercises.

Theorem 18 Bounded Monotonic Sequences are Convergent

1. Let $\{a_n\}$ be a monotonically increasing sequence that is bounded above. Then $\{a_n\}$ converges.
2. Let $\{a_n\}$ be a monotonically decreasing sequence that is bounded below. Then $\{a_n\}$ converges.

8.2: Series
Renamed the following theorem to remove the idea of "Convergence" as it isn't a test for convergence, ever. Changed the order of the list to emphasize the more important idea.

Theorem 19 n^{th}–**Term Test for Divergence**

Consider the series $\sum_{n=1}^{\infty} a_n$.

1. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.
2. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Chapter 9: Curves in the plane

9.3: Calc and parametric equations

Remove "open" from following key ideas. Elevate to Theorems?

Key Idea 1 **Finding $\frac{dy}{dx}$ with Parametric Equations.**

Let $x = f(t)$ and $y = g(t)$, where f and g are differentiable functions on an interval I and $f'(t) \neq 0$ on I. Then

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}.$$

Key Idea 2 **Finding $\frac{d^2y}{dx^2}$ with Parametric Equations**

Let $x = f(t)$ and $y = g(t)$ be twice differentiable functions on an interval I, where $f'(t) \neq 0$ on I. Then

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right] \Bigg/ \frac{dx}{dt} = \frac{d}{dt}\left[\frac{dy}{dx}\right] \Bigg/ f'(t).$$

Change wording to match the previous arc length definition.

Theorem 20 **Arc Length of Parametric Curves**

Let $x = f(t)$ and $y = g(t)$ be parametric equations with f' and g' continuous on $[t_1, t_2]$, on which the graph traces itself only once. The arc length of the graph, from $t = t_1$ to $t = t_2$, is

$$L = \int_{t_1}^{t_2} \sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

Elevate to theorem, change wording to match:

Theorem 21 **Surface Area of a Solid of Revolution**

Consider the graph of the parametric equations $x = f(t)$ and $y = g(t)$, where f' and g' are continuous on $[t_1, t_2]$ on which the graph does not cross itself.

1. The surface area of the solid formed by revolving the graph about the x-axis is (where $g(t) \geq 0$ on $[t_1, t_2]$):
$$\text{Surface Area} = 2\pi \int_{t_1}^{t_2} g(t)\sqrt{f'(t)^2 + g'(t)^2}\,dt.$$
2. The surface area of the solid formed by revolving the graph about the y-axis is (where $f(t) \geq 0$ on $[t_1, t_2]$):
$$\text{Surface Area} = 2\pi \int_{t_1}^{t_2} f(t)\sqrt{f'(t)^2 + g'(t)^2}\,dt.$$

9.5: Calc and polar functions

Elevate to theorem? Added an interval for definition of f.

Key Idea 3 **Finding $\frac{dy}{dx}$ with Polar Functions**

Let $r = f(\theta)$ be a polar function, where f is differentiable on an interval I. With $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$,
$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Elevate to theorem, as both previous statements on areas between curves and arc lengths and surface areas are theorems. Also change “open” language of arc length and surface area.

Theorem 22 **Area Between Polar Curves**

The area A of the region bounded by $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$, $\theta = \alpha$ and $\theta = \beta$, where $f_1(\theta) \leq f_2(\theta)$ on $[\alpha, \beta]$, is
$$A = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right)\,d\theta.$$

Theorem 23 **Arc Length of Polar Curves**

Let $r = f(\theta)$ be a polar function with f' continuous on $[\alpha, \beta]$, on which the graph traces itself only once. The arc length L of the graph on $[\alpha, \beta]$ is
$$L = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\,d\theta = \int_\alpha^\beta \sqrt{(r')^2 + r^2}\,d\theta.$$

Theorem 24 **Surface Area of a Solid of Revolution**

Consider the graph of the polar equation $r = f(\theta)$, where f' is continuous on $[\alpha, \beta]$ on which the graph does not cross itself.

1. The surface area of the solid formed by revolving the graph about the initial ray ($\theta = 0$) is:

$$\text{Surface Area} = 2\pi \int_\alpha^\beta f(\theta) \sin\theta \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta.$$

2. The surface area of the solid formed by revolving the graph about the line $\theta = \pi/2$ is:

$$\text{Surface Area} = 2\pi \int_\alpha^\beta f(\theta) \cos\theta \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta.$$

Chapter 11: Vector Valued Functions

11.2: Calc and vvf

Change def. of limit to remove open as well as subsequent theorem. Change wording of theorem to make it correct; $\vec{r}(t)$ may not be defined at c.

Definition 16 **Limits of Vector–Valued Functions**

Let I be an interval containing c, and let $\vec{r}(t)$ be a vector–valued function defined on I, except possibly at c. The **limit of $\vec{r}(t)$, as t approaches c, is $\vec{L}$**, expressed as

$$\lim_{t\to c} \vec{r}(t) = \vec{L},$$

means that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all t in I, $t \neq c$, if $|t - c| < \delta$, we have $||\,\vec{r}(t) - \vec{L}\,|| < \varepsilon$.

Theorem 25 **Limits of Vector–Valued Functions**

1. Let I be an interval containing c, and let $\vec{r}(t) = \langle\, f(t), g(t)\,\rangle$ be a vector–valued function in $\mathbb{R}^2$ defined on I, except possibly at c. Then

$$\lim_{t\to c} \vec{r}(t) = \left\langle \lim_{t\to c} f(t)\,,\, \lim_{t\to c} g(t) \right\rangle.$$

2. Let I be an interval containing c, and let $\vec{r}(t) = \langle\, f(t), g(t), h(t)\,\rangle$ be a vector–valued function in $\mathbb{R}^3$ defined on I, except possibly at c. Then

$$\lim_{t\to c} \vec{r}(t) = \left\langle \lim_{t\to c} f(t)\,,\, \lim_{t\to c} g(t)\,,\, \lim_{t\to c} h(t) \right\rangle$$

Definition 17 Continuity of Vector–Valued Functions

Let $\vec{r}(t)$ be a vector–valued function defined on an interval I containing c.

1. $\vec{r}(t)$ is **continuous at** c if $\lim\limits_{t\to c}\vec{r}(t) = r(c)$.
2. If $\vec{r}(t)$ is continuous at all c in I, then $\vec{r}(t)$ is **continuous on** I**.**

Theorem 26 Continuity of Vector–Valued Functions

Let $\vec{r}(t)$ be a vector–valued function defined on an interval I containing c. $\vec{r}(t)$ is continuous at c if, and only if, each of its component functions is continuous at c.

Definition 18 Derivative of a Vector–Valued Function

Let $\vec{r}(t)$ be continuous on an interval I containing c.

1. The derivative of $\vec{r}$ at $t=c$ is
$$\vec{r}\,'(c) = \lim_{h\to 0}\frac{\vec{r}(c+h)-\vec{r}(c)}{h}.$$
2. The derivative of $\vec{r}$ is
$$\vec{r}\,'(t) = \lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}.$$

Also change some of the text following this definition that uses the word “open”.

Definition 19 Tangent Vector, Tangent Line

Let $\vec{r}(t)$ be a differentiable vector–valued function on an interval I containing c, where $\vec{r}\,'(c)\neq\vec{0}$.

1. A vector $\vec{v}$ is **tangent to the graph of** $\vec{r}(t)$ **at** $t=c$ if $\vec{v}$ is parallel to $\vec{r}\,'(c)$.
2. The **tangent line** to the graph of $\vec{r}(t)$ at $t=c$ is the line through $\vec{r}(c)$ with direction parallel to $\vec{r}\,'(c)$. An equation of the tangent line is
$$\vec{\ell}(t) = \vec{r}(c) + t\,\vec{r}\,'(c).$$

Definition 20 Smooth Vector–Valued Functions

Let $\vec{r}(t)$ be a differentiable vector–valued function on an interval I. $\vec{r}(t)$ is **smooth** on I if $\vec{r}\,'(t)\neq\vec{0}$ on I.

Remove “open”, reword so it doesn’t sound like I has constant length.

Theorem 27 Vector–Valued Functions of Constant Length

Let $\vec{r}(t)$ be a differentiable vector–valued function on an interval I, where $\vec{r}(t)$ has constant length. That is, $\|\,\vec{r}(t)\,\| = c$ for all t in I (equivalently, $\vec{r}(t)\cdot\vec{r}(t) = c^2$ for all t in I). Then $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$ for all t in I.

Add continuity to requirements of f and g.

Theorem 28 Indefinite and Definite Integrals of Vector–Valued Functions

Let $\vec{r}(t) = \langle f(t), g(t)\rangle$ be a vector–valued function in $\mathbb{R}^2$, where f and g are continuous on $[a,b]$.

1. $\displaystyle\int \vec{r}(t)\;dt = \left\langle \int f(t)\;dt, \int g(t)\;dt\right\rangle$
2. $\displaystyle\int_a^b \vec{r}(t)\;dt = \left\langle \int_a^b f(t)\;dt, \int_a^b g(t)\;dt\right\rangle$

A similar statement holds for vector–valued functions in $\mathbb{R}^3$.

11.3: Calc of motion

Remove open, change to differentiable on $[a,b]$.

Key Idea 4 Average Speed, Average Velocity

Let $\vec{r}(t)$ be a differentiable position function on $[a,b]$.

The **average speed** is:

$$\frac{\text{distance traveled}}{\text{travel time}} = \frac{\int_a^b \|\,\vec{v}(t)\,\|\;dt}{b-a} = \frac{1}{b-a}\int_a^b \|\,\vec{v}(t)\,\|\;dt.$$

The **average velocity** is:

$$\frac{\text{displacement}}{\text{travel time}} = \frac{\int_a^b \vec{r}\,'(t)\;dt}{b-a} = \frac{1}{b-a}\int_a^b \vec{r}\,'(t)\;dt.$$

11.4: Unit Tangent/Normal Vectors

Remove “open” as expected by now.

Definition 21 Unit Tangent Vector

Let $\vec{r}(t)$ be a smooth function on an interval I. The unit tangent vector $\vec{T}(t)$ is

$$\vec{T}(t) = \frac{1}{\|\,\vec{r}\,'(t)\,\|}\vec{r}\,'(t).$$

Definition 22 Unit Normal Vector

Let $\vec{r}(t)$ be a vector–valued function where the unit tangent vector, $\vec{T}(t)$, is smooth on an interval I. The **unit normal vector** $\vec{N}(t)$ is

$$\vec{N}(t) = \frac{1}{\|\,\vec{T}\,'(t)\,\|}\vec{T}\,'(t).$$

Theorem 29 Unit Normal Vectors in $\mathbb{R}^2$

Let $\vec{r}(t)$ be a vector–valued function in $\mathbb{R}^2$ where $\vec{T}\,'(t)$ is smooth on an interval I. Let t_0 be in I and $\vec{T}(t_0) = \langle t_1, t_2\rangle$ Then $\vec{N}(t_0)$ is either

$$\vec{N}(t_0) = \langle -t_2, t_1\rangle \quad \text{or} \quad \vec{N}(t_0) = \langle t_2, -t_1\rangle\,,$$

whichever is the vector that points to the concave side of the graph of $\vec{r}$.

11.5: Arc Length and Curvature

Remove "open interval" - as in, the word "interval" itself is gone.

Theorem 30 Formulas for Curvature

Let C be a smooth curve in the plane or in space.

1. If C is defined by $y = f(x)$, then

$$\kappa = \frac{|f''(x)|}{\Big(1 + \big(f'(x)\big)^2\Big)^{3/2}}.$$

2. If C is defined as a vector–valued function in the plane, $\vec{r}(t) = \langle x(t), y(t)\rangle$, then

$$\kappa = \frac{|x'y'' - x''y'|}{\big((x')^2 + (y')^2\big)^{3/2}}.$$

3. If C is defined in space by a vector–valued function $\vec{r}(t)$, then

$$\kappa = \frac{\|\,\vec{T}\,'(t)\,\|}{\|\,\vec{r}\,'(t)\,\|} = \frac{\|\,\vec{r}\,'(t) \times \vec{r}\,''(t)\,\|}{\|\,\vec{r}\,'(t)\,\|^3} = \frac{\vec{a}(t)\cdot\vec{N}(t)}{\|\,\vec{v}(t)\,\|^2}.$$

Chapter 12: Functions of Several Variables

12.2: Limits and Continuity

Remove the word "open", changed wording at end to only consider points in S. Added wording to ensure there are points in S near P.

How about "every open disk centered at (x_0, y_0) contains points of S other than (x_0, y_0)."

Definition 23 Limit of a Function of Two Variables

Let S be a set containing $P = (x_0, y_0)$ where every open disk centered at P contains points in S other than P, let f be a function of two variables defined on S, except possibly at P, and let L be a real number. The **limit of** $f(x, y)$ **as** (x, y) **approaches** (x_0, y_0) **is** L, denoted

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all (x, y) in S, where $(x, y) \neq (x_0, y_0)$, if (x, y) is in the open disk centered at (x_0, y_0) with radius δ, then $|f(x, y) - L| < \varepsilon$.

Definition 24 Continuous

Let a function $f(x, y)$ be defined on a set S containing the point (x_0, y_0).

1. f is **continuous** at (x_0, y_0) if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.
2. f is **continuous on** S if f is continuous at all points in S. If f is continuous at all points in $\mathbb{R}^2$, we say that f is **continuous everywhere**.

Remove "open", change letter to S.

Theorem 31 Properties of Continuous Functions

Let f and g be continuous functions of two variables on a set S, let c be a real number, and let n be a positive integer. The following functions are continuous on S.

1. Sums/Differences: $f \pm g$
2. Constant Multiples: $c \cdot f$
3. Products: $f \cdot g$
4. Quotients: f/g (as longs as $g \neq 0$ on S)
5. Powers: f^n
6. Roots: $\sqrt[n]{f}$ (If n is even then require $f \geq 0$ on S.)
7. Compositions: Adjust the definitions of f and g to: Let f be continuous on S, where the range of f on S is J, and let g be a single variable function that is continuous on J. Then $g \circ f$, i.e., $g(f(x, y))$, is continuous on S.

Changes to limits of functions of 3 variables. Contains error: def. in part 1 should have "$<$", though book has "$=$". Add the terms "closed" and "boundary point"?

Definition 25 Open Balls, Limit, Continuous

1. An **open ball** in $\mathbb{R}^3$ centered at (x_0, y_0, z_0) with radius r is the set of all points (x, y, z) such that $\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < r$.

2. Let D be a set in $\mathbb{R}^3$ containing (x_0, y_0, z_0) where every open ball centered at (x_0, y_0, z_0) contains points of D other than (x_0, y_0, z_0), and let $f(x, y, z)$ be a function of three variables defined on D, except possibly at (x_0, y_0, z_0). The **limit** of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0) is L, denoted
$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = L,$$
means that given any $\varepsilon > 0$, there is a $\delta > 0$ such that for all (x, y, z) in D, $(x, y, z) \neq (x_0, y_0, z_0)$, if (x, y, z) is in the open ball centered at (x_0, y_0, z_0) with radius δ, then $|f(x, y, z) - L| < \varepsilon$.

3. Let $f(x, y, z)$ be defined on a set D containing (x_0, y_0, z_0). We say f is **continuous** at (x_0, y_0, z_0) if $\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0)$; if f is continuous at all points in D, we say f is **continuous on D**.

12.3: Partial Derivatives
Remove “open”

Definition 26 Partial Derivative

Let $z = f(x, y)$ be a continuous function on a set S in $\mathbb{R}^2$.

1. The **partial derivative of f with respect to x** is:
$$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

2. The **partial derivative of f with respect to y** is:
$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Remove “open”

Definition 27 Second Partial Derivative, Mixed Partial Derivative

Let $z = f(x, y)$ be continuous on a set S.

1. The **second partial derivative of f with respect to x then x** is

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \left(f_x\right)_x = f_{xx}$$

2. The **second partial derivative of f with respect to x then y** is

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \left(f_x\right)_y = f_{xy}$$

Similar definitions hold for $\dfrac{\partial^2 f}{\partial y^2} = f_{yy}$ and $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$.

The second partial derivatives f_{xy} and f_{yx} are **mixed partial derivatives**.

Remove "open"

Theorem 32 Mixed Partial Derivatives

Let f be defined such that f_{xy} and f_{yx} are continuous on a set S. Then for each point (x, y) in S, $f_{xy}(x, y) = f_{yx}(x, y)$.

Remove "open"

Definition 28 Partial Derivatives with Three Variables

Let $w = f(x, y, z)$ be a continuous function on a set D in $\mathbb{R}^3$.
The **partial derivative of f with respect to x** is:

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}.$$

Similar definitions hold for $f_y(x, y, z)$ and $f_z(x, y, z)$.

12.4: Differentiability

Add to the section the convention that "differentiable" implies "f_x and f_y exist. Fits well after the sentence "source of delight for many mathematicians." Makes life easier later on.

