

Example 282 Derivatives of vector-valued functions

Let $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$. Compute $\vec{r}'(t)$ and $\vec{r}'(\pi/2)$. Sketch $\vec{r}'(\pi/2)$ with its initial point at the origin and at $\vec{r}(\pi/2)$.

SOLUTION We compute \vec{r}' as $\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$. At $t = \pi/2$, we have $\vec{r}'(\pi/2) = \langle -1, 0, 1 \rangle$. Figure 9.10 shows two graphs of $\vec{r}(t)$, from different perspectives, with $\vec{r}'(\pi/2)$ plotted with its initial point at the origin and at $\vec{r}(\pi/2)$.

In Examples 281 and 282, sketching a particular derivative with its initial point at the origin did not seem to reveal anything significant. However, when we sketched the vector with its initial point on the corresponding point on the graph, we did see something significant: the vector appeared to be *tangent* to the graph. We have not yet defined what “tangent” means in terms of curves in space; in fact, we use the derivative to define this term.

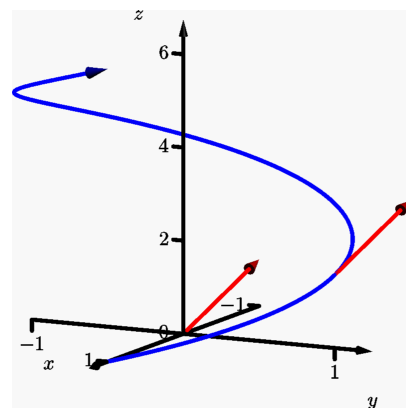


Figure 9.10: Viewing a vector-valued function, and its derivative at one point.

Definition 47 Tangent Vector, Tangent Line

Let $\vec{r}(t)$ be a differentiable vector-valued function on an open interval I containing c , where $\vec{r}'(c) \neq \vec{0}$.

1. A vector \vec{v} is **tangent to the graph of $\vec{r}(t)$ at $t = c$** if \vec{v} is parallel to $\vec{r}'(c)$.
2. The **tangent line** to the graph of $\vec{r}(t)$ at $t = c$ is the line through $\vec{r}(c)$ with direction parallel to $\vec{r}'(c)$. An equation of the tangent line is

$$\vec{\ell}(t) = \vec{r}(c) + t\vec{r}'(c).$$

Example 283 Finding tangent lines to curves in space

Let $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ on $[-1.5, 1.5]$. Find the vector equation of the line tangent to the graph of \vec{r} at $t = -1$.

SOLUTION To find the equation of a line, we need a point on the line and the line's direction. The point is given by $\vec{r}(-1) = \langle -1, 1, -1 \rangle$. (To be clear, $\langle -1, 1, -1 \rangle$ is a *vector*, not a point, but we use the point “pointed to” by this vector.)

The direction comes from $\vec{r}'(-1)$. We compute, component-wise, $\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. Thus $\vec{r}'(-1) = \langle 1, -2, 3 \rangle$.

The vector equation of the line is $\vec{\ell}(t) = \langle -1, 1, -1 \rangle + t\langle 1, -2, 3 \rangle$. This line and $\vec{r}(t)$ are sketched, from two perspectives, in Figure 9.11 (a) and (b).

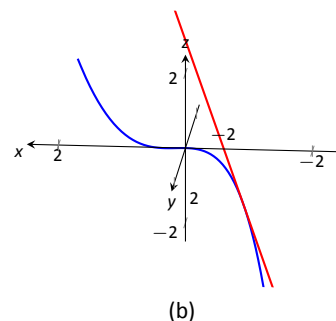
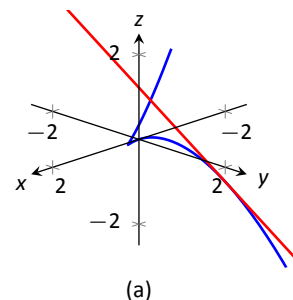


Figure 9.11: Graphing a curve in space with its tangent line.

Notes: