APEX Calculus

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Chapter 1

Limits

Calculus means "a method of calculation or reasoning." When one computes the sales tax on a purchase, one employs a simple calculus. When one finds the area of a polygonal shape by breaking it up into a set of triangles, one is using another calculus. Proving a theorem in geometry employs yet another calculus.

Despite the wonderful advances in mathematics that had taken place into the first half of the 17th century, mathematicians and scientists were keenly aware of what they *could not do.* (This is true even today.) In particular, two important concepts eluded mastery by the great thinkers of that time: area and rates of change.

Area seems innocuous enough; areas of circles, rectangles, parallelograms, etc., are standard topics of study for students today just as they were then. However, the areas of *arbitrary* shapes could not be computed, even if the boundary of the shape could be described exactly.

Rates of change were also important. When an object moves at a constant rate of change, then "distance = rate $\times$ time." But what if the rate is not constant — can distance still be computed? Or, if distance is known, can we discover the rate of change?

It turns out that these two concepts were related. Two mathematicians, Sir Isaac Newton and Gottfried Leibniz, are credited with independently formulating a system of computing that solved the above problems and showed how they were connected. Their system of reasoning was "a" calculus. However, as the power and importance of their discovery took hold, it became known to many as "the" calculus. Today, we generally shorten this to discuss "calculus."

The foundation of "the calculus" is the **limit.** It is a tool to describe a particular behavior of a function. This chapter begins our study of the limit by approximating its value graphically and numerically. After a formal definition of the limit, properties are established that make "finding limits" tractable. Once the limit is understood, then the problems of area and rates of change can be approached.

1.1 An Introduction To Limits

We begin our study of *limits* by considering examples that demonstrate key concepts that will be explained as we progress.

Consider the function $y = \frac{\sin(x)}{x}$. When x is near the value 1, what value (if any) is y near?

While our question is not precisely formed (what constitutes "near the value 1"?), the answer does not seem difficult to find. One might think first to look at a graph of this function to approximate the appropriate y values. Consider Figure **??**, where $y = \frac{\sin(x)}{x}$ is graphed. For values of x near 1, it seems that y takes on values near 0.85. In fact, when $x = 1$, then $y = \frac{\sin(1)}{1} \approx 0.84$, so it makes sense that when x is "near" 1, y will be "near" 0.84.

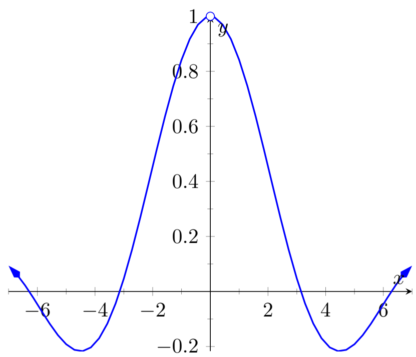


Figure 1.1.1: $\sin(x)/x$.

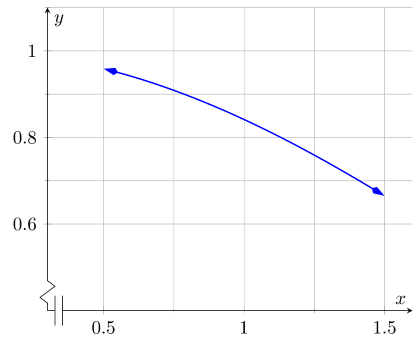


Figure 1.1.2: $\sin(x)/x$ near $x = 1$.

Consider this same function again at a different value for x. When x is near 0, what value (if any) is y near? By considering Figure **??**, one can see that it seems that y takes on values near 1. But what happens when $x = 0$? We have

$$y \to \frac{\sin(0)}{0} \to \frac{\text{âĂIJ } 0 \text{ âĂİ}}{0}.$$

The expression $0/0$ has no value; it is **indeterminate**. Such an expression gives no information about what is going on with the function nearby. We cannot find out how y behaves near $x = 0$ for this function simply by letting $x = 0$.

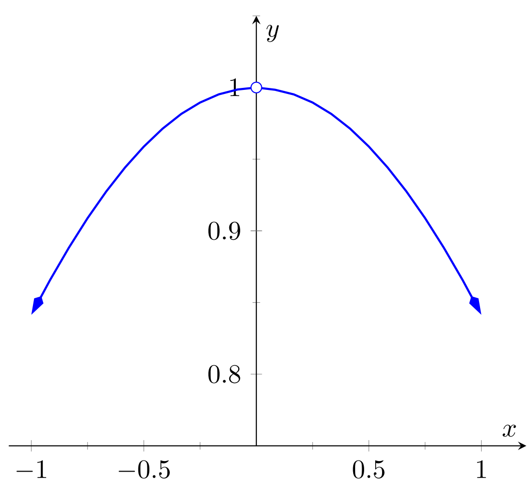


Figure 1.1.3: $\sin(x)/x$ near $x = 0$.

Finding a limit entails understanding how a function behaves near a particular value of x. Before continuing, it will be useful to establish some notation. Let $y = f(x)$; that is, let y be a function of x for some function f. The expression "the limit of y as x approaches 1" describes a number, often referred to as

L, that y nears as x nears 1. We write all this as

$$\lim_{x\to 1} y = \lim_{x\to 1} f(x) = L.$$

This is not a complete definition (that will come in the next section); this is a pseudo-definition that will allow us to explore the idea of a limit.

Above, where $f(x) = \sin(x)/x$, we approximated

$$\lim_{x\to 1} \frac{\sin(x)}{x} \approx 0.84 \text{ and } \lim_{x\to 0} \frac{\sin(x)}{x} \approx 1.$$

(We *approximated* these limits, hence used the "$\approx$" symbol, since we are working with the pseudo-definition of a limit, not the actual definition.)

Once we have the true definition of a limit, we will find limits *analytically*; that is, exactly using a variety of mathematical tools. For now, we will *approximate* limits both graphically and numerically. Graphing a function can provide a good approximation, though often not very precise. Numerical methods can provide a more accurate approximation. We have already approximated limits graphically, so we now turn our attention to numerical approximations.

Consider again $\lim_{x\to 1} \sin(x)/x$. To approximate this limit numerically, we can create a table of x and $f(x)$ values where x is "near" 1. This is done in Table **??**.

Notice that for values of x near 1, we have $\sin(x)/x$ near 0.841. The $x = 1$ row is included, but we stress the fact that when considering limits, we are *not* concerned with the value of the function at that particular x value; we are only concerned with the values of the function when x is *near* 1.

x	$\sin(x)/x$
0.9	0.870363
0.99	0.844471
0.999	0.841772
1	*0.841471*
1.001	0.841170
1.01	0.838447
1.1	0.810189

Table 1.1.4: Values of $\sin(x)/x$ with x near 1.

Now approximate $\lim_{x\to 0} \sin(x)/x$ numerically. We already approximated the value of this limit as 1 graphically in Figure **??**. Table **??** shows the value of $\sin(x)/x$ for values of x near 0. Ten places after the decimal point are shown to highlight how close to 1 the value of $\sin(x)/x$ gets as x takes on values very near 0. We include the $x = 0$ row but again stress that we are not concerned with the value of our function at $x = 0$, only on the behavior of the function *near* 0.

x	$\sin(x)/x$
-0.1	0.9983341665
-0.01	0.9999833334
-0.001	0.9999998333
0	*not defined*
0.001	0.9999998333
0.01	0.9999833334
0.1	0.9983341665

Table 1.1.5: Values of $\sin(x)/x$ with x near 0.

This numerical method gives confidence to say that 1 is a good approximation of $\lim_{x\to 0}\sin(x)/x$; that is,

$$\lim_{x\to 0}\sin(x)/x \approx 1.$$

Later we will be able to prove that the limit is *exactly* 1.

We now consider several examples that allow us explore different aspects of the limit concept.

Example 1.1.6 (Approximating the value of a limit)**.** Use graphical and numerical methods to approximate

$$\lim_{x\to 3}\frac{x^2-x-6}{6x^2-19x+3.}$$

Solution. To graphically approximate the limit, graph

$$y = (x^2-x-6)/(6x^2-19x+3)$$

on a small interval that contains 3. To numerically approximate the limit, create a table of values where the x values are near 3. This is done in Figure **??** and Table **??**, respectively.

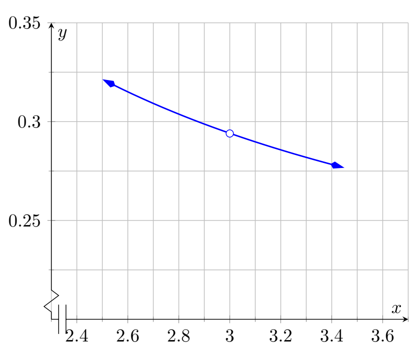


Figure 1.1.7: Graphically approximating a limit in Example **??**.

x	$\frac{x^2-x-6}{6x^2-19x+3}$
2.9	0.29878
2.99	0.294569
2.999	0.294163
3	*not defined*
3.001	0.294073
3.01	0.293669
3.1	0.289773

Table 1.1.8: Numerically approximating a limit in Example **??**.

The graph shows that when x is near 3, the value of y is very near 0.3. By considering values of x near 3, we see that $y = 0.294$ is a better approximation. The graph and the table imply that

$$\lim_{x\to 3}\frac{x^2-x-6}{6x^2-19x+3} \approx 0.294.$$

This example may bring up a few questions about approximating limits (and the nature of limits themselves).

1. If a graph does not produce as good an approximation as a table, why bother with it?

2. How many values of x in a table are "enough?" In the previous example, could we have just used $x = 3.001$ and found a fine approximation?

Graphs are useful since they give a visual understanding concerning the behavior of a function. Sometimes a function may act "erratically" near certain x values which is hard to discern numerically but very plain graphically (see Example **??**). Since graphing utilities are very accessible, it makes sense to make proper use of them.

Since tables and graphs are used only to *approximate* the value of a limit, there is not a firm answer to how many data points are "enough." Include enough so that a trend is clear, and use values (when possible) both less than and greater than the value in question. In Example **??**, we used both values less than and greater than 3. Had we used just $x = 3.001$, we might have been tempted to conclude that the limit had a value of 0.3. While this is not far off, we could do better. Using values "on both sides of 3" helps us identify trends.

Example 1.1.9 (Approximating the value of a limit)**.** Graphically and numerically approximate the limit of $f(x)$ as x approaches 0, where

$$f(x) = \begin{cases} x+1 & x<0 \\ -x^2+1 & x>0. \end{cases}$$

Solution. Again we graph $f(x)$ and create a table of its values near $x = 0$ to approximate the limit. Note that this is a piecewise defined function, so it behaves differently on either side of 0. Figure **??** shows a graph of $f(x)$, and on either side of 0 it seems the y values approach 1. Note that $f(0)$ is not actually defined, as indicated in the graph with the open circle.

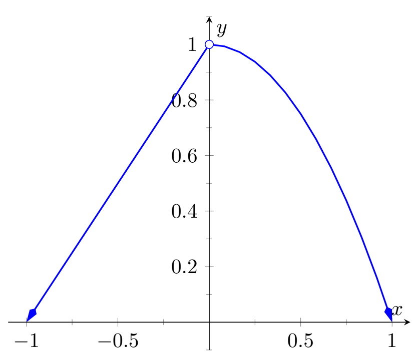


Figure 1.1.10: Graphically approximating a limit in Example **??**.

x	$f(x)$
-0.1	0.9
-0.01	0.99
-0.001	0.999
0.001	0.999999
0.01	0.9999
0.1	0.99

Table 1.1.11: Numerically approximating a limit in Example **??**.

Table **??** shows values of $f(x)$ for values of x near 0. It is clear that as x takes on values very near 0, $f(x)$ takes on values very near 1. It turns out that if we let $x = 0$ for either "piece" of $f(x)$, 1 is returned; this is significant and we'll return to this idea later.

The graph and table allow us to say that $\lim_{x\to 0} f(x) \approx 1$; in fact, we are probably very sure it *equals* 1.

1.1.1 Identifying When Limits Do Not Exist

A function may not have a limit for all values of x. That is, we cannot say $\lim_{x\to c} f(x) = L$ (where L is some real number) for all values of c, for there may not be a number that $f(x)$ is approaching. There are three ways in which a limit may fail to exist.

1. The function $f(x)$ may approach different values on either side of c.

2. The function may grow without upper or lower bound as x approaches c.

3. The function may oscillate as x approaches c.

We'll explore each of these in turn.

Example 1.1.12 (Different Values Approached From Left and Right)**.** Explore why $\lim\limits_{x\to 1} f(x)$ does not exist, where

$$f(x) = \begin{cases} x^2 - 2x + 3 & x \leq 1 \\ x & x > 1. \end{cases}$$

Solution. A graph of $f(x)$ around $x = 1$ and a table are given Figure **??** and Table **??**, respectively. It is clear that as x approaches 1, $f(x)$ does not seem to approach a single number. Instead, it seems as though $f(x)$ approaches two different numbers. When considering values of x less than 1 (approaching 1 from the left), it seems that $f(x)$ is approaching 2; when considering values of x greater than 1 (approaching 1 from the right), it seems that $f(x)$ is approaching 1. Recognizing this behavior is important; we'll study this in greater depth later. Right now, it suffices to say that the limit does not exist since $f(x)$ is approaching two *different* values as x approaches 1.

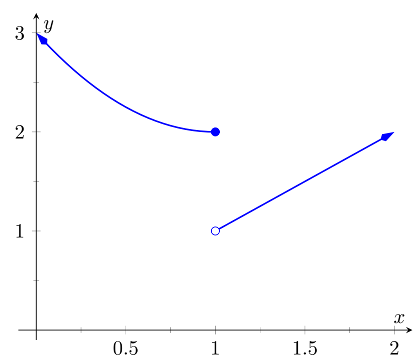


Figure 1.1.13: Observing no limit as $x \to 1$ in Example **??**.

x	$f(x)$
0.9	2.01
0.99	2.0001
0.999	2.000001
1.001	1.001
1.01	1.01
1.1	1.1

Table 1.1.14: Values of $f(x)$ near $x = 1$ in Example **??**.

Example 1.1.15 (The Function Grows Without Bound)**.** Explore why $\lim\limits_{x\to 1} 1/(x-1)^2$ does not exist.

Solution. A graph and table of $f(x) = 1/(x-1)^2$ are given in Figure **??** and Table **??**, respectively. Both show that as x approaches 1, $f(x)$ grows larger and larger.

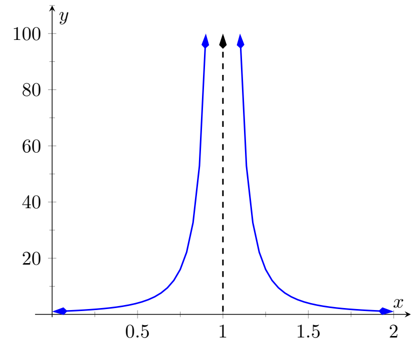


Figure 1.1.16: Observing no limit as $x \to 1$ in Example **??**.

x	$f(x)$
0.9	100.
0.99	10000.
0.999	$1. \times 10^6$
1.001	$1. \times 10^6$
1.01	10000.
1.1	100.

Table 1.1.17: Values of $f(x)$ near $x = 1$ in Example **??**.

We can deduce this on our own, without the aid of the graph and table. If x is near 1, then $(x-1)^2$ is very small, and:

$$\frac{1}{\text{very small number}} = \text{very large number} .$$

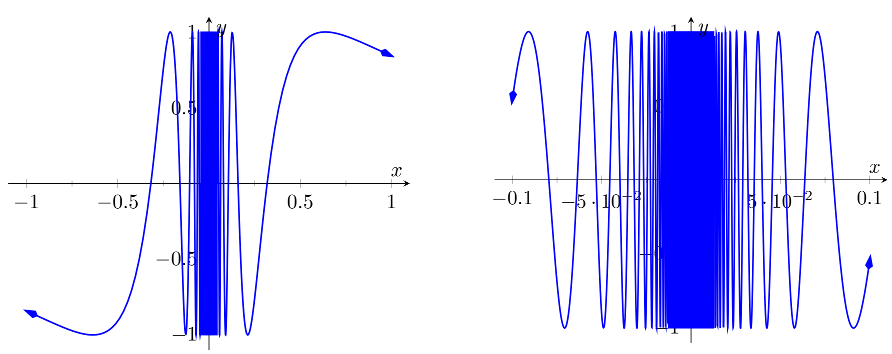
Since $f(x)$ is not approaching a single number, we conclude that

$$\lim_{x \to 1} \frac{1}{(x-1)^2}$$

does not exist.

Example 1.1.18 (The Function Oscillates)**.** Explore why $\lim_{x \to 0} \sin(1/x)$ does not exist.

Solution. Two graphs of $f(x) = \sin(1/x)$ are given in Figure **??**. Figure **??** shows $f(x)$ on the interval $[-1, 1]$; notice how $f(x)$ seems to oscillate near $x = 0$. One might think that despite the oscillation, as x approaches 0, $f(x)$ approaches 0. However, Figure **??** zooms in on $\sin(1/x)$, on the interval $[-0.1, 0.1]$. Here the oscillation is even more pronounced. Finally, in Figure **??**, we see $\sin(x)/x$ evaluated for values of x near 0. As x approaches 0, $f(x)$ does not appear to approach any value.



(a) **(b)**

Figure 1.1.19: Observing that $f(x) = \sin(1/x)$ has no limit as $x \to 0$ in Example **??**.

x	$\sin(1/x)$
0.1	-0.544021
0.01	-0.506366
0.001	0.82688
0.0001	-0.305614
$1. \times 10^{-5}$	0.0357488
$1. \times 10^{-6}$	-0.349994
$1. \times 10^{-7}$	0.420548

Table 1.1.20: Observing that $f(x) = \sin(1/x)$ has no limit as $x \to 0$ in Example **??**.

It can be shown that in reality, as x approaches 0, $\sin(1/x)$ takes on all values between -1 and 1 infinite times! Because of this oscillation, $\lim_{x\to 0} \sin(1/x)$ does not exist.

1.1.2 Limits of Difference Quotients

We have approximated limits of functions as x approached a particular number. We will consider another important kind of limit after explaining a few key ideas.

Let $f(x)$ represent the position function, in feet, of some particle that is moving in a straight line, where x is measured in seconds. Let's say that when $x = 1$, the particle is at position 10 ft., and when $x = 5$, the particle is at 20 ft. Another way of expressing this is to say

$$f(1) = 10 \text{ and } f(5) = 20.$$

Since the particle traveled 10 feet in 4 seconds, we can say the particle's **average velocity** was 2.5 ft/s. We write this calculation using a "quotient of differences," or, a **difference quotient**:

$$\frac{f(5) - f(1)}{5 - 1} \frac{\text{ft}}{\text{s}} = \frac{10 \text{ ft}}{4 \text{ s}} = 2.5 \text{ ft/s} .$$

This difference quotient can be thought of as the familiar "rise over run" used to compute the slopes of lines. In fact, that is essentially what we are doing: given two points on the graph of f, we are finding the slope of the *secant line* through those two points. See Figure **??**.

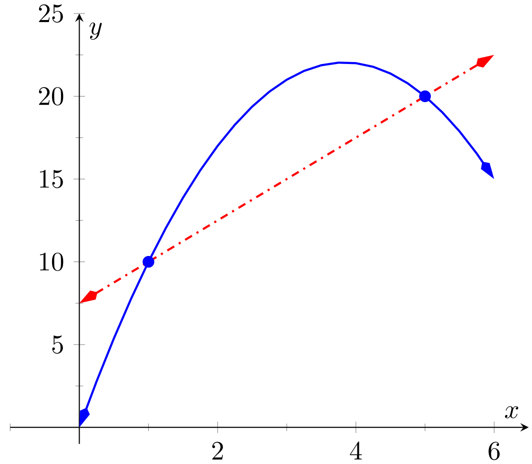


Figure 1.1.21: Interpreting a difference quotient as the slope of a secant line.

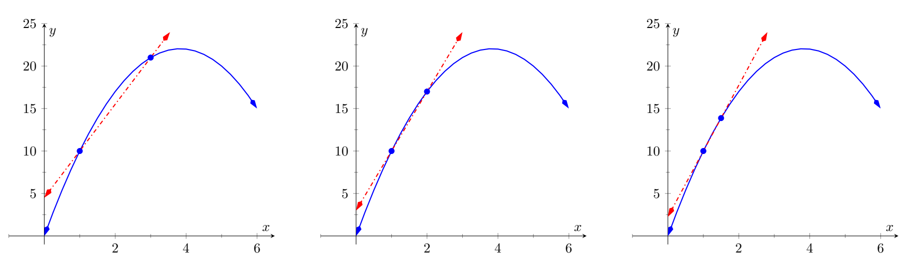
Now consider finding the average speed on another time interval. We again start at $x = 1$, but consider the position of the particle h seconds later. That is, consider the positions of the particle when $x = 1$ and when $x = 1 + h$. The difference quotient (excluding units) is now

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}.$$

Let $f(x) = -1.5x^2 + 11.5x$; note that $f(1) = 10$ and $f(5) = 20$, as in our discussion. We can compute this difference quotient for all values of h (even negative values!) except $h = 0$, for then we get "0/0," the indeterminate form introduced earlier. For all values $h \neq 0$, the difference quotient computes the average velocity of the particle over an interval of time of length h starting at $x = 1$.

For small values of h, i.e., values of h close to 0, we get average velocities over very short time periods and compute secant lines over small intervals. See Figure **??**. This leads us to wonder what the limit of the difference quotient is as h approaches 0. That is,

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \ ?$$



(a) $h = 2$ **(b)** $h = 1$ **(c)** $h = 0.5$

Figure 1.1.22: Secant lines of $f(x)$ at $x = 1$ and $x = 1 + h$, for shrinking values of h (i.e., $h \to 0$).

As we do not yet have a true definition of a limit nor an exact method for computing it, we settle for approximating the value. While we could graph the difference quotient (where the x-axis would represent h values and the y-axis would represent values of the difference quotient) we settle for making a table. See Table **??**. The table gives us reason to assume the value of the limit is about 8.5.

h	$\frac{f(1+h)-f(1)}{h}$
-0.5	9.25
-0.1	8.65
-0.01	8.515
0.01	8.485
0.1	8.35
0.5	7.75

Table 1.1.23: The difference quotient evaluated at values of h near 0.

Proper understanding of limits is key to understanding calculus. With limits, we can accomplish seemingly impossible mathematical things, like adding up an infinite number of numbers (and not get infinity) and finding the slope of a line between two points, where the "two points" are actually the same point. These are not just mathematical curiosities; they allow us to link position, velocity and acceleration together, connect cross-sectional areas to volume, find the work done by a variable force, and much more.

In the next section we give the formal definition of the limit and begin our study of finding limits analytically. In the following exercises, we continue our introduction and approximate the value of limits.

1.1.3 Exercises

Terms and Concepts

1. In your own words, what does it mean to "find the limit of $f(x)$ as x approaches 3"?

[Essay Answer]

Solution. Answers will vary.

2. An expression of the form $\frac{0}{0}$ is called ____________.

Solution. An expression of the form $\frac{0}{0}$ is called an indeterminate form.

3. True or False? The limit of $f(x)$ as x approaches 5 is $f(5)$. (Choose one: True / False)

Solution. False

4. Describe three situations where $\lim_{x\to c} f(x)$ does not exist.

[Essay Answer]

Solution. The function may approach different values from the left and right, the function may grow without bound, or the function might oscillate.

5. In your own words, what is a difference quotient?

[Essay Answer]

Solution. Answers will vary.

In the following exercises, approximate the given limits both numerically and graphically.

6. Approximate the limit numerically and graphically.

$\lim_{x \to 1} \left(x^2 + 3x - 5\right) = \boxed{\qquad\qquad}$

Solution. For a numerical approximation, make a table:

x	$x^2 + 3x - 5$
0.9	-1.49
0.99	-1.0499
0.999	-1.005
1.001	-0.994999
1.01	-0.9499
1.1	-0.49

For a graphical approximation:

webwork-6-image-1.png

It appears that when x is close to 1, that $x^2 + 3x - 5$ is close to -1. So

$$\lim_{x \to 1} \left(x^2 + 3x - 5\right) = -1.$$

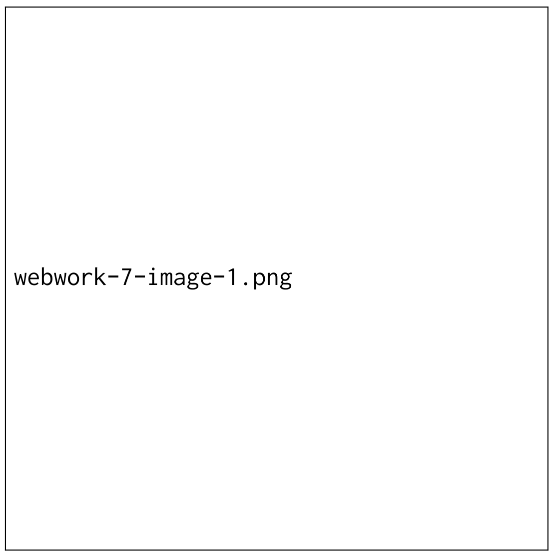
7. Approximate the limit numerically and graphically.

$\lim\limits_{x \to 0} \left(x^3 - 3x^2 + x - 5\right) = \boxed{}$

Solution. For a numerical approximation, make a table:

x	$x^3 - 3x^2 + x - 5$
-0.1	-5.131
-0.01	-5.0103
-0.001	-5.001
0.001	-4.999
0.01	-4.9903
0.1	-4.929

For a graphical approximation:



It appears that when x is close to 0, that $x^3 - 3x^2 + x - 5$ is close to -5. So

$$\lim_{x \to 0} \left(x^3 - 3x^2 + x - 5\right) = -5.$$

8. Approximate the limit numerically and graphically.

$\lim\limits_{x\to 0}\left(\frac{x+1}{x^2+3x}\right) = \boxed{\qquad\qquad}$

Solution. For a numerical approximation, make a table:

x	$\frac{x+1}{x^2+3x}$
-0.1	-3.10345
-0.01	-33.1104
-0.001	-333.111
0.001	333.555
0.01	33.5548
0.1	3.54839

For a graphical approximation:

webwork-8-image-1.png

It appears that when x is close to 0, that $\frac{x+1}{x^2+3x}$ grows without bound. So

$$\lim_{x\to 0}\left(\frac{x+1}{x^2+3x}\right) \text{ does not exist (DNE).}$$

9. Approximate the limit numerically and graphically.

$\lim\limits_{x\to 3}\left(\frac{x^2-2x-3}{x^2-4x+3}\right) = \boxed{}$

Solution. For a numerical approximation, make a table:

x	$\frac{x^2-2x-3}{x^2-4x+3}$
2.9	2.05263
2.99	2.00503
2.999	2.0005
3.001	1.9995
3.01	1.99502
3.1	1.95238

For a graphical approximation:

webwork-9-image-1.png

It appears that when x is close to 3, that $\frac{x^2-2x-3}{x^2-4x+3}$ is close to 2. So

$$\lim_{x\to 3}\left(\frac{x^2-2x-3}{x^2-4x+3}\right) = 2.$$

10. Approximate the limit numerically and graphically.

$\lim\limits_{x\to -1}\left(\frac{x^2+8x+7}{x^2+6x+5}\right) = \boxed{\qquad\qquad\qquad}$

Solution. For a numerical approximation, make a table:

x	$\frac{x^2+8x+7}{x^2+6x+5}$
-1.1	1.51282
-1.01	1.50125
-1.001	1.50013
-0.999	1.49988
-0.99	1.49875
-0.9	1.4878

For a graphical approximation:

webwork-10-image-1.png

It appears that when x is close to -1, that $\frac{x^2+8x+7}{x^2+6x+5}$ is close to $\frac{3}{2}$. So

$$\lim_{x\to -1}\left(\frac{x^2+8x+7}{x^2+6x+5}\right) = \frac{3}{2}.$$

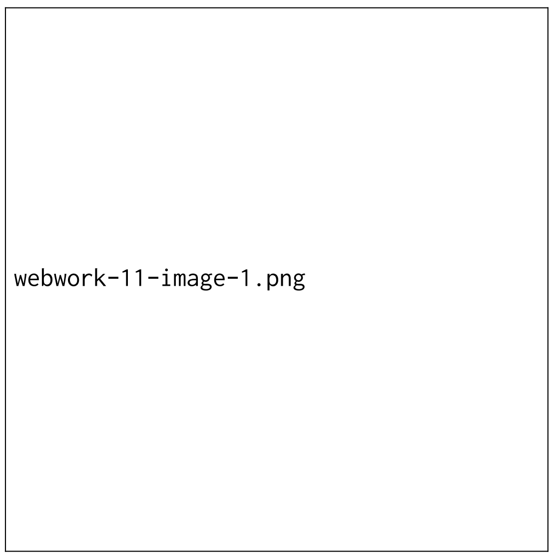
11. Approximate the limit numerically and graphically.

$\lim_{x \to 2} \left(\frac{x^2+7x+10}{x^2-4x+4} \right) = \boxed{}$

Solution. For a numerical approximation, make a table:

x	$\frac{x^2+7x+10}{x^2-4x+4}$
1.9	2691
1.99	278901
1.999	2.7989×10^7
2.001	2.8011×10^7
2.01	281101
2.1	2911

For a graphical approximation:



It appears that when x is close to 2, that $\frac{x^2+7x+10}{x^2-4x+4}$ grows without bound. So

$$\lim_{x \to 2} \left(\frac{x^2+7x+10}{x^2-4x+4} \right) \text{ does not exist (DNE).}$$

12. Approximate the limit numerically and graphically.

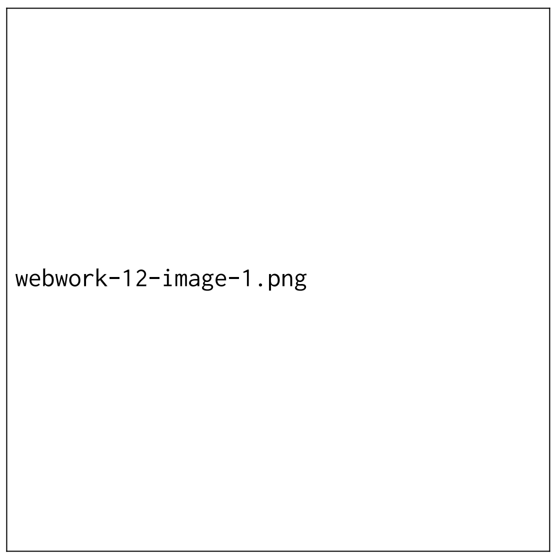
For $f(x) = \begin{cases} x+2 & x \leq 2 \\ 3x-5 & x > 2 \end{cases}$,

$\lim\limits_{x\to 2} f(x) = \boxed{}$

Solution. For a numerical approximation, make a table:

x	$f(x)$
1.9	3.9
1.99	3.99
1.999	3.999
2.001	1.003
2.01	1.03
2.1	1.3

For a graphical approximation:



It appears that when x is close to 2, that $f(x)$ approaches different values from the left and right. So

$$\lim_{x\to 2} f(x) \text{ does not exist (DNE).}$$

13. Approximate the limit numerically and graphically.

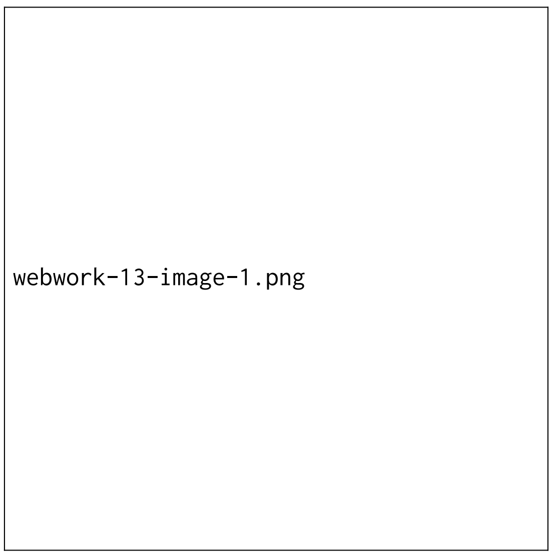
For $f(x) = \begin{cases} x^2 - x + 1 & x \leq 3 \\ 2x + 1 & x > 3 \end{cases}$,

$\lim\limits_{x \to 3} f(x) = \boxed{}$

Solution. For a numerical approximation, make a table:

x	$f(x)$
2.9	6.51
2.99	6.9501
2.999	6.995
3.001	7.002
3.01	7.02
3.1	7.2

For a graphical approximation:



It appears that when x is close to 3, that $f(x)$ approaches 7. So

$$\lim_{x \to 3} f(x) = 7.$$

14. Approximate the limit numerically and graphically.

For $f(x) = \begin{cases} cos(x) & x \leq 0 \\ x^2 + 3x + 1 & x > 0 \end{cases}$,

$\lim\limits_{x \to 0} f(x) = \boxed{\qquad\qquad}$

Solution. For a numerical approximation, make a table:

x	$f(x)$
-0.1	0.995004
-0.01	0.99995
-0.001	1
0.001	1.003
0.01	1.0301
0.1	1.31

For a graphical approximation:

webwork-14-image-1.png

It appears that when x is close to 0, that $f(x)$ approaches 1. So

$$\lim_{x \to 0} f(x) = 1.$$

15. Approximate the limit numerically and graphically.

For $f(x) = \begin{cases} \sin(x) & x \le \pi/2 \\ \cos(x) & x > \pi/2 \end{cases}$,

$\lim\limits_{x \to \pi/2} f(x) = \boxed{}$

Solution. For a numerical approximation, make a table:

x	$f(x)$
$\pi/2 - 0.1$	0.995004
$\pi/2 - 0.01$	0.99995
$\pi/2 - 0.001$	1
$\pi/2 + 0.001$	-0.001
$\pi/2 + 0.01$	-0.00999983
$\pi/2 + 0.1$	-0.0998334

For a graphical approximation:

webwork-15-image-1.png

It appears that when x is close to $\pi/2$, that $f(x)$ approaches different values from the left and right. So

$$\lim_{x \to \pi/2} f(x) \text{ does not exist (DNE).}$$

In the following exercises, a function f and a value a are given. Approximate the limit of the difference quotient, $\lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.

16. Approximate the limit of the difference quotient, $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.

$f(x) = -7x + 2$, $a = 3$

h	$\frac{f(a+h)-f(a)}{h}$
$\square$	$\square$
$\square$	$\square$
$\square$	$\square$
$\square$	$\square$

It appears that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \square$.

Solution.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-7
-0.01	-7
0.01	-7
0.1	-7

It appears that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = -7$.

17. Approximate the limit of the difference quotient, $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.

$f(x) = 9x + 0.06$, $a = -1$

h	$\frac{f(a+h)-f(a)}{h}$
$\square$	$\square$
$\square$	$\square$
$\square$	$\square$
$\square$	$\square$

It appears that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \square$.

Solution.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	9
-0.01	9
0.01	9
0.1	9

It appears that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = 9$.

18. Approximate the limit of the difference quotient, $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.

$f(x) = x^2 + 3x - 7$, $a = 1$

h	$\frac{f(a+h)-f(a)}{h}$

It appears that $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = \boxed{}$.

Solution.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	4.9
-0.01	4.99
0.01	5.01
0.1	5.1

It appears that $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = 5$.

19. Approximate the limit of the difference quotient, $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.

$f(x) = \frac{1}{x+1}$, $a = 2$

h	$\frac{f(a+h)-f(a)}{h}$

It appears that $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = \boxed{}$.

Solution.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.114943
-0.01	-0.111483
0.01	-0.110742
0.1	-0.107527

It appears that $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = -0.111111$.

20. Approximate the limit of the difference quotient, $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.

$f(x) = -4x^2 + 5x - 1$, $a = -3$

h	$\frac{f(a+h)-f(a)}{h}$
$\square$	$\square$
$\square$	$\square$
$\square$	$\square$
$\square$	$\square$

It appears that $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = \square$.

Solution.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	29.4
-0.01	29.04
0.01	28.96
0.1	28.6

It appears that $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = 29$.

21. Approximate the limit of the difference quotient, $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.

$f(x) = \ln(x)$, $a = 5$

h	$\frac{f(a+h)-f(a)}{h}$
$\square$	$\square$
$\square$	$\square$
$\square$	$\square$
$\square$	$\square$

It appears that $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = \square$.

Solution.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	0.202027
-0.01	0.2002
0.01	0.1998
0.1	0.198026

It appears that $\lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h} = 0.2$.

22. Approximate the limit of the difference quotient, $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.

$f(x) = \sin(x)$, $a = \pi$

h	$\frac{f(a+h)-f(a)}{h}$

It appears that $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h} = \boxed{}$.

Solution.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.998334
-0.01	-0.999983
0.01	-0.999983
0.1	-0.998334

It appears that $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h} = -1$.

23. Approximate the limit of the difference quotient, $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.

$f(x) = \cos(x)$, $a = \pi$

h	$\frac{f(a+h)-f(a)}{h}$

It appears that $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h} = \boxed{}$.

Solution.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	0.0499583
-0.01	0.00499996
0.01	0.00499996
0.1	0.0499583

It appears that $\lim\limits_{h\to 0}\frac{f(a+h)-f(a)}{h} = 0$.

1.2 Epsilon-Delta Definition of a Limit

This section introduces the formal definition of a limit. Many refer to this as "the epsilon–delta," definition, referring to the letters ε and δ of the Greek alphabet.

Before we give the actual definition, let's consider a few informal ways of describing a limit. Given a function $y = f(x)$ and an x-value, c, we say that "the limit of the function f, as x approaches c, is a value L":

1 if "y tends to L" as "x tends to c."

2 if "y approaches L" as "x approaches c."

3 if "y is near L" whenever "x is near c."

The problem with these definitions is that the words "tends," "approach," and especially "near" are not exact. In what way does the variable x tend to, or approach, c? How near do x and y have to be to c and L, respectively?

The definition we describe in this section comes from formalizing 3. A quick restatement gets us closer to what we want:

3âĂš If x is within a certain *tolerance level* of c, then the corresponding value $y = f(x)$ is within a certain *tolerance level* of L.

The traditional notation for the x-tolerance is the lowercase Greek letter delta, or δ, and the y-tolerance is denoted by lowercase epsilon, or ε. One more rephrasing of 3âĂš nearly gets us to the actual definition:

3âĂş If x is within δ units of c, then the corresponding value of y is within ε units of L.

We can write "x is within δ units of c" mathematically as

$$|x - c| < \delta,$$

which is equivalent to

$$c - \delta < x < c + \delta.$$

Letting the symbol " $\Longrightarrow$ " represent the word "implies," we can rewrite 3âĂş as

$$|x - c| < \delta \implies |y - L| < \varepsilon$$

or

$$c - \delta < x < c + \delta \implies L - \varepsilon < y < L + \varepsilon.$$

The point is that δ and ε, being tolerances, can be any positive (but typically small) values satisfying this implication. Finally, we have the formal definition of the limit with the notation seen in the previous section.

Definition 1.2.1 (The Limit of a Function f)**.** Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. The statement that the **limit of $f(x)$, as x approaches c, is L** is denoted by

$$\lim_{x \to c} f(x) = L,$$

and means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in I$, $x \neq c$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Mathematicians often enjoy writing ideas without using any words. Here is the wordless definition of the limit:

$$\lim_{x \to c} f(x) = L$$
$$\Longleftrightarrow$$
$$\forall\, \varepsilon > 0, \exists\, \delta > 0 \text{ s.t. } 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon.$$

Note the order in which ε and δ are given. In the definition, the y-tolerance ε is given *first* and then the limit will exist *if* we can find an x-tolerance δ that works.

An example will help us understand this definition. Note that the explanation is long, but it will take one through all steps necessary to understand the ideas.

Example 1.2.2 (Evaluating a limit using the definition)**.** Show that $\lim\limits_{x \to 4} \sqrt{x} = 2$.

Solution. Before we use the formal definition, let's try some numerical tolerances. What if the y tolerance is 0.5, or in other words $\varepsilon = 0.5$? How close to 4 does x have to be so that y is within 0.5 units of 2? That is, $1.5 < y < 2.5$? In this case, we can proceed as follows:

$$\begin{aligned}
1.5 < y &< 2.5 \\
1.5 < \sqrt{x} &< 2.5 && \text{(Let } y = \sqrt{x}\text{)} \\
1.5^2 < x &< 2.5^2 && \text{(Square the inequality)} \\
2.25 < x &< 6.25 \\
2.25 - 4 < x - 4 &< 6.25 - 4 && \text{(Subtract 4 from both sides)} \\
-1.75 < x - 4 &< 2.25
\end{aligned}$$

So, what is the desired x tolerance? Remember, we want to find a δ so that $|x - 4|$ is smaller than δ. Since $1.75 < 2.25$, then if we require $|x - 4| < 1.75$, then we have

$$\begin{aligned}
& |x - 4| < 1.75 \\
\implies & -1.75 < x - 4 < 1.75 < 2.25
\end{aligned}$$

Therefore we can have $\delta \leq 1.75$. See Figure **??**.

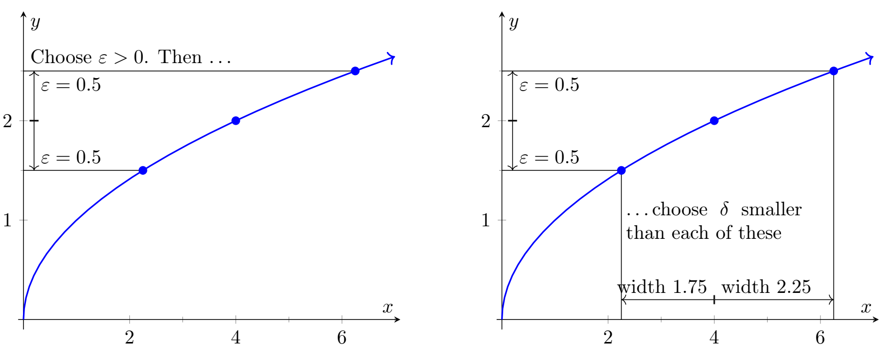


Figure 1.2.3: Illustrating the $\varepsilon - \delta$ process. With $\varepsilon = 0.5$, we pick any $\delta < 1.75$.

Given the y tolerance $\varepsilon = 0.5$, we have found an x tolerance, $\delta \leq 1.75$, such that whenever x is within δ units of 4, then y is within ε units of 2. That's what we were trying to find.

Let's try another value of ε.

What if the y tolerance is 0.01, i.e.$\varepsilon = 0.01$? How close to 4 does x have to be in order for y to be within 0.01 units of 2? (In other words for $1.99 < y < 2.01$?) Again, we just square these values to get $1.99^2 < x < 2.01^2$, or

$$\begin{aligned} 3.9601 < x &< 4.0401 \\ -0.0399 < x - 4 &< 0.0401 \end{aligned}$$

What is the desired x tolerance? In this case we must have $\delta \leq 0.0399$, which is the minimum distance from 4 of the two bounds given above.

What we have so far: if $\varepsilon = 0.5$, then $\delta \leq 1.75$ leads to $f(x)$ being less than ε from $f(4)$ and if $\varepsilon = 0.01$, then $\delta \leq 0.0399$ being less than ε from $f(4)$. A pattern is not easy to see, so we switch to general ε try to determine an adequate δ symbolically. We start by assuming $y = \sqrt{x}$ is within ε units of 2:

$$\begin{aligned} |y-2| &< \varepsilon \\ -\varepsilon < y - 2 &< \varepsilon \\ -\varepsilon < \sqrt{x} - 2 &< \varepsilon && (y = \sqrt{x}) \\ 2 - \varepsilon < \sqrt{x} &< 2 + \varepsilon && \text{(Add 2)} \\ (2-\varepsilon)^2 < x &< (2+\varepsilon)^2 && \text{(Square all)} \\ 4 - 4\varepsilon + \varepsilon^2 < x &< 4 + 4\varepsilon + \varepsilon^2 && \text{(Expand)} \\ -4\varepsilon + \varepsilon^2 < x - 4 &< 4\varepsilon + \varepsilon^2 && \text{(Subtract 4)} \end{aligned}$$

We choose the smaller of these two distances from the number 4:

$$\delta \leq \min\{4\varepsilon - \varepsilon^2, 4\varepsilon + \varepsilon^2\}.$$

Since $\varepsilon > 0$, we have $4\varepsilon - \varepsilon^2 < 4\varepsilon + \varepsilon^2$, the minimum is $\delta \leq 4\varepsilon - \varepsilon^2$. That's the formula: given an ε, set $\delta \leq 4\varepsilon - \varepsilon^2$.

We can check this for our previous values. If $\varepsilon = 0.5$, the formula gives $\delta \leq 4(0.5) - (0.5)^2 = 1.75$ and when $\varepsilon = 0.01$, the formula gives $\delta \leq 4(0.01) - (0.01)^2 = 0.399$.

So given any $\varepsilon > 0$, set $\delta \leq 4\varepsilon - \varepsilon^2$. Then if $|x - 4| < \delta$ (and $x \neq 4$), then $|f(x) - 2| < \varepsilon$, satisfying the definition of the limit. We have shown formally (and finally!) that $\lim\limits_{x\to 4} \sqrt{x} = 2$.

The previous example was a little long in that we sampled a few specific cases of ε before handling the general case. Normally this is not done. The previous example is also a bit unsatisfying in that $\sqrt{4} = 2$; why work so hard to prove something so obvious? Many ε–δ proofs are long and difficult to do. In this section, we will focus on examples where the answer is, frankly, obvious, because the non-obvious examples are even harder. In the next section we will learn some theorems that allow us to evaluate limits *analytically*, that is, without using the ε–δ definition.

Example 1.2.4 (Evaluating a limit using the definition)**.** Show that $\lim\limits_{x\to 2} x^2 = 4$.

Solution. Let's do this example symbolically from the start. Let $\varepsilon > 0$ be given; we want $|y - 4| < \varepsilon$, i.e., $\left|x^2 - 4\right| < \varepsilon$. How do we find δ such that when $|x - 2| < \delta$, we are guaranteed that $\left|x^2 - 4\right| < \varepsilon$?

This is a bit trickier than the previous example, but let's start by noticing that $\left|x^2 - 4\right| = |x - 2| \cdot |x + 2|$. Consider:

$$\left|x^2 - 4\right| < \varepsilon \implies |x-2| \cdot |x+2| < \varepsilon \implies |x - 2| < \frac{\varepsilon}{|x+2|}.$$

Could we not set $\delta = \frac{\varepsilon}{|x+2|}$?

We are close to an answer, but the catch is that δ must be a *constant* value (so it can't depend on x). There is a way to work around this, but we do have to make an assumption. Remember that ε is supposed to be a small number, which implies that δ will also be a small value. In particular, we can (probably) assume that $\delta < 1$. If this is true, then $|x-2| < \delta$ would imply that $|x-2| < 1$, giving $1 < x < 3$.

Now, back to the fraction $\frac{\varepsilon}{|x+2|}$. If $1 < x < 3$, then $3 < x+2 < 5$ (add 2 to all terms in the inequality). Taking reciprocals, we have

$$\frac{1}{5} < \frac{1}{|x+2|} < \frac{1}{3},$$

which implies

$$\frac{1}{5} < \frac{1}{|x+2|},$$

which implies

$$\frac{\varepsilon}{5} < \frac{\varepsilon}{|x+2|}. \tag{1.2.1}$$

This suggests that we set $\delta \leq \frac{\varepsilon}{5}$. To see why, let consider what follows when we assume $|x-2| < \delta$:

$$\begin{aligned}
|x-2| &< \delta \\
|x-2| &< \frac{\varepsilon}{5} && \text{(Our choice of } \delta\text{)} \\
|x-2| \cdot |x+2| &< |x+2| \cdot \frac{\varepsilon}{5} && \text{(Multiply by } |x+2|\text{)} \\
\left|x^2-4\right| &< |x+2| \cdot \frac{\varepsilon}{5} && \text{(Simplify left side)} \\
\left|x^2-4\right| &< |x+2| \cdot \frac{\varepsilon}{|x+2|} && \text{(Inequality (\textbf{??}), } \delta < 1\text{)} \\
\left|x^2-4\right| &< \varepsilon
\end{aligned}$$

We have arrived at $\left|x^2-4\right| < \varepsilon$ as desired. Note again, in order to make this happen we needed δ to first be less than 1. That is a safe assumption; we want ε to be arbitrarily small, forcing δ to also be small.

We have also picked δ to be smaller than "necessary." We could get by with a slightly larger δ, as shown in Figure **??**. The outer lines show the boundaries defined by our choice of ε. The inner lines show the boundaries defined by setting $\delta = \varepsilon/5$. Note how these dotted lines are within the dashed lines. That is perfectly fine; by choosing x within the dotted lines we are guaranteed that $f(x)$ will be within ε of 4.

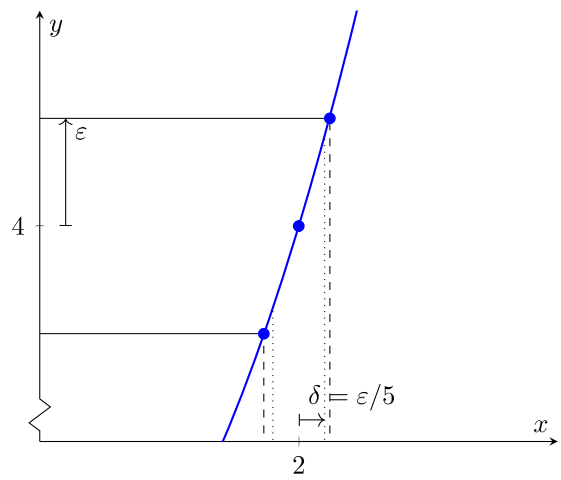


Figure 1.2.5: Choosing $\delta = \varepsilon/5$ in Example **??**.

In summary, given $\varepsilon > 0$, set $\delta = \varepsilon/5$. Then $|x-2| < \delta$ implies $\left|x^2-4\right| < \varepsilon$ (i.e.$|y-4| < \varepsilon$) as desired. This shows that $\lim\limits_{x\to 2} x^2 = 4$. Figure **??** gives a visualization of this; by restricting x to values within $\delta = \varepsilon/5$ of 2, we see that $f(x)$ is within ε of 4.

Make note of the general pattern exhibited in these last two examples. In some sense, each starts out "backwards." That is, while we want to

1. start with $|x-c| < \delta$ and conclude that
2. $|f(x)-L| < \varepsilon$,

we actually start by doing what is essentially some "scratch-work" first:

1. assume $|f(x)-L| < \varepsilon$, then perform some algebraic manipulations to give an inequality of the form
2. $|x-c| <$ *something*.

When we have properly done this, the *something* on the "greater than" side of the inequality becomes our δ. We can refer to this as the "scratch-work" phase of our proof. Once we have δ, we can formally start the actual proof with $|x-c| < \delta$ and use algebraic manipulations to conclude that $|f(x)-L| < \varepsilon$, usually by using the same steps of our "scratch-work" in reverse order.

We highlight this process in the following example.

Example 1.2.6 (Evaluating a limit using the definition)**.** Prove that $\lim\limits_{x\to 1} x^3 - 2x = -1$.

Solution. We start our scratch-work by considering $|f(x)-(-1)| < \varepsilon$:

$$\begin{aligned}
|f(x)-(-1)| &< \varepsilon \\
\left|x^3-2x+1\right| &< \varepsilon && \text{(Now factor)} \\
\left|(x-1)(x^2+x-1)\right| &< \varepsilon \\
|x-1| &< \frac{\varepsilon}{|x^2+x-1|}. && (1.2.2)
\end{aligned}$$

We are at the phase of saying that $|x-1| <$ *something*, where *something*$=$ $\varepsilon/\left|x^2+x-1\right|$. We want to turn that *something* into δ.

Since x is approaching 1, we are safe to assume that x is between 0 and 2. So

$$\begin{aligned} &0 < x < 2 \\ &0 < x^2 < 4. && \text{(squared each term)} \end{aligned}$$

Since $0 < x < 2$, we can add 0, x and 2, respectively, to each part of the inequality and maintain the inequality.

$$\begin{aligned} 0 &< x^2 + x < 6 \\ -1 &< x^2 + x - 1 < 5. && \text{(subtracted 1 from each part)} \end{aligned}$$

In Inequality (**??**), we wanted $|x-1| < \varepsilon/\left|x^2+x-1\right|$. The above shows that given any x in $[0,2]$, we know that

$$\begin{aligned} x^2 + x - 1 &< 5 && \text{which implies that} \\ \frac{1}{5} &< \frac{1}{x^2+x-1} && \text{which implies that} \\ \frac{\varepsilon}{5} &< \frac{\varepsilon}{x^2+x-1}. && (1.2.3) \end{aligned}$$

So we set $\delta \leq \varepsilon/5$. This ends our scratch-work, and we begin the formal proof (which also helps us understand why this was a good choice of δ).

Given ε, let $\delta \leq \varepsilon/5$. We want to show that when $|x-1| < \delta$, then $\left|(x^3-2x)-(-1)\right| < \varepsilon$. We start with $|x-1| < \delta$:

$$\begin{aligned} |x-1| &< \delta \\ |x-1| &< \frac{\varepsilon}{5} \\ |x-1| &< \frac{\varepsilon}{|x^2+x-1|} && \text{(Inequality (??), } x \text{ near 1)} \\ |x-1| \cdot \left|x^2+x-1\right| &< \varepsilon \\ \left|x^3-2x+1\right| &< \varepsilon \\ \left|(x^3-2x)-(-1)\right| &< \varepsilon, \end{aligned}$$

which is what we wanted to show. Thus $\lim\limits_{x\to 1} x^3 - 2x = -1$.

We illustrate evaluating limits once more.

Example 1.2.7 (Evaluating a limit using the definition)**.** Prove that $\lim\limits_{x\to 0} e^x = 1$.

Solution. Symbolically, we want to take the inequality $|e^x - 1| < \varepsilon$ and unravel it to the form $|x-0| < \delta$. Here is our scratch-work:

$$\begin{aligned} |e^x - 1| &< \varepsilon \\ -\varepsilon < e^x &- 1 < \varepsilon && \text{(Definition of absolute value)} \\ 1 - \varepsilon < e^x &< 1 + \varepsilon && \text{(Add 1)} \\ \ln(1-\varepsilon) < x &< \ln(1+\varepsilon) && \text{(Take natural logs)} \end{aligned}$$

Making the safe assumption that $\varepsilon < 1$ ensures the last inequality is valid (i.e., so that $\ln(1-\varepsilon)$ is defined). We can then set δ to be the minimum of $|\ln(1-\varepsilon)|$ and $\ln(1+\varepsilon)$; i.e.,

$$\delta = \min\{|\ln(1-\varepsilon)|, \ln(1+\varepsilon)\} = \ln(1+\varepsilon).$$

Recall $\ln(1-\varepsilon)$ is negative because $1-\varepsilon < 1$. So $|\ln(1-\varepsilon)| = -\ln(1-\varepsilon) = \ln\left(\frac{1}{1-\varepsilon}\right)$. So to determine which is smaller between $|\ln(1-\varepsilon)|$ and $\ln(1+\varepsilon)$ amounts to determining which is smaller between $\frac{1}{1-\varepsilon}$ and $1+\varepsilon$. But $(1+\varepsilon)/\left(\frac{1}{1-\varepsilon}\right) = (1+\varepsilon)(1-\varepsilon) = 1-\varepsilon^2 < 1$, so $(1+\varepsilon) < \frac{1}{1-\varepsilon}$. And therefore $\ln(1+\varepsilon) < |\ln(1-\varepsilon)|$.

Now, we work through the actual the proof:

$$\begin{aligned} |x-0| &< \delta \\ -\delta < x &< \delta && \text{(Definition of absolute value)} \\ -\ln(1+\varepsilon) < x &< \ln(1+\varepsilon) \\ \ln(1-\varepsilon) < x &< \ln(1+\varepsilon). && \text{(since } \ln(1-\varepsilon) < -\ln(1+\varepsilon)\text{)} \end{aligned}$$

The above line is true by our choice of δ and by the fact that since $|\ln(1-\varepsilon)| > \ln(1+\varepsilon)$ and $\ln(1-\varepsilon) < 0$, we know $\ln(1-\varepsilon) < -\ln(1+\varepsilon)$.

$$\begin{aligned} 1-\varepsilon < e^x &< 1+\varepsilon && \text{(Exponentiate)} \\ -\varepsilon < e^x - 1 &< \varepsilon && \text{(Subtract 1)} \end{aligned}$$

In summary, given $\varepsilon > 0$, let $\delta = \ln(1+\varepsilon)$. Then $|x-0| < \delta$ implies $|e^x - 1| < \varepsilon$ as desired. We have shown that $\lim\limits_{x\to 0} e^x = 1$.

We note that we could actually show that $\lim\limits_{x\to c} e^x = e^c$ for any constant c. We do this by factoring out e^c from both sides, leaving us to show $\lim\limits_{x\to c} e^{x-c} = 1$ instead. By using the substitution $u = x-c$, this reduces to showing $\lim\limits_{u\to 0} e^u = 1$ which we just did in the last example. As an added benefit, this shows that in fact the function $f(x) = e^x$ is *continuous* at all values of x, an important concept we will define in Section **??**.

This formal definition of the limit is not an easy concept grasp. Our examples are actually "easy" examples, using "simple" functions like polynomials, square–roots and exponentials. It is very difficult to prove, using the techniques given above, that $\lim\limits_{x\to 0}(\sin(x))/x = 1$, as we approximated in Section 1.1.

There is hope. Section **??** shows how one can evaluate complicated limits using certain basic limits as building blocks. While limits are an incredibly important part of calculus (and hence much of higher mathematics), rarely are limits evaluated using the definition. Rather, the techniques of Section **??** are employed.

1.2.1 Exercises

Terms and Concepts

1. What is wrong with the following "definition" of a limit?

> "The limit of $f(x)$, as x approaches a, is K" means that given any $\delta > 0$ there exists $\varepsilon > 0$ such that whenever $|f(x) - K| < \varepsilon$, we have $|x - a| < \delta$.

[Essay Answer]

Solution. ε should be given first, and the restriction $|x-a| < \delta$ implies $|f(x) - K| < \varepsilon$, not the other way around.

2. Which is given first in establishing a limit, the x-tolerance or the y-tolerance?

(Choose one: x-tolerance / y-tolerance)is given first.

Solution. The y–tolerance is given first.

3. True or False? ε must always be positive. (Choose one: True / False)

Solution. True

4. True or False? δ must always be positive. (Choose one: True / False)

Solution. True

In the following exercises, prove the given limit using an ε-δ proof.

5. Use an ε-δ proof to prove that

$$\lim_{x\to 5}(3-x) = -2$$

[Essay Answer]

Solution. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-5| < \delta$, $|f(x) - (-2)| < \varepsilon$.

First, some preliminary investigation to find a suitable δ. Consider $|f(x) - (-2)| < \varepsilon$:

$$\begin{aligned} |f(x)+2| &< \varepsilon \\ |(3-x)+2| &< \varepsilon \\ |5-x| &< \varepsilon \\ |x-5| &< \varepsilon \end{aligned}$$

Since we want to start with $|x-5| < \delta$, this suggests we let $\delta = \varepsilon$.

Now we can apply the definition.

$$\begin{aligned} |x-5| &< \delta \\ |x-5| &< \varepsilon \\ -\varepsilon < x-5 &< \varepsilon \\ -\varepsilon < (x-3)-2 &< \varepsilon \\ \varepsilon > (-x+3)-(-2) &> -\varepsilon \\ |(3-x)-(-2)| &< \varepsilon. \end{aligned}$$

In other words, $|x-5| < \delta$ implies $|(3-x)-(-2)| < \varepsilon$. This is what we needed to prove.

6. Use an ε-δ proof to prove that

$$\lim_{x \to 3}(x^2 - 3) = 6$$

[Essay Answer]

Solution. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x - 3| < \delta$, $|f(x) - 6| < \varepsilon$.

First, some preliminary investigation to find a suitable δ. Consider $|f(x) - 6| < \varepsilon$:

$$\begin{aligned} \left|\left(x^2 - 3\right) - 6\right| &< \varepsilon \\ \left|x^2 - 9\right| &< \varepsilon \\ |x - 3| \cdot |x + 3| &< \varepsilon \\ |x - 3| &< \frac{\varepsilon}{|x + 3|} \end{aligned}$$

Since x is near 3, we can safely assume that, for instance, $2 < x < 4$. Thus

$$\begin{gathered} 2 + 3 < x + 3 < 4 + 3 \\ 5 < x + 3 < 7 \\ \frac{1}{7} < \frac{1}{x + 3} < \frac{1}{5} \\ \frac{\varepsilon}{7} < \frac{\varepsilon}{x + 3} < \frac{\varepsilon}{5} \end{gathered}$$

Since we need to begin the actual proof with $|x - 3| < \delta$, this suggests that we take $\delta = \frac{\varepsilon}{7}$.

Now we can apply the definition.

$$\begin{aligned} |x - 3| &< \delta \\ |x - 3| &< \frac{\varepsilon}{7} \\ |x - 3| &< \frac{\varepsilon}{|x + 3|} \\ |x - 3| \cdot |x + 3| &< \varepsilon \\ \left|x^2 - 9\right| &< \varepsilon \\ \left|\left(x^2 - 3\right) - 6\right| &< \varepsilon \end{aligned}$$

In other words, $|x - 3| < \delta$ implies $\left|\left(x^2 - 3\right) - 6\right| < \varepsilon$. This is what we needed to prove.

7. Use an ε-δ proof to prove that

$$\lim_{x\to 4}(x^2+x-5)=15$$

[Essay Answer]

Solution. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-4|<\delta$, $|f(x)-15|<\varepsilon$.

First, some preliminary investigation to find a suitable δ. Consider $|f(x)-15|<\varepsilon$:

$$\begin{aligned}\left|\left(x^2+x-5\right)-15\right| &< \varepsilon\\ \left|x^2+x-20\right| &< \varepsilon\\ |x-4|\cdot|x+5| &< \varepsilon\\ |x-4| &< \frac{\varepsilon}{|x+5|}\end{aligned}$$

Since x is near 4, we can safely assume that, for instance, $3<x<5$. Thus

$$\begin{gathered}3+5<x+5<5+5\\ 8<x+5<10\\ \frac{1}{10}<\frac{1}{x+5}<\frac{1}{8}\\ \frac{\varepsilon}{10}<\frac{\varepsilon}{x+5}<\frac{\varepsilon}{8}\end{gathered}$$

Since we need to begin the actual proof with $|x-4|<\delta$, this suggests that we take $\delta=\frac{\varepsilon}{10}$.

Now we can apply the definition.

$$\begin{aligned}|x-4| &< \delta\\ |x-4| &< \frac{\varepsilon}{10}\\ |x-4| &< \frac{\varepsilon}{|x+5|}\\ |x-4|\cdot|x+5| &< \varepsilon\\ \left|x^2+x-20\right| &< \varepsilon\\ \left|\left(x^2+x-5\right)-15\right| &< \varepsilon\end{aligned}$$

In other words, $|x-4|<\delta$ implies $\left|\left(x^2+x-5\right)-15\right|<\varepsilon$. This is what we needed to prove.

8. Use an ε-δ proof to prove that

$$\lim_{x\to 2}(x^3-1)=7$$

[Essay Answer]

Solution. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-2| < \delta$, $|f(x)-7| < \varepsilon$.

First, some preliminary investigation to find a suitable δ. Consider $|f(x)-7| < \varepsilon$:

$$\begin{aligned} \left|\left(x^3-1\right)-7\right| &< \varepsilon \\ \left|x^3-8\right| &< \varepsilon \\ |x-2|\cdot\left|x^2+2x+4\right| &< \varepsilon \\ |x-2| &< \frac{\varepsilon}{|x^2+2x+4|} \end{aligned}$$

Since x is near 2, we can safely assume that, for instance, $1 < x < 3$. Thus

$$\begin{gathered} 1^2+2(1)+4 < x^2+2x+4 < 3^2+2(3)+4 \\ 7 < x^2+2x+4 < 19 \\ \frac{1}{19} < \frac{1}{x^2+2x+4} < \frac{1}{7} \\ \frac{\varepsilon}{19} < \frac{\varepsilon}{x^2+2x+4} < \frac{\varepsilon}{7} \end{gathered}$$

Since we need to begin the actual proof with $|x-2| < \delta$, this suggests that we take $\delta = \frac{\varepsilon}{19}$.

Now we can apply the definition.

$$\begin{aligned} |x-2| &< \delta \\ |x-2| &< \frac{\varepsilon}{19} \\ |x-2| &< \frac{\varepsilon}{|x^2+2x+4|} \\ |x-2|\cdot\left|x^2+2x+4\right| &< \varepsilon \\ \left|x^3-8\right| &< \varepsilon \\ \left|\left(x^3-1\right)-7\right| &< \varepsilon \end{aligned}$$

In other words, $|x-2| < \delta$ implies $\left|\left(x^3-1\right)-7\right| < \varepsilon$. This is what we needed to prove.

9. Use an ε-δ proof to prove that

$$\lim_{x \to 2} 5 = 5$$

[Essay Answer]

Solution. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x - 2| < \delta$, $|f(x) - 5| < \varepsilon$.

First, some preliminary investigation to find a suitable δ. Consider $|f(x) - 5| < \varepsilon$:

$$\begin{aligned} |5 - 5| &< \varepsilon \\ |0| &< \varepsilon \end{aligned}$$

Well, this is just plain true no matter what ε is (as long as its positive). So it really doesn't matter what δ is.

Now we can apply the definition.

$$\begin{aligned} |x - 2| &< \delta \\ |0| &< \varepsilon \\ |5 - 5| &< \varepsilon \end{aligned}$$

In other words, $|x - 2| < \delta$ (vacuously) implies $|5 - 5| < \varepsilon$. This is what we needed to prove.

10. Use an ε-δ proof to prove that

$$\lim_{x\to 0}(e^{2x}-1)=0$$

[Essay Answer]

Solution. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-0|<\delta$, $|f(x)-0|<\varepsilon$.

First, some preliminary investigation to find a suitable δ. Consider $|f(x)-0|<\varepsilon$:

$$\begin{aligned}
\left|\left(e^{2x}-1\right)-0\right| &< \varepsilon \\
\left|e^{2x}-1\right| &< \varepsilon \\
-\varepsilon < e^{2x}-1 &< \varepsilon \\
1-\varepsilon < e^{2x} &< 1+\varepsilon \\
\ln(1-\varepsilon) < 2x &< \ln(1+\varepsilon) \\
\frac{1}{2}\ln(1-\varepsilon) < x &< \frac{1}{2}\ln(1+\varepsilon) \\
\frac{1}{2}\ln(1-\varepsilon) < x-0 &< \frac{1}{2}\ln(1+\varepsilon)
\end{aligned}$$

Since we will need to start with $|x-0|<\delta$, this suggests we take $\delta = \min\left\{\frac{1}{2}\left|\ln(1-\varepsilon)\right|, \frac{1}{2}\ln(1+\varepsilon)\right\} = \frac{1}{2}\ln(1+\varepsilon)$.

Now we can apply the definition.

$$\begin{aligned}
|x-0| &< \delta \\
|x| &< \frac{1}{2}\ln(1+\varepsilon) < \frac{1}{2}\left|\ln(1-\varepsilon)\right| \\
\frac{1}{2}\ln(1-\varepsilon) < x &< \frac{1}{2}\ln(1+\varepsilon) \\
\ln(1-\varepsilon) < 2x &< \ln(1+\varepsilon) \\
1-\varepsilon < e^{2x} &< 1+\varepsilon \\
-\varepsilon < e^{2x}-1 &< \varepsilon \\
\left|e^{2x}-1\right| &< \varepsilon \\
\left|\left(e^{2x}-1\right)-0\right| &< \varepsilon
\end{aligned}$$

In other words, $|x-0|<\delta$ implies $\left|\left(e^{2x}-1\right)-0\right|<\varepsilon$. This is what we needed to prove.

11. Use an ε-δ proof to prove that

$$\lim_{x\to 0} \sin(x) = 0$$

[Essay Answer]

Hint. Use the fact that $|\sin(x)| \leq |x|$, with equality only when $x = 0$.

Solution. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x - 0| < \delta$, $|f(x) - 0| < \varepsilon$.

First, some preliminary investigation to find a suitable δ. Consider $|f(x) - 0| < \varepsilon$:

$$\begin{aligned} |\sin(x) - 0| &< \varepsilon \\ |\sin(x)| &< \varepsilon \end{aligned}$$

Using the hint that $|\sin(x)| \leq |x|$, then if $|x| < 0$, it is automatic that $|\sin(x)| < 0$. This suggests we take $\delta = \varepsilon$.

Now we can apply the definition.

$$\begin{aligned} |x - 0| &< \delta \\ |x| &< \varepsilon \\ |\sin(x)| &< \varepsilon \\ |\sin(x) - 0| &< \varepsilon \end{aligned}$$

In other words, $|x - 0| < \delta$ implies $|\sin(x) - 0| < \varepsilon$. This is what we needed to prove.

1.3 Finding Limits Analytically

In Section 1.1 we explored the concept of the limit without a strict definition, meaning we could only make approximations. In the previous section we gave the definition of the limit and demonstrated how to use it to verify our approximations were correct. Thus far, our method of finding a limit is

1. make a really good approximation either graphically or numerically, and
2. verify our approximation is correct using a ε–δ proof.

Recognizing that ε–δ proofs are cumbersome, this section gives a series of theorems which allow us to find limits much more quickly and intuitively.

Suppose that $\lim_{x\to 2} f(x) = 2$ and $\lim_{x\to 2} g(x) = 3$. What is $\lim_{x\to 2}(f(x)+g(x))$? Intuition tells us that the limit should be 5, as we expect limits to behave in a nice way. The following theorem states that already established limits do behave nicely.

Theorem 1.3.1 (Basic Limit Properties)**.** *Let b, c, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:*

$$\lim_{x\to c} f(x) = L \qquad\qquad \lim_{x\to c} g(x) = K.$$

The following limits hold.

Constant $\lim_{x\to c} b = b$

Identity $\lim_{x\to c} x = c$

Sum/Difference $\lim_{x\to c}(f(x) \pm g(x)) = L \pm K$

Scalar Multiple $\lim_{x\to c}(b \cdot f(x)) = bL$

Product $\lim_{x\to c}(f(x) \cdot g(x)) = LK$

Quotient $\lim_{x\to c}(f(x)/g(x)) = L/K$, *when* $K \neq 0$

Power $\lim_{x\to c} f(x)^n = L^n$

Root $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$

(If n is even, L must be non-negative.)

Composition *If either of the following holds:*

1. $\lim_{x\to c} f(x) = L$, $\lim_{x\to L} g(x) = K$, *and* $g(L) = K$
2. $\lim_{x\to c} f(x) = L$, $\lim_{x\to L} g(x) = K$, *and* $f(x) \neq L$ *for all* x *close to but not equal to* c

then $\lim_{x\to c} g(f(x)) = K$.

We apply the theorem to an example.

Example 1.3.2 (Using basic limit properties)**.** Let

$$\lim_{x\to 2} f(x) = 2 \qquad \lim_{x\to 2} g(x) = 3 \qquad p(x) = 3x^2 - 5x + 7.$$

Find the following limits:

1. $\lim_{x\to 2}(f(x) + g(x))$
2. $\lim_{x\to 2}(5f(x) + g(x)^2)$
3. $\lim_{x\to 2} p(x)$

Solution.

1. Using the Sum/Difference rule, we know that $\lim_{x\to 2}(f(x) + g(x)) = 2 + 3 = 5$.

2. Using the Scalar Multiple, Sum/Difference, and Power rules, we find that $\lim_{x\to 2}(5f(x) + g(x)^2) = 5 \cdot 2 + 3^2 = 19$.

3. Here we combine the Power, Scalar Multiple, Sum/Difference and Constant Rules. We show quite a few steps, but in general these can be omitted:

$$\begin{aligned}
\lim_{x\to 2} p(x) &= \lim_{x\to 2}\left(3x^2 - 5x + 7\right) \\
&= \lim_{x\to 2}\left(3x^2\right) - \lim_{x\to 2}(5x) + \lim_{x\to 2} 7 \\
&= 3 \cdot 2^2 - 5 \cdot 2 + 7 \\
&= 9
\end{aligned}$$

Part 3 of the previous example demonstrates how the limit of a quadratic polynomial can be determined using the properties of Theorem **??**. Not only that, recognize that

$$\lim_{x\to 2} p(x) = 9 = p(2);$$

i.e., the limit at 2 could have been found just by plugging 2 into the function. This holds true for all polynomials, and also for rational functions (which are quotients of polynomials), as stated in the following theorem.

Theorem 1.3.3 (Limits of Polynomial and Rational Functions)**.** *Let $p(x)$ and $q(x)$ be polynomials and c a real number. Then:*

1. $\lim_{x\to c} p(x) = p(c)$
2. $\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$*, when* $q(c) \neq 0$.

Example 1.3.4 (Finding a limit of a rational function)**.** Using Theorem **??**, find

$$\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3}.$$

Solution. Using Theorem **??**, we can quickly state that

$$\begin{aligned}\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3} &= \frac{3(-1)^2-5(-1)+1}{(-1)^4-(-1)^2+3}\\ &= 3.\end{aligned}$$

It was likely frustrating in Section **??** to do a lot of work with ε and δ to prove that

$$\lim_{x\to 2} x^2 = 4$$

as it seemed fairly obvious. The previous theorems state that many functions behave in such an "obvious" fashion, as demonstrated by the rational function in Example **??**.

Polynomial and rational functions are not the only functions to behave in such a predictable way. The following theorem gives a list of functions whose behavior is particularly "nice" in terms of limits. In the next section, we will give a formal name to these functions that behave "nicely."

Theorem 1.3.5 (Special Limits)**.** *Let c be a real number in the domain of the given function and let n be a positive integer. The following limits hold:*

1. $\lim_{x\to c} \sin(x) = \sin(c)$
2. $\lim_{x\to c} \cos(x) = \cos(c)$
3. $\lim_{x\to c} \tan(x) = \tan(c)$
4. $\lim_{x\to c} \csc(x) = \csc(c)$
5. $\lim_{x\to c} \sec(x) = \sec(c)$
6. $\lim_{x\to c} \cot(x) = \cot(c)$
7. $\lim_{x\to c} a^x = a^c$

   if $a > 0$
8. $\lim_{x\to c} \ln(x) = \ln(c)$
9. $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

   (follows from Identity and Root rules)

Example 1.3.6 (Evaluating limits analytically)**.** Evaluate the following limits.

1. $\lim_{x\to\pi} \cos(x)$
2. $\lim_{x\to 3} \big(\sec^2(x) - \tan^2(x)\big)$
3. $\lim_{x\to\pi/2} (\cos(x)\sin(x))$
4. $\lim_{x\to 1} e^{\ln(x)}$
5. $\lim_{x\to 0} \frac{\sin(x)}{x}$

Solution.

1. This is a straightforward application of Theorem **??**. $\lim_{x\to\pi} \cos(x) = \cos(\pi) = -1$.

2. We can approach this in at least two ways. First, by directly applying Theorem **??**, we have:

$$\lim_{x\to 3} \big(\sec^2(x) - \tan^2(x)\big) = \sec^2(3) - \tan^2(3).$$

   Using the Pythagorean Theorem, this last expression is 1; therefore

$$\lim_{x\to 3} \big(\sec^2(x) - \tan^2(x)\big) = 1.$$

   We can also use the Pythagorean Theorem from the start.

$$\lim_{x\to 3} \big(\sec^2(x) - \tan^2(x)\big) = \lim_{x\to 3} 1 = 1,$$

   using the Constant rule. Either way, we find the limit is 1.

3. Applying the Product rule and Theorem **??** gives

$$\lim_{x\to\pi/2} \cos(x)\sin(x) = \cos(\pi/2)\sin(\pi/2) = 0\cdot 1 = 0.$$

4. Again, we can approach this in two ways. First, we can use the exponential/logarithmic identity that $e^{\ln(x)} = x$ and evaluate $\lim_{x\to 1} e^{\ln(x)} = \lim_{x\to 1} x = 1$.

   We can also use the Composition rule. Using Theorem **??**, we have $\lim_{x\to 1} \ln(x) = \ln(1) = 0$ and $\lim_{x\to 0} e^x = e^0 = 1$, satisfying the conditions of the Composition rule. Applying this rule,

$$\lim_{x\to 1} e^{\ln(x)} = e^{\lim_{x\to 1}\ln(x)} = e^{\ln(1)} = e^0 = 1.$$

   Both approaches are valid, giving the same result.

5. We encountered this limit in Section 1.1. Applying our theorems, we attempt to find the limit as

$$\lim_{x\to 0} \frac{\sin(x)}{x} \to \frac{\sin(0)}{0}$$

   which is of the form $\frac{0}{0}$. This, of course, violates a condition of the Quotient, as the limit of the denominator is not allowed to be 0. Therefore, we are still unable to evaluate this limit with tools we currently have at hand.

The section could have been titled "Using Known Limits to Find Unknown Limits." By knowing certain limits of functions, we can find limits involving sums, products, powers, etc., of these functions. We further the development of such comparative tools with the Squeeze Theorem, a clever and intuitive way to find the value of some limits.

Before stating this theorem formally, suppose we have functions f, g and h where g always takes on values between f and h; that is, for all x in an interval,

$$f(x) \leq g(x) \leq h(x).$$

If f and h have the same limit at c, and g is always "squeezed" between them, then g must have the same limit as well. That is what the Squeeze Theorem states. This is illustrated in Figure **??**.

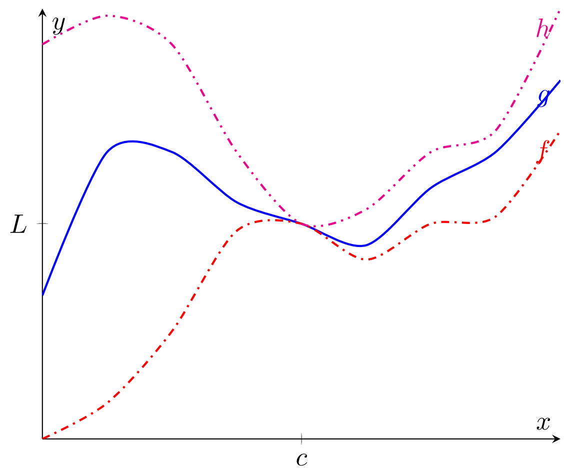


Figure 1.3.7: An illustration of the Squeeze Theorem

Theorem 1.3.8 (Squeeze Theorem)**.** *Let f, g and h be functions on an open interval I containing c such that for all x in I,*

$$f(x) \leq g(x) \leq h(x).$$

If

$$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x),$$

then

$$\lim_{x\to c} g(x) = L.$$

It can take some work to figure out appropriate functions by which to "squeeze" the given function of which you are trying to evaluate a limit. However, that is generally the only place work is necessary; the theorem makes the "evaluating the limit part" very simple.

We use the Squeeze Theorem in the following example to finally prove that $\lim\limits_{x\to 0} \frac{\sin(x)}{x} = 1$.

Example 1.3.9 (Using the Squeeze Theorem)**.** Use the Squeeze Theorem to show that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

Solution. We begin by considering the unit circle. Each point on the unit circle has coordinates $(\cos(\theta), \sin(\theta))$ for some angle θ as shown in Figure **??**.

Using similar triangles, we can extend the line from the origin through the point to the point $(1, \tan(\theta))$, as shown. (Here we are assuming that $0 \le \theta \le \pi/2$. Later we will show that we can also consider $\theta \le 0$.)

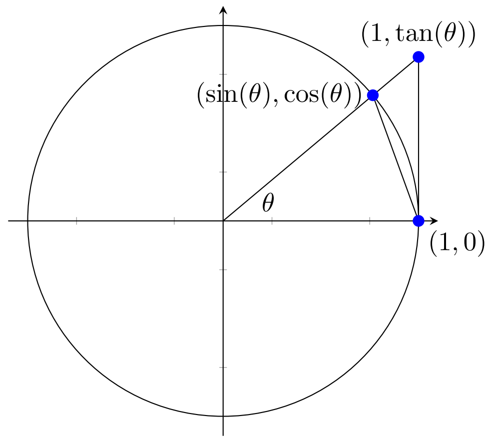


Figure 1.3.10: The unit circle and related triangles.

Figure **??** shows three regions have been constructed in the first quadrant, two triangles and a sector of a circle, which are also drawn below. The area of the large triangle is $\frac{1}{2}\tan(\theta)$; the area of the sector is $\theta/2$; the area of the triangle contained inside the sector is $\frac{1}{2}\sin(\theta)$. It is then clear from Figure **??** that

$$\frac{\tan(\theta)}{2} \ge \frac{\theta}{2} \ge \frac{\sin(\theta)}{2}.$$

(You may need to recall that the area of a sector of a circle is $\frac{1}{2}r^2\theta$ with θ measured in radians.)

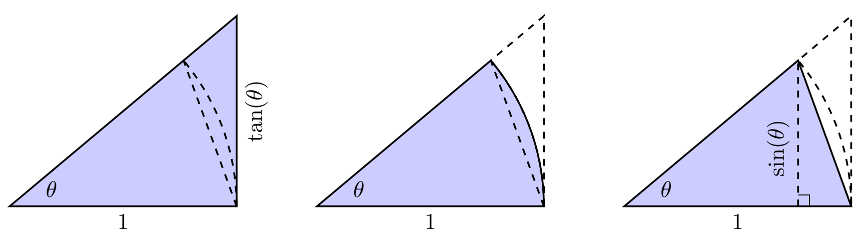


Figure 1.3.11: Bounding the sector between two triangles

Multiply all terms by $\frac{2}{\sin(\theta)}$, giving

$$\frac{1}{\cos(\theta)} \ge \frac{\theta}{\sin(\theta)} \ge 1.$$

Taking reciprocals reverses the inequalities, giving

$$\cos(\theta) \le \frac{\sin(\theta)}{\theta} \le 1.$$

(These inequalities hold for all values of θ near 0, even negative values, since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$.)

Now take limits.

$$\lim_{\theta\to 0} \cos(\theta) \le \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} \le \lim_{\theta\to 0} 1$$

$$\cos(0) \leq \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \leq 1$$
$$1 \leq \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \leq 1$$

Clearly this means that $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$.

Two notes about the Example **??** are worth mentioning. First, one might be discouraged by this application, thinking "I would *never* have come up with that on my own. This is too hard!" Don't be discouraged; within this text we will guide you in your use of the Squeeze Theorem. As one gains mathematical maturity, clever proofs like this are easier and easier to create.

Second, this limit tells us more than just that as x approaches 0, $\sin(x)/x$ approaches 1. Both x and $\sin(x)$ are approaching 0, but the *ratio* of x and $\sin(x)$ approaches 1, meaning that they are approaching 0 in essentially the same way. Another way of viewing this is: for small x, the functions $y = x$ and $y = \sin(x)$ are essentially indistinguishable.

We include this special limit, along with three others, in the following theorem.

Theorem 1.3.12 (Special Limits)**.**

1. $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$
2. $\lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0$
3. $\lim_{x \to 0} (1 + x)^{1/x} = e$
4. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

A short word on how to interpret the latter three limits. We know that as x goes to 0, $\cos(x)$ goes to 1. So, in the second limit, both the numerator and denominator are approaching 0. However, since the limit is 0, we can interpret this as saying that "$\cos(x)$ is approaching 1 faster than x is approaching 0."

In the third limit, inside the parentheses we have an expression that is approaching 1 (though never equaling 1), and we know that 1 raised to any power is still 1. At the same time, the power is growing toward infinity. What happens to a number near 1 raised to a very large power? In this particular case, the result approaches Euler's number, e, approximately 2.718.

In the fourth limit, we see that as $x \to 0$, e^x approaches 1 "just as fast" as $x \to 0$, resulting in a limit of 1.

The special limits stated in Theorem **??** are called *indeterminate forms* , in this case they are of the form 0/0, except the third limit which is of a different form. You'll learn techniques to find these limits exactly using calculus in Section **??**.

Our final theorem for this section will be motivated by the following example.

Example 1.3.13 (Using algebra to evaluate a limit)**.** Evaluate the following limit:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

Solution. We begin by attempting to apply Theorem **??** and substituting 1 for x in the quotient. This gives:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{1^2 - 1}{1 - 1}$$

which is of the form $\frac{0}{0}$, an indeterminate form. We cannot apply the theorem.

By graphing the function, as in Figure **??**, we see that the function seems to be linear, implying that the limit should be easy to evaluate. Recognize that the numerator of our quotient can be factored:

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}.$$

The function is not defined when $x = 1$, but for all other x,

$$\begin{aligned}\frac{x^2-1}{x-1} &= \frac{(x-1)(x+1)}{x-1}\\ &= \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}\\ &= x+1, \quad \text{if } x \neq 1\end{aligned}$$

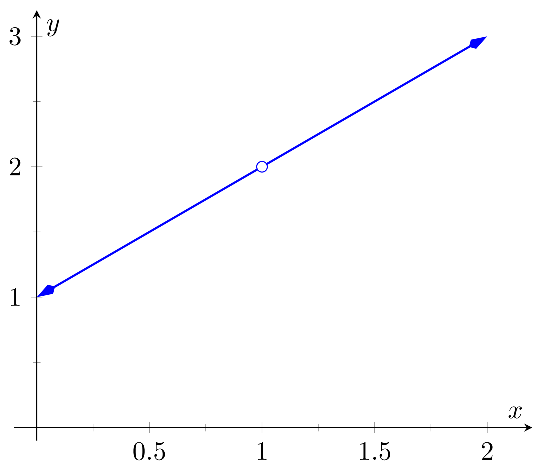


Figure 1.3.14: Graphing f in Example **??** to understand a limit.

Clearly $\lim\limits_{x\to 1}(x+1) = 2$. Recall that when considering limits, we are not concerned with the value of the function at 1, only the value the function approaches as x approaches 1. Since $(x^2-1)/(x-1)$ and $x+1$ are the same at all points except at $x = 1$, they both approach the same value as x approaches 1. Therefore we can conclude that

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-1}{x-1} &= \lim_{x\to 1}(x+1)\\ &= 2\end{aligned}$$

The key to Example **??** is that the functions $y = (x^2-1)/(x-1)$ and $y = x+1$ are identical except at $x = 1$. Since limits describe a value the function is approaching, not the value the function actually attains, the limits of the two functions are always equal.

Theorem 1.3.15 (Limits of Functions Equal At All But One Point)**.** *Let $g(x) = f(x)$ for all x in an open interval, except possibly at c, and let $\lim\limits_{x\to c} g(x) = L$ for some real number L. Then*

$$\lim_{x\to c} f(x) = L.$$

The Fundamental Theorem of Algebra tells us that when dealing with a rational function of the form $g(x)/f(x)$ and directly evaluating the limit $\lim_{x\to c} \frac{g(x)}{f(x)}$ returns "0/0", then $(x-c)$ is a factor of both $g(x)$ and $f(x)$. One can then use algebra to factor this binomial out, cancel, then apply Theorem **??**. We demonstrate this once more.

Example 1.3.16 (Evaluating a Limit with a Hole)**.** Evaluate

$$\lim_{x\to 3} \frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15}.$$

Solution. We begin by applying Theorem **??** and substituting 3 for x. This returns the familiar indeterminate form of "0/0". Since the numerator and denominator are each polynomials, we know that $(x-3)$ is factor of each. Using whatever method is most comfortable to you, factor out $(x-3)$ from each (using polynomial division, synthetic division, a computer algebra system, etc.). We find that

$$\frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15} = \frac{(x-3)\left(x^2+x-2\right)}{(x-3)\left(2x^2+9x-5\right)}.$$

We can cancel the $(x-3)$ factors as long as $x \neq 3$. Using Theorem **??** we conclude:

$$\begin{aligned}
\lim_{x\to 3} \frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15} &= \lim_{x\to 3} \frac{(x-3)\left(x^2+x-2\right)}{(x-3)\left(2x^2+9x-5\right)} \\
&= \lim_{x\to 3} \frac{x^2+x-2}{2x^2+9x-5} \\
&= \frac{10}{40} \\
&= \frac{1}{4}.
\end{aligned}$$

Example 1.3.17 (Evaluating a Limit with a Hole)**.** Evaluate

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}.$$

Solution. We begin by trying to apply the Quotient limit rule, but the denominator evaluates to zero. In fact, this limit is of the indeterminate form 0/0. We will do some algebra to resolve the indeterminate form. In this case, we multiply the numerator and denominator by the conjugate of the numerator.

$$\begin{aligned}
\frac{\sqrt{x}-3}{x-9} &= \frac{\sqrt{x}-3}{x-9} \cdot \frac{(\sqrt{x}+3)}{(\sqrt{x}+3)} \\
&= \frac{x-9}{(x-9)(\sqrt{x}+3)}
\end{aligned}$$

We can cancel the $(x-9)$ factors as long as $x \neq 9$. Using Theorem **??** we conclude:

$$\begin{aligned}
\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} &= \lim_{x\to 9} \frac{x-9}{(x-9)\left(\sqrt{x}+3\right)} \\
&= \lim_{x\to 9} \frac{1}{\sqrt{x}+3}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\lim_{x\to 9}\sqrt{x} + \lim_{x\to 9} 3} \\
&= \frac{1}{\sqrt{\lim_{x\to 9} x} + 3} \\
&= \frac{1}{\sqrt{3+3}} \\
&= \frac{1}{6}.
\end{aligned}
$$

We end this section by revisiting a limit first seen in Section 1.1, a limit of a difference quotient. Let $f(x) = -1.5x^2 + 11.5x$; we approximated the limit $\lim\limits_{h\to 0} \frac{f(1+h)-f(1)}{h} \approx 8.5$. We formally evaluate this limit in the following example.

Example 1.3.18 (Evaluating the limit of a difference quotient)**.** Let $f(x) = -1.5x^2 + 11.5x$; find $\lim\limits_{h\to 0} \frac{f(1+h)-f(1)}{h}$.

Solution. Since f is a polynomial, our first attempt should be to employ Theorem **??** and substitute 0 for h. However, we see that this gives us "0/0." Knowing that we have a rational function hints that some algebra will help. Consider the following steps:

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0} \frac{-1.5(1+h)^2 + 11.5(1+h) - \big(-1.5(1)^2 + 11.5(1)\big)}{h} \\
&= \lim_{h\to 0} \frac{-1.5(1+2h+h^2) + 11.5 + 11.5h - 10}{h} \\
&= \lim_{h\to 0} \frac{-1.5h^2 + 8.5h}{h} \\
&= \lim_{h\to 0} \frac{h(-1.5h + 8.5)}{h} \\
&= \lim_{h\to 0} (-1.5h + 8.5) \quad \text{(using Theorem ??, as } h \neq 0) \\
&= 8.5 \quad \text{(using Theorem ??)}
\end{aligned}
$$

∎

This matches our previous approximation.

This section contains several valuable tools for evaluating limits. One of the main results of this section is Theorem **??**; it states that many functions that we use regularly behave in a very nice, predictable way. In the next section we give a name to this nice behavior; we label such functions as **continuous.** Defining that term will require us to look again at what a limit is and what causes limits to not exist.

1.3.1 Exercises

Terms and Concepts

1. (Limit of a constant) Explain in your own words, without using ε–δ formality, why $\lim\limits_{x\to c} b = b$.

[Essay Answer]

Solution. Answers will vary.

2. (Limit of x) Explain in your own words, without using ε–δ formality, why $\lim_{x\to c} x = c$.

[Essay Answer]

Solution. Answers will vary.

3. (Explain "nice") What does the text mean when it says that certain functions' "behavior is 'nice' in terms of limits"? What, in particular, is "nice"?

[Essay Answer]

Solution. Answers will vary.

4. Sketch a graph that visually demonstrates the Squeeze Theorem.

5. (Limit of 0/0) You are given the following information:

(a) $\lim_{x\to 1} f(x) = 0$

(b) $\lim_{x\to 1} g(x) = 0$

(c) $\lim_{x\to 1} \frac{f(x)}{g(x)} = 2$

What can be said about the relative sizes of $f(x)$ and $g(x)$ as x approaches 1?

[Essay Answer]

Solution. As x is near 1, both f and g are near 0, but f is approximately twice the size of g. (That is, $f(x) \approx 2g(x)$.)

In the following exercises, use the following information to evaluate the given limit, when possible. If it is not possible to determine the limit, state why not.

$$\lim_{x\to 9} f(x) = 6 \qquad \lim_{x\to 6} f(x) = 9 \qquad f(9) = 6$$
$$\lim_{x\to 9} g(x) = 3 \qquad \lim_{x\to 6} g(x) = 3 \qquad g(6) = 9$$

6. (A sum) If possible, use the following information to evaluate the limit below.

$$\lim_{x\to 9} f(x) = 6 \qquad \lim_{x\to 6} f(x) = 9 \qquad f(9) = 6$$
$$\lim_{x\to 9} g(x) = 3 \qquad \lim_{x\to 6} g(x) = 3 \qquad g(6) = 9$$

If the limit doesn't exist, you may enter `DNE`. If it is not possible to know whether or not it exists, you may enter `NPK`.

$\lim_{x\to 9}(f(x) + g(x)) =$ $\boxed{}$

Solution.

7. (A quotient) If possible, use the following information to evaluate the limit below.

$$\lim_{x\to 9} f(x) = 6 \qquad \lim_{x\to 6} f(x) = 9 \qquad f(9) = 6$$
$$\lim_{x\to 9} g(x) = 3 \qquad \lim_{x\to 6} g(x) = 3 \qquad g(6) = 9$$

If the limit doesn't exist, you may enter `DNE`. If it is not possible to know whether or not it exists, you may enter `NPK`.

$\lim_{x\to 9} \left(\frac{3f(x)}{g(x)}\right) = \boxed{}$

Solution.

8. (A difference in the numerator) If possible, use the following information to evaluate the limit below.

$$\lim_{x\to 9} f(x) = 6 \qquad \lim_{x\to 6} f(x) = 9 \qquad f(9) = 6$$
$$\lim_{x\to 9} g(x) = 3 \qquad \lim_{x\to 6} g(x) = 3 \qquad g(6) = 9$$

If the limit doesn't exist, you may enter `DNE`. If it is not possible to know whether or not it exists, you may enter `NPK`.

$\lim_{x\to 9} \left(\frac{f(x)-2g(x)}{g(x)}\right) = \boxed{}$

Solution.

9. (A difference in the denominator) If possible, use the following information to evaluate the limit below.

$$\lim_{x\to 9} f(x) = 6 \qquad \lim_{x\to 6} f(x) = 9 \qquad f(9) = 6$$
$$\lim_{x\to 9} g(x) = 3 \qquad \lim_{x\to 6} g(x) = 3 \qquad g(6) = 9$$

If the limit doesn't exist, you may enter `DNE`. If it is not possible to know whether or not it exists, you may enter `NPK`.

$\lim_{x\to 6} \left(\frac{f(x)}{3-g(x)}\right) = \boxed{}$

Solution.

10. (A composition) If possible, use the following information to evaluate the limit below.

$$\lim_{x\to 9} f(x) = 6 \qquad \lim_{x\to 6} f(x) = 9 \qquad f(9) = 6$$
$$\lim_{x\to 9} g(x) = 3 \qquad \lim_{x\to 6} g(x) = 3 \qquad g(6) = 9$$

If the limit doesn't exist, you may enter `DNE`. If it is not possible to know whether or not it exists, you may enter `NPK`.

$\lim_{x\to 6} g(f(x)) =$ $\boxed{}$

Solution.

11. (A composition) If possible, use the following information to evaluate the limit below.

$$\lim_{x\to 9} f(x) = 6 \qquad \lim_{x\to 6} f(x) = 9 \qquad f(9) = 6$$
$$\lim_{x\to 9} g(x) = 3 \qquad \lim_{x\to 6} g(x) = 3 \qquad g(6) = 9$$

If the limit doesn't exist, you may enter `DNE`. If it is not possible to know whether or not it exists, you may enter `NPK`.

$\lim_{x\to 6} f(g(x)) =$ $\boxed{}$

Solution.

12. (A triple composition) If possible, use the following information to evaluate the limit below.

$$\lim_{x\to 9} f(x) = 6 \qquad \lim_{x\to 6} f(x) = 9 \qquad f(9) = 6$$
$$\lim_{x\to 9} g(x) = 3 \qquad \lim_{x\to 6} g(x) = 3 \qquad g(6) = 9$$

If the limit doesn't exist, you may enter `DNE`. If it is not possible to know whether or not it exists, you may enter `NPK`.

$\lim_{x\to 6} g(f(f(x))) =$ $\boxed{}$

Solution.

13. (Products and squares) If possible, use the following information to evaluate the limit below.

$$\lim_{x\to 9} f(x) = 6 \qquad \lim_{x\to 6} f(x) = 9 \qquad f(9) = 6$$
$$\lim_{x\to 9} g(x) = 3 \qquad \lim_{x\to 6} g(x) = 3 \qquad g(6) = 9$$

If the limit doesn't exist, you may enter `DNE`. If it is not possible to know whether or not it exists, you may enter `NPK`.

$\lim_{x\to 6} \left(f(x)g(x) - f(x)^2 + g(x)^2\right) = \boxed{}$

Solution.

In the following exercises, use the following information to evaluate the given limit, when possible. If it is not possible to determine the limit, state why not.

$$\lim_{x\to 1} f(x) = 2 \qquad \lim_{x\to 10} f(x) = 1 \qquad f(1) = 1/5$$
$$\lim_{x\to 1} g(x) = 0 \qquad \lim_{x\to 10} g(x) = \pi \qquad g(10) = \pi$$

14. (f to the g) If possible, use the following information to evaluate the limit below.

$$\lim_{x\to 1} f(x) = 2 \qquad \lim_{x\to 10} f(x) = 1 \qquad f(1) = 1/5$$
$$\lim_{x\to 1} g(x) = 0 \qquad \lim_{x\to 10} g(x) = \pi \qquad g(10) = \pi$$

If the limit doesn't exist, you may enter `DNE`. If it is not possible to know whether or not it exists, you may enter `NPK`.

$\lim_{x\to 1} f(x)^{g(x)} = \boxed{}$

Solution.

15. (cosine of g) If possible, use the following information to evaluate the limit below.

$$\lim_{x\to 1} f(x) = 2 \qquad \lim_{x\to 10} f(x) = 1 \qquad f(1) = 1/5$$
$$\lim_{x\to 1} g(x) = 0 \qquad \lim_{x\to 10} g(x) = \pi \qquad g(10) = \pi$$

If the limit doesn't exist, you may enter `DNE`. If it is not possible to know whether or not it exists, you may enter `NPK`.

$\lim_{x\to 10} \cos(g(x)) = \boxed{}$

Solution.

16. (f times g) If possible, use the following information to evaluate the limit below.

$$\lim_{x\to 1} f(x) = 2 \qquad \lim_{x\to 10} f(x) = 1 \qquad f(1) = 1/5$$
$$\lim_{x\to 1} g(x) = 0 \qquad \lim_{x\to 10} g(x) = \pi \qquad g(10) = \pi$$

If the limit doesn't exist, you may enter `DNE`. If it is not possible to know whether or not it exists, you may enter `NPK`.

$\lim_{x\to 1} (f(x)g(x)) =$ ______

Solution.

17. (A composition) If possible, use the following information to evaluate the limit below.

$$\lim_{x\to 1} f(x) = 2 \qquad \lim_{x\to 10} f(x) = 1 \qquad f(1) = 1/5$$
$$\lim_{x\to 1} g(x) = 0 \qquad \lim_{x\to 10} g(x) = \pi \qquad g(10) = \pi$$

If the limit doesn't exist, you may enter `DNE`. If it is not possible to know whether or not it exists, you may enter `NPK`.

$\lim_{x\to 1} g(5f(x)) =$ ______

Solution.

In the following exercises, use limit theorems to evaluate the given limit.

18. (Polynomial) Use limit theorems to evaluate the limit.

$\lim_{x\to 3} \left(x^2 - 3x + 7\right) =$ ______

Solution.

$$\begin{aligned}
\lim_{x\to 3} \left(x^2 - 3x + 7\right) &= \lim_{x\to 3} x^2 + \lim_{x\to 3} -3x + \lim_{x\to 3} 7 \\
&= \left(\lim_{x\to 3} x\right)^2 + \lim_{x\to 3} -3x + \lim_{x\to 3} 3 \\
&= (3)^2 + -3\,(3) + 7 \\
&= 7
\end{aligned}$$

19. (Rational and power) Use limit theorems to evaluate the limit.

$\lim\limits_{x\to\pi} \left(\frac{x-3}{x-5}\right)^7$

Solution.

$$\begin{aligned}
\lim_{x\to\pi}\left(\frac{x-3}{x-5}\right)^7 &= \left(\lim_{x\to\pi}\frac{x-3}{x-5}\right)^7 \\
&= \left(\frac{\lim\limits_{x\to\pi}(x-3)}{\lim\limits_{x\to\pi}(x-5)}\right)^7 \\
&= \left(\frac{\lim\limits_{x\to\pi}x-\lim\limits_{x\to\pi}3}{\lim\limits_{x\to\pi}x-\lim\limits_{x\to\pi}5}\right)^7 \\
&= \left(\frac{\pi-3}{\pi-5}\right)^7
\end{aligned}$$

20. (Trig) Use limit theorems to evaluate the limit.

$\lim\limits_{x\to\pi/4} \cos(x)\sin(x)$

Solution.

$$\begin{aligned}
\lim_{x\to\pi/4}\cos(x)\sin(x) &= \lim_{x\to\pi/4}\cos(x)\cdot\lim_{x\to\pi/4}\sin(x) \\
&= \cos(\pi/4)\cdot\sin(\pi/4) \\
&= \sqrt{2}/2\cdot\sqrt{2}/2 \\
&= \sqrt{3}/4
\end{aligned}$$

21. (log) Use limit theorems to evaluate the limit.

$\lim\limits_{x\to 0} \ln(x)$

Solution. This limit does not exist for at least two reasons. For one reason, $\ln(x)$ can be arbitrarily large negative if you allow x to be positive and close enough to 0. Also, since $\ln(x)$ is undefined for negative x and $\lim\limits_{x\to 0}\ln(x)$ is an expression that depends on outputs of ln using inputs *both* slightly smaller *and* slightly larger than 0, we cannot hope to give meaning to $\lim\limits_{x\to 0}\ln(x)$.

22. (Exponential) Use limit theorems to evaluate the limit.

$\lim_{x \to 3} 4^{x^3 - 8x}$

Solution.

$$\begin{aligned}\lim_{x\to 3} 4^{x^3-8x} &= 4^{\lim_{x\to 3}\left(x^3-8x\right)}\\ &= 4^{\lim_{x\to 3} x^3 - \lim_{x\to 3} 8x}\\ &= 4^{\left(\lim_{x\to 3} x\right)^3 - 8\lim_{x\to 3} x}\\ &= 4^{3^3-8\cdot 3}\\ &= 64\end{aligned}$$

23. (csc) Use limit theorems to evaluate the limit.

$\lim_{x \to \pi/6} \csc(x)$

Solution.

$$\begin{aligned}\lim_{x\to \pi/6} \csc(x) &= \csc(\pi/6)\\ &= 2\end{aligned}$$

24. (log) Use limit theorems to evaluate the limit.

$\lim_{x \to 0} \ln(1 + x)$

Solution.

25. (Rational function) Use limit theorems to evaluate the limit.

$\lim_{x\to\pi} \frac{x^2+3x+5}{5x^2-2x-2}$

Solution.

$$\begin{aligned}
\lim_{x\to\pi} \frac{x^2+3x+5}{5x^2-2x-2} &= \frac{\lim\limits_{x\to\pi} \left(x^2+3x+5\right)}{\lim\limits_{x\to\pi} (5x^2-2x-2)} \\
&= \frac{\lim\limits_{x\to\pi} x^2 + \lim\limits_{x\to\pi} (3x) + \lim\limits_{x\to\pi} (5)}{\lim\limits_{x\to\pi} (5x^2) + \lim\limits_{x\to\pi} (-2x) + \lim\limits_{x\to\pi} -2} \\
&= \frac{\lim\limits_{x\to\pi} x^2 + 3\lim\limits_{x\to\pi} x + \lim\limits_{x\to\pi} (5)}{5\lim\limits_{x\to\pi} x^2 - 2\lim\limits_{x\to\pi} x - \lim\limits_{x\to\pi} 2} \\
&= \frac{\pi^2+3\pi+5}{5\pi^2-2\pi-3}
\end{aligned}$$

26. (Rational function) Use limit theorems to evaluate the limit.

$\lim_{x\to\pi} \frac{3x+1}{1-x}$

Solution.

27. (Rational function) Use limit theorems to evaluate the limit.

$\lim_{x\to 6} \frac{x^2-4x-12}{x^2-13x+42}$

28. (Rational function) Use limit theorems to evaluate the limit.

$\lim_{x\to 0} \frac{x^2+2x}{x^2-2x}$

Solution.

29. (Rational function) Use limit theorems to evaluate the limit.

$\lim_{x\to 2} \frac{x^2+6x-16}{x^2-3x+2}$

30. (Rational function) Use limit theorems to evaluate the limit.

$\lim_{x\to 2} \frac{x^2-10x+16}{x^2-x-2}$

31. (Rational function) Use limit theorems to evaluate the limit.

$\lim_{x \to -2} \frac{x^2-5x-14}{x^2+10x+16}$

32. (Rational function) Use limit theorems to evaluate the limit.

$\lim_{x \to -1} \frac{x^2+9x+8}{x^2-6x-7}$

Use the Squeeze Theorem in the following exercises, where appropriate, to evaluate the given limit.

33. (sin) Use the Squeeze Theorem to evaluate the limit.

$\lim_{x \to 0} \left(x \sin\left(\frac{1}{x}\right)\right)$

[Essay Answer]

Solution.

34. (sin and cos) Use the Squeeze Theorem to evaluate the limit.

$\lim_{x \to 0} \left(\sin(x) \cos\left(\frac{1}{x^2}\right)\right)$

[Essay Answer]

Solution.

35. (Inequalities) Use the Squeeze Theorem to evaluate the limit.

$\lim_{x \to 1} f(x)$, where $3x - 2 \le f(x) \le x^3$

[Essay Answer]

Solution.

36. (Inequalities) Use the Squeeze Theorem to evaluate the limit.

$\lim_{x \to 3} f(x)$, where $6x - 9 \le f(x) \le x^2$

[Essay Answer]

Solution.

The following exercises challenge your understanding of limits but can be evaluated using the knowledge gained in Section **??**.

37. (sin) $\lim_{x \to 0} \frac{\sin(3x)}{x} =$

Solution.

38. (sin) $\lim\limits_{x\to 0} \frac{\sin(5x)}{8x} =$ $\boxed{}$

Solution.

39. (log) $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} =$ $\boxed{}$

Solution.

40. (sin) $\lim\limits_{x\to 0} \frac{\sin(x)}{x}$, where x is measured in degrees, not radians.

$\boxed{}$

Solution.

41. (Composition) Let $f(x) = 0$ and $g(x) = \frac{x}{x}$.

(a) Explain why $\lim\limits_{x\to 2} f(x) = 0$.
[Essay Answer]

(b) Explain why $\lim\limits_{x\to 0} g(x) = 1$.
[Essay Answer]

(c) Explain why $\lim_{x\to 2} g(f(x))$ does not exist.
[Essay Answer]

(d) Explain why the previous statement does not violate the Composition Rule of Theorem **??**.
[Essay Answer]

Solution.

(a) Apply Part 1 of Theorem **??**.

(b) Apply Theorem **??**; $g(x) = \frac{x}{x}$ is the same as $g(x) = 1$ everywhere except at $x = 0$. Thus $\lim\limits_{x\to 0} g(x) = \lim_{x\to 0} 1 = 1$.

(c) The function f always gives output 0, so $g(f(x))$ is never defined as g is not defined for an input of 0. Therefore the limit does not exist.

(d) The Composition Rule requires that $\lim\limits_{x\to 0} g(x)$ be equal to $g(0)$. They are not equal, so the conditions of the Composition Rule are not satisfied, and hence the rule is not violated.

1.4 One-Sided Limits

We introduced the concept of a limit gently, approximating their values graphically and numerically. Next came the rigorous definition of the limit, along with an admittedly tedious method for evaluating them. Section **??** gave us tools (which we call theorems) that allow us to compute limits with greater ease. Chief among the results were the facts that polynomials and rational, trigonometric, exponential and logarithmic functions (and their sums, products, etc.) all behave "nicely." In this section we rigorously define what we mean by "nicely."

In Section 1.1 we explored the three ways in which limits of functions failed to exist:

1. The function approached different values from the left and right.
2. The function grows without bound.
3. The function oscillates.

In this section we explore in depth the concepts behind 1 by introducing the *one-sided limit*. We begin with formal definitions that are very similar to the definition of the limit given in Section **??**, but the notation is slightly different and "$x \neq c$" is replaced with either "$x < c$" or "$x > c$."

Definition 1.4.1 (One-Sided Limits)**.** There is a slighlty different definition for a left-hand limit, than for a right-hand limit, but both have a lot in common with **??**.

Left-Hand Limit Let $I = (a, c)$ be an open interval, and let f be a function defined on I. The statement that the **limit of** $f(x)$**, as** x **approaches** c **from the left, is** L, (alternatively, that **the left-hand limit of** f **at** c **is** L) is denoted by

$$\lim_{x \to c^-} f(x) = L,$$

and means that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in I$, $x < c$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Right-Hand Limit Let $I = (c, b)$ be an open interval, and let f be a function defined on I. The statement that the **limit of** $f(x)$**, as** x **approaches** c **from the right, is** L, (alternatively, that **the right-hand limit of** f **at** c **is** L) is denoted by

$$\lim_{x \to c^+} f(x) = L,$$

and means that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in I$, $x > c$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Practically speaking, when evaluating a left-hand limit, we consider only values of x "to the left of c," i.e., where $x < c$. The admittedly imperfect notation $x \to c^-$ is used to imply that we look at values of x to the left of c. The notation has nothing to do with positive or negative values of either x or c. It's more like you are adding very small negative values to c to get values for x. A similar statement holds for evaluating right-hand limits; there we consider only values of x to the right of c, i.e., $x > c$. We can use the theorems from previous sections to help us evaluate these limits; we just restrict our view to one side of c.

We practice evaluating left and right-hand limits through a series of examples.

Example 1.4.2 (Evaluating one sided limits)**.** Let $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 3 - x & 1 < x < 2 \end{cases}$, as shown in Figure **??**. Find each of the following:

1. $\lim_{x \to 1^-} f(x)$
2. $\lim_{x \to 1^+} f(x)$
3. $\lim_{x \to 1} f(x)$
4. $f(1)$
5. $\lim_{x \to 0^+} f(x)$
6. $f(0)$

7. $\lim_{x\to 2^-} f(x)$

8. $f(2)$

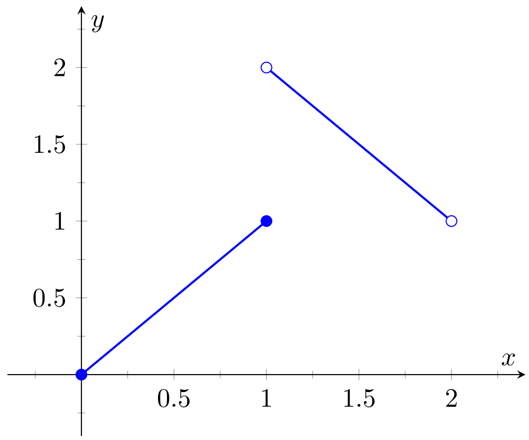


Figure 1.4.3: A graph of f in Example **??**.

Solution. For these problems, the visual aid of the graph is likely more effective in evaluating the limits than using f itself. Therefore we will refer often to the graph.

1. As x goes to 1 *from the left*, we see that $f(x)$ is approaching the value of 1. Therefore $\lim_{x\to 1^-} f(x) = 1$.

   We could of course evaluate this limit symbolically by examining the defintion of f. Since we are approaching 1 from the left, we would choose the first "piece" of the function, that is the one with domain $0 \leq x \leq 1$. So $\lim_{x\to 1^-} x = 1$.

2. As x goes to 1 *from the right*, we see that $f(x)$ is approaching the value of 2. Recall that it does not matter that there is an "open circle" there; we are evaluating a limit, not the value of the function. Therefore $\lim_{x\to 1^+} f(x) = 2$.

3. *The* limit of f as x approaches 1 does not exist, as discussed in Section 1.1. The function does not approach one particular value, but two different values from the left and the right.

4. Using the definition and by looking at the graph we see that $f(1) = 1$.

5. As x goes to 0 from the right, we see that $f(x)$ is approaching 0. Therefore $\lim_{x\to 0^+} f(x) = 0$. Note we cannot consider a left-hand limit at 0 as f is not defined for values of $x < 0$.

6. Using the definition and the graph, $f(0) = 0$.

7. As x goes to 2 from the left, we see that $f(x)$ is approaching the value of 1. Therefore $\lim_{x\to 2^-} f(x) = 1$.

8. The graph and the definition of the function show that $f(2)$ is not defined.

Note how the left- and right-hand limits were different at $x = 1$. This, of course, causes *the* limit to not exist. The following theorem states what is fairly intuitive: *the* limit exists precisely when the left- and right-hand limits are equal.

Theorem 1.4.4 (Limits and One-Sided Limits). *Let f be a function defined on an open interval I containing c. Then*

$$\lim_{x\to c} f(x) = L$$

if, and only if,

$$\lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^+} f(x) = L.$$

The phrase "if, and only if" means the two statements are *equivalent*: they are either both true or both false. If the limit equals L, then the left and right hand limits both equal L. If the limit is not equal to L, then at least one of the left and right-hand limits is not equal to L (it may not even exist).

One thing to consider in Examples **??**–**??** is that the value of the function may/may not be equal to the value(s) of its left/right-hand limits, even when these limits agree.

Example 1.4.5 (Evaluating limits of a piecewise-defined function). Let $f(x) = \begin{cases} 2-x & 0<x<1 \\ (x-2)^2 & 1<x<2 \end{cases}$ as shown in Figure **??**. Evaluate the following:

1. $\lim_{x\to 1^-} f(x)$
2. $\lim_{x\to 1^+} f(x)$
3. $\lim_{x\to 1} f(x)$
4. $f(1)$
5. $\lim_{x\to 0^+} f(x)$
6. $f(0)$
7. $\lim_{x\to 2^-} f(x)$
8. $f(2)$

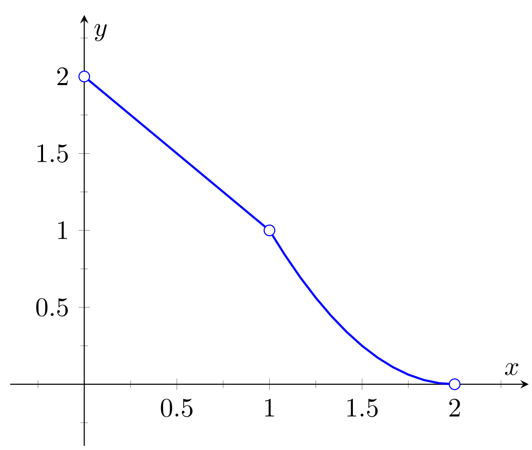


Figure 1.4.6: A graph of f from Example **??**

Solution. Again we will evaluate each using both the definition of f and its graph.

1. As x approaches 1 from the left, we see that $f(x)$ approaches 1. Therefore $\lim\limits_{x\to 1^-} f(x) = 1$.
2. As x approaches 1 from the right, we see that again $f(x)$ approaches 1. Therefore $\lim\limits_{x\to 1+} f(x) = 1$.
3. *The* limit of f as x approaches 1 exists and is 1, as f approaches 1 from both the right and left. Therefore $\lim\limits_{x\to 1} f(x) = 1$.
4. $f(1)$ is not defined. Note that 1 is not in the domain of f as defined by the problem, which is indicated on the graph by an open circle when $x = 1$.
5. As x goes to 0 from the right, $f(x)$ approaches 2. So $\lim\limits_{x\to 0^+} f(x) = 2$.
6. $f(0)$ is not defined as 0 is not in the domain of f.
7. As x goes to 2 from the left, $f(x)$ approaches 0. So $\lim\limits_{x\to 2^-} f(x) = 0$.
8. $f(2)$ is not defined as 2 is not in the domain of f.

Example 1.4.7 (Evaluating limits of a piecewise-defined function)**.** Let $f(x) = \begin{cases} (x-1)^2 & 0 \le x \le 2, x \ne 1 \\ 1 & x = 1 \end{cases}$ as shown in Figure **??**. Evaluate the following:

1. $\lim\limits_{x\to 1^-} f(x)$
2. $\lim\limits_{x\to 1^+} f(x)$
3. $\lim\limits_{x\to 1} f(x)$
4. $f(1)$

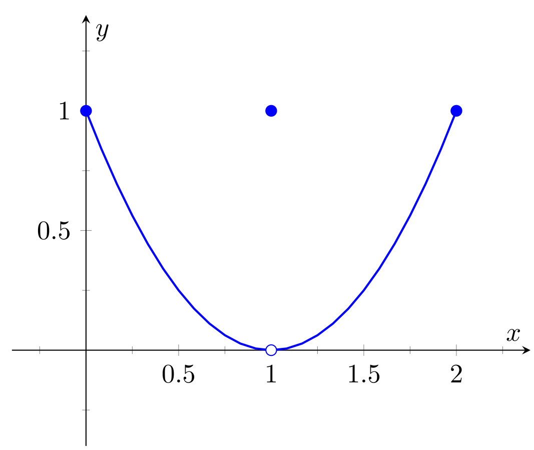


Figure 1.4.8: Graphing f in Example **??**

Solution. It is clear by looking at the graph that both the left- and right-hand limits of f, as x approaches 1, is 0. Thus it is also clear that *the* limit is 0; i.e., $\lim\limits_{x\to 1} f(x) = 0$. It is also clearly stated that $f(1) = 1$.

Example 1.4.9 (Evaluating limits of a piecewise-defined function)**.** Let $f(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ 2-x & 1 < x \le 2 \end{cases}$ as shown in Figure **??**. Evaluate the following:

1. $\lim_{x\to 1^-} f(x)$
2. $\lim_{x\to 1^+} f(x)$
3. $\lim_{x\to 1} f(x)$
4. $f(1)$

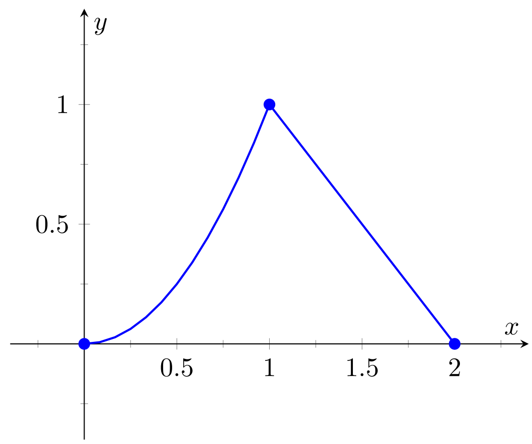


Figure 1.4.10: Graphing f in Example **??**

Solution. It is clear from the definition of the function and its graph that all of the following are equal:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1) = 1.$$

In Examples **??**–**??** we were asked to find both $\lim_{x\to 1} f(x)$ and $f(1)$. Consider the following table:

	$\lim_{x\to 1} f(x)$	$f(1)$
Example **??**	does not exist	1
Example **??**	1	not defined
Example **??**	0	1
Example **??**	1	1

Only in Example **??** do both the function and the limit exist and agree. This seems "nice;" in fact, it seems "normal." This is in fact an important situation which we explore in Section **??** entitled "Continuity." In short, a **continuous function** is one in which when a function approaches a value as $x \to c$ (i.e., when $\lim_{x\to c} f(x) = L$), it actually *attains* that value at c. Such functions behave nicely as they are very predictable.

1.4.1 Exercises

Terms and Concepts

1. What are the three ways in which a limit may fail to exist?

[Essay Answer]

Solution. The function approaches different values from the left and right; the function grows without bound; the function oscillates.

2. True or False? If $\lim\limits_{x\to 1^-} f(x) = 5$, then $\lim\limits_{x\to 1} f(x) = 5$. (Choose one: True / False)

Solution. False

3. True or False? If $\lim\limits_{x\to 1^-} f(x) = 5$, then $\lim\limits_{x\to 1^+} f(x) = 5$. (Choose one: True / False)

Solution. False

4. True or False? If $\lim\limits_{x\to 1} f(x) = 5$, then $\lim\limits_{x\to 1^-} f(x) = 5$. (Choose one: True / False)

Solution. True

In the following exercises, evaluate each expression using the given graph of $f(x)$.

5. Evaluate each expression using the given graph of f.

webwork-79-image-1.png

(a) $\lim\limits_{x\to 1^-} f(x) =$ ______

(b) $\lim\limits_{x\to 1^+} f(x) =$ ______

(c) $\lim\limits_{x\to 1} f(x) =$ ______

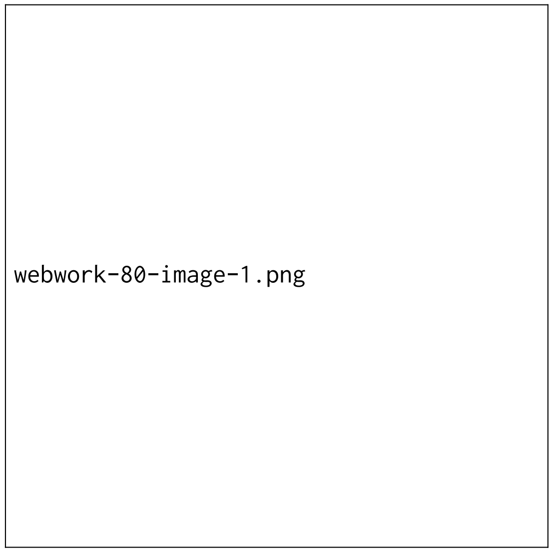
(d) $f(1) =$ ______

(e) $\lim\limits_{x\to 0^-} f(x) =$ ______

(f) $\lim\limits_{x\to 0^+} f(x) =$ ______

Solution.

6. Evaluate each expression using the given graph of f.



(a) $\lim_{x\to 1^-} f(x) =$ ______

(b) $\lim_{x\to 1^+} f(x) =$ ______

(c) $\lim_{x\to 1} f(x) =$ ______

(d) $f(1) =$ ______

(e) $\lim_{x\to 2^-} f(x) =$ ______

(f) $\lim_{x\to 2^+} f(x) =$ ______

Solution.

7. Evaluate each expression using the given graph of f.

webwork-81-image-1.png

(a) $\lim_{x\to 1^-} f(x) =$ ____

(b) $\lim_{x\to 1^+} f(x) =$ ____

(c) $\lim_{x\to 1} f(x) =$ ____

(d) $f(1) =$ ____

(e) $\lim_{x\to 2^-} f(x) =$ ____

(f) $\lim_{x\to 0^+} f(x) =$ ____

Solution.

8. Evaluate each expression using the given graph of f.

webwork-82-image-1.png

(a) $\lim_{x\to 1^-} f(x) =$ ______

(b) $\lim_{x\to 1^+} f(x) =$ ______

(c) $\lim_{x\to 1} f(x) =$ ______

(d) $f(1) =$ ______

Solution.

9. Evaluate each expression using the given graph of f.

webwork-83-image-1.png

(a) $\lim\limits_{x \to 1^-} f(x) =$ ______

(b) $\lim\limits_{x \to 1^+} f(x) =$ ______

(c) $\lim\limits_{x \to 1} f(x) =$ ______

(d) $f(1) =$ ______

Solution.

10. Evaluate each expression using the given graph of f.

webwork-84-image-1.png

(a) $\lim\limits_{x\to 1^-} f(x) =$ ______ [redacted]

(c) [redacted] $\lim\limits_{x\to 1} f(x) =$ ______ [redacted]

(b) $\lim\limits_{x\to 1^+} f(x) =$ ______ [redacted]

(d) [redacted] $f(1) =$ ______ [redacted]

Solution.

11. Evaluate each expression using the given graph of f.

webwork-85-image-1.png

(a) $\lim_{x\to -2^-} f(x) =$

(b) $\lim_{x\to -2^+} f(x) =$

(c) $\lim_{x\to -2} f(x) =$

(d) $f(-2) =$

(e) $\lim_{x\to 2^-} f(x) =$

(f) $\lim_{x\to 2^+} f(x) =$

(g) $\lim_{x\to 2} f(x) =$

(h) $f(2) =$

Solution.

12. Evaluate each expression using the given graph of f.

webwork-86-image-1.png

Let a be an integer with $-3 \le a \le 3$.

(a) $\lim\limits_{x \to a^-} f(x) =$ ______

(b) $\lim\limits_{x \to a^+} f(x) =$ ______

(c) $\lim\limits_{x \to a} f(x) =$ ______

(d) $f(a) =$ ______

Solution.

In the following exercises, evaluate the given limits of the piecewise defined functions f.

13. Evaluate the given limits of the piecewise defined function.

$$f(x) = \begin{cases} x+1 & x \le 1 \\ x^2 - 5 & x > 1 \end{cases}$$

(a) $\lim\limits_{x \to 1^-} f(x) =$ ______

(b) $\lim\limits_{x \to 1^+} f(x) =$ ______

(c) $\lim\limits_{x \to 1} f(x) =$ ______

(d) $f(1) =$ ______

Solution.

14. Evaluate the given limits of the piecewise defined function.

$$f(x) = \begin{cases} 2x^2 + 5x - 1 & x < 0 \\ \sin(x) & x \geq 0 \end{cases}$$

(a) $\lim\limits_{x\to 0^-} f(x) =$ [redacted]

(b) $\lim\limits_{x\to 0^+} f(x) =$ [redacted]

(c) [redacted] $f(x) =$ [redacted]

(d) [redacted] $(0) =$ [redacted]

Solution.

15. Evaluate the given limits of the piecewise defined function.

$$f(x) = \begin{cases} x^2 - 1 & x < -1 \\ x^3 + 1 & -1 \leq x \leq 1 \\ x^2 + 1 & x > 1 \end{cases}$$

(a) $\lim\limits_{x\to -1^-} f(x) =$ [redacted]

(b) $\lim\limits_{x\to -1^+} f(x) =$ [redacted]

(c) $\lim\limits_{x\to -1} f(x) =$ [redacted]

(d) $f(-1) =$ [redacted]

(e) [redacted] $f(x) =$ [redacted]

(f) [redacted] $f(x) =$ [redacted]

(g) [redacted] $f(x) =$ [redacted]

[redacted]h) $f(1) =$ [redacted]

Solution.

16. Evaluate the given limits of the piecewise defined function.

$$f(x) = \begin{cases} \cos(x) & x < \pi \\ \sin(x) & x \geq \pi \end{cases}$$

(a) $\lim\limits_{x\to \pi^-} f(x) =$ [redacted]

(b) $\lim\limits_{x\to \pi^+} f(x) =$ [redacted]

(c) [redacted] $f(x) =$ [redacted]

(d) [redacted] $(\pi) =$ [redacted]

Solution.

17. Evaluate the given limits of the piecewise defined function.

$$f(x) = \begin{cases} 1 - \cos^2(x) & x < a \\ \sin^2(x) & x \geq a \end{cases}$$

where a is a real number.

(a) $\lim\limits_{x \to a^-} f(x) =$ ☐ (c) $\lim\limits_{x \to a} f(x) =$ ☐

(b) $\lim\limits_{x \to a^+} f(x) =$ ☐ (d) $f(a) =$ ☐

Solution.

18. Evaluate the given limits of the piecewise defined function.

$$f(x) = \begin{cases} x + 1 & x < 1 \\ 1 & x = 1 \\ x - 1 & x > 1 \end{cases}$$

(a) $\lim\limits_{x \to 1^-} f(x) =$ ☐ (c) $\lim\limits_{x \to 1} f(x) =$ ☐

(b) $\lim\limits_{x \to 1^+} f(x) =$ ☐ (d) $f(1) =$ ☐

Solution.

19. Evaluate the given limits of the piecewise defined function.

$$f(x) = \begin{cases} x^2 & x < 2 \\ x + 1 & x = 2 \\ -x^2 + 2x + 4 & x > 2 \end{cases}$$

(a) $\lim\limits_{x \to 2^-} f(x) =$ ☐ (c) $\lim\limits_{x \to 2} f(x) =$ ☐

(b) $\lim\limits_{x \to 2^+} f(x) =$ ☐ (d) $f(2) =$ ☐

Solution.

20. Evaluate the given limits of the piecewise defined function.

$$f(x) = \begin{cases} a(x-b)^2 + c & x < b \\ a(x-b) + c & x \geq b \end{cases}$$

(a) $\lim\limits_{x \to b^-} f(x) =$

(b) $\lim\limits_{x \to b^+} f(x) =$

(c) [redacted] $\mathrm{im}_{\to b}\, f(x) =$

(d) [redacted] $(b) =$

Solution.

21. Evaluate the given limits of the piecewise defined function.

$$f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(a) $\lim\limits_{x \to 0^-} f(x) =$

(b) $\lim\limits_{x \to 0^+} f(x) =$

(c) [redacted] $\mathrm{im}_{\to 0}\, f(x) =$

(d) [redacted] $(0) =$

Solution.

The following exercises are review from prior sections.

22. Use limit theorems to evaluate the limit.

$$\lim_{x \to -1} \frac{x^2+5x+4}{x^2-3x-4}$$

23. Use limit theorems to evaluate the limit.

$$\lim_{x \to -4} \frac{x^2-16}{x^2-4x-32}$$

24. Use limit theorems to evaluate the limit.

$$\lim_{x \to -6} \frac{x^2-15x+54}{x^2-6x}$$

25. Approximate the limit numerically.

$$\lim_{x \to 0.4} \frac{x^2-4.4x+1.6}{x^2-0.4x}$$

26. Approximate the limit numerically.

$\lim\limits_{x \to 0.2} \frac{x^2+5.8x-1.2}{x^2-4.2x+0.8}$

1.5 Continuity

As we have studied limits, we have gained the intuition that limits measure "where a function is heading." That is, if $\lim\limits_{x \to 1} f(x) = 3$, then as x is close to 1, $f(x)$ is close to 3. We have seen, though, that this is not necessarily a good indicator of what $f(1)$ actually is. This can be problematic; functions can tend to one value but attain another. This section focuses on functions that *do not* exhibit such behavior.

Definition 1.5.1 (Continuous Function)**.** Let f be a function defined on an open interval I containing c.

1. f is **continuous at** c if $\lim\limits_{x \to c} f(x) = f(c)$.
2. f is **continuous on** I if f is continuous at c for all values of c in I. If f is continuous on $(-\infty, \infty)$, we say f is **continuous everywhere**.

A useful way to establish whether or not a function f is continuous at c is to verify the following three things:

1. $\lim\limits_{x \to c} f(x)$ exists,
2. $f(c)$ is defined, and
3. $\lim\limits_{x \to c} f(x) = f(c)$.

Example 1.5.2 (Finding intervals of continuity)**.** Let f be defined as shown in Figure **??**. Give the interval(s) on which f is continuous.

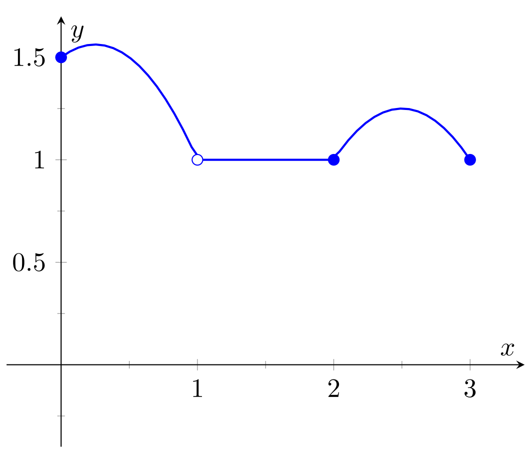


Figure 1.5.3: A graph of f in Example **??**.

Solution. We proceed by examining the three criteria for continuity.

1. The limits $\lim_{x\to c} f(x)$ exists for all c between 0 and 3.

2. $f(c)$ is defined for all c between 0 and 3, *except for* $c = 1$. We know immediately that f cannot be continuous at $x = 1$.

3. The limit $\lim_{x\to c} f(x) = f(c)$ for all c between 0 and 3, except, of course, for $c = 1$.

We conclude that f is continuous at every point of the interval $(0, 3)$ except at $x = 1$. Therefore f is continuous on $(0, 1) \cup (1, 3)$.

Example 1.5.4 (Finding intervals of continuity)**.** The *floor function*, $f(x) = \lfloor x \rfloor$, returns the largest integer smaller than the input x. (For example, $f(\pi) = \lfloor \pi \rfloor = 3$ and $f(2.5) = \lfloor 2.5 \rfloor = 2$.) The graph of f in Figure **??** demonstrates why this is often called a "step function."

Give the intervals on which f is continuous.

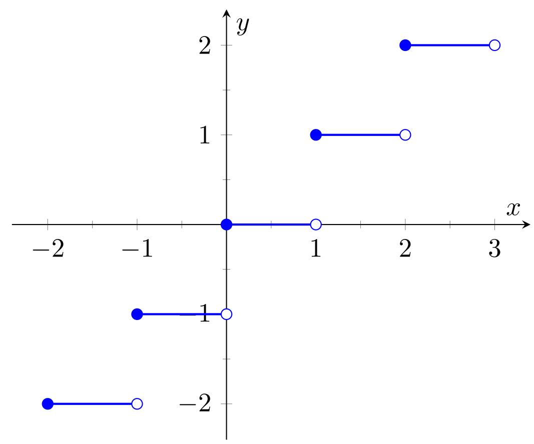


Figure 1.5.5: A graph of the step function in Example **??**.

Solution. We examine the three criteria for continuity.

1. The limits $\lim_{x\to c} f(x)$ do not exist at the jumps from one "step" to the next, which occur at all integer values of c. Therefore the limits exist for all c except when c is an integer.

2. The function is defined for all values of c.

3. The limit $\lim_{x\to c} f(x) = f(c)$ for all values of c where the limit exist, since each step consists of just a line.

We conclude that f is continuous everywhere except at integer values of c. So the intervals on which f is continuous are

$$\ldots, (-2, -1), (-1, 0), (0, 1), (1, 2), \ldots.$$

We could also say that f is continuous on all intervals of the form $(n, n + 1)$ where n is an integer.

Our definition of continuity on an interval specifies the interval is an open interval. We can extend the definition of continuity to closed intervals by considering the appropriate one-sided limits at the endpoints.

Definition 1.5.6 (Continuity on Closed Intervals)**.** Let f be defined on the closed interval $[a, b]$ for some real numbers a, b. f is **continuous on** $[a, b]$ if:

1. f is continuous on (a, b),
2. $\lim_{x \to a^+} f(x) = f(a)$ and
3. $\lim_{x \to b^-} f(x) = f(b)$.

Item 2 in Definition **??** indicates that the function is continuous *from the right* at a, while Item 3 indicates that the function is continuous *from the left* at b.

We could make the appropriate adjustments to talk about continuity on half-open intervals such as $[a, b)$ or $(a, b]$ if necessary.

Example 1.5.7 (Determining intervals on which a function is continuous)**.** For each of the following functions, give the domain of the function and the interval(s) on which it is continuous.

1. $f(x) = 1/x$
2. $f(x) = \sin(x)$
3. $f(x) = \sqrt{x}$
4. $f(x) = \sqrt{1 - x^2}$
5. $f(x) = |x|$

Solution. We examine each in turn.

1. The domain of $f(x) = 1/x$ is $(-\infty, 0) \cup (0, \infty)$. As it is a rational function, we apply Theorem **??** to recognize that f is continuous on all of its domain.
2. The domain of $f(x) = \sin(x)$ is all real numbers, or $(-\infty, \infty)$. Applying Theorem **??** shows that $\sin(x)$ is continuous everywhere.
3. The domain of $f(x) = \sqrt{x}$ is $[0, \infty)$. Applying Theorem **??** shows that $f(x) = \sqrt{x}$ is continuous on its domain of $[0, \infty)$.
4. The domain of $f(x) = \sqrt{1 - x^2}$ is $[-1, 1]$. Applying Theorems **??** and **??** shows that f is continuous on all of its domain, $[-1, 1]$.
5. The domain of $f(x) = |x|$ is $(-\infty, \infty)$. We can define the absolute value function as $f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$. Each "piece" of this piecewise defined function is continuous on all of its domain, giving that f is continuous on $(-\infty, 0)$ and $[0, \infty)$. We cannot assume this implies that f is continuous on $(-\infty, \infty)$; we need to check that $\lim_{x \to 0} f(x) = f(0)$, as $x = 0$ is the point where f transitions from one "piece" of its definition to the other. It is easy to verify that this is indeed true, hence we conclude that $f(x) = |x|$ is continuous everywhere.

Continuity is inherently tied to the properties of limits. Because of this, the properties of limits found in Theorems **??** and **??** apply to continuity as well. Further, now knowing the definition of continuity we can re-read Theorem **??** as giving a list of functions that are continuous on their domains. The following theorem states how continuous functions can be combined to form other continuous functions, followed by a theorem which formally lists functions that we know are continuous on their domains.

Theorem 1.5.8 (Properties of Continuous Functions)**.** *Let f and g be continuous functions on an interval I, let c be a real number and let n be a positive integer. The following functions are continuous on I.*

Sums/Difference $f \pm g$

Constant Multiple $c \cdot f$

Product $f \cdot g$

Quotient f/g *(as long as $g \neq 0$ on I)*

Power f^n

Root $\sqrt[n]{f}$

(If n is even then f must be ≥ 0 on I; if n is odd, then no such restriction.)

Compositions *Adjust the definitions of f and g to: Let f be continuous on I, where the range of f on I is J, and let g be continuous on J. Then $g \circ f$, i.e., $g(f(x))$, is continuous on I.*

Theorem 1.5.9 (Continuous Functions)**.** *The following functions are continuous on their domains.*

1. $f(x) = \sin(x)$
2. $f(x) = \tan(x)$
3. $f(x) = \sec(x)$
4. $f(x) = \ln(x)$
5. $f(x) = a^x$ $(a > 0)$
6. $f(x) = \cos(x)$
7. $f(x) = \cot(x)$
8. $f(x) = \csc(x)$
9. $f(x) = \sqrt[n]{x}$ *(where n is a positive integer)*

We apply these theorems in the following Example.

Example 1.5.10 (Determining intervals on which a function is continuous)**.** State the interval(s) on which each of the following functions is continuous.

1. $f(x) = \sqrt{x-1} + \sqrt{5-x}$
2. $f(x) = x\sin(x)$
3. $f(x) = \tan(x)$
4. $f(x) = \sqrt{\ln(x)}$

Solution. We examine each in turn, applying Theorems **??** and **??** as appropriate.

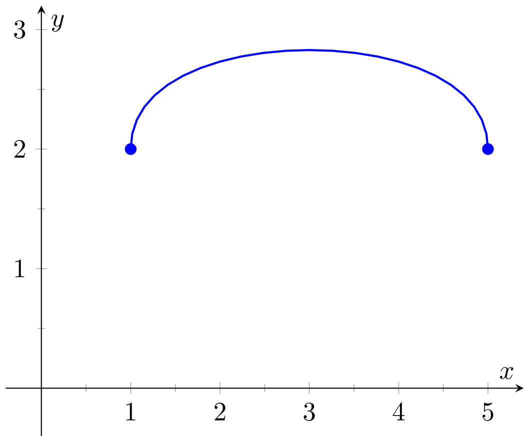


Figure 1.5.11: A graph of f

1. The square root terms are continuous on the intervals $[1, \infty)$ and $(-\infty, 5]$, respectively. As f is continuous only where each term is continuous, f is continuous on $[1, 5]$, the intersection of these two intervals. A graph of f is given in Figure **??**.

2. The functions $y = x$ and $y = \sin(x)$ are each continuous everywhere, hence their product is, too.

3. Theorem **??** states that $f(x) = \tan(x)$ is continuous *on its domain.* Its domain includes all real numbers except odd multiples of $\pi/2$. Thus $f(x) = \tan(x)$ is continuous on

$$\ldots, \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

   or, equivalently, on $D = \left\{x \in \mathbb{R} \mid x \neq (2n+1) \cdot \frac{\pi}{2}, \text{ where } n \text{ is an integer } \right\}$. $\blacksquare$

4. The range of $y = \ln(x)$ is $(-\infty, \infty)$ which includes invalid input for $\sqrt{}$. We must restrict the domain to $I = [1, \infty)$ in order to have $\ln(x)$ produce output in $J = [0, \infty)$, which can then be used as valid input into $\sqrt{}$. Since $\ln(\ )$ is continuous on I and $\sqrt{}$ is continuous on J, $f(x) = \sqrt{\ln(x)}$ is continuos on $I = [1, \infty)$.

A common way of thinking of a continuous function is that "its graph can be sketched without lifting your pencil." That is, its graph forms a "continuous" curve, without holes, breaks or jumps. This pseudo-definition glosses over some of the finer points of continuity that are beyond the scope of this text. Very strange functions are continuous that one would be hard pressed to actually sketch by hand.

However this intuitive notion of continuity does help us understand another important concept as follows. Suppose f is defined on $[1, 2]$, and $f(1) = -10$ and $f(2) = 5$. If f is continuous on $[1, 2]$ (i.e., its graph can be sketched as a continuous curve from $(1, -10)$ to $(2, 5)$) then we know intuitively that somewhere on the interval $[1, 2]$ f must be equal to -9, and -8, and $-7, -6, \ldots, 0, 1/2$, etc. In short, f takes on all *intermediate* values between -10 and 5. It may take on more values; f may actually equal 6 at some time, for instance, but we are guaranteed all values between -10 and 5.

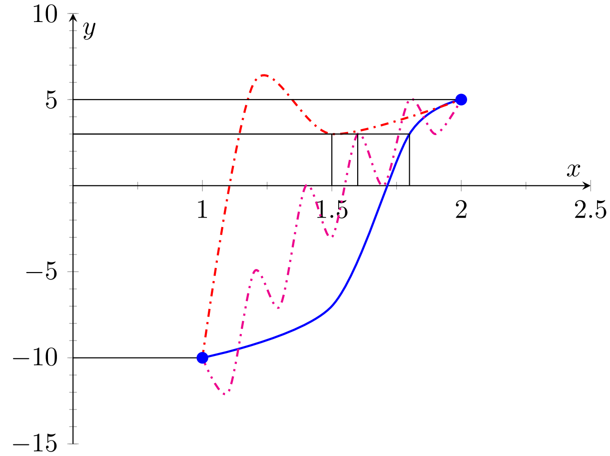


Figure 1.5.12: Illustration of the Intermediate Value Theorem: the output 3 is in between −10 and 5, and therefore any continuous function on $[1, 2]$ with $f(1) = -10$ and $f(2) = 5$ will achieve the output 3 somewhere in $[1, 2]$.

While this notion seems intuitive, it is not trivial to prove and its importance is profound. Therefore the concept is stated in the form of a theorem.

Theorem 1.5.13 (Intermediate Value Theorem)**.** *Let f be a continuous function on $[a, b]$ and, without loss of generality, let $f(a) < f(b)$. Then for every value y, where $f(a) < y < f(b)$, there is a value c in $[a, b]$ such that $f(c) = y$.*

One important application of the Intermediate Value Theorem is root finding. Given a function f, we are often interested in finding values of x where $f(x) = 0$. These roots may be very difficult to find exactly. Good approximations can be found through successive applications of this theorem. Suppose through direct computation we find that $f(a) < 0$ and $f(b) > 0$, where $a < b$. The Intermediate Value Theorem states that there is a c in $[a, b]$ such that $f(c) = 0$. The theorem does not give us any clue as to where that value is in the interval $[a, b]$, just that it exists.

There is a technique that produces a good approximation of c. Let d be the midpoint of the interval $[a, b]$, with $f(a) < 0$ and $f(b) > 0$ and consider $f(d)$. There are three possibilities:

1. $f(d) = 0$ — we got lucky and stumbled on the actual value. We stop as we found a root.

2. $f(d) < 0$. Then we know there is a root of f on the interval $[d, b]$ — we have halved the size of our interval, hence are closer to a good approximation of the root.

3. $f(d) > 0$. Then we know there is a root of f on the interval $[a, d]$ — again,we have halved the size of our interval, hence are closer to a good approximation of the root.

Successively applying this technique is called the **Bisection Method** of root finding. We continue until the interval is sufficiently small. We demonstrate this in the following example.

Example 1.5.14 (Using the Bisection Method)**.** Approximate the root of $f(x) = x - \cos(x)$, accurate to three places after the decimal.

Solution. Consider the graph of $f(x) = x - \cos(x)$, shown in Figure **??**. It is clear that the graph crosses the x-axis somewhere near $x = 0.8$. To start the Bisection Method, pick an interval that contains 0.8. We choose $[0.7, 0.9]$. Note that all we care about are signs of $f(x)$, not their actual value, so this is all we display.

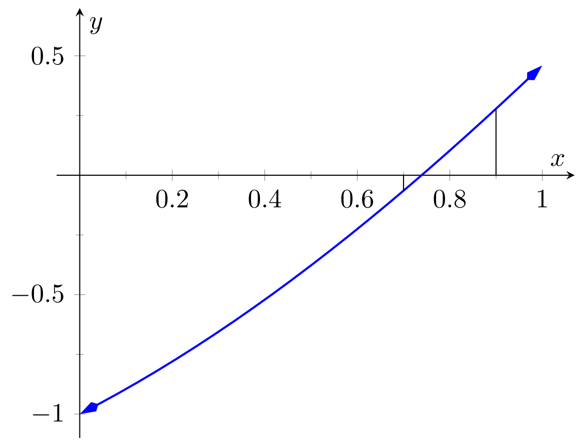


Figure 1.5.15: Graphing a root of $f(x) = x - \cos(x)$.

Iteration 1: $f(0.7) < 0$, $f(0.9) > 0$, and $f(0.8) > 0$. So replace 0.9 with 0.8 and repeat.

Iteration 2: $f(0.7) < 0$, $f(0.8) > 0$, and at the midpoint, 0.75, we have $f(0.75) > 0$. So replace 0.8 with 0.75 and repeat. Note that we don't need to continue to check the endpoints, just the midpoint. Thus we put the rest of the iterations in Table **??**.

Iteration #	Interval	Midpoint Sign
1	$[0.7, 0.9]$	$f(0.8) > 0$
2	$[0.7, 0.8]$	$f(0.75) > 0$
3	$[0.7, 0.75]$	$f(0.725) < 0$
4	$[0.725, 0.75]$	$f(0.7375) < 0$
5	$[0.7375, 0.75]$	$f(0.7438) > 0$
6	$[0.7375, 0.7438]$	$f(0.7407) > 0$
7	$[0.7375, 0.7407]$	$f(0.7391) > 0$
8	$[0.7375, 0.7391]$	$f(0.7383) < 0$
9	$[0.7383, 0.7391]$	$f(0.7387) < 0$
10	$[0.7387, 0.7391]$	$f(0.7389) < 0$
11	$[0.7389, 0.7391]$	$f(0.7390) < 0$
12	$[0.7390, 0.7391]$	

Table 1.5.16: Iterations of the Bisection Method of Root Finding

Notice that in the 12th iteration we have the endpoints of the interval each starting with 0.739. Thus we have narrowed the zero down to an accuracy of the first three places after the decimal. Using a computer, we have

$$f(0.7390) = -0.00014, f(0.7391) = 0.000024.$$

Either endpoint of the interval gives a good approximation of where f is 0. The **??** states that the actual zero is still within this interval. While we do not know its exact value, we know it starts with 0.739.

This type of exercise is rarely done by hand. Rather, it is simple to program a computer to run such an algorithm and stop when the endpoints differ by a preset small amount. One of the authors did write such a program and found the zero of f to be 0.7390851332 , accurate to 10 places after the decimal. While it took a few minutes to write the program, it took less than a thousandth of a second for the program to run the necessary 35 iterations. In less than 8 hundredths of a second, the zero was calculated to 100 decimal places (with less than 200 iterations).

It is a simple matter to extend the Bisection Method to solve problems similar to "Find x, where $f(x) = 0$." For instance, we can find x, where $f(x) = 1$. It actually works very well to define a new function g where $g(x) = f(x) - 1$. Then use the Bisection Method to solve $g(x) = 0$.

Similarly, given two functions f and g, we can use the Bisection Method to solve $f(x) = g(x)$. Once again, create a new function h where $h(x) = f(x) - g(x)$ and solve $h(x) = 0$.

In Section **??** another equation solving method will be introduced, called Newton's Method. In many cases, Newton's Method is much faster. It relies on more advanced mathematics, though, so we will wait before introducing it.

This section formally defined what it means to be a continuous function. "Most" functions that we deal with are continuous, so often it feels odd to have to formally define this concept. Regardless, it is important, and forms the basis of the next chapter.

1.5.1 Exercises

Terms and Concepts

1. In your own words, describe what it means for a function to be continuous.

[Essay Answer]

Solution. Answers will vary.

2. In your own words, describe what the Intermediate Value Theorem states.

[Essay Answer]

Solution. Answers will vary.

3. What is a "root" of a function?

[Essay Answer]

Solution. A root of a function f is a value c such that $f(c) = 0$.

4. Given functions f and g on an interval I, how can the Bisection Method be used to find a value c where $f(c) = g(c)$?

[Essay Answer]

Solution. Consider the function $h(x) = g(x) - f(x)$, and use the Bisection Method to find a root of h.

5. True or False? If f is defined on an open interval containing c, and $\lim\limits_{x\to c} f(x)$ exists, then f is continuous at c. (Choose one: True / False)

Solution. False

6. True or False? If f is continuous at c, then $\lim\limits_{x\to c} f(x)$ exists. (Choose one: True / False)

Solution. True

7. True or False? If f is continuous at c, then $\lim\limits_{x\to c^+} f(x) = f(c)$. (Choose one: True / False)

Solution. True

8. True or False? If f is continuous on $[a, b]$, then $\lim\limits_{x\to a^-} f(x) = f(a)$. (Choose one: True / False)

Solution. False

9. True or False? If f is continuous on $[0, 1)$ and $[1, 2)$, then f is continuous on $[0, 2)$. (Choose one: True / False)

Solution. False

10. True or False? The sum of continuous functions is also continuous. (Choose one: True / False)

Solution. True

In the following exercises, a graph of a function f is given along with a value a. Determine if f is continuous at a; if it is not, state why it is not.

11. Is f in the graph below continuous at $a = 1$? (Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist.)

Solution. No; the limit of f(x) as x goes to a does not equal f(a) even though both exist.

12. Is f in the graph below continuous at $a = 1$? (Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist)

Solution. No; the limit of f(x) as x goes to a does not exist.

13. Is f in the graph below continuous at $a = 1$?
(Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist)

Solution. No; f(a) is not defined.

14. Is f in the graph below continuous at $a = 0$?
(Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist)

Solution. No; the limit of f(x) as x goes to a does not exist.

15. Is f in the graph below continuous at $a = 1$?
(Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist)

Solution. Yes.

16. Is f in the graph below continuous at $a = 4$?
(Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist)

Solution. Yes.

17. Is f in the graph below continuous at $a = -2$, $a = 0$, and $a =$

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2? At $a = -2$: (Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist)

At $a = 0$: (Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist)

At $a = 2$: (Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist)

Solution. At $a = -2$: No; the limit of f(x) as x goes to a does not equal f(a) even though both exist

At $a = 0$: Yes.

At $a = 2$: No; f(a) is not defined.

In the following exercises, determine if f is continuous at the indicated values. If not, explain why.

18. Determine if f is continuous at 0 and π.

$$f(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin(x)}{x} & x \neq 0 \end{cases}$$

At $a = 0$: (Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist.)

At $a = \pi$: (Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist.)

Solution. At $a = 0$: Yes.

At $a = \pi$: Yes.

19. Determine if f is continuous at 0 and 1.

$$f(x) = \begin{cases} x^3 - x^2 & x < 1 \\ x - 2 & x \geq 1 \end{cases}$$

At $a = 0$: (Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist.)

At $a = 1$: (Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist.)

Solution. At $a = 0$: Yes.

At $a = 1$: No; the limit of f(x) as x goes to a does not equal f(a) even though both exist.

20. Determine if f is continuous at -1 and 10.

$$f(x) = \begin{cases} \frac{x^2+5x+4}{x^2+3x+2} & x \neq 1 \\ 3 & x = -1 \end{cases}$$

At $a = -1$: (Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist.)

At $a = 10$: (Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist.)

Solution. At $a = -1$: Yes.

At $a = 10$: Yes.

21. Determine if f is continuous at 0 and 8.

$$f(x) = \begin{cases} \frac{x^2-64}{x^2-11x+24} & x \neq 8 \\ 5 & x = 8 \end{cases}$$

At $a = 0$: (Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist.)

At $a = 8$: (Choose one: Yes., No; the limit of f(x) as x goes to a does not exist., No; f(a) is not defined., or No; the limit of f(x) as x goes to a does not equal f(a) even though both exist.)

Solution. At $a = 0$: Yes.

At $a = 8$: No; the limit of f(x) as x goes to a does not equal f(a) even though both exist.

In the following exercises, give the intervals on which the given function is continuous.

22. On what interval or union of intervals is $f(x) = x^2 - 3x + 9$ continuous?

Solution. Since f is a polynomial function, it is continuous on $(-\infty, \infty)$.$\blacksquare$

23. On what interval or union of intervals is $f(x) = \sqrt{x^2 - 4}$ continuous?

Solution. The domain of f is $(-\infty, 2]U[2, infty)$, and since f is a composition of continuous functions on that domain, it is continuous on $(-\infty, 2]U[2, infty)$.

24. On what interval or union of intervals is $f(x) = \sqrt{1-x} + \sqrt{x+1}$ continuous?

Solution. The domain of $\sqrt{1-x}$ is $(-\infty, 1]$, and the domain of $\sqrt{x+1}$$\blacksquare$ is $[-1, \infty)$. So the domain of f is $[-1, 1]$. And since f is the sum of two compositions of continuous functions, it is continuous on $[-1, 1]$.

25. On what interval or union of intervals is $f(t) = \sqrt{5t^2 - 30}$ continuous?

26. On what interval or union of intervals is $g(t) = \frac{1}{\sqrt{1-t^2}}$ continuous?

27. On what interval or union of intervals is $g(t) = \frac{1}{1+x^2}$ continuous?

28. On what interval or union of intervals is $f(t) = e^x$ continuous?

29. On what interval or union of intervals is $g(s) = \ln(s)$ continuous?

30. On what interval or union of intervals is $h(t) = \cos(t)$ continuous?

31. On what interval or union of intervals is $f(k) = \sqrt{1 - e^k}$ continuous?

32. On what interval or union of intervals is $f(x) = \sin\left(e^x + x^2\right)$ continuous?

33. Let f be continuous on $[1,5]$ where $f(1) = -2$ and $f(5) = -10$. Does a value $1 < c < 5$ exist such that $f(c) = -9$? Why/why not?
[Essay Answer]

Solution. Yes, by the Intermediate Value Theorem.

34. Let g be continuous on $[-3,7]$ where $g(0) = 0$ and $g(2) = 25$. Does a value $-3 < c < 7$ exist such that $g(c) = 15$? Why/why not?
[Essay Answer]

Solution. Yes, by the Intermediate Value Theorem. In fact, we can be more specific and state such a value c exists in $(0,2)$, not just in $(-3,7)$.

35. Let f be continuous on $[-1,1]$ where $f(-1) = -10$ and $f(1) = 10$. Does a value $-1 < c < 1$ exist such that $f(c) = 11$? Why/why not?
[Essay Answer]

Solution. We cannot say; the Intermediate Value Theorem only applies to function values between -10 and 10; as 11 is outside this range, we do not know.

36. Let h be a function on $[-1,1]$ where $h(-1) = -10$ and $h(1) = 10$. Does a value $-1 < c < 1$ exist such that $h(c) = 0$? Why/why not?
[Essay Answer]

Solution. We cannot say; the Intermediate Value Theorem only applies to continuous functions. As we do know know if h is continuous, we cannot say.

In the following exercises, use the Bisection Method to approximate, accurate to two decimal places, the value of the root of the given function in the given interval.

37. Use the Bisection Method to zero in on the root for $f(x) = x^2 + 2x - 4$ in the interval $[1, 1.5]$.

Iteration	Interval	Midpoint Sign
1	$[1, 1.5]$	(Choose one: $<$/ $>$)
2		(Choose one: $<$/ $>$)
3		(Choose one: $<$/ $>$)
4		(Choose one: $<$/ $>$)
5		(Choose one: $<$/ $>$)
6		(Choose one: $<$/ $>$)
7		(Choose one: $<$/ $>$)
8		

So to two decimal places, the zero is ______.

38. Use the Bisection Method to zero in on the root for $f(x) = \sin(x) - \frac{1}{2}$ in the interval $[0.5, 0.55]$.

Iteration	Interval	Midpoint Sign
1	$[0.5, 0.55]$	(Choose one: $< / >$)
2		(Choose one: $< / >$)
3		(Choose one: $< / >$)
4		(Choose one: $< / >$)
5		(Choose one: $< / >$)
6		(Choose one: $< / >$)
7		(Choose one: $< / >$)
8		

So to three decimal places, the zero is ______.

39. Use the Bisection Method to zero in on the root for $f(x) = e^x - 2$ in the interval $[0.65, 0.7]$.

Iteration	Interval	Midpoint Sign
1	$[0.65, 0.7]$	(Choose one: $< / >$)
2		(Choose one: $< / >$)
3		(Choose one: $< / >$)
4		(Choose one: $< / >$)
5		(Choose one: $< / >$)
6		(Choose one: $< / >$)
7		(Choose one: $< / >$)
8		

So to three decimal places, the zero is ______.

40. Use the Bisection Method to zero in on the root for $f(x) = \cos(x) - \sin(x)$ in the interval $[0.7, 0.8]$.

Iteration	Interval	Midpoint Sign
1	$[0.7, 0.8]$	(Choose one: $< / >$)
2		(Choose one: $< / >$)
3		(Choose one: $< / >$)
4		(Choose one: $< / >$)
5		(Choose one: $< / >$)
6		(Choose one: $< / >$)
7		(Choose one: $< / >$)
8		

So to three decimal places, the zero is ______.

The following exercises are review from prior sections.

41. Evaluate the given limits of the piecewise defined function.

$$f(x) = \begin{cases} x^2 - 5 & x < 5 \\ 5x & x \geq 5 \end{cases}$$

(a) $\lim\limits_{x\to 5^-} f(x) =$ ☐

(b) $\lim\limits_{x\to 5^+} f(x) =$ ☐

(c) [redacted]im $f(x) =$ ☐ [redacted]

(d) [redacted](5) $=$ ☐ [redacted]

42. Numerically approximate the following limits.

(a) $\lim\limits_{x\to -0.8^+} \left(\frac{x^2-8.2x-7.2}{x^2+5.8x+4}\right) =$ ☐

(b) $\lim\limits_{x\to -0.8^-} \left(\frac{x^2-8.2x-7.2}{x^2+5.8x+4}\right) =$ ☐

Solution. For a numerical approximation, make a table:

x	$\frac{x^2-8.2x-7.2}{x^2+5.8x+4}$
$-0/9$	-2.41463
-0.81	-2.34129
-0.801	-2.33413
-0.799	-2.33254
-0.79	-2.32542
-0.7	-2.25581

It appears that when x is close to -0.8 (whether slightly above or slightly below), that $\frac{x^2-8.2x-7.2}{x^2+5.8x+4}$ is close to $-\frac{7}{3}$. So

$$\lim_{x\to -0.8^+} \left(\frac{x^2-8.2x-7.2}{x^2+5.8x+4}\right) = \lim_{x\to -0.8^-} \left(\frac{x^2-8.2x-7.2}{x^2+5.8x+4}\right) = -\frac{7}{3}.$$

43. Give an example of function $f(x)$ for which $\lim\limits_{x\to 0} f(x)$ does not exist.

[Essay Answer]

Solution. Answers will vary.

1.6 Limits Involving Infinity

In Definition **??** we stated that in the equation $\lim\limits_{x\to c} f(x) = L$, both c and L were numbers. In this section we relax that definition a bit by considering situations when it makes sense to let c and/or L be "infinity."

As a motivating example, consider $f(x) = 1/x^2$, as shown in Figure **??**. Note how, as x approaches 0, $f(x)$ grows very, very large. It seems appropriate, and descriptive, to state that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

Also note that as x gets very large, $f(x)$ gets very, very small. We could represent this concept with notation such as

$$\lim_{x\to\infty} \frac{1}{x^2} = 0.$$

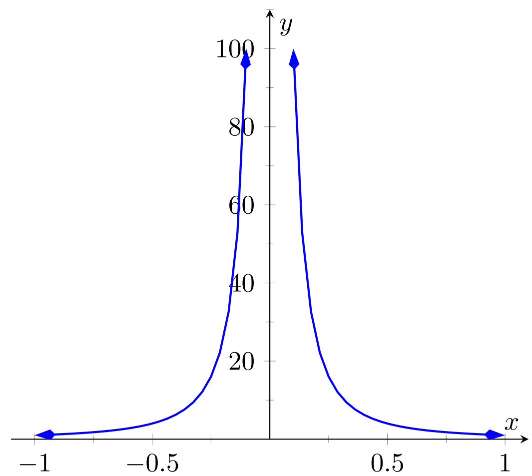


Figure 1.6.1: Graphing $f(x) = 1/x^2$ for values of x near 0.

We explore both types of use of ∞ in turn.

Definition 1.6.2 (Limit of Infinity, ∞)**.** We say $\lim\limits_{x\to c} f(x) = \infty$ if for every $M > 0$ there exists $\delta > 0$ such that for all $x \neq c$, if $|x - c| < \delta$, then $f(x) \geq M$.

This is just like the ε–δ definition in Definition **??** from Section **??**. In that definition, given any (small) value ε, if we let x get close enough to c (within δ units of c) then $f(x)$ is guaranteed to be within ε of $f(c)$. Here, given any (large) value M, if we let x get close enough to c (within δ units of c), then $f(x)$ will be at least as large as M. In other words, if we get close enough to c, then we can make $f(x)$ as large as we want. We can define limits equal to $-\infty$ in a similar way.

It is important to note that by saying $\lim\limits_{x\to c} f(x) = \infty$ we are implicitly stating that *the* limit of $f(x)$, as x approaches c, *does not exist.* A limit only exists when $f(x)$ approaches an actual numeric value. We use the concept of limits that approach infinity because it is helpful and descriptive. It is one *specific way* in which a limit can fail to exist.

Example 1.6.3 (Evaluating limits involving infinity)**.** Find $\lim\limits_{x\to 1} \frac{1}{(x-1)^2}$ as shown in Figure **??**.

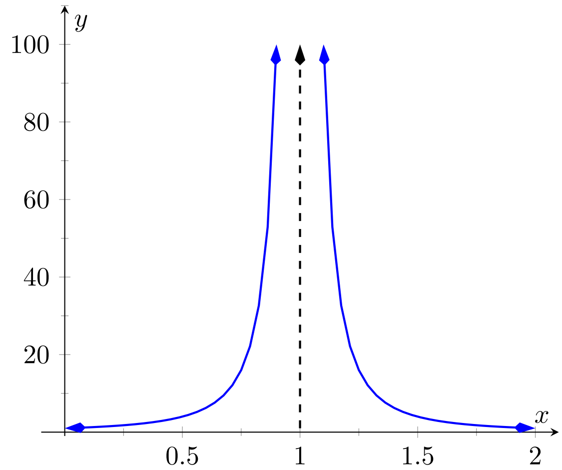


Figure 1.6.4: Observing infinite limit as $x \to 1$ in Example **??**.

Solution. In Example **??** of Section 1.1, by inspecting values of x close to 1 we concluded that this limit does not exist. That is, it cannot equal any real number. But the limit could be infinite. And in fact, we see that the function does appear to be growing larger and larger, as $f(.99) = 10^4$, $f(.999) = 10^6$, $f(.9999) = 10^8$. A similar thing happens on the other side of 1. From the graph and the numeric information, we could state $\lim\limits_{x\to 1} 1/(x-1)^2 = \infty$. We can prove this by using Definition **??**

In general, let a "large" value M be given. Let $\delta = 1/\sqrt{M}$. If x is within δ of 1, i.e., if $|x-1| < 1/\sqrt{M}$, then:

$$\begin{aligned} |x-1| &< \frac{1}{\sqrt{M}} \\ (x-1)^2 &< \frac{1}{M} \\ \frac{1}{(x-1)^2} &> M, \end{aligned}$$

which is what we wanted to show. So we may say $\lim\limits_{x\to 1} 1/(x-1)^2 = \infty$.

Example 1.6.5 (Evaluating limits involving infinity)**.** Find $\lim\limits_{x\to 0} \frac{1}{x}$, as shown in Figure **??**.

Solution. It is easy to see that the function grows without bound near 0, but it does so in different ways on different sides of 0. Since its behavior is not consistent, we cannot say that $\lim\limits_{x\to 0} \frac{1}{x} = \infty$. Instead, we will say $\lim\limits_{x\to 0} \frac{1}{x}$ does not exist. However, we can make a statement about one-sided limits. We can state that $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$.

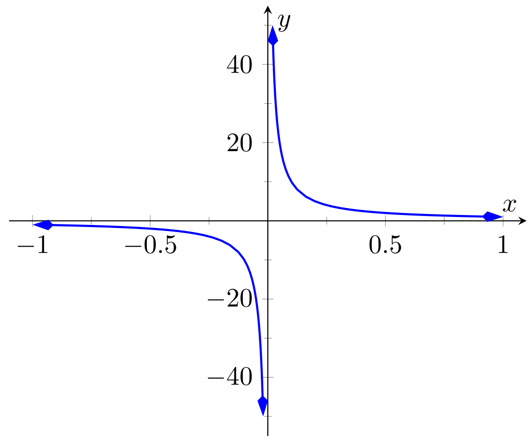


Figure 1.6.6: Evaluating $\lim_{x\to 0} \frac{1}{x}$.

1.6.1 Vertical asymptotes

If the limit of $f(x)$ as x approaches c from either the left or right (or both) is ∞ or $-\infty$, we say the function has a *vertical asymptote* at c.

Example 1.6.7 (Finding vertical asymptotes)**.** Find the vertical asymptotes of $f(x) = \frac{3x}{x^2-4}$.

Solution. Vertical asymptotes occur where the function grows without bound; this can occur at values of c where the denominator is 0. When x is near c, the denominator is small, which in turn can make the function take on large values. In the case of the given function, the denominator is 0 at $x = \pm 2$. Substituting in values of x close to 2 and -2 seems to indicate that the function tends toward ∞ or $-\infty$ at those points. We can graphically confirm this by looking at Figure **??**. Thus the vertical asymptotes are at $x = \pm 2$.

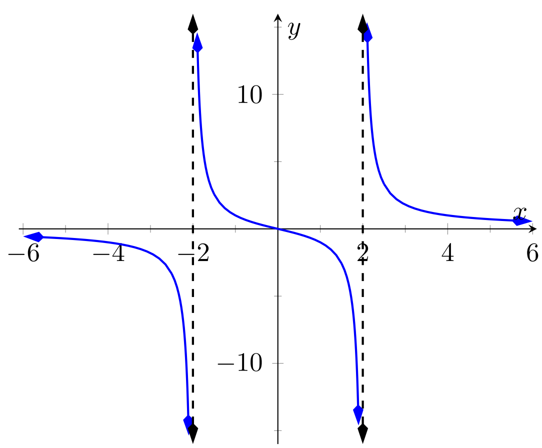


Figure 1.6.8: Graphing $f(x) = \frac{3x}{x^2-4}$.

When a rational function has a vertical asymptote at $x = c$, we can conclude that the denominator is 0 at $x = c$. However, just because the denominator is 0 at a certain point does not mean there is a vertical asymptote there. For instance, $f(x) = (x^2 - 1)/(x - 1)$ does not have a vertical asymptote at $x = 1$, as shown in Figure **??**. While the denominator does get small near $x = 1$, the numerator gets small too, matching the denominator step for step. In fact, factoring the numerator, we get

$$f(x) = \frac{(x-1)(x+1)}{x-1}.$$

Canceling the common term, we get that $f(x) = x + 1$ for $x \neq 1$. So there is clearly no asymptote, rather a hole exists in the graph at $x = 1$.

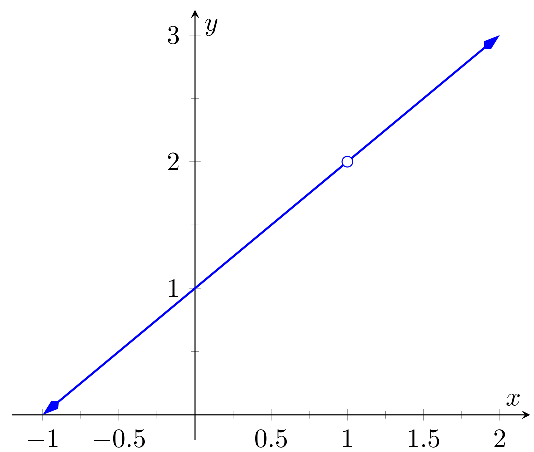


Figure 1.6.9: Graphically showing that $f(x) = \frac{x^2-1}{x-1}$ does not have an asymptote at $x = 1$.

The above example may seem a little contrived. Another example demonstrating this important concept is $f(x) = (\sin(x))/x$. We have considered this function several times in the previous sections. We found that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$; i.e., there is no vertical asymptote. No simple algebraic cancellation makes this fact obvious; we used the **??** in Section **??** to prove this.

If the denominator is 0 at a certain point but the numerator is not, then there will usually be a vertical asymptote at that point. On the other hand, if the numerator and denominator are both zero at that point, then there may or may not be a vertical asymptote at that point. This case where the numerator and denominator are both zero returns us to an important topic.