Remove "open"

Definition 29 Total Differential

Let $z = f(x, y)$ be continuous on a set S. Let dx and dy represent changes in x and y, respectively. Where the partial derivatives f_x and f_y exist, the **total differential of z** is

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

Remove “open”

Definition 30 Multivariable Differentiability

Let $z = f(x, y)$ be defined on a set S containing (x_0, y_0) where $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist. Let dz be the total differential of z at (x_0, y_0), let $\Delta z = f(x_0 + dx, y_0 + dy) - f(x_0, y_0)$, and let E_x and E_y be functions of dx and dy such that

$$\Delta z = dz + E_x dx + E_y dy.$$

1. f is **differentiable at** (x_0, y_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|| \langle dx, dy \rangle || < \delta$, then $|| \langle E_x, E_y \rangle || < \varepsilon$. That is, as dx and dy go to 0, so do E_x and E_y.
2. f is **differentiable on** S if f is differentiable at every point in S. If f is differentiable on $\mathbb{R}^2$, we say that f is **differentiable everywhere**.

Remove “open” from back to back theorems.

Theorem 33 Continuity and Differentiability of Multivariable Functions

Let $z = f(x, y)$ be defined on a set S containing (x_0, y_0). If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Theorem 34 Differentiability of Multivariable Functions

Let $z = f(x, y)$ be defined on a set S containing (x_0, y_0). If f_x and f_y are both continuous on S, then f is differentiable on S.

After the above thm in the book we’ll state the convention of what we mean by “differentiable.”
Remove “open” from 3D analogues.

Definition 31 Total Differential

Let $w = f(x, y, z)$ be continuous on a set D. Let dx, dy and dz represent changes in x, y and z, respectively. Where the partial derivatives f_x, f_y and f_z exist, the **total differential of** w is

$$dz = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz.$$

Change wording here to mimic Def 30.

Definition 32 Multivariable Differentiability

Let $w = f(x, y, z)$ be defined on a set D containing (x_0, y_0, z_0) where $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$ and $f_z(x_0, y_0, z_0)$ exist. Let dw be the total differential of w at (x_0, y_0, z_0), let $\Delta w = f(x_0 + dx, y_0 + dy, z_0 + dz) - f(x_0, y_0, z_0)$, and let E_x, E_y and E_z be functions of dx, dy and dz such that

$$\Delta w = dw + E_x dx + E_y dy + E_z dz.$$

1. f is **differentiable at** (x_0, y_0, z_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|| \langle dx, dy, dz \rangle || < \delta$, then $|| \langle E_x, E_y, E_z \rangle || < \varepsilon$.

2. f is **differentiable on** D if f is differentiable at every point in D. If f is differentiable on $\mathbb{R}^3$, we say that f is **differentiable everywhere**.

Remove "open"

Theorem 35 Continuity and Differentiability of Functions of Three Variables

Let $w = f(x, y, z)$ be defined on a set D containing (x_0, y_0, z_0).

1. If f is differentiable at (x_0, y_0, z_0), then f is continuous at (x_0, y_0, z_0).

2. If f_x, f_y and f_z are continuous on D, then f is differentiable on D.

12.5: Multi chain Rule
(Change intro to discuss path over terrain, $\frac{dz}{dt} = \nabla f \cdot \vec{r}\,'(t)$.
12.6: Directional Derivatives
remove "open" and add an "in S":

Definition 33 Directional Derivatives

Let $z = f(x, y)$ be continuous on a set S and let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. For all points (x, y) in S, the **directional derivative of** f **at** (x, y) **in the direction of** $\vec{u}$ is

$$D_{\vec{u}} f(x, y) = \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2) - f(x, y)}{h}.$$

remove "open"; (Thm always said "differentiable" though that doesn't imply the partials exist. This is what every other book I've seen does, too. The convention stated earlier covers this for us. This applies to several other def's and thm's in this section.)

Theorem 36 Directional Derivatives

Let $z = f(x, y)$ be differentiable on a set S containing (x_0, y_0), and let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. The directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2.$$

remove "open"

Definition 34 Gradient

Let $z = f(x, y)$ be differentiable on a set S containing (x_0, y_0).

1. The **gradient of** f is $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$.
2. The **gradient of** f **at** (x_0, y_0) **is** $\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$.

Remove “open”,

Theorem 37 The Gradient and Directional Derivatives

Let $z = f(x, y)$ be differentiable on a set S with gradient ∇f, let $P = (x_0, y_0)$ be a point in S and let $\vec{u}$ be a unit vector.

1. The maximum value of $D_{\vec{u}} f(x_0, y_0)$ is $|| \nabla f(x_0, y_0) ||$; the direction of maximal z increase is $\nabla f(x_0, y_0)$.
2. The minimum value of $D_{\vec{u}} f(x_0, y_0)$ is $-|| \nabla f(x_0, y_0) ||$; the direction of minimal z increase is $-\nabla f(x_0, y_0)$.
3. At P, $\nabla f(x_0, y_0)$ is orthogonal to the level curve passing through $\big(x_0, y_0, f(x_0, y_0)\big)$.

remove “open”

Definition 35 Directional Derivatives and Gradient with Three Variables

Let $w = F(x, y, z)$ be differentiable on a set D and let $\vec{u}$ be a unit vector in $\mathbb{R}^3$.

1. The **gradient** of F is $\nabla F = \langle F_x, F_y, F_z\rangle$.
2. The **directional derivative of** F **in the direction of** $\vec{u}$ is

$$D_{\vec{u}} F = \nabla F \cdot \vec{u}.$$

remove “open”

Theorem 38 The Gradient and Directional Derivatives with Three Variables

Let $w = F(x, y, z)$ be differentiable on a set D, let ∇F be the gradient of F, and let $\vec{u}$ be a unit vector.

1. The maximum value of $D_{\vec{u}} F$ is $|| \nabla F ||$, obtained when the angle between ∇F and $\vec{u}$ is 0, i.e., the direction of maximal increase is ∇F.
2. The minimum value of $D_{\vec{u}} F$ is $-|| \nabla F ||$, obtained when the angle between ∇F and $\vec{u}$ is π, i.e., the direction of minimal increase is $-\nabla F$.
3. $D_{\vec{u}} F = 0$ when ∇F and $\vec{u}$ are orthogonal.

12.7 Tangent/Normal lines, planes
remove “open”

Definition 36   Directional Tangent Line

Let $z = f(x, y)$ be differentiable on a set S containing (x_0, y_0) and let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector.

1. The line ℓ_x through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle 1, 0, f_x(x_0, y_0) \rangle$ is the **tangent line to f in the direction of x at** (x_0, y_0).
2. The line ℓ_y through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle 0, 1, f_y(x_0, y_0) \rangle$ is the **tangent line to f in the direction of y at** (x_0, y_0).
3. The line $\ell_{\vec{u}}$ through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle u_1, u_2, D_{\vec{u}} f(x_0, y_0) \rangle$ is the **tangent line to f in the direction of $\vec{u}$ at** (x_0, y_0).

remove "open"

Definition 37   Normal Line

Let $z = f(x, y)$ be differentiable on a set S containing (x_0, y_0) where

$$a = f_x(x_0, y_0) \quad \text{and} \quad b = f_y(x_0, y_0)$$

are defined.

1. A nonzero vector parallel to $\vec{n} = \langle a, b, -1 \rangle$ is **orthogonal to f at** $P = \big(x_0, y_0, f(x_0, y_0)\big)$.
2. The line ℓ_n through P with direction parallel to $\vec{n}$ is the **normal line to f at** P.

remove "open"

Definition 38   Tangent Plane

Let $z = f(x, y)$ be differentiable on a set S containing (x_0, y_0), where $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$, $\vec{n} = \langle a, b, -1 \rangle$ and $P = \big(x_0, y_0, f(x_0, y_0)\big)$.

The plane through P with normal vector $\vec{n}$ is the **tangent plane to f at** P. The standard form of this plane is

$$a(x - x_0) + b(y - y_0) - \big(z - f(x_0, y_0)\big) = 0.$$

"remove open"

Definition 39   Gradient

Let $w = F(x, y, z)$ be differentiable on a set D that contains the point (x_0, y_0, z_0).

1. The **gradient of** F is $\nabla F(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$.
2. The **gradient of** F **at** (x_0, y_0, z_0) **is**

$$\nabla F(x_0, y_0, z_0) = \langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0) \rangle .$$

remove open

Theorem 39 The Gradient and Level Surfaces

Let $w = F(x, y, z)$ be differentiable on a set D containing (x_0, y_0, z_0) with gradient ∇F, where $F(x_0, y_0, z_0) = c$.