1.6.2 Indeterminate Forms

We have seen how the limits $\lim_{x\to 0} \frac{\sin(x)}{x}$ and $\lim_{x\to 1} \frac{x^2-1}{x-1}$ each return the indeterminate form $0/0$ when we blindly plug in $x = 0$ and $x = 1$, respectively. However, $0/0$ is not a valid arithmetical expression. It gives no indication that the respective limits are 1 and 2.

With a little cleverness, one can come up with $0/0$ expressions which have a limit of ∞, 0, or any other real number. That is why this expression is called **indeterminate**.

A key concept to understand is that such limits do not really return $0/0$. Rather, keep in mind that we are taking *limits*. What is really happening is that the numerator is shrinking to 0 while the denominator is also shrinking to 0. The respective rates at which they do this are very important and determine the actual value of the limit.

An indeterminate form indicates that one needs to do more work in order to compute the limit. That work may be algebraic (such as factoring and canceling), it may involve using trigonometric identities or logarithm rules, or it may require a tool such as the Squeeze Theorem. In Section **??** we will learn yet another technique called L'HÃťpital's Rule that provides another way to handle indeterminate forms.

Some other common indeterminate forms are $\infty - \infty$, $\infty \cdot 0$, ∞/∞, 0^0, ∞^0 and 1^∞. Again, keep in mind that these are the "blind" results of directly substituting c into the expression, and each, in and of itself, has no meaning. The expression $\infty - \infty$ does not really mean "subtract infinity from infinity." Rather, it means "One quantity is subtracted from the other, but both are growing without bound." What is the result? It is possible to get every value between $-\infty$ and ∞

Note that $1/0$ and $\infty/0$ are not indeterminate forms, though they are not exactly valid mathematical expressions either. In each, the function is growing without bound, indicating that the limit will be ∞, $-\infty$, or simply not exist if the left- and right-hand limits do not match.

1.6.3 Limits at Infinity and Horizontal Asymptotes

At the beginning of this section we briefly considered what happens to $f(x) = 1/x^2$ as x grew very large. Graphically, it concerns the behavior of the function to the "far right" of the graph. We make this notion more explicit in the following definition.

Definition 1.6.10 (Limits at Infinity and Horizontal Asymptote)**.**

1. We say $\lim\limits_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M > 0$ such that if $x \geq M$, then $|f(x) - L| < \varepsilon$.

2. We say $\lim\limits_{x\to-\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M < 0$ such that if $x \leq M$, then $|f(x) - L| < \varepsilon$.

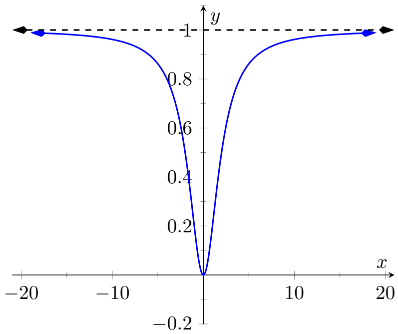
3. If $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, we say that $y = L$ is a *horizontal asymptote* of f.

We can also define limits such as $\lim\limits_{x\to\infty} f(x) = \infty$ by combining this definition with Definition **??**.

Example 1.6.11 (Approximating horizontal asymptotes)**.** Approximate the horizontal asymptote(s) of $f(x) = \frac{x^2}{x^2+4}$.

Solution. We will approximate the horizontal asymptotes by approximating the limits $\lim\limits_{x\to-\infty} \frac{x^2}{x^2+4}$ and $\lim\limits_{x\to\infty} \frac{x^2}{x^2+4}$. (A rational function can have at most *one* horizontal asymptote. So we could get away with only taking $x \to \infty$).

Figure **??** shows a sketch of f, and the table in Figure **??** gives values of $f(x)$ for large magnitude values of x. It seems reasonable to conclude from both of these sources that f has a horizontal asymptote at $y = 1$.



x	$f(x)$
10	0.9615
100	0.9996
10000	0.999996
-10	0.9615
-100	0.9996
-10000	0.999996

(a) **(b)**

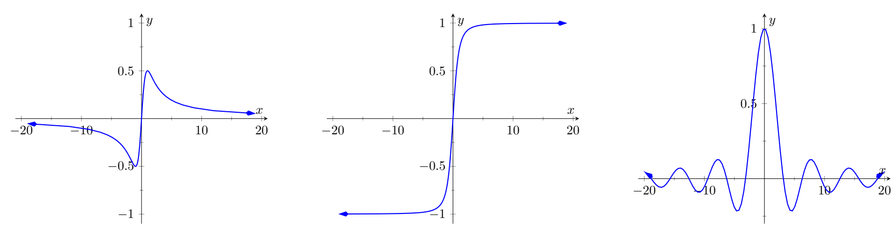
Figure 1.6.12: Using a graph and a table to approximate a horizontal asymptote in Example **??**.

Later, we will show how to determine this analytically.

Horizontal asymptotes can take on a variety of forms. Figure **??** shows that $f(x) = x/(x^2+1)$ has a horizontal asymptote of $y = 0$, where 0 is approached from both above and below.

Figure **??** shows that $f(x) = x/\sqrt{x^2+1}$ has two horizontal asymptotes; one at $y = 1$ and the other at $y = -1$.

Figure **??** shows that $f(x) = \sin(x)/x$ has even more interesting behavior than at just $x = 0$; as x approaches $\pm\infty$, $f(x)$ approaches 0, but oscillates as it does this.



(a) **(b)** **(c)**

Figure 1.6.13: Considering different types of horizontal asymptotes.

We can analytically evaluate limits at infinity for rational functions once we understand $\lim\limits_{x\to\infty} \frac{1}{x}$. As x gets larger and larger, the $1/x$ gets smaller and smaller, approaching 0. We can, in fact, make $1/x$ as small as we want by choosing a large enough value of x. Given ε, we can make $1/x < \varepsilon$ by choosing $x > 1/\varepsilon$. Thus we have $\lim_{x\to\infty} 1/x = 0$.

It is now not much of a jump to conclude the following:

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \qquad\qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Now suppose we need to compute the following limit:

$$\lim_{x\to\infty} \frac{x^3+2x+1}{4x^3-2x^2+9}.$$

A good way of approaching this is to divide through the numerator and denominator by x^3 (hence dividing by 1), which is the largest power of x to appear in the denominator. Doing this, we get

$$\begin{aligned}\lim_{x\to\infty}\frac{x^3+2x+1}{4x^3-2x^2+9} &= \lim_{x\to\infty}\frac{1/x^3}{1/x^3}\cdot\frac{x^3+2x+1}{4x^3-2x^2+9}\\ &= \lim_{x\to\infty}\frac{x^3/x^3+2x/x^3+1/x^3}{4x^3/x^3-2x^2/x^3+9/x^3}\\ &= \lim_{x\to\infty}\frac{1+2/x^2+1/x^3}{4-2/x+9/x^3}.\end{aligned}$$

Then using the rules for limits (which also hold for limits at infinity), as well as the fact about limits of $1/x^n$, we see that the limit becomes

$$\frac{1+0+0}{4-0+0}=\frac{1}{4}.$$

This procedure works for any rational function. In fact, it gives us the following theorem.

Theorem 1.6.14 (Limits of Rational Functions at Infinity). *Let $f(x)$ be a rational function of the following form:*

$$f(x)=\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0},$$

where m,n are positive integers and where any of the coefficients may be 0 except for a_n and b_m. Then:

1. *If $n=m$, then*
$$\lim_{x\to\infty}f(x)=\lim_{x\to-\infty}f(x)=\frac{a_n}{b_m}.$$
2. *If $n<m$, then*
$$\lim_{x\to\infty}f(x)=\lim_{x\to-\infty}f(x)=0.$$
3. *If $n>m$, then $\lim\limits_{x\to\infty}f(x)$ and $\lim\limits_{x\to-\infty}f(x)$ are both infinite.*

We can see why this is true. If the highest power of x is the same in both the numerator and denominator (i.e.$n=m$), we will be in a situation like the example above, where we will divide by x^n and in the limit all the terms will approach 0 except for a_nx^n/x^n and b_mx^m/x^n. Since $n=m$, this will leave us with the limit a_n/b_m. If $n<m$, then after dividing through by x^m, all the terms in the numerator will approach 0 in the limit, leaving us with $0/b_m$ or 0. If $n>m$, and we try dividing through by x^m, we end up with the denominator tending to b_m while the numerator tends to ∞.

Intuitively, as x gets very large, all the terms in the numerator are small in comparison to a_nx^n, and likewise all the terms in the denominator are small compared to b_nx^m. If $n=m$, looking only at these two important terms, we have $(a_nx^n)/(b_nx^m)$. This reduces to a_n/b_m. If $n<m$, the function behaves like $a_n/(b_mx^{m-n})$, which tends toward 0. If $n>m$, the function behaves like a_nx^{n-m}/b_m, which will tend to either ∞ or $-\infty$ depending on the values of n, m, a_n, b_m and whether you are looking for $\lim_{x\to\infty}f(x)$ or $\lim_{x\to\infty}f(x)$.

Example 1.6.15 (Finding a limit of a rational function). Confirm analytically that $y=1$ is the horizontal asymptote of $f(x)=\frac{x^2}{x^2+4}$, as approximated in Example **??**.

Solution. Before using Theorem **??**, let's use the technique of evaluating limits at infinity of rational functions that led to that theorem. The largest power of x in f is 2, so divide the numerator and denominator of f by x^2, then take limits.

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2}{x^2+4} &= \lim_{x\to\infty}\frac{x^2/x^2}{x^2/x^2+4/x^2}\\ &= \lim_{x\to\infty}\frac{1}{1+4/x^2}\\ &= \frac{1}{1+0}\\ &= 1.\end{aligned}$$

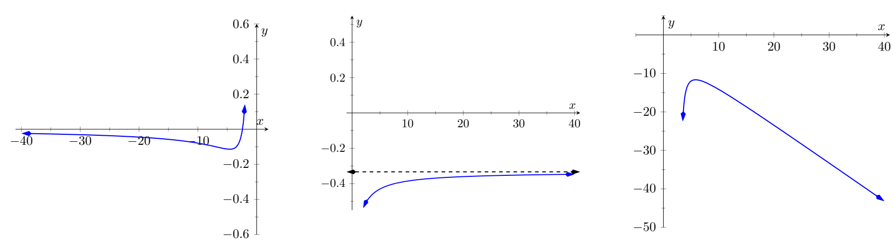
We can also use Theorem **??** directly; in this case $n=m$ so the limit is the ratio of the leading coefficients of the numerator and denominator, i.e., $1/1=1$.

Example 1.6.16 (Finding limits of rational functions)**.** Use Theorem **??** to evaluate each of the following limits.

1. $\displaystyle\lim_{x\to-\infty}\frac{x^2+2x-1}{x^3+1}$
2. $\displaystyle\lim_{x\to\infty}\frac{x^2+2x-1}{1-x-3x^2}$
3. $\displaystyle\lim_{x\to\infty}\frac{x^2-1}{3-x}$

Solution.

1. The highest power of x is in the denominator. Therefore, the limit is 0; see Figure **??**.
2. The highest power of x is x^2, which occurs in both the numerator and denominator. The limit is therefore the ratio of the coefficients of x^2, which is $-1/3$. See Figure **??**.
3. The highest power of x is in the numerator so the limit will be ∞ or $-\infty$. To see which, consider only the dominant terms from the numerator and denominator, which are x^2 and $-x$. The expression in the limit will behave like $x^2/(-x)=-x$ for large values of x. Therefore, the limit is $-\infty$. See Figure **??**.



(a) **(b)** **(c)**

Figure 1.6.17: Visualizing the functions in Example **??**.

With care, we can quickly evaluate limits at infinity for a large number of functions by considering the long run behaviour using"dominant terms" of $f(x)$. For instance, consider again $\lim\limits_{x\to\pm\infty}\frac{x}{\sqrt{x^2+1}}$, graphed in Figure **??**. The dominant terms are x in the numerator and $\sqrt{x^2}$ in the denominator. When x is very large, $x^2+1 \approx x^2$. Thus

$$\sqrt{x^2+1} \approx \sqrt{x^2} = |x| \qquad\qquad \frac{x}{\sqrt{x^2+1}} \approx \frac{x}{|x|}.$$

This expression is 1 when x is positive and -1 when x is negative. Hence we get asymptotes of $y=1$ and $y=-1$, respectively. We will show this more formally in the next example.

Example 1.6.18 (Finding a limit using dominant terms)**.** Confirm analytically that $y=1$ and $y=-1$ are the horizontal asymptote of $\lim\limits_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}}$, as graphed in Figure **??**.

Solution. The dominating term of f in the denominator is $\sqrt{x^2} = |x|$ so divide the numerator and denominator of f by $\sqrt{x^2}$, then take limits.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}} \\
&= \lim_{x\to\infty} \frac{\frac{x}{|x|}}{\sqrt{\frac{x^2+1}{x^2}}} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} \quad \text{for } x>0 \\
&= \frac{1}{\sqrt{1+0}} \\
&= 1.
\end{aligned}$$

As $x\to-\infty$, the only thing that changes is the value of $\frac{x}{|x|}$. For $x<0$, we have $\frac{x}{|x|} = -1$, making $\lim\limits_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = -1$. Therefore, the horizontal asymptotes are $y=1$ and $y=-1$.

1.6.4 Exercises

Terms and Concepts

1. True or False? If $\lim\limits_{x\to5} f(x) = \infty$, then we are implicitly stating that the limit exists. (Choose one: True / False)

Solution. False

2. True or False? If $\lim\limits_{x\to5} f(x) = 5$, then we are implicitly stating that the limit exists. (Choose one: True / False)

Solution. True

3. True or False? If $\lim\limits_{x\to1^-} f(x) = -\infty$, then $\lim\limits_{x\to1^+} f(x) = \infty$. (Choose one: True / False)

Solution. False

4. True or False? If $\lim_{x\to 5} f(x) = \infty$, then f has a vertical asymptote at $x = 5$. (Choose one: True / False)

Solution. True

5. True or False? $\infty/0$ is not an indeterminate form. (Choose one: True / False)

Solution. True

6. List five indeterminate forms.

[Essay Answer]

Solution. Answers will vary.

7. Construct a function with a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 5$.

[Essay Answer]

Solution. Answers will vary.

8. Let $\lim_{x\to 7} f(x) = \infty$. Explain how we know that f is or is not continuous at $x = 7$.

[Essay Answer]

Solution. The limit of f as x approaches 7 does not exist, hence f is not continuous. (Note: f could be defined at 7!)

In the following exercises, evaluate the given limits using the graph of the function.

9. $f(x) = \frac{1}{(x+1)^2}$ has the graph:

webwork-152-image-1.png

Use the graph of f to find:

(a) $\lim\limits_{x \to -1^-} f(x) = \boxed{\qquad\qquad}$

(b) $\lim\limits_{x \to -1^+} f(x) = \boxed{\qquad\qquad}$

10. $f(x) = \frac{1}{(x-3)(x-5)^2}$ has the graph:

webwork-153-image-1.png

Use the graph of f to find:

(a) $\lim\limits_{x\to 3^-} f(x) = \boxed{}$

(b) $\lim\limits_{x\to 3^+} f(x) = \boxed{}$

(c) $\lim\limits_{x\to 3} f(x) = \boxed{}$

(d) $\lim\limits_{x\to 5^-} f(x) = \boxed{}$

(e) $\lim\limits_{x\to 5^+} f(x) = \boxed{}$

(f) $\lim\limits_{x\to 5} f(x) = \boxed{}$

11. $f(x) = \frac{1}{e^x+1}$ has the graph:

webwork-154-image-1.png

Use the graph of f to find:

(a) $\lim\limits_{x\to-\infty} f(x) = \boxed{}$

(b) $\lim\limits_{x\to\infty} f(x) = \boxed{}$

(c) $\lim\limits_{x\to 0^-} f(x) = \boxed{}$

(d) $\lim\limits_{x\to 0^+} f(x) = \boxed{}$

12. $f(x) = x^2 \sin(\pi x)$ has the graph:

webwork-155-image-1.png

Use the graph of f to find:

(a) $\lim\limits_{x \to -\infty} f(x) =$

(b) $\lim\limits_{x \to \infty} f(x) =$

(c) $\lim\limits_{x \to 0^-} f(x) =$

(d) $\lim\limits_{x \to 0^+} f(x) =$

13. $f(x) = \cos(x)$ has the graph:

webwork-156-image-1.png

Use the graph of f to find:

(a) $\lim\limits_{x \to -\infty} f(x) = \boxed{}$

(b) $\lim\limits_{x \to \infty} f(x) = \boxed{}$

14. $f(x) = 2^x + 10$ has the graph:

webwork-157-image-1.png

Use the graph of f to find:

(a) $\lim\limits_{x\to-\infty} f(x) = \boxed{\quad\quad\quad}$

(b) $\lim\limits_{x\to\infty} f(x) = \boxed{\quad\quad\quad}$

In the following exercises, numerically approximate some limits.

15. Numerically approximate the following limits, where $f(x) = \frac{x^2-1}{x^2-x-6}$.

(a) $\displaystyle\lim_{x\to 3^-} f(x) = \boxed{}$

(b) $\displaystyle\lim_{x\to 3^+} f(x) = \boxed{}$

(c) $\displaystyle\lim_{x\to 3} f(x) = \boxed{}$

Solution. Tables will vary.

(a)

x	$f(x)$
2.9	-15.1224
2.99	-159.120
2.999	-1599.12

It seems $\displaystyle\lim_{x\to 3^-} f(x) = -\infty$.

(b)

x	$f(x)$
3.1	16.8823
3.01	160.880
3.001	1600.88

It seems $\displaystyle\lim_{x\to 3^+} f(x) = \infty$.

(c) It seems $\displaystyle\lim_{x\to 3} f(x)$does not exist.

16. Numerically approximate the following limits, where $f(x) = \frac{x^2+5x-36}{x^3-5x^2+3x+9}$.

(a) $\displaystyle\lim_{x\to 3^-} f(x) = \boxed{}$

(b) $\displaystyle\lim_{x\to 3^+} f(x) = \boxed{}$

(c) $\displaystyle\lim_{x\to 3} f(x) = \boxed{}$

Solution. Tables will vary.

(a)

x	$f(x)$
2.9	-335.641
2.99	-30350.6
2.999	-3003500

It seems $\displaystyle\lim_{x\to 3^-} f(x) = -\infty$.

(b)

x	$f(x)$
3.1	-265.609
3.01	-29650.6
3.001	-2996500

It seems $\displaystyle\lim_{x\to 3^+} f(x) = -\infty$.

(c) It seems $\displaystyle\lim_{x\to 3} f(x)$is $-\infty$.

17. Numerically approximate the following limits, where $f(x) = \frac{x^2-11x+30}{x^3-4x^2-3x+18}$.

(a) $\lim\limits_{x\to 3^-} f(x) = \boxed{}$

(b) $\lim\limits_{x\to 3^+} f(x) = \boxed{}$

(c) $\lim\limits_{x\to 3} f(x) = \boxed{}$

Solution. Tables will vary.

(a)

x	$f(x)$
2.9	132.857
2.99	12124.4
2.999	1201240

It seems $\lim\limits_{x\to 3^-} f(x) = \infty$.

(b)

x	$f(x)$
3.1	108.039
3.01	11876.4
3.001	1198760

It seems $\lim\limits_{x\to 3^+} f(x) = \infty$.

(c) It seems $\lim\limits_{x\to 3} f(x)$is ∞.

18. Numerically approximate the following limits, where $f(x) = \frac{x^2-9x+18}{x^2-x-6}$.

(a) $\lim\limits_{x\to 3^-} f(x) = \boxed{}$

(b) $\lim\limits_{x\to 3^+} f(x) = \boxed{}$

(c) $\lim\limits_{x\to 3} f(x) = \boxed{}$

Solution. Tables will vary.

(a)

x	$f(x)$
2.9	-0.632653
2.99	-0.603206
2.999	-0.600320

It seems $\lim\limits_{x\to 3^-} f(x) = 0.6$.

(b)

x	$f(x)$
3.1	-0.568627
3.01	-0.596806
3.001	-0.599680

It seems $\lim\limits_{x\to 3^+} f(x) = 0.6$.

(c) It seems $\lim\limits_{x\to 3} f(x)$is -0.6.

In the following exercises, identify the horizontal and vertical asymptotes, if any, of the given function.

19. Identify the horizontal and vertical asymptotes, if any, of f where

$$f(x) = \frac{2x^2 - 2x - 4}{x^2 + x - 20}.$$

Submit your answer as a list, using commas. Each asymptote is line of the form `x=...` or `y=...`. If there are no such asymptotes, enter an answer of `NONE`.

Solution. There are horizontal/vertical asymptotes at $y = 2, x = 4, x = -5$.

20. Identify the horizontal and vertical asymptotes, if any, of f where

$$f(x) = \frac{-3x^2 - 9x - 6}{5x^2 - 10x - 15}.$$

Submit your answer as a list, using commas. Each asymptote is line of the form `x=...` or `y=...`. If there are no such asymptotes, enter an answer of `NONE`.

Solution. There are horizontal/vertical asymptotes at $y = -\frac{3}{5}, x = 3$.

21. Identify the horizontal and vertical asymptotes, if any, of f where

$$f(x) = \frac{x^2 + x - 12}{7x^3 - 14x^2 - 21x}.$$

Submit your answer as a list, using commas. Each asymptote is line of the form `x=...` or `y=...`. If there are no such asymptotes, enter an answer of `NONE`.

Solution. There are horizontal/vertical asymptotes at $y = 0, x = 0, x = -1$.

22. Identify the horizontal and vertical asymptotes, if any, of f where

$$f(x) = \frac{x^2 - 9}{9x - 9}.$$

Submit your answer as a list, using commas. Each asymptote is line of the form `x=...` or `y=...`. If there are no such asymptotes, enter an answer of `NONE`.

Solution. There are horizontal/vertical asymptotes at $x = 1$.

23. Identify the horizontal and vertical asymptotes, if any, of f where

$$f(x) = \frac{x^2 - 9}{9x + 27}.$$

Submit your answer as a list, using commas. Each asymptote is line of the form `x=...` or `y=...`. If there are no such asymptotes, enter an answer of `NONE`.

Solution. There are horizontal/vertical asymptotes at NONE.

24. Identify the horizontal and vertical asymptotes, if any, of f where

$$f(x) = \frac{x^2 - 1}{-x^2 - 1}.$$

Submit your answer as a list, using commas. Each asymptote is line of the form `x=...` or `y=...`. If there are no such asymptotes, enter an answer of `NONE`.

Solution. There are horizontal/vertical asymptotes at $y = -1$.

In the following exercises, evaluate the given limit.

25. $\displaystyle\lim_{x\to\infty} \frac{x^3 + 2x^2 + 1}{x - 5} =$

26. $\displaystyle\lim_{x\to\infty} \frac{x^3 + 2x^2 + 1}{5 - x} =$

27. $\displaystyle\lim_{x\to-\infty} \frac{x^3 + 2x^2 + 1}{x^2 - 5} =$

28. $\displaystyle\lim_{x\to-\infty} \frac{x^3 + 2x^2 + 1}{x^2 - 5} =$

The following exercises are review from prior sections.

29. Use an ε-δ proof to prove that

$$\lim_{x\to 1}(5x-2)=3$$

[Essay Answer]

Solution. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-1| < \delta$, $|f(x)-3| < \varepsilon$.

First, some preliminary investigation to find a suitable δ. Consider $|f(x)-3| < \varepsilon$:

$$\begin{aligned} |f(x)-3| &< \varepsilon \\ |(5x-2)-3| &< \varepsilon \\ |5x-5| &< \varepsilon \\ |x-1| &< \varepsilon/5 \end{aligned}$$

Since we want to start with $|x-1| < \delta$, this suggests we let $\delta = \varepsilon/5$.

Now we can apply the definition.

$$\begin{aligned} |x-1| &< \delta \\ |x-1| &< \varepsilon/5 \\ -\varepsilon/5 < x-1 &< \varepsilon/5 \\ -\varepsilon < 5(x-1) &< \varepsilon \\ -\varepsilon < 5x-5 &< \varepsilon \\ -\varepsilon < (5x-2)-3 &< \varepsilon \\ |(5x-2)-3| &< \varepsilon. \end{aligned}$$

In other words, $|x-1| < \delta$ implies $|(5x-2)-3| < \varepsilon$. This is what we needed to prove.

30. Let $\lim_{x\to 2} f(x) = 3$ and $\lim_{x\to 2} g(x) = -1$. Evaluate the following limits.

(a) $\lim_{x\to 2}(f+g)(x) = \boxed{}$

(b) $\lim_{x\to 2}(fg)(x) = \boxed{}$

(c) $\lim_{x\to 2}(f/g)(x) = \boxed{}$

(d) $\lim_{x\to 2} f(x)^{g(x)} = \boxed{}$

31. Let $f(x) = \begin{cases} x^2-1 & x<3 \\ x+5 & x\geq 3 \end{cases}$. Is f continuous everywhere? (Choose one: yes / no)

32. Evaluate the limit:

$\lim_{x\to e} \ln(x) = \boxed{}$

Chapter Summary

In this chapter we:

- defined the limit,
- found accessible ways to approximate their values numerically and graphically,
- developed a not-so-easy method of proving the value of a limit (ε–δ proofs),
- explored when limits do not exist,
- defined continuity and explored properties of continuous functions, and
- considered limits that involved infinity.

Why? Mathematics is famous for building on itself and calculus proves to be no exception. In the next chapter we will be interested in "dividing by 0." That is, we will want to divide a quantity by a smaller and smaller number and see what value the quotient approaches. In other words, we will want to find a limit. These limits will enable us to, among other things, determine *exactly* how fast something is moving when we are only given position information.

Later, we will want to add up an infinite list of numbers. We will do so by first adding up a finite list of numbers, then take a limit as the number of things we are adding approaches infinity. Surprisingly, this sum often is finite; that is, we can add up an infinite list of numbers and get, for instance, 42.

These are just two quick examples of why we are interested in limits. Many students dislike this topic when they are first introduced to it, but over time an appreciation is often formed based on the scope of its applicability.

Chapter 2

Derivatives

Chapter 1 introduced the most fundamental of calculus topics: the limit. This chapter introduces the second most fundamental of calculus topics: the derivative. Limits describe *where* a function is going; derivatives describe *how fast* the function is going.

2.1 Instantaneous Rates of Change: The Derivative

A common amusement park ride lifts riders to a height then allows them to freefall a certain distance before safely stopping them. Suppose such a ride drops riders from a height of 150 feet. Students of physics may recall that the height (in feet) of the riders, t seconds after freefall (and ignoring air resistance, etc.) can be accurately modeled by $f(t) = -16t^2 + 150$.

Using this formula, it is easy to verify that, without intervention, the riders will hit the ground when $f(t) = 0$ so at $t = 2.5\sqrt{1.5} \approx 3.06$ seconds. Suppose the designers of the ride decide to begin slowing the riders' fall after 2 seconds (corresponding to a height of $f(2) = 86$ ft). How fast will the riders be traveling at that time?

We have been given a *position* function, but what we want to compute is a velocity at a specific point in time, i.e., we want an **instantaneous velocity**. We do not currently know how to calculate this.

However, we do know from common experience how to calculate an **average velocity**. (If we travel 60 miles in 2 hours, we know we had an average velocity of 30 mph.) We looked at this concept in Section 1.1 when we introduced the difference quotient. We have

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{\text{âĂIJriseâĂİ}}{\text{âĂIJrunâĂİ}} = \text{average velocity}.$$

We can approximate the instantaneous velocity at $t = 2$ by considering the average velocity over some time period containing $t = 2$. If we make the time interval small, we will get a good approximation. (This fact is commonly used. For instance, high speed cameras are used to track fast moving objects. Distances are measured over a fixed number of frames to generate an accurate approximation of the velocity.)

Consider the interval from $t = 2$ to $t = 3$ (just before the riders hit the ground). On that interval, the average velocity is

$$\frac{f(3) - f(2)}{3 - 2} = \frac{6 - 86}{1} = -80\,\text{ft/s},$$

where the minus sign indicates that the riders are moving *down*. By narrowing the interval we consider, we will likely get a better approximation of the instantaneous velocity. On $[2, 2.5]$ we have

$$\frac{f(2.5)-f(2)}{2.5-2} = \frac{50-86}{0.5} = -72\,\text{ft/s}.$$

Units in Calculations In the above calculations, we left off the units until the end of the problem. You should always be sure that you label your answer with the correct units. For example, if $g(x)$ gave you the cost (in \$) of producing x widgets, the units on the difference quotient would be \$/widget.

We can do this for smaller and smaller intervals of time. For instance, over a time span of one tenth of a second, i.e., on $[2, 2.1]$, we have

$$\frac{f(2.1)-f(2)}{2.1-2} = \frac{79.44-86}{0.1} = -65.6\,\text{ft/s}.$$

Over a time span of one hundredth of a second, on $[2, 2.01]$, the average velocity is

$$\frac{f(2.01)-f(2)}{2.01-2} = \frac{85.3584-86}{0.01} = -64.16\,\text{ft/s}.$$

What we are really computing is the average velocity on the interval $[2, 2+h]$ for small values of h. That is, we are computing

$$\frac{f(2+h)-f(2)}{h}$$

where h is small.

What we really want is for $h = 0$, but this, of course, returns the familiar $0/0$ indeterminate form. So we employ a limit, as we did in Section 1.1.

We can approximate the value of this limit numerically with small values of h as seen in Table ??. It looks as though the velocity is approaching -64 ft/s.

h	Average Velocity (ft/s)
1	-80
0.5	-72
0.1	-65.6
0.01	-64.16
0.001	-64.016

Table 2.1.1: Approximating the instantaneous velocity with average velocities over a small time period h.

Computing the limit directly gives

$$\begin{aligned}
\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0}\frac{-16(2+h)^2+150-(-16(2)^2+150)}{h}\\
&= \lim_{h\to 0}\frac{-16(4+4h+h^2)+150-86}{h}\\
&= \lim_{h\to 0}\frac{-64-64h-16h^2+64}{h}\\
&= \lim_{h\to 0}\frac{-64h-16h^2}{h}
\end{aligned}$$

$$= \lim_{h\to 0} -64 - 16h$$
$$= -64.$$

Graphically, we can view the average velocities we computed numerically as the slopes of secant lines on the graph of f going through the points $(2, f(2))$ and $(2+h, f(2+h))$. In Figures **??**–**??**, the secant line corresponding to $h = 1$ is shown in three contexts. Figure **??** shows a "zoomed out" version of f with its secant line. In Figure **??**, we zoom in around the points of intersection between f and the secant line. Notice how well this secant line approximates f between those two points — it is a common practice to approximate functions with straight lines.

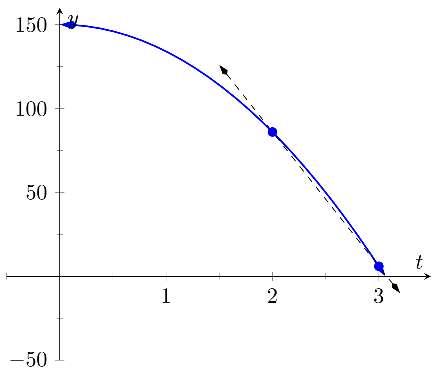


Figure 2.1.2: The function $f(x)$ and its secant line corresponding to $t = 2$ and $t = 3$.

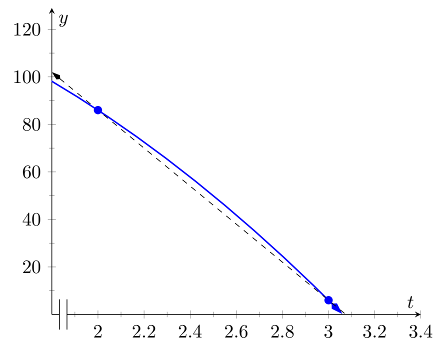


Figure 2.1.3: The function $f(x)$ and a secant line corresponding to $t = 2$ and $t = 3$, zoomed in near $t = 2$.

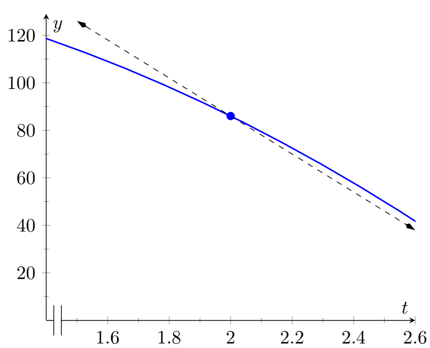


Figure 2.1.4: The function $f(x)$ with the same secant line, zoomed in further.

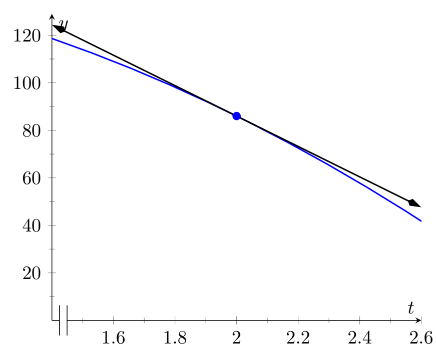


Figure 2.1.5: The function $f(x)$ with its tangent line at $t = 2$.

As $h \to 0$, these secant lines approach the **tangent line**, a line that goes through the point $(2, f(2))$ with the special slope of -64. In Figure **??** and Figure **??**, we zoom in around the point $(2, 86)$. We see the secant line, which approximates f well, but not as well the tangent line shown in Figure **??**.

We have just introduced a number of important concepts that we will flesh out more within this section. First, we formally define two of them.

Definition 2.1.6 (Derivative at a Point)**.** Let f be a continuous function on an open interval I and let c be in I. The **derivative of** f **at** c, denoted $f'(c)$, is

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h},$$

provided the limit exists. If the limit exists, we say that f **is differentiable at** c; if the limit does not exist, then f **is not differentiable at** c. If f is differentiable at every point in I, then f **is differentiable on** I.

Definition 2.1.7 (Tangent Line)**.** Let f be continuous on an open interval I and differentiable at c, for some c in I. The line with equation $\ell(x) = f'(c)(x-c)+f(c)$ is the **tangent line** to the graph of f at c; that is, it is the line through $(c, f(c))$ whose slope is the derivative of f at c.