The vector $\nabla F(x_0, y_0, z_0)$ is orthogonal to the level surface $F(x, y, z) = c$ at (x_0, y_0, z_0).

12.8 Extreme Values

Compare to single variable definitions. Change order so absolute extreme come first. Rel max/min defs are wordy but consistent. Other books are a bit sloppy here. Look at section to see what needs to be changed in accordance. Point out that abs. extrema are also rel. ext.

Definition 40 Relative and Absolute Extrema

Let $z = f(x, y)$ be defined on a set S containing the point $P = (x_0, y_0)$.

1. If $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in S, then f has an **absolute maximum** at P; if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in S, then f has an **absolute minimum** at P.
2. If there is a $\delta > 0$ such that $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in both S and the open disk of radius δ centered at P, then f has a **relative minimum** at P.

   If there is a $\delta > 0$ such that $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in both S and the open disk of radius δ centered at P, then f has a **relative maximum** at P.
3. If f has a relative maximum or minimum at P, then f has a **relative extrema** at P; if f has an absolute maximum or minimum at P, then f has a **absolute extrema** at P.

remove “open”

Definition 41 Critical Point

Let $z = f(x, y)$ be continuous on a set S. A **critical point** $P = (x_0, y_0)$ of f is a point in S such that

- $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or
- $f_x(x_0, y_0)$ and/or $f_y(x_0, y_0)$ is undefined.

This remains unchanged; true only on open sets. Already matched the revised single variable thm. Make comment later.

Theorem 40 Critical Points and Relative Extrema

Let $z = f(x, y)$ be defined on an open set S containing $P = (x_0, y_0)$. If f has a relative extrema at P, then P is a critical point of f.

Saddle point definition remains unchanged.

Fixed the thm. to ensure the point is actually a critical point. Keep defined on an open set as that is all we are concerned with; boundary points are dealt with elsewhere.

Theorem 41 Second Derivative Test

Let $z = f(x, y)$ be differentiable on an open set containing the critical point $P = (x_0, y_0)$, and let

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0).$$

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then P is a relative minimum of f.
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then P is a relative maximum of f.
3. If $D < 0$, then P is a saddle point of f.
4. If $D = 0$, the test is inconclusive.

Chapter 13: Multiple Integration

13.2 Change notation to double sum?
Add "bounded" to def. and theorem and def

Definition 42 Double Integral, Signed Volume

Let $z = f(x, y)$ be a continuous function defined over a closed, bounded region R in the x-y plane. The **signed volume** V under f over R is denoted by the **double integral**

$$V = \iint_R f(x, y)\, dA.$$

Alternate notations for the double integral are

$$\iint_R f(x, y)\, dA = \iint_R f(x, y)\, dx\, dy = \iint_R f(x, y)\, dy\, dx.$$

Theorem 42 Double Integrals and Signed Volume

Let $z = f(x, y)$ be a continuous function defined over a closed, bounded region R in the x-y plane. Then the signed volume V under f over R is

$$V = \iint_R f(x, y)\, dA = \lim_{||\Delta A|| \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i.$$

Definition 43 The Average Value of f on R

Let $z = f(x, y)$ be a continuous function defined over a closed, bounded region R in the x-y plane. The **average value of f on R** is

$$\text{average value of } f \text{ on } R = \frac{\displaystyle\iint_R f(x, y)\, dA}{\displaystyle\iint_R dA}.$$

13.4 Center of mass
Add "closed, bounded"

Definition 44 Mass of a Lamina with Vairable Density

Let $\delta(x, y)$ be a continuous density function of a lamina corresponding to a closed, bounded plane region R. The mass M of the lamina is

$$\text{mass } M = \iint_R dm = \iint_R \delta(x, y)\, dA.$$

Theorem 43 Center of Mass of a Planar Lamina, Moments

Let a planar lamina be represented by a closed, bounded region R in the x-y plane with density function $\delta(x, y)$.

1. mass: $M = \iint_R \delta(x, y)\, dA$
2. moment about the x-axis: $M_x = \iint_R y\delta(x, y)\, dA$
3. moment about the y-axis: $M_y = \iint_R x\delta(x, y)\, dA$
4. The center of mass of the lamina is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right).$$

Change summations to triple sums in triple integration theorems?