Some examples will help us understand these definitions.

Example 2.1.8 (Finding derivatives and tangent lines)**.** Let $f(x) = 3x^2 + 5x - 7$. Find:

1. $f'(1)$
2. The equation of the tangent line to the graph of f at $x = 1$.
3. $f'(3)$
4. The equation of the tangent line to the graph f at $x = 3$.

Solution.

1. We compute this directly using Definition **??**.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{3(1+h)^2+5(1+h)-7-(3(1)^2+5(1)-7)}{h} \\ &= \lim_{h\to 0} \frac{3(1+2h+h^2)+5+5h-7-1}{h} \\ &= \lim_{h\to 0} \frac{3+6h+3h^2+5+5h-8}{h} \\ &= \lim_{h\to 0} \frac{3h^2+11h}{h} \\ &= \lim_{h\to 0} 3h+11 \\ &= 11. \end{aligned}$$

2. The tangent line at $x = 1$ has slope $f'(1)$ and goes through the point $(1, f(1)) = (1, 1)$. Thus the tangent line has equation, in point-slope form, $y = 11(x-1)+1$. In slope-intercept form we have $y = 11x - 10$.

3. Again, using the definition,

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\ &= \lim_{h\to 0} \frac{3(3+h)^2+5(3+h)-7-(3(3)^2+5(3)-7)}{h} \\ &= \lim_{h\to 0} \frac{3(9+6h+h^2)+15+3h-7-35}{h} \\ &= \lim_{h\to 0} \frac{27+18h+3h^2+15+3h-42}{h} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{3h^2+23h}{h} \\
&= \lim_{h\to 0} 3h+23 \\
&= 23.
\end{aligned}
$$

4. The tangent line at $x = 3$ has slope 23 and goes through the point $(3, f(3)) = (3, 35)$. Thus the tangent line has equation $y = 23(x-3)+35 = 23x-34$.

A graph of f is given in Figure **??** along with the tangent lines at $x = 1$ and $x = 3$.

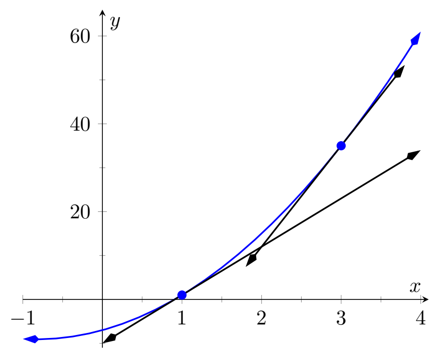


Figure 2.1.9: A graph of $f(x) = 3x^2+5x-7$ and its tangent lines at $x = 1$ and $x = 3$.

Another important line that can be created using information from the derivative is the *normal line.* It is perpendicular to the tangent line, hence its slope is the opposite–reciprocal of the tangent line's slope.

Definition 2.1.10 (Normal Line)**.** Let f be continuous on an open interval I and differentiable at c, for some c in I. The **normal line** to the graph of f at c is the line with equation

$$n(x) = \frac{-1}{f'(c)}(x-c)+f(c),$$

when $f'(c) \neq 0$. (When $f'(c) = 0$, the normal line is the vertical line through $(c, f(c))$; that is, $x = c$.)

Example 2.1.11 (Finding equations of normal lines)**.** Let $f(x) = 3x^2+5x-7$, as in Example **??**. Find the equations of the normal lines to the graph of f at $x = 1$ and $x = 3$.

Solution. In Example **??**, we found that $f'(1) = 11$. Hence at $x = 1$, the normal line will have slope $-1/11$. An equation for the normal line is

$$n(x) = \frac{-1}{11}(x-1)+1.$$

The normal line is plotted with $y = f(x)$ in Figure **??**. Note how the line looks perpendicular to f. (A key word here is "looks." Mathematically, we say that the normal line *is* perpendicular to f at $x = 1$ as the slope of the normal line is the opposite–reciprocal of the slope of the tangent line. However, normal lines may not always *look* perpendicular.

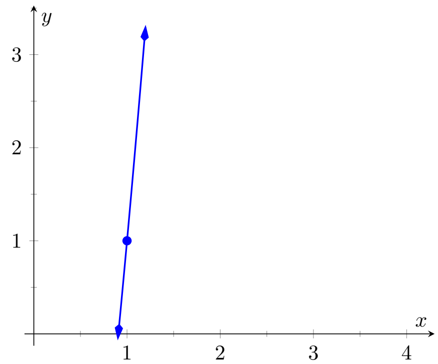


Figure 2.1.12: A graph of $f(x) = 3x^2 + 5x - 7$, along with its normal line at $x = 1$.

The aspect ratio of the picture of the graph plays a big role in this. When using graphing software, there is usually an option called `Zoom Square` that keeps the aspect ratio $1 : 1$

We also found that $f'(3) = 23$, so the normal line to the graph of f at $x = 3$ will have slope $-1/23$. An equation for the normal line is

$$n(x) = \frac{-1}{23}(x - 3) + 35.$$

Linear functions are easy to work with; many functions that arise in the course of solving real problems are not easy to work with. A common practice in mathematical problem solving is to approximate difficult functions with not-so-difficult functions. Lines are a common choice. It turns out that at any given point on the graph of a differentiable function f, the best linear approximation to f is its tangent line. That is one reason we'll spend considerable time finding tangent lines to functions.

One type of function that does not benefit from a tangent-line approximation is a line; it is rather simple to recognize that the tangent line to a line is the line itself. We look at this in the following example.

Example 2.1.13 (Finding the Derivative of a Line)**.** Consider $f(x) = 3x + 5$. Find the equation of the tangent line to f at $x = 1$ and $x = 7$.

Solution. We find the slope of the tangent line by using Definition **??**.

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \to 0} \frac{3(1+h) + 5 - (3+5)}{h} \\ &= \lim_{h \to 0} \frac{3h}{h} \\ &= \lim_{h \to 0} 3 \\ &= 3. \end{aligned}$$

We just found that $f'(1) = 3$. That is, we found the **instantaneous rate of change** of $f(x) = 3x + 5$ is 3. This is not surprising; lines are characterized by being the *only* functions with a *constant rate of change.* That rate of change is called the **slope** of the line. Since their rates of change are constant, their *instantaneous* rates of change are always the same; they are all the slope.

So given a line $f(x) = ax + b$, the derivative at any point x will be a; that is, $f'(x) = a$.

It is now easy to see that the tangent line to the graph of f at $x = 1$ is just f, with the same being true at $x = 7$.

We often desire to find the tangent line to the graph of a function without knowing the actual derivative of the function. While we will eventually be able to find derivatives of many common functions, the algebra and limit calculations on some functions are complex. Until we develop futher techniques, the best we may be able to do is approximate the tangent line. We demonstrate this in the next example.

Example 2.1.14 (Numerical Approximation of the Tangent Line)**.** Approximate the equation of the tangent line to the graph of $f(x) = \sin(x)$ at $x = 0$.

Solution. In order to find the equation of the tangent line, we need a slope and a point. The point is given to us: $(0, \sin(0)) = (0, 0)$. To compute the slope, we need the derivative. This is where we will make an approximation. Recall that

$$f'(0) \approx \frac{\sin(0 + h) - \sin(0)}{h}$$

for a small value of h. We choose (somewhat arbitrarily) to let $h = 0.1$. Thus

$$f'(0) \approx \frac{\sin(0.1) - \sin(0)}{0.1} \approx 0.9983.$$

Thus our approximation of the equation of the tangent line is $y = 0.9983(x - 0) + 0 = 0.9983x$; it is graphed in Figure **??**. The graph seems to imply the approximation is rather good.

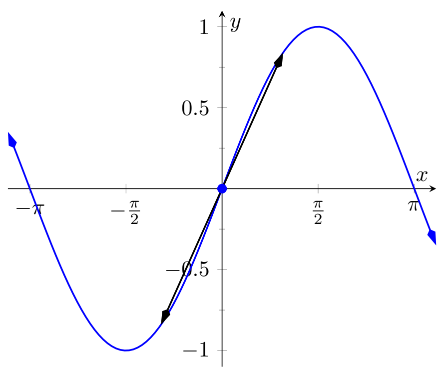


Figure 2.1.15: $f(x) = \sin(x)$ graphed with an approximation to its tangent line at $x = 0$.

Recall from Section **??** that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, meaning for values of x near 0, $\sin(x) \approx x$. Since the slope of the line $y = x$ is 1 at $x = 0$, it should seem reasonable that "the slope of $f(x) = \sin(x)$" is near 1 at $x = 0$. In fact, since we *approximated* the value of the slope to be 0.9983, we might guess the *actual value* is 1. We'll come back to this later.

Consider again Example **??**. To find the derivative of f at $x = 1$, we needed to evaluate a limit. To find the derivative of f at $x = 3$, we needed to again evaluate a limit. We have this process:

$$\begin{array}{c}\text{input specific}\\ \text{number } c\end{array} \longrightarrow \boxed{\begin{array}{c}\text{do something}\\ \text{to } f \text{ and } c\end{array}} \longrightarrow \begin{array}{c}\text{return}\\ \text{number } f'(c)\end{array}$$

This process describes a **function**; given one input (the value of c), we return exactly one output (the value of $f'(c)$). The "do something" box is where the tedious work (taking limits) of this function occurs.

Instead of applying this function repeatedly for different values of c, let us apply it just once to the variable x. We then take a limit just once. The process now looks like:

$$\begin{array}{c}\text{input}\\ \text{variable } x\end{array} \longrightarrow \boxed{\begin{array}{c}\text{do something}\\ \text{to } f \text{ and } x\end{array}} \longrightarrow \begin{array}{c}\text{return}\\ \text{function } f'(x)\end{array}$$

The output is the **derivative function,** $f'(x)$. The $f'(x)$ function will take a number c as input and return the derivative of f at c. This calls for a definition.

Definition 2.1.16 (Derivative Function)**.** Let f be a differentiable function on an open interval I. The function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

is **the derivative of** f.

Let $y = f(x)$. The following notations all represent the derivative:

$$f'(x) \;=\; y' \;=\; \frac{dy}{dx} \;=\; \frac{df}{dx} \;=\; \frac{d}{dx}(f) \;=\; \frac{d}{dx}(y).$$

Important: The notation $\frac{dy}{dx}$ is one symbol; it is *not* the fraction "dy/dx". The notation, while somewhat confusing at first, was chosen with care. A fraction-looking symbol was chosen because the derivative has many fraction-like properties. Among other places, we see these properties at work when we talk about the units of the derivative, when we discuss the Chain Rule, and when we learn about integration (topics that appear in later sections and chapters).

Examples will help us understand this definition.

Example 2.1.17 (Finding the derivative of a function)**.** Let $f(x) = 3x^2+5x-7$ as in Example **??**. Find $f'(x)$.

Solution. We apply Definition **??**.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{3(x+h)^2+5(x+h)-7-(3x^2+5x-7)}{h} \\ &= \lim_{h\to 0} \frac{3h^2+6xh+5h}{h} \\ &= \lim_{h\to 0} 3h+6x+5 \\ &= 6x+5 \end{aligned}$$

So $f'(x) = 6x+5$. Recall earlier we found that $f'(1) = 11$ and $f'(3) = 23$. Note our new computation of $f'(x)$ affirm these facts.

Example 2.1.18 (Finding the derivative of a function)**.** Let $f(x) = \frac{1}{x+1}$. Find $f'(x)$.

Solution. We apply Definition **??**.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}$$

Now find common denominator then subtract; pull $1/h$ out front to facilitate reading.

$$= \lim_{h\to 0} \frac{1}{h} \cdot \left(\frac{x+1}{(x+1)(x+h+1)} - \frac{x+h+1}{(x+1)(x+h+1)} \right)$$

Now simplify algebraically.

$$\begin{aligned} &= \lim_{h\to 0} \frac{1}{h} \cdot \left(\frac{x+1-(x+h+1)}{(x+1)(x+h+1)} \right) \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \left(\frac{-h}{(x+1)(x+h+1)} \right) \end{aligned}$$

Finally, apply the limit.

$$\begin{aligned} &= \lim_{h\to 0} \frac{-1}{(x+1)(x+h+1)} \\ &= \frac{-1}{(x+1)(x+1)} \\ &= \frac{-1}{(x+1)^2} \end{aligned}$$

So $f'(x) = \frac{-1}{(x+1)^2}$. To practice using our notation, we could also state

$$\frac{d}{dx}\left(\frac{1}{x+1}\right) = \frac{-1}{(x+1)^2}.$$

Example 2.1.19 (Finding the derivative of a function)**.** Find the derivative of $f(x) = \sin(x)$.

Solution. Before applying Definition **??**, note that once this is found, we can find the actual tangent line to $f(x) = \sin(x)$ at $x = 0$, whereas we settled for an approximation in Example **??**.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} && \text{Derivative definition} \\ &= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} && \text{Angle addition identity} \\ &= \lim_{h\to 0} \frac{\sin(x)(\cos(h)-1) + \cos(x)\sin(h)}{h} && \text{Regrouped and factored} \\ &= \lim_{h\to 0} \left(\frac{\sin(x)(\cos(h)-1)}{h} + \frac{\cos(x)\sin(h)}{h} \right) && \text{Split into two fractions} \\ &= \lim_{h\to 0} \sin(x) \cdot \lim_{h\to 0} \frac{\cos(h)-1}{h} \\ &\quad + \lim_{h\to 0} \cos(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h} && \text{Product/sum limit rules} \\ &= \sin(x) \cdot 0 + \cos(x) \cdot 1 && \text{Applied Theorem ??} \\ &= \cos(x)\ ! \end{aligned}$$

We have found that when $f(x) = \sin(x)$, $f'(x) = \cos(x)$. This should be somewhat surprising; the result of a tedious limit process on the sine function is a nice function. Then again, perhaps this is not entirely surprising. The sine function is periodic — it repeats itself on regular intervals. Therefore its rate of change also repeats itself on the same regular intervals. We should have known the derivative would be periodic; we now know exactly which periodic function it is.

Thinking back to Example **??**, we can find the slope of the tangent line to $f(x) = \sin(x)$ at $x = 0$ using our derivative. We approximated the slope as 0.9983; we now know the slope is *exactly* $\cos(0) = 1$.

Example 2.1.20 (Finding the derivative of a piecewise defined function)**.** Find the derivative of the absolute value function,

$$f(x) = |x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}.$$

See Figure **??**.

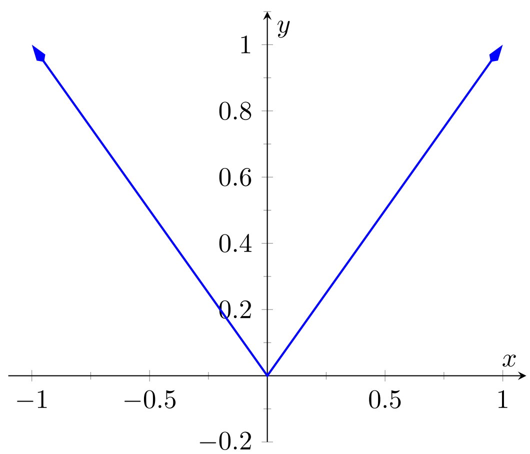


Figure 2.1.21: The absolute value function, $f(x) = |x|$. Notice how the slope of the lines (and hence the tangent lines) abruptly changes at $x = 0$.

Solution. We need to evaluate $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$. As f is piecewise-defined, we need to consider separately the limits when $x < 0$ and when $x > 0$.

When $x < 0$:

$$\begin{aligned} \frac{d}{dx}(-x) &= \lim_{h\to 0} \frac{-(x+h)-(-x)}{h} \\ &= \lim_{h\to 0} \frac{-h}{h} \\ &= \lim_{h\to 0} -1 \\ &= -1. \end{aligned}$$

When $x > 0$, a similar computation shows that $\frac{d}{dx}(x) = 1$.

We need to also find the derivative at $x = 0$. By the definition of the derivative at a point, we have

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}.$$

Since $x = 0$ is the point where our function's definition switches from one piece to other, we need to consider left and right-hand limits. Consider the following, where we compute the left and right hand limits side by side.

$$\begin{aligned} &\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0^-} \frac{-h-0}{h} \\ &= \lim_{h\to 0^-} -1 \\ &= -1 \end{aligned}$$

$$\begin{aligned} &\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0^+} \frac{h-0}{h} \\ &= \lim_{h\to 0^+} 1 \\ &= 1 \end{aligned}$$

The last lines of each column tell the story: the left and right hand limits are not equal. Therefore the limit does not exist at 0, and f is not differentiable at 0. So we have

$$f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}.$$

At $x = 0$, $f'(x)$ does not exist; there is a jump discontinuity at 0; see Figure **??**. So $f(x) = |x|$ is differentiable everywhere except at 0.

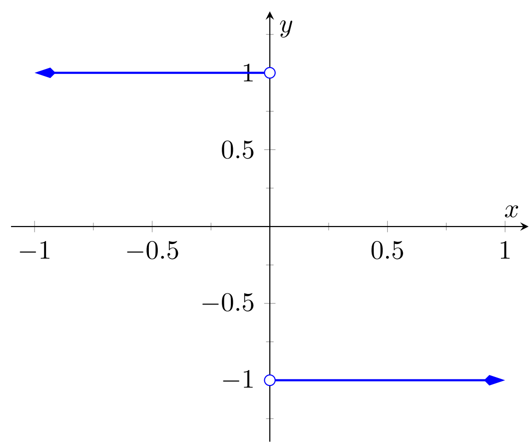


Figure 2.1.22: A graph of the derivative of $f(x) = |x|$.

The point of non-differentiability came where the piecewise defined function switched from one piece to the other. Our next example shows that this does not always cause trouble.

Example 2.1.23 (Finding the derivative of a piecewise defined function)**.** Find the derivative of $f(x)$, where $f(x) = \begin{cases} \sin(x) & x \le \pi/2 \\ 1 & x > \pi/2 \end{cases}$. See Figure **??**.

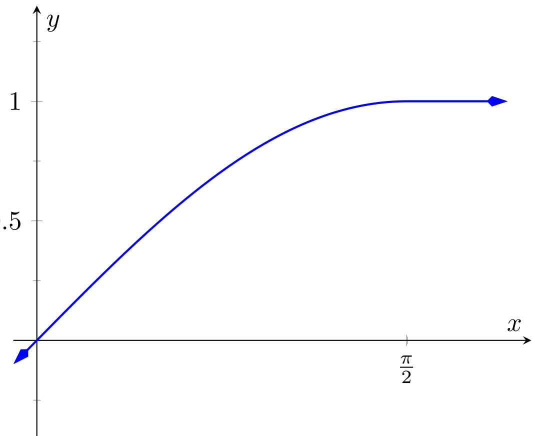


Figure 2.1.24: A graph of $f(x)$ as defined in Example **??**.

Solution. Using Example **??**, we know that when $x < \pi/2$, $f'(x) = \cos(x)$. It is easy to verify that when $x > \pi/2$, $f'(x) = 0$; consider:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1-1}{h} = \lim_{h\to 0} 0 = 0.$$

So far we have

$$f'(x) = \begin{cases} \cos(x) & x < \pi/2 \\ 0 & x > \pi/2 \end{cases}.$$

We still need to find $f'(\pi/2)$. Notice at $x = \pi/2$ that both pieces of f' are 0, meaning we can state that $f'(\pi/2) = 0$.

Being more rigorous, we can again evaluate the difference quotient limit at $x = \pi/2$, utilizing again left- and right-hand limits. We will begin with the left-hand limit:

$$\begin{aligned} &\lim_{h\to 0^-} \frac{f(\pi/2+h)-f(\pi/2)}{h} \\ &= \lim_{h\to 0^-} \frac{\sin(\pi/2+h)-\sin(\pi/2)}{h} \\ &= \lim_{h\to 0^-} \frac{\sin(\frac{\pi}{2})\cos(h)+\sin(h)\cos(\frac{\pi}{2})-\sin(\frac{\pi}{2})}{h} \\ &= \lim_{h\to 0^-} \frac{1\cdot\cos(h)+\sin(h)\cdot 0-1}{h} \\ &= \lim_{h\to 0^-} \frac{\cos(h)-1}{h} \cdot \lim_{h\to 0^-} \frac{\sin(h)}{h} \\ &= 1\cdot 0 \\ &= 0 \end{aligned}$$

Notice we used Special Limits to finally evaluate the limit.

Now we [redacted] will find the right-hand limit:

[redacted]

$$\begin{aligned} &\lim_{h\to 0^+} \frac{f(\pi/2+h)-f(\pi/2)}{h} \\ &= \lim_{h\to 0^+} \frac{1-1}{h} \\ &= \lim_{h\to 0^+} \frac{0}{h} \\ &= 0 \end{aligned}$$

Since both the left and right hand limits are 0 at $x = \pi/2$, the limit exists and $f'(\pi/2)$ exists (and is 0). Therefore we can fully write f' as

$$f'(x) = \begin{cases} \cos(x) & x \le \pi/2 \\ 0 & x > \pi/2 \end{cases}.$$

See Figure **??** for a graph of this derivative function.

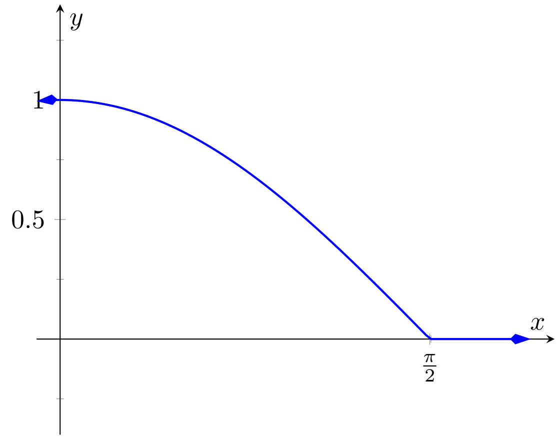


Figure 2.1.25: A graph of $f'(x)$ in Example **??**.

Recall we pseudo-defined a continuous function as one in which we could sketch its graph without lifting our pencil. We can give a pseudo-definition for differentiability as well: it is a continuous function that does not have any "sharp corners" or a vertical tangent line. One such sharp corner is shown in Figure **??**. Even though the function f in Example **??** is piecewise-defined, the transition is "smooth" hence it is differentiable. Note how in the graph of f in Figure **??** it is difficult to tell when f switches from one piece to the other; there is no "corner."

This section defined the derivative; in some sense, it answers the question of "What *is* the derivative?" Section **??** addresses the question "What does the derivative *mean*?"

2.1.1 Exercises

Terms and Concepts

1. True or False? Let f be a position function. The average rate of change on $[a, b]$ is the slope of the line through the points $(a, f(a))$ and $(b, f(b))$. (Choose one: True / False)

2. True or False? The definition of the derivative of a function at a point involves taking a limit. (Choose one: True / False)

3. In your own words, explain the difference between the average rate of change and instantaneous rate of change.

[Essay Answer]

Solution. Answers will vary.

4. In your own words, explain the difference between Definitions **??** and **??**.

[Essay Answer]

Solution. Answers will vary.

5. Let $y = f(x)$. Give three different notations equivalent to "$f'(x)$."

[Essay Answer]

Solution. Answers will vary.

In the following exercises, use the definition of the derivative to compute the derivative of the given function.

6. Use the definition of the derivative to compute the derivative of f where

$$f(x) = 6.$$

[Essay Answer]

7. Use the definition of the derivative to compute the derivative of f where

$$f(x) = 2x.$$

[Essay Answer]

8. Use the definition of the derivative to compute the derivative of f where

$$f(t) = 4 - 3t.$$

[Essay Answer]

9. Use the definition of the derivative to compute the derivative of g where

$$g(x) = x^2.$$

[Essay Answer]

10. Use the definition of the derivative to compute the derivative of f where

$$f(x) = 3x^2 - x + 4.$$

[Essay Answer]

11. Use the definition of the derivative to compute the derivative of r where

$$r(x) = \frac{1}{x}.$$

[Essay Answer]

12. Use the definition of the derivative to compute the derivative of r where

$$r(s) = \frac{1}{s-2}.$$

[Essay Answer]

In the following exercises, a function and an x-value c are given. (Note: these functions are the same as those given in Exercises 2.1.1.6 through 2.1.1.12.) Find the tangent line to the graph of the function at c, and find the normal line to the graph of the function at c.

13. Let $f(x) = 6$.

At $x = -2$, the equation of the tangent line to the graph of f is ______.

At $x = -2$, the equation of the normal line to the graph of f is ______.

14. Let $f(x) = 2x$.

At $x = 3$, the equation of the tangent line to the graph of f is ______.

At $x = 3$, the equation of the normal line to the graph of f is ______.

15. Let $f(x) = 4 - 3x$.

At $x = 7$, the equation of the tangent line to the graph of f is ______.

At $x = 7$, the equation of the normal line to the graph of f is ______.

16. Let $g(x) = x^2$.

At $x = 2$, the equation of the tangent line to the graph of g is ______.

At $x = 2$, the equation of the normal line to the graph of g is ______.

17. Let $f(x) = 3x^2 - x + 4$.

At $x = -1$, the equation of the tangent line to the graph of f is ______.

At $x = -1$, the equation of the normal line to the graph of f is ______.

18. Let $r(x) = \frac{1}{x}$.

At $x = -2$, the equation of the tangent line to the graph of r is $\boxed{}$.

At $x = -2$, the equation of the normal line to the graph of r is $\boxed{}$.

19. Let $r(x) = \frac{1}{x-2}$.

At $x = 3$, the equation of the tangent line to the graph of r is $\boxed{}$

At $x = 3$, the equation of the normal line to the graph of r is $\boxed{}$

In the following exercises, a function f and an x-value a are given. Approximate the equation of the tangent line to the graph of f at $x = a$ by numerically approximating $f'(a)$, using $h = 0.1$.

20. Let $f(x) = x^2 + 2x + 1$. Approximate the equation of the tangent line to the graph of f at $a = 3$ by numerically approximating $f'(a)$, using $h = 0.1$.

$\boxed{}$

21. Let $f(x) = \frac{10}{x+1}$. Approximate the equation of the tangent line to the graph of f at $a = 9$ by numerically approximating $f'(a)$, using $h = 0.1$.

$\boxed{}$

22. Let $f(x) = e^x$. Approximate the equation of the tangent line to the graph of f at $a = 2$ by numerically approximating $f'(a)$, using $h = 0.1$.

$\boxed{}$

23. Let $f(x) = \cos(x)$. Approximate the equation of the tangent line to the graph of f at $a = 0$ by numerically approximating $f'(a)$, using $h = 0.1$.

$\boxed{}$

24. The graph of $f(x) = x^2 - 1$ is shown.

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(a) Use the graph to approximate the slope of the tangent line to f at the following points:
At $(-1, 0)$, the slope is approximately $\boxed{\quad}$.
At $(0, -1)$, the slope is approximately $\boxed{\quad}$.
At $(2, 3)$, the slope is approximately $\boxed{\quad}$.

(b) Using the definition, $f'(x) =$ $\boxed{\quad}$.

(c) Find the slope of the tangent line at the points $(-1, 0)$, $(0, -1)$ and $(2, 3)$.
At $(-1, 0)$, the slope is $\boxed{\quad}$.
At $(0, -1)$, the slope is $\boxed{\quad}$.
At $(2, 3)$, the slope is $\boxed{\quad}$.

25. The graph of $f(x) = \frac{1}{x+1}$ is shown.

webwork-200-image-1.png

(a) Use the graph to approximate the slope of the tangent line to f at the following points:

At $(0, 1)$, the slope is approximately $\boxed{}$.

At $(1, 0.5)$, the slope is approximately $\boxed{}$.

(b) Using the definition, $f'(x) = \boxed{}$.

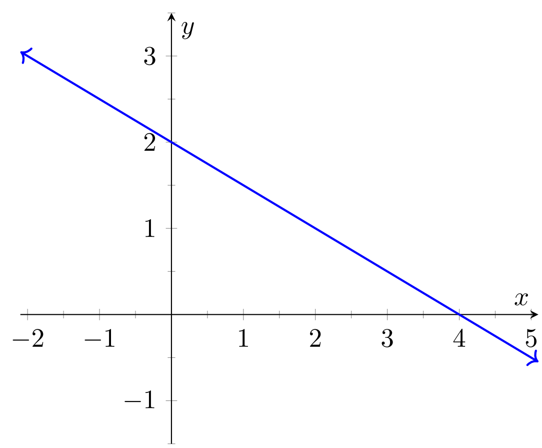
(c) Find the slope of the tangent line at the points $(0, 1)$ and $(1, 0.5)$.

At $(0, 1)$, the slope is $\boxed{}$.

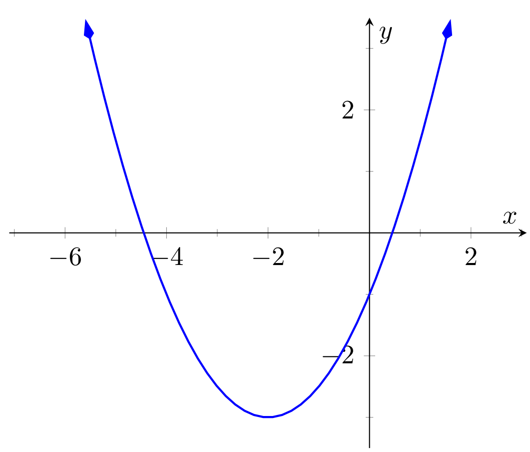
At $(1, 0.5)$, the slope is $\boxed{}$.

In the following exercises, a graph of a function $f(x)$ is given. Using the graph, sketch $f'(x)$.

26.



27.



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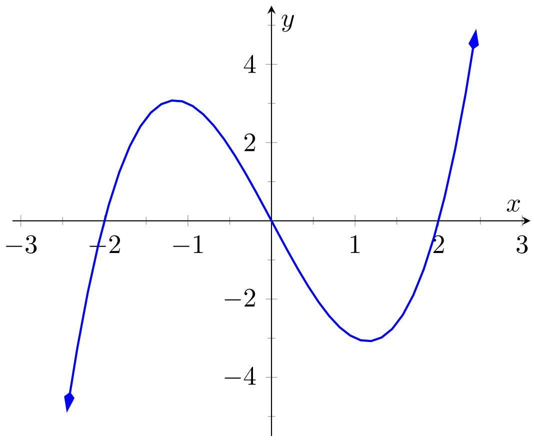

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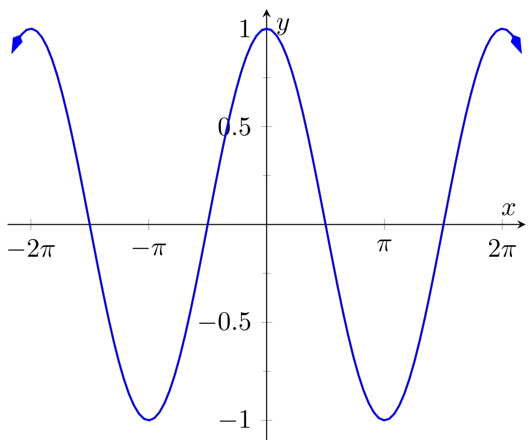
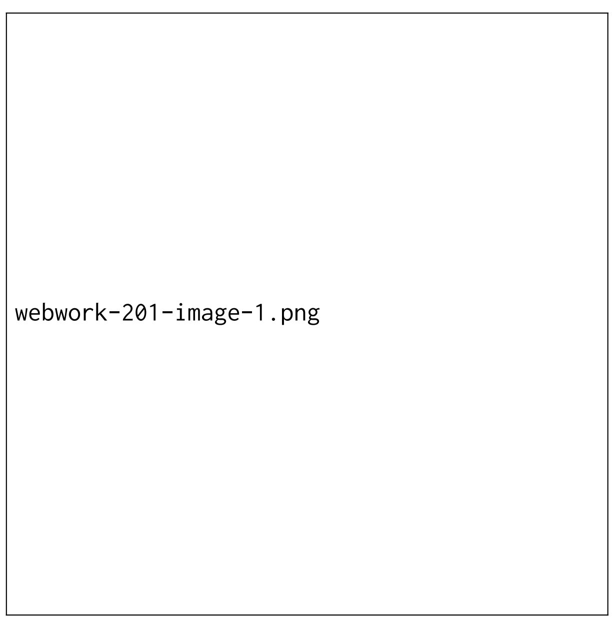


30. Using the graph of $g(x)$ below, answer the following questions.



(a) Where is $g(x) > 0$?

(b) Where is $g(x) < 0$?

(c) Where is $g(x) = 0$?

(d) Where is $g'(x) < 0$?

(e) Where is $g'(x) > 0$?

(f) Where is $g'(x) = 0$?

Review

31. Approximate $\lim_{x\to 5} \dfrac{x^2+2x-35}{x^2-10.5x+27.5}$.

32. Use the Bisection Method to approximate, accurate to two decimal places, the root of $g(x) = x^3 + x^2 + x - 1$ on $[0.5, 0.6]$.

33. Give interval or union of intervals on which each of the following functions are continuous.

(a) $\dfrac{1}{e^x+1}$

(b) $\dfrac{1}{x^2-1}$

(c) $\sqrt{5-x}$

(d) $\sqrt{5-x^2}$

34. Use the graph of $f(x)$ provided to answer the following.

webwork-205-image-1.png

(a) $\lim_{x \to -3^-} f(x) = \boxed{\quad\quad}$

(b) $\lim_{x \to -3^+} f(x) = \boxed{\quad\quad}$

(c) $\lim_{x \to -3} f(x) = \boxed{\quad\quad}$

(d) Where is f continuous? $\boxed{\quad\quad\quad\quad\quad\quad}$

2.2 Interpretations of the Derivative

Section **??** defined the derivative of a function and gave examples of how to compute it using its definition (i.e., using limits). The section also started with a brief motivation for this definition, that is, finding the instantaneous velocity of a falling object given its position function. Section **??** will give us more accessible tools for computing the derivative; tools that are easier to use than repeated use of limits.

This section falls in between the "What is the definition of the derivative?" and "How do I compute the derivative?" sections. Here we are concerned with "What does the derivative mean?", or perhaps, when read with the right emphasis, "What *is* the derivative?" We offer two interconnected interpretations of the derivative, hopefully explaining why we care about it and why it is worthy of study.

2.2.1 Interpretation of the Derivative as Instantaneous Rate of Change

Section **??** started with an example of using the position of an object (in this case, a falling amusement-park rider) to find the object's velocity. This type of example is often used when introducing the derivative because we tend to readily recognize that velocity is the *instantaneous rate of change in position.*

In general, if f is a function of x, then $f'(x)$ measures the instantaneous rate of change of f with respect to x. Put another way, the derivative answers "When x changes, at what rate does f change?" Thinking back to the amusement-park ride, we asked "When time changed, at what rate did the height change?" and found the answer to be "By -64 feet per second."

Now imagine driving a car and looking at the speedometer, which reads "60 mph." Five minutes later, you wonder how far you have traveled. Certainly, lots of things could have happened in those 5 minutes; you could have intentionally sped up significantly, you might have come to a complete stop, you might have slowed to 20 mph as you passed through construction. But suppose that you know, as the driver, none of these things happened. You know you maintained a fairly consistent speed over those 5 minutes. What is a good approximation of the distance traveled?

One could argue the *only* good approximation, given the information provided, would be based on "distance = rate × time." In this case, we assume a constant rate of 60 mph with a time of 5 minutes or 5/60 of an hour. Hence we would approximate the distance traveled as 5 miles.

Referring back to the falling amusement-park ride, knowing that at $t = 2$ the velocity was -64 ft/s, we could reasonably approximate that 1 second later the riders' height would have dropped by about 64 feet. Knowing that the riders were *accelerating* as they fell would inform us that this is an *under-approximation.* If all we knew was that $f(2) = 86$ and $f'(2) = -64$, we'd know that we'd have to stop the riders quickly otherwise they would hit the ground!

In both of these cases, we are using the instantaneous rate of change to predict future values of the output.

2.2.2 Units of the Derivative

It is useful to recognize the *units* of the derivative function. If y is a function of x, i.e., $y = f(x)$ for some function f, and y is measured in feet and x in seconds, then the units of $y' = f'$ are "feet per second," commonly written as "ft/s." In general, if y is measured in units P and x is measured in units Q, then y' will be measured in units "P per Q", or "P/Q." Here we see the fraction-like behavior of the derivative in the notation: the units of $\frac{dy}{dx}$ are $\frac{\text{units of } y}{\text{units of } x}$.

Example 2.2.1 (The meaning of the derivative: World Population)**.** Let $P(t)$ represent the world population t minutes after 12:00 a.m., January 1, 2012. It is fairly accurate to say that $P(0) = 7{,}028{,}734{,}178$ (`www.prb.org`). It is also fairly accurate to state that $P'(0) = 156$; that is, at midnight on January 1, 2012, the population of the world was growing by about 156 *people per minute* (note the units). Twenty days later (or 28,800 minutes later) we could reasonably assume the population grew by about $28{,}800 \cdot 156 = 4{,}492{,}800$ people.

Example 2.2.2 (The meaning of the derivative: Manufacturing)**.** The term *widget* is an economic term for a generic unit of manufacturing output. Suppose a company produces widgets and knows that the market supports a price of \$10 per widget. Let $P(n)$ give the profit, in dollars, earned by manufacturing and selling n widgets. The company likely cannot make a (positive) profit making just one widget; the start-up costs will likely exceed \$10. Mathematically, we would write this as $P(1) < 0$.

What do $P(1000) = 500$ and $P'(1000) = 0.25$ mean? Approximate $P(1100)$. ∎

Solution. The equation $P(1000) = 500$ means that selling 1000 widgets returns a profit of \$500. We interpret $P'(1000) = 0.25$ as meaning that when we are selling 1000 widgets, the profit is increasing at rate of \$0.25 per widget (the units are "dollars per widget.") Since we have no other information to use, our best approximation for $P(1100)$ is:

$$\begin{aligned} P(1100) &\approx P(1000) + P'(1000) \times 100 \\ &= \$500 + (100 \text{ widgets }) \cdot \$0.25/\text{widget} \\ &= \$525. \end{aligned}$$

We approximate that selling 1100 widgets returns a profit of \$525.

The previous examples made use of an important approximation tool that we first used in our previous "driving a car at 60 mph" example at the beginning of this section. Five minutes after looking at the speedometer, our best approximation for distance traveled assumed the rate of change was constant. In Examples **??** and **??** we made similar approximations. We were given rate of change information which we used to approximate total change. Notationally, we would say that

$$f(c+h) \approx f(c) + f'(c) \cdot h.$$

This approximation is best when h is "small." Small is a relative term; when dealing with the world population, $h = 22$ days $= 28{,}800$ minutes is small in comparison to years. When manufacturing widgets, 100 widgets is small when one plans to manufacture thousands.

2.2.3 The Derivative and Motion

One of the most fundamental applications of the derivative is the study of motion. Let $s(t)$ be a position function, where t is time and $s(t)$ is distance. For instance, s could measure the height of a projectile or the distance an object has traveled.

> **Convention with s** Using $s(t)$ to represent position is a fairly common mathematical convention. It is also common to use s to represent arc length.

Let's let $s(t)$ measure the distance traveled, in feet, of an object after t seconds of travel. Then $s'(t)$ has units "feet per second," and $s'(t)$ measures the *instantaneous rate of distance change with repsect to time* — it measures *velocity.*

Now consider $v(t)$, a velocity function. That is, at time t, $v(t)$ gives the velocity of an object. The derivative of v, $v'(t)$, gives the *instantaneous rate of velocity change with respect to time — acceleration.* (We often think of acceleration in terms of cars: a car may "go from 0 to 60 in 4.8 seconds." This is an *average* acceleration, a measurement of how quickly the velocity changed.) If velocity is measured in feet per second, and time is measured in seconds, then the units of acceleration (i.e., the units of $v'(t)$) are "feet per second per second," or (ft/s)/s. We often shorten this to "feet per second squared," or $\frac{\text{ft}}{\text{s}^2}$, but this tends to obscure the meaning of the units.

Perhaps the most well known acceleration is that of gravity. In this text, we use $g = 32\,\text{ft/s}^2$ or $g = 9.8\,\text{m/s}^2$. What do these numbers mean?

A constant acceleration of $32\,\frac{\text{ft/s}}{\text{s}}$ means that the velocity changes by 32 ft/s each second. For instance, let $v(t)$ measure the velocity of a ball thrown straight up into the air, where v has units ft/s and t is measured in seconds. The ball will have a positive velocity while traveling upwards and a negative velocity while falling down. The acceleration is thus $-32\,\text{ft/s}^2$. If $v(1) = 20\,\text{ft/s}$, then 1 second later, the velocity will have decreased by 32 ft/s; that is, $v(2) = -12\,\text{ft/s}$.

We can continue: $v(3) = -44\,\text{ft/s}$. Working backward, we can also figure that $v(0) = 52\,\text{ft/s}$.

These ideas are so important we write them out as a Key Idea.

Key Idea 2.2.3 (The Derivative and Motion)**.**

1. Let $s(t)$ be the position function of an object. Then $s'(t) = v(t)$ is the velocity function of the object.

2. Let $v(t)$ be the velocity function of an object. Then $v'(t) = a(t)$ is the acceleration function of the object.

2.2.4 Interpretation of the Derivative as the Slope of the Tangent Line

We now consider the second interpretation of the derivative given in this section. This interpretation is not independent from the first by any means; many of the same concepts will be stressed, just from a slightly different perspective.

Given a function $y = f(x)$, the difference quotient $\frac{f(c+h)-f(c)}{h}$ gives a change in y values divided by a change in x values; i.e., it is a measure of the "rise over run," or "slope," of the secant line that goes through two points on the graph of f: $(c, f(c))$ and $(c+h, f(c+h))$. As h shrinks to 0, these two points come close together; in the limit we find $f'(c)$, the slope of a special line called the tangent line that intersects f only once near $x = c$.

Lines have a constant rate of change, their slope. Nonlinear functions do not have a constant rate of change, but we can measure their *instantaneous rate of change* at a given x value c by computing $f'(c)$. We can get an idea of how f is behaving by looking at the slopes of its tangent lines. We explore this idea in the following example.

Example 2.2.4 (Understanding the derivative: the rate of change)**.** Consider $f(x) = x^2$ as shown in Figure **??**. It is clear that at $x = 3$ the function is growing faster than at $x = 1$, as it is steeper at $x = 3$. How much faster is it growing at 3 compared to 1?

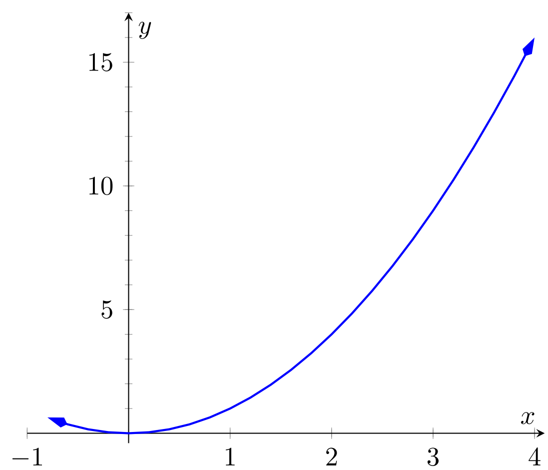


Figure 2.2.5: A graph of $f(x) = x^2$.

Solution. We can answer this exactly (and quickly) after Section **??**, where we learn to quickly compute derivatives. For now, we will answer graphically, by considering the slopes of the respective tangent lines.

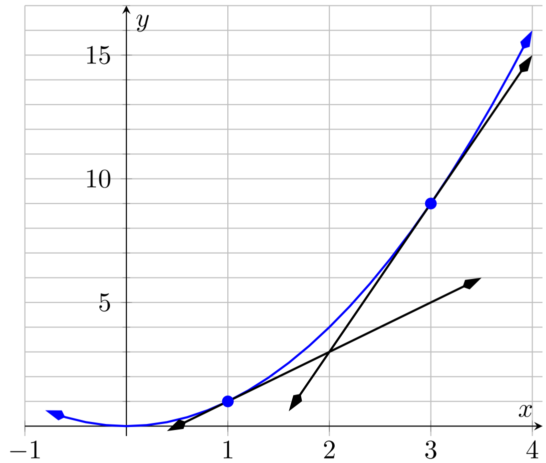


Figure 2.2.6: A graph of $f(x) = x^2$ and tangent lines at $x = 1$ and $x = 3$.

With practice, one can fairly effectively sketch tangent lines to a curve at a particular point. In Figure **??**, we have sketched the tangent lines to f at $x = 1$ and $x = 3$, along with a grid to help us measure the slopes of these lines. At $x = 1$, the slope is 2; at $x = 3$, the slope is 6. Thus we can say not only is f growing faster at $x = 3$ than at $x = 1$, it is growing *three times as fast.*

Example 2.2.7 (Understanding the graph of the derivative)**.** Consider the graph of $f(x)$ and its derivative, $f'(x)$, in Figure **??**. Use these graphs to find the slopes of the tangent lines to the graph of f at $x = 1$, $x = 2$, and $x = 3$.

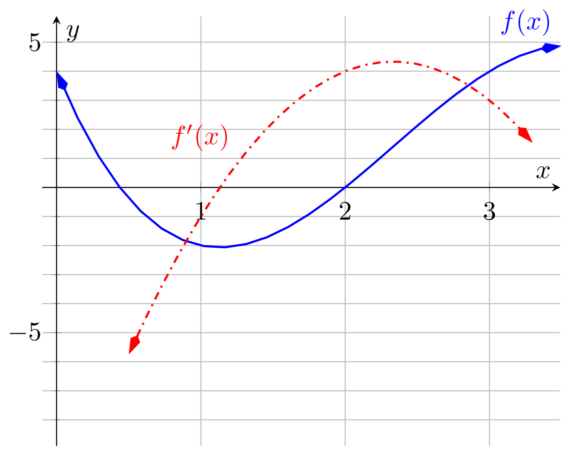


Figure 2.2.8: Graphs of f and f' in Example **??**.

Solution. To find the appropriate slopes of tangent lines to the graph of f, we need to look at the corresponding values of f'.

- The slope of the tangent line to f at $x = 1$ is $f'(1)$; this looks to be about -1.
- The slope of the tangent line to f at $x = 2$ is $f'(2)$; this looks to be about 4.
- The slope of the tangent line to f at $x = 3$ is $f'(3)$; this looks to be about 3.

Using these slopes, tangent line segments to f are sketched in Figure **??**. Included on the graph of f' in this figure are points where $x = 1$, $x = 2$ and $x = 3$ to help better visualize the y value of f' at those points.

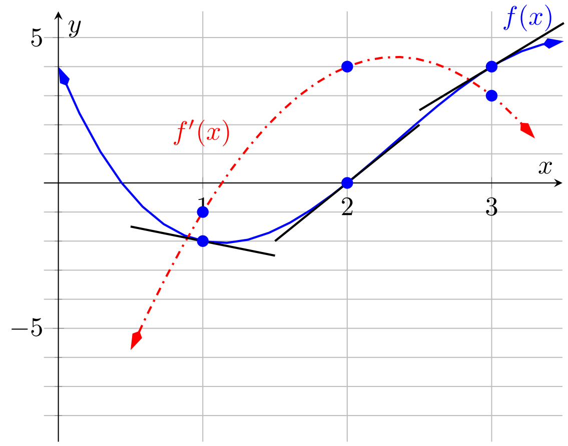


Figure 2.2.9: Graphs of f and f' in Example **??**.

Example 2.2.10 (Approximation with the derivative)**.** Consider again the graph of $f(x)$ and its derivative $f'(x)$ in Example **??**. Use the tangent line to f at $x = 3$ to approximate the value of $f(3.1)$.

Solution. Figure **??** shows the graph of f along with its tangent line, zoomed in at $x = 3$. Notice that near $x = 3$, the tangent line makes an excellent approximation of f. Since lines are easy to deal with, often it works well to approximate a function with its tangent line. (This is especially true when you don't actually know much about the function at hand, as we don't in this example.)

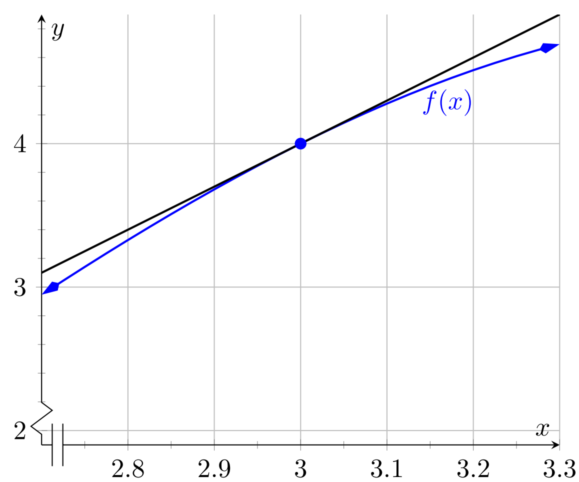


Figure 2.2.11: Zooming in on f at $x = 3$ for the function given in Examples **??** and **??**.

While the tangent line to f was drawn in Example **??**, it was not explicitly computed. Recall that the tangent line to f at $x = c$ is $y = f'(c)(x-c)+f(c)$. While f is not explicitly given, by the graph it looks like $f(3) = 4$. Recalling that $f'(3) = 3$, we can compute the tangent line to be approximately $y = 3(x-3)+4$. It is often useful to leave the tangent line in point-slope form.

To use the tangent line to approximate $f(3.1)$, we simply evaluate y at 3.1 instead of f.

$$\begin{aligned} f(3.1) &\approx y(3.1) \\ &= 3(3.1-3)+4 \\ &= 0.1 \cdot 3 + 4 \\ &= 4.3. \end{aligned}$$

We approximate $f(3.1) \approx 4.3$.

To demonstrate the accuracy of the tangent line approximation, we now state that in Example **??**, $f(x) = -x^3 + 7x^2 - 12x + 4$. We can evaluate $f(3.1) = 4.279$. Had we known f all along, certainly we could have just made this computation. In reality, we often only know two things:

1. What $f(c)$ is, for some value of c, and
2. what $f'(c)$ is.

For instance, we can easily observe the location of an object and its instantaneous velocity at a particular point in time. We do not have a "function f" for the location, just an observation. This is enough to create an approximating function for f.

This last example has a direct connection to our approximation method explained above after Example **??**. We stated there that

$$f(c+h) \approx f(c) + f'(c) \cdot h.$$

If we know $f(c)$ and $f'(c)$ for some value $x = c$, then computing the tangent line at $(c, f(c))$ is easy: $y(x) = f'(c)(x-c) + f(c)$. In Example **??**, we used the tangent line to approximate a value of f. Let's use the tangent line at $x = c$ to approximate a value of f near $x = c$; i.e., compute $y(c+h)$ to approximate $f(c+h)$, assuming again that h is "small." Note:

$$\begin{aligned} y(c+h) &= f'(c)\left((c+h)-c\right) + f(c) \\ &= f'(c) \cdot h + f(c). \end{aligned}$$

This is the exact same approximation method used above! Not only does it make intuitive sense, as explained above, it makes analytical sense, as this approximation method is simply using a tangent line to approximate a function's value.

The importance of understanding the derivative cannot be understated. When f is a function of x, $f'(x)$ measures the instantaneous rate of change of f with respect to x and gives the slope of the tangent line to f at x.

2.2.5 Exercises

Terms and Concepts

1. What is the instantaneous rate of change of position called?

$\boxed{}$

2. Given a function $y = f(x)$, in your own words describe how to find the units of $f'(x)$.

[Essay Answer]

Solution. Answers will vary.

3. What functions have a constant rate of change?

[ ]

4. Given $f(5) = 10$ and $f'(5) = 2$, approximate $f(6)$.
$f(6) \approx$ [ ]

5. Given $P(100) = -67$ and $P'(100) = 5$, approximate $P(110)$.
$P(110) \approx$ [ ]

6. Given $z(25) = 187$ and $z'(25) = 17$, approximate $z(20)$.
$z(20) \approx$ [ ]

7. Knowing $f(10) = 25$ and $f'(10) = 5$ and the methods described in this section, which approximation is likely to be most accurate: $f(10.1)$, $f(11)$, or $f(20)$? Explain your reasoning.

[Essay Answer]

Solution. $f(10.1)$ is likely most accurate, as accuracy is lost the farther from $x = 10$ we go.

8. Given $f(7) = 26$ and $f(8) = 22$, approximate $f'(7)$.
$f'(7) \approx$ [ ]

9. Given $H(0) = 17$ and $H(2) = 29$, approximate $H'(2)$.
$H'(2) \approx$ [ ]

10. Let $V(x)$ measure the volume, in decibels, measured inside a restaurant with x customers. What are the units of $V'(x)$?

[ ]

11. Let $v(t)$ measure the velocity, in ft/s, of a car moving in a straight line t seconds after starting. What are the units of $v'(t)$?

[ ]

12. The height H, in feet, of a river is recorded t hours after midnight, April 1. What are the units of $H'(t)$?

[ ]

13. P is the profit, in thousands of dollars, of producing and selling c cars.

(a) What are the units of $P'(c)$?
(b) What is likely true of $P(0)$?

[Essay Answer]

Solution.

(a) thousands of dollars per car
(b) It is likely that $P(0) < 0$. That is, negative profit for not producing any cars.

14. T is the temperature in degrees Fahrenheit, h hours after midnight on July 4 in Sidney, NE.

(a) What are the units of $T'(h)$?

(b) Is $T'(8)$ likely greater than or less than 0? Why?

(c) Is $T(8)$ likely greater than or less than 0? Why?

[Essay Answer]

Solution.

(a) degrees Fahrenheit per hour

(b) It is likely that $T'(8) > 0$ since at 8 in the morning, the temperature is likely rising.

(c) It is very likely that $T(8) > 0$, as at 8 in the morning on July 4, we would expect the temperature to be well above 0.

In the following exercises, graphs of functions f and g are given. Identify which function is the derivative of the other.

15. In the figure below, (Choose one: f is the derivative of g / g is the derivative of f).

16. In the figure below, (Choose one: f is the derivative of g / g is the derivative of f).

webwork-221-image-1.png

17. In the figure below, (Choose one: f is the derivative of g / g is the derivative of f).

webwork-222-image-1.png

18. In the figure below, (Choose one: f is the derivative of g / g is the derivative of f).

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Review

19. Use the definition of the derivative to compute the derivative of f, where $f(x) = 5x^2$.

[Essay Answer]

20. Use the definition of the derivative to compute the derivative of f, where $f(x) = (x - 2)^3$.

[Essay Answer]

21. Numerically approximate the value of $f'(\pi)$ where $f(x) = \cos(x)$.

22. Numerically approximate the value of $f'(9)$ where $f(x) = \sqrt{}(x)$.

2.3 Basic Differentiation Rules

The derivative is a powerful tool but is admittedly awkward given its reliance on limits. Fortunately, one thing mathematicians are good at is *abstraction.* For instance, instead of continually finding derivatives at a point, we abstracted and found the derivative function.

Let's practice abstraction on linear functions, $y = mx + b$. What is y'? Without limits, recognize that linear function are characterized by being functions with a constant rate of change (the slope). The derivative, y', gives the instantaneous rate of change; with a linear function, this is constant, m. Thus $y' = m$.

Let's abstract once more. Let's find the derivative of the general quadratic function, $f(x) = ax^2 + bx + c$. Using the definition of the derivative, we have:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c)}{h} \\ &= \lim_{h\to 0} \frac{ax^2 + 2ahx + ah^2 + bx + bh + c - ax^2 - bx - c)}{h} \\ &= \lim_{h\to 0} \frac{ah^2 + 2ahx + bh}{h} \\ &= \lim_{h\to 0} ah + 2ax + b \\ &= 2ax + b. \end{aligned}$$

So if $y = 6x^2 + 11x - 13$, we can immediately compute $y' = 12x + 11$.

In this section (and in some sections to follow) we will learn some of what mathematicians have already discovered about the derivatives of certain functions and how derivatives interact with arithmetic operations. We start with a theorem.

Theorem 2.3.1 (Derivatives of Common Functions)**.**

Constant Rule $\frac{d}{dx}(c) = 0$*, where* c *is a constant.*

Power Rule $\frac{d}{dx}(x^n) = nx^{n-1}$*, where* n *is an integer,* $n > 0$*.*

Other common functions $\frac{d}{dx}(\sin(x)) = \cos(x)$

$\frac{d}{dx}(\cos(x)) = -\sin(x)$

$\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$*, for* $x > 0$*.*

This theorem starts by stating an intuitive fact: constant functions have zero rate of change as they are *constant*. Therefore their derivative is 0 (they change at the rate of 0). The theorem then states some fairly amazing things. The Power Rule states that the derivatives of Power Functions (of the form $y = x^n$) are very straightforward: multiply by the power, then subtract 1 from the power. We see something incredible about the function $y = e^x$: it is its own derivative. We also see a new connection between the sine and cosine functions.

One special case of the Power Rule is when $n = 1$, i.e., when $f(x) = x$. What is $f'(x)$? According to the Power Rule,

$$f'(x) = \frac{d}{dx}(x) = \frac{d}{dx}\left(x^1\right) = 1 \cdot x^0 = 1.$$

In words, we are asking "At what rate does f change with respect to x?" Since f *is* x, we are asking "At what rate does x change with respect to x?" The answer is: 1. They change at the same rate. We can also interpret the derivative as the slope of the tangent line to the function at a point $(c, f(c))$. Since $f(x) = x$ is a linear function with constant slope 1, we can say that the derivative of $f(x) = x$ is $f'(x) = 1$.

Let's practice using this theorem.

Example 2.3.2 (Using common derivative rules to find, and use, derivatives)**.**
Let $f(x) = x^3$.

1. Find $f'(x)$.
2. Find the equation of the line tangent to the graph of f at $x = -1$.
3. Use the tangent line to approximate $(-1.1)^3$.
4. Sketch f, f' and the tangent line from 2 on the same axis.

Solution.

1. The Power Rule states that if $f(x) = x^3$, then $f'(x) = 3x^2$.
2. To find the equation of the line tangent to the graph of f at $x = -1$, we need a point and the slope. The point is $(-1, f(-1)) = (-1, -1)$. The slope is $f'(-1) = 3$. Thus the tangent line has equation $y = 3(x - (-1)) + (-1) = 3x + 2$.
3. We can use the tangent line to approximate $(-1.1)^3$ since -1.1 is close to -1. We have
$$(-1.1)^3 \approx 3(-1.1) + 2 = -1.3.$$
We can easily find the actual value: $(-1.1)^3 = -1.331$.
4. See Figure **??**.

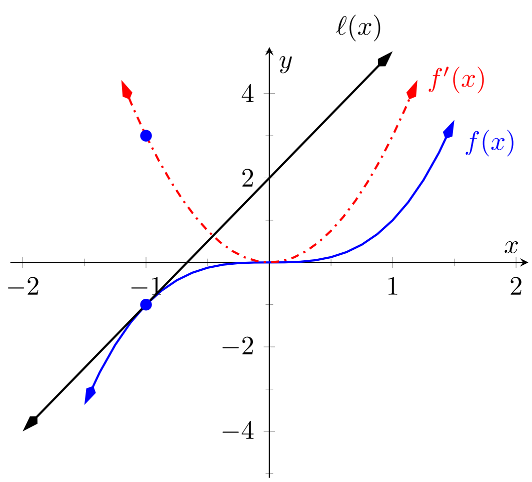

Figure 2.3.3: A graph of $f(x) = x^3$, along with its derivative $f'(x) = 3x^2$ and its tangent line at $x = -1$.

Theorem **??** gives useful information, but we will need much more. For instance, using the theorem, we can easily find the derivative of $y = x^3$, but it does not tell how to compute the derivative of $y = 2x^3$, $y = x^3 + \sin(x)$ nor $y = x^3 \sin(x)$. The following theorem helps with the first two of these examples (the third is answered in the next section).

Theorem 2.3.4 (Properties of the Derivative)**.** *Let f and g be differentiable on an open interval I and let c be a real number. Then:*

Sum/Difference Rule

$$\begin{aligned}\frac{d}{dx}(f(x) \pm g(x)) &= \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))\\ &= f'(x) \pm g'(x)\end{aligned}$$

Constant Multiple Rule

$$\begin{aligned}\frac{d}{dx}(c\cdot f(x)) &= c\cdot\frac{d}{dx}(f(x))\\ &= c\cdot f'(x).\end{aligned}$$

Theorem **??** allows us to find the derivatives of a wide variety of functions. It can be used in conjunction with the Power Rule to find the derivatives of any polynomial. Recall in Example **??** that we found, using the limit definition, the derivative of $f(x) = 3x^2 + 5x - 7$. We can now find its derivative without expressly using limits:

$$\begin{aligned}\frac{d}{dx}\big(3x^2+5x+7\big) &= 3\frac{d}{dx}\big(x^2\big) + 5\frac{d}{dx}(x) + \frac{d}{dx}(7)\\ &= 3\cdot 2x + 5\cdot 1 + 0\\ &= 6x+5.\end{aligned}$$

We were a bit pedantic here, showing every step. Normally we would do all the arithmetic and steps in our head and readily find $\frac{d}{dx}\big(3x^2+5x+7\big) = 6x+5$.

Example 2.3.5 (Using the tangent line to approximate a function value)**.** Let $f(x) = \sin(x) + 2x + 1$. Approximate $f(3)$ using an appropriate tangent line.

Solution. This problem is intentionally ambiguous; we are to *approximate* using an *appropriate* tangent line. How good of an approximation are we seeking? What does "appropriate" mean?

In the "real world," people solving problems deal with these issues all time. One must make a judgment using whatever seems reasonable. In this example, the actual answer is $f(3) = \sin(3) + 7$, where the real problem spot is $\sin(3)$. What is $\sin(3)$?

Since 3 is close to π, we can assume $\sin(3) \approx \sin(\pi) = 0$. Thus one guess is $f(3) \approx 7$. Can we do better? Let's use a tangent line as instructed and examine the results; it seems best to find the tangent line at $x = \pi$.

Using Theorem **??** we find $f'(x) = \cos(x) + 2$. The slope of the tangent line is thus $f'(\pi) = \cos(\pi) + 2 = 1$. Also, $f(\pi) = 2\pi + 1 \approx 7.28$. So the tangent line to the graph of f at $x = \pi$ is $y = 1(x-\pi) + 2\pi + 1 = x + \pi + 1 \approx x + 4.14$. Evaluated at $x = 3$, our tangent line gives $y = 3 + 4.14 = 7.14$. Using the tangent line, our final approximation is that $f(3) \approx 7.14$.

Using a calculator, we get an answer accurate to four places after the decimal: $f(3) = 7.1411$. Our initial guess was 7; our tangent line approximation was more accurate, at 7.14.

The point is *not* "Here's a cool way to do some math without a calculator." Sure, that might be handy sometime, but your phone could probably give you the answer. Rather, the point is to say that tangent lines are a good way of approximating, and many scientists, engineers and mathematicians often face problems too hard to solve directly. So they approximate.

The graphs in Figure **??** shows the graph of the function $f(x)$ along with the tangent line constructed at $x = \pi$. The graph in Figure **??** shows the same tangent line and function. Once zoomed in, you can barely distinguish the tangent line from the function. This indicates that the tangent line is a good a approximation of the function so long as we are near the point of tangency.

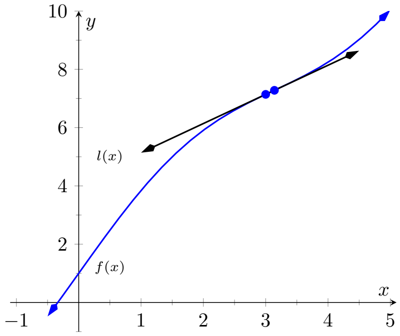

Figure 2.3.6: A graph of $f(x) = \sin(x) + 2x + 1$ along with its tangent line approximation at $x = \pi$.

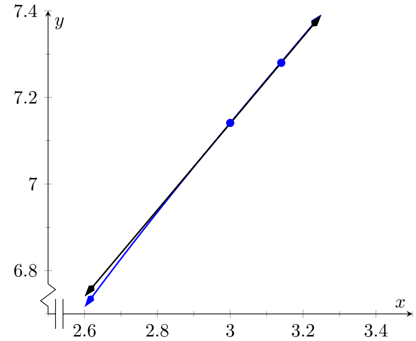

Figure 2.3.7: A graph of $f(x) = \sin(x) + 2x + 1$ along with its tangent line approximation at $x = \pi$, zoomed in.

2.3.1 Higher Order Derivatives

The derivative of a function f is itself a function, therefore we can take its derivative. The following definition gives a name to this concept and introduces its notation.

Definition 2.3.8 (Higher Order Derivatives)**.** Let $y = f(x)$ be a differentiable function on I.

1. The *second derivative* of f is:

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^y x}{d2^y} = y''.$$

2. The *third derivative* of f is:

$$f'''(x) = \frac{d}{dx}(f''(x)) = \frac{d}{dx}\left(\frac{d^y x}{d2^y}\right) = \frac{d^y x}{d3^y} = y'''.$$

3. The *nth derivative* of f is:

$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right) = \frac{d}{dx}\left(\frac{d^y x}{dn - 1^y}\right) = \frac{d^y x}{dn^y} = y^{(n)}.$$

Higher Order Derivative Caveat Definition **??** comes with the caveat "Where the corresponding limits exist." With f differentiable on I, it is possible that f' is *not* differentiable on all of I, and so on.

In general, when finding the fourth derivative and on, we resort to the $f^{(4)}(x)$ notation, not $f''''(x)$; after a while, too many ticks is too confusing.

Let's practice using this new concept.

Example 2.3.9 (Finding higher order derivatives)**.** Find the first four derivatives of the following functions:

1. $f(x) = 4x^2$
2. $f(x) = \sin(x)$
3. $f(x) = 5e^x$

Solution.

1. Using the Power and Constant Multiple Rules, we have: $f'(x) = 8x$. Continuing on, we have

$$f''(x) = \frac{d}{dx}(8x) = 8 \qquad f'''(x) = 0 \qquad f^{(4)}(x) = 0.$$

Notice how all successive derivatives will also be 0.

2. We employ Theorem **??** repeatedly.

$$\begin{aligned} f'(x) &= \cos(x) & f'''(x) &= -\cos(x) \\ f''(x) &= -\sin(x) & f^{(4)}(x) &= \sin(x) \end{aligned}$$

Note how we have come right back to $f(x)$ again. (Can you quickly figure what $f^{(23)}(x)$ is?)

3. Employing Theorem **??** and the Constant Multiple Rule, we can see that

$$f'(x) = f''(x) = f'''(x) = f^{(4)}(x) = 5e^x.$$

2.3.2 Interpreting Higher Order Derivatives

What do higher order derivatives *mean*? What is the practical interpretation?

Our first answer is a bit wordy, but is technically correct and beneficial to understand. That is,

> The second derivative of a function f is the rate of change of the rate of change of f.

One way to grasp this concept is to let f describe a position function. Then, as stated in Key Idea **??**, f' describes the rate of position change: velocity. We now consider f'', which describes the rate of velocity change. Sports car enthusiasts talk of how fast a car can go from 0 to 60 mph; they are bragging about the *acceleration* of the car.

We started this chapter with amusement-park riders free-falling with position function $f(t) = -16t^2 + 150$. It is easy to compute $f'(t) = -32t$ ft/s and $f''(t) = -32$ (ft/s)/s. We may recognize this latter constant; it is the acceleration due to gravity. In keeping with the unit notation introduced in the previous section, we say the units are "feet per second per second." This is usually shortened to "feet per second squared," written as "ft/s^2."

It can be difficult to consider the meaning of the third, and higher order, derivatives. The third derivative is "the rate of change of the rate of change of the rate of change of f." That is essentially meaningless to the uninitiated. In the context of our position/velocity/acceleration example, the third derivative is the "rate of change of acceleration," commonly referred to as "jerk."

Make no mistake: higher order derivatives have great importance even if their practical interpretations are hard (or "impossible") to understand. The mathematical topic of **series** makes extensive use of higher order derivatives.